Weak field approximations in modified theories of gravity

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What we will go through today

1. Preliminaries
   - General relativity
   - Modified gravity
2. $f(\mathcal{G})$ gravitational waves
   - Linearised $f(\mathcal{G})$ gravity
   - Test $f(\mathcal{G})$ gravities
   - Modified $A_{\mu\nu}$ and k_α
   - Closing remarks

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Linearised field equations I

Gravity is geometry

Various relativists

The Einstein field equations are well-known as

$$\underbrace{G_{\mu\nu}}_{\text{geometry}} + \underbrace{\Lambda g_{\mu\nu}}_{\text{cosmology}} = \underbrace{\kappa T_{\mu\nu}}_{\text{matter}} \tag{1}$$

Under linearised gravity, we have

$$\underbrace{g_{\mu\nu}}_{\text{metric}} = \underbrace{\eta_{\mu\nu}}_{\text{Minkowski metric}} + \underbrace{h_{\mu\nu}}_{\textit{small}\text{ perturbation}} \tag{2}$$

and one can derive the linearised field equations as

$$G_{\mu\nu} = \frac{1}{2}\left(\bar{h}_{\lambda\nu}{}_{,\mu}^{,\lambda} + \bar{h}_{\lambda\mu}{}_{,\nu}^{,\lambda} - \bar{h}_{\mu\nu}{}_{,\lambda}^{,\lambda} - \eta_{\mu\nu}\bar{h}_{\alpha\beta}{}^{,\alpha\beta}\right) = \kappa T_{\mu\nu} \tag{3}$$

Linearised field equations II

A gauge freedom exists here. The metric is notably invariant under

$$X^{\mu} \to X'^{\mu} = X^{\mu} + \xi^{\mu} \tag{4}$$

To eliminate this gauge freedom, one can impose a tensorial equivalent of the Lorenz gauge:

$$\bar{h}_{\mu\nu}{}^{,\mu} = 0 \tag{5}$$

which reduce the linearised field equations to

$$G_{\mu\nu} = -\frac{1}{2}\bar{h}_{\mu\nu}{}_{,\alpha}^{,\alpha} = \kappa T_{\mu\nu} \tag{6}$$

Gravitational waves I

Suppose we want to solve for gravitational wave solutions. $h_{\mu\nu}$ takes the standard form

$$\bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\alpha x^\alpha) \tag{7}$$

where $A_{\mu\nu}$ is a tensorial wave amplitude, k_α is a wavevector and x^α is the 4-position.
By applying the gauge condition $\bar{h}_{\mu\nu}{}^{,\mu} = 0$, we find the following constraints:

- k_α is a null (i.e. lightlike) vector:

$$k_\alpha k^\alpha = 0 \tag{8}$$

- $A_{\mu\alpha}$ is orthogonal to k_α (transverse wave):

$$A_{\mu\alpha} k^\alpha = 0 \tag{9}$$

Gravitational waves II

Finally, we have the standard form for a gravitational wave travelling in the z-direction:

$$k^{\mu} = (\omega, 0, 0, \omega) \quad A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 \\ 0 & A_{12} & -A_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

where:

- ω is clearly the angular frequency.
- A_{11} and A_{22} are the two polarisations.

Einstein-Hilbert action

The action in GR has the form

$$S = \int_{\mathcal{V}} \mathcal{L} d^4x = S_H + S_M \tag{11}$$

S_H is the Hilbert term arising from the Ricci scalar R

$$S_H = \frac{1}{16\pi} \int_{\mathcal{V}} R\sqrt{-g} d^4x \tag{12}$$

where g is the (negative) determinant of the metric $g_{\mu\nu}$ and $\sqrt{-g}d^4x$ is the proper volume element.
S_M is the *matter action* arising from the scalar matter field ϕ

$$S_M = \int_{\mathcal{V}} \mathcal{L}_M(\phi, \partial_\mu \phi, g_{\mu\nu})\sqrt{-g} d^4x \tag{13}$$

As gravitational wave solutions are vacuum solutions, we take the matter term to be zero.

$f(\mathcal{G},\mathcal{B})$ gravity

We now focus purely on the Hilbert term:

$$S = \frac{1}{16\pi}\int_{\mathcal{V}} R\sqrt{-g}d^4x \tag{14}$$

One family of modified gravity theories is $f(R)$ gravity, which modifies the action to

$$S = \frac{1}{16\pi}\int_{\mathcal{V}} f(R)\sqrt{-g}d^4x \tag{15}$$

Using the action principle:

- The field equations are recovered.
- R can be decomposed as a bulk term $\mathcal{G}$ and a boundary term $\mathcal{B}$ that does not contribute to the action:

$$R = \underbrace{\mathcal{G}}_{\text{bulk term}} + \underbrace{\mathcal{B}}_{\text{boundary term}} \tag{16}$$

$f(R)$ gravity then becomes $f(\mathcal{G},\mathcal{B})$ gravity.

$f(\mathcal{G})$ gravity

In our so-called $f(\mathcal{G},\mathcal{B})$ gravity, the action is

$$S = \frac{1}{16\pi}\int_{\mathcal{V}} f(\mathcal{G},\mathcal{B})\sqrt{-g}d^4x \tag{17}$$

Now discard the boundary term. We then have $f(\mathcal{G})$ gravity with the action

$$S = \frac{1}{16\pi}\int_{\mathcal{V}} f(\mathcal{G})\sqrt{-g}d^4x \tag{18}$$

We can then derive the field equations in $f(\mathcal{G})$ gravity:

$$f'(\mathcal{G})\left(G_{\rho\sigma}+\frac{1}{2}g_{\rho\sigma}\mathcal{G}\right)+\frac{1}{2}f''(\mathcal{G})E_{\rho\sigma}{}^{\gamma}\partial_\gamma\mathcal{G}-\frac{1}{2}g_{\rho\sigma}f(\mathcal{G})=\kappa T_{\rho\sigma} \tag{19}$$

where $E_{\rho\sigma}{}^{\gamma}$ is a connection-like term we call the pseudoscalar connection

Linearised $f(\mathcal{G})$ gravity

We already know that in the absence of gauges, the linearised Einstein tensor has the standard result

$$G_{\rho\sigma} = \frac{1}{2}\left(h_{\rho\sigma}{}_{,\alpha}^{,\alpha} - h_{\sigma\alpha}{}_{,\rho}^{,\alpha} - h_{\rho\alpha}{}_{,\sigma}^{,\alpha} + h_{,\rho\sigma} - \eta_{\rho\sigma} h_{\alpha\beta}{}^{,\alpha\beta} + \eta_{\rho\sigma} h^{,\alpha}_{,\alpha}\right) \quad (20)$$

Let us now insert the linearised metric $g_{\rho\sigma} = \eta_{\rho\sigma} + h_{\rho\sigma}$ into the whole system of $f(\mathcal{G})$ gravity. The linearised bulk term is

$$\mathcal{G} = \frac{1}{2}(h^{\mu\nu}{}_{,\alpha} h^{\alpha}_{\nu\,,\mu} - h^{\mu\,,\beta}_{\beta} h_{,\beta}) \quad (21)$$

which is identical to the action generated by a rank-2 tensor field. The linearised pseudoscalar connection is

$$E_{\rho\sigma}{}^{\gamma} = h_{\rho\sigma}{}^{,\gamma} + h^{\gamma}_{\rho\,,\sigma} - h^{\gamma}_{\sigma\,,\rho} - \frac{1}{2}\eta_{\rho\sigma} h^{,\gamma} - \frac{1}{2}\eta^{\gamma}_{\rho} h_{,\sigma} + \eta^{\gamma}_{\sigma} h_{,\rho} - \eta^{\gamma}_{\sigma} h_{\alpha\rho}{}^{,\alpha} \quad (22)$$

Integer power near-polynomial

We are now in a position to insert test $f(\mathcal{G})$s. Inspired by test $f(R)$s in historical literature, we construct a 'near-polynomial' whose terms are almost exclusively integer powers of $\mathcal{G}$:

$$f(\mathcal{G}) = \cdots + c_{-2}\mathcal{G}^{-2} + c_{-1}\mathcal{G}^{-1} + \hat{c}\log|\mathcal{G}| + c_0 + c_1\mathcal{G} + c_2\mathcal{G}^2 + \cdots \quad (23)$$

Here the important task is dimensional analysis. Due to the nature of $h_{\mu\nu}$ as a *small* perturbation, only terms without or of linear order of $h_{\mu\nu}$ should survive.

By this criteria, the integer power near-polynomial $f(\mathcal{G})$ field equations become

$$c_1 G_{\rho\sigma} + \frac{1}{2}c_0 g_{\rho\sigma} = \kappa T_{\rho\sigma} \quad (24)$$

Sadly, under the choice of parameters $c_1 = 1$ and $\frac{1}{2}c_0 = \Lambda$, this reduces to good ol' GR

$$G_{\rho\sigma} + \Lambda g_{\rho\sigma} = \kappa T_{\rho\sigma} \quad (25)$$

Half-integer power near-polynomial

Let us try a more complex $f(\mathcal{G})$, which we will call a half-integer power near-polynomial:

$$f(\mathcal{G}) = \cdots + c_{-2}\mathcal{G}^{-2} + c_{-3/2}\mathcal{G}^{-3/2} + c_{-1}\mathcal{G}^{-1} + c_{-1/2}\mathcal{G}^{-1/2} + \hat{c}\log|\mathcal{G}| \\ + c_0 + c_{1/2}\mathcal{G}^{1/2} + c_1\mathcal{G} + c_{3/2}\mathcal{G}^{3/2} + c_2\mathcal{G}^2 + \cdots \tag{26}$$

which yields the field equations

$$\underbrace{G_{\rho\sigma} + \Lambda g_{\rho\sigma}}_{\text{GR}} + \underbrace{\lambda\left(4\mathcal{G}^{-1/2}G_{\rho\sigma} - \mathcal{G}^{-3/2}E_{\rho\sigma}{}^{\gamma}\partial_{\gamma}\mathcal{G} - 2\eta_{\rho\sigma}\mathcal{G}^{1/2}\right)}_{f(\mathcal{G})\text{ contribution}} = \underbrace{\kappa T_{\rho\sigma}}_{\text{GR}} \tag{27}$$

where GR is recovered with a choice of $\lambda = 0$, and the two $O(1)$ terms within the $f(\mathcal{G})$ contribution ultimately vanish together.

Trace equation

We are left with the nice-looking

$$G_{\mu\nu} - \lambda\eta_{\rho\sigma}\mathcal{G}^{1/2} = 0 \tag{28}$$

which, upon substitution of $h_{\mu\nu}$, is

$$A_{\lambda\nu}k^\lambda k_\mu + A_{\lambda\mu}k^\lambda k_\nu - A_{\mu\nu}k^\lambda k_\lambda - \eta_{\mu\nu}A_{\alpha\beta}k^\alpha k^\beta - 2\sqrt{2}\lambda\varrho\eta_{\mu\nu} = 0 \tag{29}$$

Accounting for symmetry, this is actually 10 equations in disguise. The simplest form is found by applying $\eta^{\mu\nu}$ on both sides:

$$2A_{\lambda\mu}k^\lambda k^\mu + Ak^\lambda k_\lambda = 2\sqrt{2}\lambda\varrho \tag{30}$$

where we use the shorthand $\varrho = \sqrt{-\bar{A}^{\mu\nu}k_\alpha\bar{A}^\alpha_\mu k_\nu}$. If we assume the same $A_{\mu\nu}$ and k_α, we get $\varrho = 0$ and hence $\lambda = 0$, which returns us to GR.

Constraints revisited

Let us instead assume a modified k_α and the most general possible $A_{\mu\nu}$

$$k^\mu = (\omega, 0, 0, \omega + \hat{\omega}) \quad A_{\mu\nu} = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{01} & A_{11} & A_{12} & A_{13} \\ A_{02} & A_{12} & A_{22} & A_{23} \\ A_{03} & A_{13} & A_{23} & A_{33} \end{pmatrix} \tag{31}$$

As it turns out, this can be simplified by observing their insertion into the trace equation and sensible gauges. Their simplest form is then actually

$$k^\mu = (\omega, 0, 0, \omega) \quad A_{\mu\nu} = \begin{pmatrix} \hat{A} & 0 & 0 & 0 \\ 0 & A_{11} & A_{21} & 0 \\ 0 & A_{12} & A_{22} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{32}$$

where k^μ is identical to its GR counterpart, and $A_{\mu\nu}$ only differs from its GR counterpart by a *small* perturbation $\hat{A}$.

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Coupling constant

By (again) inserting our modified k^μ and $A_{\mu\nu}$ into the field equations, we find that the coupling constant is

$$\lambda = \frac{\omega}{2} \tag{33}$$

giving rise to the field equations

$$G_{\mu\nu} - \frac{\omega}{2}\eta_{\mu\nu}\mathcal{G}^{1/2} = 0 \tag{34}$$

What are the physical implications of this?

- A third, longitudinal polarisation
- Possible dispersion relation

Conclusion

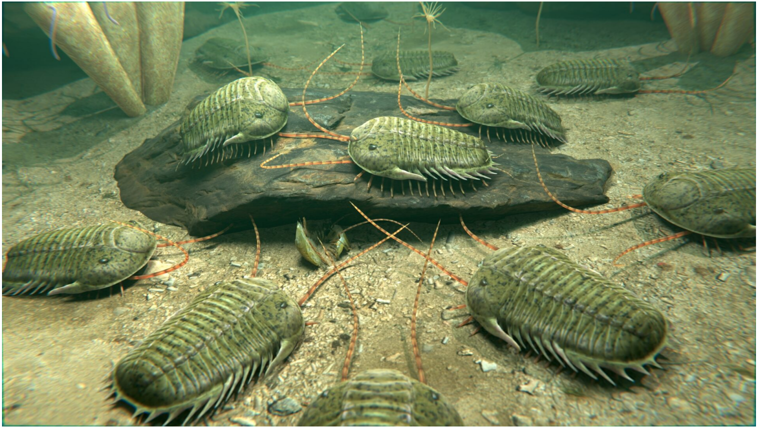

Figure: Trilobites during a parallel session.