On the existence of categorical connections

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1 Introduction

This is an extended abstract for [Bat10]. Recently there has been much interest in categorifying bundles and connections with a view to applications in Physics. In [BS04] a notion of 2-bundle and 2-connection was introduced. Faria Martins and Picken defined and studied a concrete special case which they called categorical connection on a principal bundle, where the structure group is put in the form of a crossed module.

We now describe this notion and present our result considering existence of categorical connections on a given principal bundle, which we achieve by essentially describing categorical connections as sections in an associated vector bundle.

2 Categorical connections

Definition 2.1. *A* crossed module *is a quadruple* $\mathcal{G} = (H, G, \partial : H \to G, \triangleright)$ *where* H, G *are groups,* $\triangleright : G \to Aut(H)$ *is a left action of* G *on* H *and* $\partial : H \to G$ *is an equivariant group morphism, ie*

$$\partial(g \triangleright h) = g\partial(h)g^{-1} \qquad \textit{for all } g \in G, h \in H$$

and we also require the Peiffer identity*:*

$$\partial(e) \triangleright h = ehe^{-1} \qquad \text{for all } e, h \in H.$$

When H and G are Lie groups and $\partial, \triangleright$ are smooth, $\mathcal{G}$ is said to be a Lie crossed module.

A particular kind of 2-connection is the following.

Definition 2.2. *Let $\mathcal{G} = (\partial : H \to G, \triangleright)$ be a crossed module, with associated differential crossed module $\mathfrak{G} = (\partial : \mathfrak{h} \to \mathfrak{g}, \triangleright)$. A $\mathcal{G}$-*categorical connection *on a principal G-bundle $p : E \to B$ is a pair (m, ω) where $\omega \in \Omega^1(E, \mathfrak{g})$ is a connection 1-form on E and m is an equivariant horizontal 2-form in $\Omega^2(E, \mathfrak{h})$, such that*

$$\partial(m) = \Omega \tag{1}$$

For instance, let $\mathcal{G} = (\mathrm{id} : G \to G, \triangleright)$, where $\triangleright$ is the adjoint action of G on G.

Let $p : E \to B$ be principal G-bundle with connection one-form ω. Then (ω, m) is a $\mathcal{G}$-categorical connection, where $m = \Omega$ is the curvature 2-form of ω.

2.1 Categorical connections as bundle sections

Definition 2.3. *Let (E, p, B) be a principal G-bundle. Let (ρ, V) be a representation of G in a finite dimensional vector space V. A* tensorial form *of degree k on E of type (ρ, V) is a form $\varphi \in \Omega^k(E, V)$ such that*

- *φ is G-invariant for the induced action of G on V, i.e. $R_g^* = \rho(g^{-1})\varphi$,*
- *φ is horizontal, i.e. $\varphi(X_1, \ldots, X_k) = 0$ when one of the tangent vectors X_i is vertical.*

A connection ω is a tensorial form of degree 1 in E of type $(\mathrm{Ad}, \mathfrak{g})$. By definition, m is a tensorial form of degree 2 in E of type $(\triangleright, \mathfrak{h})$. From [KN63]:

Lemma 2.4. *Let (E', q, B) be the bundle associated with the principal G-bundle (E, p, B) with fibre V, where G acts naturally by ρ. There is a one-to-one correspondence between the tensorial forms as in Definition 2.3 and sections of the bundle*

$$\begin{array}{c} \wedge^k T^* B \otimes E' \\ \big\downarrow \\ B \end{array}$$

2.2 Existence

A connection 1-form on any bundle can always be found by the means of partitions of unity, our problem lies with the form m. Our main result is the following:

Theorem 2.5. *Let $\xi = (E, p, B)$ be a principal G-bundle, and let $\mathcal{G} = (\partial : H \to G)$ a Lie crossed module. The following are equivalent:*

1. *There is a $\mathcal{G}$-categorical connection on ξ.*

2. *ξ admits a connection with curvature 2-form Ω such that $Im(\Omega) \subset \mathfrak{a}$, where $\mathfrak{a} = Im\ \partial$.*

3. *ξ admits a reduction to a structure group $G' \subset G$, with Lie algebra $\mathfrak{g}'$ contained in $\mathfrak{a}$.*

Corollary 2.6. *In the conditions of Theorem 2.5, if $\partial : \mathfrak{e} \to \mathfrak{g}$ is surjective there is always a categorical connection on a bundle with structure group G.*

To prove the Theorem 2.5, note that (1) $\Rightarrow$ (2) follows by definition, whereas (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) follow by the Ambrose-Singer theorem and (2) $\Rightarrow$ (1). For the latter, we described the categorical connection as sections of a particular bundle. Then our problem reduces to obtaining an equivariant right section of a surjective bundle morphism, which is always possible.

2.3 Example of non-existence

Let $(H, .) = (\mathbb{R}^n, +)$ and $(G, .) = (GL(n, \mathbb{R}), \circ)$. Take the trivial map $\partial : H \to G$ together with the action $f \triangleright e = f(e)$ of $GL(n, \mathbb{R})$ in $\mathbb{R}^n$. Clearly $\mathcal{G} = (\partial : H \to G, \triangleright)$ is a crossed module. Since $\partial : \mathfrak{h} \to \mathfrak{g}$ is the zero map, in order for there to be a categorical connection, the holonomy group would have to be a discrete subgroup of G. But in general that is not the case: take for instance the frame bundle over S^2, with $G = GL(2, \mathbf{R})$ and $H = (\mathbf{R}^2, +)$. If the frame bundle could be reduced to a discrete group then it would be trivial as, for any topological group, principal G-bundles over S^2 are classified by conjugacy classes in $\pi_1(G)$. In particular there would be a section of the frame bundle and hence a nowhere vanishing tangent vector field to S^2 which is impossible by the hairy ball theorem.

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