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Weak field approximations in modified theories of gravity

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Abstract

Since its formulation in the 1970s, $f(R)$ gravity has remained a prominent topic of interest in modified theories of gravity. A recent theory related to $f(R)$ gravity is $f(\mathcal{G})$ gravity, where the Ricci scalar R is decomposed to a bulk term $\mathcal{G}$ and a boundary term $\mathcal{B}$ that does not contribute to the action. In the Einstein-Hilbert action, $\mathcal{B}$ is removed from R, and the remaining $\mathcal{G}$ is mapped to a function $f(\mathcal{G})$.

In this thesis, we will investigate linearised $f(\mathcal{G})$ gravity and several of its interesting aspects. Firstly, there is a correspondence between the linearised $\mathcal{G}$ and the Lagrangian of a rank-2 tensor field. Secondly, we discover that the standard GR field equations can be recovered from the linearised $f(\mathcal{G})$ field equations by specific choices of parameters in $f(\mathcal{G})$.

We then select a more generalised $f(\mathcal{G})$ that yields only one modified gravity term in the field equations, whose simplest physically meaningful gravitational wave solution shows a proportional relationship between the coupling constant λ of the modified gravity term and the angular frequency ω of the gravitational wave. Finally, this thesis closes with physical discussions of this solution.

Contents

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Chapter 1

Preliminaries

Quote 1.1 Gravity is geometry

Various relativists

1.1 Introduction

First formulated in 1915 [1], general relativity (GR) remains the best-accepted theory of gravitation to date. The central idea of GR is the *Einstein field equations*, or simply the so-called *field equations*.

Definition 1.1 (Einstein field equations)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.1}$$

where:

- The first term is the geometry term. $G_{\mu\nu}$ is the *Einstein tensor*, which is the trace-reverse of the *Ricci tensor* and encodes roughly half of the information on curvature.
- The second term is the cosmological term. Λ is the *cosmological constant* and $g_{\mu\nu}$ is the *metric*, which defines the spacetime of an exact solution of the field equations.
- The third term is the matter term. κ is the *coupling constant* and $T_{\mu\nu}$ is the *stress-energy tensor*, which encodes the matter-energy content.

Without considering the cosmological term, the field equations illustrate the equivalence between curvature and matter-energy content, giving rise to the well-known aphorism ‘Space tells matter how to move; Matter tells space how to curve’ by Wheeler [2].

Einstein’s original derivation of the field equations in 1915 was mostly based on physical intuition [3]. We begin with the expressions for force and acceleration of a two-body system in gravitation

$$F = \frac{GMm}{r^2} \qquad a_m = \frac{GM}{r^2} \tag{1.2}$$

where a_m is the acceleration of the mass m. Distinct from other forces, gravitation has an acceleration a_m that is independent of m or any other intrinsic quantities of the object. In other words:

- When an object is dropped inside an elevator stationary on the Earth’s surface, the gravitational acceleration is g, independent of mass.
- When an object is released inside an elevator accelerating upwards at exactly g in a vacuum, the gravitational acceleration is also g, independent of mass.

This is Einstein’s famous ‘elevator experiment’, which motivated Einstein’s idea that gravity is not a force in the traditional sense, but rather a result of the curvature of spacetime.

However, all equations of motion in physics, with a few exceptions, such as in thermodynamics, are ultimately derived via the *action principle*. Such an action-based derivation of the field equations was formulated by Hilbert in the same year [4], and remains the standard derivation of the field equations today. The generalised action in GR is known as the *Einstein-Hilbert action* and has two terms [5]:

Definition 1.2 (Einstein-Hilbert action)

$$S = \int_{\mathcal{V}} \mathcal{L} d^4x = S_H + S_M \tag{1.3}$$

The terms are defined as follows:

- S_H is the *Hilbert term* arising from the *Ricci scalar* R:

$$S_H = \frac{1}{16\pi} \int_{\mathcal{V}} R\sqrt{-g} d^4x \tag{1.4}$$

  where g is the (negative) determinant of the metric $g_{\mu\nu}$ and $\sqrt{-g}d^4x$ is the proper volume element.

- S_M is the *matter action* arising from the scalar matter field ϕ:

$$S_M = \int_{\mathcal{V}} \mathcal{L}_M(\phi, \partial_\mu \phi, g_{\mu\nu})\sqrt{-g} d^4x \tag{1.5}$$

The longevity of GR and against classical tests is well known [6]. Nonetheless, modified theories of gravity have consistently been proposed [7]. Some famous examples over the years are introduced below:

- The so-called *Einstein-Cartan theory* assumes a non-zero torsion tensor. It reduces to GR in vacuum, and is hence is also well supported by classical tests of GR [8][9][10].

- *Massive gravity* endows a non-zero mass to the graviton. As this theory predicts timelike gravitational waves, very stringent constraints have been placed on it in recent years since the detection of gravitational waves [11].

- *Extended theories of gravity* start with modifying the Einstein-Hilbert action from their GR counterpart [12]:

  - $f(R)$ gravity, which was proposed in 1970 by Buchdahl [13], maps R in the Einstein-Hilbert action to $f(R)$, a function of itself.
  - $f(T)$ gravity, whose extensions were developed by Lobo and Harko [14], maps R in the Einstein-Hilbert action to $f(T)$, where T is the torsion scalar in teleparallel gravity [15].
  - $f(Q)$ gravity maps R in the Einstein-Hilbert action to $f(Q)$, where Q is the nonmetricity scalar in symmetric teleparallel gravity [16].
  - $f(\mathcal{G})$ gravity, a specific subset of $f(R)$ gravity developed by Böhmer and Jensko in 2021 [17] that will be the theory of interest in this thesis.

The main theme of this thesis is gravitational waves in modified gravity. Despite its relatively new status, some research in this emerging area has already been made [18][19]. Future work is also promising for several reasons. Current observations of gravitational waves confirm GR up to a tiny margin of error [20][21]. As such, we already expect very stringent constraints for any deviations from GR. With future collaborations like LISA [22], which offer greater precision in gravitational wave detection, observational gravitational wave physics has become a strong candidate for testing modified gravity theories and imposing greater constraints on them.

This thesis is structured as follows:

- Chapter 1 provides a brief review of gravitational waves and $f(\mathcal{G})$ gravity.

- Chapter 2 develops the weak field expansions in the framework of $f(\mathcal{G})$ gravity.

- Chapter 3 inserts test $f(\mathcal{G})$s into our linearised $f(\mathcal{G})$ gravity and solves for the simplest $f(\mathcal{G})$ field equations that do not reduce to GR.
- Chapter 4 solves for the wavevector k^μ and the tensorial amplitude $A_{\mu\nu}$ of gravitational waves in this set of field equations.
- Chapter 5 summarises the thesis and suggests potential future work.

The standard Einstein summation convention is employed, where Greek letters denote coordinate indices. The metric has the standard GR signature $(-+++)$.

1.2 Linearised field equations

We begin with the field equations. Outside of cosmology, the cosmological term is usually omitted, leaving

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.6}$$

This seemingly simple equation is, in fact, highly non-linear. Like in many cases, however, the field equations can be linearised.

Suppose that the spacetime is nearly flat but very slightly curved. Such a spacetime can be represented by decomposing the geometry into flat (Minkowski) spacetime and a *small* perturbation, or *gauge transformation* ξ. In effect, the metric is then the sum of the Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,0,0,0)$ and the *small* perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}((h_{\mu\nu})^2) \quad \text{where} \quad |h_{\mu\nu}| \ll 1 \tag{1.7}$$

This process is called *linearised gravity* [23]. As both the metric $g_{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$ are symmetric, $h_{\mu\nu}$ is also symmetric.

We can now derive the linearised field equations. We begin by writing out the inverse metric under the linearised gravity regime

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{1.8}$$

The Christoffels are [2]

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\eta^{\mu\nu}(h_{\alpha\mu,\beta} + h_{\beta\nu,\alpha} - h_{\alpha\beta,\nu}) = \frac{1}{2}(h^\mu_{\alpha,\beta} + h^\mu_{\beta,\alpha} - h_{\alpha\beta}{}^{,\mu}) \tag{1.9}$$

As we have $\eta_{\mu\nu} \approx g_{\mu\nu}$, we have forgoed $g_{\mu\nu}$ for the Minkowski metric in our derivations.

From definition, the linearised *Ricci tensor* is hence

$$R_{\mu\nu} = \frac{1}{2}(h^\alpha_{\mu,\nu\alpha} + h^\alpha_{\nu,\mu\alpha} - h_{\mu\nu,\alpha}^{;\alpha} - h_{,\mu\nu}) \tag{1.10}$$

The last term is effectively a 'scalar perturbation' $h = \eta^{\alpha\beta}h_{\alpha\beta}$ differentiated with respect to the indices μ and ν. By contracting the linearised Ricci tensor, one finds the linearised Ricci scalar

$$R = h^{;\mu\nu}_{\mu\nu} - h^{;\rho}_{,\rho} \tag{1.11}$$

This allows us to perform the trace reverse, which gives the linearised Einstein tensor as [6]

$$G_{\mu\nu} = \frac{1}{2}\left(\bar{h}_{\lambda\nu}{}^{;\lambda}_{,\mu} + \bar{h}_{\lambda\mu}{}^{;\lambda}_{,\nu} - \bar{h}_{\mu\nu}{}^{;\lambda}_{,\lambda} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta}\right) \tag{1.12}$$

While this might look very bulky, one can actually reduce the expression by considering gauge freedom introduced by the newly added perturbation $h_{\mu\nu}$. Consider the *small* coordinate (gauge) transformation

$$X^\mu \to X'^\mu = X^\mu + \xi^\mu \tag{1.13}$$

where ξ^μ is an infinitesimal vector field. Under this transformation, the metric is invariant, while the metric perturbation $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \xi_{\nu,\mu} - \xi_{\mu,\nu} \tag{1.14}$$

As the metric is gauge-invariant under this gauge transformation, a *gauge freedom* has arisen in that many metric perturbations correspond to the same metric. To eliminate some of the gauge freedom, one can impose a gauge condition. A convenient choice is the tensorial form of the *Lorenz gauge*:

$$\bar{h}_{\mu\nu}{}^{,\mu} = 0 \tag{1.15}$$

where we have the trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \tag{1.16}$$

This fixes the gauge freedom by restricting the allowed vector fields ξ^μ in the gauge transformation. The transformed perturbation $h'_{\mu\nu}$ must also satisfy the Lorenz gauge condition, which imposes a constraint on ξ^μ:

$$\xi^{\mu,\alpha}{}_{,\alpha} = 0 \tag{1.17}$$

In this tensorial Lorenz gauge, the linearised Einstein tensor is [6]

$$G_{\mu\nu} = -\frac{1}{2}\bar{h}_{\mu\nu}{}^{,\alpha}{}_{,\alpha} \tag{1.18}$$

This yields the linearised Einstein field equations:

Theorem 1.1 (Linearised field equations)

$$G_{\mu\nu} = -\frac{1}{2}\bar{h}_{\mu\nu}{}^{,\alpha}{}_{,\alpha} = \kappa T_{\mu\nu} \tag{1.19}$$

For completeness, we also note that the linearised field equations with the cosmological constant take the form

$$G_{\mu\nu} + \Lambda h_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.20}$$

This is a result we will recover later in this thesis.

1.3 Gravitational wave solutions

Derivation of gravitational wave solutions take place under linearised gravity, and we start with the linearised field equations as seen in (1.19). As gravitational waves are a non-matter source, we also set the stress-energy tensor (i.e. matter content) as zero. This makes gravitational wave solutions vacuum solutions:

Theorem 1.2 (Linearised vacuum field equations)

$$\bar{h}_{\mu\nu}{}^{,\alpha}{}_{,\alpha} = 0 \tag{1.21}$$

Noting that this is effectively a wave equation, we can reasonably set up a trial solution for the *gravitational wave*

Definition 1.3 (Gravitational wave)

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\alpha x^\alpha) \tag{1.22}$$

where k_α is the wavevector, x^α is the 4-position and $A_{\mu\nu}$ is some tensorial amplitude. Both k_α and $A_{\mu\nu}$ are constants. Much like in electromagnetic waves, only the real part of the phase term is physical.

By inserting this $\bar{h}_{\mu\nu}$ into the gauge $\bar{h}^{\mu\alpha}{}_{,\alpha} = 0$, we can find the following constraints on k_α [24]:

- k_α is a null (i.e. lightlike) vector:

$$k_\alpha k^\alpha = 0 \tag{1.23}$$

- $A_{\mu\alpha}$ is orthogonal to k_α (transverse wave):

$$A_{\mu\alpha}k^\alpha = 0 \tag{1.24}$$

Here the physical significance is clearly seen. Expectedly, k_α corresponds to the angular frequency ω.

We can now impose further gauge conditions by adjusting the initial data for the Lorenz gauge equations. For a given 4-velocity u_ν, we impose an additional gauge condition, which is the traceless tensorial amplitude:

$$A^\mu_\mu = 0 \tag{1.25}$$

This, combined with the tensorial Lorenz gauge, is the so-called *transverse-traceless gauge* or the *TT gauge* [6].

From $A^{0\nu} = 0$, we can see that the first row and the first column vanishes. As $A^{\mu\nu}$ is established to be traceless, we also have

$$A^{11} + A^{22} + A^{33} = 0 \tag{1.26}$$

Considering also that $A^{\mu\nu}$ is symmetric, the most general matrix that satisfies these conditions leaves only two independent wave amplitudes out of the original 10:

$$A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.27}$$

$A_+ = 0$ and $A_\times = 0$ represent two different polarisations of gravitational waves:

- The A_+ mode or the *plus polarisation* (this means $A_\times = 0$) describes stretching and squeezing along axes aligned with the x- and y-axes. When a gravitational wave in this mode passes through, it elongates spacetime along one axis (say, the x-axis) while contracting along the perpendicular axis (y-axis), and then alternates this behaviour.

- The $A_\times$ mode or the *cross polarisation* (this means $A_+ = 0$) describes stretching and squeezing along axes rotated by 45 degrees relative to the x- and y-axes. i.e., along lines like

$$x' = (x+y)/\sqrt{2} \quad y' = (x-y)/\sqrt{2} \tag{1.28}$$

The deformation pattern is the same as the A_+ mode, but the axes of elongation and contraction are rotated by 45 degrees.

This is analogous to polarisations in EM waves, which are separated by 90°. The angles are different because EM waves correspond to oscillations of EM fields, which are vector fields in orthogonal directions, whereas gravitational waves correspond to tensorial deformations of spacetime that are rotations of each other by 45 degrees in the transverse plane.

1.4 Decomposition of the Ricci scalar

We apply the action principle to the Hilbert term in 1.4, which stipulates that the time derivative of the action of an isolated system is zero. As is well known, doing so to any system yields, through integration by parts, two different integrals, one of which is the so-called *boundary term* which does not contribute to the action [25].

In the case of the Hilbert term, this means one can decompose the Ricci scalar, which takes the full form

$$R = g^{\mu\nu}\left(\Gamma^\lambda_{\mu\nu,\lambda} - \Gamma^\lambda_{\mu\lambda,\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\lambda_{\lambda\sigma} - \Gamma^\sigma_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}\right) \tag{1.29}$$

into two parts: the *bulk term* $\mathcal{G}$ and the *boundary term* $\mathcal{B}$ [17]:

$$R = \mathcal{G} + \mathcal{B} \tag{1.30}$$

where $\mathcal{G}$ and $\mathcal{B}$ are respectively defined as

Definition 1.4 (Bulk term)

$$\mathcal{G} = g^{\mu\nu}(\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\lambda\sigma}) \tag{1.31}$$

Definition 1.5 (Boundary term)

$$\mathcal{B} = \nabla_{\sigma}B^{\sigma} \tag{1.32}$$

where we use the shorthand

$$B^{\sigma} = g^{\mu\nu}\Gamma^{\sigma}_{\mu\nu} - g^{\sigma\nu}\Gamma^{\lambda}_{\lambda\nu} \tag{1.33}$$

This is a *pseudovector*.

By construction, we see that the two terms are constructed from Christoffel symbols, which have rank 3 but are not tensors. As such, while the terms have rank 0, they do not transform tensorially and are therefore *not* scalars. As such, they are called *pseudoscalars*. Likewise, the rank-1 pseudovector is so-called as it does not transform tensorially either.

1.5 Transformation of $\mathcal{G}$ and $\mathcal{B}$

> **Quote 1.2** They arise naturally and there is no point to name them after people.
>
> *Christian G. Böhmer, on $M^{\mu\nu}_{\lambda}$ and $E^{\mu\nu\lambda}$, November 2024*

Consider the infinitesimally small transformation ξ:

$$X^{\mu} \to X'^{\mu} = X^{\mu} + \xi^{\mu}(X) \tag{1.34}$$

The general coordinate transformations become

$$\frac{\partial X'^{\mu}}{\partial X^{\nu}} = \delta^{\mu}_{\nu} + \partial_{\nu}\xi^{\mu} \quad \frac{\partial X^{\mu}}{\partial X'^{\nu}} = \delta^{\mu}_{\nu} - \partial_{\nu}\xi^{\mu} \tag{1.35}$$

Yielding the metric, inverse metric and Christoffel transformations: Using the above, along with the usual transformation laws, the metric, inverse metric and Christoffel symbol transform as

$$g'_{\mu\nu}(X') = g_{\mu\nu} - \partial_{\mu}\xi^{\lambda}g_{\lambda\nu} - \partial_{\nu}\xi^{\lambda}g_{\mu\lambda} + \mathcal{O}(\xi^2) \tag{1.36}$$

$$g'^{\mu\nu}(X') = g^{\mu\nu} + \partial_{\lambda}\xi^{\mu}g^{\lambda\nu} + \partial_{\lambda}\xi^{\nu}g^{\mu\lambda} + \mathcal{O}(\xi^2) \tag{1.37}$$

$$\Gamma'^{\gamma}_{\mu\nu}(X') = \Gamma^{\gamma}_{\mu\nu} + \partial_{\lambda}\xi^{\gamma}\Gamma^{\lambda}_{\mu\nu} - \partial_{\mu}\xi^{\lambda}\Gamma^{\gamma}_{\nu\lambda} - \partial_{\nu}\xi^{\lambda}\Gamma^{\gamma}_{\mu\lambda} - \partial_{\mu}\partial_{\nu}\xi^{\gamma} + \mathcal{O}(\xi^2) \tag{1.38}$$

From these, we can derive the transformations of $\mathcal{G}$ and $\mathcal{B}$

$$\mathcal{G}'(X') = \mathcal{G}(X) - M^{\alpha\beta}_{\gamma}\partial_{\alpha}\partial_{\beta}\xi^{\gamma} \tag{1.39}$$

$$\mathcal{B}'(X') = \mathcal{B}(X) + M^{\alpha\beta}_{\gamma}\partial_{\alpha}\partial_{\beta}\xi^{\gamma} \tag{1.40}$$

Where we have defined a non-tensorial object that we will call the *pseudoscalar deviation*. The object represents the 'degree' of failure of pseudoscalars $\mathcal{G}$ and $\mathcal{B}$ to transform as scalars [17].

Definition 1.6 (Pseudoscalar deviation)

$$M^{\mu\nu}_{\lambda} := \frac{\delta\mathcal{G}}{\delta\Gamma^{\lambda}_{\mu\nu}} = 2g^{\rho(\nu}\Gamma^{\mu)}_{\lambda\rho} - g^{\mu\nu}\Gamma^{\rho}_{\rho\lambda} - g^{\rho\sigma}\delta^{(\nu}_{\lambda}\Gamma^{\mu)}_{\rho\sigma} \tag{1.41}$$

where the brackets around the indices is the so-called *symmetriser* that symmetrises the expression[1]. For example:

$$g^{\rho(\nu}\Gamma^{\mu)}_{\lambda\rho} = \frac{1}{2}(g^{\rho\nu}\Gamma^{\mu}_{\lambda\rho} + g^{\rho\mu}\Gamma^{\nu}_{\lambda\rho}) \tag{1.42}$$

The invariance of the Ricci scalar under general coordinate transformations is well-known. By adding together the already transformed $\mathcal{G}'$ and $\mathcal{B}'$, we can find that

$$\mathcal{G}' + \mathcal{B}' = \mathcal{G}(X) - M^{\alpha\beta}_{\gamma}\partial_{\alpha}\partial_{\beta}\xi^{\gamma} + \mathcal{B}(X) + M^{\alpha\beta}\gamma\partial_{\alpha}\partial_{\beta}\xi^{\gamma} = \mathcal{G} + \mathcal{B} = R \tag{1.43}$$

The pseudoscalar deviations cancel out, and (1.30) is recovered.

[1]Conversely, the square brackets [and] is the *antisymmetriser*. For example: $g_{b[c}R_{d]a} = \frac{1}{2}g_{bc}R_{da} - g_{bd}R_{ca}$.

1.6 Modified gravity

In $f(R)$ gravity, the Hilbert term of the gravitational action is modified by replacing R with an arbitrary function of it:

$$S_H = \frac{1}{16\pi}\int_{\mathcal{V}} f(R)\sqrt{-g}d^4x \tag{1.44}$$

Specifically, $f(R)$ gravity is related to the scalar-tensor family of theories. The Brans-Dicke theory, a scalar-tensor theory, reduces to $f(R)$ gravity under specific choices of variables [26].

In recent years, more specific subcategories of $f(R)$ gravity have been developed [6]. We know, from (1.30) which we just proved, that $R = \mathcal{G} + \mathcal{B}$. We can then write our $f(R)$ as

$$f(R) = f(\mathcal{G}, \mathcal{B}) \tag{1.45}$$

This is the premise of the so-called $f(\mathcal{G}, \mathcal{B})$ *gravity* and, assuming no matter-energy content, gives rise to the following Einstein-Hilbert action:

$$S = \frac{1}{2\kappa}\int f(\mathcal{G}, \mathcal{B})\sqrt{-g}d^4x \tag{1.46}$$

By applying the action principle, one finds

Theorem 1.3 ($f(\mathcal{G}, \mathcal{B})$ field equations)

$$\begin{aligned}\frac{\partial f}{\partial \mathcal{G}}\left(G_{\rho\sigma} + \frac{1}{2}g_{\rho\sigma}\mathcal{G}\right) + \frac{1}{2}E_{\rho\sigma}{}^{\gamma}\partial_\gamma\left(\frac{\partial f}{\partial \mathcal{G}}\right) - \frac{1}{2}g_{\rho\sigma}f(\mathcal{G}, \mathcal{B}) + \frac{1}{2}\frac{\partial f}{\partial \mathcal{B}}\,g_{\rho\sigma}\mathcal{B} + g_{\rho\sigma}\partial^\mu\partial_\mu\left(\frac{\partial f}{\partial \mathcal{B}}\right) - \\ \partial_\rho\partial_\sigma\left(\frac{\partial f}{\partial \mathcal{B}}\right) + \frac{1}{2}g_{\rho\sigma}\partial_\mu(g^{\mu\nu})\partial_\nu\left(\frac{\partial f}{\partial \mathcal{B}}\right) + \frac{1}{\sqrt{-g}}\partial_{(\rho}(\sqrt{-g})\partial_{\sigma)}\left(\frac{\partial f}{\partial \mathcal{B}}\right) =& \kappa T_{\rho\sigma}\end{aligned} \tag{1.47}$$

The boundary term does not contribute to the action [27]. For this reason, it is generally safe to discard it. Hence, the only remaining component of the Ricci tensor is the pseudoscalar $\mathcal{G}$. When we do so to $f(\mathcal{G}, \mathcal{B})$ gravity, it reduces to $f(\mathcal{G})$ *gravity*, whose Einstein-Hilbert action is

$$S = \frac{1}{2\kappa}\int f(\mathcal{G})\sqrt{-g}d^4x \tag{1.48}$$

This yields the field equations [17]

Theorem 1.4 ($f(\mathcal{G})$ field equations)

$$f'(\mathcal{G})\left(G_{\rho\sigma} + \frac{1}{2}g_{\rho\sigma}\mathcal{G}\right) + \frac{1}{2}f''(\mathcal{G})E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - \frac{1}{2}g_{\rho\sigma}f(\mathcal{G}) = \kappa T_{\rho\sigma} \tag{1.49}$$

where $E_{\rho\sigma}{}^{\gamma}$ is a term we will call the *pseudoscalar connection*. This is a connection term made up of three index-permuted pseudoscalar deviations [17]:

Definition 1.7 (Pseudoscalar connection)

$$\begin{aligned}E^{\mu\nu\lambda} := M^{\{\lambda\mu\nu\}} =& M^{\lambda\mu\nu} + M^{\nu\lambda\mu} - M^{\mu\nu\lambda} \\ =& 2g^{\rho\mu}g^{\nu\sigma}\Gamma^{\lambda}_{\rho\sigma} - 2g^{\lambda(\mu}g^{\nu)\sigma}\Gamma^{\rho}_{\rho\sigma} + g^{\mu\nu}g^{\lambda\rho}\Gamma^{\sigma}_{\sigma\rho} - g^{\mu\nu}g^{\rho\sigma}\Gamma^{\lambda}_{\rho\sigma}\end{aligned} \tag{1.50}$$

Chapter 2

Linearisation of $f(\mathcal{G})$ gravity

> **Quote 2.1** Sure, the abyss is great for staring into. But if screaming is your thing, you'll want to go with the void.
>
> *Illustration from the website of Christian G. Böhmer*

2.1 Bulk term

As the higher orders of $h_{\mu\nu}$ vanish, we can safely replace $g_{\mu\nu}$ by $\eta_{\mu\nu}$ and employ the linearised Christoffels in (1.9):

$$\begin{aligned}\mathcal{G} =& \frac{1}{4}\eta^{\mu\nu}((h^{\lambda}_{\mu,\sigma} + h^{\lambda}_{\sigma,\mu} - h_{\mu\sigma}{}^{,\lambda})(h^{\sigma}_{\lambda,\nu} + h^{\sigma}_{\nu,\lambda} - h_{\lambda\nu}{}^{,\sigma}) \\ &- (h^{\sigma}_{\mu,\nu} + h^{\sigma}_{\nu,\mu} - h_{\mu\nu}{}^{,\sigma})(h^{\lambda}_{\lambda,\sigma} + h^{\lambda}_{\sigma,\lambda} - h_{\lambda\sigma}{}^{,\lambda})) + O(h^3)\end{aligned} \tag{2.1}$$

where O is the so-called *big-O notation*.

Even though the lowest order terms are quadratic in $h_{\mu\nu}$, $\mathcal{G}$ corresponds to the part of the Ricci scalar R that contributes to the action, and its vanishing would be clearly unphysical. This is also justified by the fact that most action in physics are at least to the quadratic order.

Another justification to preserve the second-order terms comes from (1.49): if the trial function $f(\mathcal{G})$ includes terms of $\mathcal{G}^{1/2}$, $\mathcal{G}^{-1/2}$ and so on, we might end up with an expression involving square roots of $\mathcal{G}$, and with that, linear orders of $h_{\mu\nu}$.

$$\begin{aligned}\mathcal{G} = \frac{1}{4}\eta^{\mu\nu}(& h^{\lambda}_{\mu,\sigma}h^{\sigma}_{\lambda,\nu} + h^{\lambda}_{\mu,\sigma}h^{\sigma}_{\nu,\lambda} - h^{\lambda}_{\mu,\sigma}h_{\lambda\nu}{}^{,\sigma} + h^{\lambda}_{\sigma,\mu}h^{\sigma}_{\lambda,\nu} + h^{\lambda}_{\sigma,\mu}h^{\sigma}_{\nu,\lambda} - h^{\lambda}_{\sigma,\mu}h_{\lambda\nu}{}^{,\sigma} \\ & -h_{\mu\sigma}{}^{,\lambda}h^{\sigma}_{\lambda,\nu} - h_{\mu\sigma}{}^{,\lambda}h^{\sigma}_{\nu,\lambda} + h_{\mu\sigma}{}^{,\lambda}h_{\lambda\nu}{}^{,\sigma} - h^{\sigma}_{\mu,\nu}h^{\lambda}_{\lambda,\sigma} - h^{\sigma}_{\mu,\nu}h^{\lambda}_{\sigma,\lambda} + h^{\sigma}_{\mu,\nu}h_{\lambda\sigma}{}^{,\lambda} \\ & -h^{\sigma}_{\nu,\mu}h^{\lambda}_{\lambda,\sigma} - h^{\sigma}_{\nu,\mu}h^{\lambda}_{\sigma,\lambda} + h^{\sigma}_{\nu,\mu}h_{\lambda\sigma}{}^{,\lambda} + h_{\mu\nu}{}^{,\sigma}h^{\lambda}_{\lambda,\sigma} + h_{\mu\nu}{}^{,\sigma}h^{\lambda}_{\sigma,\lambda} - h_{\mu\nu}{}^{,\sigma}h_{\lambda\sigma}{}^{,\lambda}) + O(h^3)\end{aligned} \tag{2.2}$$

Some indices are contracted:

$$\begin{aligned}\mathcal{G} = \frac{1}{4}\eta^{\mu\nu}(& h^{\lambda}_{\mu,\sigma}h^{\sigma}_{\lambda,\nu} + h^{\lambda}_{\mu,\sigma}h^{\sigma}_{\nu,\lambda} - h^{\lambda}_{\mu,\sigma}h_{\lambda\nu}{}^{,\sigma} + h_{,\mu}h_{,\nu} + h^{\lambda}_{\sigma,\mu}h^{\sigma}_{\nu,\lambda} - h^{\lambda}_{\sigma,\mu}h_{\lambda\nu}{}^{,\sigma} \\ & -h_{\mu\sigma}{}^{,\lambda}h^{\sigma}_{\lambda,\nu} - h_{\mu\sigma}{}^{,\lambda}h^{\sigma}_{\nu,\lambda} + h_{\mu\sigma}{}^{,\lambda}h_{\lambda\nu}{}^{,\sigma} - h^{\sigma}_{\mu,\nu}h_{,\sigma} - h^{\sigma}_{\mu,\nu}h^{\lambda}_{\sigma,\lambda} + h^{\sigma}_{\mu,\nu}h_{\lambda\sigma}{}^{,\lambda} \\ & -h^{\sigma}_{\nu,\mu}h_{,\sigma} - h^{\sigma}_{\nu,\mu}h^{\lambda}_{\sigma,\lambda} + h^{\sigma}_{\nu,\mu}h_{\lambda\sigma}{}^{,\lambda} + h_{\mu\nu}{}^{,\sigma}h_{,\sigma} + h_{\mu\nu}{}^{,\sigma}h^{\lambda}_{\sigma,\lambda} - h_{\mu\nu}{}^{,\sigma}h_{\lambda\sigma}{}^{,\lambda}) + O(h^3)\end{aligned} \tag{2.3}$$

We apply the Minkowski metric, which raises some indices:

$$\begin{aligned}\mathcal{G} = \frac{1}{4}(& h^{\lambda\nu}{}_{,\sigma}h^{\sigma}_{\lambda,\nu} + h^{\lambda\nu}{}_{,\sigma}h^{\sigma}_{\nu,\lambda} - h^{\lambda\nu}{}_{,\sigma}h_{\lambda\nu}{}^{,\sigma} + h^{,\nu}h_{,\nu} + h^{\lambda,\nu}_{\sigma}h^{\sigma}_{\nu,\lambda} - h^{\lambda,\nu}_{\sigma}h_{\lambda\nu}{}^{,\sigma} \\ & -h^{\nu,\lambda}_{\sigma}h^{\sigma}_{\lambda,\nu} - h^{\nu,\lambda}_{\sigma}h^{\sigma}_{\nu,\lambda} + h^{\nu,\lambda}_{\sigma}h_{\lambda\nu}{}^{,\sigma} - h^{\sigma,\mu}_{\mu}h_{,\sigma} - h^{\sigma,\nu}_{\mu}h^{\lambda}_{\sigma,\lambda} + h^{\sigma,\mu}_{\mu}h_{\lambda\sigma}{}^{,\lambda} \\ & -h^{\sigma\mu}{}_{,\mu}h_{,\sigma} - h^{\sigma\mu}{}_{,\mu}h^{\lambda}_{\sigma,\lambda} + h^{\sigma\mu}{}_{,\mu}h_{\lambda\sigma}{}^{,\lambda} + h^{,\sigma}h_{,\sigma} + h^{,\sigma}h^{\lambda}_{\sigma,\lambda} - h^{,\sigma}h_{\lambda\sigma}{}^{,\lambda}) + O(h^3)\end{aligned} \tag{2.4}$$

We then note that $h_{\mu\nu}$ is symmetric. As all indices are free, and that

$$h^{\mu\nu}{}_{,\mu} = h^{\nu,\mu}_{\mu} \tag{2.5}$$

we can proceed to simplify $\mathcal{G}$ as

Linearised result 1 (Bulk term)

$$\mathcal{G} = \frac{1}{2}(h^{\lambda\nu}{}_{,\sigma}h^{\sigma}_{\lambda,\nu} - h^{\sigma\mu}{}_{,\mu}h_{,\sigma}) + O(h^3) \tag{2.6}$$

2.2 Recovery of the tensor field Lagrangian

The action of a rank-2 free tensor field is well known as [28]

Definition 2.1 (Tensor field action)

$$S = \int \left(\frac{1}{2}h^{\mu\nu,\sigma}h_{\mu\nu,\sigma} - h^{\mu\nu}{}_{,\nu}h^{\sigma}_{\mu,\sigma} + 1h^{\mu\nu}{}_{,\nu}h^{\sigma}_{\sigma,\mu} - \frac{1}{2}h^{\nu}_{\nu,\mu}h^{\sigma,\mu}_{\sigma} - \lambda T^{\mu\nu}h_{\mu\nu}\right) d\tau \tag{2.7}$$

This corresponds to the Lagrangian

$$L = \frac{1}{2}h^{\mu\nu,\sigma}h_{\mu\nu,\sigma} - h^{\mu\nu}{}_{,\nu}h^{\sigma}_{\mu,\sigma} + h^{\mu\nu}{}_{,\nu}h^{\sigma}_{\sigma,\mu} - \frac{1}{2}h^{\nu}_{\nu,\mu}h^{\sigma,\mu}_{\sigma} - \lambda T^{\mu\nu}h_{\mu\nu} \tag{2.8}$$

As we are investigating the vacuum field equations, the stress-energy term vanishes:

$$L = \frac{1}{2}h^{\mu\nu,\sigma}h_{\mu\nu,\sigma} - h^{\mu\nu}{}_{,\nu}h^{\sigma}_{\mu,\sigma} + h^{\mu\nu}{}_{,\nu}h^{\sigma}_{\sigma,\mu} - \frac{1}{2}h^{\nu}_{\nu,\mu}h^{\sigma,\mu}_{\sigma} \tag{2.9}$$

Cancelling out indices yields

$$L = \frac{1}{2}h^{,\mu}h_{,\mu} - h^{\mu\nu}{}_{,\nu}h^{\sigma}_{\mu,\sigma} + h^{\mu\nu}{}_{,\nu}h_{,\mu} - \frac{1}{2}h_{,\mu}h^{,\mu} \tag{2.10}$$

Noting that the first and last terms cancel out, we have

$$L = \frac{1}{2}h^{,\mu}h_{,\mu} - h^{\mu\nu}{}_{,\nu}h^{\sigma}_{\mu,\sigma} + h^{\mu\nu}{}_{,\nu}h_{,\mu} - \frac{1}{2}h_{,\mu}h^{,\mu} = h^{\lambda\nu}{}_{,\sigma}h^{\sigma}_{\lambda,\nu} - h^{\sigma\mu}{}_{,\mu}h_{,\sigma} \tag{2.11}$$

which, amazingly, is identical to the linearised bulk term $\mathcal{G}$.

From classical mechanics, it is understood that the boundary term does not contribute to the action. Once boundary conditions have been established, boundary terms can be added or subtracted from the Lagrangian without loss of generality, in that the resultant equations of motion are not altered [29]. This justifies our choice of $f(\mathcal{G})$ over $f(\mathcal{G},\mathcal{B})$ or $f(R)$: Discarding the boundary term leaves us with $\mathcal{G}$, the 'physically meaningful' part of R which happens to be the Lagrangian generated from a rank-2 tensor field, without affecting the equations of motion (i.e. the field equations).

In GR, this tensor field is well known as the graviton field, which is the metric perturbation $h_{\mu\nu}$ [30]. Theoretically, the graviton is ultimately sourced from the rank-2 stress-energy tensor, which makes it a so-called *tensor boson* [31]. Since the formulation of the ADM formalism in the 1960s [32], there have been considerable efforts to develop a spin-2 field theory of gravitation. However, attempts so far have consistently failed for one reason. Under a quantised treatment, GR has a superficial degree of divergence of

$$D = 2L + 2 \tag{2.12}$$

where L is the number of graviton loops. At $1, 2, 3, \cdots$ loops, we have $D = 4, 6, 8\cdots$. That is to say, at each higher order, we require more counterterms, which are ultimately infinite. Thus, a quantum formulation of GR, or quantum gravity, is non-renormalisable.

2.3 Boundary term

Performing the same substitutions as $\mathcal{G}$ yield

$$\mathcal{B} = \frac{1}{2}\nabla_\sigma \left(\eta^{\mu\nu}(h^\sigma_{\mu,\nu} + h^\sigma_{\nu,\mu} - h_{\mu\nu}{}^{,\sigma}) - \eta^{\sigma\nu}(h^\lambda_{\lambda,\nu} + h^\lambda_{\nu,\lambda} - h_{\lambda\nu}{}^{,\lambda})\right) \tag{2.13}$$

With respect to the covariant derivative, we note that:

- The covariant derivative of the Minkowski metric yields zero.
- The covariant derivative of an object in Minkowski space simply reduces to the partial derivative.

Using the chain rule:

$$\mathcal{B} = \frac{1}{2}\eta^{\mu\nu}\nabla_\sigma(h^\sigma_{\mu,\nu} + h^\sigma_{\nu,\mu} - h_{\mu\nu}{}^{,\sigma}) - \frac{1}{2}\eta^{\sigma\nu}\nabla_\sigma(h^\lambda_{\lambda,\nu} + h^\lambda_{\nu,\lambda} - h_{\lambda\nu}{}^{,\lambda}) \tag{2.14}$$

In flat space, the covariant derivative reduces to an ordinary partial derivative

$$\mathcal{B} = \frac{1}{2}\eta^{\mu\nu}\partial_\sigma(h^\sigma_{\mu,\nu} + h^\sigma_{\nu,\mu} - h_{\mu\nu}{}^{,\sigma}) - \frac{1}{2}\eta^{\sigma\nu}\partial_\sigma(h^\lambda_{\lambda,\nu} + h^\lambda_{\nu,\lambda} - h_{\lambda\nu}{}^{,\lambda}) \tag{2.15}$$

We then differentiate the perturbation terms

$$\mathcal{B} = \frac{1}{2}\eta^{\mu\nu}(h^\sigma_{\mu,\nu\sigma} + h^\sigma_{\nu,\mu\sigma} - h_{\mu\nu}{}^{,\sigma}_{,\sigma}) - \frac{1}{2}\eta^{\sigma\nu}(h^\lambda_{\lambda,\nu\sigma} + h^\lambda_{\nu,\lambda\sigma} - h_{\lambda\nu}{}^{,\lambda}_{,\sigma}) \tag{2.16}$$

Using the metric to shift indices, we find

$$\mathcal{B} = \frac{1}{2}(2h^{\mu\sigma}{}_{,\mu\sigma} - h^{,\sigma}_{,\sigma} - h^{\sigma\mu}{}_{,\mu\sigma} - h^{\lambda\sigma}{}_{,\lambda\sigma} + h^{\sigma,\lambda}_{\lambda,\sigma}) \tag{2.17}$$

Finally, we again note that $h_{\mu\nu}$ is symmetric. As all indices are free, one can further simplify this as

Linearised result 2 (Boundary term)

$$\mathcal{B} = \frac{1}{2}(h^{\mu,\nu}_{\nu,\mu} - h^{,\mu}_{,\mu}) \tag{2.18}$$

2.4 Pseudoscalar deviation

Starting with (2.19), the expression for the pseudoscalar deviation, one expands the symmetrisation terms and finds:

$$M^{\mu\nu}_\lambda = g^{\rho\nu}\Gamma^\mu_{\lambda\rho} + g^{\rho\mu}\Gamma^\nu_{\lambda\rho} - g^{\mu\nu}\Gamma^\rho_{\rho\lambda} - \frac{1}{2}g^{\rho\sigma}\delta^\nu_\lambda\Gamma^\mu_{\rho\sigma} - \frac{1}{2}g^{\rho\sigma}\delta^\mu_\lambda\Gamma^\nu_{\rho\sigma} \tag{2.19}$$

Note that most indices are no longer free due to $M^{\mu\nu}_\lambda$ also having 3 indices. The only free indices are ρ and σ, and the other indices should be treated carefully, especially with respect to the Kronecker deltas.

We recall the expressions for the linearised inverse metric and the Christoffels

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \Gamma^\mu_{\alpha\beta} = \frac{1}{2}(h^\mu_{\alpha,\beta} + h^\mu_{\beta,\alpha} - h_{\alpha\beta}{}^{,\mu})$$

Inserting the two expressions, we can expand all five terms on the RHS of (2.19), here labelled with circled numbers:

$$\begin{aligned}
①^{\mu\nu}_\lambda &= \frac{1}{2}(\eta^{\rho\nu} - h^{\rho\nu})(h^\mu_{\lambda,\rho} + h^\mu_{\rho,\lambda} - h_{\lambda\rho}{}^{,\mu}) \\
②^{\mu\nu}_\lambda &= \frac{1}{2}(\eta^{\rho\mu} - h^{\rho\mu})(h^\nu_{\lambda,\rho} + h^\nu_{\rho,\lambda} - h_{\lambda\rho}{}^{,\nu}) \\
③^{\mu\nu}_\lambda &= -\frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})(h^\rho_{\rho,\lambda} + h^\rho_{\lambda,\rho} - h_{\rho\lambda}{}^{,\rho}) \\
④^{\mu\nu}_\lambda &= -\frac{1}{4}\delta^\nu_\lambda(\eta^{\rho\sigma} - h^{\rho\sigma})(h^\mu_{\rho,\sigma} + h^\mu_{\sigma,\rho} - h_{\rho\sigma}{}^{,\mu}) \\
⑤^{\mu\nu}_\lambda &= -\frac{1}{4}\delta^\mu_\lambda(\eta^{\rho\sigma} - h^{\rho\sigma})(h^\nu_{\rho,\sigma} + h^\nu_{\sigma,\rho} - h_{\rho\sigma}{}^{,\nu})
\end{aligned} \tag{2.20}$$

Expanding the expressions yield

$$\begin{aligned}
①^{\mu\nu}_{\lambda} &= \frac{1}{2}(h^{\mu}_{\lambda}{}^{,\nu} + h^{\mu\nu}{}_{,\lambda} - h^{\nu}_{\lambda}{}^{,\mu}) + O(h^2) \\
②^{\mu\nu}_{\lambda} &= \frac{1}{2}(h^{\nu}_{\lambda}{}^{,\mu} + h^{\nu\mu}{}_{,\lambda} - h^{\mu}_{\lambda}{}^{,\nu}) + O(h^2) \\
③^{\mu\nu}_{\lambda} &= \frac{1}{2}(-\eta^{\mu\nu}h_{,\lambda} - \eta^{\mu\nu}h^{\rho}_{\lambda,\rho} + \eta^{\mu\nu}h_{\rho\lambda}{}^{,\rho}) + O(h^2) \\
④^{\mu\nu}_{\lambda} &= \frac{1}{4}\delta^{\nu}_{\lambda}(-h^{\mu}_{\rho}{}^{,\rho} - h^{\mu}_{\sigma}{}^{,\sigma} + h^{,\mu}) + O(h^2) \\
⑤^{\mu\nu}_{\lambda} &= \frac{1}{4}\delta^{\mu}_{\lambda}(-h^{\nu}_{\rho}{}^{,\rho} - h^{\nu}_{\sigma}{}^{,\sigma} + h^{,\nu}) + O(h^2)
\end{aligned} \tag{2.21}$$

From symmetries and (2.5), $①^{\mu\nu}_{\lambda}$, $②^{\mu\nu}_{\lambda}$ and $③^{\mu\nu}_{\lambda}$ can be simplified as

$$(① + ② + ③)^{\mu\nu}_{\lambda} = h^{\mu\nu}{}_{,\lambda} - \frac{1}{2}\eta^{\mu\nu}h_{,\lambda} + O(h^2) \tag{2.22}$$

$④^{\mu\nu}_{\lambda}$ and $⑤^{\mu\nu}_{\lambda}$ are slightly tricky. To begin with, we recognise that there is no term where the free indices ρ and σ coexist. Hence, there is nothing stopping us from labelling σ as ρ. This gives

$$\begin{aligned}
④^{\mu\nu}_{\lambda} &= \frac{1}{4}\delta^{\nu}_{\lambda}(-2h^{\mu}_{\rho}{}^{,\rho} + h^{,\mu}) + O(h^2) \\
⑤^{\mu\nu}_{\lambda} &= \frac{1}{4}\delta^{\mu}_{\lambda}(-2h^{\nu}_{\rho}{}^{,\rho} + h^{,\nu}) + O(h^2)
\end{aligned} \tag{2.23}$$

Note that so far we have not eliminated the 2^{nd}-order terms outright even though there exist 1^{st}-order terms. This is not without good reason. In most cases we investigate, $f(\mathcal{G})$ will be a polynomial of $\mathcal{G}$ for simplicity, among other reasons. Now suppose we have, in $f(\mathcal{G})$, negative orders of $\mathcal{G}$. This might reduce any 2^{nd}-order or 3^{rd}-order terms down to 0^{th}- or linear order terms.

We are then in a position to write out the entire expression for the linearised $M^{\mu\nu}_{\lambda}$:

Linearised result 3 (Pseudoscalar deviation)

$$M^{\mu\nu}_{\lambda} = h^{\mu\nu}{}_{,\lambda} - \frac{1}{2}\eta^{\mu\nu}h_{,\lambda} + \frac{1}{4}\delta^{\nu}_{\lambda}(h^{,\mu} - 2h^{\mu}_{\rho}{}^{,\rho}) + \frac{1}{4}\delta^{\mu}_{\lambda}(h^{,\nu} - 2h^{\nu}_{\rho}{}^{,\rho}) + O(h^2) \tag{2.24}$$

2.5 Pseudoscalar connection

Again, starting from the original expression for the term in (1.50), one can expand the symmetrisation and find

$$E^{\mu\nu\lambda} = 2g^{\rho\mu}g^{\nu\sigma}\Gamma^{\lambda}_{\rho\sigma} - g^{\lambda\mu}g^{\nu\sigma}\Gamma^{\rho}_{\rho\sigma} - g^{\lambda\nu}g^{\mu\sigma}\Gamma^{\rho}_{\rho\sigma} + g^{\mu\nu}g^{\lambda\rho}\Gamma^{\sigma}_{\sigma\rho} - g^{\mu\nu}g^{\rho\sigma}\Gamma^{\lambda}_{\rho\sigma} \tag{2.25}$$

For clarity, we dedicate three subsections for the five terms, again recalling the linearised inverse metric and Christoffels:

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \quad \Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}(h^{\mu}_{\alpha,\beta} + h^{\mu}_{\beta,\alpha} - h_{\alpha\beta}{}^{,\mu})$$

We label the 1^{st} term $2g^{\rho\mu}g^{\nu\sigma}\Gamma^{\lambda}_{\rho\sigma}$ as $①^{\mu\nu\lambda}$. Writing out the term in full gives

$$①^{\mu\nu\lambda} = (\eta^{\rho\mu} - h^{\rho\mu})(\eta^{\nu\sigma} - h^{\nu\sigma})(h^{\lambda}_{\rho,\sigma} + h^{\lambda}_{\sigma,\rho} - h_{\rho\sigma}{}^{,\lambda}) \tag{2.26}$$

Expanding yields

$$①^{\mu\nu\lambda} = (\eta^{\rho\mu} - h^{\rho\mu})(h^{\lambda}_{\rho}{}^{,\nu} + h^{\lambda\nu}{}_{,\rho} - h^{\nu}_{\rho}{}^{,\lambda}) + O(h^2) \tag{2.27}$$

One can further expand the expression as

$$①^{\mu\nu\lambda} = h^{\lambda\mu,\nu} + h^{\lambda\nu,\mu} - h^{\nu\mu,\lambda} + O(h^2) \tag{2.28}$$

We label the 2^{nd} term $-g^{\lambda\mu}g^{\nu\sigma}\Gamma^{\rho}_{\rho\sigma}$ as $②^{\mu\nu\lambda}$. Writing out the term in full gives

$$②^{\mu\nu\lambda} = -\frac{1}{2}(\eta^{\lambda\mu} - h^{\lambda\mu})(\eta^{\nu\sigma} - h^{\nu\sigma})(h^{\rho}_{\rho,\sigma} + h^{\rho}_{\sigma,\rho} - h_{\rho\sigma}{}^{,\rho}) \tag{2.29}$$

Expanding yields

$$②^{\mu\nu\lambda} = -\frac{1}{2}(\eta^{\lambda\mu} - h^{\lambda\mu})(h^{,\nu} + h^{\rho\nu}{}_{,\rho} - h_\rho^{\nu,\rho}) + O(h^2) \tag{2.30}$$

From symmetries and (2.5), the last two terms cancel out:

$$②^{\mu\nu\lambda} = -\frac{1}{2}(\eta^{\lambda\mu} - h^{\lambda\mu})h^{,\nu} + O(h^2) \tag{2.31}$$

One can further expand the expression as

$$②^{\mu\nu\lambda} = -\frac{1}{2}\eta^{\lambda\mu}h^{,\nu} + O(h^2) \tag{2.32}$$

Now we label the 3rd term $2g^{\lambda\nu}g^{\mu\sigma}\Gamma^\rho_{\rho\sigma}$ as $③^{\mu\nu\lambda}$. To our great relief, $③^{\mu\nu\lambda}$ is simply $②^{\mu\nu\lambda}$ with the positions of the non-free indices μ and ν switched. Hence, we can immediately write it as

$$③^{\mu\nu\lambda} = -\frac{1}{2}\eta^{\lambda\nu}h^{,\mu} + O(h^2) \tag{2.33}$$

We then label the 4th term $g^{\mu\nu}g^{\lambda\rho}\Gamma^\sigma_{\sigma\rho}$ as $④^{\mu\nu\lambda}$. Again, miraculously, we find that it is $④^{\mu\nu\lambda}$ is simply $②^{\mu\nu\lambda}$ with an inverse sign the positions of the free and non-free indices exchanged among themselves. Hence, we can immediately write it as

$$④^{\mu\nu\lambda} = \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} + O(h^2) \tag{2.34}$$

Combining these terms, we have

$$(②^{\mu\nu\lambda} + ③^{\mu\nu\lambda} + ④^{\mu\nu\lambda}) = -\frac{1}{2}\eta^{\lambda\mu}h^{,\nu} - \frac{1}{2}\eta^{\lambda\nu}h^{,\mu} + \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} + O(h^2) \tag{2.35}$$

We label the 5th term $g^{\mu\nu}g^{\rho\sigma}\Gamma^\lambda_{\rho\sigma}$ as $⑤^{\mu\nu\lambda}$. Writing out the term in full gives

$$⑤^{\mu\nu\lambda} = -\frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})(\eta^{\rho\sigma} - h^{\rho\sigma})(h^\lambda_{\rho,\sigma} + h^\lambda_{\sigma,\rho} - h_{\rho\sigma}{}^{,\lambda}) \tag{2.36}$$

Expanding yields

$$⑤^{\mu\nu\lambda} = -\frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})(h_\rho^{\lambda,\rho} + h_\sigma^{\lambda,\sigma} - h^{,\lambda}) + O(h^2) \tag{2.37}$$

Again, as ρ and σ are free indices, we can convert ρ to σ for terms in which ρ appears alone or swap ρ and σ in terms where they appear together. Then, recognising the symmetry of $h_{\mu\nu}$, we have

$$⑤^{\mu\nu\lambda} = -\frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})(2h_\rho^{\lambda,\rho} - h^{,\lambda}) \tag{2.38}$$

One can further expand the expression as

$$⑤^{\mu\nu\lambda} = -\eta^{\mu\nu}h_\rho^{\lambda,\rho} + \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} + O(h^2) \tag{2.39}$$

Finally, combining the five terms yields

$$E^{\mu\nu\lambda} = h^{\lambda\mu,\nu} + h^{\lambda\nu,\mu} - h^{\nu\mu,\lambda} - \frac{1}{2}\eta^{\lambda\mu}h^{,\nu} - \frac{1}{2}\eta^{\lambda\nu}h^{,\mu} + \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} - \eta^{\mu\nu}h_\rho^{\lambda,\rho} + \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} + O(h^2) \tag{2.40}$$

which reduces slightly to

Linearised result 4 (Pseudoscalar connection)

$$E^{\mu\nu\lambda} = h^{\lambda\mu,\nu} + h^{\lambda\nu,\mu} - h^{\nu\mu,\lambda} - \frac{1}{2}\eta^{\lambda\mu}h^{,\nu} - \frac{1}{2}\eta^{\lambda\nu}h^{,\mu} + \eta^{\mu\nu}h^{,\lambda} - \eta^{\mu\nu}h_\rho^{\lambda,\rho} + O(h^2) \tag{2.41}$$

2.6 Recovery via the pseudoscalar deviation

By relabelling one index in the expression of Linearised result 3, one finds

$$M^{\mu\nu}_{\sigma} = h^{\mu\nu}{}_{,\sigma} - \frac{1}{2}\eta^{\mu\nu}h_{,\sigma} + \frac{1}{4}\delta^{\nu}_{\sigma}(h^{,\mu} - 2h^{\mu}_{\rho}{}^{,\rho}) + \frac{1}{4}\delta^{\mu}_{\sigma}(h^{,\nu} - 2h^{\nu}_{\rho}{}^{,\rho}) + O(h^2) \tag{2.42}$$

We can now solve for the entirely contravariant version of $M^{\mu\nu}_{\sigma}$ by calculating

$$M^{\lambda\mu\nu} = g^{\lambda\sigma}M^{\mu\nu}_{\sigma} = (\eta^{\lambda\sigma} + h^{\lambda\sigma})\left(h^{\mu\nu}{}_{,\sigma} - \frac{1}{2}\eta^{\mu\nu}h_{,\sigma} + \frac{1}{4}\delta^{\nu}_{\sigma}(h^{,\mu} - 2h^{\mu}_{\rho}{}^{,\rho}) + \frac{1}{4}\delta^{\mu}_{\sigma}(h^{,\nu} - 2h^{\nu}_{\rho}{}^{,\rho})\right) + O(h^2) \tag{2.43}$$

Expansion then yields

$$M^{\lambda\mu\nu} = h^{\mu\nu,\lambda} - \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} + \frac{1}{4}\eta^{\nu\lambda}\left(h^{,\mu} - 2h^{\mu}_{\rho}{}^{,\rho}\right) + \frac{1}{4}\eta^{\mu\lambda}\left(h^{,\nu} - 2h^{\nu}_{\rho}{}^{,\rho}\right) + O(h^2) \tag{2.44}$$

Recalling (1.50), we permute the indices thrice and solve for

$$E^{\mu\nu\lambda} = M^{\lambda\mu\nu} + M^{\nu\lambda\mu} - M^{\mu\nu\lambda} \tag{2.45}$$

which gives

$$\begin{aligned} E^{\mu\nu\lambda} =& h^{\mu\nu,\lambda} - \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} + \frac{1}{4}\eta^{\nu\lambda}\left(h^{,\mu} - 2h^{\mu}_{\rho}{}^{,\rho}\right) + \frac{1}{4}\eta^{\mu\lambda}\left(h^{,\nu} - 2h^{\nu}_{\rho}{}^{,\rho}\right) + \\ & h^{\lambda\mu,\nu} - \frac{1}{2}\eta^{\lambda\mu}h^{,\nu} + \frac{1}{4}\eta^{\mu\nu}\left(h^{,\lambda} - 2h^{\lambda}_{\rho}{}^{,\rho}\right) + \frac{1}{4}\eta^{\lambda\nu}\left(h^{,\mu} - 2h^{\mu}_{\rho}{}^{,\rho}\right) - \\ & h^{\nu\lambda,\mu} + \frac{1}{2}\eta^{\nu\lambda}h^{,\mu} - \frac{1}{4}\eta^{\lambda\mu}\left(h^{,\nu} - 2h^{\nu}_{\rho}{}^{,\rho}\right) - \frac{1}{4}\eta^{\nu\mu}\left(h^{,\lambda} - 2h^{\lambda}_{\rho}{}^{,\rho}\right) + O(h^2) \end{aligned} \tag{2.46}$$

Cancelling out terms, and we have

$$E^{\mu\nu\lambda} = h^{\mu\nu,\lambda} + h^{\lambda\mu,\nu} - h^{\nu\lambda,\mu} - \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} - \frac{1}{2}\eta^{\lambda\mu}h^{,\nu} + \frac{1}{2}\eta^{\nu\lambda}h^{,\mu} + \frac{1}{2}\eta^{\nu\lambda}h^{,\mu} - \eta^{\nu\lambda}h^{\mu}_{\rho}{}^{,\rho} + O(h^2) \tag{2.47}$$

Simplifying even further, we find

$$E^{\mu\nu\lambda} = h^{\mu\nu,\lambda} + h^{\lambda\mu,\nu} - h^{\nu\lambda,\mu} - \frac{1}{2}\eta^{\mu\nu}h^{,\lambda} - \frac{1}{2}\eta^{\lambda\mu}h^{,\nu} + \eta^{\nu\lambda}h^{,\mu} - \eta^{\nu\lambda}h^{\mu}_{\rho}{}^{,\rho} + O(h^2) \tag{2.48}$$

which is identical to Linearised result 3.

Finally, to convert $E^{\mu\nu\lambda}$ to its form seen in (1.49), we need to attack two of its indices by moving them down:

$$\begin{aligned} E_{\rho\sigma}{}^{\gamma} =& g_{\rho\mu}g_{\sigma\nu}E^{\mu\nu\gamma} = \eta_{\rho\mu}\eta_{\sigma\nu}E^{\mu\nu\gamma} + O(h^2) \\ =& \eta_{\rho\mu}\eta_{\sigma\nu}\left(h^{\mu\nu,\gamma} + h^{\gamma\mu,\nu} - h^{\nu\gamma,\mu} - \frac{1}{2}\eta^{\mu\nu}h^{,\gamma} - \frac{1}{2}\eta^{\gamma\mu}h^{,\nu} + \eta^{\nu\gamma}h^{,\mu} - \eta^{\nu\gamma}h^{\mu}_{\alpha}{}^{,\alpha}\right) + O(h^2) \\ =& h_{\rho\sigma}{}^{,\gamma} + h^{\gamma}_{\rho,\sigma} - h^{\gamma}_{\sigma,\rho} - \frac{1}{2}\eta_{\rho\sigma}h^{,\gamma} - \frac{1}{2}\eta^{\gamma}_{\rho}h_{,\sigma} + \eta^{\gamma}_{\sigma}h_{,\rho} - \eta^{\gamma}_{\sigma}h_{\alpha\rho}{}^{,\alpha} + O(h^2) \end{aligned} \tag{2.49}$$

At this point, we have linearised all the objects appearing in (1.49). For ease of reading, we display all of them below, with the big-O symbols omitted:

$$\begin{aligned} g_{\rho\sigma} =& \eta_{\rho\sigma} + h_{\rho\sigma} \\ G_{\rho\sigma} =& \frac{1}{2}\left(h_{\rho\sigma}{}_{,\alpha}^{,\alpha} - h_{\sigma\alpha}{}_{,\rho}^{,\alpha} - h_{\rho\alpha}{}_{,\sigma}^{,\alpha} + h_{,\rho\sigma}\right) - \frac{1}{2}\eta_{\rho\sigma}\left(h_{\alpha\beta}{}^{,\alpha\beta} - h_{,\alpha}^{,\alpha}\right) \\ =& \frac{1}{2}\left(h_{\rho\sigma}{}_{,\alpha}^{,\alpha} - h_{\sigma\alpha}{}_{,\rho}^{,\alpha} - h_{\rho\alpha}{}_{,\sigma}^{,\alpha} + h_{,\rho\sigma} - \eta_{\rho\sigma}h_{\alpha\beta}{}^{,\alpha\beta} + \eta_{\rho\sigma}h_{,\alpha}^{,\alpha}\right) \\ \mathcal{G} =& h^{\mu\nu}{}_{,\alpha}h^{\alpha}_{\nu,\mu} - h^{\mu,\beta}_{\beta}h_{,\beta} \\ E_{\rho\sigma}{}^{\gamma} =& h_{\rho\sigma}{}^{,\gamma} + h^{\gamma}_{\rho,\sigma} - h^{\gamma}_{\sigma,\rho} - \frac{1}{2}\eta_{\rho\sigma}h^{,\gamma} - \frac{1}{2}\eta^{\gamma}_{\rho}h_{,\sigma} + \eta^{\gamma}_{\sigma}h_{,\rho} - \eta^{\gamma}_{\sigma}h_{\alpha\rho}{}^{,\alpha} + O(h^2) \end{aligned}$$

Chapter 3

Test $f(\mathcal{G})$ gravities

Quote 3.1 I can't think of a good motivation why this is a good idea.

Christian G. Böhmer, on the trace reverse, 23 November 2023

3.1 Dimensional constraints

We again consider the mathematical implications when solving the linearised $f(\mathcal{G})$ field equations: while the final result should consist of linear orders of $h_{\mu\nu}$ only. As it turns out, this is quite intuitive: As discussed previously, we have taken all second-order and higher terms of $h_{\mu\nu}$ to vanish, with the sole exception of $\mathcal{G}$ whose existence hinges solely on second-order terms.

In the last chapter, we have effectively derived a linearised version of $f(\mathcal{G})$ gravity. This puts us in a position to solve for the gravitational wave solutions for a given $f(\mathcal{G})$. Intuitively, inspired by simple test $f(R)$s in existing literature [33], a good toy model for $f(\mathcal{G})$ is an infinite series of polynomials, where each term has a different order.

$$f(\mathcal{G}) = \cdots + c_{-2}\mathcal{G}^{-2} + c_{-1}\mathcal{G}^{-1} + c_0 + c_1\mathcal{G} + c_2\mathcal{G}^2 + \cdots \tag{3.1}$$

While simple, this candidate has one problem. We investigate the first term of (1.49), which can be expanded as

$$f'(\mathcal{G})\left(G_{\rho\sigma} + \frac{1}{2}g_{\rho\sigma}\mathcal{G}\right) = f'(\mathcal{G})G_{\rho\sigma} + \frac{1}{2}f'(\mathcal{G})(\eta_{\rho\sigma} + h_{\rho\sigma})\mathcal{G} \tag{3.2}$$

The first derivative of a purely polynomial $f(\mathcal{G})$ will not include a term of order $\mathcal{G}_{-1}$. To make sure that a $\mathcal{G}_{-1}$ term appears in $f'(\mathcal{G})$, we add a term of $\log|\mathcal{G}|$ in $f(\mathcal{G})$. We will call this resultant $f(\mathcal{G})$ a *near-polynomial*. In this thesis, we will restrict our discussions to near-polynomials to keep calculations manageable.

3.2 Integer power near-polynomial

The simplest form of our near-polynomial is hence shown below:

Definition 3.1 (Integer power near-polynomial)

$$f(\mathcal{G}) = \cdots + c_{-2}\mathcal{G}^{-2} + c_{-1}\mathcal{G}^{-1} + \hat{c}\log|\mathcal{G}| + c_0 + c_1\mathcal{G} + c_2\mathcal{G}^2 + \cdots \tag{3.3}$$

where c_n and $\hat{c}$[a] are a series of constants.

[a] This constant is not distinct from the others. It has a hat merely because we have run out of subscripts to assign.

The first derivative of this test near-polynomial is

$$f'(\mathcal{G}) = \cdots - 2c_{-2}\mathcal{G}^{-3} - c_{-1}\mathcal{G}^{-2} + \hat{c}\mathcal{G}^{-1} + c_1 + 2c_2\mathcal{G} + \cdots \tag{3.4}$$

Here, differentiating $\log|\mathcal{G}|$ term in $f(\mathcal{G})$ has resulted in a $\mathcal{G}^{-1}$ term in $f'(\mathcal{G})$, and the significance of the $\log|\mathcal{G}|$ term is verified.

The second derivative is then

$$f''(\mathcal{G}) = \cdots + 6c_{-2}\mathcal{G}^{-4} + 2c_{-1}\mathcal{G}^{-3} - \hat{c}\mathcal{G}^{-2} + 2c_2 + 6c_3\mathcal{G} + \cdots \tag{3.5}$$

Now consider the fully expanded $f(\mathcal{G})$ field equations:

$$\underbrace{f'(\mathcal{G})G_{\rho\sigma}}_{①} + \underbrace{\frac{1}{2}f'(\mathcal{G})(\eta_{\rho\sigma} + h_{\rho\sigma})\mathcal{G}}_{②} + \underbrace{\frac{1}{2}f''(\mathcal{G})E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G}}_{③} - \underbrace{\frac{1}{2}(\eta_{\rho\sigma} + h_{\rho\sigma})f(\mathcal{G})}_{④} = \kappa T_{\rho\sigma}$$

We analyse the dimensionality of each term, noting that only terms of linear or inverse linear orders $h_{\mu\nu}$ should survive in the final expression, while all other terms perish:

- ①: $G_{\rho\sigma}$ is of linear order $h_{\mu\nu}$. As such, $f'(\mathcal{G})$ effectively reduces to $\hat{c}\mathcal{G}^{-1} + c_1$, and the term becomes

$$f'(\mathcal{G})G_{\rho\sigma} = (\hat{c}\mathcal{G}^{-1} + c_1)G_{\rho\sigma} \tag{3.6}$$

- ②: Due to the metric, $\mathcal{G}$ evolves into two groups of terms of second and third orders of $h_{\mu\nu}$ respectively. The second-order group can be reduced to zeroth order by $\mathcal{G}^{-1}$, while the third-order group can be either reduced to linear order by the $\mathcal{G}^{-1}$ term or inverse linear order by $\mathcal{G}^{-2}$. As such, the term becomes

$$\frac{1}{2}f'(\mathcal{G})(\eta_{\rho\sigma} + h_{\rho\sigma})\mathcal{G} = \frac{1}{2}\hat{c}\mathcal{G}^{-1}\eta_{\rho\sigma}\mathcal{G} + \frac{1}{2}(-c_{-1}\mathcal{G}^{-2} + \hat{c}\mathcal{G}^{-1})h_{\rho\sigma}\mathcal{G} \tag{3.7}$$

- ③: As was previously seen, $E_{\rho\sigma}{}^{\gamma}$ is of linear order $h_{\mu\nu}$. As such, $E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G}$ goes up to the third order $h_{\mu\nu}$. We thus admit, from $f''(\mathcal{G})$, only the term $-\hat{c}\mathcal{G}^{-2}$. The term then becomes

$$\frac{1}{2}f''(\mathcal{G})E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} = -\frac{1}{2}\hat{c}\mathcal{G}^{-2}E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} \tag{3.8}$$

- ④: We follow a similar train of thought to ②, which gives

$$\frac{1}{2}(\eta_{\rho\sigma} + h_{\rho\sigma})f(\mathcal{G}) = \frac{1}{2}\eta_{\rho\sigma}(\hat{c}\log|\mathcal{G}| + c_0) + \frac{1}{2}h_{\rho\sigma}(c_{-1}\mathcal{G}^{-1} + \hat{c}\log|\mathcal{G}| + c_0) \tag{3.9}$$

We can thus construct a preliminary form of the integer power near-polynomial $f(\mathcal{G})$ gravity field equations:

$$\begin{aligned}(\hat{c}\mathcal{G}^{-1} + c_1)G_{\rho\sigma} + \frac{1}{2}\hat{c}\eta_{\rho\sigma} + \frac{1}{2}(-c_{-1}\mathcal{G}^{-1} + \hat{c})h_{\rho\sigma} - \frac{1}{2}\hat{c}\mathcal{G}^{-2}E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - \\ \frac{1}{2}\eta_{\rho\sigma}(\hat{c}\log|\mathcal{G}| + c_0) - \frac{1}{2}h_{\rho\sigma}(c_{-1}\mathcal{G}^{-1} + \hat{c}\log|\mathcal{G}| + c_0) = \kappa T_{\rho\sigma}\end{aligned} \tag{3.10}$$

Two terms cancel out, which reduces the equation to

$$\underbrace{(\hat{c}\mathcal{G}^{-1} + c_1)G_{\rho\sigma}}_{①} + \frac{1}{2}\hat{c}(\eta_{\rho\sigma} + h_{\rho\sigma}) - \frac{1}{2}\hat{c}\mathcal{G}^{-2}E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - \frac{1}{2}\eta_{\rho\sigma}(\hat{c}\log|\mathcal{G}| + c_0) - \frac{1}{2}h_{\rho\sigma}(\hat{c}\log|\mathcal{G}| + c_0) = \kappa T_{\rho\sigma} \tag{3.11}$$

One final simplification we can make concerns the term ①. The Einstein tensor is of linear order $h_{\mu\nu}$, while $\mathcal{G}^{-1}$ is of order $h_{\mu\nu}^{-2}$. As such, we would expect $\hat{c}\mathcal{G}^{-1}G_{\rho\sigma}$ (which is of order $h_{\mu\nu}^{-1}$) to be *large*. This is clearly unphysical. To eliminate this, we must set

$$\hat{c} = 0 \tag{3.12}$$

The field equations then become

Theorem 3.1 (Integer power near-polynomial linearised $f(\mathcal{G})$ field equations)

$$c_1 G_{\rho\sigma} + \frac{1}{2} c_0 g_{\rho\sigma} = \kappa T_{\rho\sigma} \tag{3.13}$$

But this looks a bit familiar, doesn't it?

Suppose we fix the constants c_1 and c_0 to the following values:

$$c_1 = 1 \quad \frac{1}{2} c_0 = \Lambda \tag{3.14}$$

whence (3.13) reads

$$G_{\rho\sigma} + \Lambda g_{\rho\sigma} = \kappa T_{\rho\sigma} \tag{3.15}$$

This is identical to (1.1)[1]. And so, we have recovered our good friend, the field equations with the cosmological constant.

Two comments are in order here:

- GR as we know today is actually a form of $f(\mathcal{G})$ gravity with a very specific choice of $f(\mathcal{G})$ and its parameters. We will see this again in the next section.
- Unfortunately at the same time, this means that an integer power near-polynomial returns us exactly to GR.

3.3 Half-integer power near-polynomial

One way to prevent the linearised $f(\mathcal{G})$ field equations from reducing to GR is to introduce half-integer power terms, like so:

Definition 3.2 (Half-integer power near-polynomial)

$$\begin{aligned} f(\mathcal{G}) = \cdots &+ c_{-2}\mathcal{G}^{-2} + c_{-3/2}\mathcal{G}^{-3/2} + c_{-1}\mathcal{G}^{-1} + c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| \\ &+ c_0 + c_{1/2}\mathcal{G}^{1/2} + c_1\mathcal{G} + c_{3/2}\mathcal{G}^{3/2} + c_2\mathcal{G}^2 + \cdots \end{aligned} \tag{3.16}$$

where we have expanded the constants to include half-integers.

The first derivative of this test near-polynomial is

$$\begin{aligned} f'(\mathcal{G}) = \cdots &- 2c_{-2}\mathcal{G}^{-3} - \frac{3}{2}c_{-3/2}\mathcal{G}^{-5/2} - c_{-1}\mathcal{G}^{-2} - \frac{1}{2}c_{-1/2}\mathcal{G}^{-3/2} + \hat{c}\mathcal{G}^{-1} + \frac{1}{2}c_{1/2}\mathcal{G}^{-1/2} \\ &+ c_1 + \frac{3}{2}c_{3/2}\mathcal{G}^{1/2} + 2c_2\mathcal{G} + \frac{5}{2}c_{5/2}\mathcal{G}^{3/2} + \cdots \end{aligned} \tag{3.17}$$

The second derivative is then

$$\begin{aligned} f''(\mathcal{G}) = \cdots &+ 6c_{-2}\mathcal{G}^{-4} + \frac{15}{4}c_{-3/2}\mathcal{G}^{-7/2} + 2c_{-1}\mathcal{G}^{-3} + \frac{3}{4}c_{-1/2}\mathcal{G}^{-5/2} - \hat{c}\mathcal{G}^{-2} - \frac{1}{4}c_{1/2}\mathcal{G}^{-3/2} + \frac{3}{4}c_{3/2}\mathcal{G}^{-1/2} \\ &+ 2c_2 + \frac{15}{4}c_{5/2}\mathcal{G}^{1/2} + 6c_3\mathcal{G} + \cdots \end{aligned} \tag{3.18}$$

We again consider the fully expanded $f(\mathcal{G})$ field equations:

$$\underbrace{f'(\mathcal{G})G_{\rho\sigma}}_{①} + \underbrace{\frac{1}{2}f'(\mathcal{G})(\eta_{\rho\sigma} + h_{\rho\sigma})\mathcal{G}}_{②} + \underbrace{\frac{1}{2}f''(\mathcal{G})E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G}}_{③} - \underbrace{\frac{1}{2}(\eta_{\rho\sigma} + h_{\rho\sigma})f(\mathcal{G})}_{④} = \kappa T_{\rho\sigma}$$

where only terms of linear or inverse linear orders of $h_{\mu\nu}$ should survive in the final expression. Hence:

[1] At this point, one might wonder why the result we have recovered is not (1.19). This is because we have assumed the Lorenz gauge in the linearised field equations, which we are yet to do here.

- ①: $G_{\rho\sigma}$ is of linear order $h_{\mu\nu}$. As such, $f'(\mathcal{G})$ effectively reduces to $\hat{c}\mathcal{G}^{-1} + \frac{1}{2}c_{1/2}\mathcal{G}^{-1/2} + c_1$, and the term becomes
$$f'(\mathcal{G})G_{\rho\sigma} = \left(\hat{c}\mathcal{G}^{-1} + \frac{1}{2}c_{1/2}\mathcal{G}^{-1/2} + c_1\right) G_{\rho\sigma} \tag{3.19}$$

- ②: Due to the metric, $\mathcal{G}$ evolves into two groups of terms of second and third orders of $h_{\mu\nu}$ respectively. The second-order terms can be reduced to zeroth order by $\mathcal{G}^{-1}$, linear order by $\mathcal{G}^{-1/2}$ or inverse linear order by $\mathcal{G}^{-3/2}$, while the third-order terms can be reduced to either linear order by $\mathcal{G}^{-1}$, zeroth order by $\mathcal{G}^{-3/2}$ or inverse linear order by $\mathcal{G}^{-2}$. As such, the term becomes
$$\begin{aligned}\frac{1}{2}f'(\mathcal{G})(\eta_{\rho\sigma} + h_{\rho\sigma})\mathcal{G} =& \frac{1}{2}\left(-\frac{1}{2}c_{-1/2}\mathcal{G}^{-3/2} + \hat{c}\mathcal{G}^{-1} + \frac{1}{2}c_{1/2}\mathcal{G}^{-1/2}\right)\eta_{\rho\sigma}\mathcal{G} + \\ & \frac{1}{2}\left(-c_{-1}\mathcal{G}^{-2} - \frac{1}{2}c_{-1/2}\mathcal{G}^{-3/2} + \hat{c}\mathcal{G}^{-1}\right) h_{\rho\sigma}\mathcal{G}\end{aligned} \tag{3.20}$$

- ③: As was previously seen, $E_{\rho\sigma}{}^{\gamma}$ is of linear order $h_{\mu\nu}$. As such, $E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G}$ goes up to the third order $h_{\mu\nu}$. We thus admit, from $f''(\mathcal{G})$, the terms $-\hat{c}\mathcal{G}^{-2} - \frac{1}{4}c_{1/2}\mathcal{G}^{-3/2}$. The term then becomes
$$\frac{1}{2}f''(\mathcal{G})E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} = -\frac{1}{2}\left(\hat{c}\mathcal{G}^{-2} + \frac{1}{4}c_{1/2}\mathcal{G}^{-3/2}\right) E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} \tag{3.21}$$

- ④: We follow a similar train of thought to ②, which gives
$$\begin{aligned}\frac{1}{2}(\eta_{\rho\sigma} + h_{\rho\sigma})f(\mathcal{G}) =& \frac{1}{2}\eta_{\rho\sigma}\left(c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| + c_0 + c_{1/2}\mathcal{G}^{1/2}\right) + \\ & \frac{1}{2}h_{\rho\sigma}\left(c_{-1}\mathcal{G}^{-1} + c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| + c_0\right)\end{aligned} \tag{3.22}$$

We can thus construct a preliminary form of the half-integer power near-polynomial $f(\mathcal{G})$ gravity field equations:
$$\begin{aligned}&\left(\hat{c}\mathcal{G}^{-1} + \frac{1}{2}c_{1/2}\mathcal{G}^{-1/2} + c_1\right) G_{\rho\sigma} + \frac{1}{2}\left(-\frac{1}{2}c_{-1/2}\mathcal{G}^{-1/2} + \hat{c} + \frac{1}{2}c_{1/2}\mathcal{G}^{1/2}\right)\eta_{\rho\sigma} + \\ &\frac{1}{2}\left(-c_{-1}\mathcal{G}^{-1} - \frac{1}{2}c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\right) h_{\rho\sigma} - \frac{1}{2}\left(\hat{c}\mathcal{G}^{-2} + \frac{1}{4}c_{1/2}\mathcal{G}^{-3/2}\right) E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - \\ &\frac{1}{2}\left(c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| + c_0 + c_{1/2}\mathcal{G}^{1/2}\right)\eta_{\rho\sigma} - \frac{1}{2}\left(c_{-1}\mathcal{G}^{-1} + c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| + c_0\right) h_{\rho\sigma} = \kappa T_{\rho\sigma}\end{aligned} \tag{3.23}$$

Some terms cancel out:
$$\begin{aligned}&\left(\hat{c}\mathcal{G}^{-1} + \frac{1}{2}c_{1/2}\mathcal{G}^{-1/2} + c_1\right) G_{\rho\sigma} - \frac{1}{2}\left(\hat{c}\mathcal{G}^{-2} + \frac{1}{4}c_{1/2}\mathcal{G}^{-3/2}\right) E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - \\ &\frac{1}{2}\eta_{\rho\sigma}\left(\frac{3}{2}c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| + c_0 + \frac{1}{2}c_{1/2}\mathcal{G}^{1/2} - \hat{c}\right) - \frac{1}{2}h_{\rho\sigma}\left(2c_{-1}\mathcal{G}^{-1} + \frac{3}{2}c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| + c_0 - \hat{c}\right) = \kappa T_{\rho\sigma}\end{aligned} \tag{3.24}$$

Again, physicality must be preserved in that no terms should blow up to infinity. Terms of order $h_{\mu\nu}^{-1}$ and below must be eliminated. This forces some constants to be zero:
$$c_{-1} = c_{-1/2} = \hat{c} = 0 \tag{3.25}$$

Hence
$$\left(\frac{1}{2}c_{1/2}\mathcal{G}^{-1/2} + c_1\right) G_{\rho\sigma} - \frac{1}{2}\left(\frac{1}{4}c_{1/2}\mathcal{G}^{-3/2}\right) E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - \frac{1}{2}\eta_{\rho\sigma}\left(c_0 + \frac{1}{2}c_{1/2}\mathcal{G}^{1/2}\right) - \frac{1}{2}c_0 h_{\rho\sigma} = \kappa T_{\rho\sigma} \tag{3.26}$$

and
$$\left(\frac{1}{2}c_{1/2}\mathcal{G}^{-1/2} + c_1\right) G_{\rho\sigma} - \frac{1}{8}c_{1/2}\mathcal{G}^{-3/2}E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - \frac{1}{2}c_0 g_{\rho\sigma} - \frac{1}{4}\eta_{\rho\sigma}c_{1/2}\mathcal{G}^{1/2} = \kappa T_{\rho\sigma} \tag{3.27}$$

Rescaling the constants yields

Theorem 3.2 (Half-integer power near-polynomial linearised $f(\mathcal{G})$ field equations)

$$\lambda_1 G_{\rho\sigma} + \lambda_3 g_{\rho\sigma} + \lambda_2(4\mathcal{G}^{-1/2}G_{\rho\sigma} - \mathcal{G}^{-3/2}E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - 2\eta_{\rho\sigma}\mathcal{G}^{1/2}) = \kappa T_{\rho\sigma} \tag{3.28}$$

Specific constants return us to something close to GR. If we set

$$\lambda_1 = 1 \quad \lambda_3 = \Lambda \quad \lambda_2 = \lambda \tag{3.29}$$

we will find that

$$\underbrace{G_{\rho\sigma} + \Lambda g_{\rho\sigma}}_{\text{GR}} + \lambda \underbrace{\left(4\mathcal{G}^{-1/2}G_{\rho\sigma} - \mathcal{G}^{-3/2}E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G} - 2\eta_{\rho\sigma}\mathcal{G}^{1/2}\right)}_{f(\mathcal{G})\text{ contribution}} = \underbrace{\kappa T_{\rho\sigma}}_{\text{GR}} \tag{3.30}$$

where GR is recovered with a choice of $\lambda = 0$.

3.4 Gravitational wave equations

Again, we note that gravitational waves are a non-matter source. By considering the vacuum field equations without the cosmological term, we find:

$$\underbrace{G_{\rho\sigma}}_{O(h)} + \lambda(\underbrace{4\mathcal{G}^{-1/2}G_{\rho\sigma} - \mathcal{G}^{-3/2}E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G}}_{O(1)} - \underbrace{2\eta_{\rho\sigma}\mathcal{G}^{1/2}}_{O(h)}) = 0 \tag{3.31}$$

The LHS of the equation is comprised of two terms of linear order h and two terms of order 1. It can immediately be seen that the $O(1)$ terms must cancel out[2], and we naturally find the following nice-looking equation

$$G_{\mu\nu} - \lambda\eta_{\rho\sigma}\mathcal{G}^{1/2} = 0 \tag{3.32}$$

where λ, which we shall call the $f(\mathcal{G})$ *coupling constant*, has been rescaled.

Inserting (1.22), the linearised bulk term becomes

$$\begin{aligned}
\mathcal{G} &= \frac{1}{2}(h^{\mu\nu}{}_{,\alpha}h^{\alpha}_{\mu,\nu} - h^{\mu\alpha}{}_{,\alpha}h_{,\mu}) \\
&= \frac{1}{2}(\bar{A}^{\mu\nu}\exp(ik_\sigma x^\sigma)ik_\alpha\bar{A}^{\alpha}_{\mu}\exp(ik_\sigma x^\sigma)ik_\nu - \bar{A}^{\mu\alpha}\exp(ik_\sigma x^\sigma)ik_\alpha\bar{A}\exp(ik_\sigma x^\sigma)ik_\mu) \\
&= \frac{1}{2}(\bar{A}^{\mu\alpha}k_\alpha\bar{A}k_\mu - \bar{A}^{\mu\nu}k_\alpha\bar{A}^{\alpha}_{\mu}k_\nu)\exp(2ik_\sigma x^\sigma)
\end{aligned} \tag{3.33}$$

where we have not differentiated the almighty $\bar{A}^{\mu\nu}$ due to assuming it to be a constant.

The square root of $\mathcal{G}$ is then simply

$$\sqrt{\mathcal{G}} = 2^{-1/2}\exp(ik_\sigma x^\sigma)\sqrt{\bar{A}^{\mu\alpha}k_\alpha\bar{A}k_\mu - \bar{A}^{\mu\nu}k_\alpha\bar{A}^{\alpha}_{\mu}k_\nu} \tag{3.34}$$

By definition, the trace of the trace-reverse of a tensor always yields zero. Happily, we then have

$$\bar{A} = 0 \to \sqrt{\mathcal{G}} = 2^{-1/2}\exp(ik_\sigma x^\sigma)\sqrt{-\bar{A}^{\mu\nu}k_\alpha\bar{A}^{\alpha}_{\mu}k_\nu} \tag{3.35}$$

For simplicity, let us make the following shorthand:

$$\varrho = \sqrt{-\bar{A}^{\mu\nu}k_\alpha\bar{A}^{\alpha}_{\mu}k_\nu} \tag{3.36}$$

Substituting the linearised objects in (1.12) and (1.16), (3.32) becomes

$$\bar{h}_{\lambda\nu;\mu}{}^{;\lambda} + \bar{h}_{\lambda\mu;\nu}{}^{;\lambda} - \bar{h}_{\mu\nu;\lambda}{}^{;\lambda} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta} - 2^{-1/2}\lambda\varrho\exp(ik_\sigma x^\sigma)\eta_{\mu\nu} = 0 \tag{3.37}$$

[2]Otherwise, we have a term that is *large* compared to the $O(h)$ terms, and the LHS would be non-zero.

From (1.22), one can identify the trace-reversed perturbation as

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\sigma x^\sigma) \tag{3.38}$$

Inserting this gives

$$(A_{\lambda\nu}k^\lambda k_\mu + A_{\lambda\mu}k^\lambda k_\nu - A_{\mu\nu}k^\lambda k_\lambda - \eta_{\mu\nu}A_{\alpha\beta}k^\alpha k^\beta - 2^{-1/2}\lambda\varrho\eta_{\mu\nu}) \exp(ik_\sigma x^\sigma) = 0 \tag{3.39}$$

where we intuitively factor out the exponential and find

Theorem 3.3 (Linearised half-integer power near polynomial $f(\mathcal{G})$ gravitational wave equations)

$$A_{\lambda\nu}k^\lambda k_\mu + A_{\lambda\mu}k^\lambda k_\nu - A_{\mu\nu}k^\lambda k_\lambda - \eta_{\mu\nu}A_{\alpha\beta}k^\alpha k^\beta - 2^{-1/2}\lambda\varrho\eta_{\mu\nu} = 0 \tag{3.40}$$

which is a result without loss of generality!

Chapter 4

$f(\mathcal{G})$ gravitational waves

> **Quote 4.1** You shouldn't be able to pump solids, but you can pump peanut butter.
>
> *Christian G. Böhmer, 22 November 2023*

4.1 Trace equation

We now proceed from where we left off, which is (3.40).

$$A_{\lambda\nu}k^{\lambda}k_{\mu} + A_{\lambda\mu}k^{\lambda}k_{\nu} - A_{\mu\nu}k^{\lambda}k_{\lambda} - \eta_{\mu\nu}A_{\alpha\beta}k^{\alpha}k^{\beta} - 2^{-1/2}\lambda\varrho\eta_{\mu\nu} = 0$$

Note that this is actually a collection of 10 equations in disguise, with each corresponding to a degree of freedom in the (symmetric) perturbation[1].

$$A_{\lambda 0}k^{\lambda}k_{0} + A_{\lambda 0}k^{\lambda}k_{0} - A_{00}k^{\lambda}k_{\lambda} + A_{\alpha\beta}k^{\alpha}k^{\beta} + 2^{-1/2}\lambda\varrho = 0 \tag{4.1}$$

$$A_{\lambda 1}k^{\lambda}k_{0} + A_{\lambda 0}k^{\lambda}k_{1} - A_{01}k^{\lambda}k_{\lambda} = 0 \tag{4.2}$$

$$A_{\lambda 2}k^{\lambda}k_{0} + A_{\lambda 0}k^{\lambda}k_{2} - A_{02}k^{\lambda}k_{\lambda} = 0 \tag{4.3}$$

$$A_{\lambda 3}k^{\lambda}k_{0} + A_{\lambda 0}k^{\lambda}k_{3} - A_{03}k^{\lambda}k_{\lambda} = 0 \tag{4.4}$$

$$A_{\lambda 1}k^{\lambda}k_{1} + A_{\lambda 1}k^{\lambda}k_{1} - A_{11}k^{\lambda}k_{\lambda} - A_{\alpha\beta}k^{\alpha}k^{\beta} - 2^{-1/2}\lambda\varrho = 0 \tag{4.5}$$

$$A_{\lambda 2}k^{\lambda}k_{1} + A_{\lambda 1}k^{\lambda}k_{2} - A_{12}k^{\lambda}k_{\lambda} = 0 \tag{4.6}$$

$$A_{\lambda 3}k^{\lambda}k_{1} + A_{\lambda 1}k^{\lambda}k_{3} - A_{13}k^{\lambda}k_{\lambda} = 0 \tag{4.7}$$

$$A_{\lambda 2}k^{\lambda}k_{2} + A_{\lambda 2}k^{\lambda}k_{2} - A_{22}k^{\lambda}k_{\lambda} - A_{\alpha\beta}k^{\alpha}k^{\beta} - 2^{-1/2}\lambda\varrho = 0 \tag{4.8}$$

$$A_{\lambda 3}k^{\lambda}k_{2} + A_{\lambda 2}k^{\lambda}k_{3} - A_{23}k^{\lambda}k_{\lambda} = 0 \tag{4.9}$$

$$A_{\lambda 3}k^{\lambda}k_{3} + A_{\lambda 3}k^{\lambda}k_{3} - A_{33}k^{\lambda}k_{\lambda} - A_{\alpha\beta}k^{\alpha}k^{\beta} - 2^{-1/2}\lambda\varrho = 0 \tag{4.10}$$

which summarises to

$$A_{\lambda\nu}k^{\lambda}k_{\mu} + A_{\lambda\mu}k^{\lambda}k_{\nu} - A_{\mu\nu}k^{\lambda}k_{\lambda} = 0 \quad \mu < \nu \tag{4.11}$$

$$A_{\lambda\nu}k^{\lambda}k_{\mu} + A_{\lambda\mu}k^{\lambda}k_{\nu} - A_{\mu\nu}k^{\lambda}k_{\lambda} + A_{\alpha\beta}k^{\alpha}k^{\beta} + 2^{-1/2}\lambda\varrho = 0 \quad \mu = \nu = 0 \tag{4.12}$$

$$A_{\lambda\nu}k^{\lambda}k_{\mu} + A_{\lambda\mu}k^{\lambda}k_{\nu} - A_{\mu\nu}k^{\lambda}k_{\lambda} - A_{\alpha\beta}k^{\alpha}k^{\beta} - 2^{-1/2}\lambda\varrho = 0 \quad \mu = \nu = 1,2,3 \tag{4.13}$$

The simplest form these equations can take is the trace form, which we can derive applying the inverse Minkowski metric $\eta^{\mu\nu}$ on both sides:

$$\eta^{\mu\nu}A_{\lambda\nu}k^{\lambda}k_{\mu} + \eta^{\mu\nu}A_{\lambda\mu}k^{\lambda}k_{\nu} - \eta^{\mu\nu}A_{\mu\nu}k^{\lambda}k_{\lambda} - \eta^{\mu\nu}\eta_{\mu\nu}A_{\alpha\beta}k^{\alpha}k^{\beta} - \eta^{\mu\nu}2^{-1/2}\lambda\varrho\eta_{\mu\nu} = 0 \tag{4.14}$$

which reduces to

$$2A_{\lambda\mu}k^{\lambda}k^{\mu} - Ak^{\lambda}k_{\lambda} - 4A_{\alpha\beta}k^{\alpha}k^{\beta} - 2\sqrt{2}\lambda\varrho = 0 \tag{4.15}$$

All indices are free in this equation, and we naturally have

[1] Remember that only 10 out of 16 components of the perturbation are free as a result of symmetry.

Theorem 4.1 (Trace equation)

$$2A_{\lambda\mu}k^\lambda k^\mu + Ak^\lambda k_\lambda = 2\sqrt{2}\lambda\varrho \tag{4.16}$$

4.2 Modified k^μ and $A_{\mu\nu}$

Now let us discuss the physical meaning of (4.16). We recall that the tensorial amplitude $A_{\mu\nu}$ and the wavevector k^μ are subject to two constraints in GR in the form of (1.23) and (1.24). Applying these constraints to (4.16) yields

$$2\sqrt{2}\lambda\varrho = 0 \tag{4.17}$$

where $\lambda = 0$ and we return to GR. In other words, no effects of modified gravity are shown under the two constraints.

Consider the implications of this very carefully. To begin with, we recall the simplest possible form of k^μ and $A_{\mu\nu}$ in standard GR [2]:

$$k^\mu = (\omega_0, 0, 0, \omega_3) \quad A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.18}$$

where, importantly, one has

- Null wavevector:
$$\omega_0 = \omega_3 \tag{4.19}$$
- Transverse wave (see (1.25)).
- Traceless tensorial amplitude:
$$A_{11} = -A_{22} \tag{4.20}$$

For a solution that displays the effects of our modified gravity, we will not assume these constraints. The simplest modifications we can make are

- Timelike wavevector:
$$k^\mu = (\omega, 0, 0, \omega + \hat{\omega}) \tag{4.21}$$
where $\hat{\omega}$ is *small*. Importantly, we also set $\hat{\omega} < 0$ to prevent the unphysical case of spacelike wavevectors.
- Traceful wave amplitude: We begin with the most generalised symmetric tensorial amplitude
$$A_{\mu\nu} = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{01} & A_{11} & A_{12} & A_{13} \\ A_{02} & A_{12} & A_{22} & A_{23} \\ A_{03} & A_{13} & A_{23} & A_{33} \end{pmatrix} \tag{4.22}$$
where we have a *small* trace
$$\mathrm{Tr}(A_{\mu\nu}) = \hat{A} = A_{00} + A_{11} + A_{22} + A_{33} \tag{4.23}$$

4.3 $f(\mathcal{G})$ coupling constant

Now we want to solve (4.16) under the modifications. We begin by computing k^μ, A^μ_ν and $A^{\mu\nu}$ by applying Minkowski metrics, which gives:

$$k_\mu = \begin{pmatrix} -\omega_0 \\ 0 \\ 0 \\ \omega_0 + \hat{\omega} \end{pmatrix} \quad A^\mu_\nu = \begin{pmatrix} -A_{00} & -A_{01} & -A_{02} & -A_{03} \\ A_{01} & A_{11} & A_{12} & A_{13} \\ A_{02} & A_{12} & A_{22} & A_{23} \\ A_{03} & A_{13} & A_{23} & A_{33} \end{pmatrix} \quad A^{\mu\nu} = \begin{pmatrix} A_{00} & -A_{01} & -A_{02} & -A_{03} \\ -A_{01} & A_{11} & A_{12} & A_{13} \\ -A_{02} & A_{12} & A_{22} & A_{23} \\ -A_{03} & A_{13} & A_{23} & A_{33} \end{pmatrix} \tag{4.24}$$

The LHS of (4.16) becomes

$$\text{LHS} = 2A_{00}\omega^2 + 4\hat{A}\omega^2 - 4A_{03}\omega(\omega+\hat{\omega}) + 2A_{33}(\omega+\hat{\omega})^2 - 4\hat{A}(\omega+\hat{\omega})^2 \tag{4.25}$$

As the RHS of (4.16) involves a square root, we square both sides of the equation to solve it, keeping in mind that we will eventually take the square root of it.

The LHS squares to

$$\begin{aligned}\text{LHS}^2 = {}& 4A_{00}^2\omega^4 - 16A_{00}A_{03}\omega^4 + 16A_{03}^2\omega^4 + 8A_{00}A_{33}\omega^4 - 16A_{03}A_{33}\omega^4 + 4A_{33}^2\omega^4 - 16A_{00}A_{03}\omega^3\hat{\omega}+\\ & 32A_{03}^2\omega^3\hat{\omega} + 16A_{00}A_{33}\omega^3\hat{\omega} - 48A_{03}A_{33}\omega^3\hat{\omega} + 16A_{33}^2\omega^3\hat{\omega} - 32A_{00}\hat{A}\omega^3\hat{\omega} + 64A_{03}\hat{A}\omega^3\hat{\omega}-\\ & 32A_{33}\hat{A}\omega^3\hat{\omega} + 16A_{03}^2\omega^2\hat{\omega}^2 + 8A_{00}A_{33}\omega^2\hat{\omega}^2 - 48A_{03}A_{33}\omega^2\hat{\omega}^2 + 24A_{33}^2\omega^2\hat{\omega}^2 - 16A_{00}\hat{A}\omega^2\hat{\omega}^2+\\ & 96A_{03}\hat{A}\omega^2\hat{\omega}^2 - 80A_{33}\hat{A}\omega^2\hat{\omega}^2 + 64\hat{A}^2\omega^2\hat{\omega}^2 - 16A_{03}A_{33}\omega\hat{\omega}^3 + 16A_{33}^2\omega\hat{\omega}^3 + 32A_{03}\hat{A}\omega\hat{\omega}^3-\\ & 64A_{33}\hat{A}\omega\hat{\omega}^3 + 64\hat{A}^2\omega\hat{\omega}^3 + 4A_{33}^2\hat{\omega}^4 - 16A_{33}\hat{A}\hat{\omega}^4 + 16\hat{A}^2\hat{\omega}^4\end{aligned} \tag{4.26}$$

The RHS squares to

$$\begin{aligned}\text{RHS}^2 = 16\lambda^2\Big(& 2A_{00}^2\omega^2 - A_{01}^2\omega^2 - A_{02}^2\omega^2 - 4A_{00}A_{03}\omega^2 + A_{00}A_{11}\omega^2 - 2A_{03}A_{11}\omega^2+\\ & 2A_{01}A_{13}\omega^2 - A_{13}^2\omega^2 + A_{00}A_{22}\omega^2 - 2A_{03}A_{22}\omega^2 + 2A_{02}A_{23}\omega^2 - A_{23}^2\omega^2+\\ & 2A_{00}A_{33}\omega^2 + A_{11}A_{33}\omega^2 + A_{22}A_{33}\omega^2 - \frac{1}{2}A_{00}^2\omega\hat{\omega} - 4A_{00}A_{03}\omega\hat{\omega} + 2A_{03}^2\omega\hat{\omega} - A_{00}A_{11}\omega\hat{\omega}-\\ & 2A_{03}A_{11}\omega\hat{\omega} - \frac{1}{2}A_{11}^2\omega\hat{\omega} + 2A_{01}A_{13}\omega\hat{\omega} - 2A_{13}^2\omega\hat{\omega} - A_{00}A_{22}\omega\hat{\omega} - 2A_{03}A_{22}\omega\hat{\omega}-\\ & A_{11}A_{22}\omega\hat{\omega} - \frac{1}{2}A_{22}^2\omega\hat{\omega} + 2A_{02}A_{23}\omega\hat{\omega} - 2A_{23}^2\omega\hat{\omega} + A_{00}A_{33}\omega\hat{\omega} + A_{11}A_{33}\omega\hat{\omega}+\\ & A_{22}A_{33}\omega\hat{\omega} - \frac{1}{2}A_{33}^2\omega\hat{\omega} - \frac{A_{00}^2\hat{\omega}^2}{4} + A_{03}^2\hat{\omega}^2 - \frac{1}{2}A_{00}A_{11}\hat{\omega}^2 - \frac{A_{11}^2\hat{\omega}^2}{4} - A_{13}^2\hat{\omega}^2 - \frac{1}{2}A_{00}A_{22}\hat{\omega}^2-\\ & \frac{1}{2}A_{11}A_{22}\hat{\omega}^2 - \frac{A_{22}^2\hat{\omega}^2}{4} - A_{23}^2\hat{\omega}^2 + \frac{1}{2}A_{00}A_{33}\hat{\omega}^2 + \frac{1}{2}A_{11}A_{33}\hat{\omega}^2 + \frac{1}{2}A_{22}A_{33}\hat{\omega}^2 - \frac{A_{33}^2\hat{\omega}^2}{4}\Big)\end{aligned} \tag{4.27}$$

Both expressions are quite unwieldy. However, we can solve them by extracting the terms of zeroth order perturbation[2]. The equations then reduce to

$$\text{LHS}^2_{O(1)} = 4A_{00}^2\omega^4 - 16A_{00}A_{03}\omega^4 + 16A_{03}^2\omega^4 + 8A_{00}A_{33}\omega^4 - 16A_{03}A_{33}\omega^4 + 4A_{33}^2\omega^4 \tag{4.28}$$

$$\begin{aligned}\text{RHS}^2_{O(1)} = 16\lambda^2\Big(& 2A_{00}^2\omega^2 - A_{01}^2\omega^2 - A_{02}^2\omega^2 - 4A_{00}A_{03}\omega^2 + A_{00}A_{11}\omega^2 - 2A_{03}A_{11}\omega^2+\\ & 2A_{01}A_{13}\omega^2 - A_{13}^2\omega^2 + A_{00}A_{22}\omega^2 - 2A_{03}A_{22}\omega^2 + 2A_{02}A_{23}\omega^2 - A_{23}^2\omega^2+\\ & 2A_{00}A_{33}\omega^2 + A_{11}A_{33}\omega^2 + A_{22}A_{33}\omega^2\Big)\end{aligned} \tag{4.29}$$

We stop for a moment and review the tensorial amplitude $A_{\mu\nu}$ itself. In (4.23), we have intentionally defined the most general possible form of $A_{\mu\nu}$. This does not mean, however, that all components of $A_{\mu\nu}$ are non-zero. By inspecting the structure of (4.28 and (4.29), we can see that a possible $A_{\mu\nu}$ is

$$A_{\mu\nu} = \begin{pmatrix} A_{00} & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & A_{33} \end{pmatrix} \tag{4.30}$$

To keep the structure of our $A_{\mu\nu}$ as close to its standard GR form as possible, we retain the assumption that $A_{11} = -A_{22}$. Amazingly, (4.16) is then

$$4A_{00}^2\omega^2 + 8A_{00}A_{33}\omega^2 + 4A_{33}\omega^2 = 16\lambda^2(A_{00}^2 + 2A_{00}A_{33}) \tag{4.31}$$

[2]Why we extract them instead of the terms of order 2 perturbation - which reduce to linear order after taking the square root - will soon be seen.

This reduces to

$$\lambda = \frac{\omega}{2}\sqrt{1 + \frac{A_{33}}{A_{00}}} \tag{4.32}$$

which is surprisingly simple!

Some comments can be made regarding this relationship:

- First, we note that λ is independent of the perturbation $\hat{\omega}$. In fact, our whole derivation is independent of $\hat{\omega}$. If one is to assume a wavevector identical to GR, they would identically recover the zero-order terms and the relation in (4.32). This suggests that the simplest wavevector is actually its standard GR form

  $$k^\mu = (\omega, 0, 0, \omega) \tag{4.33}$$

  which we did not dare assume initially.

- Also of interest is the parameter $\frac{A_{33}}{A_{00}}$. It is dimensionless by construction, which is expected if one considers the fact that λ itself should be dimensionless as well. At the same time, it implies that in addition to ω, λ is also dependent on the two extra polarisations A_{00} and A_{33}.

4.4 A_{00} as perturbation

So far, the relationship we have derived in (4.32) remains problematic. The only real constraint on our $A_{\mu\nu}$ is that $\mathrm{Tr}(A_{\mu\nu}) = A_{00} + A_{33}$ must be *small*. Hence, we can, in theory, admit *large* values for A_{00} and A_{33} so long as $A_{00} + A_{33}$ remains *small*.

Again, the problem lies in the fact that we have attempted to keep our assumption of $A_{\mu\nu}$ in (4.30) as generalised as possible. Let us instead assume the simplest possible form of $A_{\mu\nu}$ that retains the effects of our modified gravity

$$A_{\mu\nu} = \begin{pmatrix} A_{00} & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.34}$$

where we have only one extra component compared to GR. (4.32) then becomes

$$\lambda = \frac{\omega}{2} \tag{4.35}$$

This simplified result is satisfying for reasons one can see by investigating A_{00}. comparing the simplest forms of our $f(\mathcal{G})$ tensorial amplitude and the standard GR tensorial amplitude:

$$A_{\mu\nu,f(\mathcal{G})} = \begin{pmatrix} A_{00} & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_{\mu\nu,\mathrm{GR}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We know that current observational results of gravitational waves confirm GR up to a *tiny* margin of error [21], which means that the gravitational wave tensorial amplitude of any viable modified theory of gravity must exhibit only *small* deviations from GR. Hence, for our theory to be viable, A_{00} must be *small*. It is then intuitive to declare that A_{00} is the only component that differs from GR in our $f(\mathcal{G})$ gravity. That is to say

$$A_{11} = -A_{22}$$

remains true.

At this point, it should be obvious that A_{00} is the trace of the tensorial amplitude itself:

$$A_{00} = \hat{A} \tag{4.36}$$

Thus recovering our $f(\mathcal{G})$ tensorial amplitude

$$A_{\mu\nu} = \begin{pmatrix} \hat{A} & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.37}$$

where $\hat{A}$ is *small*, and A_{11}, A_{12} and A_{22} are identical to their standard GR counterparts.

An important observation is that the choice of $A_{\mu\nu}$ in 4.37 justifies our previous choice to only extract the part of the equation of zero-order perturbation. To begin, we note that under the tensorial amplitude in (4.37), the 'zero order' terms we previously extracted from the squared (4.16) are actually

$$\hat{A}^2\omega^2 = 8\lambda^2\hat{A}^2 \tag{4.38}$$

which is not of zero-order perturbation, but of second-order. After taking the square root and therefore restoring the dimensionality of our original (4.37), we find

$$\hat{A}\omega = 2\sqrt{2}\lambda\hat{A} \tag{4.39}$$

which is, very nicely, of first-order perturbation!

Let us summarise our previous discussion. Initially, when solving for (4.37), we have extracted the order zero perturbation terms, seemingly without justification. However, if we retroactively assume the simplest possible tensorial amplitude in (4.37) and substitute the individual components into the zero-order terms of the squared (4.16), these zero-order terms actually become terms of linear order $\hat{A}$. This result then justifies our choice of extracting the seemingly 'zero-order' terms, which turn out to be linear-order perturbation terms.

Hence, the gravitational wave field equations of our $f(\mathcal{G})$ gravity is

$$G_{\mu\nu} - \frac{\omega}{2}\eta_{\mu\nu}\mathcal{G}^{1/2} = 0 \tag{4.40}$$

Finally, we are in a position to make some brief discussions on the physical meaning of our findings.

4.5 Physical discussions

Longitudinal polarisation. Let us begin by considering our good friend, the perturbation $\hat{A}$. In standard GR, only two gravitational wave polarisations exist: the tensor modes A_{11} and A_{12} [34]. In our $f(\mathcal{G})$ gravity, a *third* polarisation exists in the form of $\hat{A}$. While A_{11} and A_{12} are transverse modes, $\hat{A}$ is instead a longitudinal (i.e. scalar) mode. That is to say, it represents a compression or expansion along the direction of propagation.

In principle, interferometric detectors such as LIGO, Virgo, KAGRA, or the future LISA are sensitive to any extra polarisation components [20]. So far, no clear evidence for extra polarisation states has emerged [20], although future sensitivity improvements or additional detectors (with different arm orientations and baselines) could set further constraints on such modes.

Dispersion. In (4.40), we have seen that the coupling constant is actually a rescaled version of the angular frequency ω. This suggests that the effects of the modified gravity term become more pronounced as the angular frequency rises. In other words, signatures hinting at our $f(\mathcal{G})$ gravity are more likely to emerge in high-frequency gravitational wave observations.

This relationship could give rise to dispersion. That is to say, a frequency-dependent phase shift will exist over long distances. Experimentally, such a dispersion can be tested by measuring whether high-frequency components of the wave arrive out of phase with low-frequency components.

Chapter 5

Conclusion and outlook

> **Quote 5.1** I would be really interested in an effective field theory treatment of a quantized version of this.
>
> *Paulina Schlachter, on this thesis, 16 February 2025*

5.1 Summary

We now summarise the main ideas of this thesis. Assuming no matter-energy content, the Einstein-Hilbert action is

$$S_H = \frac{1}{16\pi} \int_{\mathcal{V}} R\sqrt{-g} d^4x$$

Noting the action principle, we can decompose the Ricci scalar into two parts

$$R = \mathcal{G} + \mathcal{B}$$

where $\mathcal{B}$ is a boundary term that does not contribute to the action, and the remaining $\mathcal{G}$ is

$$\mathcal{G} = g^{\mu\nu}(\Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\lambda\nu} - \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\lambda\sigma})$$

A modified theory of gravity arises if one discards $\mathcal{B}$ is discarded and maps $\mathcal{G}$ to a function of itself $f(\mathcal{G})$. This modified gravity, closely related to $f(R)$ gravity, is known as $f(\mathcal{G})$ gravity (not to be confused with the so-called Gauss-Bonnet gravity) as we have effectively replaced R in the Einstein-Hilbert action with an $f(\mathcal{G})$.

The weak field approximations in $f(\mathcal{G})$ gravity can then be considered. For the standard form perturbation $h_{\mu\nu}$, the linearised $\mathcal{G}$ is

$$\mathcal{G} = \frac{1}{2}(h^{\lambda\nu}{}_{,\sigma}h^{\sigma}_{\lambda,\nu} - h^{\sigma\mu}{}_{,\mu}h_{,\sigma})$$

It is discovered that the simplest choice of $f(\mathcal{G})$ that does not reduce to GR is the 'near-polynomial'

$$\begin{aligned} f(\mathcal{G}) = \cdots &+ c_{-2}\mathcal{G}^{-2} + c_{-3/2}\mathcal{G}^{-3/2} + c_{-1}\mathcal{G}^{-1} + c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| \\ &+ c_0 + c_{1/2}\mathcal{G}^{1/2} + c_1\mathcal{G} + c_{3/2}\mathcal{G}^{3/2} + c_2\mathcal{G}^2 + \cdots \end{aligned}$$

Under this, the linearised field equations are

$$G_{\mu\nu} - \lambda\eta_{\rho\sigma}\mathcal{G}^{1/2} = 0$$

where λ is a parameter that resembles a coupling constant.

We can now solve for the gravitational wave solutions in $f(\mathcal{G})$ gravity. As we know, in the context of gravitational waves, the trace-reverse of the perturbation is

$$\bar{h}_{\mu\nu} = A_{\mu\nu}\exp(ik_\alpha x^\alpha)$$

It is discovered that to prevent us from returning to standard GR, we must assume that gravitational waves in this modified gravity theory are not transverse. The wavevector and the tensorial amplitude thus take the form

$$k^\mu = (\omega, 0, 0, \omega) \quad A_{\mu\nu} = \begin{pmatrix} \hat{A} & 0 & 0 & 0 \\ 0 & A_{11} & A_{21} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where k^μ is identical to its standard GR counterpart, and $\hat{A}$ represents a *small* deviation from standard GR.

By inserting k^μ and $A_{\mu\nu}$ into the field equations, it is then found that

$$\lambda = \frac{\omega}{2}$$

In $f(\mathcal{G})$ gravity, the modified gravity term of the gravitational wave field equations is then directly related to the temporal component of the wavevector

$$G_{\mu\nu} - \frac{\omega}{2}\eta_{\mu\nu}\mathcal{G}^{1/2} = 0$$

implying a dispersion relation.

5.2 Future work

Finally, we can briefly discuss future work based on this thesis that can be potentially carried out in the foreseeable future, both experimentally and theoretically.

As mentioned previously, current observational data already place very stringent constraints on $f(\mathcal{G})$ gravity. Therefore, the viability of our $f(\mathcal{G})$ gravity is strongly dependent on future observational data from collaborations like LISA. From here, there are two possibilities. The historical longevity of GR suggests that the likely scenario is the confirmation of GR up to an even *smaller* margin of error. This would place even stricter constraints on our theory of $f(\mathcal{G})$ gravity, possibly de facto ruling it out as a viable theory. Though unlikely, there nonetheless remains the possibility that a longitudinal polarisation node is indeed discovered. Such a node, however *small*, would suggest a degree of viability to $f(\mathcal{G})$ gravity.

Theoretical work following from this thesis presents opportunities for more immediately obtainable results. Although work has been done on spherically symmetric and cosmological solutions in $f(\mathcal{G})$ gravity [17], it remains one of the less studied modified gravity theories due to its nature as a very recent offshoot of $f(R)$ gravity. However, $f(\mathcal{G})$ gravity does hold a few aspects that make it unique among current modified gravity theories. Firstly, like $f(R)$ gravity, $f(\mathcal{G})$ gravity presents no significant deviations from the geometry of standard GR, which makes it much more concise when compared to the Einstein-Cartan, teleparallel and nonmetricity theories. Secondly, the fact that $f(\mathcal{G})$ gravity discards the 'useless' boundary term $\mathcal{B}$ also makes it a strong alternative to $f(R)$ gravity, its closest relative.

It is worth noting that the usefulness of the near-polynomials employed in this thesis, which nominally have an infinite number of terms, is mostly restricted to weak gravity due to the vanishing of higher-order terms in this regime. Other exact solutions in $f(\mathcal{G})$ gravity, especially those in strong gravity, remain to be studied. As a fitting end to this thesis, we will lay down some open questions:

- What are the $f(\mathcal{G})$s that are consistent with the classical tests of GR?
- What is the behaviour of prominent exact solutions (e.g. Schwarzschild, Reissner-Nordström, Kerr, Kerr-Newman) in $f(\mathcal{G})$ gravity?
- How can $f(\mathcal{G})$ gravity be formulated in the ADM formalism?
- What are some possible extensions to $f(\mathcal{G})$ gravity?

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