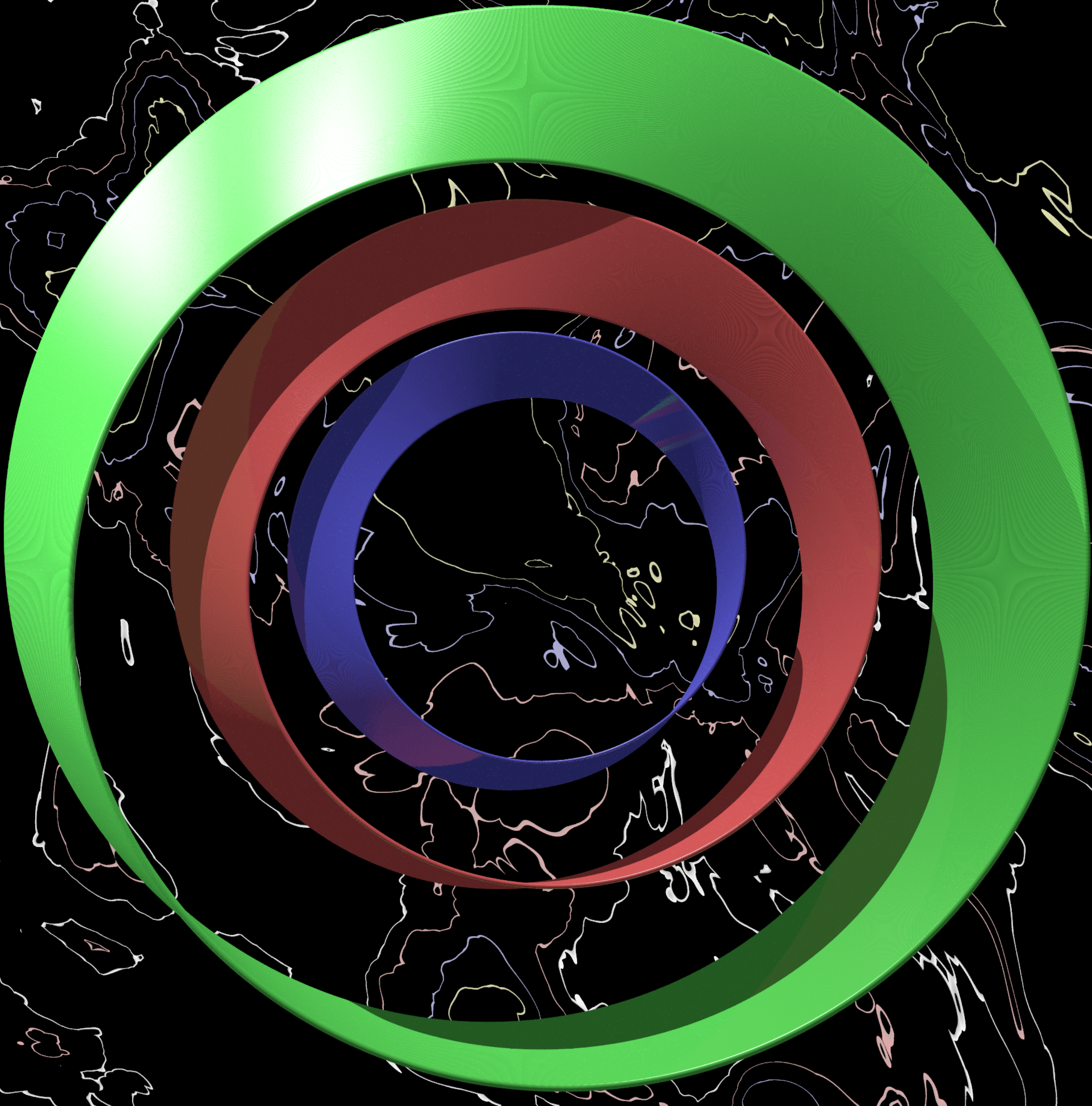
Spinors & Symmetries U(1)



N. Booker

To my parents

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Chapter 1

Instead of a foreword

1.1 Acknowledgements

Spinors & Symmetries began as chapter of another book before being expanded into a standalone book including Lie theory. It serves as an introduction to Lie groups, Lie algebras and spinors.

I would like to thank Prof. Betti Hartmann, whose summer studentship opening allowed me to investigate this topic. I am also eternally indebted to Abhijeet Vats, under whose guidance I was able to develop my LaTeX skills to a satisfactory level. Without them, this book would undoubtedly not have been in its current form.

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Chapter 2

Naive Lie theory

2.1 Lie derivatives

Definition 2.1 (Lie derivative) The *Lie derivative* evaluates the change of a tensor field along the flow defined by another vector field. For two vectors U and V, the Lie derivative of U^i with respect to (i.e. along) V^i is

$$\mathcal{L}_V U = V^j \partial_j U^i - U^j \partial_j V^i \tag{2.1}$$

Here:

- The first term represents the directional derivative of U along V. i.e. how U changes along the flow of U.
- The second term is associated with the change in the vector field V as it moves along U.

Remark 2.1 By comparing this to the directional derivative in vector calculus, it would be intuitive that the Lie derivative likewise transforms as a vector. Conceptually, the Lie derivative is the derivative of U along the flow generated by V.

Remark 2.2 Much like the covariant derivative, the Lie derivative illustrated how tensor fields change when 'dragged' along the flow generated by a vector field. Unlike the covariant derivative, however, it does not consider the underlying connection or curvature.

Definition 2.2 (Lie bracket) In some literature, the Lie derivative is written as the so-called *Lie bracket* instead:

$$\underbrace{[V, U]}_{\text{Lie bracket}} := \underbrace{\mathcal{L}_V U}_{\text{Lie derivative}} \tag{2.2}$$

Theorem 2.1 (Linearity of the Lie derivative) The Lie derivative is a linear operator, i.e. for vectors U, V and W:

$$\mathcal{L}_V (U + W) = \mathcal{L}_V U + \mathcal{L}_V W \tag{2.3}$$

$$\mathcal{L}_V (UW) = (\mathcal{L}_V U)W + U(\mathcal{L}_V W) \tag{2.4}$$

Remark 2.3 The vectors V and W can likewise be replaced by arbitrary functions f and g, and the Lie derivative stay the same by definition.

Derivation 2.1 (Higher-rank Lie derivatives) We can also derive the Lie derivative of a rank-2 tensor:

$$\mathcal{L}_V W_i^j = V^k \partial_k W_i^j + (\partial_i V^k) W_k^j - (\partial_k V^j) W_i^k \tag{2.5}$$

Remark 2.4 Here we see the tendency of the operator $\partial_i V^j$ to sacrifice one of its indices for the sake of the partial derivative as well as the target tensor, much like the poor Christoffel symbol in *Metric's Destiny*. However, we find that unlike the covariant derivative, terms which have arbitrary indices assuming covariant positions are *positive*.

2.2 Lie algebras

Definition 2.3 (Bilinear) For a vectors $u, v, w \in V$ and scalars $a, b \in F$, where V is a vector space over a field F[a], a *bilinear*, *bilinear product* or a *bilinear form* is a map $B : V \times V \to F$ that are linear with respect to both arguments

$$B(au + bv, w) = aB(u, w) + bB(v, w) \quad B(u, av + bw) = aB(u, v) + bB(u, w) \tag{2.6}$$

[a] e.g. real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

Remark 2.5 Dot products, matrix multiplications and inner products in vector spaces are all bilinear products.

Definition 2.4 (Algebra) A vector space V with a bilinear defined on it is known as an *algebra*.

Definition 2.5 (Lie algebra) A vector space V with a Lie bracket $[x, y]$, where $x, y \in V$ defined on it is known as a *Lie algebra*.

2.3 Lie groups

Before we proceed to Lie groups and Lie algrbras, it is intuitive to review the definition of a *group*.

Definition 2.6 (Group) A non-empty set G with a binary operation $\circ$ is a group if it satisfies the following properties:

- Closure:
$$a, b \in G \to a \circ b \in G \tag{2.7}$$
- Associativity:
$$(a \circ b) \circ c = a \circ (b \circ c) \tag{2.8}$$
- Identity:
$$\exists e \in G \quad \text{such that} \quad a \circ e = e \circ a = a \tag{2.9}$$
- Inverse:
$$\forall a \in G, \exists a^{-1} \in G \quad \text{such that} \quad a \circ a^{-1} = a^{-1} \circ a = e \tag{2.10}$$

Definition 2.7 (Lie group) The manifold on which the Lie algebra rests is called a *Lie group*, which is a group that is also a differentiable manifold.

Remark 2.6 Alternatively, one can regard a Lie algebra as the *tangent space* of a Lie group. If the Lie group is 3D, the Lie algebra is then a tangent plane.
Remark 2.7 A Lie group's corresponding Lie algebra is usually represented in *Fraktur*. For example, SO(3)'s Lie algebra is $\mathfrak{so}(3)$.

Definition 2.8 (Generator) In practice, we can think of Lie algebras as *generators* of a Lie group:

$$R = e^{\theta M} \tag{2.11}$$

where R are members of the Lie group and M (called the *generator*) are members of the corresponding Lie algebra and θ is the rotation angle.

Derivation 2.2 (Generator) Differentiating Equation 2.11 by θ yields

$$\partial_\theta R = g e^{\theta g} \tag{2.12}$$

Identifying θ as merely a phase, we set $\theta = 0$ for convenience and find that

$$\frac{\partial R}{\partial \theta}_{\theta=0} = g e^0 \tag{2.13}$$

Hence the generator can be so-created:

Theorem 2.2 (Generator)

$$g = \frac{\partial R}{\partial \theta}_{\theta=0} \tag{2.14}$$

2.4 Orthogonal and unitary groups

In starting the topic of spinors, we actually consider something quite simple: matrices. One can recall, perhaps from their quantum mechanics course, that

Definition 2.9 (Orthogonal operator) An operator is *orthogonal* if its Hermitian conjugate is its inverse:

$$\hat{A}^T = \hat{A}^{-1} \quad \text{or} \quad \hat{A}^T \hat{A} = 1 \tag{2.15}$$

Remark 2.8 A more satisfying definition is that the columns and rows forming the matrices are orthonormal vectors (i.e. the columns form an orthonormal basis).
Here, we can loosely equivalate a *matrix* and an operator. If one considers a *group* of such matrices[1], they will find a Lie group called the *orthogonal group*.

Definition 2.10 (Orthogonal group) For an n-dimensional space, we can denote the group of distance-preserving transformations by $\mathrm{O}(n)$.

Likewise we can define

Definition 2.11 (Unitary operator) An operator is *unitary* if its Hermitian conjugate is its inverse:

$$\hat{A}^\dagger = \hat{A}^{-1} \quad \text{or} \quad \hat{A}^\dagger \hat{A} = 1 \tag{2.16}$$

Again, we denote a group of such matrices as a *unitary group* which is likewise a Lie group.

Definition 2.12 (Unitary group) For an n-dimensional space, we can denote the group of all unitary $n \times n$ matrices by $\mathrm{U}(n)$.

Remark 2.9 All orthogonal matrices are unitary. i.e. orthogonal matrices are a specific case of unitary matrices. The significance of this will soon be apparent.

Definition 2.13 (Special group) For an n-dimensional space, we can denote the group of all unitary matrices with determinant 1 (i.e. $\det A = 1$) by $\mathrm{S}(n)$. These matrices are known as *unimodular matrices*.

Combining this with the previously mentioned Lie groups yields two more sophisticated Lie groups:

Definition 2.14 ($\mathrm{SO}(n)$ and $\mathrm{SU}(n)$ groups) By considering unimodular matrices only, we can define the following:

- A $\mathrm{O}(n)$ group whose elements are unimodular is called a *special orthogonal group* or a $\mathrm{SO}(n)$ group.
- A $\mathrm{U}(n)$ group whose elements are unimodular is called a *special unitary group* or a $\mathrm{SU}(n)$ group.

2.5 Sneak peek of representation theory

We end our formalism with an overview of group representations, a part of representation theory. As physicists, we are merely satisfied with the layman's definition: a *representation* of some abstract mathematical object is the description of its elements in terms of matrices and algebraic operations. In a more rigorous definition:

[1] In the scope of this book, you do not need to have learned group theory before for this.

Definition 2.15 (Group representation) A *group representation* is some map ϕ that gives, from a n-dimensional group member G, an invertible $n \times n$ matrix. ρ preserves the algebraic operations $\circ$ defined on the group:

$$\forall A, B \in G \quad \rho(A \circ B) = \rho(A)\rho(B) \tag{2.17}$$

Remark 2.10 A consequence of this is that the identity of the group must be mapped to an identity matrix $\mathbb{I}$:

$$\rho(I) = \mathbb{I} \tag{2.18}$$

Chapter 3

Specific Lie groups

3.1 General ideas

Before starting any specific derivations, we first note down some general ideas which will be reflected in later derivations:

- The generators we derive are merely the whole list of generators *that are independent from each other*[1].
- Addition or subtraction between two generators always yields an antisymmetric (and traceless if SU(n)) matrix which is then another generator.
- Multiplication between two generators does not always yield another generator. The result is not necessarily antisymmetric (but always traceless for SU(n)).
- We do not consider division as algebras only need to have addition, subtraction and multiplication defined.
- In physics, the generators, which are usually anti-Hermitian, used are often multiplied by $-i$ to make them Hermitian.

3.2 SO(n) groups

Before considering the SO(3) group (or indeed, a generic special orthogonal group SO(n)), it is expedient to look at a general case, which is the orthogonal group O(n).
O(n) consists of all $n \times n$ matrices R such that

$$R^T R = \mathbb{I}_n, \quad \det(R) = \pm 1 \tag{3.1}$$

In terms of its generators A, R is predictably represented by

$$R = e^A, \quad A \in \mathfrak{o}(n). \tag{3.2}$$

where A are the collection of $n \times n$ skew-symmetric matrices A (i.e., $A^T = -A$) and make up the Lie algebra $\mathfrak{o}(n)$.
A group SO(n) is identical to O(n) with one exception: its elements S can only have determinant $+1$ instead of ± 1:

$$R^T R = \mathbb{I}_n, \quad \det(R) = 1 \tag{3.3}$$

again, its generators A for which $R = e^A$ are members of the Lie algebra $\mathfrak{so}(n)$.
Remark 3.1 Importantly, the physical implication of SO(n)'s extra rescriction is that SO(n) includes only the rotation components of O(n), excluding reflections.

[1]This is like how we say a 4D Ricci tensor has only 10 (independent) components instead of 16 due to symmetry.

Derivation 3.1 ($\mathrm{O}(1)$ and $\mathrm{SO}(1)$ groups) The group $\mathrm{O}(1)$ consists of all 1×1 orthogonal matrices, which implies that $A^2 = 1$ and $\det A = \pm 1$. Possible As are hence

$$A = \pm 1 \to \mathrm{O}(1) = \{1, -1\} \tag{3.4}$$

-1 and 1 correspond to reflection and identity respectively.
Now we consider $\mathrm{SO}(1)$, which consists of all 1×1 orthogonal matrices with determinant 1. As it turnns out, there is only one candidate that fits the description:

$$A = 1 \to \mathrm{SO}(1) = \{1\} \tag{3.5}$$

Thus, the $\mathrm{SO}(1)$ group is trivial, containing only the identity element.

Derivation 3.2 ($\mathrm{O}(2)$ and $\mathrm{SO}(2)$ groups) The $\mathrm{O}(2)$ group or the *group of planar isometries* consists of all 2×2 orthogonal matrices A which then satisfy $A^T A = \mathbb{I}_2$ and $\det A = \pm 1$, which ensures that A preserves lengths and angles. In 2 dimensions, $\mathrm{O}(2)$ can be described as the group of rotations and reflections in the plane:

- Rotations:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.6}$$

- Reflections[a]:

$$M(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{3.7}$$

where $\theta \in [0, 2\pi)$. Thus, $\mathrm{O}(2)$ can be expressed as

$$\mathrm{O}(2) = \{R(\theta) \mid \theta \in [0, 2\pi)\} \cup \{M(\theta) \mid \theta \in [0, 2\pi)\} \tag{3.8}$$

Now we consider the $\mathrm{SO}(2)$ group or the *circle group*, which consists of all 2×2 orthogonal matrices with determinant 1. This condition means that reflections are excluded, and the only transformations remaining are (pure) rotations. Thus, $\mathrm{SO}(2)$ can be expressed as:

$$\mathrm{SO}(2) = \{R(\theta) \mid \theta \in [0, 2\pi)\} \tag{3.9}$$

[a] Also known as *improper rotations*.

Derivation 3.3 ($\mathrm{O}(3)$ and $\mathrm{SO}(3)$ groups) The group $\mathrm{O}(3)$ or the *group of 3D isometries* consists of all 3×3 orthogonal matrices, which then satidfy $A^T A = \mathbb{I}_3$ and $\det A = \pm 1$, where $\det(A) = 1$ represent rotations while $\det(A) = -1$ represent reflections. Thus, $\mathrm{O}(3)$ can be written as:

$$\mathrm{O}(3) = \{A \in \mathbb{R}^{3\times 3} \mid A^T A = I, \det(A) = \pm 1\}. \tag{3.10}$$

The $\mathrm{SO}(3)$ group or the *rotation group* has only As with determinant $+1$, which are proper rotations. As such, we have

$$\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det(R) = 1\}. \tag{3.11}$$

$\mathrm{SO}(3)$ can be parameterized in terms of a rotation axis and angle:

Definition 3.1 ($\mathrm{SO}(3)$ rotation matrices)

$$R_{xy}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_{yz}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \quad R_{zx}(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \tag{3.12}$$

to the following generators that belong to the $\mathfrak{so}(3)$ Lie algebra

Definition 3.2 ($\mathfrak{so}(3)$ generators)

$$g_{xy} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad g_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad g_{zx} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{3.13}$$

Like we discussed before, we can make the generators Hermitian by applying a factor of i, which gives

$$g_{xy} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad g_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad g_{zx} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{3.14}$$

3.3 SU(n) groups

SU(n) groups are often used in high energy physics:

- SU(2) encodes spin and isospin (i.e. QED).
- SU(3) encodes QCD.
- SU(2) × SU(2) encodes the (Euclidian) Lorentz groups.

Each SU(n) group has $n^2 - 1$ real group parameters (i.e. generators M_i), which are Hermitian and traceless. They form the Lie algebra $\mathfrak{su}(n)$. To study SU(n), it is useful to begin by looking at the U(n) groups. Unitary transformations stem from U(n) where n is the dimensionality/a constant associated with the degrees of freedom. Each set of transformations is then said to be generated[2] by a set of *generators*:

$$R_i = e^{iM_i\theta} \tag{3.15}$$

Remark 3.2 The U(n) group has n^2 generators M_i which are $n \times n$ matrices.
For some dimension n, infinitesimal spin rotation (or equivalents) can be approximated as a phase:

$$\lim_{n\to\infty} \left(1 + \frac{i\alpha}{n} M_i\right)^n \approx e^{iM_i\theta} = R_i \tag{3.16}$$

This leads to the generalised phase change $\phi \to \phi' = \phi e^{iM_i \cdot \theta}$, which allows for us to represent these spin rotations as a Lie group.

Derivation 3.4 (U(1) and SU(1) groups) A general phase change $e^{i\theta}$ is governed by U(1), also called the *circle group*. In other words, the elements of U(1) are $e^{i\theta}$ for all possible θs:

$$U(1) = \{e^{i\theta} | \theta \in [0, 2\pi)\} \tag{3.17}$$

This group only has one generator: the 1D unit matrix $\mathbb{I}_1$. This makes up the Lie algebra $\mathfrak{u}(1)$.

$$\mathfrak{u}(1) = \mathbb{I}_1 = (1) \tag{3.18}$$

Remark 3.3 U(1) is isomorphic to SO(2) and is also called the circle group.
Now we look at the SU(1) group. We impose the condition $\det R = 1$, after which the only remaining element is the number 1. This makes SU(1) quite a trivial group as no transformation happen at all. Even more so is its Lie algebra $\mathfrak{su}(1)$, which consists only of the number 0 - the generator of 1.

Derivation 3.5 (U(2) and SU(2) groups) These are the groups that govern spin-$\frac{1}{2}$. If we recall quantum mechanics, we will see that the wavefunction is transformed by $\psi \to \psi' = \psi e^{iS\theta}$:

$$U(2) = \{e^{iS\theta} | \theta \in [0, 2\pi)\} \tag{3.19}$$

Hence for U(2), one has the set of 4 generators $\vec{S} = \frac{1}{2}\vec{\sigma}$, where σ_i are the infamous Pauli matrices:

[2] To put it unimaginatively...

Definition 3.3 (Pauli matrices)

$$\sigma_0 = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.20}$$

After halving, these matrices make up the Lie algebra $\mathfrak{u}(2)$. σ_0 corresponds to a general phase change while the rest correspond to rotations on the yz, xz and xy planes.
One might recall that in many sources σ_0 is omitted, as it is merely the 2D identity matrix in a new coat of paint. There is, however, another reason. For SU(2), we have only 3 generators: the universally accepted Pauli matrices σ_1, σ_2 and σ_3. This is because $e^{i\sigma_0\theta}$ fails the conditon $\det R = 1$, which is required by SU(2). Finally, the Lie algebra $\mathfrak{su}(2)$ are made up of Σ_1, Σ_2 and Σ_3, which are the Pauli matrices σ_1, σ_2 and σ_3 with a prefactor of 1/2. They satisfy

$$[\Sigma_j, \Sigma_k] = i\epsilon_{jlk}\Sigma_l \tag{3.21}$$

Remark 3.4 Here we make a significant observation: In 2D and above, the generators of $\mathrm{U}(n)$ is always the combination of those of $\mathrm{SU}(n)$ and U(1). One then says that $\mathrm{U}(n)$ is isomorphic to the direct product of $\mathrm{SU}(n)$ and U(1). i.e.

Theorem 3.1 ($\mathrm{U}(n)$-$\mathrm{SU}(n)$ relation)

$$\mathrm{U}(n) = \mathrm{SU}(n) \times \mathrm{U}(1) \tag{3.22}$$

Derivation 3.6 (U(3) and SU(3) groups) For QCD we have one more dimension. The transformations are $R_i = e^{i\frac{1}{2}\lambda_i\theta}$:

$$U(2) = \{e^{i\frac{1}{2}\lambda\theta} | \theta \in [0, 2\pi)\} \tag{3.23}$$

$\frac{1}{2}\lambda_i$ are the generators for U(3) called the *Gell-Mann matrices*:

Definition 3.4 (Gell-Mann matrices)

$$\begin{gathered}
\lambda_0 = \mathbb{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \\
\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{gathered} \tag{3.24}$$

Again, all $\frac{1}{2}\lambda_i$ matrices make up the Lie algebra $\mathfrak{u}(3)$.
To preserve $\det R = 1$, the identity matrix in disguise λ_0 is eliminated for $\mathfrak{su}(3)$. Hence SU(3) has only 8 generators λ_1 to λ_8 - the universally accepted Gell-Mann matrices.

The Gell-Mann matrices are the SU(3) equivalent of the SU(2) Pauli matrices. While the SU(2) eigenvectors correspond to a physical quantity (spin-up and spin-down states in spin-1/2 systems), the Gell-Mann matrices' eigenstates do not directly correspond to physical quantities.
Instead, one can think of them as different 'basis states' in the *colour space*, and linear combinations of these states describe physical colour configurations of quarks and gluons.

Definition 3.5 (Colour basis) The eigenvectors of the Gell-Mann matrices are known as the *colour basis*:

$$|\mathrm{red}\rangle = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad |\mathrm{green}\rangle = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad |\mathrm{blue}\rangle = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{3.25}$$

From the 3 eigenstates, we have 3 colours: **red**, **green** and **blue**. They are complemented by the three antiquark colours: **antired**, **antigreen** and **antiblue**. A general *colour state* can therefore be represented as

$$c = \alpha c_1 + \beta c_2 + \gamma c_3 \tag{3.26}$$

where α, β and γ are complex numbers. The complex numbers represent probabilities and phase relationships in quantum states, same as complex spin states.
Just like how Pauli matrices can be combined to form the so-called *ladder operators* to move between spin states, we can also combine the Gell-Mann matrices to form ladder operators to move between colour states:

Definition 3.6 (QCD ladder operators)

$$\underbrace{T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)}_{\text{red} \to \text{green}} \quad \underbrace{V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5)}_{\text{red} \to \text{blue}} \quad \underbrace{U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)}_{\text{green} \to \text{blue}} \tag{3.27}$$

3.4 SO(1,3) group

We now consider the SO(1,3) group or the so-called *Lorentz group*, which includes all Lorentz transformations (i.e. rotations and boosts) in Minkowski spacetime[3].

Definition 3.7 (Isometry) A coordinate transformation $X^a \to X'^b$ is called an *isometry* or a *local isometry* if the metric (i.e. the line element) is invariant.

The Lorentz group encode all Minkowski spacetime isometries *under which the origin is invariant.* A generalisation of the Lorentz group (or in this context, the *homogeneous Lorentz group*) that includes all isometries is hence the *inhomogeneous Lorentz group* or the *inhomogeneous special orthogonal group* or the *Poincaré group*.
Remark 3.5 The only difference between the two is that the Poincaré group includes isometries that change the position of the origin. Now what is another name for this?
If you have translations in mind, then you would be right.

Definition 3.8 (Poincaré group) A Poincaré group ISO(1,3)[a] is the full symmetry group of Minkowski spacetime (i.e. SR).

[a] As you may have guessed, ISO stands for inhomogeneous special orthogonal group.

It is then intuitive that an element of the Poincaré group would be called the *Poincaré transformation.* An would be a combination of a Lorentz transformation $\lambda_{\mu\nu}$ and a translation vector a_μ:

$$x_\mu \to x'\mu = \Lambda_{\mu\nu} x^\nu + a_\mu \tag{3.28}$$

or in wavefunction form

$$\phi(x) \to \phi(\Lambda x + a_\mu) \tag{3.29}$$

Remark 3.6 In 3D, the Poincaré group reduces to an *(inhomogeneous) Galiean group* ISO(3), also known as a *Euclidian group* E(3).

Derivation 3.7 (SO(1,3) group) To construct the $\mathfrak{so}(1,3)$ algebra, one first parameterise the generators of the SO(1,3) group. Instead of the usual notation J^a for generators, we use $\omega_{\rho\sigma}J^{\rho\sigma}$ where $\omega_{\rho\sigma}$ are a series of parameters. It then follows that, for some metric trace g

$$(e^{i\omega_{\rho\sigma}J^{\rho\sigma}})^T g e^{i\omega_{\rho\sigma}J^{\rho\sigma}}) = g \tag{3.30}$$

which gives

$$i\omega_{\rho\sigma}((J^{\rho\sigma})^T g J^{\rho\sigma}) = 0 \tag{3.31}$$

Removing the prefactor and unpacking the metric trace yields

$$J^{\rho\sigma}_{\alpha\nu} g^{\alpha\mu} + g_{\nu\alpha} J^{\rho\sigma\alpha\mu} = J^{\rho\sigma\mu}_{\nu} + J^{\rho\sigma\nu}_{\mu} = 0 \tag{3.32}$$

[3] Sometimes one includes reflections, and the Lorentz group simply becomes O(3). Its identity element, SO(1,3), is then called the *rescricted Lorentz group*.

which means that this rank-4 J must be antisymmetric in μ and ν. It then makes sense for us to choose it to be also antisymmetric in ρ and σ. This yields the expression

$$J_{\nu}^{\rho\sigma\mu} = i(g^{\rho\mu}\delta_{\nu}^{\sigma} - g^{\sigma\mu}\delta_{\nu}^{\rho}) \tag{3.33}$$

To recover the rank-2 $J^{\rho\sigma}$, we set $\mu = \nu$ and contract them. Solving the resultant expression gives the list of generators:

Definition 3.9 ($\mathfrak{so}(1,3)$ generators)

$$\begin{gathered}
J^{10} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^{20} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^{30} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\
J^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad J^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}
\end{gathered} \tag{3.34}$$

Theorem 3.2 ($\mathfrak{so}(1,3)$ generator commutation relations) By recovering the $\mathfrak{su}(2)$ generators as $\Sigma_j = \frac{1}{2}\epsilon_{jkl}J^{kl}$, we find the $\mathfrak{so}(1,3)$ generator commutation relations:

$$[J^{j0}, J^{k0}] = -i\epsilon_{jkl}J^{l0} \quad [\Sigma_j, j^{k0}] = i\epsilon_{jkm}\Sigma_l \tag{3.35}$$

where the first and second item is simply the $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ generator commutation relations respectively.

In fact, we can note that the generators in Equation 3.34 include the generators in Equation 3.13 in the form of J^{12}, J^{13} and J^{23}, where we have expanded into the time coordinate by adding one row on top and one column to the left. The purposes of the generators are hence clear:

- *Rotations* are generated by J^{12}, J^{13} and J^{23}.
- *Boosts* are generated by J^{10}, J^{20} and J^{30}.

Chapter 4

Spinors

4.1 Emergence of spinors

But what does this look like in practice? We can, for example, consider the innocent 2D space. Both SO(2) and SU(2) reduce to the form of

$$U = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{4.1}$$

There is, however, one crucial difference: for SO(2), both a and b are real, while for SU(2) they are complex. In fact, we can make a further simplification here. In the case of SO(2), the group of matrices reduce further, and contains only 2D rotation matrices which will presumably look quite familiar:

$$U_{\mathrm{SO}(2)} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{4.2}$$

Remark 4.1 This is rotation around a point. The SO(3) group is not much worse, consisting only of rotations on a line. For this reason, SO(n) groups are also called *rotation groups*.
We remain in 2D space for a bit more and investigate how the elements of SO(2) and SU(2) operate. Quite intuitively, SO(2) operates on real vectors:

$$\vec{v}' = U_{\mathrm{SO}(2)}\vec{v} \tag{4.3}$$

For SU(2), the transformation matrix is a bit more tricky. Consider a rotation by θ:

$$U_{\mathrm{SU}(2)} = \exp\left(-\frac{i\theta}{2}X \cdot \boldsymbol{\sigma}\right) \tag{4.4}$$

where $X = (x, y, z)$ are the unit vectors and $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.
Nonetheless, we would ideally observe the same stratagem, but with the *complex* vectors:

$$\psi' = U_{\mathrm{SU}(2)}\psi \tag{4.5}$$

This ψ is called a *spinor*. Here, we might be tempted to commit the same old mistake we did in the very beginning and presume that spinors are just tensors with complex components.
However, one can immediately see that this is not the case even easier than how we realised that tensors are more than glorified matrices. This is because tensors, on their own, can be complex. Instead, spinors are defined with respect to orthogonal rotation groups. Spinors are not something 'physical' because they are not invariant under coordinate transforms.

Definition 4.1 (Spinor) For the transformation matrix $U \in \mathrm{SO}(n)$ in n dimensions, a spinor[a] S transforms as

$$\psi' = U\psi \tag{4.6}$$

[a] Pronounced like 'spinner'.

Spinors in 3D space transform invariantly under SO(3), and spinors in 4D spacetime transform invariantly under SO(1, 3). We will see this in detail in the Pauli and Weyl spinors.

Remark 4.2 Here we note that SO(1, 3) is distinct from SO(4). The former is defined with respect to the 4D Lorentian spacetime metric (i.e. Minkowski metric), while the latter is defined with respect to a metric with 4 spatial dimensions and signature $(+,+,+,+)$.

4.2 Pauli spinor

Derivation 4.1 (Factoring of matrices) We first revise a bit of mathematics: We can *factor* a range of matrices into a vector-dual vector pair. For example:

$$\begin{pmatrix} 1 & 100 \\ 4 & 400 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 100 \end{pmatrix} \tag{4.7}$$

where $\otimes$ is the *tensor product*. As a reminder, we can see how the components of the pair are determined by writing the tensor product more intuitively:

$$\begin{array}{cc} & \begin{pmatrix} 1 & 100 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 4 \end{pmatrix} & \begin{pmatrix} 1 & 100 \\ 4 & 400 \end{pmatrix} \end{array} \tag{4.8}$$

Note that not all matrices can be factored directly: One example is $\begin{pmatrix} 1 & 100 \\ 4 & 500 \end{pmatrix}$. For direct factoring, the matrix must satisfy:

- Columns must be multiples of each other.
- Rows must be multiples of each other.
- The determinant of the matrix must be 0.

Still we can factor other matrices. We can, for example, 'break down' the matrix into components:

$$\begin{pmatrix} 1 & 100 \\ 4 & 500 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 100 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 500 \end{pmatrix} \tag{4.9}$$

Now we can calculate vector-dual vector pair for each matrix. The end result is the sum of a series of pairs:

$$\begin{pmatrix} 1 & 100 \\ 4 & 500 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} + 100 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} + 500 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{4.10}$$

Remark 4.3 We can multiply one vector/dual vector by some number m and divide the other by m. The resulting pair will still be a solution. However, for all solutions $\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix}$, the ration $\frac{a}{b}$ must be the same.

Normally, from a set of x, y and z coordinates, one represents a vector by $\vec{v} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$.
While this is indeed the most intuitive way to write down a vector, one should note that $\vec{v} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ is merely a formalism. We use it because it is the simplest formalism that successfully encodes information about x, y and z. For example, one can always encode 3D coordinates using a system of three equations with three unknowns (and only one solution for each), but we are not motivated to do this as it is very ineffective.
However, for reasons which will be apparent later on, we often want to encode a 3-vector with Pauli matrices instead. We first recall the Pauli matrices:

Definition 4.2 (Pauli matrices)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.11}$$

As it turns out, a combination of all three Pauli matrices, or rather a *Pauli vector*, can encode any

3-vector. Consider a set of coordinates x, y and z, we can represent them with a so-called *Pauli vector*:

Definition 4.3 (Pauli vector)

$$V = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{4.12}$$

Remark 4.4 Even though it is mathematically a 2×2 matrix, it still physically represents the same 3 coordinates as its 3-vector counterpart. In fact, the determinant of the Pauli vector is the negative of the magnitude of the vector squared:

$$\det V = -||\vec{v}||^2 \tag{4.13}$$

Theorem 4.1 (Conjugation of Pauli vectors) We can flip a Pauli vector by the i direction by using *conjugation*:

$$V' = -\sigma_i V \sigma_i \tag{4.14}$$

Noting that the Pauli matrices are unitary, we are somewhat inspired to find a connection between them and the aforementioned SU groups. If we consider a rotation by π on the xy plane, we simply do that by using

$$V' = -\sigma_y(-\sigma_x V \sigma_x)\sigma_y = \sigma_y \sigma_x V (\sigma_y \sigma_x)^\dagger \tag{4.15}$$

We see that this is not quite the same as how spinors rotate - this is because we are so far one step away from spinors. But we can formulate how Pauli vectors rotate.

Theorem 4.2 (Rotation of Pauli vectors) Assuming a rotation matrix A:

$$V' = AVA^\dagger \tag{4.16}$$

where A and $A^\dagger$ are each a so-called *half-rotation*.

A is 2D, has determinant 1 and is unitary (i.e. $V' = V$). Thus, it must belong to the SU(2) group. Every two elements in SU(2) correspond to (i.e. map to) one element in SO(3). This is known as the the double cover SU(2) $\to$ SO(3).

Theorem 4.3 (Double cover invariance) In n dimensions, a spinor is invariant under any transformation group that is a double cover of the SO(n) group.

Definition 4.4 (Pauli spinor) A Pauli vector can be decomposed into two *Pauli spinors* or a spinor-dual spinor pair. Due to its unique structure, the spinors simplify:

$$\begin{pmatrix} z & x - iy \\ x + iy & z \end{pmatrix} \to \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \begin{pmatrix} \zeta_1 & \zeta_2 \end{pmatrix} \tag{4.17}$$

where

$$\xi^1 = \zeta_2 = \sqrt{x - yi} \quad \xi^2 = -\zeta_1 = -i\sqrt{x + yi} \tag{4.18}$$

Remark 4.5 Again, ξ^1 and ξ^2 are not related to Killing vectors.
We can plug the pair form back into the rotation of Pauli vectors:

$$V' = A \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \begin{pmatrix} \zeta_1 & \zeta_2 \end{pmatrix} A^\dagger \tag{4.19}$$

Here, it can be seen that A operates on the spinor $\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$, while $A^\dagger$ operates on the dual spinor $\begin{pmatrix} \zeta_1 & \zeta_2 \end{pmatrix}$. The nominal significance of half-rotations is thus seen. Now consider the equivalence of a 3D vector v_i and a Pauli vector V^a_b. We can expand this to higher ranks, and thus represent all tensors with spinors. We can do so by the Pauli matrices, which, as a matter of fact, each have three indices: σ^a_{ib}, where $_i$ is the *tensor index* ^a_b is the *spinor index*.

$$g_{ij}\sigma^a_{ib}\sigma^c_{jd} = g^{ac}_{bd} \tag{4.20}$$

Remark 4.6 Every two spinor indices correspond to one tensor index. Thus, a spinor is often informally called a rank $\frac{1}{2}$ tensor. This is, however, a very misleading statement as a spinor of dimension n does *not*

have $\sqrt{n}$ components. As we have just seen, a 3D spinor[1] has 2 components.
We can further sit on this by considering the following: If we take both vector and spinor indices into account, σ^c_{jd} is effectively a *map* which takes us from the 3D vector space to the 4D spinor space, which is the tensor product of the 2D spaces of the covariant and contravariant spinors.

4.3 Weyl spinor

As the whole idea of Pauli spinors is based on the Pauli matrices in x, y and z directions, we can comfortably conclude that Pauli spinors are associated with 3D space. In GR, where we consider 4D spacetime, *Weyl spinors*, which also have 2 components, are used instead.
We can consider an analogue of the Pauli vector in 4D spacetime. This is the *Weyl vector*.

Definition 4.5 (Weyl vector)

$$W = t\sigma_t + x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \tag{4.21}$$

where σ_t is simply I, the identity matrix.

Remark 4.7 The determinant of the Weyl vector is the proper distance squared. i.e. the spacetime interval:

$$\det V = s^2 \tag{4.22}$$

Derivation 4.2 (Lorentz transformation) One notable example is the Lorentz transformation or the *Lorentz boost*. In $\mathrm{SO}(1,3)$, one has 6 transformations in total:

- Rotation in the xy, yz and zx planes
- Boosts in the x, y and z directions

For a general Lorentz transformation $\Lambda \in \mathrm{SO}(1,3)$, we have the corresponding *special linear group* $\mathrm{SL}(2,\mathbb{C})$[a] transformation L which acts on Weyl spinors. Every two elements in SL(2) correspond to (i.e. map to) one element in $\mathrm{SO}(1,3)$. Again, this is a double cover $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)$.
A Weyl spinor ψ transforms under a Lorentz transformation as $\psi' = L\psi$, where the precise form of L is yet to be determined. Again as $L \in \mathrm{SL}(2,\mathbb{C})$, we have $\psi' = \psi$
We define the concept of *rapidity*:

Definition 4.6 (Rapidity)

$$w = \tanh^{-1}(v/c) \tag{4.23}$$

For a boost in some direction i by rapidity w:

$$S_i = \exp\left(\frac{w}{2}\sigma_i\right) \tag{4.24}$$

[a] $\mathbb{C}$ reminds us that we are dealing with complex numbers.

Definition 4.7 (Weyl spinor) A Weyl vector can be decomposed into two *Weyl spinors* or a spinor-dual spinor pair:

$$\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \to \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \begin{pmatrix} \psi^{1*} & \psi^{2*} \end{pmatrix} \tag{4.25}$$

where

$$|\psi^1| = \sqrt{ct+z} \quad |\psi^2| = \sqrt{ct-z} \tag{4.26}$$

Now we consider inner products. We want them to be invariant under rotations. This is simple for Pauli spinors as their rotations are unitary. Thus the inner product of two Pauli spinors ξ and χ are simply $\xi^\dagger\chi$. For two Weyl spinors, however, the linear transformations are not always unitary. So we introduce a 'correction' matrix to ensure that the inner product is preserved under transformations.

[1] Even though a Pauli spinor is itself 2D, its corresponding vector Lies in 3D space.

Theorem 4.4 (Weyl spinor inner products) The inner product between two Weyl spinors ψ and ϕ is $\psi^T \epsilon \phi$ where we have the *spinor metric*

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.27}$$

Note 4.1 (Dual spinor form of a Weyl spinor) The dual spinor form of a spinor is hence given by $\psi^T \epsilon$.

Remark 4.8 Note that this leads to $L^T \epsilon = \epsilon L^{-1}$.
Due to this lack of direct correspondence there is an extra element called *chariality* which essentially denotes the *handedness* of a Weyl spinor. Each physical transformation of Weyl spinors corresponds to two matrices in the $\mathrm{SL}(2, \mathbb{C})$ group, if one adjusts the axes of the first matrix, they find the second matrix, making both physically equivalent. One is then said to be *left-handed* and the other is said to be *right-handed*. Mathematically, the two matrices are complex conjugates of each other and give rise to left-handed and right-handed Weyl spinors.
Remark 4.9 By convention, we take the previously seen Weyl spinors as left-handed.

Definition 4.8 (Right-handed Weyl spinors) We can derive the right-handed Weyl spinors from their left-handed counterparts:

- The left dual spinor $\psi^{\dot{a}}$ is the complex conjugate of the right spinor ψ_a.
- The right dual spinor $\psi_{\dot{a}}$ is the complex conjugate of the left spinor ψ^a.

We can thus summarise the Weyl and dual Weyl spinors:

Weyl and dual Weyl spinors

Type	Lorentz transformation	Notation	In terms of left spinor components
left	$\psi \to L\psi$	$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$	self
left dual	$\psi^T \epsilon \to \psi^T \epsilon L^{-1}$	$\begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$	$\begin{pmatrix} -\psi^2 & \psi^1 \end{pmatrix}$
right	$\psi^\dagger \epsilon \to \psi^\dagger \epsilon (L^{-1})^*$	$\begin{pmatrix} \psi_{\dot{1}} & \psi_{\dot{2}} \end{pmatrix}$	$\begin{pmatrix} -\psi^{2*} & \psi^{1*} \end{pmatrix}$
right dual	$\psi^* \to L^* \psi^*$	$\begin{pmatrix} \psi^{\dot{1}} \\ \psi^{\dot{2}} \end{pmatrix}$	$\begin{pmatrix} \psi^{1*} \\ \psi^{2*} \end{pmatrix}$

Figure 4.1: Hermann and Helene Weyl (Konrad Jacobs, March 1913)

4.4 Dirac spinor

When dealing with spin-$\frac{1}{2}$ particles in SR, we need to keep track of both charalities. As such we employ the *Dirac spinor*.

Definition 4.9 (Dirac spinor) The Dirac spinor is simply a left-chiral Weyl spinor stacked on top of a right-chiral Weyl spinor which is switched from row representation to column representation. As such, it has 4 components.

Definition 4.10 (Double cover of SO(3) by SU(2)) e recall that a Pauli vector V transforms as

$$V' = AVA^{\dagger} \tag{4.28}$$

where $R \in \mathrm{SO}(3)$. For the equivalent 3D vector $\vec{v}$, the equivalent is

$$\vec{v}' = R\vec{v} \tag{4.29}$$

where $R \in \mathrm{SO}(3)$. As we have to use both A and $A^{\dagger}$ to accomplish what R did,

In real life, we are primarily concerned with Pauli, Weyl and Dirac spinors due to the dimensionality of our own spacetime. However, we ultimately want to find a mechanism to generate spinors in any number of dimensions. This mechanism is the *Clifford algebra*, which we will strive to arrive at in the next few sections.

4.5 Motivating examples in physics

Remark 4.10 A famous motivating example is usually the comparison of physical space and state space. In physical space, the spin-up and spin-down states $|\uparrow\rangle$ and $|\downarrow\rangle$ are π apart. In state space, as they must be orthogonal, they are $\pi/2$ apart. In state space, a state returns itself after one spin (2π). A similar relation happens in polarisation with respect to physical space and polarisation space. After half a rotation in physical space or a whole rotation in polarised space, the wave is phase-shifted by π and has to do another rotation in polarised space to return to its original phase.
Fun fact 4.1 In fact, spinors came from the idea of the Dirac equation and the theory of complex potentials.
Remark 4.11 In particle physics:

- Spin-0 particles (e.g. Higgs bosons) are represented by scalars.
- Spin-$\frac{1}{2}$ particles (e.g. quarks, electrons, neutrinos) are represented by spinors.
- Spin-1 particles (e.g. photons, gluons, W and Z bosons) are represented by vectors.
- Spin-2 particles (e.g. gravitons) are represented by matrices (in the case of the graviton, the stress-energy tensor).

Chapter 5

Algebras

5.1 Grassmann algebra

> **Quote 5.1** In this context, we're discussing "an algebra" which refers to something like a number system - a mathematic structure that includes some sort of number and a multiplication operation between them. This is different from the general term "algebra" which describes math involving variables.
>
> *Ian Dunn and Zoë Wood, in the* Graphics Programming Compendium*.*

Before Clifford algebras, we first need to understand the so-called *Grassmann algebras*.

Definition 5.1 (Multivector & wedge product) For any vectors u, v, w, etc., we can define the *wedge product*

$$u \wedge v \wedge w \wedge \cdots \tag{5.1}$$

The result of a wedge product of k vectors is called a k*-vector* or a k*-blade*. k is called the *grade* which is analogous to the tensorial rank.

Remark 5.1 A sum of k-vectors of different grades is called a *multivector*. To get an intuitive understanding, we look at the 2-vector (also called the *bivector* or the *antivector*) and the 3-vector.

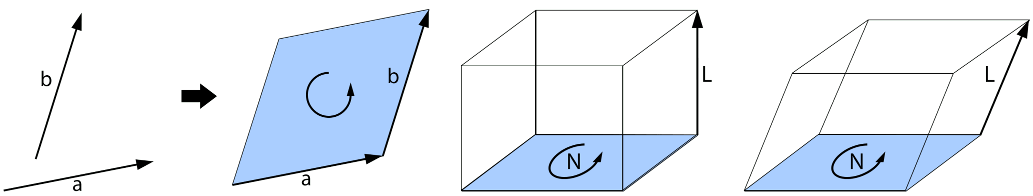

Figure 5.1: Windings of a 2-vector (left) and two similar 3-vectors (right).

Aside from the 2D area in blue[1], the bivector also contains an orientation, or rather a *winding*, which is the direction as shown by the anticlockwise arrow. The inclusion of the winding distinguishes a bivector from simply an area.

Derivation 5.1 (2-vectors and 3-vectors) Suppose we have two vectors $u = u_1e_1 + u_2e_2 + u_3e_3$ and $v = v_1e_1 + v_2e_2 + v_3e_3$. The wedge product is then

$$\begin{aligned} u \wedge v &= (u_1e_1 + u_2e_2 + u_3e_3) \wedge (v_1e_1 + v_2e_2 + v_3e_3) \\ &= (u_2v_3 - u_3v_2)(e_2 \wedge e_3) + (u_3v_1 - u_1v_3)(e_3 \wedge e_1) + (u_1v_2 - u_2v_1)(e_1 \wedge e_2) \end{aligned} \tag{5.2}$$

[1]The equivalent of this for 3-vectors is the 3D volume seen on the right.

To simplify the expression, the following shorthand exists:

$$e_{12} = e_1 \wedge e_2 \quad e_{23} = e_2 \wedge e_3 \quad e_{31} = e_3 \wedge e_1 \quad e_{123} = e_1 \wedge e_2 \wedge e_3 \tag{5.3}$$

Which gives

$$u \wedge v = (u_2v_3 - u_3v_2)e_{23} + (u_3v_1 - u_1v_3)e_{31} + (u_1v_2 - u_2v_1)e_{12} \tag{5.4}$$

While this might look like an outer product, the wedge product is associative, while the outer product is not.
We now consider a 3-vector:

$$u \wedge v \wedge w = (u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2 - u_1v_3w_2 - u_2v_1w_3 - u_3v_2w_1)e_{123} \tag{5.5}$$

There is only one component e_{123}. A 3-vector changes sign under a mirror reflection due to the presence of the winding. As such, it is also called an *antiscalar* or *pseudoscalar* when we work in 3D space[a].

[a] The implication being that a n-vector is only an antiscalar in n-dimensional space.

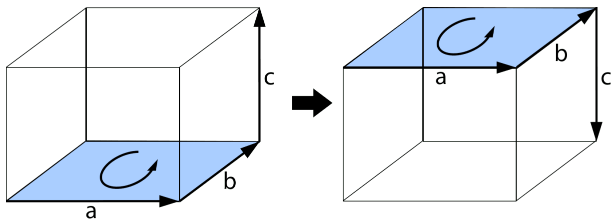

Figure 5.2: Change of sign of the antiscalar due to opposite windings.

Remark 5.2 To reverse the winding of a trivector, simply reverse the windings of all its bivector faces.

Definition 5.2 (Exterior product) The generalisation of the wedge product to multivectors of any grade is called the *exterior product*. Consider two differential forms α and β. In index notation, their exterior product $\alpha \wedge \beta$ can be represented as

$$(\alpha \wedge \beta)_{i_1 \ldots i_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[i_1 \ldots i_p} \beta_{i_{p+1} \ldots i_{p+q}]} \tag{5.6}$$

where $_{[i_1 \ldots i_p} \beta_{i_{p+1} \ldots i_{p+q}]}$ denotes the antisymmetrisation of the indices[a]. The winding is shown in the signs of each term. Consider the wedge product of a 1-form α_i and a 2-form β_{jk}. The resultant 3-form is

$$(\alpha \wedge \beta)_{ijk} = 3\alpha_{[i}\beta_{jk]} = \alpha_i \beta_{jk} - \alpha_j \beta_{ik} + \alpha_k \beta_{ij} \tag{5.7}$$

[a] You have hopefully seen this in *Metric's Destiny*.

Theorem 5.1 (Exterior product properties) The exterior product observes the following properties:

- If one term is a scalar, the wedge product involving that term reduces to scalar multiplication. For scalar a and some n-form B:

$$a \wedge B = B \wedge a = aB \tag{5.8}$$

- Linearity. For scalars a and b:

$$u \wedge (av + bw) = au \wedge v + bu \wedge w \tag{5.9}$$

- The exterior product of a vector against itself is meaningless:

$$u \wedge u = -u \wedge u = 0 \tag{5.10}$$

Definition 5.3 (Grassmann algebra) As the exterior product is linear, the resultant vector space of the resultant n-form $\Lambda^n(V)$ is an algebra. This algebra is known as the *exterior algebra* or the *Grassmann*[a] *algebra.*

[a] This is the international spelling. The German spelling is *Graßmann.*

Quote 5.2 I think it uses the ß (or ss in international spelling)

Paul Kothgasser, on a different name, 29 September 2024

Quote 5.3 the ß is my favourite letter
i even managed to weasel it into my bsc thesis even tho that was in english

Paulina Schlachter, 29 September 2024

The Generic Grass Man | Hermann Grassmann | The Grassmann Algebra

Figure 5.3: Three types of Grassmann.

Remark 5.3 Get it? *Grass*mann? *Blade*?

Note 5.1 The fact that $u \wedge u = 0$ is what makes the wedge product different from the tensor product. Otherwise, the corresponding Grassmann algebra simply reduces to the so-called *tensor algebra* which defines the tensor product.

5.2 Clifford algebra

Definition 5.4 (Clifford algebra) The Clifford algebra $\mathcal{C}(m, n)$ is a collection of matrices (which we denote as *symbols*) with m elements squaring to 1 and n elements squaring to -1. The symbols anti-commute:

$$s_i s_j = -s_i s_j \quad \text{for} \quad i \neq j \tag{5.11}$$

Remark 5.4 For example, we can interpret the number 1 and -1 as the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the negative identity $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The imaginary number i, interpreted as the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is then a Clifford algebra denoted as $\mathcal{C}(0, 1)$.

Definition 5.5 (Clifford product) The rather annoyingly notationless *Clifford product* between vectors $\vec{u}$ and $\vec{v}$ is defined as

$$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v} \tag{5.12}$$

The object resulting from a Clifford product of vectors is called a *versor.*

Remark 5.5 For orthogonal vectors, $\vec{u}\vec{v}$ reduces to $\vec{u} \wedge \vec{v}$, and for parallel vectors, $\vec{u}\vec{v}$ reduces to $\vec{u} \cdot \vec{v}$. Now we introduce the so-called *Grassmann numbers.*

Definition 5.6 (Grassmann number) The Grassmann numbers θ_i satisfy

$$\theta_i \theta_j = -\theta_j \theta_i \quad \text{and} \quad \theta_i^2 = 0 \tag{5.13}$$

Grassmann numbers are related to Clifford algebra elements by

$$\theta^i = \frac{1}{\sqrt{2}}(s^{2n} + i s^{2n+1}) \tag{5.14}$$

Chapter 6

Rotations

6.1 Spin groups

Previously, we have seen that the imaginary number i can be represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This idea can in fact be generalised into 3 symbols i, j and k which (implicitly along with the identity $\mathbb{I}$ or 1) are called the *quarternions*, which can be seen as an extension of the concept of complex numbers.

Definition 6.1 (Quaternion) The *quaternion* is number system a number system that is an extension of complex numbers. It is of the form

$$a + bi + cj_dk \tag{6.1}$$

where a, b, c and d are coefficients. Its members, the quaternions i, j and k satisfy:

$$i^2 = -1 \quad j^2 = -1 \quad k^2 = -1 \quad ijk = -1 \tag{6.2}$$

Derivation 6.1 (Quarternion relationships) We can multiply both sides of $ijk = -1$ by k:

$$ijkk = -k \rightarrow ij(-1) = -k \rightarrow ij = k \tag{6.3}$$

All other relationships then follow:

Theorem 6.1 (Quarternion relationships) i, j and k are related by

$$ij = k \quad jk = i \quad ki = j \tag{6.4}$$

or equivalently

$$ji = -k \quad kj = -i \quad ik = -j \tag{6.5}$$

Remark 6.1 Any pair out of i, j and k are hence anticommutative.

Definition 6.2 (Quaternion conjugate) The *quaternion conjugate* q^* is likewise an extension of the complex conjugate. For a quaternion q, it is

$$q^* = a - bi - cj - dk \tag{6.6}$$

where a, b, c and d are coefficients. Its members, the quaternions i, j and k satisfy:

$$i^2 = -1 \quad j^2 = -1 \quad k^2 = -1 \quad ijk = -1 \tag{6.7}$$

Definition 6.3 (Quaternion inverse) The *quaternion inverse* is

$$q^{-1} = \frac{q^*}{||q||^2} \tag{6.8}$$

Theorem 6.2 (Quarternion-Pauli matrices equivalence) Each quaternion corresponds to a matrix that can be represented by Pauli matrices:

$$i \leftrightarrow -\sigma_y\sigma_z \quad j \leftrightarrow -\sigma_z\sigma_x \quad k \leftrightarrow -\sigma_x\sigma_y \tag{6.9}$$

Remark 6.2 Mathematically, we say that each quaternion and each of the RHS Pauli matrix products are *isomorphic*. In fact, from the RHS equivalents, we can think of i, j and k as the x, y and z axes.

Theorem 6.3 (Reflections) The reflection in some direction i is governed by

$$V' = -UVU^{-1} \tag{6.10}$$

U is the versor corresponding to the dimension. In 3D this is σ_i. In 4D this is γ_i where we have the extended Pauli matrices or the γ *matrices* $\gamma_t = \gamma_0 = \mathbb{I}$ and $\gamma_i = \sigma_i$ for $i = 1, 2, 3$.

Remark 6.3 A 'negative reflection' yields the same result as the negative signs for $-U$ and $-U^{-1}$ cancel out.

Theorem 6.4 (Rotations in 3D space and 4D spacetime) A rotation in a 3D space $\mathcal{C}(3,0)$ in the ij plane for some indices i and j by an angle of θ can be represented by

$$V' = e^{-\sigma_i\sigma_j\frac{\theta}{2}}Ve^{\sigma_i\sigma_j\frac{\theta}{2}} \tag{6.11}$$

In a 4D spacetime $\mathcal{C}(1,3)$, this is

$$V' = e^{-\gamma_i\gamma_j\frac{\theta}{2}}Ve^{\gamma_i\gamma_j\frac{\theta}{2}} \tag{6.12}$$

A Lorentz boost in some non-temporal direction k is

$$V' = e^{-\gamma_t\gamma_k\frac{\theta}{2}}Ve^{\gamma_t\gamma_k\frac{\theta}{2}} \tag{6.13}$$

where t is the time direction.

To generalise, a rotation in some space $\mathcal{C}(p,q)$ can be written as

$$V' = e^{-B\frac{\theta}{2}}Ve^{B\frac{\theta}{2}} \tag{6.14}$$

where B is the corresponding versor. We then say that $e^{-B\frac{\theta}{2}}$ and $e^{B\frac{\theta}{2}}$ are members of the $\mathbf{Spin}(p,q)$ group.

Remark 6.4 In the meantime, we note that a rotation U_1U_2 is simply two reflections.
Again we generalise this to transformations of

$$V' = (U_1 \cdots U_k)V(U_1 \cdots U_k)^{-1} \tag{6.15}$$

Definition 6.4 (Pin group in space) Members of $\mathcal{C}(n)$ that are *normalised* versors make up the $\mathrm{Pin}(n)$ group. They represent all rotations/reflected rotations in n-dimensional space.

Definition 6.5 (Spin group in space) Members of $\mathcal{C}(n)$ that are *normalised* versors *of even length* (i.e. k is even) make up the $\mathrm{Spin}(n)$ group. They represent all rotations[a] in n-dimensional space.

[a]But not reflected rotations as any odd-length elements are eliminated.

Remark 6.5 Here we observe two double covers:

- The $\mathrm{Pin}(n)$ group is a double cover of the $\mathrm{O}(n)$ group.
- The $\mathrm{Spin}(n)$ group is a double cover of the $\mathrm{SO}(n)$ group.

We now incorporate time dimensions

Definition 6.6 (Spin group in space) Members of $\mathcal{C}(p,q)$ that are versors *of even length* (i.e. k is even) and observe $U_i^2 = \pm 1$[a] make up the $\mathrm{Spin}(p,q)$ group. They represent all rotations[b] in a spacetime with p temporal dimensions and q spatial dimensions.

[a] U_i is spacelike if $U_i^2 = -1$ and timelike if $U_i^2 = 1$
[b] But not reflected rotations as any odd-length elements are eliminated.

6.2 Spin

Previously we have seen how 2×2 matrices that square to $\mathbb{I}$ can represent i. This can be expanded to matrices of any size. Such matrices are known as *representations* of i. From this concept of representation we consider analogues for SO(3) and $\mathfrak{so}(3)$.
As it turns out the previous SO(3) rotation matrices and $\mathfrak{so}(3)$ generators we have derived are those of spin-1. This is the *spin-1 representation* of SO(3) and $\mathfrak{so}(3)$. If we recall an undergrad course:

Spin	Corresponding object	Example particle
0	scalar	Higgs boson
1/2	spinor	quarks & leptons
1	vector	gluon, photon & W and Z bosons
2	matrix	graviton
n	rank-n tensor	N/A

In spin-0, each generator is simply a number, that being 0. Each rotation matrix is likewise a number, this time 1. This is known as the *spin-0 representation* or the *trivial representation* due to how utterly simple it is.
Remark 6.6 Using these rotation 'matrices', we recover the fact that scalars undergo no change under rotation.
However, for spin-$\frac{1}{2}$ particles, we have a problem as no 2×2 matrices satisfy the conditions for generators. That is, no *spin-$\frac{1}{2}$ representation* of SO(3) and $\mathfrak{so}(3)$ exist. There is nonetheless a workaround: An equivalent representation exists in the SU(2) Lie group, which double-covers SO(3).In the same vein, while we cannot acquire spin-$\frac{1}{2}$ representation of $\mathrm{SO}^+(1,3)$, we can find its equivalent in $\mathrm{SL}(2,\mathbb{C})$ which double covers to $\mathrm{SO}^+(1,3)$.