Scenes from a Monopoly: Quickest Detection of Ecological Regimes*

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Abstract

We study the stochastic dynamics of a renewable resource harvested by a monopolist facing a downward sloping demand curve. We introduce a framework where harvesting sequentially affects the resource's potential to regenerate, resulting in an endogenous ecological regime shift. In a multi-period setting, the firm's objective is to find the profit-maximizing harvesting policy while simultaneously detecting in the quickest time possible the change in regime. Encapsulating the idea of environmental surveillance, the use of quickest detection method allows us to easily translate our framework to real-time detection. Solving analytically, we show that a negative regime shift induces an aggressive extraction behaviour due to a combination of faster detection, a sense of urgency, and higher markups. Precautionary behaviour can result due to increasing resource rent. We study the probability of extinction and show the emergence of catastrophe risk which can be both reversible and irreversible.

JEL Codes: D42, Q21, Q57

1 Introduction

The exploitation of renewable resources, such as overfishing of the North Sea cod, deforestation in the Amazon and soil degradation due to unsustainable agricultural practices, is an issue that is receiving considerable attention. The dynamic management of such resources often involves decisions concerning optimal extraction policies under *ecological* uncertainty, defined by Pindyck (2002) as uncertainty over the evolution of the relevant ecosystem. This raises pertinent economic questions about the behaviour of a firm harvesting these resources, especially if the dynamics driving the resource growth change. One way that the current literature captures this source of uncertainty is by means of stochastic bio-economic models, reflected in the variance of the fluctuations. Another way is to focus on ecological regime shifts, defined as an abrupt change in the structure of the natural ecosystems supplying the resource or a change in the system dynamics such

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as intrinsic growth rate or the carrying capacity of the resource (Polasky et al. (2011); Arvaniti et al. (2019)).

There already exists a large literature studying the impact of stochastic fluctuations on extraction activities, mainly utilizing real options theory (Andersen and Sutinen (1984); Pindyck (1984); Reed (1988); Reed and Clarke (1990); Saphores (2003); Alvarez and Koskela (2007); Pizarro and Schwartz (2018)). An emerging literature builds on this to integrate resource management with a variety of regime shifts, such as Polasky et al. (2011), Ren and Polasky (2014), Baggio and Fackler (2016), de Zeeuw and He (2017) and Arvaniti et al. (2019).[1] These studies, however, are limited in two respects. With the exception of Pindyck (1984), none of these works incorporate a market structure and take the price as fixed or exogenous. This is done for tractability reasons but leads to results that may underestimate the crucial role of a market structure, which often indeed drives firms' harvesting decisions. Furthermore, the literature on regime shifts implicitly assumes the firm to be able to discern the change in resource dynamics and subsequently make the appropriate extraction decision.[2] In this paper we take a different approach from these studies and pose the following research questions: within a resource market where prices are endogenously determined, how does an ecological regime shift influence a monopolist firm's harvesting decisions? What is the profit-maximizing policy of this firm who also wants to *detect* this shift, in a framework where this change in regime is endogenously determined by the firm's extraction activity?

We build a model of a monopolist firm, facing a downward sloping demand curve, who faces two sources of uncertainty in the resource dynamics. The first source is the natural randomness of the environmental conditions, here represented by Gaussian noise, and the second is the *timing* of the ecological regime shift. This shift, defined as a change in the resource's ability to grow, is made dependent on the firm's own extraction efforts. In our model, the firm knows with certainty that a regime shift will eventually occur: what matters for the firm's harvesting decisions is when it will take place. In a multi-period setting, the monopolist wants to detect this shift as soon as possible, and this detection procedure is explicitly incorporated in its profit maximizing actions. The resource dynamics are assumed to be monitored by the monopolist through sequential observations and we model the firm's detection process by means of a *quickest detection* method. This method extends the classical hypothesis testing and change-point problems to a sequential framework and an optimal stopping problem. The aim is to detect a change in the resource growth, if one occurs, with the shortest delay possible. Using the sequential nature of the detection process we incorporate non-stationary dynamics.

Our model allows for fully analytical solutions and finds an optimal extraction policy which is not only a function of market preferences and the current state of the resource stock but is also dependent on the detection (optimal stopping) time, which in turn is dependent on the magnitude of the endogenous regime shift. This implies that, within a multi-period framework and non-stationary dynamics, the firm's decision horizon varies for each period and is updated according to the size of the regime shift. We find that in the event of a negative regime shift, for low stock levels, the firm adopts a precautionary policy by reducing extraction. This is because the change in regime creates a physical

[1] Refer to Li et al. (2018) for an overview.

[2] We use the words extraction and harvesting interchangeably.

scarcity of the resource which in turn increases the resource rent for the monopolist, leading to reduced extraction levels. For higher stock levels, however, this effect is outweighed by an interplay of two factors: (i) market preferences, where an elastic demand allows the monopolist to have increasing expected rate of growth of its marginal revenue, compared to pre-regime shift period, and (ii) an updated decision horizon due to the detection of the regime shift which for larger regime changes maybe short, creating a sense of urgency. Both these factors together reduce the resource rent and prompt the firm to adopt an aggressive behaviour by increasing extraction. Lastly, we define the risk of catastrophe as the situation in which the growth rate of the resource becomes negative, thus exhibiting a net tendency for the resource to reach extinction. We further distinguish between the scenarios of irreversible and reversible catastrophe, based on whether the firm can avert the resource extinction by reducing or stopping extraction by studying the distribution of the catastrophe's hitting time. We find that this hitting time in fact follows an inverse Gaussian distribution with larger magnitudes of regime shifts resulting in thicker tails.

An important element of our model is assimilating the idea of environmental monitoring of the resource to detect for changes in the stock and its structure, which is in fact quite common in real-world resource management. The use of *quickest detection* method to capture monitoring allows us to easily translate our framework to real-time detection which we discuss in further detail in section 4. Klemas (2013) talks about how remote sensing techniques, in near-real time, help detect changes that affect recruitment, distribution patterns and survival of fish stocks. These techniques, combined with *in situ* measurements, constitute the most effective ways for efficient management and controlled exploitation of marine resources. Shimabukuro et al. (2019) details the monitoring of deforestation and forest degradation in the Brazilian Amazon. In ecology, using real-time remote sensing data is increasingly common, especially with indicators of approaching thresholds, or impending collapse in ecosystems.[3] Thus our model is especially relevant to understand how firms operate in the modern day resource market while incorporating a relevant form of monitoring of the ecological dynamics.

In section 2 we lay out the different building blocks of the model, section 3 describes profit maximization within a sequential framework and additionally we define the risk and first passage time to catastrophe. In section 4 we discuss how our model could be applied in a situation where the firm monitors the resource process in real time and section 5 discusses the model solution and its economic implications. Section 6 concludes.

2 The Model

2.1 Resource Dynamics

We start by modeling the evolution of the renewable resource stock X_t. Let X_t be the stock at time t, which behaves according to the stochastic differential equation

$$dX_t = \big(\mu - q_t\big)dt + \sigma dW_t \tag{1}$$

where $q_t \in \mathbb{R}^+$ is the resource extraction chosen by the firm, $\sigma \in \mathbb{R}^+$ is the intensity of noise in the evolution of the resource stock, $\mu \in \mathbb{R}^+$ is the constant growth rate of

[3]See Porter et al. (2012); Batt et al. (2013); Carpenter et al. (2014); Scheffer et al. (2015)

the resource and $X_t \geq 0$[4]. Finally, W_t is the standard Brownian motion in the filtered probability space $(\Omega, \mathcal{F}, P)$.

In order to capture the regime shift that the dynamic system can undergo, we describe two alternative scenarios faced by the firm: one in which the resource evolves according to equation (1), and an alternative one in which the stock's ability to regenerate - the drift - changes. This is consistent with Polasky et al. (2011) in which a regime shift is defined as a change in the system dynamics such as intrinsic growth rate or the carrying capacity of the resource. The evolution for the resource stock then becomes

$$dX_t = \big(\mu + \lambda - q_t\big)dt + \sigma dW_t, \tag{2}$$

where $\lambda \in \mathbb{R}$ is the change in resource growth. If $\lambda < 0$, the growth rate of the resource is reduced, and vice versa. To provide some intuition, we give two examples where we would observe such a change in drift. In the context of fisheries, the term $\big(\mu + \lambda\big)$, where $\lambda < 0$, may indicate recruitment overfishing which occurs when the parent stock (spawning biomass) is depleted to a level where it no longer has the reproductive capacity to replenish itself, not having enough adults to produce recruits thus changing the genetic makeup of the population over time (Pauly (1983)). The collapse of the Atlantic northern cod in the early Nineties in Newfoundland, Canada, was attributed to gross overestimation of stock sizes and the failure to recognize that recruitment overfishing was a definite possibility (Walters and Maguire (1996)). Similarly, logging and timber production have a direct impact on forest recovery and tree recruitment and growth. This is seen in the tropical forests of Ghana which have been unsustainably logged. Hawthorne et al. (2012) suggest that post-harvest forest regeneration may have been affected and that full recovery of tree stocks is unlikely, even in three further felling cycles.

Equation (2) implies that the firm's harvesting activities do not affect the resource's ability to regenerate in any way. However, this assumption does not seem grounded in empirical observation and it can be seen from the examples above that often the firm's harvesting decisions influence the resource's recruitment and growth process. We therefore rewrite (2) as:

$$dX_t = \big(\mu + \lambda(q_{ex}) - q_t\big)dt + \sigma dW_t, \tag{3}$$

where ex is the past time period that determines the magnitude of λ. We therefore study a framework in which *past extraction decisions* determine the future changes in resource growth. We want to model the scenario in which at a given change point in time θ, which is happening with certainty but at time unknown, the stochastic differential equation (SDE) driving the resource stock will switch between drifts, and the growth rate of the resource will change:

$$dX_t = \begin{cases} \big(\mu - q_t\big)dt + \sigma dW_t & t < \theta \\ \big(\mu + \lambda(q_{ex}) - q_t\big)dt + \sigma dW_t & t \geq \theta. \end{cases} \tag{4}$$

The sign of $\lambda(q_{ex})$ can be both positive or negative, which represents the fact that the effect of firm extraction on the resource growth can be both positive or negative. This

[4]This positivity constraint allows the problem to have reasonable implications and a relatively simple solution, at the expense of an increase of the hidden mathematical requirements for the solution to be sufficient and unique.

also implies that the firm's actions influences the magnitude of the change of regime. Note that since the occurrence of θ is certain, the question faced by the monopolist is not *if* a regime shift will occur but rather *when.* This framework seems appropriate for today, since the focus has moved from questions regarding the probability of the occurrences of collapses and regime shifts, to the question of when and how such occurrences will have to be dealt with.

The firm now faces two sources of uncertainty when choosing the harvesting policy that maximizes its profits. The first is the variance of the Gaussian noise source σ^2, which is the variation inherent to the natural randomness of environmental conditions: we choose the diffusion coefficient σ to be independent of the state X_t (i.e. a drifted Brownian motion) in order to include the possibility that the exogenous environmental shocks may drive the resource to extinction, something that log-normal fluctuations in a geometric Brownian motion by construction cannot represent. The second source is the *timing* θ of the shift, at which the resource's drift changes from μ to $\mu + \lambda(q_{ex})$.

2.2 Firm Dynamics

We consider a risk-neutral monopolist facing a linear inverse demand function of the form $p(q) = a - bq$, with quadratic harvesting costs $cq^2/2$, and with fixed costs F where $c, F \geq 0$. The harvesting rate is chosen by the firm in order to maximize the expected value of the sum of discounted profits subject to the constraint (4), and the profit function takes the form

$$\Pi(q) = \left[(a - bq)q - c\frac{q^2}{2} - F\right] \tag{5}$$

We assume a profit function not directly depending on the stock level X but only on the harvested quantity: this implies a marginal cost function linear in harvesting, rather than the stock level, and fixed operating costs. This assumption can be relaxed, at the expense of an optimal extraction function only available in numerical form.

2.3 Optimal Detection

The firm's problem now involves the *detection* of the change in drift of X_t, as seen in (4). The monopolist monitors the resource stock via sequential observations and uses quickest detection method to detect the regime change. This comprises of three variables: a stochastic process under observation (the evolution of the renewable resource), a change point at which the statistical properties of the process undergo a change (a regime shift), and a decision maker that observes the stochastic process and aims to detect this change (the monopolist). This method builds on change-point problems and extends it to the sequential framework where as long as the behavior of the observations is consistent with the initial state, one is content to let the process continue. However, if the state changes, then the observer would like to detect it as soon as possible after its occurrence. The gist is to produce a detection policy that minimizes the average delay to detection subject to a bound on the average frequency of false alarms (Tartakovsky et al. (2014)).[5]

[5] For a short introduction to quickest detection methods refer to Polunchenko et al. (2013) For a more detailed review we refer to Poor and Hadjiliadis (2008)

The firm therefore searches for a "rule" (an optimal stopping time) τ adapted to the filtration $\mathcal{F}_t$, at which it detects the change point θ, so it may reassess its harvesting decisions given the change of environment in which it operates. In the period before θ, the dynamics of the resource X_t are determined by the (possibly nonlinear) SDE

$$dX_t = \big(\mu - q_t\big)dt + \sigma dW_t.$$

Girsanov theory tells us that the process

$$M_t = \exp\left(-\int_0^t \frac{\mu - q_s}{\sigma} dW_s - \frac{1}{2}\int_0^t \frac{(\mu - q_s)^2}{\sigma^2} ds\right)$$

is a P-martingale. Therefore, the process

$$\tilde{W}_t = W_t + \int_0^t \frac{\mu - q_s}{\sigma} ds$$

is a Q-Brownian motion, where one obtains the new probability measure by $Q = \mathbb{E}_P(M_t)$. The process X_t therefore admits the representation

$$X_t = x_0 + \int_0^t d\tilde{W}_s$$

and is therefore a Brownian motion under the measure Q. The firm's detection problem now becomes

$$dX_t = \begin{cases} d\tilde{W}_t & t < \theta \\ \lambda(q_{ex}) + d\tilde{W}_t & t \geq \theta. \end{cases} \tag{6}$$

If the period ex that determines λ is outside the interval $[0, t]$, then the firm's detection problem reverts exactly to the *Brownian disorder* problem, which is the detection of the change between a martingale and a sub/supermartingale, depending on the sign of λ. This requires that the harvesting decisions, that define both sign and magnitude of the change in resource growth, be set strictly before the time of the initial condition on X (here normalized to 0, i.e. X_0).

Change-point detection in the disorder problem involves the optimization of the trade off between two measures, one being the delay between the time a change occurs and it is detected i.e. $(\tau - \theta)^+$, and the other being a measure of the frequency of false alarms for events of the type $(\tau < \theta)$. This problem has been first studied by Shiryaev (1963), and the procedure of the cumulative sum process (CUSUM) has been proven to be optimal by Shiryaev (1996) and in the case of multiple drifts by Hadjiliadis and Moustakides (2006). The firm minimizes the worst possible detection delay over all possible realizations of paths of X_t before the change and over all possible change points θ. This is given by

$$J(\tau) = \sup_\theta \; \operatorname{ess\,sup} \mathbb{E}_\theta[(\tau - \theta)^+ | \mathcal{F}_\theta] \tag{7}$$

and the stopping rule is obtained by minimizing (7) under a "false alarm" constraint. This stochastic control problem is given by

$$\min_\tau J(\tau) \quad \text{s.t.} \quad \mathbb{E}_{\theta=\infty}[\tau] = T.$$

This constraint gives the class of stopping times τ, for which the mean time $\mathbb{E}_{\theta=\infty}[\tau]$ until giving a (false) alarm is equal to T. It can be interpreted as a measure of the "quality" of the detection system, since it fixes the expected delay in the detection under a false alarm, i.e. when $\theta = \infty$ (the process never actually changes regime).

It is shown by Hadjiliadis and Moustakides (2006) that one can only focus on the constraints that bind with equality. The CUSUM procedure involves first observing the process given by the logarithm of the likelihood ratio (the Radon-Nikodym derivative) of the process X_t (note that we are under the measure Q) under the two regimes and comparing it with its minimum observed value. Define

$$u_t(\beta) = \log \frac{dQ_{\theta=0}}{dQ_{\theta=\infty}} = \lambda(q_{ex})X_t - \frac{\lambda(q_{ex})^2}{2}t.$$

The CUSUM statistic process is then given by the difference at any instant $s \leq t$ between u_t and its minimum obtained value up to that instant, namely

$$CS_t(\lambda(q_{ex})) = u_t(\lambda(q_{ex})) - \inf_{0 \leq s \leq t} u_t(\lambda(q_{ex})) \geq 0.$$

This can be interpreted simply by noticing that if the two regimes are very similar (i.e. $|\lambda|$ is very small), then the Radon-Nikodym derivative will be close to unity, implying that the CUSUM process will be most of the time close to zero, and unless the diffusion parameter is very small it will be difficult to detect the presence of such a small drift. If on the other hand the two regimes are rather different, then one should be able to detect more easily when the regime changes, and the CUSUM process should reflect this change as it increases. One would therefore expect to search for a threshold in order to determine when the CUSUM process is "large enough" to reflect the change of regime: this is indeed the case. Shiryaev (1996) and Hadjiliadis and Moustakides (2006) show that the optimal CUSUM stopping rule is given by the stopping time

$$\tau(\lambda(q_{ex}), \nu) = \inf\{t \geq 0; CS_t \geq \nu), \tag{8}$$

where the threshold ν is given by the root of the equation

$$\frac{2}{\lambda(q_{ex})^2}(e^{\nu} - \nu - 1) = T.$$

It can be shown that the delay function of this procedure is given by

$$\mathbb{E}[\tau(\lambda(q_{ex}), \nu)] = \frac{2}{\lambda(q_{ex})^2}(e^{-\nu} + \nu - 1). \tag{9}$$

At the stopping time τ, therefore, the firm will detect the change in drift of λ in (6), which means that the firm will have detected a change from a Q-martingale to a Q-sub/supermartingale. Note immediately that the larger the change in drift λ, the smaller the threshold ν and the "earlier" one expects the CUSUM process to hit the threshold after the change occurred. If λ is very small, then ν will be very large and the firm may wait for much longer before detecting a change of regime: in such a case it may be that $\tau(\lambda(q_{ex}), \nu) \geq T$, and once T is reached the firm will assume that the regime has changed.

The effective time period in which the firm optimizes is therefore between $t = 0$ and the final time given by the minimum between T and $\tau(\lambda(q_{ex}), \nu)$, the actual time at which the regime shift occurs plus the delay of detection. In other words, the firm programs its profit maximization assuming that the non-controlled part of the drift in the SDE driving X_t is given by μ, and subsequently by $(\mu + \lambda(q_{ex}))$. The "tolerance" T is chosen by the firm; however, $\tau(\lambda(q_{ex}, \nu)$ is a random variable. Since the firm knows the average delay time of detection, as given by (9), it can assume as time horizon the sum of the expectations of both change-point and delay, which is equivalent to taking a time interval $[0, \min\{T, \tau_c = \mathbb{E}[\theta] + \mathbb{E}[\tau(\lambda(q_{ex}), \nu)]\}]$. In the baseline detection case the firm has an uniform prior on the time of the regime shift: this implies that simply $\mathbb{E}[\theta] = T/2$.

3 Profit Maximization

If the firm's harvesting activities do not impact the growth of the resource, as seen in (2), then the regime shift is entirely exogenous from the firm's point of view and the stochastic control problem is:

$$\begin{aligned} \sup_{q \in Q} \quad & \mathbb{E}_0 \int_0^{\tau_c} \Pi(q_s) e^{-\rho s} ds + \mathbb{E}_{\tau_c} \int_{\tau_c}^{\infty} \Pi(q_s) e^{-\rho s} ds \\ & dX_t = \begin{cases} \big(\mu - q_t\big) dt + \sigma dW_t, & t < \tau_c \\ \big(\mu + \lambda - q_t\big) dt + \sigma dW_t, & t \geq \tau_c \end{cases} \\ & X_t \geq 0 \quad \forall t \\ & \tau_c = \min\left\{T \ , \ \mathbb{E}[\theta] + \mathbb{E}[\tau(\lambda, \nu)]\right\} \end{aligned} \tag{10}$$

The simplest way of modeling a regime shift is to assume that the shift occurs only once, as in Polasky et al. (2011) and Ren and Polasky (2014). However, as pointed by Sakamoto (2014), regime shifts are better modeled as open-ended processes. An example being the Pacific ecosystem, where in the mid-1970s, the Pacific changed from a cool "anchovy regime" to a warm "sardine regime" and a shift back to an anchovy regime occurred in the middle to late 1990s (Chavez et al. (2003)). The above model can be straightforwardly extended to a multi-period setting, in which the firm detects multiple regime changes throughout subsequent periods and adjusts its optimal harvesting policy accordingly. The problem will then read:

$$\begin{aligned} \sup_{q \in Q} \quad & \sum_{i=0}^{\infty} \mathbb{E}_{\tau_i} \int_{\tau_i}^{\tau_{i+1}} \Pi(t, q_t) e^{-\rho t} dt \\ & dX_t = \begin{cases} \big(\mu + \lambda_i - q_t\big) dt + \sigma dW_t, & t \in [\tau_i, \tau_{i+1}) \\ \big(\mu + \lambda_{i+1} - q_t\big) dt + \sigma dW_t, & t \geq \tau_{i+1}, \ i \in \mathbb{N}. \end{cases} \\ & X_t \geq 0 \quad \forall t \\ & \tau_i = \min\left\{T \ , \ \mathbb{E}[\theta] + \mathbb{E}[\tau(\lambda_i), \nu)]\right\} \end{aligned} \tag{11}$$

where $i \in \mathbb{N}$ are the different periods, and the harvesting policy exists among the class of admissible controls Q. Here $\lambda_0 = 0$ and τ_i, λ_i are the subsequent periods and relative changes in resource growth. We assume $\tau_0 = 0$ for simplicity.

To incorporate the firm's actions in a way that it affects the resource's ability to grow

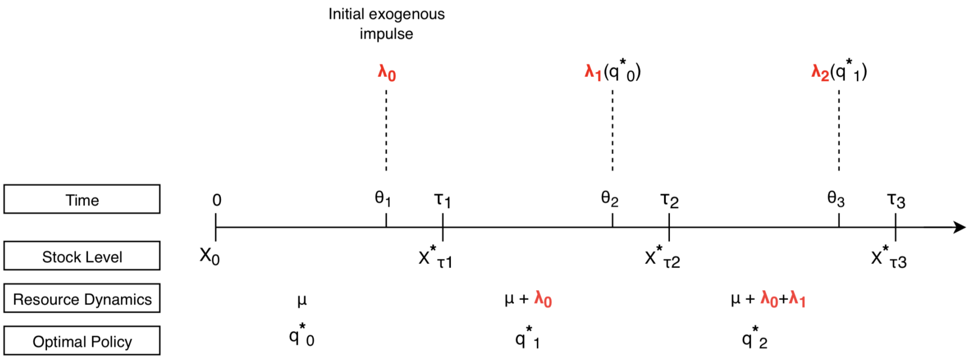


Figure 1: Sequential Detection

and regenerate, as in (3), we formalize the structure of the firm's harvesting decisions in a sequential manner, where the firm assumes a constant $\lambda(q_{ex})$ for each period[6]. To analyse the firm's optimization problem in a sequential detection scenario we work through a three period example. A schematic representation of which can be seen in Figure 1. This can easily be extended to a multi-period.

3.1 Period $[0, \tau_1]$

At time $t = 0$ the firm believes that the resource is driven by a diffusion process with the natural growth rate μ and and begins harvesting activity at level $q^*(0, x_0)$. At a random time $\theta_1 \in [0, T]$, there is an initial exogenous change[7] in the resource dynamics with $\lambda_0 < 0$. Until the detection of this change, the firm operates in an environment where the resource evolves according to the process

$$dX_t = (\mu - q_0^*(t, X_t))dt + \sigma dW_t, \quad t \in [0, \tau(\lambda_0, \nu)], \tag{12}$$

where $\tau(\lambda_0, \nu) \leq T$ is the detection time. The final time of the period which the firm uses as a reference for its decisions is given by

$$\tau_1 = \mathbb{E}[\theta] + \mathbb{E}[\tau(\lambda_0, \nu)] = \mathbb{E}[\theta] + \frac{2}{\lambda_0^2}\left(e^{-\nu} + \nu - 1\right) \tag{13}$$

where the threshold ν solves $\frac{2}{\lambda_0^2}(e^{\nu} - \nu - 1) = T$. Within this time interval $[0, \tau_1]$, the value of the firm is given by

$$\begin{aligned} V(0, X_0) \quad &= \quad \sup_{q \in Q} \mathbb{E}_0 \int_0^{\tau_1} \Pi(q) e^{-\rho t} dt \\ &\text{s.t.} \quad dX_t = (\mu - q)dt + \sigma dW_t, \\ &\qquad X_t \geq 0. \end{aligned} \tag{14}$$

[6]The explicit dependence of the stopping time τ on λ makes the control variable q and the limit of integration τ_1 simultaneous, and the model becomes intractable. In order to circumvent this issue, we model the firm to detect a change in drift $\lambda(q_{ex})$ which is determined by extraction in the *previous* period

[7]The first change is exogenous so as to start the process of subsequent adjustment.

Before solving the problem, let us first characterize the solution given the positivity constraint. The Hamilton-Jacobi-Bellman (HJB) equation for the firm's optimization problem reads

$$0 = V_t - \rho V + \max_{q \in Q} \left\{ (a - bq)q - \frac{c}{2}q^2 - F - qV_x \right\} + \mu V_x + \frac{\sigma^2}{2} V_{xx}, \quad X_t \geq 0. \tag{15}$$

where Q is the set of admissible Markov controls for which $q^* \geq 0, X * (t, q^*) \geq 0$[8]. Once solved, this problem will yield a control in the feedback form $q(t, X_t)$. Because of the constraint $X_t \geq 0 \; \forall t \in [0, \tau_1]$, the value function $V(t,x)$ is not necessarily always differentiable. Using viscosity solutions, as first shown in the fundamental work by Crandall and Lions (1981), we show in the appendix that the value function V is a weak solution of the optimization problem (15), and once we obtain a solution for V we can conclude it will solve the firm's problem (in a weak sense).

Equation (15) implies an optimal extraction policy given by

$$q^*(t, X_t) = \left[\frac{a - V_x}{2b + c} \right]_+ . \tag{16}$$

Note that this implies that in order for extraction to stay positive, $V_x \geq a$, meaning the resource rent cannot exceed the demand intercept parameter. This is clearly a consequence of the assumption of linear demand, which results in a quadratic criterion. It will be clear in what follows that the solution will be naturally constrained by the boundary conditions to satisfy this requirement. Substituting in (15) and grouping terms, we obtain the following partial differential equation:

$$0 = V_t - \rho V + AV_x + BV_x^2 + \frac{\sigma^2}{2} V_{xx} + C \tag{17}$$

where the constants A, B and C are given by

$$\begin{aligned} A &= \mu - \frac{a}{2b + c}, \\ B &= \frac{1}{2(2b + c)}, \\ C &= \frac{a^2}{2(2b + c)} - F. \end{aligned}$$

The natural boundary conditions of this problem are given by

$$V(t, x) = 0 \text{ for } x < 0, \; V(t, 0) = 0, \; q(t, 0) = 0 \tag{18}$$

without imposing a smooth pasting condition because of the viscosity argument.

Because of the homogeneous form of the profit function, we guess a solution of the HJB equation of the form

[8] See Fleming and Soner (2006) for the full definition of control admissibility.

$$V(t, x) = e^{\rho(t-\tau_1)}V(x)$$

and we linearize it with the nonlinear change of variable

$$V'(x) = \frac{\sigma^2}{2B}\frac{\psi'(x)}{\psi(x)} = e^{-\rho(t-\tau_1)}V_x(t, x)$$

where $\psi(.)$ is a general twice differentiable function on $\mathbb{R}$. By this linearization, one can easily obtain the general solution

$$\psi_g(x) = c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}. \tag{19}$$

where $\alpha_{1,2} = \frac{-A \pm \sqrt{A^2 - 4BC}}{\sigma^2}$ and $\alpha_2 < \alpha_1$. The constants are given by the boundary conditions (18), after noticing that $V(t, 0) = 0$ implies $\psi(0) = 1$. The particular solution can be computed in closed form, but its expression is lengthy and therefore omitted, and henceforth only referred to as $\psi(x)$. The optimal harvesting policy in feedback form is therefore

$$q^*(t, x) = q^m - \sigma^2 \frac{\psi'(x)}{\psi(x)} e^{-\rho(\tau_1 - t)}, \tag{20}$$

It consists of two parts: one driven purely by market preferences as seen in $q^m = \frac{a}{2b+c}$. This is the quantity at which the monopolist's marginal revenue equals marginal cost, it's the profit maximizing harvesting policy the monopolist would choose if there were no fluctuations in the evolution of the resource (i.e. if $\sigma = 0$). The second part not only consists of market preferences but is variable and explicitly dependent on state X_t and modulated by the distance between present and the detection time, representing the time horizon of the firm. Observe that V_x here is the rent associated with a unit of the resource stock. It is the scarcity value or the market value of the marginal unit of *in situ* stock. From (16) we obtain the resource rent for the monopolist:

$$V_x = \sigma^2 \frac{\psi'(x)}{\psi(x)} e^{-\rho(\tau_1 - t)}(2b + c) \tag{21}$$

Note that when the rent of the resource rises, q^* decreases[9]. The "instantaneous" drift of

[9] The optimal harvesting function exhibits a sigmoid-like form. Assume for the sake of exposition $t = \tau_c$, $\sigma = 1$ and $a = 2b + c$, one obtains

$$q^* = \frac{c_1(1 - \alpha_1)e^{\alpha_1 x} + c_2(1 - \alpha_2)e^{\alpha_2 x}}{c_1 e^{\alpha_1 x} + c_2 e^{\alpha_2 x}}.$$

If we have $c_1 = c_2$, $\alpha_2 < 1 < \alpha_1$, we obtain a shifted hyperbolic tangent function, directly related to the logistic function. For general parameter values, therefore, the optimal extraction policy has a modulated sigmoid form. This results in the following limiting behavior:

$$\begin{aligned}
\lim_{x\to\infty} q^*(t, x) &= q^m - \nu(\Theta)e^{-\rho(\tau_1 - t)}, \\
\lim_{\tau_c\to\infty} q^*(t, x) &= q^m.
\end{aligned}$$

where ν is a general continuous and bounded function of the model parameters. This result shows that for any time $t \in [0, \tau_1]$, there is a maximum harvesting level given by a fixed amount, generated by market conditions, minus a parameter which incorporates the dynamics of the resource stock and the time horizon of the firm. If this horizon is long enough, all resource-related parameters are ignored and the monopolist's optimal harvest is entirely driven by the market.

the optimally controlled stock is given by

$$\mu(x,t) = \mu - q^m + \sigma^2 \frac{\psi'(x)}{\psi(x)} e^{-\rho(\tau_1 - t)}$$

and as $t \to \tau_1$ the effective discount rate reduces and the drift increases. At the end of the period, the optimally controlled stock will be given by

$$X^*_{\tau_1} = X_0 + [\mu - q^m]\,\tau_1 + \sigma^2 \int_0^{\tau_1} \frac{\psi'(X_t)}{\psi(X_t)} e^{-\rho(\tau_1 - t)} dt + \sigma \int_0^{\tau_1} dW_t. \tag{22}$$

where the second integral is to be interpreted in the Itô sense. Similar to (20), the overall dynamics of the optimally controlled resource stock also comprise of a fixed growth part, given by the natural growth μ and market preferences, and a variable growth part.

3.2 Period $[\tau_1, \tau_2]$

Once the new regime is detected at $t = \tau_1$, the firm then immediately reassesses its optimal policy to $q_1^*(t,x)$, as the dynamics of the resource stock are now

$$dX_t = (\mu - \lambda_0 - q_1^*(t, X_t))dt + \sigma dW_t, \quad t \in [\tau_1, \tau_2]$$

The optimal policy for this period is easily seen to have the same form as (20). Normalizing time to $t_0 = \tau_1$, one recognizes that the two problems are equivalent, with a change in drift from μ to $\mu - \lambda_0$. We therefore have

$$q_1^*(t,x) = q^m - \sigma^2 \frac{\tilde{\psi}'(x)}{\tilde{\psi}(x)} e^{-\rho(\tau_2 - t)}, \tag{23}$$

where the exponents $\tilde{\psi}(x)$ include the new drift in the coefficient A. In the meantime, however, while the firm assumes a constant λ_0, its past decisions start to catch up. At a random time θ_2 the growth of the "new" process will modify as a function of the past harvesting actions, yielding a change in drift $\lambda_1(q_0^*)$ given by

$$\lambda_1(q_0^*) = \begin{cases} \mu \, \Delta X_0 & \Delta X_0 < 0 \\ \mu \, \sqrt{\Delta X_0} & \Delta X_0 > 0 \end{cases} \tag{24}$$

$$\text{where} \quad \Delta X_0 = \frac{X^*_{\tau_1} - X_0}{X_0} \tag{25}$$

Equations (24) and (25) indicate that the magnitude of change in drift λ depends on how much the resource stock has deviated from its initial value. The observed sign will depend on whether the firm's harvesting actions have generated a net increase or decrease in the total stock of the resource. Note that the effect the net change has on the resource's capacity to regenerate is not symmetric. When $\lambda_1(q_0^*)$ is negative, it has a linear impact on the growth rate. However when $\lambda_1(q_0^*)$ is positive, the effect is concave. This is to capture the fact that an ecosystem is more vulnerable to negative shocks. The sign and magnitude of this regime shift is assumed known by the firm, but *when* it occurs is uncertain and to be detected. At $\tau_2(\lambda_1(q_0^*), \nu, T)$ the monopolist will (on average) detect this change in regime of the resource dynamics. Similar to (22), the resource stock at τ_2 will be

$$X^*_{\tau_2} = X^*_{\tau_1} + [\mu - \lambda_0 - q^m]\,\tau_2 + \sigma^2 \int_{\tau_1}^{\tau_1+\tau_2} \frac{\tilde{\psi}'(X_t)}{\tilde{\psi}(X_t)} e^{-\rho(\tau_2 - t)} dt + \sigma \int_{\tau_1}^{\tau_1+\tau_2} dW_t.$$

3.3 Period $[\tau_2, \tau_3]$

In this period we illustrate the emergence of catastrophe risk. After the new regime is detected at $t = \tau_2$, the firm will reassess its optimal policy to $q^*_2(t,x)$ as the dynamics of the resource stock are now[10]

$$dX_t = (\mu - \lambda_0 - \lambda_1(q^*_0) - q^*_2(t, X_t))dt + \sigma dW_t, \quad t \in [\tau_2, \tau_3]$$

Let us suppose that $(\mu - \lambda_0 - \lambda_1(q^*_0)) > 0$ so that the firm does not find itself under risk. The firm at this point begins the detection process for the next change of regime and if $\lambda_2 < 0$, the firm will realize the future emergence of catastrophe risk if

$$\mu + \sum_{j=0}^{i=2} \lambda_j < 0,$$

noting that at the next detection time τ_3 the new regime will be one in which the drift of the resource stock process will be negative, meaning that the resource will have a net tendency to be driven towards an extinction state $(X = 0)$.

We define the **risk of catastrophe** as the situation in which the instantaneous drift of the resource stock X_t is negative in period i at any time $t \in [\tau_i, \tau_{i+1}]$:

$$\mu + \sum_{j=0}^{i-1} \lambda_j - q^m + \sigma^2 \int_{\tau_i}^{t} \frac{\tilde{\psi}'(X_s)}{\tilde{\psi}(X_s)} e^{-\rho(\tau_{i+1} - s)} ds < 0, \tag{26}$$

which implies that $P(\lim_{t\to\infty} X_t = 0) = 1$.

First passage time to catastrophe: At this moment the firm may have to reassess its extraction policies, due to the fact that the resource growth rate has been affected by its past extraction decisions to a point where extinction is likely. In fact, the probability of the resource being zero in infinite time is unity, which means that the resource eventually *will* be depleted. The firm, however, can now exploit the non-stationary nature of the time intervals in which it operates: a first immediate analysis should be what happens if it stops extracting. Normalizing time to $\tau_i = 0$, we define the probability of extinction as

$$\phi(x) = \Pr\left[\inf_{t\in\mathbb{R}^+} X_t \le 0 \,\middle|\, X_0 = X^*_{\tau_i}, q^*(t, X_t) = 0\right] \tag{27}$$

and the first time to catastrophe as

$$\tau_c = \inf[\, t | X_t \le 0, X_0 = X^*_{\tau_i}, q^*(t, X_t) = 0]. \tag{28}$$

[10] λ_1 can be positive as well but for exposition we assume a negative regime shift.

Then X_t follows simply a drifted Brownian motion and the problem is equivalent of finding where a standard Brownian motion crosses the line $x - \mu - \sum_{j=0}^{i} \lambda_j$ (remember that $\mu + \sum_{j=0}^{i} \lambda_j$ is negative). It's a classic stochastic analysis problem, and it allows the firm to realize that if it stops extracting the expected time to catastrophe is

$$\mathbb{E}\tau_c = \frac{X^*_{\tau_i}}{\left|\mu + \sum_{j=0}^{i} \lambda_j\right|}. \tag{29}$$

and the probability of extinction is

$$\phi(x) = \exp\left(-\frac{2\left(|\mu + \sum_{j=0}^{i} \lambda_j|\right)}{\sigma^2} x\right). \tag{30}$$

If $\mathbb{E}\tau_c \leq \tau_{i+1}$, on average the resource will be depleted within the detection period even if the firm stops extracting altogether: we are therefore in a situation of **irreversible catastrophe**, where even the most precautionary of extraction behavior cannot avoid on average the resource from being depleted. In other words, since extraction always reduces the drift, (29) gives the upper bound on all first times to catastrophe. Since deviation from the optimal policy is costly, it is likely that the firm will continue its extraction policy until extinction. If the firm stops extracting, then the first passage time to catastrophe τ_{cat}, for a resource stock starting at $X^*_{\tau_i}$, will be distributed according to the following density:

$$\begin{aligned} \mathrm{P}\{\tau_{cat} \in dt\} &= \frac{X^*_{\tau_i}}{\sqrt{2\pi\sigma^2 t^3}} \exp\left(-\frac{(X^*_{\tau_i} - (\mu + \sum_{j=0}^{i} \lambda_j)t)^2}{2\sigma^2 t}\right) dt, \\ &= IG\left(\left|\frac{X^*_{\tau_i}}{\mu + \sum_{j=0}^{i} \lambda_j}\right|, \left(\frac{X^*_{\tau_i}}{\sigma}\right)^2\right), \end{aligned} \tag{31}$$

which follows an inverse Gaussian distribution, as seen in Figure 2.

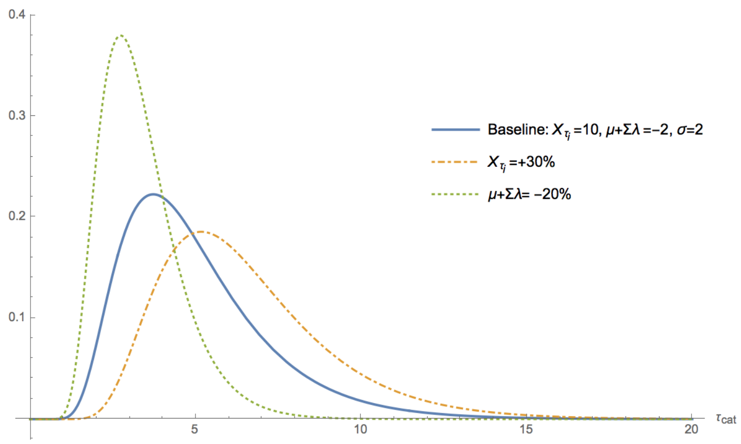


Figure 2: Distribution of the time to catastrophe and effect of a higher initial level of stock (dot-dashed) and of a larger regime shift magnitude (dashed).

Because of the stochastic fluctuations, the firm cannot know with certainty whether the first passage time will happen before the next regime change, but it can have an average measurement of its probability. If $\mathbb{E}\tau_c \geq \tau_{i+1}$, so if $X_{\tau_i} > \mu\tau_{i+1}$, catastrophe is on average avoidable within the first detection period if the firm stops extraction, therefore the firm can study whether its optimal extraction policy allows to avoid it as well. In other words, the firm wants to check whether

$$\begin{aligned} \mathbb{E}\tau_c &\leq \tau_{i+1}, \\ \tau_c &= \inf[\, t|X_t \leq 0, t \in [0, \tau_{i+1}], X_0 = X^*_{\tau_i}]. \end{aligned}$$

Define $\psi(t) = \psi(t; X_{\tau_i}, 0)$ the density function of the first time to catastrophe: then we have that

$$1 - \psi(t) = 1 - \phi(0, t), \tag{32}$$

where $\phi(x, t)$ is the probability that the optimally controlled resource stock X^*_t hits the absorbing barrier at 0, and can be written as

$$\phi(x, t) = \Pr\left[\inf_{s\in[t,\tau_{i+1}]} X^*_s \leq 0 \middle| X_t = x\right],$$

for $0 \leq t \leq \tau_{i+1}$. The firm therefore has to solve the Kolmogorov forward equation given by

$$\frac{\partial}{\partial t}\phi(x, t) + \frac{\partial}{\partial x}\phi(x, t)\left(\mu + \sum_{j=1}^{i-1}\lambda_j - q^*(t, x)\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}\phi(x, t) = 0 \tag{33}$$

with absorbing boundary conditions given by

$$\begin{cases} \phi(x, \tau_{i+1}) = 1 & x \leq 0 \\ \phi(x, \tau_{i+1}) = 0 & x > 0, \end{cases} \tag{34}$$
$$\phi(0, t) = 1,$$
$$\phi(t, \infty) = 0.$$

The KFE for this problem has no closed form solution, given the dependence of the extraction policy on both x and t, and needs to be solved numerically with standard methods. Once the solution is obtained, the firm can recover the density of the first time to catastrophe τ_c from (32) and compute its numerical first moment: if $\mathbb{E}\tau_c \geq \tau_{i+1}$ the firm continues its optimal extraction policy.

3.4 Period $[\tau_i, \tau_{i+1}]$

More generally, for the period $[\tau_i, \tau_{i+1}]$ where $i = 1, 2...n$ we can model the optimal extraction policy as:

$$q^*_i(t, x, \lambda_{i-1}) = q^m - \underbrace{\sigma^2\frac{\psi'(x, \lambda_{i-1})}{\psi(x, \lambda_{i-1})}e^{-\rho(\tau_{i+1}-t)}}_{q^v(t,x,\lambda_{i-1})} \tag{35}$$

the resource rent as:

$$V_x(t, x, \lambda_{i-1}) = q^v(t, x, \lambda_{i-1})(2b + c) \tag{36}$$

the change in the growth of the resource dependent on its past harvesting actions as[11]:

[11] Note that in the first period $[0, \tau_1]$ the change in drift, λ_0 is assumed to be exogenous and not dependent on past harvest efforts.

$$\lambda_i = \begin{cases} \left(\mu + \sum_{j=0}^{i-1} \lambda_j - \lambda_{i-1}\right) \frac{X^*_{\tau_i} - X^*_{\tau_{i-1}}}{X^*_{\tau_{i-1}}} & \Delta X^*_{\tau_{i-1}} < 0 \\ \\ \left(\mu + \sum_{j=0}^{i-1} \lambda_j - \lambda_{i-1}\right) \sqrt{\frac{X^*_{\tau_i} - X^*_{\tau_{i-1}}}{X^*_{\tau_{i-1}}}} & \Delta X^*_{\tau_{i-1}} > 0 \\ \end{cases} \tag{37}$$

and the firm will (on average) detect the regime shift at:

$$\mathbb{E}[\tau_{i+1}] = \mathbb{E}[\theta] + \mathbb{E}[\tau(\lambda(q_i^*), \nu, T)] \tag{38}$$

4 Real-time detection and optimal extraction

The optimal extraction policy in each time interval $[\tau_i, \tau_{i+1}]$ is obtained by assuming as time horizon the expectation of the optimal stopping time $\tau_{i+1} = \min\{T, \mathbb{E}[\theta] + \mathbb{E}[\tau(-\lambda, \nu)]\}$. This is therefore an *ex ante* policy: the actual detection of when the regime shift happens is only represented via a first-order stochastic criterion. The time θ at which the regime changes, however, is a random variable: the firm therefore will use the expected detection time (13) to evaluate the boundary conditions, but simultaneously observe continuously the optimally controlled level of stock X_t, change to the measure Q and compute the Radon-Nikodym derivative of the two measures (before and after the regime change) and check whether its value exceeds the threshold value ν. If the threshold is reached *before* the expected detection time τ_{i+1}, then the firm simply switches to the subsequent period with the modified drift, since the regime shift has been detected. If the expected detection time τ_{i+1} is reached and the threshold has not yet been reached, implying that the regime has not yet shifted, the firm continues the optimal extraction assuming the same underlying resource dynamics, but now in the an infinitesimal time interval as horizon. In other words, the infinitesimal optimal extraction policy if the expected detection time is exceeded is given by

$$q_i^*(t, x, \lambda_{i-1}) = q^m - \sigma^2 \frac{\psi'(x, \lambda_{i-1})}{\psi(x, \lambda_{i-1})} e^{-\rho(T-t)}, \qquad t \in [\tau_{i+1}, T], \tag{39}$$

until either the Radon-Nikodym derivative of the measures of the two regimes (under the measure Q by which X_t is a $Q-$Brownian motion) reaches the threshold ν, or until the firm's tolerance time limit T is reached.

This notion of observation under measure changes might appear as a mathematical abstraction: we note, however, that the disorder problem (6) based on the observation of the resource stock X_t is equivalent in probability to the disorder problem based on the observation of the *residual process* given by

$$Y_t = X_t - \int_0^t \frac{\mu + \sum_{j=1}^{i-1} \lambda_j - q^*(s, X_s)}{\sigma} ds \tag{40}$$

under the original measure. In other words, the firm can detect the change by either observing the resource stock and changing measure appropriately, or by extracting residuals from the stock variation, the growth rate and the optimal extraction policy and then studying the original P-Brownian motion. The computational difference between the two is marginal if the extraction policy is of simple form, such as the constant part of the extraction sigmoid such that the resulting controlled resource stock effectively remains Gaussian, but the second strategy could be of substantially simpler implementation for when the extraction policy is in its nonlinear part. If real-time observations are not continuous but rather arrive at discrete times $t_i, i \in \mathbb{N}$, and assuming a constant frequency between times Δt, then the residual process on which the firm has to apply the detection procedure is the stationary process $X_{t_i} - X_{t_i - \Delta t} - (\mu + \sum_{j=1}^{i-1} \lambda_j - q^*(t, X_{t-\Delta t}))\Delta t$ (after standardization of the diffusive part).

5 Characteristics of the Solution

Observe in (35)-(38), in each period the final level of resource, $X(t)$, is a random variable, and therefore so is the impact on the new growth rate, however it is continuously dependent on the optimal harvesting policy. The variation in $X(t)$ is conserved in the *magnitude*, the absolute value of the percentage change in the resource stock translates directly to a change in drift. This is observed in Figure 3 where we show ten simulated time paths of an the optimally controlled stock of resource. Here the first detection time is common to all but subsequent detections are extraction-dependent.

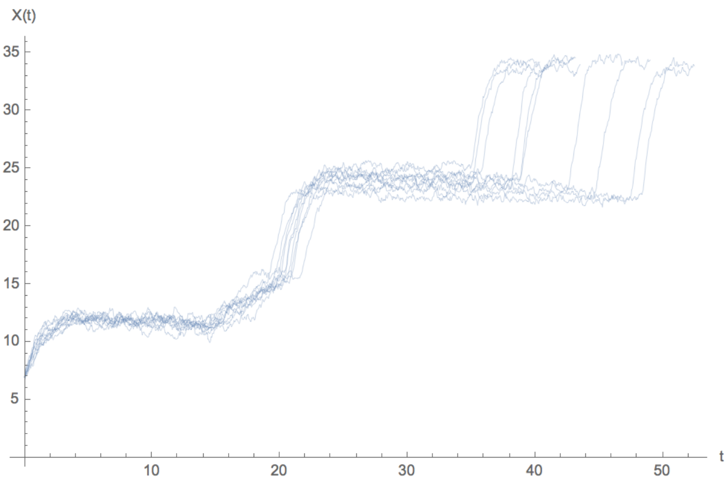
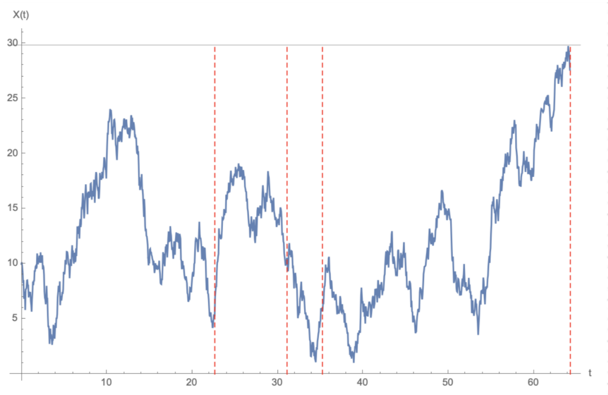
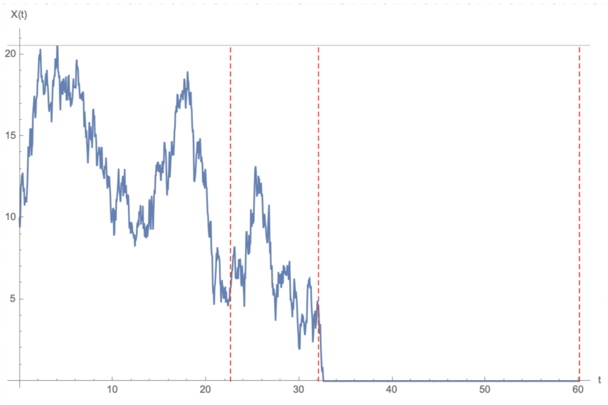


Figure 3: Simulation of 10 time paths with same parameters resulting in different extraction-dependent detection times. All simulations are done with a Shoji-Ozaki discretization method for the time-dependent drift.

Due to the sequential nature of the detection process and the stochastic dynamics of the resource, there is no steady state in our model. The system is non stationary and is randomly changing and as a result optimal harvest must be specified for every state that can possibly occur. Additionally, in a multi-period setting, the detection of each regime shift alters the monopolist's time horizon. The larger is the difference between the initial and final level of stock, the larger will be the magnitude of the change in resource growth rate λ_i. This implies, on average, an *earlier* expected time of detection. Once the firm detects the regime shift, the magnitude of change in the resource growth will either increase or decrease the probability of extinction of the resource by entering the SDE drift with the same sign as the difference between initial and final level of resource biomass. This is evident in panel (a) of Figure 4, which shows a possible time path of



(a)

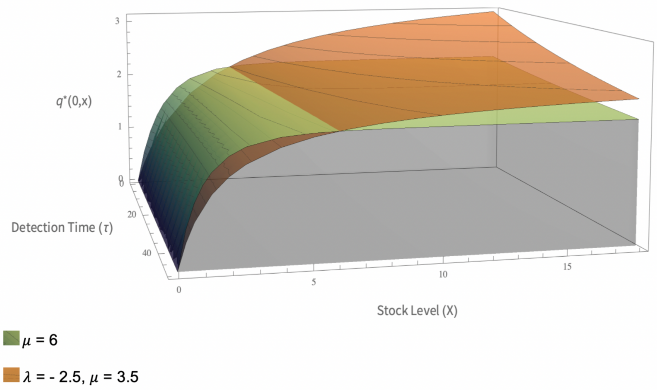


(b)

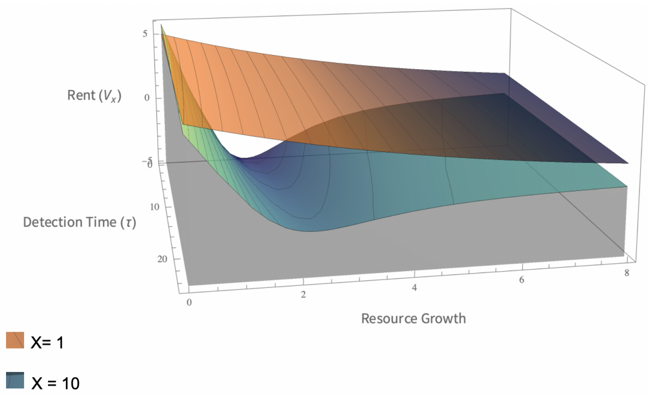
Figure 4: Simulated Time Paths of the Optimally Controlled Stock Biomass. Red dashed lines represent detection times. Panels (a) and (b): For a demand function of the form $p(q) = 3 - 0.1q$, variable cost function $c(q) = 0.5q^2/2$ and fixed cost $F = 0.25$. The natural growth of the resource $\mu = 5$, variance $\sigma = 3$ and $T = 50$. $\lambda_0 = -1.5$ and $X_0 = 10$. The values are in thousand tonnes and years. Panel (b) results in resource extinction.

the stock biomass, for the first four periods, being harvested under the profit maximizing policies of the firm. As the monopolist detects each regime shift, represented by the red dashed lines, it's horizon for the period changes and it pursues the appropriate optimal policy. Panel (b) shows an example of the firm extracting the resource to extinction, with a collapse occurring in the third period. The varying time horizon of each period plays into the firm's extraction decisions.

With (35) and (36), we can now examine how a change in regime and its detection affects the firm decisions. To do this we choose a range of values for the model parameters, which are meant to be largely illustrative. Figure 5 shows the optimal extraction policy of the firm in the first period $[0, \tau_1]$ as shown in (20). The resource has a natural growth $\mu = 6000$ *tonnes/year* and $\sigma = 3250$ *tonnes/year*. The firm incurs a variable cost with $c = 1.25$, a fixed cost of 500 *\$/year* and applies a discount rate of $\rho = 0.02$. We illustrate the case of a negative regime shift as it is of more interest and relevance today. Suppose now that the ecological system undergoes a change, resulting in a modified drift with $\lambda = -2500$ *tonnes/year*. In Panel (a) of Figure 5, we observe that at lower stock levels



(a)



(b)

Figure 5: Panel (a): Optimal extraction policies for the monopolist. For a demand function of the form $p(q) = 5 - 0.75q$, variable cost function $c(q) = 1.25q^2/2$ and fixed cost $F = 0.5$. The natural growth of the resource $\mu = 6$, variance $\sigma = 3.25$. The values are in thousand tonnes and years. Panel (b): Resource rent for the monopolist at low $(X = 1)$ and high $(X = 10)$ stock level.

the firm adopts a precautionary behaviour for all detection periods. This is due to the negative regime shift reducing the growth rate of the resource and creating a physical scarcity. Consecutively, this scarcity increases the resource rent and reduces the extraction levels by the monopolist. Panel (b) depicts the resource rent for the monopolist at various detection periods for two different stock levels. Note that for a low stock, a decline in resource growth i.e. a negative regime shift always leads to an increase in the value of the marginal unit of *in situ* stock.

On the contrary, for higher stock levels the monopolist adopts an aggressive approach and increases its extraction. A straightforward application of Ito's lemma allows to express the total differential of the marginal revenue of the monopolist as

$$\frac{1}{dt}\,\mathbb{E}_t\,dMR = 2bq_x(t,x)q^* - 2bq_x(t,x)\mu - \sigma^2 bq_{xx}(t,x) + 2\rho bq^v(t,x). \tag{41}$$

In (41) the first term is positive, the second term is negative and since extraction is a concave function of the stock level x, the curvature term is positive. Higher levels of stock have the effect of increasing the expected rate of growth of marginal revenue. Therefore the scarcity effect is outweighed by the increase in the markup over the marginal cost which reduces the resource rent and results in higher extraction by the monopolist. There is another factor driving the increased extraction efforts by the monopolist - the decision horizon of the firm. In panel (a) observe the increase in extraction is not uniform across all detection periods. As stated above, one of the model implications is that the larger is the regime shift, the quicker a firm is likely to detect it and the shorter is the time horizon for that period. If the monopolist faces a short time horizon and high levels of resource stock, it increases its extraction even more than it would for the same stock levels and a longer detection period. This is because the monopolist believes that another shift in regime could happen very soon and resource extinction or collapse may be impending. In Panel (b) the behaviour of the resource rent for a higher stock level is more varied compared to that of a lower stock. A decline in resource growth rate reduces the rent which increases the firm's extraction. This effect is amplified for shorter detection periods. However, if the regime shift is exceptionally large, the rent starts to increase and the firm may decide to lower its extraction. For this reason, at high stock levels, the extraction decisions of the firm is a combination of market dynamics and the altered time horizon from the detection of the regime. The interplay between these two factors is evident in Figure 6. The resource undergoes a regime shift similar to Figure 5 and additionally the market preferences change as the slope of the monopolist's demand curve becomes steeper, indicating that the demand for the resource is now less elastic. The extraction levels for longer detection periods and all stock levels continue to be lower than the pre-regime shift levels. However, for short detection periods and higher stock levels, the firm increases its extraction, and continues to charge a higher price. This highlights the effect of urgency - or impending doom - that drives the monopolist's decisions.

6 Concluding Remarks

We introduce a model of a monopolist firm that operates in a resource market where the prices are endogenously determined and in which ecological uncertainty takes the form of both Gaussian noise and regime shifts. These shifts are allowed to be dependent on the the monopolist's extraction efforts: unlike the previous literature, we explicitly model the

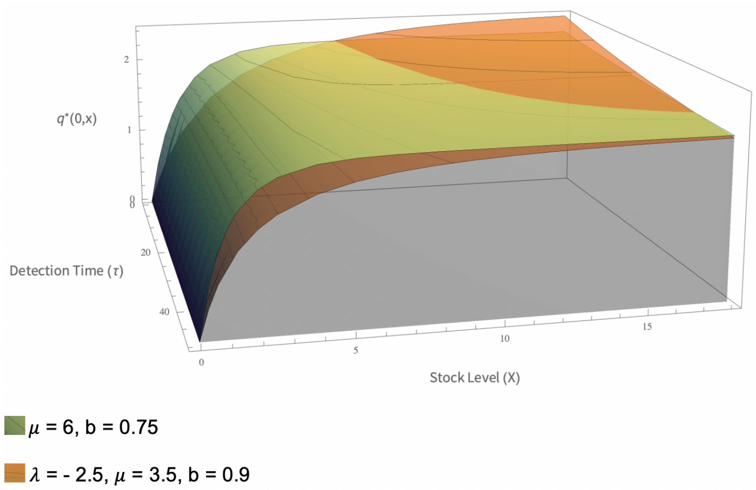


Figure 6: Optimal extraction policies for the monopolist facing a regime shift in the resource dynamics and a change in the slope of the market demand.

firm's detection process of the regime change and incorporate it in the profit-maximizing policies. Our closed form solutions help us pin down the economic mechanisms that drive the extraction behaviour of the firm. In the event of a negative regime shift, for low resource stock levels, an increase in the resource rent results in the firm adopting a precautionary policy by reducing extraction. For higher stock levels, a regime shift leads to an increase in extraction due to an altered and relatively shorter time horizon and demand elasticity - which reduces the resource rent and results in the monopolist adopting an aggressive behaviour.

To conclude, some caveats are in order. Our model framework is intentionally simple and stylized in order to be able to obtain analytical solutions, allowing us to characterize the importance of the market structure whilst still allowing for a rich solution behavior. Furthermore, we have made two simplifying assumptions in the form a constant growth rate and cost function that is not directly dependent on stock. Both these assumptions can be relaxed at the expense of obtaining an extraction policy only in numerical form. Lastly, a potential criticism may be the assumption of a monopoly: a pure monopoly is rare, and a game theoretic approach of several powerful players interacting could be more appropriate to the renewable resource market. However, our primary aim is to see how a firm, whose prices are not exogenously determined, adjusts its extraction levels in the presence a regime shift that it can attempt to detect in real time. The case of a monopoly can then be used as a first step towards richer competition structures.

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A Viscosity solutions

In all that follows we will use as a reference Fleming and Soner (2006) as well as follow its notations. What we want to achieve is to show that the value function V is a weak solution of the optimization problem (15), and if we obtain a form of V we can conclude it solves the firm's problem (in a weak sense).

We write the HJB equation in form of its infinitesimal generator. Define the set $\mathcal{D} \in C([0, \tau_c] \times \mathbb{R})$. Then $V(t, x) \in \mathcal{D}$ is a classical solution of the optimization problem (15) if it satisfies the equation

$$-\frac{\partial}{\partial t}V + A_t[V(t, .)](x) = 0, \tag{42}$$

where A is the generator of the HJB equation. If X_t were modeled as a geometric Brownian motion, the state constraint would not need to apply, since the multiplicative nature of the noise would naturally allow the resource stock to be positive, and because of the well-behaving nature of the functional forms of the problem we expect a smooth solution for all $X_t > 0$. But imposing $X_t \geq 0$ does not imply that that the value function has to be differentiable at $X = 0$. Now, define a continuous function $\mathcal{H}$ (the Hamiltonian) such that

$$A_t[\phi](x) = \mathcal{H}(t, x, D\phi(x), D^2\phi(x))$$

and consider the equation

$$-\frac{\partial}{\partial t}W(t, x) + \mathcal{H}(t, x, DW(t, x), D^2W(t, x)) = 0. \tag{43}$$

A function $V(t, x) \in \mathcal{C}([0, \tau_c] \times \mathbb{R})$ is a viscosity subsolution of (43) if for all $v \in C^\infty(\mathcal{D})$

$$-\frac{\partial}{\partial t}v(\bar{t}, \bar{x}) + \mathcal{H}(\bar{t}, \bar{x}, Dv(\bar{t}, \bar{x}), D^2v(\bar{t}, \bar{x})) \leq 0$$

for every point $(\bar{t}, \bar{x})$ which is a local maximum of $V - v$. Similarly, $V(t, x)$ is a viscosity supersolution of (43) if if for all $v \in C^\infty(\mathcal{D})$

$$-\frac{\partial}{\partial t}v(\bar{t}, \bar{x}) + \mathcal{H}(\bar{t}, \bar{x}, Dv(\bar{t}, \bar{x}), D^2v(\bar{t}, \bar{x})) \geq 0.$$

for every point $(\bar{t}, \bar{x}) \in \mathcal{D}$ which is a local minimum of $V - v$. The function $V(t, x)$ is a viscosity solution of the equation (43) if it is both a viscosity subsolution and a viscosity supersolution. This implies that the function $V(t, x)$ is a weak solution of the optimization problem (15). Let us now show that V is a viscosity solution of our problem (15).

Let $v \in C^2([0, \tau_c] \times \mathbb{R})$, let $V - v$ be maximized at the point $(\bar{t}, \bar{x}) \in ([0, \tau_c] \times \mathbb{R})$ and let us fix an optimal control (extraction rate) $q \in Q$. Let $X(.) = X(.;t,q)$ be the controlled stochastic process that drives the resource stock. For every time $\tau > \bar{t}$ for which $X_\tau > 0$, we have, using Ito's lemma and Bellman's principle of optimality,

$$
\begin{aligned}
0 &\leq \frac{\mathbb{E}_{\bar{t}}\big[V(\bar{t},\bar{x}) - v(\bar{t},\bar{x}) - V(\tau, x(\tau)) + v(\tau, x(\tau))\big]}{\tau - \bar{t}} \\
0 &\leq \frac{1}{\tau - \bar{t}}\mathbb{E}_{\bar{t}}\left[\int_{\bar{t}}^{\tau} \Pi(t,x,q)dt - v(\bar{t},\bar{x}) + v(\tau, x(\tau))\right].
\end{aligned}
$$

This implies

$$0 \leq v_t(\bar{t},\bar{x}) + \Pi(\bar{t},x,q) + v_x(\mu + q) + \frac{\sigma^2}{2}v_{xx}$$

for all $q \in Q$: we can then write

$$
\begin{aligned}
0 &\leq v_t(\bar{t},\bar{x}) + \sup_{q\in Q}\left[\Pi(\bar{t},x,q) + v_x(\mu - q) + \frac{\sigma^2}{2}v_{xx}\right] \\
0 &\leq v_t - \mathcal{H}(\bar{t},\bar{x}, Dv(\bar{t},\bar{x}), D^2v(\bar{t},\bar{x})).
\end{aligned}
$$

This proves that V is a viscosity subsolution of the problem (15). Proceeding similarly proves that V is a viscosity supersolution of the problem: if $V - v$ attains a minimum at $(\bar{t},\bar{x})$ then for any $\epsilon > 0$ and $\tau > \bar{t}$ we can find a control $q \in Q$ such that

$$0 \geq -\epsilon(\tau - \bar{t}) + \mathbb{E}\left[\int_{\bar{t}}^{\tau} \Pi(t,x,q)dt - v(\bar{t},\bar{x}) + v(\tau, x(\tau))\right]$$

which implies

$$\epsilon \geq \frac{1}{\tau - \bar{t}}\mathbb{E}_{\bar{t}}\left[\int_{\bar{t}}^{\tau} \Pi(t,x,q)dt - v(\bar{t},\bar{x}) + v(\tau, x(\tau))\right].$$

Proceeding equivalently as before, one shows that V is a viscosity supersolution of (15). We can conclude that V is a viscosity solution of (15). Note that for every time $\tau_e \in [0, \tau_c]$ for which $X_\tau > 0$, since for optimality we have $\Pi_q(., q^*) - V_x = 0$ and Π is continuous and twice differentiable in q, it can be easily shown that the inequalities of the definition of sub- and supersolution are satisfied with equality, which means that $V(t,x)$ is also a classical solution of (15) for each $t = \tau_e$. We now need to deal with the positivity constraint. Given the "feasible" set $\mathcal{D}' = ([0,\tau_c] \times O \subset \mathbb{R}^+)$, we cannot impose that the value function $V(t,x)$ is differentiable (or continuous, for that matter) at 0 at the left boundary of $\partial\mathcal{D}'$. Following Fleming and Soner (2006), we need to impose a boundary inequality, which does not require neither V nor the boundary $\partial\mathcal{D}'$ to be differentiable at 0. This implies that the value function $V(t,0)$ must be a viscosity subsolution of (15). Following the previous definitions, we must have

$$
\begin{aligned}
v_t(t,0) &\leq -\mathcal{H}(t,0,Dv,D^2v) && (44)\\
&\leq \sup_{q\in Q}\left\{\Pi(t,x,q) + v_x(0)(\mu - q) + v_{xx}(0)\frac{\sigma^2}{2}\right\} && (45)
\end{aligned}
$$

for all continuous functions for which $V - v$ is locally maximized around $x = 0$. Given a natural boundary condition given by the fact that when the resource is zero, the extraction must be zero and consequently the objective Π must be zero. Since $V - v$ has to be maximized around 0, we have

$$\mathcal{H}(t, 0, a, a_x) \geq \mathcal{H}(t, 0, v_x(t, 0), v_{xx}(t, 0)) \quad \forall a \geq v_x(t, 0).$$

The proof is simple, one just needs to write $\mathcal{H}(t, 0, \alpha, \alpha_x)) = \sup_{q \in Q} \Pi(t, q) + \alpha(\mu + q) + \alpha_x \frac{\sigma^2}{2}$ and use $\alpha \geq v_x(t, 0)$ to show the inequality holds. Given this result, condition (45) is easily seen to be satisfied by $V(t, 0) = 0$, which we choose because of its immediate intuitive economic interpretation. We therefore can say that the constrained viscosity solution given by

$$\begin{array}{rcl} V_x(t,0) & \geq & \Pi(t,0,q) \qquad (46) \\ V(t,0) & = & 0 \qquad (47) \\ V(t,x) & \text{solves} & V_t - \mathcal{H}(t, x, DV(t,x), D^2V(t,x)) = 0 \quad x \in [(0, \tau_c] \times \mathbb{R}] \end{array}$$

is a solution to the problem (15). Uniqueness of the solution is proven by means of the comparison principle, and since the proof follows closely the one provided by Crandall et al. (1992), is omitted.