Time-Varying Formation Control for General Linear Multi-agent Systems with Switching Directed Topologies

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EE261 2020 Spring

Motivation

Broad Application

- Light show performance
- Communication relay
- Contour mapping

Limitation of Previous Works

- Time-invariant topology
- No theoretical guarantee
- No physical experiment

Graph Representation

Elements of A Graph

- Graph

$$G = \{Q, E, W\}$$

- Node set

$$Q = \{q_1, \ldots, q_N\}, \ |Q| = N$$

- Edge set

$$E = \{(q_i, q_j) : q_i, q_j \in Q\}$$

- Weights

$$w_{ji} > 0 \iff q_{ij} = (q_i, q_j) \in E$$

Graph Representation

- Weighted adjacency matrix

$$W = [w_{ij}] \in \mathbb{R}^{N \times N}$$

- Degree matrix $D \in \mathbb{R}^{N \times N}$ with $\deg_{\text{in}}(q_i) = \sum_{j=1}^{N} w_{ij}$

$$D = \begin{bmatrix} \deg_{\text{in}}(q_1) & & \\ & \ddots & \\ & & \deg_{\text{in}}(q_N) \end{bmatrix}$$

Graph Laplacian

$$L = D - W \qquad \mathcal{L} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} W D^{-\frac{1}{2}}$$

Intuition: Diffusion & Connectivity

$$Lx = \underbrace{\sum_{q_{ij} \in E} (x(j) - x(i))}_{\text{sum of outgoing derivatives for } q_i} \qquad x^{\mathsf{T}} L x = \underbrace{\sum_{q_{ij} \in E} (x(i) - x(j))^2 \geq 0}_{L \text{ is positive semi-definite}}$$

- $0 \in \lambda(L)$ with associated eigenvector $\mathbf{1}_N$ for any graph

For better intuiton, vector x has elements v_i representing node i's value.

- $\begin{cases} 0 \text{ algebraic multiplicity: } 1 \\ L \text{ positive semi-definite} \end{cases} \Rightarrow 2^{\text{nd}}$ min. eigenval. $\lambda_2 > 0$ (all-connected)
- 0 algebraic multiplicity: $n \Rightarrow (n-1)$ connected components

Problem Formulation: Scenario

Agent Dynamics

Each agent $i = 1, 2, \ldots, N$ has the general linear dynamics

$$\dot{x}_i = Ax_i(t) + Bu_i(t) \qquad x_i(t) \in \mathbb{R}^3,\ u_i(t) \in \mathbb{R}^m$$

Time-Varying Graph

There are p topology graphs in total.

- $\sigma(t)\colon [0, \infty) \mapsto \{1, 2, \ldots, p\}$ maps time to a specific topology
- Each graph $G_{\sigma(t)}$ has its Laplacian matrix $L_{\sigma(t)}$
- Each agent node i in $G_{\sigma(t)}$ has a neighbor set $N^i_{\sigma(t)}$

Problem Formulation: Related Definitions

Definition (Formation Achievement)

A multi-agent system is said to achieve time-varying formation $h(t)$ if $\forall$ bounded initial states, $\exists$ a vector-valued function $r(t) \in \mathbb{R}^3$ s.t.

$$\lim_{t \to \infty} (x_i(t) - h_i(t) - r(t)) = 0 \qquad i = 1, \ldots, N$$

where $r(t)$ is called the formation reference function.

Closed-loop System

Control Law

Let $\phi_i(t) = x_i(t) - h_i(t)$, the paper proposed the control law

$$u_i(t) = K_1 x_i(t) + K_2 \phi_i(t) + \alpha K_3 \sum_{j \in N_{\sigma(t)}}^{i} w_{ij} \left(\phi_j(t) - \phi_i(t) \right) + v_i(t)$$

where

- α: coupling strength, bound determined by graph, design parameter
- $v_i(t)$: formation compensation, dependent on $h_i(t)$, expands feasible set

Closed-loop System

$$\begin{aligned} \dot{x}(t) = &\left(I_N \otimes (A + BK_1 + BK_2) - \alpha L_{\sigma(t)} \otimes BK_3 \right) x(t) \\ &+ \left(\alpha L_{\sigma(t)} \otimes BK_3 - I_N \otimes BK_2 \right) h(t) + (I_N \otimes B) v(t) \end{aligned}$$

Basic Goals As Problems

Convergence to Desired Formation

Can we have bounded $\phi(t) = x_i(t) - h_i(t)$?

Explicit Trajectory of the Formation Reference Function

$$\lim_{t\to\infty} (\phi_i(t) - r(t)) = 0 \qquad i = 1, \ldots, N$$

Can we have the expression of $r(t)$, the “residual error”?

Topology Switching Frequency

How fast can we change the topology while keeping the system stable?
How the structure of the graph affects this “minimum time”?

Analysis: Basic Goals As Problems

Dynamics of Formation Error

- Deviation from the ideal formation

$$\phi_i(t) = x_i(t) - h_i(t)$$

- Using the closed-loop dynamic of x, $\dot{\phi}_i = \dot{x}_i - \dot{h}_i$

$$\begin{aligned}\dot{\phi}(t) = &\left(I_N \otimes (A + BK_1 + BK_2) - \alpha L_{\sigma(t)} \otimes BK_3\right) \phi(t) \\ &+ (I_N \otimes (A + BK_1))h(t) + (I_N \otimes B)\, v(t) \\ &- (I_N \otimes I_n)\, \dot{h}(t)\end{aligned}$$

- We want $\phi(t)$ to be bounded when t is sufficiently large

Analysis: $\phi(t) = x(t) - h(t)$ and $r(t)$

Using Property of Laplacian

We construct nonsingular matrix $U \in \mathbb{R}^{n\times n}$ s.t.

$$U^{-1} = \left[\begin{array}{c} - \quad \bar{u}_1 \quad - \\ \hline \\ \bar{U} \\ \\ \end{array}\right] \qquad U = \left[\begin{array}{c|c} | & \\ \tilde{u}_1 & \tilde{U} \\ | & \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ \vdots & \tilde{U} \\ 1 & \end{array}\right]$$

For eigenvectors x of the Laplacian L, $Lx = \lambda x$

$$U^{-1} L_{\sigma(t)} U = \left[\begin{array}{c|c} 0 & \bar{u}_1 L_{\sigma(t)} \tilde{U} \\ \hline 0 & \bar{U} L_{\sigma(t)} \tilde{U} \\ \end{array}\right]$$

Analysis: $\phi(t) = x(t) - h(t)$ and $r(t)$

Transformed States

$$\theta(t) = \left(U^{-1} \otimes I_n\right) \phi(t) = \begin{bmatrix} \begin{matrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{matrix} & \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} & \vartheta \end{bmatrix}$$

Decompose Complete Dynamics

$$\begin{aligned} \dot{\phi}(t) = &\left(I_N \otimes (A + BK_1 + BK_2) - \alpha L_{\sigma(t)} \otimes BK_3\right) \phi(t) \\ &+ \left(I_N \otimes (A + BK_1)\right) h(t) + \left(I_N \otimes B\right) v(t) \\ &- \left(I_N \otimes I_n\right) \dot{h}(t) \end{aligned}$$

Analysis: $\phi(t) = x(t) - h(t)$ and $r(t)$, Break into θ_1 and ϑ

$$
\begin{aligned}
\dot{\phi}(t) =& (I_N \otimes (A + BK_1 + BK_2) \\
&- \alpha L_{\sigma(t)} \otimes BK_3)\phi(t) \\
&+ (I_N \otimes (A + BK_1))\, h(t) \\
&+ (I_N \otimes B)\, v(t) \\
&- (I_N \otimes I_n)\, \dot{h}(t)
\end{aligned}
$$

$$
\theta(t) = \begin{bmatrix} \left.\begin{matrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{matrix}\right\} & \vartheta \end{bmatrix}
$$

$$
\begin{aligned}
\dot{\theta}_1(t) =& (A + BK_1 + BK_2)\theta_1(t) \\
&- \alpha(\bar{u}_1 L_{\sigma(t)} \tilde{U}) \otimes BK_3 \vartheta(t) \\
&+ (\bar{u}_1 \otimes (A + BK_1))\, h(t) \\
&+ (\bar{u}_1 \otimes B)\, v(t) - (\bar{u}_1 \otimes I_n)\, \dot{h}(t)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\vartheta}(t) =& (I_{N-1} \otimes (A + BK_1 + BK_2) \\
&- \alpha(\bar{u}_1 L_{\sigma(t)} \tilde{U}) \otimes BK_3)\vartheta(t) \\
&+ (\bar{U} \otimes (A + BK_1))\, h(t) \\
&+ (\bar{U} \otimes B)\, v(t) - (\bar{U} \otimes I_n)\, \dot{h}(t)
\end{aligned}
$$

Analysis: $\phi(t) = x(t) - h(t)$ and $r(t)$, Numerical Illustration

Consider number of agents $N = 3$. A possible construction of U

$$U = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & & 1 \end{bmatrix} \quad \Rightarrow \quad U^{-1} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ -1 & & 1 \end{bmatrix}$$

Therefore, consider in 3D xyz space where $n = 3$,

$$\theta = \left(U^{-1} \otimes I_n\right)\phi = \left(\begin{bmatrix} 1 & & \\ -1 & 1 & \\ -1 & & 1 \end{bmatrix} \otimes I_3\right) \left[\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \hline \phi_4 \\ \vdots \\ \phi_9 \end{array}\right] = \left[\begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \hline \phi_4 - \phi_1 \\ \vdots \\ \phi_9 - \phi_3 \end{array}\right] = \left[\begin{array}{c} \theta_1 \in \mathbb{R}^{3\times 1} \\ \hline \vartheta \in \mathbb{R}^{3\times 2} \end{array}\right]$$

Analysis: $\phi(t) = x(t) - h(t)$ and $r(t)$, Decomposition

We further construct auxiliary variables ϕ_c and $\phi_{\bar{c}}$ s.t. $\phi = \phi_c + \phi_{\bar{c}}$

$$\begin{aligned}\phi_c(t) &= (U \otimes I_n)\begin{bmatrix}\theta_1(t) \\ \mathbf{0}\end{bmatrix} = (U \otimes I_n)(\mathbf{e}_1 \otimes \theta_1(t)) \\ &= (U\mathbf{e}_1) \otimes (I_n \theta_1(t)) = \mathbf{1}_N \otimes \theta_1(t) \\ \phi_{\bar{c}}(t) &= (U \otimes I_n)\begin{bmatrix}\mathbf{0} \\ \vartheta(t)\end{bmatrix}\end{aligned}$$

Then

$$\phi_{\bar{c}} = \phi - \mathbf{1}_N \otimes \theta_1 = x_i - h_i - \theta_1$$

According to definition of formation achievement, we want $\lim_{t\to\infty} \phi_{\bar{c}}(t) = 0$, due to non-singularity of U,

$$\lim_{t\to\infty} \phi_{\bar{c}}(t) = 0 \iff \lim_{t\to\infty} \vartheta(t) = 0$$

We can choose $r(t) = \theta_1(t)$

Analysis: $\phi(t) = x(t) - h(t)$ and $r(t)$, Results

$$\begin{aligned}\dot{\vartheta}(t) =& (I_{N-1} \otimes (A + BK_1 + BK_2) - \alpha(\bar{u}_1 L_{\sigma(t)} \tilde{U}) \otimes BK_3)\vartheta(t) \\ &+ \left(\bar{U} \otimes (A + BK_1)\right) h(t) + \left(\bar{U} \otimes B\right) v(t) - \left(\bar{U} \otimes I_n\right) \dot{h}(t)\end{aligned}$$

To have $\lim_{t\to\infty} \vartheta(t) = 0$,

1. Zero-input part is A.S. Let $\bar{\vartheta}$ denote zero-input system.
2. Zero-state part $\to 0$ as $t \to \infty$

Find Way to Solve for $v_i(t)$

Choose $T = \begin{bmatrix} \tilde{B} \\ \hat{B} \end{bmatrix}$ s.t. $TB = \begin{bmatrix} I_{n_u} \\ \mathbf{0} \end{bmatrix}$, from 2 it can be proved that

$$\lim_{t\to\infty} (\bar{B}A(h_i - h_j) - \bar{B}(\dot{h}_i - \dot{h}_j)) = 0$$

$$\lim_{t\to\infty} \left(\bar{B}(A + BK_1)(h_i - h_j) - \bar{B}(\dot{h}_i - \dot{h}_j + \bar{B}(v_i - v_j))\right) = 0$$

Analysis: $\phi(t) = x(t) - h(t)$ and $r(t)$, Results (Cont'd)

$r(t)$

$$\begin{aligned}\dot{\theta}_1(t) =& (A + BK_1 + BK_2)\theta_1(t) - \alpha(\bar{u}_1 L_{\sigma(t)} \tilde{U}) \otimes BK_3 \vartheta(t) \\ & + (\bar{u}_1 \otimes (A + BK_1))\, h(t) + (\bar{u}_1 \otimes B)\, v(t) - (\bar{u}_1 \otimes I_n)\, \dot{h}(t)\end{aligned}$$

$r(t)$ is dependent on x, v and h.

Analysis: Dwell Time

- Choose $\beta > 0$, solve for $P = P^{\mathsf{T}} \succ 0$

$$(A + BK_1 + BK_2)P + P(A + BK_1 + BK_2)^{\mathsf{T}} - BB^T + \beta P < 0$$

- For each graph, given $\hat{\mu}_{\sigma(t)}$ (min. real part of eigenvalues of $\bar{U} L_{\sigma(t)} \tilde{U}$) for $\mu_{\sigma(t)} \in (0, \hat{\mu}_{\sigma(t)})$, choose $\Xi_{\sigma(t)} = \Xi_{\sigma(t)}{}^{\mathsf{T}} \succ 0$ s.t.

$$\left(\bar{U} L_{\sigma(t)} \tilde{U}\right)^{\mathsf{T}} \Xi_{\sigma(t)} + \Xi_{\sigma(t)} \left(\bar{U} L_{\sigma(t)} \tilde{U}\right) > 2\mu_{\sigma(t)} \Xi_{\sigma(t)}$$

- Let $\gamma = \max \lambda_{\max}\left(\Xi_i^{-1} \Xi_j\right)$, it can be shown that the Lyapunov function

$$V = \bar{\vartheta}^{\mathsf{T}} (\Xi_{\sigma(t)} \otimes P^{-1}) \bar{\vartheta}$$

satisfies $\dot{V} = -\beta V$, which shows exponential decay of V, dwell time $\tau_0 > \frac{\ln(\gamma)}{\beta}$

Experiment Setting

$$A = \begin{bmatrix} 4 & -2 & 2 \\ 1 & 3 & 5 \\ 2 & 7 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$h_i = \begin{bmatrix} r \sin\left(\omega t + \frac{(i-1)\pi}{3}\right) \\ 2r \sin\left(\omega t + \frac{(i-1)\pi}{3}\right) \\ r \cos\left(\omega t + \frac{(i-1)\pi}{3}\right) \end{bmatrix} \qquad r = 6, \ \omega = 2$$

Implementation video:

`https://www.bilibili.com/video/BV13f4y1m7XZ`

Limitations

- No collision avoidance exists
- The assignment of the eigenvalues are not "optimal"

Bibliography

- DOI: 10.1016/j.automatica.2016.06.024 (Analysis Part, not successfully implemented and contains potential error, such as wrong $v(t)$ calculation)
- DOI: 10.1049/iet-cta.2013.1007 (The actual implemented paper)