The ADM Formalism: From Tetrads to Quantum Gravity

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What we will go through today

1. Preliminaries
   - Geometry and tensors
   - Einstein field equations
   - Tetrads
2. ADM formalism
   - 3+1 decomposition of spacetime
   - Extrinsic curvature
   - Hamiltonian formulation
   - Wheeler-DeWitt equation
   - Closing remarks

Motivation

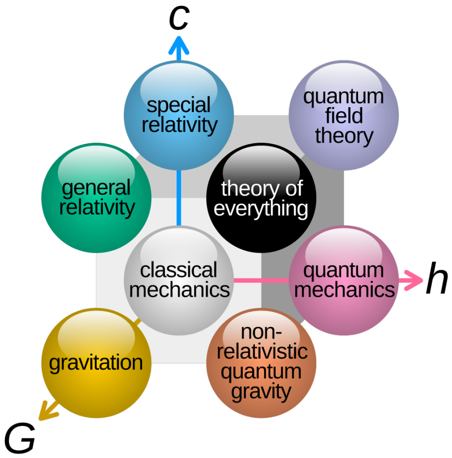

Figure: The so-called *cGh* cube

Tensors

Some well-known examples:

- A scalar is a rank-0 tensor.
- A vector is a rank-1 tensor.
- A matrix is a rank-2 tensor.

A *tensor* is something that transforms as a tensor.

Well...

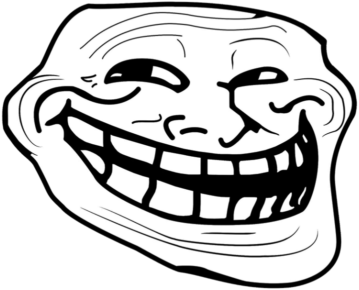

Figure: Ain't I a stinker?

Tensors, take 2

A *tensor* with p top indices and q bottom indices is said to be a rank-$p+q$ tensor. It transforms according to the general coordinate transformations

$$T'^{a_1 \dots a_p}_{b_1 \dots b_q} = \frac{\partial X'^{a_1}}{\partial X^{c_1}} \dots \frac{\partial X'^{a_p}}{\partial X^{c_p}} \frac{\partial X^{d_1}}{\partial X'^{b_1}} \dots \frac{\partial X^{d_q}}{\partial X'^{b_q}} T^{c_1 \dots c_p}_{d_1 \dots d_q} \tag{1}$$

Remark 1

Tensors are *not* glorified matrices!

Einstein field equations

The *Einstein field equations* or *field equations* for the cosmological cosnstant Λ and coupling constant κ is

$$G_{ij} + \Lambda g_{ij} = \kappa T_{ij} \tag{2}$$

This equation has three terms:

- The geometry term G_{ij}, consisting solely of the *Einstein tensor*
- The cosmological term Λg_{ij}, where g_{ij} is the metric tensor or simply the *metric*
- The matter term κT_{ij}, where T_{ij} is the *stress-energy tensor*

Stress-energy tensor

The so-called *stress-energy tensor* encodes the matter content of the spacetime of interest.

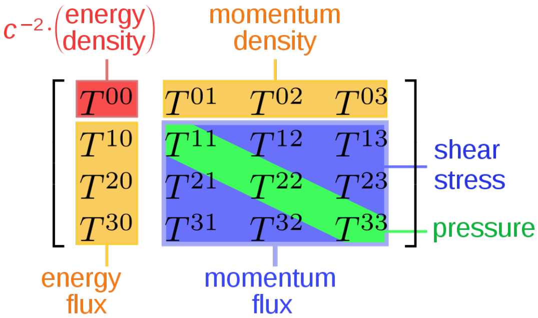

Figure: Components of the stress-energy tensor

Physical meaning of the stress-energy tensor

Some interesting examples:

- Stress-energy tensor in a perfect fluid (no viscosity or heat conduction)

$$T_{ab} = \rho u_a u_b - p(\eta_{ab} - u_a u_b) \tag{3}$$

- Stress-energy tensor in an EM field

$$T_{ab} = \frac{1}{\mu_0}\left(F_{ac}F^c_b - \frac{1}{4}\eta_{ab}F_{mn}F^{mn}\right) \tag{4}$$

- Stress-energy tensor in a scalar field Ψ

$$T_{ab} = -\partial_a\Psi\partial_b\Psi + \eta_{ab}\frac{1}{2}\left(\eta^{cd}\partial_c\Psi\partial_d\Psi - m^2\Psi^2\right) \tag{5}$$

From metric to Einstein: metric

The *metric* is a representation of the geometry we operate on - or a 'metric' of the geometry (ha ha ha).

- 3D Euclidian space (flat)

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{6}$$

- 4D Minkowski spacetime (SR, nonetheless flat)

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{7}$$

or in HEP formalism

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{8}$$

From metric to Einstein: curvature

The complete information on curvature is given by the *Riemann tensor*

$$R^c_{bad} = \Gamma^c_{bd,a} - \Gamma^c_{ad,b} + \Gamma^e_{bd}\Gamma^c_{ea} - \Gamma^e_{ad}\Gamma^c_{eb} \tag{9}$$

where Γ^a_{bc} are *Christoffel symbols*. Contracting the Riemann tensor yields the *Ricci tensor*, a form we use more often:

$$R_{ab} = R^c_{acb} \tag{10}$$

This leaves us with the curvature generated by matter content only.

Remark 2

In a flat spacetime, Christoffels are always zero. Hence both the Riemann and Ricci tensors vanish.

From metric to Einstein: trace-reverse

Figure: The trace-reverse as seen in 'The N$^{\mathrm{th}}$ Degree'

From metric to Einstein: Einstein tensor

Ignoring the cosmological term, the field equations are

$$R_{ij} - \frac{1}{2} g_{ij} R = G_{ij} = \kappa T_{ij} \tag{11}$$

where $R = g^{ij} R_{ij}$ is the *Ricci scalar*.
Significance of the field equations:

- Gravity is geometry (various GR contributors)
- Space tells matter how to move, matter tells space how to curve (John Archibald Wheeler)
- Recovery of the so-called 'table cloth' analogy

'Table cloth' analogy

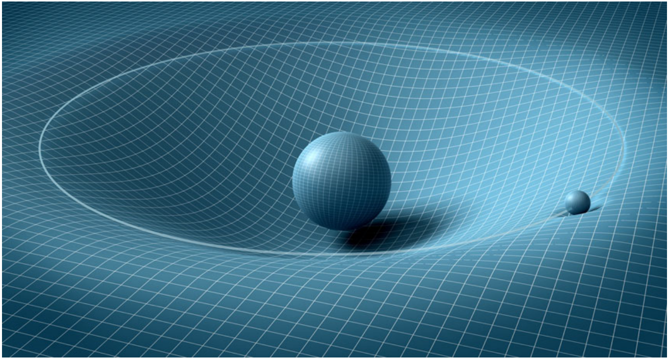

Figure: The infamous 'table cloth' analogy in popular science

Etymology

The name *tetrad* comes 'four' in Greek. It is also called a *Vierbein* (four legs).

Figure: Vierbein

Tetrad formalism

Coordinate and tetrad bases

$$X_\mu = (t, x, y, z) \quad \gamma_a = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \tag{12}$$

Remark 3

To distinguish between coordinate and tetrad indices, we use Greek letters for the former and Latin letters for the latter.

The tetrad metric is then $\gamma_a \gamma_b = \gamma_{ab}$. In GR, the goal is to produce locally flat frames, so we always have

$$\gamma_{ab} = \eta_{ab} \tag{13}$$

Motivation

Why introduce tetrads?

1. The physics is more transparent when expressed in a locally inertial frame (or some other frame adapted to the physics), as opposed to the coordinate frame, where Salvador Dali rules.
2. If you want to consider spin-$\frac{1}{2}$ particles and quantum physics, you better work with tetrads.
3. For good reason, much of the general relativistic literature works with tetrads, so it's useful to understand them.

-Andrew J. S. Hamilton

Tetrad field

The so-called tetrad field or *frame field* e_a^μ converts the coordinate basis X_μ to the tetrad basis γ_a:

$$\gamma_a = e_a^\mu X_\mu \tag{14}$$

It undergoes both *general coordinate transformations* and *local Lorentz transformations*:

$$e'^a_\mu = \Lambda^a_b e^b_\nu \frac{\partial X^\nu}{\partial X'^\mu} \tag{15}$$

Hence, a tetrad field is *not* a tensor.
Lorentz transformations are observed by *spinors* as part of the $\mathrm{SO}(1,3)$ group - but we won't discuss that.

Example: The Schwarzschild metric

Practically, the tetrad fields are very similar to change in coordinate bases. For example, a tetrad field with the following components

$$e^0_\mu dX^\mu = \left(1 - \frac{2M}{r}\right)^{1/2} dt \tag{16}$$

$$e^1_\mu dX^\mu = \left(1 - \frac{2M}{r}\right)^{-1/2} dr \tag{17}$$

$$e^2_\mu dX^\mu = r d\theta \tag{18}$$

$$e^3_\mu dX^\mu = r \sin\theta d\Psi \tag{19}$$

transforms the (coordinate) Schwarzschild metric into the (tetrad) Minkowski metric η_{ab}.

Now that wasn't so hard, was it?

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                                        You keeping us on course,
                                          Little buddy?           \

      Yes, Skipper \                                 __________________________
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Figure: Famous last words by relativists

Motivation

Schrödinger equation

$$i\hbar\partial_t|\Psi\rangle = \hat{H}|\Psi\rangle \tag{20}$$

Hamilton's equations

$$\frac{\partial H}{\partial X_i} = -\dot{p}_i \quad \frac{\partial H}{\partial p_i} = \dot{X}_i \tag{21}$$

Remark 4

In classical field theory, the generalised 3-velocity is used. In GR, the generalised 4-velocity $\dot{X}^i$ is not well-defined!

Constant time hypersurfaces

We solve this by 'granting previlige' to one of the four coordinates (e.g. t) so that we can define the generalised velocity using it.
Decomposition of X^μ into a t and the 3D x^α

$$X^\mu = (t, x^\alpha) \quad \text{where} \quad \alpha = 1, 2, 3 \tag{22}$$

This allows for the introduction of the *constant time hypersurfaces*:

- In an n-dimensional manifold, it is possible to *embed* an $n-1$-dimesional hypersurface Σ_t
- A 'slice' of a higher dimension

Constant time hypersurfaces

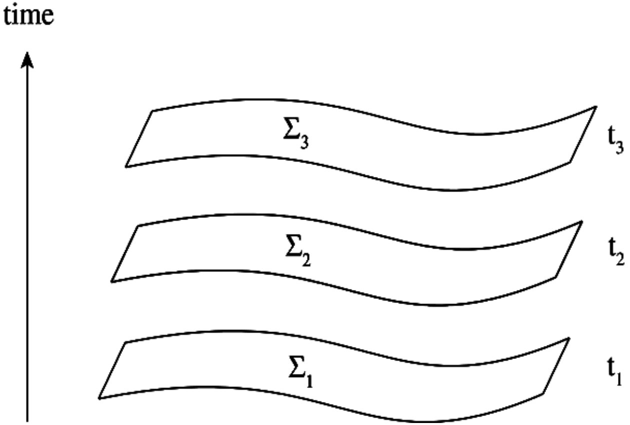

Figure: Constant time hypersurfaces (spatial dimensions reduced to two for visualisation)

Splitting the tetrad

A general tetrad frame can be written as

$$\gamma_m = (\xi_0, \xi_a) \quad \text{where} \quad a = 1, 2, 3 \tag{23}$$

where γ_0 satisfies:

$$\xi_0 \cdot \xi_0 = -1 \qquad \underbrace{\xi_0 \cdot \xi_a = 0}_{\text{3+1 decomposition}} \tag{24}$$

3+1 tetrad metric:

$$\gamma_{mn} = \begin{pmatrix} -1 & 0 \\ 0 & \gamma_{ab} \end{pmatrix} \tag{25}$$

γ_{ab} is the so-called *spatial (tetrad) metric*.

Spatial tetrad field

Conversion from the spatial coordinate metric $g_{\alpha\beta}$ to the spatial tetrad metric γ_{ab}:

$$g_{\alpha\beta} = \gamma_{ab} e^a_\alpha e^b_\beta \tag{26}$$

where we have defined the *spatial tetrad field* and the *inverse spatial tetrad field* as

$$e^a_\alpha = \begin{pmatrix} \alpha & 0 \\ -e^a_\alpha \beta^\alpha & e^a_\alpha \end{pmatrix} \quad e^\alpha_a = \begin{pmatrix} 1/\alpha & \beta^\alpha/\alpha \\ 0 & e^\alpha_a \end{pmatrix} \tag{27}$$

Two important quantities emerge: α and β_α.

Contents of the spatial tetrad field

Lapse function: the rate at which the proper time τ of the tetrad rest frame advances per unit coordinate time t

$$\alpha = \frac{d\tau}{dt} \tag{28}$$

Shift vector: the velocity at which the tetrad rest frame moves through the spatial coordinate X^α per unit coordinate time t.

$$\beta^\alpha = \frac{dX^\alpha}{dt} \tag{29}$$

4D metric

Using the previously discussed objects, the good ol' 4D metric can be represented as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_\mu \beta^\mu & \beta_\nu \\ \beta_\mu & \gamma_{\mu\nu} \end{pmatrix} \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^\nu}{\alpha^2} \\ \frac{\beta^\mu}{\alpha^2} & \gamma_{\mu\nu} - \frac{\beta^\mu \beta^\nu}{\alpha^2} \end{pmatrix} \tag{30}$$

or in line element form

$$ds^2 = (-\alpha^2 + \beta_\mu \beta^\mu) dt^2 + 2\beta_\mu dt dx^\mu + \gamma_{\mu\nu} dx^\mu dx^\nu \tag{31}$$

Extended covariant derivative

Normal vector w.r.t. the hypersurface:

$$n^\mu = (1/\alpha, -\beta^i/\alpha) \tag{32}$$

The goal is to place objects that live on manifolds onto our hypersurfaces. This is accomplished by the so-called *projection operator*:

$$P^\alpha_\beta = \delta^\alpha_\beta + n^\alpha n_\beta \tag{33}$$

Quite satisfyingly, the *extended covariant derivative* is then simply

$$D_\mu = P^\alpha_\mu \nabla_\alpha \tag{34}$$

This is an analogue of ∇_α rescricted to the hypersurface.

Curvature on the hypersurface?

Intrinsic curvature - the curvature encoded by the Riemann, Ricci and Weyl tensors. Variation of the *tangent vector* along the geodesic.
Exteinsic curvature - variation of the *normal vector* along the *tangent vector*:

$$K_{ij} = -n_\mu \nabla_j e_i^\mu \tag{35}$$

ADM Hamilton's equations

Canonical - conforming to Hamiltonian mechanics.
The *ADM canonical momentum* is

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = -\sqrt{\gamma}(K^{ij} - \gamma^{ij} K) \tag{36}$$

where δ is the *functional derivative*.
ADM Hamilton's equations:

$$\frac{\delta H}{\delta \pi^{ij}} = \dot{\gamma}_{ij} \quad \frac{\delta H}{\delta \gamma^{ij}} = -\dot{\pi}_{ij} \tag{37}$$

Time evolution

Using action principles (or the principle of stationary action) we can find the following *evolution equations*:

- **Spatial metric**:

$$\dot{\gamma}_{ij} = D_i \beta_j + D_j \beta_i - 2\alpha K_{ij} \tag{38}$$

- **Canonical momentum**:

$$\begin{aligned}\dot{\pi}^{ij} = &- N\sqrt{\gamma}\left(R^{ij} - \frac{1}{2}\gamma^{ij}R\right) + \frac{N}{2\sqrt{\gamma}}\left(\pi_{cd}\pi^{cd} - \frac{\pi^2}{2}\right)\gamma^{ij} \\ &- \frac{2N}{\sqrt{\gamma}}\left(\pi^{ic}\pi_c{}^j - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{\gamma}\left(D^i D^j N - \gamma^{ij} D_c D^c N\right) \\ &+ D_c\left(\pi^{ij} N^c\right) - \pi^{ic} D_c N^j - \pi^{jc} D_c N^i\end{aligned} \tag{39}$$

GR operators

Wavefunctions $|\Psi\rangle$ become *wave functionals* $\Psi[\gamma_{ab}]$:

$$\hat{\gamma}_{ij}(t, x^k)|\Psi\rangle \to \gamma_{ij}(t, x^k)\Psi[\gamma_{ab}] \tag{40}$$

$$\hat{\pi}^{ij}(t, x^k)|\Psi\rangle \to -i\frac{\delta}{\delta\gamma_{ij}(t, x^k)}\Psi[\gamma_{ab}] \tag{41}$$

Emergence of quantum gravity

The evolution equations become

$$\hat{R}^0|\Psi\rangle \doteq -\left[\sqrt{\gamma}R + \frac{1}{\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right)\right]\Psi\left[\gamma_{kl}\right] = 0 \tag{42}$$

$$\hat{R}^i|\Psi\rangle \doteq -2D_j\pi^{ij}\Psi\left[\gamma_{kl}\right] = 0 \tag{43}$$

The first one can be rewritten as

$$\left[\sqrt{\gamma}R - \frac{\hbar^2}{\sqrt{\gamma}}\left(\frac{1}{2}\gamma_{ab}\gamma_{cd} - \gamma_{ac}\gamma_{bd}\right)\frac{\delta}{\delta\gamma_{ab}}\frac{\delta}{\delta\gamma_{cd}}\right]\Psi\left[\gamma_{kl}\right] = 0 \tag{44}$$

This is the *Wheeler-DeWitt equation*, the analogue to the Schrödinger equation in quantised GR.

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Conclusion

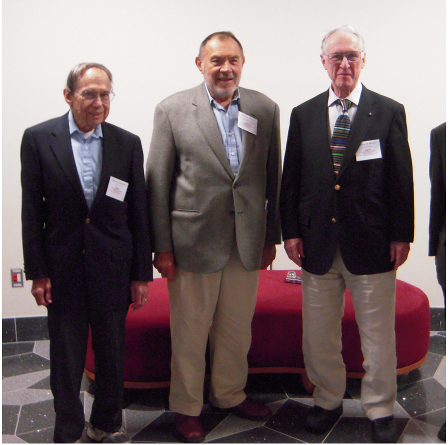

Figure: Arnowitt, Deser and Misner (2009)