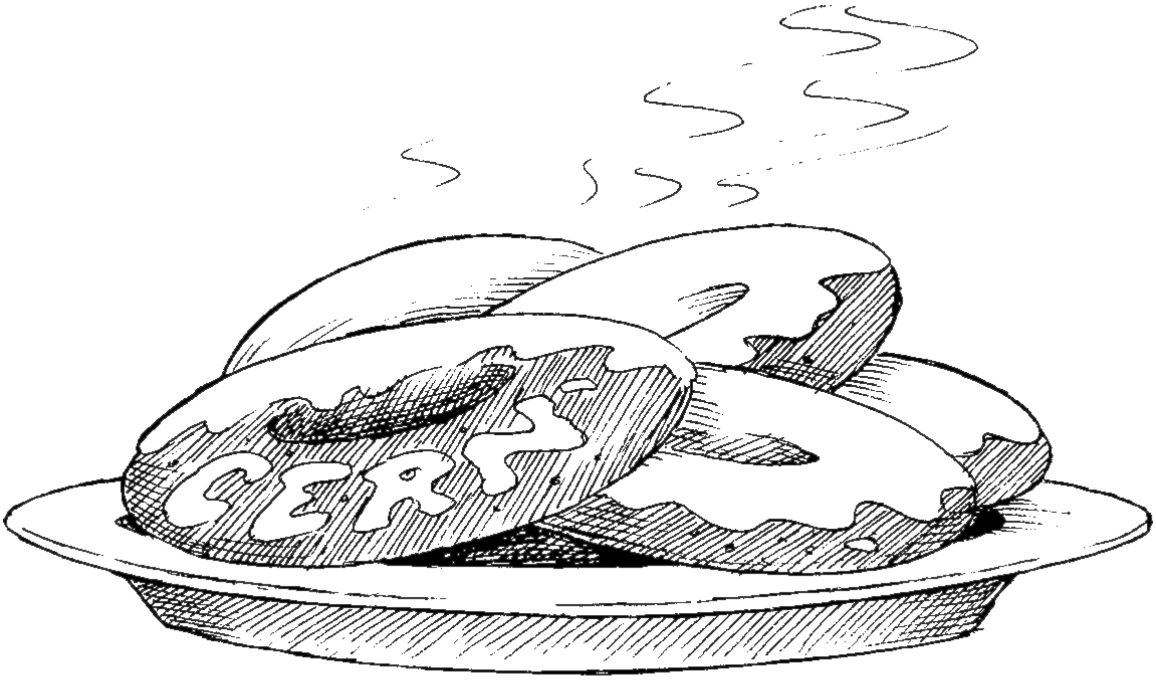
Electron's
$\mathbb{II}$
Destiny
The Standard Model



A Very Short SM Book by
N. Booker

To my parents

Contents

1 Instead of a foreword **4**
1.1 Special relativity . . . 4
1.2 Standard model . . . 5

2 Particle interactions **8**
2.1 Fermi's golden rule . . . 8
2.2 Dirac equation . . . 13
2.3 Angels and demons . . . 15

3 Quantum electrodynamics **17**
3.1 Spin, helicity and chirality . . . 17
3.2 Gauge invariance . . . 19
3.3 General QED calculations . . . 20
3.4 $2 \to 2$ QED calculations . . . 22

4 Quantum chromodynamics **24**
4.1 Colour . . . 24
4.2 General QCD calculations . . . 25

5 Weak interaction **26**
5.1 Parity . . . 26
5.2 Weak force . . . 26
5.3 Pion . . . 26
5.4 W boson . . . 26

6 Flavour physics **27**
6.1 Quark mixing . . . 27
6.2 CPT symmetry . . . 27
6.3 Kaon . . . 27
6.4 Z boson . . . 27
6.5 electroweak unification . . . 27

7 Higgs boson and the destiny of HEP **28**
7.1 Scalar field . . . 28
7.2 Higgs mechanism . . . 28
7.3 Beyond the standard model . . . 28

Chapter 1

Instead of a foreword

About the notes

Electron's Destiny III emerged from a series of lecture notes based on the standard model course at University College London and its lecture notes in the 2024-25 year. Other sources used for the book include:

- *QCD and Hadron Physics* and *Kern- und Teilchenphysik* by Gernot Eichmann (University of Graz).
- *Theoretical Particle Physics* by Axel Maas (University of Graz).
- *A Course in Modern Mathematical Physics* by Peter Szekeres (University of Adelaide).

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1.1 Special relativity

As HEP calculations mostly fall into relativistic limits, it is therefore not surprising that we will make use of SR in HEP. Special relativity as seen in HEP follows largely the same formalism (e.g. $c = 1$), but with a few key differences:

- **In stark contrast to GR**, the Minkowski metric is written with the reverse signature

$$g^{ij} = g_{ij} = \text{diag}(1, -1, -1, -1) \tag{1.1}$$

- A 4-vector is likewise x^μ. However, we often write $x^\mu = (x^0, x^k) = (x^0, \vec{x})$, where $x^k = \vec{x}$ is the corresponding 3-vector. Its scalar equivalent is simply x. The contravariant 4-position, 4-momentum, 4-current and 4-differential are thus

$$x^\mu = (t, \vec{x}) \quad p^\mu = (E, \vec{p}) \quad j^\mu = (\rho, \vec{J}) \quad \partial^\mu = (\partial^t, -\vec{\nabla}) \tag{1.2}$$

  where we recall that ρ is the charge density and $\vec{J}$ is the 3-current density. Using the inverse metric, we find that their covariant counterparts are

$$x_\mu = (t, -\vec{x}) \quad p_\mu = (E, -\vec{p}) \quad j_\mu = (\rho, -\vec{J}) \quad \partial_\mu = (\partial^t, \vec{\nabla}) \tag{1.3}$$

  Remark 1.1 When using the 4-differential, a covariant 4-differential always operates on a contravariant 4-vector and vice versa. Hence the negative signs cancel out.

The *Lorentz boost* is

$$P^j = \Lambda^i_j P^i = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^i \tag{1.4}$$

where

$$\beta = \frac{v}{c} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} \quad p = \beta m \gamma c = \gamma m v \tag{1.5}$$

m is the rest mass or the *invariant mass*. The inverse Lorentz transformation is

$$P^j = \Lambda^i_j P^i = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P^i \tag{1.6}$$

Theorem 1.1 (4-momentum)

$$p^2 = p \cdot p = p_i p^i = E^2 - |\vec{p}|^2 = m^2 \tag{1.7}$$

Note 1.1 (Massless assumption) Quite often, we will be working with scenarios in which $m = 0$, which leads to $E = p$. This assumption can be made when:

- **Massless particles:** This is self-explaintory.
- **Ultra-relativistic limit of massive particles:** When the speed of a massive particle approaches c, we have $E = \sqrt{m^2c^4 + p^2c^2} \ll mc^2$[a], making the m negligible compared to the E.

[a] Here E expectedly accounts for both the rest mass-energy and the kinetic energy.

1.2 Standard model

The central concept is quantum fields, whose vibrations we refer to as *particles*. We recall *fermions* and *bosons*:

- Fermions are 'matter particles'. They have half-spin and obey the Pauli exclusion principle.
- Bosons are 'force particles'. They have integer spin.

Three *generations* exist:

Generations of particles

Generation	Quark	Lepton[1]
1	Up/down	Electron
2	Charm/strange	Muon
3	Top/bottom	Tau

Remark 1.2 The first generation makes up ordinary matter, and the second (and third) generations are their heavier (and even heavier) counterparts.

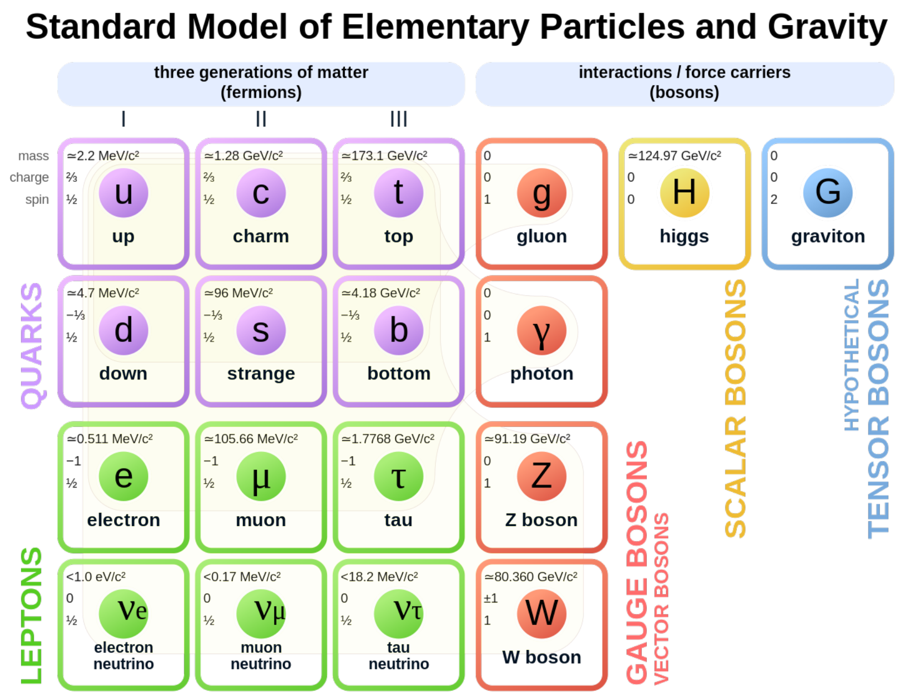

Figure 1.1: Standard model (with gravitons).

Now that we have reviewed elementary particles, we reminds ourselves that particle interactions can be illustrated by *Feynman diagrams*.

Theorem 1.2 (Feynman diagrams) Feynman diagrams are complicated but nonetheless doable:

- List the particle contents of both sides, with any particles consisting of quarks broken down to their quark contents.
- Link all particles that appear on both sides.
- Quark changes are noted by emissions or absorptions of W bosons, which almost invariably transform into other arbitrary particles.
  - An end-product of $+e$ charge (e.g. u and $\bar{d}$) indicates a W^+ boson.
  - An end-product of $-e$ charge (e.g. $\bar{c}$ and s) indicates a W^- boson.
- A photon can only couple to particles that carry EM charge. i.e. it could produce a quark-antiquark/W^+W^-/charged lepton-antilepton pair, and would find its other end at another particle-antiparticle pair. This makes sense, as photons are produced in annihilation. However, due to its nature, a photon **must** have a charged particle involved.
- A gluon can only couple to particles that carry colour charge. i.e. it can only produce gluon pairs and quark-antiquark pairs. It would find itself emitted by a particle that appears on both sides. The selection is completely arbitrary.
- A W boson can also produce a charge lepton-antineutrino/charge antilepton-neutrino pair, with its sign depending on the generated pair. If both ends of a W boson have one particle on each side, we do not denote the sign of the W boson as it can go either way. Sometimes a W is called a ‘charged current’.
- The Z boson, capable of producing neutrino-antineutrino pairs and quark-antiquark pairs, is largely uninteresting (see previous remark on why). However, if there is a neutrino-antineutrino pair, it **must** come from a Z boson. Sometimes a Z is called a ‘neutral current’.

- If you have a vertex connecting a lepton to a quark, then there is definitely something wrong!

For particles bumping into each other, we have annihilation and scattering. Unless we have a particle-antiparticle pair (annihilation), it is always scattering.

- For annihilation, the exchange particle's line is drawn horizontally.
- For scattering, the exchange particle's line is drawn vertically. In scattering:
  - Cuarged currents make the particles change their positions.
  - Neutral currents make the particles stay at their positions.

And that's it!

Chapter 2

Particle interactions

2.1 Fermi's golden rule

Before deriving Fermi's golden rule, we first introduce the so-called the *Dirac delta* or the *Dirac delta function* if you're boring at parties. The 'function' was introduced by Dirac as a continuous form of the discrete *Kronecker delta* which we have seen before.

Definition 2.1 (Dirac delta) The Dirac delta has unit area (i.e. an area of 1). When $x \neq 0$, one has $\delta = 0$. Hence, for meaningful calculations, x must be fixed to 0. A general version of the Dirac delta fixed to some a is

$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1 \qquad \int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a) \tag{2.1}$$

Remark 2.1 The importance of $f(a) = \int_{-\infty}^{\infty} f(x)\delta(x-a)dx$ is that it effectively allows us to eliminate integrals, just like how Krocker delta eliminates indices[1].
Remark 2.2 Any function that satisfies these conditions can be a Dirac delta, likes an infinitely narrow Gaussian $\lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{x^2}{2\sigma^2}}$. Another way to put it is

$$\int_{y_1}^{y_2} \delta(y)dy = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases} \tag{2.2}$$

Now consider again the Dirac delta of a function $f(x)$. This time, we have an x_0 such that $f(x_0) = 0$. By performing the integral $\int_{x_1}^{x_2} \delta(f(x)) \frac{df}{dx} dx$ we find that

$$\delta(f(x)) = \left|\frac{df}{dx}\right|^{-1} \delta(x, x_0) \tag{2.3}$$

Derivation 2.1 (The simplest derivation) Consider a normalised solution to the TISE

$$\hat{H}_0 \phi_k = E_k \phi_k \quad \text{where} \quad \langle \phi_j | \phi_i \rangle = \delta_{ij} \tag{2.4}$$

The implication is that the states are stable with no transitions between them. Now we introduce a Hamiltonian perturbation $\hat{H}'(t, \hat{x})$ which represents transitions between states. The TISE becomes

$$i\partial_t \psi = (\hat{H}_0 + \hat{H}')\psi \tag{2.5}$$

$|\psi_k\rangle$ itself is a complete basis, allowing us to represent ψ in a superposition of basis states:

$$\psi(t, \vec{x}) = \sum_k c_k(t) \phi_k(\vec{x}) e^{-iE_k t} \tag{2.6}$$

[1] And replaces them with other indices.

where the time-evolution coefficients $c_k(t)$ represent the transitions between states[a]. We substitute this back into the TISE and find

$$i = \sum_l \partial_t c_k \phi + k = \sum_k \hat{H}' c_k(t) \phi_k e^{-iE_k t} \tag{2.7}$$

We assume that at $t = 0$, the initial state is $|\phi_i\rangle$ or in wavefunction formalism ϕ_i, at which point

$$\psi(0) = \underbrace{|\phi_i\rangle}_{①} = \underbrace{\sum_k c_k(0)|\phi_k\rangle}_{②} \tag{2.8}$$

As ① = ②, the coefficient $c_k(0)$ must be δ_{ik}. As δ_{ik} is a Kronecker delta, we have essentially fixed the right-hand side.

$$i \sum_l \partial_t c_k \phi_k = \hat{H}' \phi_i e^{-iE_i t} \tag{2.9}$$

Taking the inner product of both sides with the final state $|\phi_f\rangle$ or in wavefunction formalism $\phi_f(x)$:

$$\partial_t c_f = -i\langle \phi_f | \hat{H}' | \phi_i \rangle e^{i(E_f - E_i)t} \tag{2.10}$$

where we define the time-independent *transition matrix element* with units of energy

$$T_{fi} = \langle \phi_f | \hat{H}' | \phi_i \rangle = \int_V \phi_f^*(x) \hat{H} \phi_i(x) d^3x \tag{2.11}$$

Integrating the previous differential equation yields c_f after some time T[b] yields

$$c_f(T) = -iT_{fi} \int_0^T e^{i(E_f - E_i)t} dt \tag{2.12}$$

As $c_f(T)$ represents transition, the transition probability P_{fi} observes

$$P_{fi} = c_f(T) c_f^*(T) \tag{2.13}$$

The differential of the transition rate $d\Gamma_{fi}$ is then

$$d\Gamma_{fi} = \frac{P_{fi}}{T} \tag{2.14}$$

Solving for the integral (which we will leave as an exercise for the mathematically inclined) gives

Exercise 2.1 Derive that

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \left| \frac{dn}{dE_f} \right|_{E_i} \tag{2.15}$$

Hint: The Dirac delta $\delta(E_f - E_i)$ performs the fixture $E_f = E_i$.

By recalling undergrad thermodynamics, one can determine that $\left| \frac{dn}{dE_f} \right|_{E_i}$ is simply the *density of states* $\rho(E_i)$. We hence have Fermi's golden rule to the 1^{st}-order

Theorem 2.1 (1^{st}-order Fermi's golden rule)

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i) \tag{2.16}$$

[a] When a perturbation is introduced, it can induce transitions between different energy eigenstates. Hence $c_k(t)$ represents the probability amplitude for the system to be in state $|\phi_k\rangle$ at time t.

[b] Note that this is not T_{fi}

Derivation 2.2 (2nd-order terms) Previously we have assumed $c_k(t)$ to be a Kronecker delta. i.e. $c_{k\neq i}(t) = 0$. This causes any $\hat{H}$ terms from the 2nd-order to be *smol*. In the 2nd-order perturbation theory, we need the $\hat{H}$ terms from the 3rd-order to be *smol*.
An improved approximation which preserves 2nd-order terms can thus be made by assuming $c_i(t) = 1$. The original differential equation is then

$$\partial_t c_f = \underbrace{-i\langle\phi_f|\hat{H}|\phi_i\rangle e^{i(E_f-E_i)t}}_{1^{\text{st}}\text{-order terms}} + \underbrace{(-i)^2 \sum_{k\neq i} \langle\phi_f|\hat{H}'|\phi_k\rangle e^{i(E_f-E_k)t} \int_0^t \langle\phi_k|\hat{H}'|\phi_i\rangle e^{i(E_k-E_i)t'} dt'}_{2^{\text{nd}}\text{-order terms}} \tag{2.17}$$

Again solving T_{fi} like the 1st-order solution, we find

Theorem 2.2 (2nd-order Fermi's golden rule)

$$T_{fi} = \langle\phi_f|\hat{H}|i\rangle + \sum_{k\neq i} \frac{\langle\phi_f|\hat{H}'|\phi_k\rangle\langle\langle|\hat{H}'|\phi_i\rangle}{E_i - E_k} \tag{2.18}$$

Derivation 2.3 (SR correction) We now modify Fermi's golden rule for SR conditions. To simplify calculations involving Lorentz transformations, calculate the integral in momentum space instead of the position space. Recalling the density of states and the wavevector one can then use the transformation $dx = dp/2\pi$. We then consider the simple case where the final state consists of 2 particles, say, particles 1 and 2. In this process, both energy and momentum must be conserved:

$$E_1 + E_2 = E_i \quad \text{and} \quad \vec{p}_1 + \vec{p}_2 = \vec{p}_i \tag{2.19}$$

This can be enforced using the following Dirac deltas:

$$\int \ldots \delta\left(E_i - E_1 - E_2\right) dE \quad \text{and} \quad \int \ldots \delta\left(\vec{p}_i - \vec{p}_1 - \vec{p}_2\right) d^3\vec{p} \tag{2.20}$$

Hence $\Gamma_{fi} = 2\pi|T_{fi}|^2\rho(E_i)$ becomes

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta\left(E_i - E_1 - E_2\right)\left(\vec{p}_i - \vec{p}_1 - \vec{p}_2\right) \frac{d\vec{p}_1^3}{(2\pi)^3} \frac{d\vec{p}_2^3}{(2\pi)^3} \tag{2.21}$$

Remark 2.3 Here $\frac{d\vec{p}_1^3}{(2\pi)^3}\frac{d\vec{p}_2^3}{(2\pi)^3}$ indicates that we integrate all possible momentums.
Now we need the expression to be Lorentz-invariant. This is accomplished by the Lorentz invariant *transition matrix* $\mathcal{M}_{fi}$, which describes transitions from $|\phi_i\rangle$ to $|\phi_f\rangle$:

$$\mathcal{M}_{fi} = \langle\phi_f|\hat{H}|\phi_i\rangle \tag{2.22}$$

However, we are currently looking at relativistic states. To complete our calculation, we must return to our original non-relativistic states. Relativistic states are normalised differently from non-relativistic states. The initial and final states normalised in non-relativistic conditions are given by

$$|\phi_{i(\text{NR})}\rangle = \frac{|\phi_i\rangle}{\sqrt{2E_i}} \quad \text{and} \quad |\phi_{f(\text{NR})}\rangle = \frac{|\phi_f\rangle}{\sqrt{2E_f}} \tag{2.23}$$

Hence

$$\mathcal{M}_{fi} = \sqrt{2E_f \cdot 2E_i}\langle\phi_{f(\text{NR})}|\hat{H}|\phi_{i(\text{NR})}\rangle = \sqrt{2E_f \cdot 2E_i}T_{fi} \tag{2.24}$$

The Lorentz invariant form of Fermi's golden rule in SR conditions is then

Theorem 2.3 (Fermi's golden rule in SR limits)

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |T_{fi}|^2 \delta\left(E_i - E_1 - E_2\right)\left(\vec{p}_i - \vec{p}_1 - \vec{p}_2\right) \frac{d\vec{p}_1^3}{(2\pi)^3 2E_1} \frac{d\vec{p}_2^3}{(2\pi)^3 2E_2} \tag{2.25}$$

Remark 2.4 While both $\mathcal{M}_{fi}$ and the integral as a whole are Lorentz-invariant, the reaction rate Γ_{fi} is not due to the introduction of the extra terms $1/2E_i$.

In specific scenarios, Fermi's golden rule yields commonly used results:

Derivation 2.4 (Two-body decay) We start with Fermi's golden rule in SR limits. The Lorentz invariance of the integral means that it can be evaluated in any frame, and the most convenient frame is the centre-of-mass frame where we have

$$E_i = m_i \quad \vec{p}_i = 0 \tag{2.26}$$

Hence, from the Dirac deltas, we perform the following fixtures:

$$m_i - E_1 - E_2 = 0 \quad \vec{p}_1 + \vec{p}_2 = 0 \tag{2.27}$$

At the same time, we note the energy-momentum relation $E_{1,2} = \sqrt{|\vec{p}_1|^2 + m_{1,2}^2}$. As we are in the centre-of-mass frame, spherical symmetry holds, and the integral can be rewritten as

$$\mathrm{d}\vec{p}^3 = |\vec{p}_1|^2 \mathrm{d}|\vec{p}_1| \sin\theta \mathrm{d}\theta \mathrm{d}\phi = |\vec{p}_1|^2 \mathrm{d}|\vec{p}_1| \mathrm{d}\Omega \tag{2.28}$$

Evaluating the new integral using substitutions and the properties of the Dirac delta yields

Theorem 2.4 (Fermi's golden rule for two-body decay) For *any* two-body decay, the following version of Fermi's golden rule is valid:

$$\frac{1}{\tau} = \Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |\mathcal{M}_{fi}|^2 \mathrm{d}\Omega \tag{2.29}$$

where, due to the energy fixture from the Dirac delta, we know that

$$|\vec{p}^*| = \frac{1}{2m_i} \sqrt{[m_i^2 - (m_1 + m_2)^2][m_i^2 - (m_1 - m_2)^2]} \tag{2.30}$$

Derivation 2.5 (2→2 elastic scattering (centre-of-mass frame)) The general picture is again simple. Particles 1 and 2 bump into each other and produce particles 3 and 4. As we have 2 initial particles, the initial values can be split into two, with the Dirac deltas being

$$\delta\left(E_1 + E_2 - E_3 - E_4\right)\left(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4\right) \tag{2.31}$$

We can adapt $\sigma = \frac{\Gamma_{fi}}{\Phi}$ to our scenario:

$$\sigma = \frac{\Gamma_{fi}}{\Phi} = \frac{\Gamma_{fi}}{v_1 + v_2} \tag{2.32}$$

where v_1 and v_2 are velocities.
Evaluating the integral in the centre-of-mass frame yields

Theorem 2.5 (Differential cross-section in the centre-of-mass frame)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2 s} |\mathcal{M}_{fi}|^2 \tag{2.33}$$

where $s = (E_1 + E_2)^2$ is the so-called *Mandelstam variable* you will see later.

A Lorentz-invariant form of this is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s |\vec{p}_1^*|^2} |\mathcal{M}_{fi}|^2 \tag{2.34}$$

Derivation 2.6 (2→2 scattering (lab frame))

Theorem 2.6 (Differential cross-section in the lab frame)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |\mathcal{M}_{fi}|^2 \tag{2.35}$$

Finally, we complete the derivation by deriving the interaction matrix. Consider the most generic and simplest example:

Derivation 2.7 ($\mathcal{M}_{fi}$ in 2→2 scattering) For simplicity, we only consider *leading order* interactions[a]. A simple interaction in which particles a and b produce c and d via a virtual particle x can have two orderings:

- a emits the particle x which then interacts with b, producing c and d.
- b emits the antiparticle $\bar{x}$ which then interacts with a, producing c and d.

Remark 2.5 Like the interaction matrix itself (as we will see almost immediately), the Feynman diagram sums over these two possibilities. Hence the virtual particle is represented by a vertical line with no direction.

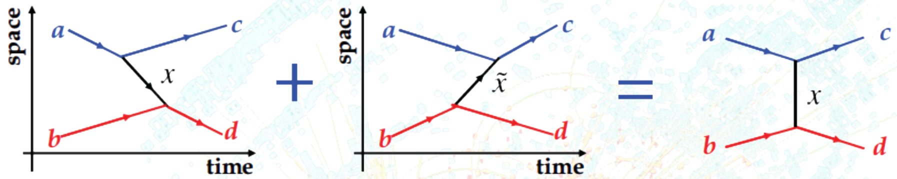

Figure 2.1: The final Feynman diagram is the sum of all possible time orderings.

Recalling the interaction matrix $T_{fi}^{ab} = \sum_{i\neq j} \frac{\langle f|\hat{H}_I|j\rangle\langle j|\hat{H}_I|i\rangle}{E_i - E_j}$ and determining the initial and final states and energies gives

$$T_{fi}^{ab} = zfrac\langle d|\hat{H}_I|b+x\rangle\langle c+x|\hat{H}_I|a\rangle(E_a+E_b) - (E_c+E_x+E_b) \tag{2.36}$$

We then recall how wavefunction normalisation differs in relativistic and non-relativistic wavefunctions

$$\langle c+x|V|a\rangle = \frac{\mathcal{M}_{a\to c+x}}{(2E_a 2E_c 2E_x)^{1/2}} \quad \langle d|V|b+x\rangle = \frac{\mathcal{M}_{b+x\to d}}{(2E_b 2E_x 2E_d)^{1/2}} \tag{2.37}$$

One can then simplfy T_{fi}^{ab} and derive the Lorentz invariant form:

$$\mathcal{M}_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab} = \frac{1}{2E_x}\cdot\frac{\mathcal{M}_{b+x\to d}\mathcal{M}_{a\to c+x}}{(E_a - E_c - E_x)} \quad \mathcal{M}_{fi}^{ba} = \frac{1}{2E_x}\cdot\frac{\mathcal{M}_{a+x\to c}\mathcal{M}_{b\to d+x}}{(E_b - E_d - E_x)} \tag{2.38}$$

But we also have

$$\mathcal{M}_{a+x\to c} = \mathcal{M}_{a\to c+x} = \langle c|\hat{H}_I|a\rangle \quad \mathcal{M}_{b+x\to d} = \mathcal{M}_{b\to d+x} = \langle d|\hat{H}_I|b\rangle \tag{2.39}$$

where $\hat{H}_I$ is the interaction Hamiltonian. Summing over and simplifying:

$$\mathcal{M}_{fi} = \mathcal{M}_{fi}^{ab} = \frac{\langle d|\hat{H}_I|b\rangle\langle c|\hat{H}_I|a\rangle}{(p_a - p_c)^2 - m_x^2} \tag{2.40}$$

where $\frac{1}{(p_a-p_c)^2 - m_x^2} = \frac{1}{q^2 - m_x^2}$ is the *propagator term.* q^2 is the 4-momentum transfer.

[a]Going beyond the lowest order often yields infinite solutions.

2.2 Dirac equation

At this point, the Schrödinger equation should be very familiar:

$$i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi + V\psi \tag{2.41}$$

In particle physics, we need to modify the Schrödinger equation as it is not Lorentz-invariant. Historically, the first attempt at yielding a Lorentz-invariant version of the Schrödinger equation manifested in the so-called *Klein-Gordon equation*:

Derivation 2.8 (Klein-Gordon equation) We recall the energy-momentum relationship (again with $c = 1$)

$$E^2 = p^2 + m^2 \tag{2.42}$$

In QM, E and p become operators $\hat{E} = -\partial_t$ and $\hat{p} = -i\nabla$. Using this on a wavefunction yields

Theorem 2.7 (Klein-Gordon equation)

$$\partial_t^2\psi = \nabla^2\psi - m^2\psi \quad\rightarrow\quad \underbrace{(\partial^i\partial_i + m^2)\psi = 0}_{\text{index notation}} \tag{2.43}$$

This equation has the following plane wave solution:

$$\psi(x,t) = Ne^{i(p\cdot x - Et)} \tag{2.44}$$

Remark 2.6 What could possibly go wrong?

As it turns out, the Klein-Gordon equation predicts negative mass (i.e. energy) with *associated* negative probability amplitudes. While the negative energy solutions can be discarded in classical scenarios, in quantum mechanics, all solutions are required to form a complete set of states.

Derivation 2.9 (Dirac equation) It wasn't until Dirac came around that everything was saved. The Klein-Gordon equation has second derivatives, which Dirac replaced with first derivatives:

Theorem 2.8 (Dirac equation)

$$\hat{E}\phi = (\alpha\cdot\hat{p} + \beta m)\phi \quad\rightarrow\quad i\partial_t\psi = (\beta m - i\alpha_i\partial_i)\psi \tag{2.45}$$

where α and β are coefficients and i sums over the spatial dimensions.

At the same time, any solution must also satisfy the Klein-Gordon equation, as it is essentially a rephrasing of Einstein's SR energy conditions. To prove that the Dirac equation does, we square the indexed equation, which eventually becomes the Klein-Gordon equation when certain conditions are imposed on the coefficients α and β:

Theorem 2.9 (Dirac equation coefficient properties)

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = I \quad \alpha_i\alpha_j + \alpha_j\alpha_i = 0 \quad \alpha_i\beta + \beta\alpha_i = 0 \tag{2.46}$$

where I is unity (1).

To satisfy these conditions, α_i and β must be anti-commuting 4×4 Hermitian[a] matrices. The wave-function is thus a so-called *Dirac spinor*[b]:

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}\phi_1\\ \phi_2\\ \chi_3\\ \chi_2\end{pmatrix} \tag{2.47}$$

Remark 2.7 For matter, the Dirac spinor u is made up of the left-chiral Weyl (2-)spinor ϕ and the right-chiral Weyl spinor χ with ϕ stacked on top of χ. ϕ corresponds to the spin-state while χ encodes relativistic effects. For antimatter this is flipped. You will find this intuitive after a few examples.

[a] So that the Hamiltonian is also Hermitian.

[b]For more on spinors, see *Metric's Destiny II.*

In the Dirac equation, one should be able to recognise an implicit covariant derivative[2]. We can take advantage of this by introducing the so-called γ matrices.

$$\gamma^0 \equiv \beta \quad \gamma^1 \equiv \beta\alpha_x \quad \gamma^2 \equiv \beta\alpha_y \quad \gamma^3 \equiv \beta\alpha_z \tag{2.48}$$

The Dirac equation in index notation is thus

Theorem 2.10 (Dirac equation in index notation)

$$(i\gamma^i\partial_i - m)\psi = 0 \tag{2.49}$$

One can also derive, from the conditions for α^i and β, the anti-commutator of γ. Other properties exist:

Theorem 2.11 (γ matrix properties) The γ matrix observes the following properties:

- γ matrices anti-commute:
$$\{\gamma^i, \gamma^i\} = \gamma^i\gamma^j + \gamma^i\gamma^j = 2g^{ij} \tag{2.50}$$
- γ^0 is Hermitian: $\gamma^{0\dagger} = \gamma^0$
- γ^k where $k = 1, 2, 3$ is anti-Hermitian: $\gamma^{k\dagger} = -\gamma^k$

Using the Clifford algebra (as seen in the appendix), one can then write out the γ matrices:

Definition 2.2 (γ matrices)

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \tag{2.51}$$

where I is the identity matrix and σ^k are the Pauli matrices.

Remark 2.8 Even though the Dirac equation still has negative energy solutions, negative probability amplitudes are eliminated.

Derivation 2.10 (4-current) As we consider the quantum mechanical picture, the charge density has become a probability density $\rho = \psi^\dagger\psi$. From this, the 3-current can be written as

$$\vec{J} = \psi^\dagger\vec{\alpha}\psi \tag{2.52}$$

which gives the 4-current

$$j^i = (\rho, \vec{J}) = \left(\psi^\dagger\psi, \psi^\dagger\vec{\alpha}\psi\right) = \psi^\dagger\gamma^0\gamma^i\psi \tag{2.53}$$

For simplicity, we define the so-called *Dirac adjoint*

Definition 2.3 (Dirac adjoint) The adjoint of some Dirac spinor ψ is $\bar{\psi} = \psi^\dagger\gamma^0$

The 4-current is then

$$j^\mu = \bar{\psi}\gamma^\mu\psi \tag{2.54}$$

Plugging the plane wave solution

$$\psi = Nu(p^\mu)e^{-ip_\mu x^\mu} \tag{2.55}$$

into the Dirac equation, one can cancel the wavefunctions out and find

Theorem 2.12 (Dirac equation in momentum form)

$$(\gamma^i p_i - m)u = 0 \tag{2.56}$$

where u is a four-Dirac spinor dependent only on energy and momentum. It is the spinor analogue to the wavefunction/state vector which accounts for relativistic energy-momentum conservation.

Now the quantity we want to solve becomes u.

[2]Note that the covariant derivative in Minkowski spacetime is just the ordinary 4-derivative as there is no curvature, and all Christoffels vanish.

Derivation 2.11 (The simplest solution) The simple solution assumes the particle to be at rest at $t = 0$. i.e. momentum at $t = 0$ is zero. The equation reduces to

$$(\gamma^0 E - m)u = 0 \tag{2.57}$$

which has 4 solutions.

$$\underbrace{u_1(m,0) = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad u_2(m,0) = N\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{E=m} \quad \underbrace{u_3(m,0) = N\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad u_4(m,0) = N\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{E=-m} \tag{2.58}$$

Remark 2.9 $E = -m$ indicates negative energy. With the benefit of hindsight, we can easily surmise that the negative energy cases denote antimatter.

Derivation 2.12 (General solution) We recall the Dirac equation in momentum form $(\gamma^i p_i - m)u = 0$. Noting the summation convention, we plug in the γ matrices and the 4-vector and find

$$\left[\begin{pmatrix}I & 0\\0 & -I\end{pmatrix}E - \begin{pmatrix}0 & \vec{\sigma}\\-\vec{\sigma} & 0\end{pmatrix}\cdot\vec{p} - m\begin{pmatrix}I & 0\\0 & I\end{pmatrix}\right]u = 0 \quad \begin{pmatrix}(E-m)I & -\vec{\sigma}\cdot\vec{p}\\ \vec{\sigma}\cdot\vec{p} & (-E-m)I\end{pmatrix}\begin{pmatrix}\phi\\ \chi\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{2.59}$$

where $\vec{\sigma}$ is a 'vector' of the 3 Pauli matrices, and ϕ and χ are both (two-component) Weyl spinors. Recognising this as a system of two equations, we have

$$u = \begin{pmatrix}\phi\\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\phi\end{pmatrix} \tag{2.60}$$

Remark 2.10 Here we note that χ is a scaled down (or *suppressed*) version of ϕ. Hence ϕ is also called the *large component* while χ is called the *small component*. In non-relativistic limits, the large component dominates. The small component contains the relativistic corrections.
For matter, the two simplest solutions for the large component are the spin-up and spin-down states along the z axis (i.e. the eigenstates of σ_z): $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$. Plugging in yields

$$u_1 = N_1\begin{pmatrix}1\\0\\ \frac{p_z}{E+m}\\ \frac{p_x+ip_y}{E+m}\end{pmatrix} \quad u_2 = N_2\begin{pmatrix}0\\1\\ \frac{p_x-ip_y}{E+m}\\ \frac{-p_z}{E+m}\end{pmatrix} \tag{2.61}$$

For antimatter the equation becomes $(\gamma^\mu p_\mu + m)v = 0$, and the solutions are

$$v_1 = N_1\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\ \frac{-p_z}{E+m}\\0\\1\end{pmatrix} \quad v_2 = N_2\begin{pmatrix}\frac{p_z}{E+m}\\ \frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix} \tag{2.62}$$

Note 2.1 (Feynman slash notation) In some literature, one will see the so-called *Feynman slash notation* (rarely called the *Dirac slash notation*), which is a simplification similar to the d'Alembertial. For some A_γ, its Feynman slash notation denotes

$$\not{A} = \gamma^\mu A_\mu \tag{2.63}$$

2.3 Angels and demons

To wrap up, we make some brief comments on antimatter. First, we need a more rigorous definition of antimatter:

Definition 2.4 (Feynman-Stückelberg interpretation) We consider the exponential term which has the following equivalence:

$$e^{-i(-E)(-t)} = e^{-iEt} \tag{2.64}$$

As such, we can understand negative energy solutions in two equally valid ways:

- A positive energy antiparticle travelling forward in time.
  Remark 2.11 This implies antimatter and is the physically useful interpretation.

  A negative energy particle travelling backwards in time.
  Remark 2.12 This is the practically useful interpretation.

Remark 2.13 From the second interpretation, we represent antiparticles as travelling backwards in time on Feynman diagrams (thus assuming negative energy as well).

Figure 2.2: A canister of antimatter as infamously seen in *Angels and Demons* (2009).

Another implication is that from u-spinors which correspond to matter, we could define a v-spinor which corresponds to antimatter:

$$u_4(-E,-\vec{p})e^{-i((-E)t-(-\vec{p})\cdot\vec{x})} \to v_1(E,\vec{p})e^{i(Et-\vec{p}\cdot\vec{x})} \quad u_3(-E,-\vec{p})e^{-i((-E)t-(-\vec{p})\cdot\vec{x})} \to v_2(E,\vec{p})e^{i(Et-\vec{p}\cdot\vec{x})} \tag{2.65}$$

where $-i$ and i imply forward time and backwards time respectively. We hence have the Dirac equation for antimatter:

Theorem 2.13 (Dirac equation in momentum form for antimatter)

$$(\gamma^i p_i + m)v = 0 \tag{2.66}$$

Chapter 3

Quantum electrodynamics

This chapter discusses quantum electrodynamics (QED), which studies the electromagnetic force.

3.1 Spin, helicity and chirality

The 'vector of matrices' for the 4-spinor spin state is defined as

$$\hat{\vec{S}} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{3.1}$$

where we have defined the Σ matrix as $\Sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$.

The spinors for particles at rest are eigenstates of $\hat{S}_z$ with eigenvalues $\pm 1/2$:

$$\hat{S}_z u_1 = +\frac{1}{2}u_1 \quad \hat{S}_z^v v_1 = -\hat{S}_z v_1 = +\frac{1}{2}v_1 \tag{3.2}$$

$$\hat{S}_z u_2 = -\frac{1}{2}u_2 \quad \hat{S}_z^v v_2 = -\hat{S}_z v_2 = -\frac{1}{2}v_2 \tag{3.3}$$

It might be surprising to you, but on its own, spin is not a very useful quantity. Rather we want to look at its projection along the particle's direction of momentum. This projection is known as *helicity*:

Definition 3.1 (Helicity operator)

$$\hat{h} = \frac{\vec{S}\cdot\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}\cdot\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|} \tag{3.4}$$

On a Dirac spinor, $\hat{h}$ has eigenvalues $h = \pm 1$.

Definition 3.2 (Chirality) Chirality (from 'hand' in Greek) denotes the *handedness* of a particle:

- When $h = 1$, helicity is positive as the particle's spin direction is the same as its direction of motion.
  Chirality is *left-handed*. This orientation is so-called as it observes the right-hand rule: Align the right thumb in the momentum's direction, and the curled fingers should align with the spin direction.

- When $h = -1$, helicity is negative as the particle's spin direction is opposite from its direction of motion.
  Chirality is *right-handed*. This orientation is so-called as it observes the left-hand rule: Align the left thumb in the momentum's direction, and the curled fingers should align with the spin direction.

Helicity is compatible with the symmetries of the Dirac equation, as such, the eigenstates of $\hat{h}$ should also be solutions to the Dirac equation, each representing the particle's solution in a different helicity.

Theorem 3.1 (Helicity eigenstates) For a massive spin-$\frac{1}{2}$ fermions which are free particles, the helicity eigenstates of matter propagating in the (θ, ϕ) direction are

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \\ \frac{|\vec{p}|}{E+m}\cos(\theta/2) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin(\theta/2) \end{pmatrix} \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -\sin(\theta/2) \\ e^{i\phi}\cos(\theta/2) \\ \frac{|\vec{p}|}{E+m}\sin(\theta/2) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos(\theta/2) \end{pmatrix} \tag{3.5}$$

The states observe

$$\hat{h}u_\uparrow = +u_\uparrow \quad \hat{h}u_\downarrow = -u_\downarrow \tag{3.6}$$

For antimatter, the helicity operator $\hat{h}^v = -\hat{h}$ is the negative of its matter counterpart, and one has the eigenstates

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin(\theta/2) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos(\theta/2) \\ -\sin(\theta/2) \\ e^{i\phi}\cos(\theta/2) \end{pmatrix} \quad v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos(\theta/2) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin(\theta/2) \\ \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{3.7}$$

which observe

$$\hat{h}^v v_\uparrow = +v_\uparrow \quad \hat{h}^v v_\downarrow = -v_\downarrow \tag{3.8}$$

Chirality is conserved by $\bar{u}\gamma^\mu u$. It is encoded by the so-called *chiral states*.

Definition 3.3 (γ^5 matrix) Chiral states are eigenstates of the γ^5 matrix:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{3.9}$$

which observes the following properties:

$$(\gamma^5)^2 = I \quad \gamma^{5\dagger} = \gamma^5 \quad \{\gamma^5, \gamma^\mu\} = \gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0 \tag{3.10}$$

Remark 3.1 The general action of the γ^5 matrix on a 4-vector/4-spinor is

$$\gamma^5\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} C \\ D \\ A \\ B \end{pmatrix} \tag{3.11}$$

Definition 3.4 (Chiral states) The right-handed and left-handed chiral states for matter and antimatter are thus

$$\gamma^5 u_R = +u_R \quad \gamma^5 u_L = -u_L \quad \gamma^5 v_R = -v_R \quad \gamma^5 v_L = +v_L \tag{3.12}$$

As it turns out, the chiral states $u_\uparrow$, $u_\downarrow$, $v_\uparrow$ and $v_\downarrow$ are simply the helicity states u_R, u_L, v_R and v_L in the relativistic limit. i.e. when $E \ll m$ $(m \to 0)$. The only change is that the prefactor $\sqrt{E+m}$ becomes $\sqrt{E}$.

So far we have failed to define the chiral states *sui generis*, which we will do now. Chiral states can be said to be defined by the γ^5 operator:

Definition 3.5 (Chiral projection operators) The so-called *chiral projection operators* are defined as

$$\hat{P}_R = \tfrac{1}{2}(1+\gamma^5) \quad \hat{P}_L = \tfrac{1}{2}(1-\gamma^5) \tag{3.13}$$

They generate chiral states from spinors:

$$\hat{P}_R u_R = u_R \quad \hat{P}_L u_R = 0 \quad \hat{P}_R u_L = 0 \quad \hat{P}_L u_L = u_L \tag{3.14}$$

$$\hat{P}_R v_R = 0 \quad \hat{P}_L v_R = v_R \quad \hat{P}_R v_L = v_L \quad \hat{P}_L v_L = 0 \tag{3.15}$$

Remark 3.2 Inevitably one finds

$$(\hat{P}_R + \hat{P}_L)u = u_L + u_R = u \tag{3.16}$$

Theorem 3.2 (Conservation of chirality) Chirality is conserved in any interaction which includes currents $\bar{\psi}\gamma^\mu\phi$.

$$\begin{aligned}\bar{\psi}\gamma^\mu\phi &= \bar{\psi}_R\gamma^\mu\phi_R + \bar{\psi}_R\gamma^\mu\phi_L + \bar{\psi}_L\gamma^\mu\phi_R + \bar{\psi}_L\gamma^\mu\phi_L \\ &= \bar{\psi}_R\gamma^\mu\phi_R + 0 + 0 + \bar{\psi}_L\gamma^\mu\phi_L\end{aligned} \tag{3.17}$$

Note 3.1 Two implications exist due to the conservation of chirality:

- Right-handed particles only interact with right-handed particles, and left-handed particles only interact with left-handed particles. It is impossible to turn a left-handed chiral state into a right-handed chiral state through the exchange of a virtual particle, and vice versa.
- The chirality of an electron cannot be changed when it interacts with a photon.

3.2 Gauge invariance

So far we have dealt with the Hamiltonian formalism of QM. This however does not provide a complete picture as it hides Lorentz invariance of the Dirac equation (by separating space and time components). In HEP we employ the Lagrangian formalism instead.

Derivation 3.1 (Recovering the Dirac equation) By using the Lagrangian density

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_i\psi - m\bar{\psi}\psi \tag{3.18}$$

and plugging it into the Euler-Lagrange equation

$$\partial_i\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_i\psi_j\right)}\right) - \frac{\partial\mathcal{L}}{\partial\psi_j} = 0 \tag{3.19}$$

one recovers the Dirac equation

$$i\gamma^i\partial_i\psi - m\psi = 0 \tag{3.20}$$

Theorem 3.3 (Noether's theorem) Every symmetry leads to a conservation law.

Noether's theorem leads to *gauge invariance*, which states that there should be phase symmetry - i.e. quantities should not change due to a *local* phase change. Indeed, our current Lagrangian density is invariant under a *global* phase change

$$\psi(x) \to \psi'(x) = e^{i\chi\theta}\psi(x) \tag{3.21}$$

However, $\mathcal{L}$ does change under a *local* phase change

$$\psi(x) \to \psi'(x) = e^{i\chi(x)\theta}\psi(x) \tag{3.22}$$

To ensure gauge invariance, we introduce a vector field or a so-called *gauge field* A_i. It transforms as

Theorem 3.4 (Gauge field transformations)

$$A_i' = A_i - \partial_i\chi(x) \tag{3.23}$$

Remark 3.3 This A_i is identical to the 4-potential we have seen in electromagnetism.
The gauge field allows us to modify the derivative to the *covariant derivative*

Definition 3.6 (Gauge theory covariant derivative)

$$\partial_i \to D_i = \partial_i + iqA_i(x) \tag{3.24}$$

Remark 3.4 It can clearly be seen that this covariant derivative is distinct from the covariant derivative seen in GR. However, the motivation is quite similar. Both covariant derivatives are designed to preserve the form of physical laws under local transformations:

- In GR, the covariant derivative is introduced to preserve general coordinate transformations.
- In QED, the covariant derivative is introduced to preserve Noether's theorem in local phase transformations.

The gauge field A_i acts as a *connection* on the *principal fiber bundle* and is analogous to the Christoffel symbol, which is a connection on the *tangent bundles*.
We now replace the ordinary derivative with the covariant derivative. The Lagrangian density becomes

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_i\psi - m\bar{\psi}\psi \tag{3.25}$$

Additionally, we need to include the dynamics of the gauge field $A_i(x)$. This is done by adding the electromagnetic tensor or Faraday tensor F_{ij} and its corresponding Lagrangian density:

$$F_{ij} = \partial_i A_j - \partial_j A_i \quad \mathcal{L}_{\text{EM}} = -\frac{1}{4}F_{ij}F^{ji} \tag{3.26}$$

The total Lagrangian density with QED correction is then

$$\mathcal{L} = i\bar{\psi}\gamma^i D_j\psi - m\bar{\psi}\psi - \frac{1}{4}F_{ij}F^{ji} \tag{3.27}$$

Remark 3.5 By enforcing local gauge invariance, we have introduced the electromagnetic field and its interaction with the fermion field ψ. $A_i(x)$ mediates the electromagnetic force, and its quantisation leads to the photon, the force carrier in QED[1].
Now we can expans the QED Lagrangian and write out its full form:

Definition 3.7 (QED Lagrangian)

$$\mathcal{L} = \underbrace{i\bar{\psi}\gamma^i\partial_i\psi}_{\text{fermion kinetic}} - \underbrace{m_e\bar{\psi}\psi}_{\text{fermion mass}} - \underbrace{q\bar{\psi}\gamma^i A_i\psi}_{\text{interaction}} - \underbrace{\frac{1}{4}F_{ij}F^{ij}}_{\text{photon kinetic}} + \underbrace{\frac{1}{2}m_\gamma^2 A_i A^i}_{\text{photon mass}} \tag{3.28}$$

Remark 3.6 Note that there is an extra *photon mass* term $\frac{1}{2}m_\gamma^2 A_\mu A^\mu$. This term breaks gauge symmetry and is thus non-existent in QED.

3.3 General QED calculations

Definition 3.8 (Propagator) Each virtual particle has a *propagator*, which is the function associated with the particle one writes down in using Fermi's golden rule.

Theorem 3.5 (Feynman rules for QED) The Feynman rules for QED allow us to construct a Lorentz invariant transition matrix. On the LHS we have $-i\mathcal{M}$. On the RHS, we multiply the following terms one by one:

- **Initial and final particles:**
  - For a particle the initial and final states are $u(p)$ and $\bar{u}(p)$.
  - For antiparticles, which are travelling in reverse time, the initial and final states are $\bar{v}(p)$ and $v(p)$.
- **Vertices:** At every vertex, we write

$$-iq\gamma^\mu \tag{3.29}$$

  where q is the *charge*.
- **Virtual particles:**
  - For each virtual photon we write the *photon propagator*

$$-ig_{ij}/q^2 \tag{3.30}$$

[1] In gauge theories, forces arise from requiring local gauge invariance, leading to the introduction of gauge fields mediating interactions (e.g., photons in QED).

- For each virtual fermion we write the *fermion propagator*

$$i(\not{q} + m)/(q^2 - m^2) \tag{3.31}$$

where q^2 is the four-momentum exchange.

- One should also note:
  - When writing down spinors, do so in the direction *against the Feynman diagram arrows* of their corresponding particles and with the γ matrix in between. This applies to both the initial and final set of arrows (i.e. states).
  - The first index of the metric should always correspond to the γ index on the LHS, and the second index is the right one.

Note 3.2 (Setting up QED calculations) In calculations, one should then follow these steps:

- Determine the initial and final state.
- Draw the (lowest-order) Feynman diagram.
- Choose a reference frame.
- Choose the axes and draw a cartoon of the kinematics. Usually the direction of particle motion of the z-axis. In $2 \to 2$ processes, one can always choose $\phi = 0$ or $\phi = \pi$.
- Write down the 4-vectors.
- Find the formula for the matrix $\mathcal{M}$ from the Feynman diagram:
  - Break the Feynman diagram into fermions (currents) and virtual particles (propagators).
  - Write down the adjoint spinor at each vertex (i.e. working backwards along the arrows).
  - Make sure that each vertex has a different index (e.g. (i and j)), with the metric g_{ij} connecting the indices (if the exchange particle is a photon).
- Evaluate the matrix from the spinors, and calculate $|\mathcal{M}_{fi}|^2$:
  - If the initial and final states are **different**[a], square the matrix elements before summing. This is possible as the probability amplitude of the states cannot interfere with each other.

$$|\mathcal{M}_{fi}|^2 = \sum_{\text{final states}} |\mathcal{M}_{fi}(\text{state})|^2 \tag{3.32}$$

  - If the initial and final states are **identical**, sum the matrix elements before squaring. Superposition resulting from identical states means that there is interference (constructive or destructive) between the states, which we account for by summing before squaring.

$$|\mathcal{M}_{fi}|^2 = \left| \sum_{\text{indistinguishable paths}} \mathcal{M}_{fi}(\text{path}) \right|^2 \tag{3.33}$$

  Average over the different initial states if necessary.
- Apply the phase space equation to get the decay rate.

[a] e.g. different helicities or polarisations.

3.4 2→2 QED calculations

Derivation 3.2 (General 2→2 QED calculation) We now consider a simple $2 \to 2$ interaction with initial momenta p_1 and p_2 and final momenta p_3 and p_4. For the charge q (not to be confused with the 4-momentum transfer q^2), the Lorentz invariant matrix is

$$-i\mathcal{M} = (\overline{v}(p_2)(iq\gamma^\mu)u(p_1))\,\frac{-ig_{\mu\nu}}{q^2}\,(\overline{u}(p_3)(iq\gamma^\nu)v(p_4)) \tag{3.34}$$

which one can rewrite as

$$\mathcal{M} = \langle\psi(p_3)|\hat{H}_I|\psi(p_1)\rangle\frac{1}{q^2-m^2}\langle\psi(p_4)|\hat{H}_I|\psi(p_2)\rangle \tag{3.35}$$

We now consider QED effects. This can be done by slightly perturbing the 4-momentum:

$$p_i = (E - q_e\phi, \vec{p} - q_e\vec{A}) \tag{3.36}$$

where q_e is the electric charge of the fermion, *not* the exchange 4-momentum. Now we write out the 3-momentum and energy in QM operator form:

$$\vec{p} = -i\vec{\nabla} \quad E = i\partial/\partial t \tag{3.37}$$

The 4-differential $i\partial_i$ then becomes $i\partial_i - q_eA_i$. The Dirac equation then becomes

$$(i\gamma^i\partial_i - m)\psi - q_e\gamma^iA_i\psi = 0 \tag{3.38}$$

Multiplying by γ^0 yields

$$\hat{H}\psi = i\frac{\partial\psi}{\partial t} = \underbrace{\gamma^0 m\psi}_{\text{mass term}} - \underbrace{i\gamma^0\vec{\gamma}\cdot\vec{\nabla}}_{\text{kinetic term}} + \underbrace{q_e\gamma^0\gamma^\mu A_\mu\psi}_{\text{EM potential term}} \tag{3.39}$$

This recovers the Hamiltonian, which we want to plug into the expression for the interaction matrix. But first, we rewrite the 4-potential in its plane wave solution $A_i = \varepsilon_i^j e^{i(\vec{p}\cdot\vec{x}-Et)}$. where ϵ^j are the polarisation states

$$\varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \tag{3.40}$$

which is related to the metric by $\sum_k \varepsilon_i^k(\varepsilon_j^k)^* = -g_{ij}$. With this form of the 4-potential we compute the Hamiltonian terms of the interaction matrix $\mathcal{M}$

$$\langle\psi(p_3)|\hat{H}_I|\psi(p_1)\rangle = q_e\overline{u}(p_3)\gamma^i u(p_1)\varepsilon_i^j \tag{3.41}$$

and the $\mathcal{M}$ reduces to

$$\mathcal{M} = -q_1 j_e^i\frac{g_{ij}}{q^2}q_2 j_2^j = -q_1q_2\frac{j_1\cdot j_2}{q^2} \tag{3.42}$$

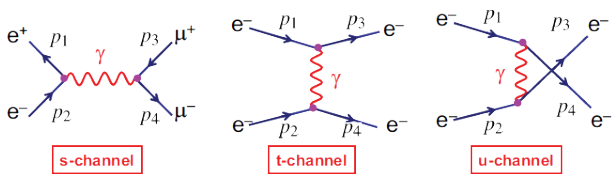

Figure 3.1: s-, t- and u- channels.

For one $2 \to 2$ interaction, we have three possible channels:

- The s-channel is timelike and represents particles 1 and 2 joining and forming a virtual particle before splitting. It is so-called because the relevant variable is the centre-of-mass energy $s = (p_1 + p_2)^2 = m^2$
- The t-channel is spacelike and represents one of the initial particles emitting a virtual particle, which is then absorbed by the other initial particle.
- The u-channel is also spacelike and is identical to the t-channel, except that the final particles are exchanged with each other in their positions on the Feynman diagram. This is seen in u in that p^4 replaces p_3 in t.

A useful shorthand when considering any $2 \to 2$ process (e.g. annihilation, scattering) is the Mandelstam variables:

Definition 3.9 (Mandelstam variables) The Mandelstam variables s, t and u correspond to the s-, t- and u- channels respectively.

$$s = (p_1 + p_2)^2 \quad t = (p_1 - p_3)^2 \quad u = (p_1 - p_4)^2 \tag{3.43}$$

s, t and u are equal to the four-momentum exchange q^2 in their own channels. They are Lorentz invariant and satisfy

$$s > 0 \quad t < 0 \quad u < 0 \quad s + t + u = 2m_i^2 + 2m_f^2 \tag{3.44}$$

Theorem 3.6 (Crossing symmetry) The so-called *crossing symmetry* dictates that different processes can actually be related by exchanging particles with anti-particles and reversing their momentum flow.

Remark 3.7 Essentially, one can rotate a Feynman diagram by 90 degrees and get a different process with 2 of the 4 particles becoming antiparticles and 2 of the 4 momentums becoming negative due to the same arrows being interpreted differently after the rotation.

Chapter 4

Quantum chromodynamics

This chapter discusses quantum chromodynamics (QCD), which studies the strong force.

4.1 Colour

Even though the colours (and anticolours) reflect the colour states, the actual colour charges we use in calculations are the *colour isospin* I_3^c and the *colour hypercharge* Y^c. The naming corresponds to the isospin I_3 and the practically obsolete hypercharge Y in flavour physics.

Colour charge in each unit colour

Charge	**red**	**green**	**blue**	**antired**	**antigreen**	**antiblue**
I_3^c	$1/2$	$1/2$	0	$-1/2$	$-1/2$	0
Y^c	$1/3$	$1/3$	$-2/3$	$-1/3$	$-1/3$	$2/3$

Due to Noether's theorem, colour charge is conserved.
From gauge transformations, an analogy can be made between QED and QCD:

- In QED, we have the gauge transformation

$$\psi \to \psi' = \psi e^{iq\chi(x)} \tag{4.1}$$

  where $\chi(x)$ is a local function and q is the charge/coupling strength. This gives rise to a term in the Dirac Lagrangian $\bar{\psi}q\gamma^i A_i \psi$ which couples the fermion to the photon field A_i. The photon is the virtual particle in QED interactions.

- In QCD, we have the gauge transformation

$$\psi \to \psi' = \psi e^{ig_s \frac{1}{2}\lambda_i \theta_i(x)} \tag{4.2}$$

  where $\theta_(x)$ are a collection of 8 local functions and g_s is the coupling strength. This gives rise to a term in the Dirac Lagrangian $\bar{\psi} g_s \frac{1}{2}\lambda^j \gamma^i G_i^j \psi$ which couples the fermion to the gluon field G_i^j. The gluon is the virtual particle in QCD interactions.

We have just defined the gluon, which carries the *strong force.*
Remark 4.1 Gluons only go one way. As colour charge is conserved, it must carry both charge and anticharge.

Theorem 4.1 (Colour confinement) Free particles must be charge neutral. i.e. they must satisfy $I_3^c = Y^c = 0$

In practice, this means that:

- Gluons are not free particles as they carry colour charge. Rather, the strong force exists as a short-range force.
- Quarks cannot be isolated, although quark-antiquark jets can. The formation of these jets, which are hadrons, is called *hadronisation*.

4.2 General QCD calculations

Theorem 4.2 (Feynman QCD rules) For QCD, there are a few additions to the Feynman rules:

- For each QED vertex μ, ν, $\cdots$, one has a corresponding QCD vertex a, b, $\cdots$.
- For each virtual gluon we have the *gluon propagator*, which, compared to the photon propagator, has the extra term δ_{ab}:

  $$-ig_{\mu\nu}\delta_{ab}/q^2 \tag{4.3}$$

  This makes sure that the same gluon is emitted at one end and absorbed at the other and effectively imposes colour conservation.

Chapter 5

Weak interaction

5.1 Parity

5.2 Weak force

5.3 Pion

5.4 W boson

Chapter 6

Flavour physics

Flavour physics studies processes related to quark and lepton flavour, which are often mediated by the weak force. It is connected to weak interactions but not synonymous with studying the weak force itself. Instead, the weak force is described by the electroweak theory, which we will investigate in the next chapter.

6.1 Quark mixing

6.2 CPT symmetry

6.3 Kaon

6.4 Z boson

6.5 electroweak unification

Chapter 7

Higgs boson and the destiny of HEP

7.1 Scalar field

7.2 Higgs mechanism

7.3 Beyond the standard model