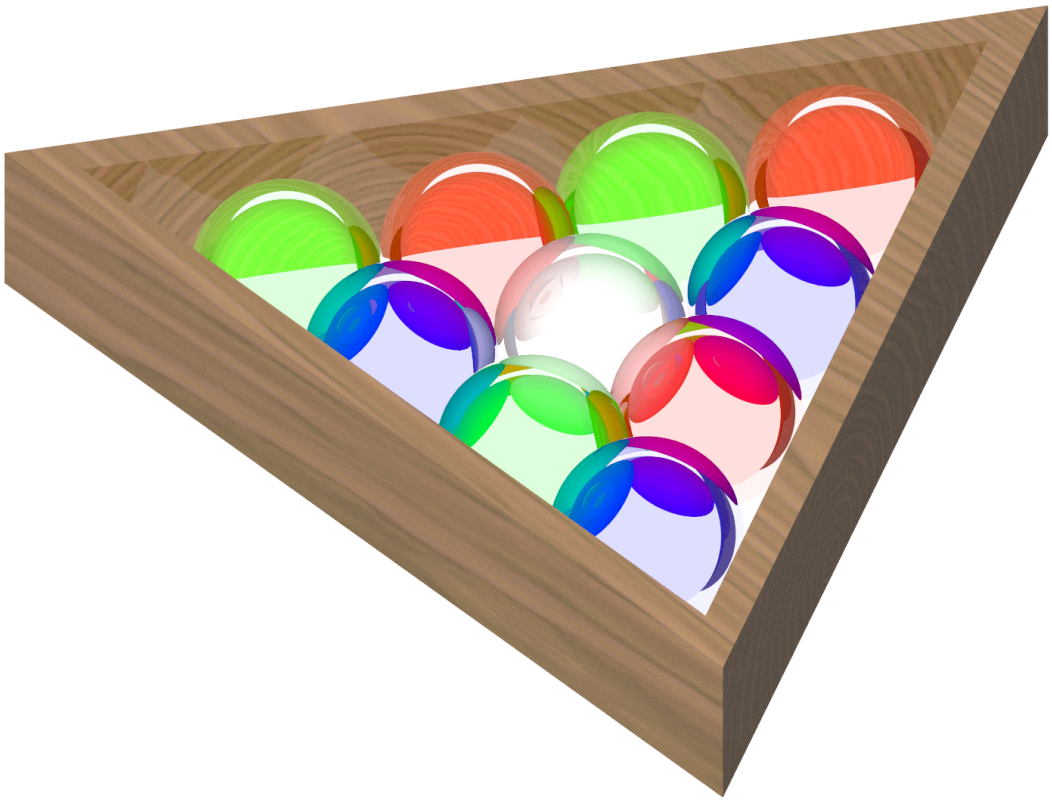
Particles & Wizards

Quantum Field Theory



A Very Short QFT Book by

N. Booker

To my parents

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Chapter 1

Instead of a foreword

1.1 Acknowledgements

> **Quote 1.1** Credit for those who work hard, I just happen to know some things.
>
> *Felix Halbwedl, in his infinite humility, 22 December 2024*

Particles & Wizards: Quantum Field theory emerged from a series of lecture notes based on the quantum field theory course at University College London lectured by Prof. Alessio 'Wizard'[1] Serafini as well as his own notes for the course in the 2023-24 and 2024-25 years, which the first two chapters of this book are mostly based on. Other sources used for the book include:

- *Quantum Field Theory I* by Niklas Beisert (ETH Zürich)
- *Quantum Fields* by Nikolay Bogoliubov (JINR) and Dmitry Shirkov (JINR)
- *Quantum Field Theory* by Gernot Eichmann (Technische Universität Graz)
- *Quantum Field Theory for the Gifted Amateur* by Tom Lancaster (University of Durham) and Stephen J. Blundell (University of Oxford)
- *Quantum Field Theory I* by Axel Maas (Technische Universität Graz)
- *Quantum Field Theory 1* by Douglas Ross (University of Southampton)
- *Quantum Field Theory* by David Tong (University of Cambridge)

I want to extend my gratitude to Prof. Serafini for answering many questions on QFT I had throughout my master's year and Felix Halbwedl for stumulating physical discussions and his consistent advice in improving the contents and the formatting this book. I am also eternally indebted to Abhijeet Vats, under whose guidance I was able to develop my LaTeX skills to a satisfactory level. Without them, this book would undoubtedly not have been in its current form.
Finally, I would like to thank my colleagues who learned QFT at roughly the same time as I did - K. Batrakov, P. Kothgasser, K. Lang, L. Ross, P. Schlachter, F. Silva, M. Sokołowski and E. Szymańska - for their encouragement and support.

1.2 How to use this book

> **Quote 1.2** What could possible go wrong?
>
> *Alessio Serafini, 16 January 2024*

[1] The nickname is so-bestowed to his character appearing as a wizard in the UCL PandA Day plays.

Quote 1.3 but QFT is not conceptually difficult
its just classical field theory with commutators, at least if one stays clear of more esoteric things like supersymmetry or string theory
if you already have experience in CFT, QFT should pose no difficulties to you at all

Paulina Schlachter, 29 September 2024

Quote 1.4 When conquering QFT you'll find yourself in a similar position as Napoleon with Russia.

Felix Halbwedl, 20 October 2024

QFT is the unification of quantum mechanics and special relativity. It is not a theory of quantum gravity because it still operates within the completely flat Minkowski space. While the development of HEP preceded that of QFT historically, QFT actually forms the theoretical basis of HEP.
Our ultimate objective in QFT is thus to calculate the final state from some initial state. This is accomplished by applying the scattering matrix S_{fi} to the initial state. The interaction-relevant part of S_{fi} is another matrix $\mathcal{M}_{fi}$ known as the transition amplitude, which is significant in its own right and can be derived from the Feynman rules. We will spend the majority of this book deriving the Feynman rules of various field theories.
With this in mind, this book currently consists of three parts, one of which is complete and two in progress:

- Part I resembles a standard 'Quantum Field Theory I' course in most universities, covering canonical quantisation up to the QED Feynman rules.
- Part II resembles the 'Quantum Field Theory I' course in a few universities that begin with path integrals instead of canonical quantisation, and the intersection between 'Quantum Field Theory I' and 'Quantum Field Theory II' course in most other universities. It rederives the conclusions in Part I using path integrals.
- Part III also resembles the intersection between 'Quantum Field Theory I' and 'Quantum Field Theory II' course, covering renormalisation and regularisation in loop Feynman diagrams.
- More parts on QCD and gauge theories are planned.

For any comments, suggestions or typos, please e-mail `zcapxix(at)ucl(dot)ac(dot)uk`.

Legend

For chapters that are not fully completed, a filled square is placed in front its title to denote its completion status:

- ■ - mostly complete
- ■ - in progress
- ■ - empty

Chapter 2

Preliminaries

2.1 Quantum mechanics of mixed states

> **Quote 2.1** Fortunately, quantum mechanics is easy and can be summarised in a few lines.
>
> *Alessio Serafini*

A quantum state can always be represented by a Hermitian, positive semi-definite operator with trace 1 ϱ[1] (i.e. all eigenvalues of ϱ are positive semi-definite and add up to 1).

Definition 2.1 (Positive definiteness and positive semi-definiteness) A positive definite operator ϱ always yields a positive expectation value

$$\langle\psi|\varrho|\psi\rangle > 0 \tag{2.1}$$

A positive semi-definite operator ϱ always yields a non-negative expectation value

$$\langle\psi|\varrho|\psi\rangle \leq 0 \tag{2.2}$$

We now introduce the so-called *Sylvester's criterion*.

Definition 2.2 (Minor) A *minor* of some matrix is the determinant of the resultant matrix after deleting an arbitrary number of rows and columns from the initial matrix. For a square matrix, a minor is called a *principal minor* when the indices of the deleted rows and those of the deleted columns are *identical*[a].

[a] This is significant in that if one deletes, say, the 3^{rd} row and the 4^{th} column, the resulting matrix is a minor but not a principal minor.

Theorem 2.1 (Sylvester's criterion) One can use minors of a matrix to test positive definiteness and positive semi-definiteness. For a Hermitian $n \times n$ matrix:

- Positive definiteness holds if all the *leading* principal minors are positive. i.e. if the determinants of the top-left $1 \times 1, \cdots, n \times n$ sub-matrices are positive.
- Positive semi-definiteness holds if *all* principal minors are non-negative.

While a pure quantum state is simply a bra or a ket, the bra-ket notation is insufficient for a *mixed state*, which is described by a *density matrix* or a *density operator*. Unlike a state vector, which is an element *of* the Hilbert space, the density matrix is an operator *on* the Hilbert space.
Physically, a mixed state is a statistical mixture (i.e. ensemble) of i different pure states $|\Psi_i\rangle$ with probabilities p_i, such that:

$$\varrho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \tag{2.3}$$

[1] Note that this might not be the *density operator* ρ.

In this way, the density matrix extends the concept of quantum state to mixed states, systems where we do not have complete knowledge.

Derivation 2.1 (Von Neumann equation) We recall the *theorem of Liouville*[a] in classical field theory, which states that the phase space distribution function $\rho(p, q)$ is constant along the trajectories of the system:

Theorem 2.2 (Theorem of Liouville)

$$\partial_t \rho = \{H, \rho\} \tag{2.4}$$

where H is the Hamiltonian and $\{\}$ is the *Poission bracket*, which for functions f and g and phase space coordinates (q_i, p_i) satisfy

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{2.5}$$

In QFT, the density operator ρ is functionally equivalent to the phase space distribution function. Hence we have an equivalent for the theorem of Liouville

Theorem 2.3 (Von Neumann equation)

$$\dot{\rho} = -i[H, \rho] \tag{2.6}$$

where we have assumed that $\hbar = 1$.

Due to the aforementioned equivalence, the equation is also called the *quantum Liouville equation* or the *Liouville-von Neumann equation*. It is also the mixed state equivalent of the Schrödinger equation, which deals with pure states.

[a] Or more commonly in the English-speaking world, *Liouville's theorem*.

Derivation 2.2 (Reduction to pure state) To prove the last statement, we consider a pure state, where we only have one possible i. The density matrix is

$$\rho = |\Psi\rangle\langle\Psi| \tag{2.7}$$

Substituting this into the LHS of the von Neumann equation gives

$$\partial_t \rho = (\partial_t |\Psi\rangle)\langle\Psi| + |\Psi\rangle(\partial_t \langle\Psi|) \tag{2.8}$$

Substituting into the RHS yields

$$-i[H, \rho] = -iH|\Psi\rangle\langle\Psi| + i|\Psi\rangle\langle\Psi|H \tag{2.9}$$

We now equate the two sides:

$$(\partial_t |\Psi\rangle)\langle\Psi| + |\Psi\rangle(\partial_t \langle\Psi|) = -iH|\Psi\rangle\langle\Psi| + i|\Psi\rangle\langle\Psi|H \tag{2.10}$$

where we have recovered the Schrödinger equation and its adjoint form

$$\partial_t |\Psi\rangle = -iH|\Psi\rangle \quad \partial_t \langle\Psi| = i\langle\Psi|H \tag{2.11}$$

Now we remind ourselves of how measurements work. We have already seen the so-called *projection-valued measures* or *PVMs*. Previously, they have been known to us as *projectors*.

Definition 2.3 (Projection-valued measure) A projection-valued measure P_i is a linear and positive semi-definite operator that satisfies, for the density operator ρ:

- Normalisation condition:

$$\sum_i P_i = \mathbb{I} \quad \text{or} \quad \sum_i \mathrm{Tr}(\rho P_i) = 1 \tag{2.12}$$

where $\mathbb{I}$ is the unit matrix of appropriate dimension.

- Orthogonality condition:
$$P_i P_j = \delta_{ij} P_i \quad \text{or} \quad P_i^2 = P_i \tag{2.13}$$

Functionally, it maps a quantity in a vector space V into a subspace $W \subset V$. Each P_i corresponds to an eigenvalue of an observable (e.g. position or spin).

Theorem 2.4 (Born rule) The probability of obtaining the outcome i is, for a mixed state:
$$p_j = \mathrm{Tr}(\rho P_j) = \mathrm{Tr}(|j\rangle\langle\phi|k\rangle\langle j|) = \langle j|\phi\rangle\langle\phi|j\rangle = |\langle j|\phi\rangle|^2 \tag{2.14}$$

For a pure state, this simplifies to
$$p_i = \langle\Psi|P_i|\Psi\rangle \tag{2.15}$$

Remark 2.1 Here we see the significance of Equation 2.12, which is that all probabilities expectedly sum up to 1.
So far, we have been working with PVMs, which are utterly perfect, innocent and idealised measurements. In real life, measurement devices are not ideal, giving rise to noise. These imperfect (or rather *generalised*) measurements are described by *positive operator-valued measures* or *POVMs*.

> **Quote 2.2** 'Positive Operator Valued Measure', an acronym fabricated by mathematical physicists to scare all others away.
>
> *Alessio Serafini*

> **Quote 2.3** It's a bit like the Marlinspike village band. You know: PO-POM, PO-POM, PO-PO-POM, PO-POM, PO-POM, PO-PO-POM...
>
> *Captain Haddock, in* Tintin and Alph-Art

Definition 2.4 (Positive operator-valued measure) A positive operator-valued measure $\prod_i$ is a linear and positive semi-definite operator that satisfies the normalisation condition only:
$$\sum_i \prod_i = \mathbb{I} \quad \text{or} \quad \sum_i \mathrm{Tr}(\rho \prod_i) = 1 \tag{2.16}$$

By imposing the condition
$$\mathrm{Tr}[\prod_i \prod_j] = \delta_{ij} \tag{2.17}$$

POVMs reduce to PVMs/projectors.
Remark 2.2 The Born rule is the same as for PVMs, save for the nominal replacement of P_is by $\prod_i$s.

2.2 Classical field theory

Note 2.1 (Metric signature) Unlike GR, the Minkowski 4-metric in HEP has the signature $(+,-,-,-)$. That is, the line element has the form
$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{2.18}$$

We now briefly discuss the *variational formalism*. Two equivalent formulations of the variational formalism exist - Lagrangian and Hamiltonian mechanics. In classical mechanics, the central quantities are the 4-position x (or often q) and momentum p.

Note 2.2 (Reference frames) We can choose certain frames that simplify calculations:

- For a spacelike separation $(x-y)^2 < 0$, one can always, without loss of generality, choose a frame to set $(x^0 - y^0) = 0$.
- For a timelike separation $(x-y)^2 > 0$, one can always, without loss of generality, choose a frame to set $(\vec{x} - \vec{y}) = 0$.

In field theories, the 4-position x is replaced with a 4-field $\phi(x) = (\phi_0, \phi_1, \phi_2, \phi_3)$. ϕ_1, ϕ_2 and ϕ_3 are simply the spatial components of the corresponding 3-field, while ϕ_0 is a *scalar* or *time-like* component of the 4-field[2].

Definition 2.5 (Action) For a set of fields ϕ_i with the 4-position x^i, the *action* is defined as

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x = \int L dx^0 = \int L dt \tag{2.19}$$

where L is the Lagrangian and $\mathcal{L}$ is the *Lagrangian density*[a].

[a] Often also simply called the *Lagrangian*, although you will be able to tell the difference by looking at the notation.

Theorem 2.5 (Action principle) The *action principle* is simply another name for the *principle of stationary action*, which is itself often erroneously known as the *principle of least action*[a]. This simply means that the time derivative of the action of an isolated system is zero.

[a] This is because the principle states that instead of at a minimum, action tends to stay *stationary*, be it a maximum, a minimum or a saddle point.

Remark 2.3 As the principle can be used for action generated by *any* field, it is usually mentioned as '*an* action principle' instead of '*the* action principle'.
Now we look at what a field Lagrangian actually looks like. The simplest Lagrangian is that of a free massless scalar field, which can be used to model particles like massless scalar bosons[3]. It is given by

Definition 2.6 (Free massless scalar field Lagrangian)

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \tag{2.20}$$

The sole term is the *kinetic energy density*, which arises from the variation of the field ϕ over the 4-coordinates.

Remark 2.4 A *free field* is so-called as it has no interactions, which manifests in extra terms in the Lagrangian.

> **Quote 2.4** We're free! We're free!
>
> *Slave, when freed, in* Yuri's Revenge

Figure 2.1: Slave Miner

One can introduce mass to the free field. Note that this does not mean that the field itself is massive (which makes little sense) but rather that the particle that generates the field is massive. With the addition of a mass term, the Lagrangian becomes

Definition 2.7 (Free massive scalar field Lagrangian)

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 \tag{2.21}$$

where the second term is the *potential energy density*.

Remark 2.5 This Lagrangian is actually a reduced form of the free massive complex scalar field La-

[2] For example, in the electromagnetic 4-potential, ϕ_0 is the electric scalar potential.
[3] We see it more often in approximate models as fundamental massless scalar particles with zero mass are rare.

grangian[4], which is

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi^* \partial_\mu \phi - \frac{1}{2}m^2|\phi|^2 \tag{2.22}$$

Derivation 2.3 (Euler-Lagrange equations and the boundary term) To find the equations of motion via the action principle, we vary the action S given in Equation 2.19 with respect to ϕ, which involves integration by parts:

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right] \tag{2.23}$$

Noting that $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$, we can write

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi)\right] \tag{2.24}$$

We can apply integration by parts to the term involving $\partial\mu(\delta\phi)$. The variation of the action is thus

$$\delta S = \int d^4x \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi}_{①} + \int d^4x \underbrace{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial\phi\right)}_{②} \tag{2.25}$$

Through this process, we have exposed the so-called *boundary term* ②, which is a total derivative and does not contribute to the equations of motion. This is because it can be converted into a surface integral over the boundary of the integration region using Gauss's law[a] In contrast, ① is known as the *bulk term*[b].
There are two scenarios in which the boundary term can be ignored:

- The field and its derivative vanishes at the boundary.
- The boundary extends into infinity.

Assuming the first point and applying the action principle leads to

Theorem 2.6 (Euler-Lagrange equations)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{2.26}$$

By recognising that ∂_μ is just the derivative over 4-coordinates or the *4-derivative*, we can see that this is the Euler-Lagrange equations that we have previously seen.

[a] As such, the boundary term is also called the *surface term*.
[b] The designations 'bulk term' and 'boundary term' are more common in general relativity with respect to the Einstein-Hilbert action, but are nice to remember nonetheless.

Before proceeding, we make a few comments on the bulk and boundary terms:

- The bulk term is so-called because it integrates over the entire volume of spacetime. When an action principle is imposed, the integrand of the bulk term vanishes, as seen in Equation 2.26.
- Hence, the bulk term reflects how the action responds to changes in the field ϕ in the 'bulk' or the 'interior' (i.e. everything minus the boundary) of the spacetime we consider. Under an action principle, the bulk term defines the allowed configurations for ϕ via the Euler-Lagrange equations.
- The boundary term reflects the influence of boundary conditions for the action. By imposing an action principle, we have also assumed that $\delta\phi = 0$ on the boundary - a boundary condition.
- This is why boundary conditions (like Dirichlet or Neumann conditions) are usually imposed in variational problems to ensure well-defined dynamics in the bulk term.

The *Hamiltonian* is essentially a Legendre transformation of the Lagrangian:

[4] Boy, is that a mouthful!

Definition 2.8 (Hamiltonian and Hamiltonian density) The *Hamiltonian* H is

$$H = \int \mathcal{H}(\phi, \pi, \partial_\mu \phi) d^3x \tag{2.27}$$

which is the volume integral of the *Hamilton density* $\mathcal{H}$. Also simply called the *Hamiltonian*, it is given by

$$\mathcal{H}(\phi_i, \pi_i, t) = \sum_i \pi_i \dot{\phi}_i(\phi_j, \pi_j) - \mathcal{L}(\dot{x}_k(x_j, p_j), x_k, t) \tag{2.28}$$

where ϕ is the field and π is the and *canonical momentum*, the equivalent of momentum in field theory.

Remark 2.6 As it turns out, the quantity which we have been led to believe to be the Hamiltonian as undergrads is actually the Hamiltonian density $\mathcal{H}$.

Derivation 2.4 (Hamilton's equations) By taking the variation of Equation 2.28, one finds

$$\delta\mathcal{H} = \sum_i \delta\pi_i \dot{\phi}^i(\phi_j, \pi_j) - \delta\pi^i \frac{\partial \mathcal{L}}{\partial \pi^i} = \sum_i \delta\pi_i \dot{\phi}^i(\phi_j, \pi_j) - \sum_i \delta\phi^i(\phi_j, \pi_j)\dot{\pi}_i \tag{2.29}$$

Now compare this against the general variation:

$$\delta\mathcal{H} = \delta\phi^i(\phi_j, \pi_j)\frac{\partial \mathcal{H}}{\partial \phi^i(\phi_j, \pi_j)} + \delta\pi_i \frac{\partial \mathcal{H}}{\partial \pi_i} \tag{2.30}$$

By equating the two expressions for $\delta\mathcal{H}$, we recover the so-called *Hamilton's equations*:

Theorem 2.7 (Hamilton's equations)

$$\frac{\partial \mathcal{H}}{\partial \phi_i} = -\dot{\pi}_i \quad \frac{\partial \mathcal{H}}{\partial \pi_i} = \dot{\phi}_i \tag{2.31}$$

Remark 2.7 From this, we can rewrite Hamilton's equations in terms of Poisson brackets:

$$\dot{x}_i = \{x_i, H\} \quad \dot{p}_i = \{p_i, H\} \tag{2.32}$$

The quantum version of the first equation is simply the Schrödinger equation in the Heisenberg picture.

2.3 Gauge symmetries

Definition 2.9 (Covariant derivative) The *covariant derivative* is the extension of the 4-derivative in the presence of a 4-vector field A^μ. It is

$$D^\mu = \partial^\mu + iA^\mu \tag{2.33}$$

Definition 2.10 (Probability current) The 4-vector equivalent of the probability density ϱ is the so-called *probability 4-current*, *Noether current* or *conserved current* J^μ. Its zeroth component is simply the good ol' probability density, and its 3 other components are the *probability (3-)current* $\mathbf{J}^\alpha$. It is given by

$$J_\mu = i(\phi^* D_\mu \phi - \phi D_\mu \phi^*) \tag{2.34}$$

The probability 3-current $\mathbf{J}^\alpha$ represents the flow or movement of probability in space. i.e. how the probability associated with a quantum particle's position is distributed and 'flows' over time. This is analogous to the relationship between energy density and 3-momentum as components of 4-momentum, where 3-momentum represents a flow of energy.
From the Noether current, one can derive a charge-like quantity called the *total probability*, the *probability charge* or the *conserved charge* as it is conserved with respect to time:

Definition 2.11 (Probability charge)

$$Q = \int d^3x J^0 \tag{2.35}$$

In most physical contexts, Q is expected to be normalised to 1. Further doing a time integral yields the flux of the conserved current through a 3D constant time hypersurface.
Remark 2.8 A parallel here can be made with the Hamiltonian, which observes $H = \int d^3x p^0$.
As probability is conserved, J^μ is Lorentz-invariant and satisfies the continuity equation

Theorem 2.8 (Continuity equation)

$$\partial_\mu J^\mu = 0 \tag{2.36}$$

Note 2.3 This implies the conservation of the charge associated with the probability current:

$$\frac{dQ}{dt} = \int d^3x \partial_0 J^0 = -\int d^3x \nabla \cdot \mathbf{J} = 0 \tag{2.37}$$

Exercise 2.1 Let ϕ be a free scalar field obeying the Klein-Gordon equation (Equation 3.2), and let J^μ be the associated density and current 4-vector. Derive the continuity equation 2.36.

So far, we have heard about the layman's version of Noether's theorem 'All symmetries lead to conservation laws.' But what exactly does a 'symmetry' mean? As it turns out, symmetry does *not* imply invariance under gauge transformations. Rather, it is defined with respect to an action principle:

Definition 2.12 (Gauge symmetry) A transformation consisting of a *gauge symmetry* or a *quasi-symmetry* is the variation of the Lagrangian under the transformation $\delta\mathcal{L}$ is the *total differential* of a current-like quantity K^μ that is a vector field whose significance we will soon see. i.e.

$$\delta\mathcal{L} = \partial_\mu K^\mu \tag{2.38}$$

Remark 2.9 Gauge symmetries include many important symmetries in physics, such as translations and gauge transformations.

Derivation 2.5 (Action principle and gauge symmetry) Importantly, K^μ is a boundary term, which we will demonstrate now. We can solve for the variation of the action by integrating the variation of the Lagrangian in Equation 2.38, which gives:

$$\delta S = \int d^4x \delta\mathcal{L} = \int d^4x \partial_\mu K^\mu \tag{2.39}$$

Using the divergence theorem, this integral can be converted into a surface integral over the boundary of the spacetime region ∂V:

$$\delta S = \int_{\partial V} d^3x K^\mu n_\mu \tag{2.40}$$

where n_μ is the normal vector to the boundary. Importantly, under gauge symmetry, only a boundary term exists, and *the bulk term vanishes.*
Again, by using an action principle, we easily see that

$$\int_S d^3x K^\mu n_\mu = 0 \tag{2.41}$$

Assuming a vanishing boundary term as we have previously done, we will inevitably conclude that a gauge symmetry means *an invariant action* $\delta S = 0$ (i.e. an action principle).

Derivation 2.6 (Noether's theorem) We can now finally derive Noether's theorem. The variation of the Lagrangian can be written as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi \tag{2.42}$$

where $\delta\phi$ is the variation of the field ϕ under the symmetry transformation.
We can use the product rule on the second term, yielding

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi \tag{2.43}$$

Using the Euler-Lagrange equations, this becomes

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_j)}\delta_\theta\phi_j\right) \tag{2.44}$$

By substituting this into Equation 2.38, we find

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu K^\mu \tag{2.45}$$

Finally, by consulting the continuity equation seen in Equation 2.36, one can identify the Noether current and hence recover Noether's theorem:

Theorem 2.9 (Noether's theorem)

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - K^\mu \tag{2.46}$$

Quote 2.5 Yes, but it is the same sound

David Steiner, comparing the 'oe' in 'Noether' with 'ö', 21 November 2024

As an example, we now derive the Noether current under Poincaré transformations, which, as seen in *Spinors & Symmetries*, includes translations and Lorentz transformations.

Derivation 2.7 (Translation) An infinitesimal 4-coordinate translation and its corresponding field transformation are

$$x'^\mu = x^\mu + \epsilon^\mu \quad \phi'(x) = \phi(x) + \epsilon^\nu\partial_\nu\phi \tag{2.47}$$

By using Noether's theorem, we can find that the contribution to the Noether current is the stress-energy tensor:

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu}\mathcal{L} \tag{2.48}$$

In a general translation, one has $\delta x = \epsilon^\nu\partial_\nu\phi$. Inserting this into Equation 2.42, we can find that $K^\mu = 0$. Hence, the Noether current is simply

$$J^\mu = T^{\mu\nu}\epsilon_\nu \tag{2.49}$$

By inserting the stress-energy tensor into the Euler-Lagrange equations, it is possible to find the conservation law

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.50}$$

Note 2.4 Equation 2.50 merely states that $T^{\mu\nu}$ is invariant when differentiated over *4-coordinates*. As such, it describes the conservation of energy-momentum *density* as opposed to energy and momentum themselves. In cosmology, where inflation exists, the total energy of the universe is *not* conserved. Rather, as inflation gives rise to the expansion of spacetime, the energy *increases* to preserve the invariance of energy-momentum density.

Derivation 2.8 (Lorentz transformation) A Lorentz transformation can be either a rotation or a Lorentz boost. Assuming the parameterisation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \quad \omega^{\mu\nu} = -\omega^{\nu\mu} \tag{2.51}$$

where $\omega^{\nu\mu}$ is some parameter, the coordinate and field transformations are

$$x'^{\mu} = x^{\mu} + \Lambda^{\mu}_{\nu} x^{\nu} \quad \phi'(x) = \phi(x) + \frac{1}{2}\omega^{\rho\sigma}\Sigma_{\rho\sigma}\phi \tag{2.52}$$

where $\Sigma_{\rho\sigma}$ are the generators of the Lorentz group in the representation of ϕ (e.g., for scalars $\Sigma_{\rho\sigma} = 0$, for vectors $\Sigma_{\rho\sigma}$ corresponds to antisymmetric tensors, and so on).
Again, by using Noether's theorem, we can find that the contribution to the Noether current is the total angular momentum, which includes the orbital angular momentum and spin:

$$M^{\mu\rho\sigma} = x^{\rho}T^{\mu\sigma} - x^{\sigma}T^{\mu\rho} + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}\Sigma^{\rho\sigma}\phi \tag{2.53}$$

For Lorentz transformations, $K^{\mu} = \partial_{\nu}(x^{\nu}J^{\mu} - x^{\mu}J^{\nu})$, and the Noether current becomes:

$$J^{\mu} = M^{\mu\rho\sigma}\omega_{\rho\sigma} \tag{2.54}$$

Hence, combining both types of symmetries, the most general form of the Noether current under Poincaré transformations can be written as:

$$J^{\mu} = T^{\mu\nu}\epsilon_{\nu} + \frac{1}{2}M^{\mu\rho\sigma}\omega_{\rho\sigma}, \tag{2.55}$$

where $T^{\mu\nu}$ represents energy-momentum contributions, and $M^{\mu\rho\sigma}$ represents both orbital and intrinsic angular momentum contributions.

Part I

Canonical quantisation

Chapter 3

Free fields

In this chapter, we make an overview of the so-called Klein-Gordon equation, an intuitive attempt at developing a special relativstic quantum theory, as well as its failures. By quantising the Klein-Gordon field, we will get a first taste of canonical quantisation.

3.1 Klein-Gordon equation and its demise

The infamous *Klein-Gordon equation* was an earlier attempt at unifying quantum mechanics with SR. At that point, QFT was still not developed, and it was naively assumed that the Schrödinger equation could be modified to be Lorentz-invariant by purely applying the wavefunction ϕ to both sides of the equivalence

$$E_p^2 = m^2 + |p|^2 \to -\partial_t^2 = m^2 - \nabla^2 \tag{3.1}$$

where, for convenience, we have set $c = \hbar = 1$. $E_p^2 = m^2 + |p|^2$ is known as the *on-shell condition*[1]. This gives

Theorem 3.1 (Klein-Gordon equation) The Klein-Gordon equation describes scalar (spin-0) particles in a relativistic framework:

$$(\Box + m^2)\phi = 0 \tag{3.2}$$

where we have modified the d'Alembertian $\Box = D^\mu D_\mu$ such that the covariant derivative replaces the partial derivative. $(\Box + m^2)$ is known as the *Klein-Gordon operator*.

Remark 3.1 As free fields are solutions to the Klein-Gordon equation, a Klein-Gordon operator acting on a free field always gives 0.

Exercise 3.1 Show that the Klein-Gordon equation can be recovered by using an action principle on the free massive scalar field Lagrangian in Equation 2.21.

The Klein-Gordon equation has a plane wave general solution

$$\phi(x) = Ne^{-iE_p t - p\cdot x} \tag{3.3}$$

where N is a normalisation constant.

Derivation 3.1 (Demise) Consider a simple 1D potential barrier of the form

$$V(x) = \begin{cases} 0 & x < 0 \\ V & x \geq 0 \end{cases} \tag{3.4}$$

According to the Klein-Gordon equation, the simplest solution would be

$$\phi(t,x) = \begin{cases} e^{-i(E_p t - px)} + ae^{-i(E_p t + px)} & x < 0 \\ be^{-i(E_p t + kx)} & x \geq 0 \end{cases} \tag{3.5}$$

[1] In reference to the so-called *mass shell* in momentum space, a surface where the energy and momentum satisfy the on-shell condition.

where p is the momentum, $k = \sqrt{(E_p - V_0)^2 - m^2}$, and

- $e^{-i(E_p t - px)}$ is the part of the field travelling at the +ve x-direction that has not yet reached the potential barrier.
- $ae^{-i(E_p t + px)}$ is the part of the field reflected at the barrier travelling at the −ve x-direction.
- $be^{-i(E_p t + kx)}$ is the part of the field transmitted through the barrier travelling at the +ve x-direction.

Intuitively, both $\phi(x)$ and $\partial_x \phi(x)$ are continuous at $x = 0$, from which we find the parameters

$$a = \frac{p-k}{p+k} \quad b = \frac{2p}{p+k} \tag{3.6}$$

By inserting the $x < 0$ solution into the Klein-Gordon equation, we find a dispersion relation

$$p = \pm\sqrt{E_p^2 + m^2} \tag{3.7}$$

To reflect the forward-travelling nature of $e^{-i(E_p t - px)}$m the group velocity $v_g = \partial_p E_p$ must be positive. This forces us to adopt the positive solution.
Now we insert the $x \geq 0$ solution into the Klein-Gordon equation. Due to the non-zero potential V, we replace the partial derivatives with covariant derivatives:

$$i\partial_t \to i\partial_t - V \quad \partial_t \to \partial_t + iV \tag{3.8}$$

which gives

$$k = \mp\sqrt{(E_p - V)^2 - m^2} \tag{3.9}$$

Again, to reflect the forward-travelling nature of $be^{-i(E_p t + kx)}$, the group velocity or its inverse $\frac{1}{v_g} = \frac{\partial k}{\partial E_p} = \mp\frac{E_p - V}{|k|}$ must be positive. Now consider the case $V > E_p$. From the group velocity condition, we are forced to adopt the negative solution.
One can find a negative energy solution for each positive energy solution. However, this can be handwaved, as we will see much, much later, as antimatter. The real problem lies with the probability density, which we recall to be the 0^{th} component of the conserved current:

$$\varrho = i(\phi^*(\partial_t + iV)\phi - \phi(i\partial_t - V)\phi^*) \tag{3.10}$$

which, in this case, is simply

$$\varrho = 2b^2(E_p - V) \tag{3.11}$$

For $E < V$, this probability density is always negative.
Remark 3.2 Wait, what?

A negative probability density is always unphysical, which can be resolved by turning the (classical) Klein-Gordon field into an operator. This is the beginning of quantum field theory.
Historically, the quantisation of fields/operators as the so-called *field operator* is known as *second quantisation*, in contrast to quantised particles, which was known as the *first quantisation*[2]. Today, we call both *canonical quantisation* as canonical commutation relations are utilised in both quantisation processes.

3.2 Fock space

A generic state in QFT is essentially a linear combination of k particle states for some arbitrary k. This is significant in that k is not fixed - particles might be created and annihilated. Hilbert spaces, which have a fixed number of particles, fail to describe QFT. Rather, the vector space QFT lies in is known as a *Fock space*:

[2] In fact, second quantisation is a slight misnomer as quantising operators is also possible (although unnecessary) in QM.

Definition 3.1 (Fock space) The Fock[a] space $\mathcal{F}(\mathcal{H}_1)$ is the direct sum of all n-particle Hilbert spaces:

$$\mathcal{F}(\mathcal{H}_1) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \tag{3.12}$$

where $\mathcal{H}_0 = \mathbb{C}$ is the vacuum state space (zero particles), and $\mathcal{H}_n = \mathcal{H}_1^{\otimes n}$ is the n-particle Hilbert space.

[a] Named after Vladimir Fock, or *Fok* in scientific transliteration.

Definition 3.2 (Creation and annihilation operators) Fock spaces are equipped with creation and annihilation operators $a^\dagger$ and a that adds a particle to the final state and removes a particle from the initial state respectively:

$$a^\dagger|n\rangle = |n+1\rangle \quad a|n\rangle = |n-1\rangle \tag{3.13}$$

Remark 3.3 Bosonic creation and annihilation operators are near-identical the good ol' ladder operators in QM as they obey the same commutation relations. Fermionic creation and annihilation operators observe anticommutation relations instead.

Theorem 3.2 (Bosonic and fermionic operator commutations)

- Bosonic operators commute:
$$[B_i, B_j] = B_iB_j - B_jB_i = 0 \tag{3.14}$$
- Fermionic operators anticommute:
$$\{F_i, F_j\} = F_iF_j + F_jF_i = 0 \tag{3.15}$$
- A bosonic operator commutes with a fermionic operator:
$$[B, F] = BF - FB = 0 \tag{3.16}$$

3.3 Quantisation of the Klein-Gordon field

A momentum space is a generalisation of the *reciprocal space* or *wavevector space* you may have seen before in crystallography. The momentum and position (or physical) spaces are Fourier transforms of each other.

Theorem 3.3 (Fourier transform identity) From the delta function property $\int_{-\infty}^{\infty} dk\, f(k)\delta(k) = f(0)$, one has

$$\int_{-\infty}^{\infty} e^{-ikx} dx = 2\pi\delta(k) \tag{3.17}$$

So far we have been working in position space, which is, informally speaking, the collection of all possible positional vectors. In the following derivation, even though the field and canonical momentum operators are still in position space, their expressions are in momentum space, which we Fourier-transform back to position space.

Derivation 3.2 (Field operator) The Klein-Gordon equation general solution in Equation 3.3, can be rewritten to account for negative energy solutions:

$$\phi(x,t) = \int d^3pN_p \left(f_p e^{-i(E_pt - p\cdot x)} + f_p^* e^{i(E_pt - p\cdot x)}\right) \tag{3.18}$$

where N_p, a real function of p, is the previously seen normalisation factor and f_p is a complex function of p (and hence based in momentum space). Importantly, as the field operator $\phi(x,t)$ is based in position space, we must perform a Fourier transform $\int d^3pN_p$ to convert f_p from momentum space to position space.
To ensure that the resultant quantised Hamiltonian will evolve with time in the same way, we replace f_p and f_p^* with the annihilation and creation operators a_p and $a_p^\dagger$. As the Klein-Gordon field is a scalar

field (spin-0), it is a bosonic field and its components commute under field quantisation:

Theorem 3.4 (Bosonic creation and annigilation operator commutations) For two arbitrary momenta p and q in bosonic fields, their creation and annihilation operators a_p, $a_p^\dagger$, a_q and $a_q^\dagger$ observe:

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta^3(p-q) \tag{3.19}$$

$$[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0 \tag{3.20}$$

Recalling that special relativity must be observed, we must choose an N_p that makes $\phi(x)$ Lorentz-invariant. The Lorentz-invariant phase space volume element for a single particle is given by

Definition 3.3 (Lorentz-invariant phase space volume element)

$$dV = \frac{dp^3}{(2\pi)^3 2E_p} \tag{3.21}$$

For the field operator, we can remove a factor of $1/\sqrt{2E_p}$ to this volume element so that no factors of E_p emerge in the field and momentum operator commutation relations[a]. Taking $N_p = 1/((2\pi)^3\sqrt{2E_p})$ the Klein-Gordon field operator is then written as

Definition 3.4 (Klein-Gordon field operator)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right) \tag{3.22}$$

[a] Lorentz invariance is preserved as the factor $1/\sqrt{2E_p}$ is not really gone, but rather absorbed into the field amplitude.

Note 3.1 (Two formalisms of $\phi(p)$) Intuitively, the field operator in momentum space is the Fourier transform of its position space counterpart

$$\phi(p) = \int d^3x e^{-ip\cdot x}\phi(x) \tag{3.23}$$

However, the notation $\phi(p)$ is overloaded: a second formalism exists in some literature where $\phi(p)$ denotes the *momentum space contribution* to the field operator seen in Equation 3.22:

$$\phi(p)_{\text{alt}} = \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right) \tag{3.24}$$

which gives rise the (equally correct) formula for the position space field operator

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\phi(p)_{\text{alt}} \tag{3.25}$$

In this book, we use the first formalism.

The momentum operator can then be written by adding a factor of iE_p and changing the sign on the first term

Definition 3.5 (Klein-Gordon momentum operator)

$$\pi(x) = -\int \frac{d^3p}{(2\pi)^3\sqrt{2}} \sqrt{E_p} \left(-a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right) \tag{3.26}$$

The commutation relations for bosonic fields are then

Theorem 3.5 (Bosonic field and momentum operator commutations) For some arbitrary spacetime coordinates x and y

$$[\phi(x), \pi(y)] = i\delta^3(p-q) \tag{3.27}$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0 \tag{3.28}$$

By recalling the definition of the Hamiltonian, we can derive the Klein-Gordon Hamiltonian density from the Klein-Gordon Lagrangian density as

$$\mathcal{H} = \frac{1}{2}(\pi^2 + |\nabla\phi|^2 + m^2\phi^2) \tag{3.29}$$

We then find the (time-independent) Hamiltonian operator:

$$H = \int \frac{d^3p}{(2\pi)^3 2E_p} E_p^2 \left(a_p^\dagger a_p + a_p a_p^\dagger\right) = \underbrace{\int \frac{d^3p}{(2\pi)^3} E_p a_p^\dagger a_p}_{①} + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} \left[a_p, a_p^\dagger\right]}_{②} \tag{3.30}$$

However, as $\left[a_p, a_p^\dagger\right]$ is proportional to the identity, so is the term ②, and we can neglect it for convenience. Due to the ordering of its creation and annihilation operators, the remaining term ① is known as the *normal-ordered* or *Wick-ordered*[3] Hamiltonian:

Definition 3.6 (Normal-ordered Klein-Gordon Hamiltonian)

$$:H: = \int \frac{d^3p}{(2\pi)^3} E_p a_p^\dagger a_p \tag{3.31}$$

Here we have introduced the concept of *normal ordering*:

Definition 3.7 (Normal ordering) For some operator O that can be expressed as a polynomial of creation and annihilation operators, its normal ordering or *Wick ordering* $:O:$ is the same polynomial, but with all creation operators to the left of all annihilation operators. This is often necessary in that it eliminates unphysicality created by vacuum fluctuations like $a_p^\dagger a_p$.

Remark 3.4 For example, given some $O = a_p a_q a_p^\dagger$, we have $:O: = a_p^\dagger a_p a_q$.

> **Quote 3.1** You wanted me back... I'm back!
>
> *John Wick, in* John Wick: Chapter 2

By considering Equation 3.31 as the entire Hamiltonian[4], we are able to easily diagonalise it. Therefore, we can find that $:H:$ admits the eigenvectors $|n\rangle$ or $a_p|n\rangle$, with the eigenvalues being

$$:H:a_p|n\rangle = (-n - E_p)a_p|n\rangle \tag{3.32}$$

Remark 3.5 Conversely, *antinormal ordering* places creation operators to the right instead.

Theorem 3.6 (Vacuum expectation value) The vacuum expectation value of any normal-ordered expression yields zero.

[3] Named after Gian Carlo Wick.

[4] An advantage of considering a quantity's normal-ordered counterpart as itself is that we eliminate any uninteresting constant terms. This often simplifies calculations.

Chapter 4

Interacting fields I: Preliminaries

We now investigate particle interactions as well as the two matrices S_{fi} and $\mathcal{M}_{fi}$, the first of which physically governs the probability a certain interaction will take place. The Feynman rules, which we use to build the formula for calculating elements of $\mathcal{M}_{fi}$, are then derived for ϕ^4 theory, a simple toy model. Some concluding discussions on their use in high energy physics are then made.

4.1 Dynamic pictures and time evolution

We are now in a position to expand our field theory beyond free particles and into particle interactions. To this end, we modify our Hamiltonian to include an interaction term known as the *interaction Hamiltonian* H_{int}:

$$H = H_0 + H_{\text{int}} \tag{4.1}$$

where H_0 is the *free Hamiltonian*.
In QM, there are three *dynamical pictures* or *representations*. Effectively, they are different formalisms through which one can represent time evolution. Now we revisit them in the context of scattering, where quantum states are slightly different. We have 'in' states $|\psi, \text{in}\rangle$ which denote *prepared* or *incoming* particles and 'out' states $\langle\alpha, \text{out}|$ which denote *detected* or *outgoing* particles.
Remark 4.1 Significantly, 'out' states are treated as half of the density operator. i.e. they are effectively regarded as operators, not states:

$$P_\alpha = |\alpha, \text{out}\rangle\langle\alpha, \text{out}| \tag{4.2}$$

Definition 4.1 (Schrödinger picture) The *Schrödinger picture* is the representation we have encountered in undergrad QM. Time evolution is represented as follows:

- Operators are time-invariant.
- 'in' states evolve under the *total Hamiltonian* H:

$$|\psi, t\rangle = e^{-iH(t-t_0)}|\psi, \text{in}\rangle \tag{4.3}$$

- 'out' states are time-invariant.

Definition 4.2 (Heisenberg picture) The *Heisenberg picture* is the opposite of the Schrödinger picture:

- Operators evolve under the *total Hamiltonian* H:

$$O_H = e^{iH(t-t_0)} O e^{-iH(t-t_0)} \tag{4.4}$$

- 'in' states are time-invariant.
- 'out' states evolve under the *total Hamiltonian* H:

$$_H\langle\alpha, t| = \langle\alpha, \text{out}|e^{-iH(t-t_0)} \tag{4.5}$$

Definition 4.3 (Interaction picture) The *interaction picture* lies between the Schrödinger and Heisenberg pictures:

- Operators evolve under the *free Hamiltonian* H_0:

$$O_H = e^{iH_0(t-t_0)} O e^{-iH_0(t-t_0)} \tag{4.6}$$

- 'in' states evolve under both the *total* and *free Hamiltonians*:

$$|\psi, t\rangle = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} |\psi, \text{in}\rangle \tag{4.7}$$

  where we often label *interaction picture evolution operator* $U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$.

- 'out' states evolve under the *free Hamiltonian* H_0:

$${}_I\langle \alpha, t| = \langle \alpha, \text{out}| e^{-iH_0(t-t_0)} \tag{4.8}$$

Note 4.1 A few points of interest should be mentioned:

- There is a 'conservation of time evolution terms': Multiplying all 3 time-evolved terms should leave only $e^{-iH(t-t_0)}$, where H is expectedly the *total Hamiltonian*.
- t_0 refers to the time when the state is prepared as $|\psi, \text{in}\rangle$. At this time, all dynamical pictures are identical.
- While the Heisenberg picture generally simplifies calculations, the interaction picture is advantageous when the Hamiltonian includes an interaction term.

The evolution operator in the interaction picture is

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \tag{4.9}$$

Theorem 4.1 (Interaction picture evolution operator properties) The interaction picture evolution operator has several properties:

- Under zero time evolution, it returns the identity:

$$U(t_0, t_0) = \mathbb{I} \tag{4.10}$$

- Inverse:

$$U^{-1}(t_1, t_2) = U(t_2, t_1) \tag{4.11}$$

- Stacking:

$$U(t_3, t_1) = U(t_3, t_2) U(t_2, t_1) \tag{4.12}$$

Derivation 4.1 (Time ordering) By differentiating Equation 4.9 against t and using the initial condition in Equation 4.10, one can derive an alternate expression of $U(t, t_0)$ in terms of itself at a different point in time:

$$U(t, t_0) = \mathbb{I} - i \int_{t_0}^{t} dt_1 H_{\text{int},I}(t_1) U(t_1, t_0) \tag{4.13}$$

By inserting $U(t, t_0)$ into $U(t_1, t_0)$ over and over and over again, one yields the so-called *Dyson series*:

$$U(t, t_0) = \mathbb{I} + \sum_{j=1}^{\infty} (-i)^j \int_{t_0}^{t} dt_j \cdots \int_{t_0}^{t} dt_1 H_{\text{int},I}(t_j) \cdots H_{\text{int},I}(t_1) \tag{4.14}$$

To ensure that the intergrations are performed at the correct temporal order, we introduce the so-called *time ordering symbol*:

Definition 4.4 (Time ordering symbol) Consider a series operators $A_1(x_1)\cdots A_n(x_n)$, each of which can be represented in the form of creation and annihilation operators like $A_n(x_n) = A_n^+(x_n) + A_n^-(x_n)$. The time ordering symbol is a loosely defined convenience which reorders the operators according to their 4-position:

$$T[A_1(t_1)\cdots A_n(t_n)] = (-1)^p A_{i_1}(t_{i_1})\cdots A_{i_n}(t_{i_n}) \quad \text{for} \quad x_{i_1} \leq \cdots \leq x_{i_n} \tag{4.15}$$

where p, the parity, is determined as follows:

- Bosonic-bosonic swaps do not contribute to p. No sign change occurs as bosonic operators commute.
- Bosonic-fermionic swaps do not contribute to p. No sign change occurs as bosonic and fermionic operators commute.
- Due to anticommutation, fermionic-fermionic swaps contribute to p by the following:

$$p = \text{num. of fermionic-fermionic swaps} \mod 2 \tag{4.16}$$

Remark 4.2 When a system is entirely comprised of bosons or fermions, this simplifies:

- For bosonic operators, $p = 0$.
- For bosonic operators, $p = 0$ if the number of swaps is even and $p = 1$ if the number of swaps is odd.

Noting that the Dyson series can be represented compactly via an exponential, we represent $U(t, t_0)$ as a time-ordered exponential:

$$U(t, t_0) = T\left[\exp\left(-i\int_{t_0}^{t} dt' H_{\text{int},I}(t')\right)\right] \tag{4.17}$$

In simplified sceanrios, T can also be represented mathematically with the *Heaviside step function* $\theta(t)$:

$$T[A_1(t_1)A_2(t_2)] = \theta(t_1 - t_2)A_1(t_1)A_2(t_2) \pm \theta(t_2 - t_1)A_2(t_2)A_1(t_1) \tag{4.18}$$

where:

Definition 4.5 (Heaviside step function)

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{4.19}$$

Theorem 4.2 (Heaviside step function properties)

$$\partial_t \theta(t) = \delta(t) \tag{4.20}$$

$$(\partial_t \theta(t))\phi(t) = -\delta(t)(\partial_t \phi(t)) \tag{4.21}$$

4.2 Scattering matrix

One way in which a field can interact with its environment is scattering. In scattering, we have the *S-matrix* or the *scattering matrix*, which encodes all the information about the probabilities of different scattering processes. It can be derived by taking limits of $U(t, t_0)$, where time evolution covers the entire history of the system, from the infinite past to the infinite future:

$$S = \lim_{t\to\infty} \lim_{t_0\to 0\infty} U(t, t_0) \tag{4.22}$$

Definition 4.6 (S-matrix operator) For the initial/incoming state $|\psi, \text{in}\rangle$ and the final/outgoing state $\langle\alpha, \text{out}|$, one can find the S-matrix element S_{fi} via the S-matrix operator S:

$$S_{fi} = \langle\alpha, \text{out}|S|\psi, \text{in}\rangle \tag{4.23}$$

Like in HEP, S_{fi} represents the probability amplitude that $|\psi, \text{in}\rangle$ evolves into $\langle\alpha, \text{out}|$.

Now consider a highly idealised system of n particles which are prepared with momenta $\{p_i, i = 1, \cdots, n\}$ at time $t_0 \to -\infty$. They interact with (i.e. scatter in) a perturbed Hamiltonian $H = H_0 + H_{\text{int}}$, reaching a set of final momenta $\{q_i, i = 1, \cdots, n\}$ time $t \to \infty$. One can represent the initial and final states with

$$|p, \text{in}\rangle = \prod_{i=1}^{n} a^\dagger_{p_i}|0\rangle \quad \langle q, \text{out}| = \langle 0|\prod_{i=1}^{n} a^\dagger_{q_i} \tag{4.24}$$

where $|0\rangle$ is the ground state. In this scenario, the S-matrix elements are

$$S_{qp} = \langle q, \text{out}|e^{-iH(t-t_0)}|p, \text{in}\rangle \tag{4.25}$$

To evaluate the elements of S_{qp}, we need to diagonalise the full Hamiltonian H. As this is typically impossible, we must use a perturbative approach to deal with H_{int}, where we assume that the interaction Hamiltonian $H_{\text{int}}(t)$ is zero at $t \to -\infty$ and $t \to \infty$.

Derivation 4.2 (Scattering in the Heisenberg picture) One can recall the so-called *Heisenberg equation*, which is the Schrödinger equation in the Heisenberg picture. Here, the classical Poisson bracket is replaced with a commutator:

Theorem 4.3 (Heisenberg equation)

$$\dot{\phi} = i[:H:, \phi] \tag{4.26}$$

Remark 4.3 At $t = 0$, the Heisenberg equation is identical to the Schrödinger equation.
The same equation can be applied to a_p:

$$\dot{a}_p = i[:H:, a_p] \to \dot{a}_p = iE_p a_p \tag{4.27}$$

Plugging this result into the field operator yields

$$\phi(x)_H = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a^\dagger_p e^{ip\cdot x}\right) \tag{4.28}$$

where one has the additional on-shell condition

$$p^0 = \sqrt{|p|^2 + m^2} \tag{4.29}$$

From this, one can represent the creation and annihilation operators in terms of the wavefunctions.

$$ia^\dagger_{p,H} = \int \frac{\mathrm{d}^3\mathbf{x}}{\sqrt{2E_p}}\left(\mathrm{e}^{-ip\cdot x}\left(\partial_0\varphi_H(x)\right) - \varphi_H(x)\left(\partial_0\mathrm{e}^{-ip\cdot x}\right)\right) \tag{4.30}$$

By noting that the Heisenberg picture S-matrix element is

Definition 4.7 (Heisenberg picture S-matrix element)

$$S_{qp,H} = \langle 0|a_{q_1,H}\cdots a_{q_m,H}a^\dagger_{p_1,H}\cdot a^\dagger_{p_n,H}|0\rangle \tag{4.31}$$

We can plug in Equation 4.30, which yields the so-called *LSZ formula* or the *LSZ reduction formula* for S-matrix elements, named after Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann:

Theorem 4.4 (LSZ reduction formula for S-matrix elements) For a series of spacetime coordinates x_i, the scattering matrix elements can be represented by

$$S_{qp,H} = \int \frac{d^4x_1}{\sqrt{2E_{p_1}}} \cdots \int \frac{d^4x_{m+n}}{\sqrt{2E_{q_m}}} e^{-i\sum_{j=1}^{n} p_j\cdot x_j + i\sum_{j=1}^{m} q_j\cdot x_{n+j}} \prod_{j=1}^{m+n} (\partial_{x_j}^2 + m^2)\langle 0|T\left(\phi_H\left(x_1\right)\cdots\phi_H\left(x_{m+n}\right)\right)|0\rangle \tag{4.32}$$

We investigate the physical significance of each term:

- The integral terms $\int \frac{d^4x_j}{\sqrt{2E_{p_j}}}$. The energies E_{p_j} associated with momenta p_j are integrated over the spacetime coordinates x_j for each external particle. As the LSZ formula is normalised with respect to single-particle states, one has the normalisation factors $\sqrt{2E_{p_j}}$
- The exponential phases $e^{-i\sum_{j=1}^{n} p_j\cdot x_j + i\sum_{j=1}^{m} q_j\cdot x_{n+j}}$ correspond to plane waves representing the incoming and outgoing particles with momenta p_j and q_j respectively.
- The Klein-Gordon operators $(\partial_{x_j}^2 + m^2)$ enforce that the external particles are on-shell.
- The time-ordered expectation value is labelled G as the time ordering symbol is a $(m+n)$-point Green's function:
$$G_{m+n} = \langle 0|T[\phi_H(x_1)\cdots\phi_H(x_{m+n})]|0\rangle \tag{4.33}$$
This is effectively a vacuum expectation value.

As of now, we cannot yet solve the LSZ reduction formula. This is because we do not have an expression for the Heisenberg picture fields γ_H. We can switch to the interaction picture (denoted by the subscript $_I$), which yields the expression

Theorem 4.5 (Interacting time-ordered propagator)

$$G_{m+n,I} = \frac{\langle 0|T[\phi_I(x_1)\cdots\phi_I(x_{m+n})]S|0\rangle}{\langle 0|S|0\rangle} \tag{4.34}$$

It is also[a] called the *interacting Green's function*, the *$(m+n)$-point Green's function* or the *correlation function*[b].

[a] The reason behind the name 'propagator' will be seen when we arrive at Feynman diagrams

[b] So-called as it is used to study correlations between field operators at different spacetime points in the interacting vacuum.

We note the following for this propagator:

- The *interacting time-ordered propagator* is so-called as it represents the probability amplitude for scattering processes involving $m+n$ field insertions (i.e. $m+n$ points in spacetime where the fields are evaluated). i.e. the probability amplitude for a particle (or more generally, a field excitation) to travel or 'propagate' from one point to another in spacetime.
- The 'interacting' does not refer to the dynamic picture. Instead, it is used to describe the presence of an interaction term in the Hamiltonian.
- Generally, one can use this expression without complications as it covers the whole term G_{m+n} and no operators are lost.

4.3 Self-interaction: ϕ^4 theory

We now introduce the concept of *self-interaction*, which is the interaction between a particle and its own field. A good toy model is the so-called ϕ^4 *theory*, which can be found by adding a *quartic interaction term* $-\frac{\lambda}{4!}\phi^4$ to Equation 2.21:

Definition 4.8 (ϕ^4 theory massive scalar field Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{4.35}$$

where λ is some dimensionless coupling constant that determines the strength of the interaction, and the $1/4!$ normalisation factor compensates for the fact that there are 4! ways to permute the (4 identical) fields in ϕ^4 theory.

ϕ^4 theory which is among a family of theories known as ϕ^n theory. It is superior to all other ϕ^n theories for two reasons:

- **Energetic stability:** ϕ^4 *theory* has the second-simplest interaction term that respects the symmetry $\phi \to -\phi$[1]:
  - The general form for the potential energy in ϕ^3 theory is $V(\phi) = m^2\phi^2 + g\phi^3$. Due to the odd exponential in $g\phi^3$, one can get a negative Hamiltonian expectation for a large, negative ϕ, even if the coupling constant g is positive.
  - In contrast, the $\lambda\phi^4$ term ϕ^4 theory is positive as long as λ is positive, ensuring a stable minimum, which is crucial for physical systems.
- **Renormalisability:** In certain dimensions, ϕ^3 theory is non-renormalisable, unlike ϕ^4 theory.

By integrating the normal-ordered interaction term $:\phi^4:$[2], we find the so-called *interaction Hamiltonian density*:

$$\mathcal{H}_{\text{int}} = -\frac{\lambda}{4!}:\phi^4: \tag{4.36}$$

The Hamiltonian is thus

$$H_{\text{int}} = \int d^3x\mathcal{H}_{\text{int}} = \frac{\lambda}{4!}\int d^3x:\phi^4: \tag{4.37}$$

which yields the S-matrix operator

$$S = T\left[\sum_{j=0}^{\infty}\frac{(-i)^j}{j!}\left(\frac{\lambda}{4!}\int d^4:\phi(x)^4:\right)\right] \tag{4.38}$$

Next, we will see how Equation 4.34 is a Taylor expansion of λ in the ϕ^4 theory. But before this, we introduce one last bit of formalism.

Definition 4.9 (Wick contraction) For operators A and B, the *Wick contraction*[a] or simply *contraction* is simply its vacuum expectation value

$$\overbrace{AB} = \langle 0|AB|0\rangle \tag{4.39}$$

[a] So-called as just like the contraction of indices in GR (which, in the case of a rank-2 tensor, starts with two indices and ends with a scalar), it starts with two operators and ends with a number.

Derivation 4.3 (Alternative forms of the Wick contraction) From the definition of Wick contractions, we can final several equivalences:

- The product of two operators can always be split into a part that contributes to the vacuum expectation value and a part that does not, which is actually the normal ordering[a] $:AB:$.

  Theorem 4.6 (Wick's first theorem)

  $$AB = \langle 0|:AB:|0\rangle + :AB: = \overbrace{AB} + :AB: \tag{4.40}$$

  In some literature, a rearranged version of *Wick's first theorem* is actually used as a less intuitive

[1] ϕ^2 theory has the simplest interaction term. But this term is simply the previously seen the mass term $\frac{1}{2}m^2\phi^2$.
[2] Recall that this is necessary to remove vacuum divergences like $\langle 0|aa^\dagger aa^\dagger|0\rangle$, which are not normal-ordered.

definition of Wick contractions.

$$\overbracket{AB} = AB - :AB: \tag{4.41}$$

- If A and B are the fields $\phi(x)$ and $\phi(y)$, the RHS becomes the Feynman propagator $D_F(x-y)$:

$$\overbracket{\phi(x)\phi(y)} = D_F(x-y) \tag{4.42}$$

- The vacuum expectation value is inherently time-ordered, so we can even make the equivalence

$$\overbracket{AB} = \langle 0|AB|0\rangle = \langle 0|T[AB]|0\rangle \tag{4.43}$$

[a] As normal-ordered operators always have creation operators before annihilation operators, their vacuum expectation value $\langle 0|:AB:|0\rangle$ is always zero.

Often, it is more convenient to use a very similar operation called the *time-ordered pairing* instead of Wick contractions.

Definition 4.10 (Time-ordered pairing)

$$\underbracket{A(x)B(y)} = \begin{cases} \overbracket{A(x)B(y)} & x^0 > y^0 \\ (-1)^p \overbracket{B(x)A(y)} & y^0 > x^0 \end{cases} \tag{4.44}$$

where p, last seen in Definition 4.4, is our good friend, the parity.

> **Quote 4.1** In more accurate books like Bogoliubov's, the time dependent contractions are written down with bottom brackets.
>
> *Felix Halbwedl, 22 December 2024*

Theorem 4.7 (Wick's second theorem) The time-ordering $T[A_1A_2A_3A_4A_5A_6\cdots]$, where all operators are made up of creation and annihilation operators like $A_i = A_i^+ + A_i^-$, can be expressed in terms of time-ordered pairings:

$$T[A_1A_2A_3A_4A_5A_6\cdots] = :A_1A_2A_3A_4A_5A_6\cdots: + \underbrace{\sum_{\text{single}} :\underbracket{A_1}A_2A_3A_4A_5A_6\cdots:}_{①} + \underbrace{\sum_{\text{double}} :\underbracket{A_1}A_2\underbracket{A_3}A_4A_5A_6\cdots:}_{②} + \cdots \tag{4.45}$$

where:

- ① denotes the sum of all the possible results of $A_1A_2A_3A_4A_5A_6\cdots$ undergoing one Wick contraction somewhere in the expression:

$$\sum_{\text{single}} :\underbracket{A_1}A_2A_3A_4A_5A_6\cdots: = :\underbracket{A_1}A_2A_3A_4A_5A_6\cdots: + :\underbracket{A_1}A_3A_2A_4A_5A_6\cdots: + \cdots \tag{4.46}$$

- ② denotes the sum of all the possible results of $A_1A_2A_3A_4A_5A_6\cdots$ undergoing two Wick contractions somewhere in the expression:

$$\sum_{\text{double}} :\underbracket{A_1}A_2\underbracket{A_3}A_4A_5A_6\cdots: = :\underbracket{A_1}A_2\underbracket{A_3}A_4A_5A_6\cdots: + :\underbracket{A_1}A_3\underbracket{A_2}A_4A_5A_6\cdots: + \cdots \tag{4.47}$$

- ...and so on.

An alternative, non-time-ordered version of *Wick's second theorem* is

$$A_1A_2A_3A_4A_5A_6\cdots = :A_1A_2A_3A_4A_5A_6\cdots: + \sum_{\text{single}} :\overbracket{A_1}A_2A_3A_4A_5A_6\cdots: + \sum_{\text{double}} :\overbracket{A_1}A_2\overbracket{A_3}A_4A_5A_6\cdots: + \cdots \tag{4.48}$$

where we simply use the normal contraction.

Finally, a nice trick is the so-called *Wick's third theorem* or *Wick's theorem for vacuum expectation values*:

Theorem 4.8 (Wick's third theorem) For operators A, B_1, $\cdots$, B_n, the following is observed:

$$\langle 0|T[AB_1 \cdots B_n]|0\rangle = \sum_i \langle 0|T[\overbrace{AB_1 \cdots B_i}\cdots B_n]|0\rangle \tag{4.49}$$

4.4 Feynman diagrams

Now we can evaluate the multi-point Green's function by inserting Wick's second theorem. Luckily for us, in doing so, many terms cancel out, and Equation 4.34 reduces to a two-point Green's function or the *Feynman propagator* of some spacetime coordinates x and y, labelled $D_F(x-y)$:

$$D_F(x-y) = \langle 0|T(\phi_I(x)\phi_I(y))|0\rangle \tag{4.50}$$

As we are still in the interaction picture, the interaction picture field operators $\phi_I(x)$ evolve through time under the *free* Hamiltonian H_0, just as we have seen before. This simplifies calculations quite a bit when one substitutes the interaction picture field operator $\phi_I(x)$ and yields

Definition 4.11 (Feynman propagator) The Feynman propagator is the probability amplitude for a scalar particle to propagate from the spacetime point x to y, taking into account quantum fluctuations:

$$D_F(x-y) = \lim_{\epsilon\to 0+} i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \tag{4.51}$$

where $i\epsilon$ ensures causal propagation. The on-shell condition is enforced by $p^0 = E_p$.

Note 4.2 The Feynman propagator is essentially the Green's function for the Klein-Gordon operator:

$$(\partial_x^2+m^2)D_F(x-y) = \lim_{\epsilon\to 0+} i\int \frac{d^4p}{(2\pi)^4}\frac{(\partial_x^2+m^2)e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} = -i\delta^4(x-y) \tag{4.52}$$

Now we introduce the *transition amplitude* or *scattering amplitude* $\mathcal{M}_{fi}$, a matrix related to the S-matrix whose physical significance we will see later.

Definition 4.12 (Transition amplitude)

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(p_f-p_i)\mathcal{M}_{fi} \tag{4.53}$$

where $\delta^4(P_i-P_f)$ enforces momentum conservation.

Remark 4.4 The physical significance of $\mathcal{M}_{fi}$ can be quickly found: The Kronecker delta δ_{fi} is essentially the identity that accounts for the case where there is no interaction. Hence, $\mathcal{M}_{fi}$ is a rescaled version of the interaction-dependent part of S_{fi}.
One can construct $\mathcal{M}_{fi}$ (and by that, S_{fi}) of a given field theory through its Feynman rules. These rules make use of the so-called *Feynman diagrams*, which is effectively the graphical representation of a S-matrix:

- **External legs**: The initial and final particle states, known as *external legs*, are the starting and end points on the left and right sides. For a total number of n such points, one has an n-point Feynman diagram.
- **Vertices:** Denoted by visible round dots. The number of vertices, known as the *order*, corresponds to the order of the coupling constant of the field theory.
- **Propagators**: The intermediate lines and loops represent abstract 'paths' the particles take or virtual particles whose terms and Feynman diagram representations are known as *propagators*[3].

[3] As ϕ^4 theory is purely self-interacting and does not involve any specific particles, we will use plain lines for all particles and propagators, which is almost always not the case in real life.

A Feynman diagram of the 0^{th} order has no vertices:

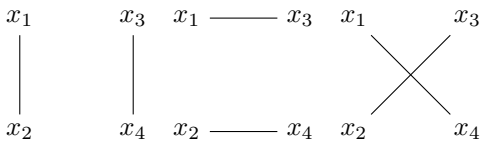

Figure 4.1: 0^{th}-order Feynman diagrams

A Feynman diagram of the 1^{st} order has a single vertex:

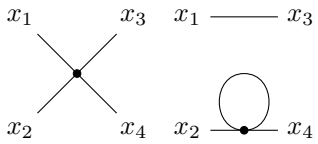

Figure 4.2: 1^{st}-order Feynman diagrams

Note the existence of the loop in the diagram on the right. Functionally, this diagram is identical to the middle entry in the 0^{th}-order diagrams, with the only difference being the removal of the loop. Hence, it is not the simplest expression this particular interaction can take. The middle 0^{th}-order diagram is then known as the *leading order*[4] Feynman diagram with respect to the interaction it represents as it is the most reduced form of the interaction.
A Feynman diagram of the 2^{nd} order has two vertices:

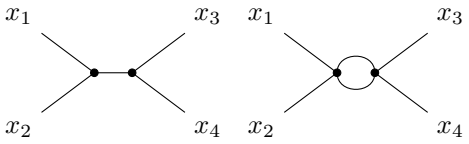

Figure 4.3: 2^{nd}-order Feynman diagrams

All of the diagrams above are 4-point Feynman diagrams.

> **Quote 4.2** You can still insert some hand drawings. Hand drawings are better than no drawings, and if you don't do it now, it eventually never happens.
>
> *Felix Halbwedl, encouraging the author to finish the Feynman diagram illustrations, 22 November 2024*

Note 4.3 (Vanishing Feynman diagrams) Some Feynman diagrams only exist on paper: a Feynman diagram vanishes if its corresponding mathematical expression involves taking the vacuum expectation value of a normal-ordered operator.

Remark 4.5 In this chapter and the next, we will restrict ourselves to leading order Feynman diagrams. The introduction of loops will often bring about divergences to infinity that must be eliminated via *renormalisation*.

4.5 ϕ^4 theory Feynman rules

Aside from the previously seen initial and final states, the two important elements in a Feynman diagram are external legs and (internal) virtual particles:

[4] This is not always tree level, which merely denotes an absence of loops.

Definition 4.13 (External leg) In a Feynman diagram, an *external leg* or more boringly an *external point* is an incoming or outgoing particle. Specifically:

- An *incoming external leg* is an initial state (i.e. incoming) particle, typically on the left side.
- An *outgoing external leg* is a final state (i.e. outgoing) particle, typically on the right side.

Figure 4.4: 2 incoming external legs

Definition 4.14 (External and internal propagators) In a Feynman diagram, two types of propagators exist:

- *External propagators* or *external leg propagators* represent *probability amplitudes* of incoming or outgoing particles in a scattering process.
- *Internal propagators* represent *probability amplitudes* of virtual particles created at some time and then annihilated at a later time.

A Feynman diagram with m incoming external legs and n outgoing external legs is represented a $(m+n)$-point Green's function, which itself is made up of Feynman propagators. As the Feynman propagator has a S_{fi} term, it can be written as a perturbative expansion like S_{fi} in Equation 4.38:

$$G_{m+n} = \sum_k G^{(k)}_{m+n} \tag{4.54}$$

where:

- The 0^{th} term represents the Feynman propagator where no interactions happen. It has only external leg propagators.
- For $k > 0$, the k^{th} term represents the Feynman propagator where k interaction happens (represented by a Feynman diagram of order k). It has both external and internal propagators.

Derivation 4.4 ($2 \to 2$ processes) Putting it all together, we now look at the example of a $(2+2)$-point Green's function with incoming external legs x_1 and x_2, outgoing external legs x_3 and x_4 and no interaction in the middle of the Feynman diagram can be represented in terms of Feynman propagators as

$$G^{(0)}_{2+2} = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3) \tag{4.55}$$

If 1 interaction happens at point y, we have the extra term

$$G^{(1)}_{2+2} = -i\frac{\lambda}{4!}4! \int d^4y D_F(x_1 - y)D_F(x_2 - y)D_F(x_3 - y)D_F(x_4 - y) \tag{4.56}$$

where λ and $1/4!$ are the coupling constant and the normalisation term seen in Definition 4.8. The 4! factor that follows accounts for the fact that each vertex has four ϕ-fields, and there are 4! ways to contract these (four) fields with external legs or propagators.
If 2 interactions happen at points y_1 and y_2 on the left and right sides respectively, we have the extra term, where the factors λ, $1/4!$ and 4! are squared:

$$G^{(2)}_{2+2} = -\left(\frac{\lambda}{4!}\right)^2 \frac{4!^2}{2} \int d^4y_1 \int d^4y_2 D_F(x_1 - y1)D_F(x_2 - y_1)D_F(x_3 - y_2)D_F(x_4 - y_2) \tag{4.57}$$

and so on.

Already, we can roughly see a pattern emerging. This can be generalised by the so-called ϕ^4 theory Feynman rules. However, before we introduce them, we first have to go through a few more formalities:

Note 4.4 (Momentum space) In analysing propagators and Feynman diagrams, it is again convenient to perform a Fourier transform to momentum space. We can relate the position space field $\phi(x)$ and its counterpart $\tilde{\phi}(p)$ in momentum space as follows:

$$\tilde{\phi}(p) = \int d^4x e^{ip\cdot x}\phi(x) \quad \phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\tilde{\phi}(p) \tag{4.58}$$

where $p \cdot x = p^\mu x_\mu = Et - \vec{p}\cdot\vec{x}$.

From the definition of S_{fi} and the specifics of ϕ^4 theory, we can derive (which we will not do here) that specifically in ϕ^4 theory, S_{fi} and $\mathcal{M}^2_{fi}$ are related by

$$S_{qp} = \mathcal{M}_{fi} i \prod_{j=1}^{n} \frac{1}{2E_{p_j}} \prod_{j=1}^{m} \frac{1}{2E_{q_j}} (2\pi)^4 \delta^4(P_i - P_f) \tag{4.59}$$

where m and n are indices for initial and final particles, $P_i = \sum_{j=1}^n p_j$ and $P_f = \sum_{j=1}^n q_j$
Now we introduce the ϕ^4 theory Feynman rules:

Theorem 4.9 (ϕ^4 theory Feynman rules) For a given Feynman diagram in ϕ^4 theory, the transition amplitude matrix elements $\mathcal{M}_{fi}$ is constructed as follows:

ϕ^4 theory Feynman rules (partial)

For each	Assign	Physical meaning
Internal line	k_j	Propagator momenta
Incoming external line	p_j	Incoming momenta
Outgoing external line	q_j	Outgoing momenta
Internal line	$\frac{i}{k_j^2 - m^2}$	Internal propagator
Vertex	$-i\frac{\lambda}{4!}(2\pi)^4\delta^4 \sum_j p_j$	Vertex propagator[a]

Also:

- Introduce a factor C for the number of contractions leading to the same diagram.
- Include the $1/k!$ factor from the Taylor expansion, where k is the perturbative order.
- Remove a factor of $(2\pi)^4\delta^3(p - q)$, where p and q are the total initial and detected momenta, as it is usually already accounted for in the definition of the cross-section or decay rate in terms of the matrix elements.

[a] The sum is over all lines exiting the vertex and forces four-momentum conservation at the vertex.

Remark 4.6 For $2 \to 2$ interactions ($m = n = 2$), $\mathcal{M}_{fi}$ reduces very nicely to $-i\lambda$. Here the nature of ϕ^4

theory as a useful toy model becomes clear.
We conclude this subsection with this next bit, which is mostly formalism. Return to position space and consider again m incoming external legs and n outgoing external legs:

Definition 4.15 (Amputated propagator) It is convenient to write all 'internal' or 'interior' parts of the propagator as a so-called *amputated propagator* $\bar{G}(y_1, \cdots, y_l)$, giving the whole propagator as

$$G_{m+n} = D_F(x_1 - y_1) \cdots D_F(x_{n+m} - y_l) \times \bar{G}(y_1, \cdots y_l) \tag{4.60}$$

where, $l \leq m + n$, given that more than one leg may couple to the same vertex.

Remark 4.7 As the amputated propagator is effectively the whole propagator with all external leg propagators removed, it is known to be 'amputated.' Conversely, one can construct the whole propagator G_{m+n} by multiplying all external legs, and finally the amputated propagator.

4.6 Beginnings of HEP

We continue our discussion in momentum space and investigate a few experimental HEP-adjacent concepts: the *transition rate*, the *decay rate* and the *cross-section*.

Derivation 4.5 (Probability density) We first adopt the shorthand notation

$$A_p = \prod_{j=1}^{n} a_{p_j} \quad B_q = \prod_{j=1}^{n} a_{q_j, H} \tag{4.61}$$

From the Born rule, the probability density of scattering involving m particles is

$$dP(q_1, \cdots, q_m) = \mathrm{Tr}[\rho d \prod_{q_1} \cdots d\pi_{q_m}] \tag{4.62}$$

In momentum space, we have

$$d\prod_q = \frac{d^3q}{(2\pi)^3} a^\dagger |0\rangle\langle 0| a_q \tag{4.63}$$

where $1/(2\pi)^3$ is a normalisation factor resulting from normalising the previously unnormalised plane wave solution ϕ

$$\langle 0| a_p a_p^\dagger |0\rangle = (2\pi)^3 \delta(0) \tag{4.64}$$

In any case, by inserting Equation 4.63, Equation 4.62 becomes

$$\begin{aligned} dP(q_1, \cdots, q_m) &= V^{-n} \int \frac{d^{3n}k}{(2\pi)^{3n}} \frac{d^{3m}q}{(2\pi)^{3m}} \langle 0|A_k B_q^\dagger|0\rangle\langle 0|B_q A_p^\dagger|0\rangle\langle 0|A_p A_k^\dagger|0\rangle \\ &= V^{-n} \frac{d^{3m}q}{(2\pi)^{3m}} \langle 0|A_k B_q^\dagger|0\rangle\langle 0|B_q A_p^\dagger|0\rangle \\ &= \frac{1}{V^n} \frac{d^{3m}q}{(2\pi)^{3m}} |S_{qp}|^2 \end{aligned} \tag{4.65}$$

which is the simplified form of the probability density.

A problem in Equation 4.59 that we have left unaddressed is the momentum conservation-enforcing term $\delta^4(P_i - P_f)$. In it, the momentum states exist throughout the entirety of the space-time. In a real experiment, however, incoming and outgoing states are localised. To deal with this, we assume the interaction happens over a time of T in a system localised in some volume V[5]. First, we can perform the rewrite

$$(2\pi)^4 \delta^4(P_i - P_f) = \int_{VT} d^4x e^{i(P_f - P_i)x} \tag{4.66}$$

[5]This is not a problem, as both T and V ultimately disappear.

Taking the square modulus gives

$$|(2\pi)^4\delta^4(P_i - P_f)|^2 = \approx (2\pi)^4\delta^4(P_i - P_f)|\int_{VT} d^4x e^{i(P_f - P_i)x}| = VT(2\pi)^4\delta^4(P_i - P_f) \tag{4.67}$$

The transition rate, which is the probability per unit time, is denoted by W.

Definition 4.16 (Transition rate differential) The differential of the transition rate is given by

$$dW = |\mathcal{M}_{fi}^2|V\prod_{j=1}^{n}\frac{1}{2E_{p_j}V}\,\mathrm{LIPS}(m) \tag{4.68}$$

In this expression, we have crammed all Lorentz-invariant terms together as the so-called *Lorentz invariant phase space*, which is defined with respect to m particles in the final state:

Definition 4.17 (Lorentz invariant phase space)

$$\mathrm{LIPS}(m) \equiv (2\pi)^4\delta^4(P_i - P_f)|\prod_{k=1}^{m}\frac{d^3q_k}{(2\pi)^3 2E_k} \tag{4.69}$$

Now we consider decays. Unlike other interactions, there is only one initial particle in decays. As such, the decay rate Γ_{if}, which is the transition rate for decays, is

Definition 4.18 (Decay rate)

$$\Gamma_{if} = \frac{1}{2m}\int |\mathcal{M}_{fi}|^2\,\mathrm{LIPS}(m) \tag{4.70}$$

In the case where the end product consists of 2 particles, the decay rate reduces to

$$\Gamma_{if} = \frac{1}{2m}\int |\mathcal{M}_{fi}|^2 q_f d\Omega \tag{4.71}$$

where Ω is the so-called *solid angle*.
A quantity ultimately related to the transition rate is the cross-section. We begin with the *particle flux*:

Definition 4.19 (Particle flux) The particle flux for a beam with velocity v_1 and a density of particles of $1/V$[a] and a target with velovity v_2 is

$$F = \frac{|v_1 - v_2|}{V} \tag{4.72}$$

[a] i.e. 1 particle in a volume of V.

This is simply the number of particles per unit area which run past each other per unit time.
As the *cross section* is the transition rate for a single particle *per unit beam flux*, we can find the differential cross section by dividing the transition rate by the flux:

$$d\sigma = \frac{dW}{F} = \frac{1}{|v_1 - v_2|}\frac{1}{4E_1E_2}|\mathcal{M}_{fi}|^2\,\mathrm{LIPS}(m) \tag{4.73}$$

This expression is Lorentz-invariant. We then integrate and find

Definition 4.20 (Cross section)

$$\sigma = \frac{1}{4\sqrt{(p_1\cdot p_2)^2 - m_1^2m_2^2}}\int |\mathcal{M}_{fi}|^2\,\mathrm{LIPS}(m) \tag{4.74}$$

Remark 4.8 If the final particles are identical, $\mathrm{LIPS}(m)$ is divided by $m!$.
A useful shorthand when considering any $2 \to 2$ process (e.g. annihilation, scattering) is the Mandelstam variables:

Definition 4.21 (Mandelstam variables) The Mandelstam variables s, t and u correspond to the s-,

t- and u- channels respectively.

$$s = (p_1 + p_2)^2 \quad t = (p_1 - p_3)^2 \quad u = (p_1 - p_4)^2 \tag{4.75}$$

s, t and u are equal to the four-momentum exchange q^2 in their own channels. They are Lorentz invariant and satisfy

$$s > 0 \quad t < 0 \quad u < 0 \quad s + t + u = m_1^2 + m_1^2 + m_3^2 + m_4^2 \tag{4.76}$$

Figure 4.5: Feynman's bongos

Chapter 5

Interacting fields II: QED

A generalised and actually physical version of the Klein-Gordon equation is the Dirac equation. Using canonical quantisation, we will quantise the Dirac field and develop a Lagrangian for quantum electrodynamics, which accounts for fields generated by both electrons/positrons and photons.

5.1 Dirac equation

The ill-fated Klein-Gordon equation, which we have found to be *kaputt*, is a Lorentz-invariant 2^{nd}-order DE. We now propose a better candidate in the form of a Lorentz-invariant 1^{st}-order DE, whose most general form would be known as the *Dirac equation*

Theorem 5.1 (Dirac equation)

$$(i\gamma^\mu\partial_\mu - m)\phi = (i\not{\partial} - m)\psi = 0 \tag{5.1}$$

γ^μ is a yet undetermined 4-vector, and we have defined the so-called *Feynman slash notation* for some four-vector a

$$\not{a} = \gamma^\mu a_\mu \tag{5.2}$$

Quote 5.1 The equation was more intelligent than its author.

Paul Dirac, on his equation[a] *(disputed)*

[a] The rationale behind the quote, as per Victor Weisskopf, was that 'A great deal more was hidden in the Dirac equation than the author had expected when he wrote it down in 1928'.

Remark 5.1 The Dirac equation does not directly conflict with the Klein-Gordon equation. In fact, every solution to the Dirac equation is also a solution to the Klein-Gordon equation[1]. However, the reverse is not true: the spinorial nature of the solution means that the probability $\rho = \psi^\dagger\psi$ will always be non-negative, and the Dirac equation excludes negative probability states that can be admitted as solutions to the Klein-Gordon equation.

Derivation 5.1 (γ matrices) The previous remark allows us to determine the γ matrices by applying the differential operator $i\gamma^\mu\partial_\mu + m$ to the Dirac equation[a] and equating it with the Klein-Gordon equation

$$\underbrace{-(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi}_{\text{Application of } i\gamma^\mu\partial_\mu + m} = \underbrace{-(\partial^\mu\partial_\mu - m^2)\psi}_{\text{Klein-Gordon equation}} = 0 \tag{5.3}$$

By equating the two, we have seen that the term $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu$ must be equal to $\partial^\mu\partial_\mu$. Effectively, through $\gamma^\mu\gamma^\nu$, one index was shifted up. Hence, a sensible guess of $\gamma^\mu\gamma^\nu$ would be the metric:

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} \tag{5.4}$$

However, this is wrong for the reason that the off-diagonal components of the 4-metric are zero, thus

[1] This is because the Klein-Gordon equation must still be satisfied to fulfil the SR energy-momentum relation.

implying

$$\gamma^0\gamma^1 = 0 \quad \text{and} \quad (\gamma^0)^2 = -(\gamma^1)^2 = 1 \tag{5.5}$$

at the same time. In fact, such conditions can never be satisfied as long as the components of γ^μ are mere numbers. However, if one switches the indices on the LHS of Equation 5.3 and adds this otherwise identical expression to Equation 5.3, they will find

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \gamma^\nu\gamma^\mu\partial_\nu\partial_\mu = 2\partial^\mu\partial_\mu \tag{5.6}$$

This becomes

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{5.7}$$

where one has the anticommutator

Definition 5.1 (Anticommutator)

$$\{a, b\} = ab + ba \tag{5.8}$$

One can see from Equation that γ^μ are elements of a *Clifford algebra*[b], from the definition of which it is clear that elements of γ^μ must be matrices.

Definition 5.2 (γ matrices) There exist 4[a] γ matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \tag{5.9}$$

for $j = 1, 2, 3$. σ_j are the Pauli matrices we know and love.

[a] There is a 5^{th} γ matrix, but it is defined purely for our convenience and is actually not a part of the γ matrices.

[a] Note the $+$ sign of the second term of this operator!
[b] This is covered in *Spinors & Symmetries*.

Interestingly, the Pauli matrices satisfy the following relation:

Theorem 5.2 (Pauli matrices property)

$$\sigma_j\sigma_k = i\epsilon_{jkl}\sigma_l + \delta_{jk}\mathbb{I}_2 \tag{5.10}$$

where we have once again encountered our good friends, the Kronecker delta δ_{jk} and the Levi-Civita symbol ϵ_{jkl}.

Definition 5.3 (Chiral representation) In the so-called *chiral representation*, the γ matrices are represented by

$$\gamma^0_{\text{ch}} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \quad \gamma^j_{\text{ch}} = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \tag{5.11}$$

Finally, we conclude with a list of useful formulae for γ matrices:

Theorem 5.3 (Commonly used γ matrix formulae)

$$\begin{aligned}
\operatorname{Tr}(\gamma^\mu\gamma^\nu) &= 4g^{\mu\nu} && (5.12)\\
\operatorname{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2n}}) &= g^{\mu_1\mu_2}\operatorname{Tr}(\gamma^{\mu_3}\dots\gamma^{\mu_{2n}}) - g^{\mu_1\mu_3}\operatorname{Tr}(\gamma^{\mu_2}\gamma^{\mu_4}\dots\gamma^{\mu_{2n}}) + \dots + \\
&\quad g^{\mu_1\mu_n}\operatorname{Tr}(\gamma^{\mu_2}\dots\gamma^{\mu_{2n-1}}) && (5.13)\\
\operatorname{Tr}(\gamma^{\mu_1}\dots\gamma^{\mu_{2n+1}}) &= 0 && (5.14)\\
\operatorname{Tr}(\not{a}\not{b}) &= 4a\cdot b && (5.15)\\
\operatorname{Tr}(\not{a}\not{b}\not{c}\not{d}) &= 4(a\cdot b c\cdot d - a\cdot c b\cdot d + a\cdot d b\cdot c) && (5.16)\\
\gamma^\alpha\gamma^\mu\gamma_\alpha &= -2\gamma^\mu && (5.17)\\
\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha &= 4g^{\mu\nu} && (5.18)\\
\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha &= -2\gamma^\rho\gamma^\nu\gamma^\mu && (5.19)
\end{aligned}$$

Remark 5.2 As the γ matrices are 4×4, the wave solution of the Dirac equation ϕ has 4 components. However, it is important to note that this ψ is *not* a vector due to it not transforming under general coordinate transformations. Rather, it is a *spinor* which one can better understand by reading the companion book *Spinors & Symmetries*. While it is possible to simply regard ψ as a 4-vector in some regards, doing so would be quite morally questionable.

Derivation 5.2 (Dirac equation general solution) Despite ψ being a spinor[a], a general plane wave solution is again of the form $\psi = ue^{-ip\cdot x}$. From this and the Klein-Gordon on-shell condition, the eigenvalue equation is

$$(\not{p} - m)u = 0 \tag{5.20}$$

The slashed momentum matrix is of the form

$$\not{p} = \begin{pmatrix} p^0\mathbb{I}_2 & p\cdot\sigma \\ p\cdot\sigma & -p^0\mathbb{I}_2 \end{pmatrix} \tag{5.21}$$

where p is the 3-vector and σ is a 3D vector whose elements are the Pauli matrices.
Remark 5.3 Note that we are still living in momentum space.
The eigenspinors, called *Dirac spinors*, are

$$u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{p\cdot\sigma}{E_p+m}\chi_s \end{pmatrix} \quad v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{p\cdot\sigma}{E_p+m}\chi_s \end{pmatrix} \tag{5.22}$$

for $s = 1, 2$. χ_s are 2-component spinors (or so-called *Weyl spinors*), and are defined by

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.23}$$

By noting that

$$(p\cdot\sigma)^2 = |p|^2 = E_p^2 - m^2 \tag{5.24}$$

it can be seen that the eigenvectors are normalised as

$$u_r^\dagger(p)u_s(p) = v_r^\dagger(p)v_s(p) = \delta_{rs}2E_p \tag{5.25}$$

As $\not{p}$ is not Hermitian, the eigenvectors are not orthogonal. However, we do have a metric relation that is quite similar to orthogonality

$$u_r^\dagger(p)v_s(-p) = 0 \tag{5.26}$$

Equation 5.25 and Equation 5.26 can be represented in Lorentz invariant form:

$$\bar{u}_r(p)u_s(p) = -\bar{v}_r(p)\bar{v}_s(p) = \sigma_{rs}2m \quad \bar{u}_r(p)v_s(p) = 0 \tag{5.27}$$

where the bar on top denotes the *Dirac adjoint*:

Definition 5.4 (Dirac adjoint) The Dirac adjoint $\bar{\psi}$ of some ψ is

$$\bar{\psi} = \psi^\dagger\gamma^0 \tag{5.28}$$

Finally, we are in a position to write down the full general solution of the Dirac equation, which represents fermions like electrons and positions:

Theorem 5.4 (Dirac equation general solution)

$$\psi = \int d^3p \sum_{s=1}^{2}(b_s(p)u_s(p)e^{-ip\cdot x} + d_s(p)v_s(p)e^{ip\cdot x}) \tag{5.29}$$

where $b_s(p)$ and $d_s(p)$ are 4 constants.

From their Dirac equation, spinors $u_s(p)$ and $v_s(p)$ further satisfy

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \quad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{5.30}$$

Remark 5.4 The on-shell condition is enforced by $p^0 = E_p$. The $u_s(p)$ term accounts for positive energy solutions which represent particles, while the $v_s(p)$ term accounts for negative energy solutions which represent antiparticles[b].

[a] The horror!
[b] Here, the emergence of antiparticles have justified the negative energy solutions as physical.

We can now write down the Lagrangian and Hamiltonian densities for the (free field) Dirac equation:

Definition 5.5 (Dirac Lagrangian)

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{5.31}$$

which, by noting the Dirac equation itself, is always zero.

Definition 5.6 (Dirac Hamiltonian)

$$\mathcal{H} = i\psi^\dagger\partial_0\psi - \mathcal{L} = i\psi^\dagger\partial_0\psi = \bar{\psi}(-i\gamma^j\partial_j + m)\psi \tag{5.32}$$

Remark 5.5 The canonical momentum is $\pi = i\psi^\dagger$ by dint of $\gamma^0\gamma^0 = \mathbb{I}_4$.

5.2 Story of a spinor

Previously, we have said that spinors, unlike tensors, do not undergo general coordinate transformations. Instead, they undergo rotation-like transformations defined by Lie groups[2]. In the case of the Dirac equation solutions, the rotation transformations are defined by the SU(2) group, whose representation is

$$\hat{S} = \frac{1}{2}\Sigma = \frac{1}{2}\begin{pmatrix}\sigma & 0\\ 0 & \sigma\end{pmatrix} \tag{5.33}$$

which is actually a 3-vector whose components are operators. We have the following commutation relations

$$[\hat{S}_i, H] = i\epsilon_{ijk}p^i\gamma_j \tag{5.34}$$

Such a commutation relation does not preserve Lorentz invariance. However, if we recall LS coupling, we will remember that the *total* angular momentum is both the orbital angular momentum $\hat{L}$ and the spin $\hat{S}$, and it just so happens that there is the following commutation relation

$$[\hat{L} + \hat{S}_i, H] = 0 \tag{5.35}$$

from which we recover the Lorentz invariance of the total angular momentum.

Derivation 5.3 (Lorentz group generators) Given a Lorentz transformation $x \to x' = \Lambda x$, we expect the Dirac equation to be Lorentz invariant:

$$(i\gamma^\mu\partial_\mu - m)\psi(x) \to (i\gamma^\mu\partial'_\mu - m)\psi'(x') \tag{5.36}$$

where the wavefunction transforms according to the (internal) spinor transformation $T(\Lambda)$, defined by

$$\psi'(x') = T(\Lambda)\psi(x) = T(\Lambda)\psi(\Lambda^{-1}x') \tag{5.37}$$

Now we try to determine $T(\Lambda)$. One can write out the transformation in terms of indices:

$$x'^\mu = \Lambda^\nu_\mu x^\mu \quad \partial_\mu\Lambda^\nu_\mu\partial'_\nu \tag{5.38}$$

The original, untransformed and final, transformed equations can then be written as

$$(i\gamma^\nu\Lambda^\mu_\nu\partial'_\nu - m)\psi(\Lambda^{-1}x') = 0 \quad T\Lambda^{-1}(i\gamma^\mu\partial'_\mu - m)T(\Lambda)\psi(\Lambda^{-1}x') = 0 \tag{5.39}$$

By equating the two[a], one finds

$$T(\Lambda)^{-1}\gamma^\mu T(\Lambda) = \lambda^\mu_\nu\gamma^\nu \tag{5.40}$$

[2] Again see *Spinors & Symmetries*.

Using the parameterisation of generators in *Spinors & Symmetries*, an infinitesimal Lorentz transformation may be parameterised as

$$\Lambda^\mu_\nu = \delta^\mu_\nu - \omega_{\rho\sigma}(g^{\rho\nu}\delta^\sigma_\nu - g^{\sigma\mu}\delta^\rho_\nu) + o(\omega) \tag{5.41}$$

and a transformation may be given as

$$T(\Lambda) = e^{i\omega_{\rho\sigma}s^{\rho\sigma}} = \mathbb{I}_4 + i\omega_{\rho\sigma}s^{\rho\sigma} + o(\omega) \tag{5.42}$$

where $s^{\rho\sigma}$ are the all-too-familiar generators that we have parameterised via $\omega_{\rho\sigma}$ in *Spinors & Symmetries*. Plugging the two expressions into Equation 5.40 gives

$$i[s^{\rho\sigma}, \gamma^\mu] = g^{\rho\mu}\gamma^\sigma - g^{\sigma\mu}\gamma^\rho \tag{5.43}$$

which surprisingly reduces, via (anti)commutation relations to the simple expression

Definition 5.7 (Lorentz group generators) The following generators define the spinor representation of the Lorentz group

$$s^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma] \tag{5.44}$$

[a] One can do so as both equations apply for all ψs.

Remark 5.6 We conclude by saying that the transformations are $T(\Lambda) = e^{i\omega_{\rho\sigma}s^{\rho\sigma}}$.

Definition 5.8 (Proper and improper Lorentz transformations) There exists two kinds of Lorentz transformations:

- *Proper Lorentz transformations* have matrices with determinant 1.
- *Improper Lorentz transformations* have matrices with determinant -1.

Remark 5.7 For example, the parity and time reversal Lorentz transformations Λ_P and Λ_T are improper:

$$\Lambda_P = \text{diag}(1, -1, -1, -1) \quad \Lambda_T = \text{diag}(-1, 1, 1, 1) \tag{5.45}$$

One can find more improper transformations by multiplying proper ones with them. Their spinorial representations are

$$T(\Lambda_P) = \gamma^0 \quad T(\Lambda_T)\psi(x) = -\gamma^1\gamma^3\psi(\Lambda_T x)^* \tag{5.46}$$

Derivation 5.4 (Bilinears) The so-called *bilinears* are useful for defining quantities with particular properties under Lorentz transformations and appear in Lagrangians for fermion field theories.
We recall from *Spinors & Symmetries* that mathematically, a bilinear is a map that is linear in both its arguments. For Lorentz groups, these 2 arguments are ψ and $\bar{\psi}$, and a Lorentz group bilinear is essentially a map in which sandwiches some quantity between $\bar{\psi}$ and ψ. When we feed it ψ and $\bar{\psi}$ (both of which it is linear in), it will return a result that transforms in a certain way.
Before we construct them, we first introduce the previously teased γ^5 matrix:

Definition 5.9 (γ^5 matrix)

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.47}$$

Theorem 5.5 (γ^5 matrix properties) The γ^5 matrix satisfies

$$\gamma^{5\dagger} = \gamma^5 \quad \{\gamma^5, \gamma^\mu\} = 0 \quad (\gamma^5)^2 = 1 \tag{5.48}$$

Finally, γ^5 changes sign for improper Lorentz transformations. e.g.

$$\gamma^5\gamma^0 = -\gamma^0\gamma^5 \tag{5.49}$$

With the γ^5 finally set up, we can define the basis for all possible γ matrix products, which consists of a lofty 16 matrices:

$$\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\} \tag{5.50}$$

Here $\sigma^{\mu\nu}$ is defined as

$$\sigma^{\mu\nu} = 2s^{\mu\nu} \tag{5.51}$$

in which $s^{\mu\nu}$ is the Lorentz group generators defined in Equation 5.44.
There are a total of 16 Lorentz group bilinears. Depending on the quantity we sandwich with $\bar{\psi}$ and ψ, we can categorise them according to their transformation properties:

- **Scalar:** This quantity remains unchanged under any Lorentz transformation. An example is the mass m.
$$\bar{\psi}\psi \to \bar{\psi}\psi \tag{5.52}$$
- **Pseudoscalar**[a]**:** This quantity accounts for parity by changing signs under spatial inversion (parity transformation). It is associated with chirality.
$$\bar{\psi}\gamma^5\psi \to \det(\Lambda)\bar{\psi}\gamma^5\psi \tag{5.53}$$
- **Vector:** This quantity transforms as a 4-vector under Lorentz transformations.
$$\bar{\psi}\gamma^\mu\psi \to \Lambda^\mu_\nu\bar{\psi}\gamma^\mu\psi \tag{5.54}$$
It can represent conserved currents[b] and obeys the continuity equation to enforce charge conservation:
$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \tag{5.55}$$
- **Pseudovector:** This quantity transforms identically to a 4-vector, except that it switches sign under parity. It appears in theories involving axial currents[c] and chiral symmetry.
$$\bar{\psi}\gamma^\mu\gamma^5\psi \to \det(\Lambda)\Lambda^\mu_\nu\bar{\psi}\gamma^\mu\gamma^5\psi \tag{5.56}$$
- **Rank-2 tensor:** This quantity transforms as a rank-2 antisymmetric tensor.
$$\bar{\psi}\sigma^{\mu\nu}\psi \to \Lambda^\nu_\lambda\Lambda^\nu_\sigma\bar{\psi}\sigma^{\lambda\sigma}\psi \tag{5.57}$$

We can verify that all quantities are linear in both arguments, which is skippable if taken for granted:

- Scalar:
$$\bar{\psi}(a\psi_1 + b\psi_2) = a\bar{\psi}\psi_1 + b\bar{\psi}\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\psi = a\bar{\psi}_1\gamma^5\psi + b\bar{\psi}_2\psi \tag{5.58}$$
- Pseudoscalar:
$$\bar{\psi}(a\psi_1 + b\psi_2)\gamma^5 = a\bar{\psi}\gamma^5\psi_1 + b\bar{\psi}\gamma^5\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\gamma^5\psi = a\bar{\psi}_1\gamma^5\psi + b\bar{\psi}_2\gamma^5\psi \tag{5.59}$$
- Vector:
$$\bar{\psi}(a\psi_1 + b\psi_2)\gamma^\mu = a\bar{\psi}\gamma^\mu\psi_1 + b\bar{\psi}\gamma^\mu\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\gamma^\mu\psi = a\bar{\psi}_1\gamma^\mu\psi + b\bar{\psi}_2\gamma^\mu\psi \tag{5.60}$$
- Pseudovector:
$$\bar{\psi}(a\psi_1+b\psi_2)\gamma^\mu\gamma^5 = a\bar{\psi}\gamma^\mu\gamma^5\psi_1+b\bar{\psi}\gamma^\mu\gamma^5\psi_2 \quad (a\bar{\psi}_1+b\bar{\psi}_2)\gamma^\mu\gamma^5\psi = a\bar{\psi}_1\gamma^\mu\gamma^5\psi+b\bar{\psi}_2\gamma^\mu\gamma^5\psi \tag{5.61}$$
- Rank-2 tensor:
$$\bar{\psi}(a\psi_1 + b\psi_2)\sigma^{\mu\nu} = a\bar{\psi}\sigma^{\mu\nu}\psi_1 + b\bar{\psi}\sigma^{\mu\nu}\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\sigma^{\mu\nu}\psi = a\bar{\psi}_1\sigma^{\mu\nu}\psi + b\bar{\psi}_2\sigma^{\mu\nu}\psi \tag{5.62}$$

[a] So-called as it switches sign under parity, *not* because it has zero rank (which it doesn't).
[b] In fact, the electromagnetic 4-current can be written as $J^\mu = \bar{\psi}\gamma^\mu\psi$.
[c] As such, it is also called a *axial vector*. For example, the axial current in weak interactions is $\bar{\psi}\gamma^\mu\gamma^5\psi$ and plays a role in describing the *handedness* of particles.

Yet another symmetry of the Dirac field is the so-called *charge conjugation*, which is the sign-flip of all charges:

Definition 5.10 (Helicity operator) For a spinor ϕ, the charge conjugation C is defined as

$$T(C)\phi = i\gamma^2\phi^* \tag{5.63}$$

5.3 Quantisation of the Dirac field

Having acquired the the Dirac field as the general solution of the Dirac equation in Equation 5.29, we now attempt to quantise it, which allows us to quantise several significant quantities using it. In quantising the Klein-Gordon equation, we replaced f_p and f_p^* with creation and annihilation operators (see Derivation 3.2). Here we start less ambitiously. We order that $b_s(p)$ become an operator. For $d_s(p)$, we do the same but replace it with $d_s^\dagger(p)$. The general solution then reads

$$\psi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} (b_s(p)u_s(p)e^{-ip\cdot x} + d_s^\dagger(p)v_s(p)e^{ip\cdot x}) \tag{5.64}$$

where again, for the sake of convenient normalisations, we have added a normalisation factor of $\frac{1}{(2\pi)^3\sqrt{2E_p}}$, previously seen in Equation 3.21. The canonical momentum can likewise be found:

Definition 5.11 (Dirac equation canonical momentum)

$$\pi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} (b_s^\dagger(p)u_s^\dagger(p)e^{ip\cdot x} + d_s(p)v_s^\dagger(p)e^{-ip\cdot x}) \tag{5.65}$$

Before mindlessly assigning the bosonic commutation relations to $b_s(p)$ and $d_s(p)$, we stop for a moment and realise that the Dirac field is not actually a bosonic field. The solution to the Dirac equation is not a scalar, but a spinor. Hence, it describes spin-$\frac{1}{2}$ particles[3] and is a fermionic field instead. As such, we instead impose the following *fermionic anticommutation relations* for some indices i and j:

Theorem 5.6 (Fermionic creation and annihilation operator anticommutations)

$$\{b_i(p), b_j^\dagger(q)\} = \{d_i(p), d_j^\dagger(q)\} = \delta_{ij}(2\pi)^2\delta^3(p-q) \tag{5.66}$$

$$\{b_i(p), b_j(q)\} = \{d_i(p), d_j(q)\} = \{b_i(p), d_j(q)\} = 0 \tag{5.67}$$

Suddenly recalling the useful relation in Equation 5.30 for no reason whatsoever, we can rewrite it using the definition of the Dirac adjoint in Equation 5.28 as

$$\sum_s u_s(p)_\alpha u_s^\dagger(p)_\beta = (E_p + m)\delta_{\alpha\beta} - (p\cdot\gamma\gamma^0)_{\alpha\beta} \quad \sum_s v_s(p)_\alpha v_s^\dagger(p)_\beta = (E_p - m)\delta_{\alpha\beta} - (p\cdot\gamma\gamma^0)_{\alpha\beta} \tag{5.68}$$

Putting Equation 5.65, Equation 5.66, Equation 5.67 and Equation 5.68 all together, we can solve for the commutator $[\phi_\alpha(x), \pi_\beta(y)]$ and find the following anticommutation relation

Theorem 5.7 (Fermionic field and momentum operator anticommutations)

$$\{\psi_\alpha(x), \pi_\beta(y)\} = i\delta_{\alpha\beta}\delta^3(x-y) \tag{5.69}$$

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\pi_\alpha(x), \pi_\beta(y)\} = 0 \tag{5.70}$$

By integrating Equation 5.32 and then using Equation 5.25, we can find the normal-ordered Hamiltonian

Definition 5.12 (Dirac equation normal-ordered Hamiltonian)

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=1}^{2} (b_s^\dagger(p)b_s(p) + d_s^\dagger(p)d_s(p)) \tag{5.71}$$

[3] Here we see why a spinor is often called a 'rank-half tensor'.

We can likewise find the charge by integrating the 0^{th} component 4-current J^0, given by the vector bilinear in Equation 5.54 as $\psi^\dagger\psi$:

$$Q = e\int d^3x{:}\psi^\dagger\psi{:} = e\int \frac{d^3p}{(2\pi)^3}\sum_{s=1}^{2}(b_s^\dagger(p)b_s(p) - d_s^\dagger(p)d_s(p)) \tag{5.72}$$

where the normal ordering is used to ensure the result is physical.
Remark 5.8 From this, one can verify a particle and its antiparticle carry equal but opposite charges.

5.4 Quantisation of the electromagnetic field

Derivation 5.5 (Recovery of Maxwell's equations) By setting the (ultimately zero) Lagrangian density as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu \tag{5.73}$$

where one has the Faraday tensor in terms of the 4-potential

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{5.74}$$

We can apply the Euler-Lagrange equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) - \frac{\partial\mathcal{L}}{\partial A^\nu} = 0 \tag{5.75}$$

where A^ν, which we ultimately recognise as a field variable, replaces ψ.
We then recover Maxwell's equations in index notation

Theorem 5.8 (Maxwell's equations)

$$\partial_\mu F^{\nu\mu} = J^\nu \tag{5.76}$$

In gauge theory, there exist physical observables are invariant under certain transformations of the potentials. In the case of electromagnetism, this manifests in the invariance of the EM fields E and B under the following gauge transformation of the 4-potential[4]

$$A_\mu(x) \to A_\mu(x) + \partial_\mu\Lambda(x) \tag{5.77}$$

where $\Lambda(x)$ is a scalar field. This gauge invariance of E and B leads to one redundant degree of freedom or a *gauge freedom*: an infinity of A^μs corresponds to the same set of E and B. If such redundancies are not eliminated, calculations involving A^μ could mistakenly count multiple configurations of A^μs as distinct and end up with erroneous results.

Derivation 5.6 (Fixing the gauge) We then perform *gauge fixing*, which is a mathematical convenience that eliminates the extra degrees of freedom while having no effect on the physical variables of interest - in this case, E and B. Like in classical EM, we attempt to eliminate this gauge freedom in Equation 5.77 by the Lorentz-invariant Lorenz gauge.

> **Quote 5.2** Amazingly, the missing "t" is not a typo here.
>
> *Alessio Serafini*

$$\partial_\mu A^\mu = 0 \tag{5.78}$$

However, there exists one more degree of freedom even after this Lorenz-like gauge, brought about by the U(1) transformations:

$$A^\mu(x) \to A^\mu(x) + \partial^\mu\chi(x) \quad \psi \to e^{-ie\chi}\psi \tag{5.79}$$

where $\chi(x)$ satisfy the wave equation $\partial_\mu\partial^\mu = 0$. We eliminate this gauge freedom by performing R_ξ

[4] One can verify this by noting that the conditions $\frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0}$ (where $F^{\mu 0}$ are the canonically conjugate EM fields) and $\frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = 0$ simultaneously hold.

Landau gauge, which adds a term to the Lagrangian:

Theorem 5.9 (R_ξ Landau gauge)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{5.80}$$

The simplest R_ξ Landau gauge is the Feynman-'t Hooft gauge, which is used in most QFT calculations:

Theorem 5.10 (Feynman-'t Hooft gauge)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{5.81}$$

We can write the Faraday tensors in the first term in terms of 4-potentials and expand out the last term:

$$-\frac{1}{2}(\partial_\mu A^\mu)^2 = -\frac{1}{2}\partial_\mu(A^\mu\partial_\nu A^\nu) + \frac{1}{2}\partial_\mu\partial_\nu(A^\mu A^\nu) - \frac{1}{2}\partial_\nu A^\mu \partial_\mu A^\nu \tag{5.82}$$

where the total derivatives are boundary terms that do not contribute to the action, and can hence be vanished. The photon field Lagrangian, arising from electromagnetism, hence reduces to

Definition 5.13 (Photon field Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial_\mu A^\nu \partial^\mu A^\nu \tag{5.83}$$

Fun fact 5.1 We can also use an alternative 4-potential, comprising of the magnetic scalar potential and the electric vector potential. However, this is rarely used due to the absence of observed magnetic monopoles. We can now quantise the 4-potential, which is a real, massless Klein-Gordon (scalar) field. The classical wave solution is

$$A^\mu(x) = \int d^3p \sum_{\lambda=0}^{3}(\epsilon_\lambda^\mu(p) f_\lambda(p) e^{-ip\cdot x} + \epsilon_\lambda^{\mu*}(p) f_\lambda^*(p) e^{ip\cdot x}) \tag{5.84}$$

where ϵ^μ is a *polarisation versor*, a 4-versor. One can recall from *Spinors & Symmetries* that a versor is simply a unit quaternion (i.e. it has norm 1). Significantly, the polarisation versor holds the following summation property:

Theorem 5.11 (Polarisation versor property)

$$\sum_{\lambda=0}^{3} g_{\lambda\lambda}\epsilon_\lambda^\mu(p)\epsilon_\lambda^{\nu*}(p) = \sum_{\lambda=0}^{3} g_{\lambda\lambda}\epsilon_\lambda^{\nu*}(p)\epsilon_\lambda^\mu(p) = g^{\mu\nu} \tag{5.85}$$

where the index λ labels the polarisation states of the photon.

One notes this to quite resemble the *tetrad fields* in general relativity.

Derivation 5.7 (Polarisation and the Lorentz gauge) For a photon, there are four possible indices, but not all are physical:

- $\lambda = 0$: This is a *longitudinal polarisation* that is often unphysical.
- $\lambda = 1, 2$: They are the two physical *transverse polarisations*[a] of the photon.
- $\lambda = 3$: This is a *scalar polarisation* that is also often unphysical.

As photons in QED are gauge bosons, the choice of polarisation versors is not unique. This gauge freedom can be removed by applying Lorenz gauge:

$$p_\mu \epsilon_\lambda^\mu = 0 \tag{5.86}$$

where p^μ is the photon's 4-momentum.
In doing so, the two unphysical components of the 4-versor have been eliminated due to them being unphysical under the gauge, and only the two transverse polarisation states remain physical for photons.

[a] i.e. they are perpendicular to the direction of propagation and to each other.

Using the same procedure we have done before, we insert the normalisation factor into Equation 5.87 and replace $f_\lambda(p)$ and $f^*_\lambda(p)$ with creation and annihilation operators

Definition 5.14 (Photon field 4-potential)

$$A^\mu(x) = \int \frac{d^3p}{(2\pi^3)\sqrt{2E_p}} \sum_{\lambda=0}^{3} (\epsilon^\mu_\lambda(p) a_\lambda(p) e^{-ip\cdot x} + \epsilon^{\mu *}_\lambda(p) a^\dagger_\lambda(p) e^{ip\cdot x}) \tag{5.87}$$

where the creation and annihilation operators observe the commutation relation

$$[a_\lambda(p), a^\dagger_{\lambda'}(q)] = -g_{\lambda\lambda'}(2\pi)^3\delta^3(p-q) \tag{5.88}$$

We then derive the canonical momentum of a photon field. Noting from previously that it satisfies

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} \tag{5.89}$$

we can further say that, by inserting Equation 5.83

$$\pi^\mu = \partial_0 A^\mu \tag{5.90}$$

Finally, plugging in Equation 5.87 gives

Definition 5.15 (Photon field canonical momentum)

$$\pi^\mu = -\int \frac{d^3p}{(2\pi^3)\sqrt{2E_p}} \sum_{\lambda=0}^{3} (\epsilon^{\mu *}_\lambda(p) a^\dagger_\lambda(p) e^{ip\cdot x} - \epsilon^\mu_\lambda(p) a_\lambda(p) e^{-ip\cdot x}) \tag{5.91}$$

Again we consider the nature of the photon. It has spin-1, and is thus a boson. The standard bosonic commutations thus apply:

$$[A^\mu(x), \pi^\nu(u)] = -ig^{\mu\nu}\delta^3(x-y) \tag{5.92}$$

$$[A^\mu(x), A^\nu(u)] = [\pi^\mu(x), \pi^\nu(u)] = 0 \tag{5.93}$$

One final loose end is the Hamiltonian. By inserting Equation 5.90 into Equation 2.28 (where, notably, the field is A^μ instead of ϕ), the photon field Hamiltonian reads

Definition 5.16 (Photon field Hamiltonian)

$$\mathcal{H} = \frac{1}{2}\dot{A}^\nu \dot{A}_\nu + \frac{1}{2} DA^\nu DA_\nu \tag{5.94}$$

5.5 QED Feynman rules

We now begin constructing the QED Lagrangian by coupling the field Lagrangian from the Dirac equation general solution to the photon field Lagrangian, seen in Equation 5.31 and Equation 5.83 respectively[5]. However, we cannot simply add the two together due to having to first reconcile the U(1) transformations seen in Equation eq:u1transformations with $\partial_\mu\psi$. To this end, we replace the partial derivative with the previously seen covariant derivative, albeit modified with a factor of e on the last term:

$$D_\mu = \partial_\mu + ieA_\mu \tag{5.95}$$

This allows us to write the QED Lagrangian as

Definition 5.17 (QED Lagrangian)

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{2}\partial_\mu A^\nu \partial^\mu A^\nu \tag{5.96}$$

[5] Note that they are not the free and interaction Lagrangians, as we will see almost immediately.

which one can decompose into a *free part* $\mathcal{L}_{\text{free}}$ and a *interaction part* $\mathcal{L}_{\text{int}}$ by decomposing the QED covariant derivative according to Equation 5.95

$$\mathcal{L}_{\text{free}}\bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{2}\partial_\mu A^\nu \partial^\mu A^\nu \quad \mathcal{L}_{\text{int}} = -eA_\mu\bar{\psi}\gamma^\mu\psi \tag{5.97}$$

Derivation 5.8 (Fermionic field creation and annihilation operators) One can also represent the fermionic field creation and annihilation operators $b_s(p)$ and $d_s^\dagger(p)$ in terms of wavefunctions, much like the interacting field creation and annihilation operators in Equation 4.30:

$$-id_s^\dagger(p) = \int \frac{d^3x}{\sqrt{2E_p}}\frac{\bar{v}_s}{2m}(e^{-ip\cdot x}\partial_0\psi - \psi\partial_0 e^{-ip\cdot x}) \tag{5.98}$$

$$ib_s^\dagger(p) = \int \frac{d^3x}{\sqrt{2E_p}}(e^{-ip\cdot x}\partial_0\bar{\psi} - \bar{\psi}\partial_0 e^{-ip\cdot x})\frac{u_s}{2m} \tag{5.99}$$

where the factor of $1/2m$ arises from the normalisation condition in Equation 5.27. The photon field creation and annihilation operators can likewise be represented by

$$ia_\lambda^\dagger(p) = \int \frac{d^3x}{\sqrt{2E_p}} g_{\lambda\lambda}\epsilon_\lambda^\mu(p)(e^{-ip\cdot x}\partial_0 A_\mu - A_\mu\partial_0 e^{-ip\cdot x}) \tag{5.100}$$

Derivation 5.9 (Propagators) We can now calculate the QED propagators. Using Equation 5.85 which eliminates those pesky 4-versors, the photon field propagator is

Definition 5.18 (Photon field propagator)

$$\langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle = -i\lim_{\epsilon\to 0+}\int \frac{d^4}{(2\pi)^4}\frac{g^{\mu\nu}e^{-ip\cdot(x-y)}}{p^2+i\epsilon} \tag{5.101}$$

and the fermionic propagator is

Definition 5.19 (Fermionic propagator)

$$\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle = -i\lim_{\epsilon\to 0+}\int \frac{d^4}{(2\pi)^4}\frac{i(p\!\!\!/ + m)e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{5.102}$$

$$\langle 0|T[\psi_\alpha(x)\psi_\beta(y)]|0\rangle = \langle 0|T[\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)]|0\rangle = 0 \tag{5.103}$$

which is a Green's function of the Dirac operator:

$$(p\!\!\!/ - m)\lim_{\epsilon\to 0+}\int \frac{d^4}{(2\pi)^4}\frac{i(p\!\!\!/ + m)e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} = i\delta^4(z-y) \tag{5.104}$$

Finally, we can formulate the QED Feynman rules:

Theorem 5.12 (QED Feynman rules) For a given Feynman diagram in QED, the transition amplitude matrix elements $\mathcal{M}_{fi}$ is constructed as follows:

QED Feynman rules (partial)

For each	Assign	Physical meaning
Incoming and outgoing electron	$\bar{u}_\alpha(s,p)$ and $u_\alpha(s,p)$	Electron 4-spinor
Incoming and outgoing positron	$v_\alpha(s,p)$ and $\bar{v}_\alpha(s,p)$	Position 4-spinor
Incoming and outgoing photon	$\epsilon^{*\mu}(\lambda,p)$ and $\epsilon^{\mu}(\lambda,p)$	Photon 4-versor
Internal photon line	$\frac{-ig^{\mu\nu}}{p^2}$	Photon propagator
Internal fermion line	$\frac{i(\not{p}+m)e^{-ip\cdot(x-y)}}{p^2-m^2}$	Fermion propagator
Internal loop	$\int d^4k_n/(2\pi)^4$	Internal loop propagator
Vertex	$-ie\gamma^n_{\alpha\beta}$	Vertex propagator[a]
Vertex	$(2\pi)^4\delta^3(k_i-k_f)$	Term enforcing 4-momentum conservation

where the incoming and outgoing photon indices are μ and ν, the incoming and outgoing fermion indices are α and β, and the incoming and outgoing 4-momenta are k_i and k_f. Each internal loop has a so-called *internal momenta* k_n.
Finally, before taking a well-deserved break, remove a factor of $(2\pi)^4\delta^3(p-q)$, where p and q are the total initial and detected momenta.

[a] The external leg indices, in the order along the direction of the external leg arrows, are α and β while the propagator index (photon *or* fermion) is n.

Note 5.1 One should note the following points:

- The symbols s and λ seen in the electrons, positrons and photons are actually indices, which are put into the bracket purely for ease of viewing.
- In QED, the previously seen factor C observes $C = k!$, and C and $1/k!$ cancel out.
- The simplest Feynman diagrams are tree-level diagrams, in which there are no loops and thus no integration over internal momenta.
- When the scattering involves more than one diagram, Wick's theorem will be used, in which the exchange of two fermion operators changes the sign of the expression.

Unlike ϕ^4 theory, QED is not a toy model and concerns itself with real particles - in specific, fermions and photons. One can represent them in a Feynman diagram as follows:

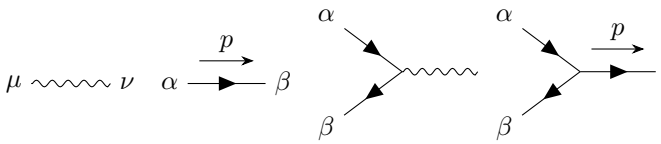

Figure 5.1: QED Feynman diagram elements. L-R: photon propagator, fermion propagator, vertex leading to photon propagator, vertex leading to fermion propagator

Quote 5.3 Our two statements actually describe the same, but are written down on two different sides of the same medal.
As they[a] said, it's not the difficulty of the territory. You can avoid the most dangerous cliffs if you take the right route. It's more the sheer size which bends your knee, as you have to cross long distances in the realm of QFT.

Felix Halbwedl, on Quote 1.3 and Quote 1.4, 17 November 2024

[a] Author of Quote 1.3, not an impersonal 'they'.

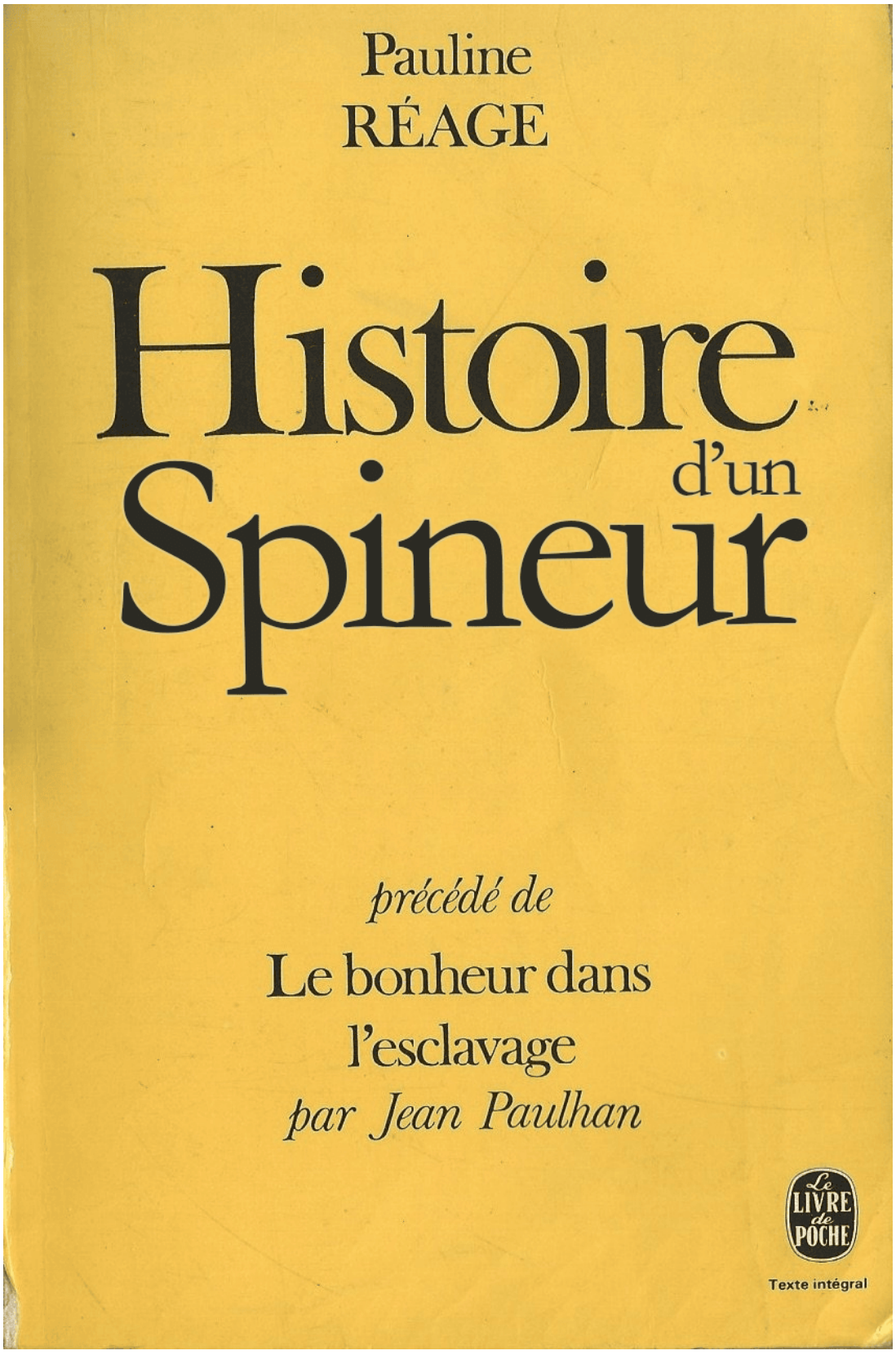

Figure 5.2: Story of a Spinor

Part II

Path integrals

Chapter 6

Free fields

Quote 6.1 Nata rana inne tanana nara na
Olina kanana tanene tuti
Eh pirun
Tara ra oy
Nare neira ara gaka
Ara eire ganana orange juice
I got lidelele orange juice
I got a little orange juice
I gotta have a bit orange juice
I gotta have a bit orrrrrange juice
I gotta have my orange juice
Juice juice juice juice juice
Juice juice juice juice juice
Juice juice juice juice juice
Gotta have a little bit of orange juice
I got haaaaaaave my orange juice
Just a little bit of orange juice
Gotta go got juice
Gotta have a little bit of orange juice
Oh oh oh oh orange juice
I gotta have my daily orange juice
I want my orange orange orange
Orange juuuuuuice
Juice
Juice
Gimme orange juice
I gotta have my orange juice
I gotta have my orange juice
I gotta have my orange juice
I gotta have my orange orange orange
Ooooooooraaaaange juice
Ey!

Richard Feynman, playing the bongos, September 1981

While canonical quantisation is the more intuitive approach to developing field theories, it suffers from certain drawbacks, especially with respect to extra degrees of gauge freedom (as we have seen in deriving the QED Feynman rules). In developing more complicated field theories like QCD, we will make use of the other formalism: path integrals.

6.1 Scalar particles

Unsurprisingly, the end goal in path integrals is the same as that in canonical quantisation. The end goals remain the same: the S-matrix S_{fi}, the transition amplitude $\mathcal{M}_{fi}$ and the Feynman rules of a given field theory that constructs $\mathcal{M}_{fi}$.
The central idea of path integrals is that a particle in motion can and *will* take every possible trajectory or *path*. We postulate that each path contributes a factor of e^{iS} to the time evolution operator $U(t,t_0)$:

$$U(t,t_0) = \sum_{\text{all paths}} e^{iS} \tag{6.1}$$

where S is the action.
In innocent quantum mechanics, $U(t,t_0)$ is given by

$$U(t,t_0) = \langle f|e^{-iHT}|1\rangle \tag{6.2}$$

where 1 is the initial state[1] and $T = t - t_0$ is the time interval.
One can solve the contribution to $U(t,t_0)$ of a certain path by slicing the path's time interval into *smol* time steps of ϵ:

Derivation 6.1 (Time-slicing) Recall the *Lie product formula* we have seen in *Electron's Destiny*. For any operators or square matrices $\hat{A}$ and $\hat{B}$, one has

$$e^{\hat{A}+\hat{B}} = \lim_{N\to\infty}\left(e^{\hat{A}/N}e^{\hat{B}/N}\right)^N = \lim_{N\to\infty}\left(e^{\hat{B}/N}e^{\hat{A}/N}\right)^N \tag{6.3}$$

where N is the so-called *Trotter number*.
Practically, this has an alternative formulation. Suppose that, instead of solving directly for $e^{\hat{A}+\hat{B}}$, we solve it segment by segment, solving first a $e^{\epsilon(\hat{A}+\hat{B})}$ and then calculate $(e^{\epsilon(\hat{A}+\hat{B})})^{1/\epsilon}$, where the segment $e^{\epsilon(\hat{A}+\hat{B})}$ has the form

Theorem 6.1 (Lie product formula)

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} + O(\epsilon^2) \tag{6.4}$$

where, as $\epsilon \to 0$, $O(\epsilon^2)$ vanishes.

Our good friend, the unitary time evolution operator, can then be approximated as

$$e^{-iHT} = \left(e^{-iH\epsilon}\right)^N \tag{6.5}$$

where $N = T/\epsilon$ is again the Trotter number.
From this, we are in a position to construct the generic time evolution operator $U(t,t_0)$. We write it as the sum of a series of steps N:

$$\begin{aligned} U(t,t_0) &= \sum_k \sum_{i_j} \langle f,t_N|e^{-iH\epsilon}|N-1,t_{N-1}\rangle \cdots \langle 1,t_1|e^{-iH\epsilon}|i,t_0\rangle \\ &= \int \prod_i dx_i \langle x_f,t_N|e^{-iH\epsilon}|x_{N-1},t_{N-1}\rangle \cdots \langle x_1,t_1|e^{-iH\epsilon}|x_i,t_0\rangle \end{aligned} \tag{6.6}$$

where $\epsilon = T/N$ is again a single time step.
Remark 6.1 Note that the second line is not a single integral but $N-1$ integrals.

Now assume that the Hamiltonian can be decomposed as

$$H = \frac{1}{2}p_i^2 + V(q_i) \tag{6.7}$$

where we recall p_i and q_i to be generalised momenta and coordinates.
Importantly, as neither p_i nor q_i are scalars, we *cannot* simply say that $e^{-iH\epsilon} = e^{-i\epsilon(p_i^2/2+V(q_i))} = e^{-i\epsilon p_i^2/2}e^{-i\epsilon V(q_i))}$. A trick must be used to solve for the decomposed result:

[1] We have avoided writing i to prevent confusion with indices that will appear later.

Theorem 6.2 (Baker-Campbell-Hausdorff formula) Suppose one has the known matrices X and Y and the unknown matrix Z which satisfy $e^X e^Y = e^Z$. Z can be solved by

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots \tag{6.8}$$

where square brackets are commutators.

Amazingly, this yields the convenient approximation

$$e^{-iH\epsilon} \approx e^{-i\epsilon p_i^2/2} e^{-i\epsilon V(q_i))} \tag{6.9}$$

which is *almost* identical to the result if p_i and q_i *were* scalars. We can then say that, for some arbitrary step i, that

$$\langle q_{i+1}, t_{i+1}| \, e^{-iH\epsilon} \, |q_i, t_i\rangle = e^{-\epsilon V(q_i)} \int \prod_j \frac{dp_j^i}{2\pi} \prod_k e^{-i\epsilon\left(\frac{p_k^{i2}}{2} - ip_k \frac{q_k^{i+1}-q_k^i}{\epsilon}\right)} \tag{6.10}$$

Now we define the so-called *integration measure*, which is nothing but a convenient shorthand:

Definition 6.1 (Integration measure)

$$\int \mathcal{D}q = \prod_i^N \prod_j^M \frac{dq_j^i}{\sqrt{2\pi\epsilon}} \quad \text{where} \quad d_t q_j^i = \frac{q_k^{i+1} - q_k^i}{\epsilon} \tag{6.11}$$

M is the number of independent coordinates.

The path integral becomes

$$U(t,t_0) = \int \mathcal{D}p\mathcal{D}q e^{-\epsilon p_k (d_t q_j^i)} e^{-i\epsilon H} \tag{6.12}$$

The momenta integral is a Gaussian integral, which is defined as $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$. Hence, integrating over p_j^i gives

$$U(t,t_0) = \int \mathcal{D}q e^{i\sum^N \epsilon L} \tag{6.13}$$

If one sends N to infinity, we get

$$U(t,t_0) = \int \mathcal{D}q e^{iS} = \int \mathcal{D}q e^{i\int dt L} \tag{6.14}$$

where S is the action. This verifies our postulate in Equation 6.1[2]. Now we turn this theory into a field theory:

- The Lagrangian L is replaced by the Lagrangian density $\mathcal{L}$.
- The coordinates q are replaced with the fields ϕ[3].

which gives, after introducing sensible limits

$$U(t,t_0) = \int_{\phi(t_0)}^{\phi(t)} \mathcal{D}\phi e^{i\int d^4x \mathcal{L}} \tag{6.15}$$

Theorem 6.3 (Integration measure properties)

$$\int \mathcal{D}\phi = \phi(x) \tag{6.16}$$

$$\int \mathcal{D}\phi\phi = \frac{\phi^2(x)}{2} \tag{6.17}$$

[2]Importantly, time ordering is not a concern as the integration the Lagrangian naturally preserves the time order. The same can be seen in the time steps in Equation 6.6.

[3]Hence, the shorthand differential $\mathcal{D}q$ becomes $\mathcal{D}\phi$.

6.2 Sources

Expectedly, fields are emitted by sources which couple to them. We thus introduce, for every field $\phi(x)$, a source $J(x)$. This modifies the time evolution operator, the result of which we will call the almighty *generating functional* $Z(J(x))$. It is so-called as it is used in path integrals to generate Green's functions.

Definition 6.2 (Generating functional)

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}(\phi(x))+J(x)\phi(x))} \tag{6.18}$$

Remark 6.2 But this looks a bit familiar, doesn't it?
If you share this opinion, you'd be right. As $Z(J(x))$ integrates over all possible field configurations, it is the quantum analogue of the well-known *partition function* in statistical physics[4].

Derivation 6.2 (Free massive scalar field) We are now in a position to derive the generating functional of some free field. Consider the following Lagrangian of a free massive scalar field[a]

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{6.19}$$

The generating functional is hence

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x\frac{1}{2}\partial_\mu\phi\partial^\mu\phi+\frac{1}{2}m^2\phi^2+i\int d^4xJ(x)\phi(x)} \tag{6.20}$$

Let us first analyse the exponential

$$i\int d^4x\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + i\int d^4xJ(x)\phi(x) \tag{6.21}$$

Now we can introduce the two-point Green's function or the propogator, which, as seen before, satisfies

$$(\partial^2 + m^2)G(x-y) = \delta^4(x-y) \tag{6.22}$$

For our convenience, the field $\phi(x)$ can be decomposed

$$\phi(x) = \phi_c(x) + \phi_q(x) \tag{6.23}$$

where:

- $\phi_c(x)$ satisfies the classical equation of motion involving the source

$$(\partial^2 + m^2)\phi_c(x) = J(x) \tag{6.24}$$

- $\phi_q(x)$ is the quantum fluctuation around the classical solution.

The exponential term is hence

$$-\frac{1}{2}i\int d^4x\phi_q(\partial^2+m^2)\phi_q + i\int d^4xd^4yJ(x)G(x-y)J(y) \tag{6.25}$$

So far, this generating functional remains unnormalised and diverges into infinity. We thus introduce the so-called *normalised generating functional* $Z_0[J]$, which has the form

Definition 6.3 (Normalised generating functional)

$$Z_0[J] = \frac{Z[J]}{Z[J=0]} \tag{6.26}$$

In our case, this is

$$Z_0[J(x)] = \frac{\int \mathcal{D}\phi e^{-\frac{1}{2}i\int d^4x(\partial^2+m^2)\phi+i\int d^4xJ(x)\phi(x)}}{\int \mathcal{D}\phi e^{-\frac{1}{2}i\int d^4x(\partial^2+m^2)\phi}} \tag{6.27}$$

[4] In fact, it is simply called the *partition function* in some literature.

The numerator integrates as

$$e^{-\frac{1}{2}\int d^4x d^4y J(x)G(x-y)J(y)} \underbrace{\int \mathcal{D}\phi_q e^{-\frac{1}{2}i\int d^4x \phi_q(\partial^2+m^2)\phi_q}}_{①} \tag{6.28}$$

The denominator is a Gaussian normalization integral where, by definition, $J = 0$ (see Equation 6.26). As such, the $\phi_c(x)$ has no effect, and we can effectively rewrite $\phi(x)$ as $\phi_q(x)$. This makes the denominator equivalent to ①, and we are left with

$$Z_0[J(x)] = e^{-\frac{1}{2}\int d^4x d^4y J(x)G(x-y)J(y)} \tag{6.29}$$

where again, $G(x-y)$ is the all-too-familiar two-point Green's function:

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{6.30}$$

[a] This is identical to Equation 2.21 save for the minus sign.

From this, we are almost capable of establishing a relationship between the propogatoe and the generating functional. The only missing piece is the so-called *functional derivative* δ, which is the analogue of a normal derivative for a function.

Theorem 6.4 (Functional derivative properties)

$$\frac{\delta 1}{\delta\phi(x)} = 0 \tag{6.31}$$

$$\frac{\delta\phi(y)}{\delta\phi(x)} = \delta(x-y) \tag{6.32}$$

$$\frac{\delta}{\delta\phi(x)}(\alpha(y)\beta(z)) = \frac{\delta\alpha(y)}{\delta\phi(x)}\beta(z) + \alpha(x)\frac{\delta\beta(z)}{\delta\phi(x)} \tag{6.33}$$

A general relation between propagators and the normalised generating functional in free fields can now be made. For a system of n 4-positions, we have

Theorem 6.5 (Propagator-generating functional relation)

$$G_0^{(n)} = \langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle = \frac{1}{i^n}\frac{\delta^n Z_0(J)}{\delta J(x_1)\cdots\delta J(x_n)}\bigg|_{J=0} \tag{6.34}$$

This then allows us to reconstruct the generating functional in terms of the propagator:

$$Z[J] = \sum_{n=0}^{\infty}\int d^dx_1\ldots d^dx_n\langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle J(x_1)\cdots J(x_n) \tag{6.35}$$

6.3 Ward-Takahashi identities

The Ward-Takahashi identities are the quantum analogue of Noether's theorem, and impose constraints relationships between different Green's functions to preserve symmetry. Suppose a field undergoes the following gauge transformation:

$$\phi' = \phi + \delta\phi = \phi + \epsilon f(\phi, x) \tag{6.36}$$

where ϵ is *smol*. For some general function $F(\phi)$, such a gauge transformation results in

$$\delta F(\phi) = \frac{\delta F}{\delta\phi}\delta\phi = \frac{\delta F}{\delta\phi}\epsilon f(\phi, x) \tag{6.37}$$

For the generating functional, this becomes

$$\delta Z(\phi) = \frac{\delta Z}{\delta \phi}\delta\phi = \frac{\delta Z}{\delta \phi}\epsilon f(\phi, x) = \int \mathcal{D}\phi\epsilon\Bigg(\underbrace{i\left(\frac{\delta S}{\delta \phi} + J\right)f}_{\text{action}} + \underbrace{\frac{\delta f}{\delta \phi} + O(\epsilon)^2}_{\text{measure}}\Bigg)e^{iS+i\int d^dxJ\phi} = 0 \tag{6.38}$$

We should see $\delta Z = 0$ as the gauge transformation should leave the generating functional invariant.
We now differentiate this expression with respect to J and set $J = 0$ afterwards. This yields

$$\left\langle T\phi(y)\frac{\delta f(\phi, x)}{\delta\phi(x)}\right\rangle + i\left\langle \phi(y)\frac{\delta S}{\delta\phi(x)}f\right\rangle + \langle Tf\rangle = 0 \tag{6.39}$$

where we have used the shorthand $\langle\rangle$ to represent vacuum expectation values. This is the Ward-Takahashi identity for one field.
For the general case, which has many fields, the expression becomes

Theorem 6.6 (Ward-Takahashi identities for many fields) The Ward-Takahashi identities relate different Green's functions by eliminating redundancies introduced by the global symmetry:

$$\left\langle T\left[\prod_{l=1}^{n}\phi_{i_l}(y_l)\frac{\delta f_k}{\delta\phi_k(x)}\right]\right\rangle + i\left\langle T\left[\prod_{l=1}^{n}\phi_{i_l}(y_l)\frac{\delta S}{\delta_k\phi(x)}f_k\right]\right\rangle + \sum_{m=1}^{n}\left\langle\prod_{l=1}^{m-1}\phi_{i_l}(y_l)f_{i_m}\prod_{r=m+1}^{n}\phi_{i_r}(y_r)\right\rangle = 0 \tag{6.40}$$

where i denotes specific (field) components and n is the total number of fields.

Chapter 7

Interacting fields I: Preliminaries

Chapter 8

Interacting fields II: QED

Part III

Renormalisation and regularisation

Chapter 9

Renormalisation

9.1 UV singularity

Previously, we noted that everything done so far has been restricted to the tree level. This is obviously problematic because loop interactions emerge in real life. In the author's infinite benevolence, the reader has been led to believe, up to this point of the book, that the result from the Feynman rules is the entirety of the transition amplitude $\mathcal{M}$, whereas in reality, the full $\mathcal{M}$ follows the form

$$\mathcal{M} = \alpha\mathcal{M}_0 + \alpha^2\mathcal{M}_1 + \alpha^2\mathcal{M}_2 + \cdots \tag{9.1}$$

in which α is some coupling constant. $\alpha\mathcal{M}_0$ is the transition amplitude term constructed from the Feynman rules that we have blindly believed to be the full $\alpha\mathcal{M}$ up to now. The total transition amplitude is the sum of all possible $\alpha\mathcal{M}_i$s stemming from all possible processes. As the order goes up, the coupling constant becomes increasingly *small* to the point that its corresponding term practically vanishes.
This is not the end of the story. Rather, one can see that some higher-order terms diverge into infinity. Consider the following vertex correction for $e^+e^- \to \mu^+\mu^-$ scattering:

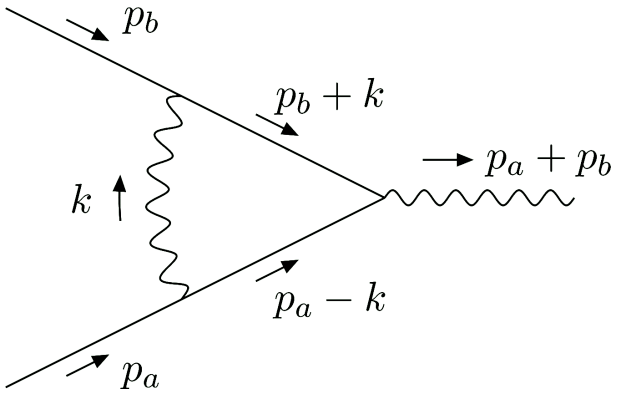

Figure 9.1: Vertex correction in $e^+e^- \to \mu^+\mu^-$ scattering.

Using the QED Feynman rules, the correction terms involving k are, in their entirety

$$\text{terms} = \int \frac{d^4k}{(2\pi)^4} \frac{k^2}{k^2((p_b+k)^2 - m^2)((p_b-k)^2 - m^2)} \tag{9.2}$$

In the limit where $k \ll p_a$ and $k \ll p_b$ (effectively $k \to \infty$), this becomes

$$\text{terms} = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k_4} = \infty \tag{9.3}$$

As they happen under $k \to \infty$, these singularities[1] are called *ultraviolet singularities* or simply *UV singularities*. However, this divergence of constituent $\alpha\mathcal{M}_i$ terms up to infinity is not unique to UV singularities.

[1] The meaning of the term is very similar to the meaning of singularities in GR.

9.2 Loop corrections

Fortunately, there are no infinities in real-world physics. Rather, we have mistakenly assumed that coupling constants do not change with energy scales. The key point lies in realising that the mass m and the charge e we have employed in the QED Lagrangian are actually merely the *tree-level* versions of the terms they claim to represent. In higher orders, mass and charge have correction terms as well. This extends to the scalar and vector fields ψ and A which are dependent on e and m.
We will now denote the previously seen m, e, ϕ and A as m_0, e_0, ψ_0 and A_0. For clarity, the actual (read: renormalised) quantities will be denoted as m_R, e_R, ψ_R and A_R, and are related to their tree-level counterparts by the so-called *renormalisation factors* Z_1, Z_2, Z_3 and Z_m:

Definition 9.1 (Renormalisation factors)

$$m_R = \frac{m_0}{Z_m} \quad e_m = \frac{Z_2\sqrt{Z_3}e_0}{Z_1} \quad \psi_R = \frac{\psi_0}{\sqrt{Z_2}} \quad A_R = \frac{A_0}{\sqrt{Z_3}} \tag{9.4}$$

Chapter 10

Regularisation