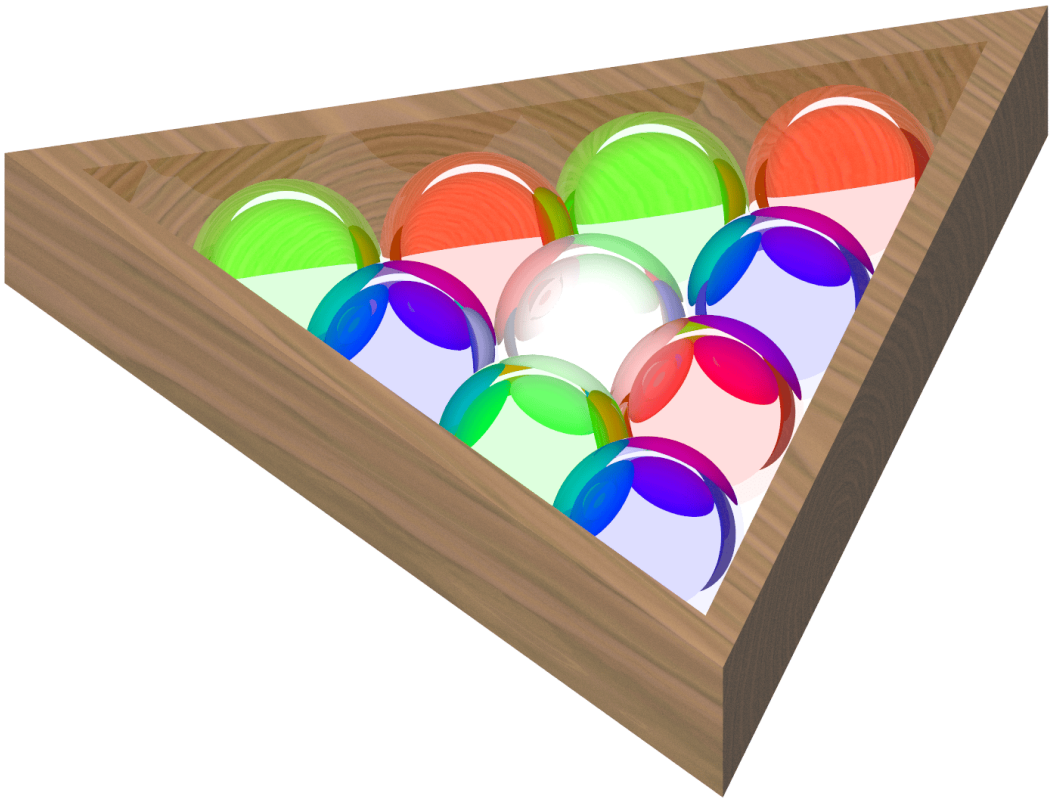
Particles & Wizards

Introduction to Quantum Field Theory



A Very Short Proto-book by

N. Booker

To my parents

Contents

Chapter 1

Instead of a foreword

1.1 How to use this book

Quote 1.1 But QFT is not conceptually difficult. It's just classical field theory with commutators, at least if one stays clear of more esoteric things like supersymmetry or string theory. If you already have experience in CFT, QFT should pose no difficulties to you at all.

Paulina Schlachter, 29 September 2024

Quote 1.2 When conquering QFT you'll find yourself in a similar position as Napoleon with Russia.

Felix Halbwedl, 20 October 2024

Quantum field theory is the unification of quantum mechanics and special relativity. It is not a theory of quantum gravity because it still operates within the completely flat Minkowski space. While the development of high-energy physics preceded that of QFT historically, QFT actually forms the theoretical basis of HEP.
Our ultimate objective in QFT is thus to calculate the final state from some initial state. This is accomplished by applying the scattering matrix S_{fi} to the initial state. The interaction-relevant part of S_{fi} is another matrix $\mathcal{M}_{fi}$ known as the transition amplitude, which is significant in its own right.
The construction $\mathcal{M}_{fi}$ is made easy thanks to Feynman diagrams. As it turns out, every process in high energy physics can be represented by a Feynman diagram, which consists of three types of objects: lines, vertices and loops, which are nothing but lines coming back to themselves[1]. The mathematical expression of the Feynman diagram, which happens to be $\mathcal{M}_{fi}$, can then be constructed by the Feynman rules of the field theory of interest, which assign each type of object to a mathematical expression and multiply them together.
To derive these rules is the heart of quantum field theory. We start with the Lagrangian of our field theory of interest, which can always be split into a free part and an interacting part. The free part corresponds to lines, while the interacting part corresponds to vertices. There are two ways to go from the Lagrangian to the Feynman rules: the canonical quantisation formulation and the path integral formulation.
This book is structured as follows:

- While knowledge of special relativity is assumed, we provide a brief overview of classical field theory and non-relativistic quantum mechanics in Part I. The two topics directly precede the formulation of QFT and usually constitute the beginning of a standard 'Quantum Field Theory I' course in most universities. An experienced reader should have no problem reading this part like a novel.

- Part II covers canonical quantisation, in which fields are quantised. Historically, it is also called 'second quantisation' as it builds on 'first quantisation' in quantum mechanics, where physical quantities are quantised. For this reason, it is the more intuitive approach, and is the staple of a standard 'Quantum Field Theory I' course in most universities.

[1] Mathematically, this means we integrate the expression of a line over the internal loop momenta.

- Part III covers the path integral formulation, in which, from the action S of a system, we derive a generating functional $Z[J]$ that takes the form of a path integral. Propagators are then derived by taking functional derivatives of $Z[J]$. This approach is somewhat easier but less intuitive. As such, it is usually at the intersection between 'Quantum Field Theory I' and 'Quantum Field Theory II' courses in most universities.

- As it turns out, both formulations, which yield the same Feynman rules, fail in structures beyond the tree level of Feynman diagrams due to the emergence of infinities. The elimination of these unphysical infinities requires the theory to be renormalised, which can only be done by first performing the nice mathematical trick of regularisation. The two topics, which are covered in Part IV, are also usually at the intersection between 'Quantum Field Theory I' and 'Quantum Field Theory II' courses in most universities.

Many parts of the book require knowledge of spinors, symmetries and Lie theory. These topics will not be covered in this book. Rather, the reader is encouraged to consult the companion book *Spinors & Symmetries* or other standard Lie theory texts.
 For any comments, suggestions or typos, please e-mail the following address:

`neil(dot)booker(dot)21(at)ucl(dot)ac(dot)uk`

Quote 1.3 🐈, aber sicher doch. Ich mache auch Fehler.

Felix Halbwedl, 8 February 2025

1.2 Acknowledgements

Quote 1.4 Credit for those who work hard, I just happen to know some things.

Felix Halbwedl, in his infinite humility, 22 December 2024

Quote 1.5 Not a word of Halbwedl and not a thought of Booker.

The Author, on this book, 11 April 2025

Work on *Particles & Wizards: Introduction to Quantum Field theory* started shortly before the 2024-25 quantum field theory course at University College London lectured by Prof. Alessio Serafini[2]. I would like to thank him for answering the many questions on QFT I had throughout my master's year and for his consistent support.
I want to extend my gratitude to Felix Halbwedl, who stimulated many physical discussions on various topics in QFT and offered much advice on the contents and the formatting of this book. I also thank him for the many quotes he has contributed to the book as well as his moral support. I am also thankful to Abhijeet Vats, under whose guidance I was able to develop my LaTeX skills to a satisfactory level. Without them, this book would undoubtedly not have been in its current form.

1.3 References

- *Introduction to Gauge Field Theory* by David Bailin (University of Sussex) and Alexander Love (University of Sussex)
- *Quantum Field Theory I* by Niklas Beisert (ETH Zürich)
- *Quantum Field Theory II* by Niklas Beisert (ETH Zürich)
- *Quantum Fields* by Nikolay Bogoliubov (JINR) and Dmitry Shirkov (JINR)
- *Quantum Field Theory* by Gernot Eichmann (Technische Universität Graz)

[2]Known lovingly as the 'Wizard' due to his character appearing as a wizard in the UCL PandA Day plays.

- *Quantenfeldtheorie* by Matthias Gaberdiel (ETH Zürich)
- *Quantum Field Theory II* by Matthias Gaberdiel (ETH Zürich) and Aude Gehrmann-De Ridder (ETH Zürich)
- *Quantum Field Theory I* by Axel Maas (Technische Universität Graz)
- *Quantum Field Theory II: Gauge Theories* by Axel Maas (Technische Universität Graz)
- *Quantum Field Theory* by Alessio Serafini (University College London)
- *Notes on Quantum Field Theory I* by Marco Serone (SISSA) An Introduction to Quantum Field Theory by Michael E. Peskin (SLAC) and Daniel V. Schroeder (Weber State University)

Part I

Preliminaries

Chapter 2

Classical field theory

Up to the mid-20$^{\text{th}}$ century, significant equations of motion had been usually derived by intuition, and this (understandably) carries over when these equations are introduces in undergraduate physics. We will begin with the action principle, which is possibly the most important idea in the entirety of physics from which the Euler-Lagrange equations then follow as a result of the vanishing of the boundary term. Then, it can be seen that the equations of motion of any area in physics can be derived systematically by inserting its Lagrangian[1] into the *Euler-Lagrange equations*.

2.1 Action principle

The two elements leading to the action principle are the action itself and the concept of symmetries. We begin by transitioning from classical mechanics to classical field theory and work slowly towards the action.

Note 2.1 (Metric signature) Unlike GR, convention dictates that the Minkowski 4-metric in HEP has the signature $(+,-,-,-)$. That is, the line element has the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{2.1}$$

We now briefly discuss the *variational formalism*. Two equivalent formulations of the variational formalism exist - Lagrangian and Hamiltonian mechanics. In classical mechanics, the central quantities are the 4-position x (or often q) and momentum p.

Note 2.2 (Reference frames) We can choose certain frames that simplify calculations:

- For a spacelike separation $(x-y)^2 < 0$, one can always, without loss of generality, choose a frame to set $(x^0 - y^0) = 0$.
- For a timelike separation $(x-y)^2 > 0$, one can always, without loss of generality, choose a frame to set $(\vec{x} - \vec{y}) = 0$.

We now take the monumental step of actually understanding what a field theory actually means. The central point is migrating from a coordinate-centric system we have seen up to this point to a field-centric system. In field theories, the 4-position x is replaced with a 4-field $\phi(x) = (\phi_0, \phi_1, \phi_2, \phi_3)$. ϕ_1, ϕ_2 and ϕ_3 are simply the spatial components of the corresponding 3-field, while ϕ_0 is a *scalar* or *time-like* component of the 4-field[2]. For example, we consider the *Lagrangian density*[3] with only one *kinetic term*. Previously, this would merely be the kinetic energy:

$$\mathcal{L} = \frac{mv^2}{2} \tag{2.2}$$

In a field theory, we turn the velocity to derivative over 4-coordinates or the *4-derivatives* of the field and absorb the coupling-like mass m to a normalised value of 1. This then constructs the simplest field

[1] Which, regrettably, are ultimately formulated via intuition as there is no other way to do so.
[2] For example, in the electromagnetic 4-potential, ϕ_0 is the electric scalar potential.
[3] Often also simply called the *Lagrangian*, although you will be able to tell the difference by looking at the notation.

theory Lagrangian, which is that of a free massless scalar field, which can be used to model particles like massless scalar bosons[4]:

Definition 2.1 (Free massless scalar field Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{2.3}$$

The sole term is the *kinetic energy density*, which arises from the variation of the field ϕ over the 4-coordinates.

The only term in this Lagrangian is a so-called kinetic term, which we can now properly define as a term consisting of field derivatives. In analogy to classical mechanics, the rest of the terms make up the *potential*.
One way we can introduce a potential to the Lagrangian is to make the free field massive instead of massless. Note that this does not mean that the field itself is massive (which makes little sense) but rather that the particle that generates the field is massive. This gives rise to a mass coupling[5] term, and the Lagrangian becomes the standard Lagrangian of a free massive scalar field, which we use to derive the Klein-Gordon equation we will see later:

Definition 2.2 (Free massive scalar field Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \tag{2.4}$$

where the second term is the *potential energy density*.

Let us compare this CFT result with its classical mechanics analogue, which is the Lagrangian of a harmonic oscillator:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \tag{2.5}$$

Essentially, $\frac{1}{2}m^2|\phi|^2$ represents the field-theory version of Hooke's law. It is a potential energy term that determines how the field oscillates about its vacuum. The mass term m^2 replaces the spring constant k and sets the curvature (strength) of the potential.
From the Lagrangian, we can formulate how quantities related to it are defined in CFT. In classical mechanics, the *canonical momentum* is defined as

$$p = \frac{dL}{d\dot{q}} \tag{2.6}$$

where q is the generalised 4-coordinates. In CFT, this becomes

Definition 2.3 (Canonical momentum)

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} \tag{2.7}$$

The action in CFT is unchanged from its CM counterpart, but uses the CFT Lagrangian instead of the CM Lagrangian:

Definition 2.4 (Action) For a set of fields ϕ_i with the 4-position x^i, the *action* is defined as

$$S = \int \mathcal{L}(\phi, \partial_\mu\phi)d^4x = \int Ldx^0 = \int Ldt \tag{2.8}$$

where L is the Lagrangian and $\mathcal{L}$ is the Lagrangian density.

We are now in a position to discuss symmetries. Physicists use the word 'symmetry' as a more sophisticated way of saying that a physical quantity we are interested in stays invariant under the change of

[4]We see it more often in approximate models as fundamental massless scalar particles with zero mass are rare.
[5]We will often see the word 'coupling'. Sometimes, it is the short form of the *coupling constant*, which denotes any physical constant that a term in the Lagrangian may have. We then speak of the field *coupling* to whatever the physical quantity this coupling constant represents. For example, in a mass term, the coupling constant is the mass, and we say that in the term, the field couples to mass.

some other 'background' quantity. If this physical quantity of interest instead changes with the background quantity, we then sadly say that symmetry is 'broken'. Broadly speaking, we are interested in the following symmetries:

- **External and internal symmetries:**
  - *External symmetries* are transformations that involve changes to the spacetime coordinates themselves. One example is Poincaré symmetry.
  - *Internal symmetries* are transformations that act on internal degrees of freedom of fields (e.g. charge, spin, etc.), leaving spacetime coordinates unchanged.
- **Discrete and continuous symmetries:**
  - *Continuous symmetries* are governed by transformation parameters that can admit a continuous range of values. One example is Poincaré symmetry.
  - *Discrete symmetries* involve transformations that take on only specific values. Examples are the C, P and T symmetries.

So far, we have heard about the layman's version of Noether's theorem 'All symmetries lead to conservation laws'. We have made a conceptual overview of symmetries, but what are their mathematical implications? As it turns out, symmetries are defined with respect to an action principle. Consider an *infinitesimal coordinate transformation*

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu \tag{2.9}$$

Up to the first order expansion, a generic field in x^μ undergoes the corresponding transformation:

$$\phi(x) \to \phi'(x') = \phi(x) - \epsilon^\mu \partial_\mu \phi(x) \tag{2.10}$$

The variation of the field can then be written as

$$\delta\phi(x) = \phi'(x') - \phi(x) = -\epsilon^\mu \partial_\mu \phi(x) \tag{2.11}$$

An important point of note concerns the Lagrangian (density). While it has rank 0, the Lagrangian $\mathcal{L}$ in some theories might not transform as a scalar. Rather, its variation takes the general form of the total derivative of a current-like vector field K^μ. Without considering fields, the variation of the Lagrangian *arising from a symmetry transformation* has the general form

$$\delta\mathcal{L} = \partial_\mu K^\mu(\phi, \partial_\mu \phi) \tag{2.12}$$

where:

- Physically, $K^\mu(\phi, \partial_\mu\phi)$ is the measure of the failure of $\mathcal{L}$ to transform as a scalar.
- $\partial_\mu K^\mu(\phi, \partial_\mu\phi)$ is then a 'total derivative' of $K^\mu(\phi, \partial_\mu\phi)$, which depends on both the field ϕ and the field's 4-derivative $\partial_\mu\phi$.

Derivation 2.1 (Action principle) We can solve for the variation of the action by integrating the variation of the Lagrangian in Equation 2.12, which gives:

$$\delta S = \int d^4x \delta\mathcal{L} = \int d^4x \partial_\mu K^\mu(\phi, \partial_\mu\phi) \tag{2.13}$$

Using the divergence theorem, this integral can be converted into a surface integral over the boundary $\mathcal{S} = \partial V$ of the spacetime region V:

$$\delta S = \int_{\partial V} d^3x K^\mu(\phi, \partial_\mu\phi) n_\mu \tag{2.14}$$

where n_μ is the normal vector to the boundary. We are left with a *boundary term* that is exactly $\partial_\mu K^\mu(\phi, \partial_\mu\phi)n_\mu$ where n_μ is a directional 4-vector.
There are two scenarios in which this boundary term can be ignored:

- The variation $\delta\phi$ (and with that, ϕ and its derivative) vanishes on the boundary[a].
- The boundary extends into infinity.

The important step now is to *assume the first point*, which can be justified if we impose boundary conditions. Depending on the physical scenario, we usually use one of the two main boundary conditions.

Definition 2.5 (Dirichlet boundary condition) The *Dirichlet boundary condition* or the *boundary condition of the first type* sets the position to be time-invariant at the boundary:

$$\dot{q}|_{\partial V} = 0 \quad \partial_\mu \phi|_{\partial V} = 0 \tag{2.15}$$

Definition 2.6 (Neumann boundary condition) The *Neumann boundary condition* or the *boundary condition of the second type* sets the momentum is time-invariant at the boundary:

$$\left.\frac{\partial L}{\partial \dot{q}}\right|_{\partial V} = 0 \quad \left.\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right|_{\partial V} = 0 \tag{2.16}$$

where we see that this boundary condition is realised by exploiting Hamilton's equations[a].

[a]We only derive Hamilton's equations later on. However, our current lack of knowledge of Hamilton's equations does not prevent us from imposing the Neumann boundary condition. Rather, it only temporarily prevents us from interpretating the LHS as the momentum time derivative - which we have revealed as a spoiler anyway.

Either way, the term $\partial_\mu K^\mu(\phi, \partial_\mu\phi)n_\mu$ then vanishes due to its dependency on both ϕ and $\partial_\mu\phi$:

$$\int_{\mathcal{S}} d^3x K^\mu(\phi, \partial_\mu\phi)n_\mu = 0 \tag{2.17}$$

In fact, assuming that we choose suitable boundary conditions, all total derivative terms of the form $\partial_\mu K^\mu(\phi, \partial_\mu\phi)n_\mu$ are boundary terms, and hence identically vanish.

Note 2.3 Boundary terms do not contribute to the equations of motion. As such, we can always add or subtract boundary terms to/from the Lagrangian arbitrarily[a].

[a]The same applies to zero terms for the simple reason that they are zero

From this, we see that the action is invariant under the symmetry:

Theorem 2.1 (Action principle)

$$\delta S = 0 \tag{2.18}$$

This is the almighty *action principle*.

[a]This is typically justified in physical field theories where fields and their variations vanish at spatial or temporal infinity.

Remark 2.1 The *action principle* is simply another name for the *principle of stationary action*, which is itself often erroneously known as the *principle of least action*[6]. This simply means that the time derivative of the action of an isolated system is zero. As the principle can be used for action generated by *any* field, it is often mentioned as '*an* action principle' instead of '*the* action principle'.

2.2 Equations of motion

The physical significance of our previous derivation is not immediately obvious, especially with respect to how the boundary term vanishes by dint of the Dirichlet and Neumann boundary condition. One can lift this shrowd of confusion by realising that the action principle allows us to derive the equations of motion. Let us show this with the general example in classical mechanics.

[6]This is because the principle states that instead of at a minimum, action tends to stay *stationary*, be it a maximum, a minimum or a saddle point.

Derivation 2.2 (Euler-Lagrange equations) Now that we have assumed the vanishing of the boundary term, let us evaluate the Lagrangian variation $\delta\mathcal{L}(\phi, \partial_\mu\phi)$ explicitly. We consider an infinitesimal variation of the field:

$$\phi(x) \to \phi(x) + \delta\phi(x) \tag{2.19}$$

The variation of the Lagrangian is exactly analogous to differentiating a function of two variables:

$$f(x,y) \to f(x+\delta x, y+\delta y) \to \delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \tag{2.20}$$

As $\mathcal{L}$ has two variables, the field ϕ and the field derivative ∂_μ, its total variation is given by

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \tag{2.21}$$

Using the product rule gives

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \tag{2.22}$$

Noting that $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$, we can write

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \tag{2.23}$$

Inserting this result into Equation 2.13 gives

$$\delta S = \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi)\right] \tag{2.24}$$

We can apply integration by parts to the second term and apply Gauss's law[a]. This gives

$$\delta S = \int d^4x \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big|_{\partial V}}_{\text{boundary term}} - \int d^4x \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi \tag{2.25}$$

where we recall that ∂V denotes the boundary.
Through this process, we have explicitly exposed the so-called boundary term. But there is something very peculiar about this boundary term, isn't there? Let us inspect the two components:

- $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}$ is nothing but the time derivative of the momentum, which is zero at the boundry under the Neumann boundary condition in Equation 2.15.
- $\delta\phi$ is nothing but the time derivative (variation) of the position, which is zero at the boundary under the Dirichlet boundary condition in Equation 2.16.

So you see, no matter which boundary condition we take, this boundary term is ultimately nothing but zero. The important conclusion you should yield from our discussion is the following:

Note 2.4 Due to its vanishing, the boundary term *does not contribute to the equations of motion.*

In contrast, the rest of the expression, which contribute to the equations of motion, is known as the *bulk term*[b].
The final step we have left is to simply apply the action principle, which then leads to the famous Euler-Lagrange equations:

Theorem 2.2 (Euler-Lagrange equations)

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0 \tag{2.26}$$

[a] As such, the boundary term is also called the *surface term*.
[b] This expression is rarely used, and almost always in gravitation.

Before proceeding, we make some comments on the bulk and boundary terms:

- The bulk term is so-called because it integrates over the entire volume of spacetime. It is the term that contributes to the action. When an action principle is imposed, the integrand of the bulk term vanishes, as seen in Equation 2.26.
- The boundary term reflects the influence of boundary conditions for the action. It is the term that does not contribute to the action. By imposing an action principle, we have also assumed that $\delta\phi = 0$ on the boundary - a boundary condition.

Like in CM, the *Hamiltonian* is essentially a Legendre transformation of the Lagrangian:

Definition 2.7 (Hamiltonian and Hamiltonian density) The *Hamiltonian* H is

$$H = \int \mathcal{H}(\phi, \pi, \partial_\mu \phi) d^3x \tag{2.27}$$

which is the volume integral of the *Hamilton density* $\mathcal{H}$. Also simply called the *Hamiltonian*, it is given by

$$\mathcal{H}(\phi_i, \pi_i, t) = \sum_i \pi_i \dot{\phi}_i(\phi_j, \pi_j) - \mathcal{L}(\dot{x}_k(x_j, p_j), x_k, t) \tag{2.28}$$

where ϕ is the field and π is the canonical momentum.

Remark 2.2 As it turns out, the quantity which we have been led to believe to be the Hamiltonian as undergrads is actually the Hamiltonian density $\mathcal{H}$.

Derivation 2.3 (Hamilton's equations) By taking the variation of Equation 2.28, one finds

$$\delta\mathcal{H} = \sum_i \delta\pi_i \dot{\phi}^i(\phi_j, \pi_j) - \delta\pi^i \frac{\partial \mathcal{L}}{\partial \pi^i} = \sum_i \delta\pi_i \dot{\phi}^i(\phi_j, \pi_j) - \sum_i \delta\phi^i(\phi_j, \pi_j)\dot{\pi}_i \tag{2.29}$$

Now compare this against the general variation:

$$\delta\mathcal{H} = \delta\phi^i(\phi_j, \pi_j) \frac{\partial \mathcal{H}}{\partial \phi^i(\phi_j, \pi_j)} + \delta\pi_i \frac{\partial \mathcal{H}}{\partial \pi_i} \tag{2.30}$$

By equating the two expressions for $\delta\mathcal{H}$, we recover the so-called *Hamilton's equations*:

Theorem 2.3 (Hamilton's equations)

$$\frac{\partial \mathcal{H}}{\partial \phi_i} = -\dot{\pi}_i \quad \frac{\partial \mathcal{H}}{\partial \pi_i} = \dot{\phi}_i \tag{2.31}$$

Defining the Poisson bracket $\{f, g\}$ of some two quantities f and g as

Definition 2.8 (Poisson bracket)

$$\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{2.32}$$

we can rewrite Hamilton's equations as

$$\dot{x}_i = \{x_i, H\} \quad \dot{p}_i = \{p_i, H\} \tag{2.33}$$

As we will see later, the quantum version of the first equation is simply the Heisenberg equation, or the Schrödinger equation in the Heisenberg picture.

2.3 Noether's theorem

We can now finally derive Noether's theorem. The first step is to realise that the variation of the Lagrangian has been written in two formulations in Equation 2.21 and Equation 2.12 respectively. Com-

bining them yields

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu K^\mu \to \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu K^\mu = 0 \tag{2.34}$$

Now let us define another current-like quantity called the *Noether current*:

Definition 2.9 (Noether current) The 4-vector equivalent of the probability density ϱ is the so-called Noether current, *probability 4-current* or *conserved current* J^μ. Its zeroth component is simply the good ol' probability density, and its 3 other components are the *probability (3-)current* $\mathbf{J}^\alpha$.

From the Noether current, one can derive a charge-like quantity representing the total probability called the *Noether charge*, the *probability charge* or the *conserved charge* as it is conserved with respect to time:

Definition 2.10 (Probability charge)

$$Q = \int d^3x J^0 \tag{2.35}$$

As probability is conserved, J^μ is Lorentz-invariant and satisfies the *continuity equation*:

Theorem 2.4 (Continuity equation)

$$\partial_\mu J^\mu = 0 \tag{2.36}$$

Rather cheatingly, we can now equate Equation 2.34 and Equation 2.36:

$$\partial_\mu J^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu K^\mu = 0 \tag{2.37}$$

By removing the partial derivatives, we recover the expression for the Noether current, known as Noether's theorem:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - K^\mu \tag{2.38}$$

In the most common case, the Lagrangian is strictly invariant, which leads to the vanishing of the vector field K^μ. We then have:

Theorem 2.5 (Noether's theorem)

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - K^\mu \tag{2.39}$$

> **Quote 2.1** Yes, but it is the same sound
>
> *David Steiner, comparing the 'oe' in 'Noether' with 'ö', 21 November 2024*

Note 2.5 Noether's theorem implies the conservation of the charge associated with the probability current:

$$\frac{dQ}{dt} = \int d^3x \partial_0 J^0 = 0 \tag{2.40}$$

Finally, we can directly relate the Noether current and the action, from Equation 2.25 and Equation 2.26, one can see that

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\delta S}{\delta \phi} \tag{2.41}$$

which, by insertion into Equation 2.37, gives

$$\partial_\mu J^\mu = \delta\phi \frac{\delta S}{\delta \phi} = \frac{\delta \phi}{\delta \theta} \frac{\delta S}{\delta \phi} \tag{2.42}$$

where θ is the parameter for an infinitesimal transformation satisfying

$$x \to x' = x + \theta \delta x \tag{2.43}$$

Exercise 2.1 Let ϕ be a free scalar field obeying the Klein-Gordon equation (Equation 4.2), and let J^μ be the associated density and current 4-vector. Derive the continuity equation (Equation 2.36).

Remark 2.3 One essential type of symmetry in QFT is the so-called *gauge symmetries*, which is are both internal symmetries and continuous symmetries. Gauge symmetries are governed by Lie groups we have seen in *Spinors & Symmetry*.

2.4 Poincaré transformations

As an example, we now derive the Noether current under Poincaré transformations, which, as seen in *Spinors & Symmetries*, includes translations and Lorentz transformations.

Derivation 2.4 (Translation) Let us assume the same transformations as in Equation 2.9 and Equation 2.10. The field variation is then shown in Equation 2.11, which we substitute into the transformation of $\mathcal{L}$ in Equation 2.21. We obtain

$$\delta\mathcal{L} = -\epsilon^\nu \left(\frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \partial_\nu \phi \right) \tag{2.44}$$

Now substitute this into Equation 2.13:

$$\delta S = -\epsilon^\nu \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \partial_\nu \phi \right) \tag{2.45}$$

By inserting the equivalence from the Euler-Lagrange equations, we can rewrite the integral as

$$\delta S = -\epsilon^\nu \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \tag{2.46}$$

where the terms within the bracket is the canonical stress-energy tensor. A simple shift of indices gives its contravariant form:

Definition 2.11 (Stress-energy tensor)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \tag{2.47}$$

Importantly, by comparing Equation 2.13 and Equation 2.46, one can identify

$$\delta\mathcal{L} = \partial_\mu(\epsilon_\nu T^{\mu\nu}) = \partial_\mu K^\mu \tag{2.48}$$

In other words, our new friend $\epsilon_\nu T^{\mu\nu}$ resembles the previously seen boundary/surface term K^μ. Finally, if one takes a partial derivative of the stress-energy tensor and applies the Euler-Lagrange equations, they will find that

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.49}$$

This shows the well-known conservation of matter-energy content, and follows directly from Noether's theorem applied to spacetime translations.

Note 2.6 Equation 2.49 merely states that $T^{\mu\nu}$ is invariant when differentiated[a] over *4-coordinates*. As such, it describes the conservation of energy-momentum *density* as opposed to energy and momentum themselves. In cosmology, where inflation exists, the total energy of the universe is *not* conserved. Rather, as inflation gives rise to the expansion of spacetime, the energy *increases* to preserve the invariance of energy-momentum density.

[a] In curved space, the partial derivative becomes the covariant derivative in GR.

Derivation 2.5 (Lorentz transformation) A Lorentz transformation can be either a rotation or a Lorentz boost. Assuming the parameterisation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \quad \omega^{\mu\nu} = -\omega^{\nu\mu} \tag{2.50}$$

where $\omega^{\nu\mu}$ is some parameter, the coordinate and field transformations are

$$x^\mu \to x'^\mu = x^\mu + \Lambda^\mu_\nu x^\nu \quad \phi(x) \to \phi'(x) = \phi(x) + \frac{1}{2}\omega^{\rho\sigma}\Sigma_{\rho\sigma}\phi \tag{2.51}$$

where $\Sigma_{\rho\sigma}$ are the generators of the representations of the Lorentz group abstract elements corresponding to ϕ (e.g., for scalars $\Sigma_{\rho\sigma} = 0$, for vectors $\Sigma_{\rho\sigma}$ corresponds to antisymmetric tensors, and so on). Again, by using Noether's theorem, we can find that the contribution to the Noether current is the total angular momentum, which includes the orbital angular momentum and spin:

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Sigma^{\rho\sigma}\phi \tag{2.52}$$

For Lorentz transformations, $K^\mu = \partial_\nu(x^\nu J^\mu - x^\mu J^\nu)$, and the Noether current becomes:

$$J^\mu = \omega_{\rho\sigma}M^{\mu\rho\sigma} \tag{2.53}$$

where the nature of $\omega_{\rho\sigma}M^{\mu\rho\sigma}$ as a boundary term is easily seen.

Hence, combining both types of symmetries, the most general form of the Noether current under Poincaré transformations can be written as:

$$J^\mu = \epsilon_\nu T^{\mu\nu} + \frac{1}{2}\omega_{\rho\sigma}M^{\mu\rho\sigma} \tag{2.54}$$

where $T^{\mu\nu}$ represents energy-momentum contributions, and $M^{\mu\rho\sigma}$ represents both orbital and intrinsic angular momentum contributions.
We will now discuss an important point that will pop up over and over again in the rest of the book. As should be well understood at this point, QFT is a quantum theory that incorporates SR, which is based on Lorentz transforms:

- A quantity is *Lorentz-covariant* if it transforms under the Lorentz group representation corresponding to its type (scalar, vector, axial vector, spinor, rank-2 tensor, etc.).
- A quantity is *Lorentz-invariant* if it is invariant under Lorentz transformations.
- A equation is Lorentz-covariant/invariant if all its quantities are Lorentz-covariant/invariant.

While Lorentz invariance is intuitive, we find it prudent to define Lorentz covariance rigorously:

Theorem 2.6 (Lorentz covariance) A field $\phi(x)$ transforms under a Lorentz transformation Λ as:

$$\phi'(x') = \rho(\Lambda)\phi(x)\rho(\Lambda)^{-1} \tag{2.55}$$

where the dash represents the new field/coordinate and $\rho(\Lambda)$ is a *representation* of the Lorentz group ($\mathrm{SO}(1,3)$) acting on the field.
As seen in *Spinors & Symmetries*, this representation differs:

- A scalar field $\phi(x)$ transforms trivially:

$$\phi'(x') = \phi(x) \tag{2.56}$$

- A vector field $A^\mu(x)$, transforms as:

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \tag{2.57}$$

- A spinor field $\psi(x)$, transforms as

$$\psi'(x') = T(\Lambda)\psi(x) \tag{2.58}$$

  where $T(\Lambda)$ is a spinorial representation we will derive in Derivation 6.3.

We end with some physical comments. Consider a quantum field theory *within the standard model*:

- Fields are Lorentz-covariant because they transform under representations of the Lorentz group.
- Equations of motion and physical laws are Lorentz-covariant as they involve only Lorentz-covariant quantities. This Lorentz covariance is effectively enforced due to SR.
- The Lagrangian is Lorentz-invariant, ensuring that the theory as a whole respects Lorentz symmetry.

A more complete overview of the representation theory of the Lorentz group can be seen in *Spinors & Symmetries*.

Chapter 3

Non-relativistic quantum mechanics

> **Quote 3.1** Fortunately, quantum mechanics is easy and can be summarised in a few lines.
>
> *Alessio Serafini*

The other prerequisite of QFT is non-relativistic quantum mechanics, which is actually a special case of QFT where there is only one dimension - time.

3.1 Quantum mechanics of mixed states

A quantum state can always be represented by a Hermitian, positive semi-definite operator with trace 1 ϱ[1] (i.e. all eigenvalues of ϱ are positive semi-definite and add up to 1).

Definition 3.1 (Positive definiteness and positive semi-definiteness) A positive definite operator ϱ always yields a positive expectation value

$$\langle\psi|\varrho|\psi\rangle > 0 \tag{3.1}$$

A positive semi-definite operator ϱ always yields a non-negative expectation value

$$\langle\psi|\varrho|\psi\rangle \leq 0 \tag{3.2}$$

We now introduce the so-called *Sylvester's criterion*.

Definition 3.2 (Minor) A *minor* of some matrix is the determinant of the resultant matrix after deleting an arbitrary number of rows and columns from the initial matrix. For a square matrix, a minor is called a *principal minor* when the indices of the deleted rows and those of the deleted columns are *identical*[a].

[a] This is significant in that if one deletes, say, the 3^{rd} row and the 4^{th} column, the resulting matrix is a minor but not a principal minor.

Theorem 3.1 (Sylvester's criterion) One can use minors of a matrix to test positive definiteness and positive semi-definiteness. For a Hermitian $n \times n$ matrix:

- Positive definiteness holds if all the *leading* principal minors are positive. i.e. if the determinants of the top-left $1 \times 1, \cdots, n \times n$ sub-matrices are positive.
- Positive semi-definiteness holds if *all* principal minors are non-negative.

While a pure quantum state is simply a bra or a ket, the bra-ket notation is insufficient for a *mixed state*, which is described by a *density matrix* or a *density operator*. Unlike a state vector, which is an element *of* the Hilbert space, the density matrix is an operator *on* the Hilbert space.
Physically, a mixed state is a statistical mixture (i.e. ensemble) of i different pure states $|\Psi_i\rangle$ with

[1] Note that this might not be the *density operator* ρ.

probabilities p_i, such that:

$$\varrho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \tag{3.3}$$

In this way, the density matrix extends the concept of quantum state to mixed states, systems where we do not have complete knowledge.

Derivation 3.1 (Von Neumann equation) Let us first derive the *theorem of Liouville*[a] in classical field theory, which states that the phase space distribution function $\rho(p, q, t)$ is constant along the trajectories of the system:

$$\frac{d\rho}{dt} = 0 \tag{3.4}$$

This expression is not useful on its own. Rather, we write down the total derivative of ρ against t explicitly

$$\frac{d\rho}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial t} - \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t} \right) + \frac{\partial \rho}{\partial t} \tag{3.5}$$

Using Hamilton's equations in Equation 2.31, we can rewrite the total derivative as

$$\frac{d\rho}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} \right) + \frac{\partial \rho}{\partial t} \tag{3.6}$$

We can absorb the first term on the RHS into the Poisson bracket. This gives us

Theorem 3.2 (Theorem of Liouville)

$$\frac{d\rho}{dt} = \{\rho, \mathcal{H}\} + \frac{\partial \rho}{\partial t} = 0 \tag{3.7}$$

where the final RHS term vanishes when

When ρ has no *explicit* time dependence (like in Hamiltonian systems), the term $\frac{\partial \rho}{\partial t}$ vanishes, and we simply have

$$\{\rho, \mathcal{H}\} = 0 \tag{3.8}$$

In QFT, the density operator ρ is functionally equivalent to the phase space distribution function. Hence we have an equivalent for the theorem of Liouville, which, importantly, does *not* generally give $\frac{d\rho}{dt} = 0$:

Theorem 3.3 (Von Neumann equation)

$$i\frac{d\rho}{dt} = [H, \rho] + i\frac{\partial \rho}{\partial t} \tag{3.9}$$

where, again, the last RHS term vanishes when ρ has no *explicit* time dependence.

Due to the aforementioned equivalence, the equation is also called the *quantum Liouville equation* or the *Liouville-von Neumann equation*. It is also the mixed state equivalent of the Schrödinger equation, which deals with pure states.

Remark 3.1 In QM, the Hamiltonian H is an operator, and the Hamiltonian density does not exist.

[a] Or more commonly in the English-speaking world, *Liouville's theorem*.

Derivation 3.2 (Reduction to pure state) To prove the last statement, we consider a pure state which has no no explicit dependence on time and where we only have one possible i. The density matrix is

$$\rho = |\Psi\rangle\langle\Psi| \tag{3.10}$$

Substituting this into the LHS of the von Neumann equation gives

$$\partial_t \rho = (\partial_t |\Psi\rangle)\langle\Psi| + |\Psi\rangle(\partial_t \langle\Psi|) \tag{3.11}$$

Substituting into the RHS yields

$$-i[H, \rho] = -iH|\Psi\rangle\langle\Psi| + i|\Psi\rangle\langle\Psi|H \tag{3.12}$$

We now equate the two sides:

$$(\partial_t|\Psi\rangle)\langle\Psi| + |\Psi\rangle(\partial_t\langle\Psi|) = -iH|\Psi\rangle\langle\Psi| + i|\Psi\rangle\langle\Psi|H \tag{3.13}$$

where we have recovered the Schrödinger equation and its adjoint form

$$\partial_t|\Psi\rangle = -iH|\Psi\rangle \quad \partial_t\langle\Psi| = i\langle\Psi|H \tag{3.14}$$

Now we remind ourselves of how measurements work. We have already seen the so-called *projection-valued measures* or *PVMs*. Previously, they have been known to us as *projectors*.

Definition 3.3 (Projection-valued measure) A projection-valued measure P_i is a linear and positive semi-definite operator that satisfies, for the density operator ρ:

- Normalisation condition:
$$\sum_i P_i = \mathbb{I} \quad \text{or} \quad \sum_i \mathrm{Tr}(\rho P_i) = 1 \tag{3.15}$$
where $\mathbb{I}$ is the unit matrix of appropriate dimension.
- Orthogonality condition:
$$P_i P_j = \delta_{ij} P_i \quad \text{or} \quad P_i^2 = P_i \tag{3.16}$$

Functionally, it maps a quantity in a vector space V into a subspace $W \subset V$. Each P_i corresponds to an eigenvalue of an observable (e.g. position or spin).

Theorem 3.4 (Born rule) The probability of obtaining the outcome i is, for a mixed state:

$$p_j = \mathrm{Tr}(\rho P_j) = \mathrm{Tr}(|j\rangle\langle\phi|k\rangle\langle j|) = \langle j|\phi\rangle\langle\phi|j\rangle = |\langle j|\phi\rangle|^2 \tag{3.17}$$

For a pure state, this simplifies to

$$p_i = \langle\Psi|P_i|\Psi\rangle \tag{3.18}$$

Remark 3.2 Here we see the significance of Equation 3.15, which is that all probabilities expectedly sum up to 1.
So far, we have been working with PVMs, which are utterly perfect, innocent and idealised measurements. In real life, measurement devices are not ideal, giving rise to noise. These imperfect (or rather *generalised*) measurements are described by *positive operator-valued measures* or *POVMs*.

> **Quote 3.2** 'Positive Operator Valued Measure', an acronym fabricated by mathematical physicists to scare all others away.
>
> *Alessio Serafini*

Definition 3.4 (Positive operator-valued measure) A positive operator-valued measure $\prod_i$ is a linear and positive semi-definite operator that satisfies the normalisation condition only:

$$\sum_i \prod_i = \mathbb{I} \quad \text{or} \quad \sum_i \mathrm{Tr}(\rho \prod_i) = 1 \tag{3.19}$$

By imposing the condition

$$\mathrm{Tr}[\prod_i \prod_j] = \delta_{ij} \tag{3.20}$$

POVMs reduce to PVMs/projectors.
Remark 3.3 The Born rule is the same as for PVMs, save for the nominal replacement of P_is by $\prod_i$s.

3.2 Time evolution and dynamic pictures

In innocent non-relativistic quantum mechanics, the generic Hamiltonian consists of the kinetic energy and the potential:

$$H = \frac{p^2}{2m} + V(\mathbf{x}) \tag{3.21}$$

In field theories, we can generalise this. The kinetic energy, which represents a free particle, becomes the *free Hamiltonian* H_0 while the potential energy, which represents particle interactions, becomes the *interaction Hamiltonian* H_{int}. The total Hamiltonian hence reads

$$H = H_0 + H_{\text{int}} \tag{3.22}$$

The splitting of our beloved Hamiltionian into so-called free and interacting parts gives rise to some convenient tricks in calculations. This is possible because our interpretation on which quantity is evolving in time in quantum mechanics is ultimately philosophically arbitrary. Depending on which term(s) we believe to be actually evolving with time, there exists three *dynamical pictures* or *representations*: the *Schrödinger picture*, the *Heisenberg picture* and *interaction picture*. We consider:

- A series of 'in' states $|\psi, \text{in}\rangle$ which denote *prepared* or *incoming* particles at some initial time t_0.
- A series of 'out' states $\langle\alpha, \text{out}|$ which denote *detected* or *outgoing* particles at current time t. These states are the same 'in' states, but *after* undergoing time evolution.
- Finally, for simplicity, we assume that the only operation that has happened between the 'in' and 'out' states is unitary time evolution whose operator is $U(t, t_0)$.

If $t = t_0$, time evolution has not happened, and the three pictures are identical.

Definition 3.5 (Schrödinger picture) The representation we have used throughout undergrad QM is the Schrödinger picture, where the unitary time evolution operator in natural units is

$$U(t, t_0) = e^{-iH(t-t_0)} \tag{3.23}$$

Time evolution is represented as follows:

- Operators are time-invariant.
- 'in' states evolve under the *total* Hamiltonian H:

$$|\psi, t\rangle = e^{-iH(t-t_0)}|\psi, \text{in}\rangle = U(t, t_0)|\psi, \text{in}\rangle \tag{3.24}$$

- 'out' states are time-invariant. This is because they are actually 'in' states that have *already* undergone time evolution in disguise. That is to say, $U(t, t_0)$ is absorbed into the 'out' state by definition:

$$\langle\alpha, \text{out}| = \langle\psi, \text{in}|e^{iH(t-t_0)} = \langle\psi, \text{in}|U^\dagger(t, t_0) \tag{3.25}$$

As we all know, the Schrödinger picture equation of motion is simply the *Schrödinger equation*. By differentiating ψ by time and accounting for $U(t, t_0)$, we naturally find

Theorem 3.5 (Schrödinger equation)

$$i\frac{d\psi}{dt} = H\psi \tag{3.26}$$

Definition 3.6 (Heisenberg picture) The *Heisenberg picture* is the opposite of the Schrödinger picture. While $U(t, t_0)$ has the same form, it is the operators that evolve instead of states. Time evolution is represented as follows:

- Operators evolve under the *total Hamiltonian* H:

$$O_H = e^{iH(t-t_0)}Oe^{-iH(t-t_0)} = U^\dagger(t, t_0)OU(t, t_0) \tag{3.27}$$

- 'in' states are time-invariant.

- 'out' states evolve under the *total Hamiltonian* H:

$$ {}_H\langle \alpha, t| = \langle \alpha, \text{out}|e^{-iH(t-t_0)} = \langle \alpha, \text{out}|U(t,t_0) \tag{3.28} $$

Note 3.1 ('Out' state as half an operator) There is an important subtlety here which only emerges when we inspect the Heisenberg picture. Even though our 'out' state is nominally a state, it is effectively regarded as 'half of a density operator' instead of a state:

$$ P_\alpha = |\alpha, \text{out}\rangle\langle \alpha, \text{out}| \tag{3.29} $$

Derivation 3.3 (Heisenberg equation) The Heisenberg picture equivalent of the Schrödinger equation is the so-called *Heisenberg equation*. Instead of differentiating ψ by time, we differentiate O_H since it is now the time-dependent quantity.

$$ \frac{\partial O_H}{\partial t} = \frac{\partial U^\dagger}{\partial t} O_S U + U^\dagger O_S \frac{\partial U}{\partial t} + \frac{\partial O_H}{\partial t} = U^\dagger H O_S U - U^\dagger O_S H U + \frac{\partial O_H}{\partial t} = U^\dagger [H, O_S] U + \frac{\partial O_H}{\partial t} \tag{3.30} $$

Reabsorbing U and $U^\dagger$ gives us

Theorem 3.6 (Heisenberg equation)

$$ i\frac{dO_H}{dt} = [O_H, H] + \frac{\partial O_H}{\partial t} \tag{3.31} $$

where, again, the last RHS term vanishes when O_H has no explicit dependence on time.

Finally, we note that this equation then corresponds well with Equation 3.7 and Equation 3.9.

Remark 3.4 Expectedly, the Heisenberg and Schrödinger equations are identical at $t = t_0$.
The *interaction picture* lies between the Schrödinger and Heisenberg pictures. The central idea is to decompose the Hamiltonian into two parts, the *free Hamiltonian* H_0 and the *interacting Hamiltonian* H_I. The total Hamiltonian H is then

$$ H = H_0 + H_I \tag{3.32} $$

Note 3.2 (Determining H_0 and H_I) H_0 is usually number operators like $a^\dagger a$, $b^\dagger b$, etc. (representing the energy of independent modes) while H_I is usually mixed terms like $a^\dagger b$, $ab^\dagger$, $a^\dagger a b^\dagger b$, etc. (representing products of operators from different modes or nonlinearities) and external driving terms $a^\dagger e^{-i\omega t}$ + h.c. (representing interaction with classical fields). This will make more sense when you finish reading this part.

Definition 3.7 (Interaction picture) The evolution operator in the interaction picture is

$$ U(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \tag{3.33} $$

Time evolution is represented as follows:

- Operators evolve under the *free Hamiltonian* H_0:

$$ O_H = e^{iH_0(t-t_0)} O e^{-iH_0(t-t_0)} \tag{3.34} $$

  Importantly, however, H_0 itself is always time-independent even though it is an operator. This is because the interaction picture is built *around* the dynamics of H_0.

- 'in' states evolve under both the *total* and *free Hamiltonians*:

$$ |\psi, t\rangle = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} |\psi, \text{in}\rangle \tag{3.35} $$

  where we often label *interaction picture evolution operator* $U(t,t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$.

- 'out' states evolve under the *free Hamiltonian* H_0:

$$ {}_I\langle \alpha, t| = \langle \alpha, \text{out}|e^{-iH_0(t-t_0)} \tag{3.36} $$

Note 3.3 As noted previously, the three pictures are functionally identical. To verify this, simply acknowledge that:

- If we write the 'out' states as-is, there is a 'conservation of time evolution terms': Multiplying all 3 time-evolved terms should leave only a prefactor of $e^{-iH(t-t_0)}$, where H is expectedly the *total Hamiltonian*.
- If we write out the 'out' states as a time evolution of 'in' states, then there is a prefactor of 1.

While the Heisenberg picture generally simplifies calculations, the interaction picture is advantageous when the Hamiltonian includes an interaction term.

It is now possible to define *symmetries of concrete systems* in all three dynamical pictures.

Theorem 3.7 (Conserved quantity) An observable represented by some operator $\hat{A}$ is said to be a conserved quantity if the time derivative of its expectation value is zero:

$$\frac{d}{dt}\left\langle \hat{A} \right\rangle = 0 \quad \text{or} \quad \partial_t \hat{A} = 0 \tag{3.37}$$

This has two equivalent statements in the form of commutators:

- In the Schrödinger and Heisenberg pictures:

$$[\hat{A}, \hat{H}] = 0 \tag{3.38}$$

- In the interaction picture:

$$[\hat{A}_I(t), \hat{H}_0] = 0 \tag{3.39}$$

Derivation 3.4 (Time ordering) By differentiating Equation 3.33 against t, one can derive an alternate expression of $U(t, t_0)$ in terms of itself at a different point in time:

$$U(t, t_0) = \mathbb{I} - i \int_{t_0}^{t} dt_1 H_{\text{int},I}(t_1) U(t_1, t_0) \tag{3.40}$$

By inserting $U(t, t_0)$ into $U(t_1, t_0)$ over and over and over again, one yields the so-called *Dyson series*:

$$U(t, t_0) = \mathbb{I} + \sum_{j=1}^{\infty} (-i)^j \int_{t_0}^{t} dt_j \cdots \int_{t_0}^{t} dt_1 H_{\text{int},I}(t_j) \cdots H_{\text{int},I}(t_1) \tag{3.41}$$

To ensure that the intergrations are performed at the correct temporal order, we introduce the so-called *time ordering symbol*:

Definition 3.8 (Time ordering symbol) Consider a series operators $A_1(x_1) \cdots A_n(x_n)$, each of which can be represented in the form of creation and annihilation operators like $A_n(x_n) = A_n^+(x_n) + A_n^-(x_n)$. The time ordering symbol is a loosely defined convenience which reorders the operators according to their 4-position:

$$T[A_1(t_1) \cdots A_n(t_n)] = (-1)^p A_{i_1}(t_{i_1}) \cdots A_{i_n}(t_{i_n}) \quad \text{for} \quad x_{i_1} \leq \cdots \leq x_{i_n} \tag{3.42}$$

where p, the parity, can be represented by Grassmann parity[a]:

$$p = \sum_{i<j} \pi(A_i)\pi(A_j) \mod 2 \tag{3.43}$$

where the Grassmann parity $\pi(A_i)$ observes

- $\pi(A_i) = 0$ for bosonic operators, which commute as they are Grassmann-even.
- $\pi(A_i) = 1$ for fermionic operators, which anticommute as they are Grassmann-odd.

[a] See *Spinors & Symmetries*.

Remark 3.5 One can also define this without Grassmann mathematics, albeit less satisfyingly:

- Bosonic-bosonic swaps do not contribute to p. No sign change occurs as bosonic operators commute.
- Bosonic-fermionic swaps do not contribute to p. No sign change occurs as bosonic and fermionic operators commute.
- Due to anticommutation, fermionic-fermionic swaps contribute to p by the following:

$$p = \text{num. of fermionic-fermionic swaps} \mod 2 \tag{3.44}$$

Remark 3.6 When a system is entirely comprised of bosons or fermions, this simplifies:

- For bosonic operators, $p = 0$.
- For fermionic operators, $p = 0$ if the number of swaps is even and $p = 1$ if the number of swaps is odd.

Noting that the Dyson series can be represented compactly via an exponential, we represent $U(t, t_0)$ as a time-ordered exponential:

$$U(t, t_0) = T\left[\exp\left(-i\int_{t_0}^{t} dt' H_{\text{int},I}(t')\right)\right] \tag{3.45}$$

In simplified sceanrios, T can also be represented mathematically with the *Heaviside step function* $\theta(t)$:

$$T[A_1(t_1)A_2(t_2)] = \theta(t_1 - t_2)A_1(t_1)A_2(t_2) \pm \theta(t_2 - t_1)A_2(t_2)A_1(t_1) \tag{3.46}$$

where:

Definition 3.9 (Heaviside step function)

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{3.47}$$

Theorem 3.8 (Heaviside step function properties)

$$\partial_t \theta(t) = \delta(t) \tag{3.48}$$

$$(\partial_t \theta(t))\phi(t) = -\delta(t)(\partial_t \phi(t)) \tag{3.49}$$

Part II

Canonical quantisation

Chapter 4

Free fields

> **Quote 4.1** What could possible go wrong?
>
> *Alessio Serafini, 16 January 2024*

In this chapter, we make an overview of the so-called Klein-Gordon equation, an intuitive attempt at developing a special relativistic quantum theory, as well as its failures. By quantising the Klein-Gordon field, we will get a first taste of canonical quantisation.

4.1 Klein-Gordon equation and its demise

In quantum mechanics, there is only one dimension, which is time. This becomes a problem in high-energy physics, where we take into account the movement of the particles. Also, we must note that particle interactions in high-energy physics often take place in the relativistic limit. As such, any candidates for a theory of HEP must satisfy the following:

- The theory must be in 4D (3 spatial dimensions and 1 temporal dimension).
- The theory must incorporate special relativity.

An earlier attempt at constructing such a theory is the infamous *Klein-Gordon equation*. At that point, QFT was still not developed, and it was naively assumed that the Schrödinger equation could be modified to be Lorentz-invariant by simply applying the wavefunction ϕ to both sides of the equivalence

Theorem 4.1 (On-shell condition)

$$E_p^2 = m^2 + |p|^2 \quad \text{or} \quad p^\mu p_\mu = m^2 \tag{4.1}$$

where, for convenience, we have set $c = \hbar = 1$.

This is known as the *on-shell condition*[1]. Intuitively, this gives

Theorem 4.2 (Klein-Gordon equation) The Klein-Gordon equation describes scalar (spin-0) particles in a relativistic framework:

$$(\Box + m^2)\phi = 0 \tag{4.2}$$

where $(\Box + m^2)$ is known as the *Klein-Gordon operator*.

Remark 4.1 Here we make a historical comment. The Klein-Gordon equation (and the Dirac equation, as we will see later) was historically derived by intuition, like we have done here. However, the modern standard derivation involves inserting the free massive scalar field Lagrangian in Equation 2.4 (which we ultimately resign to formulating by intuition) into the Euler-Lagrange equations.

Remark 4.2 As free fields are solutions to the Klein-Gordon equation, a Klein-Gordon operator acting on a free field always gives 0.

[1] So-called in reference to the so-called *mass shell*. Essentially, *on-shell* refers to what satisfies the classical equations of motion, whereas *off-shell* refers those that do not.

Exercise 4.1 Show that the Klein-Gordon equation can be recovered by using an action principle on the free massive scalar field Lagrangian in Equation 2.4.

The Klein-Gordon equation has a plane wave general solution

$$\phi(x) = Ne^{-iE_p t - p\cdot x} \tag{4.3}$$

where N is a normalisation constant.

Derivation 4.1 (Demise) Consider a simple 1D potential barrier of the form

$$V(x) = \begin{cases} 0 & x < 0 \\ V & x \geq 0 \end{cases} \tag{4.4}$$

According to the Klein-Gordon equation, the simplest solution would be

$$\phi(t,x) = \begin{cases} e^{-i(E_p t - px)} + ae^{-i(E_p t + px)} & x < 0 \\ be^{-i(E_p t + kx)} & x \geq 0 \end{cases} \tag{4.5}$$

where p is the momentum, $k = \sqrt{(E_p - V_0)^2 - m^2}$, and

- $e^{-i(E_p t - px)}$ is the part of the field travelling at the +ve x-direction that has not yet reached the potential barrier.
- $ae^{-i(E_p t + px)}$ is the part of the field reflected at the barrier travelling at the −ve x-direction.
- $be^{-i(E_p t + kx)}$ is the part of the field transmitted through the barrier travelling at the +ve x-direction.

Intuitively, both $\phi(x)$ and $\partial_x\phi(x)$ are continuous at $x = 0$, from which we find the parameters

$$a = \frac{p-k}{p+k} \quad b = \frac{2p}{p+k} \tag{4.6}$$

By inserting the $x < 0$ solution into the Klein-Gordon equation, we find a dispersion relation

$$p = \pm\sqrt{E_p^2 + m^2} \tag{4.7}$$

To reflect the forward-travelling nature of $e^{-i(E_p t - px)}$, the group velocity $v_g = \partial_p E_p$ must be positive. This forces us to adopt the positive ($x \geq 0$) solution, which we insert into the Klein-Gordon equation. Due to the non-zero potential V, we replace the partial derivatives with covariant derivatives[a]:

Definition 4.1 (Scalar field covariant derivative)

$$i\partial_t \to i\partial_t - V \quad \partial_t \to \partial_t + iV \tag{4.8}$$

This gives

$$k = \mp\sqrt{(E_p - V)^2 - m^2} \tag{4.9}$$

Again, to reflect the forward-travelling nature of $be^{-i(E_p t + kx)}$, the group velocity or its inverse $\frac{1}{v_g} = \frac{\partial k}{\partial E_p} = \mp\frac{E_p - V}{|k|}$ must be positive. Now consider the case $V > E_p$. From the group velocity condition, we are forced to adopt the negative solution.
One can find a negative energy solution for each positive energy solution. However, this can be handwaved, as we will see much, much later, as antimatter. The real problem lies with the probability density, which we recall to be the 0^{th} component of the conserved current:

$$\varrho = i(\phi^*(\partial_t + iV)\phi - \phi(i\partial_t - V)\phi^*) \tag{4.10}$$

which, in this case, is simply

$$\varrho = 2b^2(E_p - V) \tag{4.11}$$

For $E < V$, this probability density is always negative.
Remark 4.3 Wait, what?

[a] We will justify this when we reach Equation 6.13.

A negative probability density is always unphysical, which can be resolved by turning the (classical) Klein-Gordon field into an operator. This is the beginning of quantum field theory.
Historically, the quantisation of fields/operators as the so-called *field operator* is known as *second quantisation*, in contrast to quantised particles, which was known as the *first quantisation*[2]. Today, we call both *canonical quantisation* as canonical commutation relations are utilised in both quantisation processes.

4.2 Fock space

A generic state in QFT is essentially a linear combination of k particle states for some arbitrary k. This is significant in that k is not fixed - particles might be created and annihilated. Hilbert spaces, which have a fixed number of particles, fail to describe QFT. Rather, the vector space QFT lies in is known as a *Fock space*:

Definition 4.2 (Fock space) The Fock[a] space $\mathcal{F}(\mathcal{H}_1)$ is the direct sum of all n-particle Hilbert spaces:

$$\mathcal{F}(\mathcal{H}_1) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \quad n \in \mathbb{Z} \tag{4.12}$$

where $\mathcal{H}_0 = \mathbb{C}$ is the vacuum state space (zero particles), and $\mathcal{H}_n = \mathcal{H}_1^{\otimes n}$ is the n-particle Hilbert space.

[a] Named after Vladimir Fock, or *Fok* in scientific transliteration.

Remark 4.4 In mathematics, this decomposition is known as a $\mathbb{Z}$-grading as $n \in \mathbb{Z}$.

Definition 4.3 (Creation and annihilation operators) Fock spaces are equipped with creation and annihilation operators $a^\dagger$ and a that adds a particle to the final state and removes a particle from the initial state respectively:

$$a^\dagger|n\rangle = |n+1\rangle \quad a|n\rangle = |n-1\rangle \tag{4.13}$$

Remark 4.5 Bosonic creation and annihilation operators are near-identical the good ol' ladder operators in QM as they obey the same commutation relations. Fermionic creation and annihilation operators observe anticommutation relations, which are powered by Grassmann mathematics covered in *Spinors & Symmetries*.

Theorem 4.3 (Bosonic and fermionic operator commutations)

- Bosonic operators commute as they are Grassmann-even:

$$[B_i, B_j] = B_i B_j - B_j B_i = 0 \tag{4.14}$$

- Fermionic operators anticommute[a] as they are Grassmann-odd:

$$\{F_i, F_j\} = F_i F_j + F_j F_i = 0 \tag{4.15}$$

- A bosonic operator commutes with a fermionic operator:

$$[B, F] = BF - FB = 0 \tag{4.16}$$

[a] The physical justification is that if they commute, the Hamiltonian would be unbounded from below and causality would be violated.

Let us revise the properties of commutators and anticommutators.

Theorem 4.4 (Commutator and anticommutator properties) Commutators:

- **Bilinearity:** The commutator is linear in both the first and second arguments

$$[\alpha A + \beta C, B] = \alpha[A, B] + \beta[C, B] \quad [A, \alpha B + \beta C] = \alpha[A, B] + \beta[A, C] \tag{4.17}$$

[2] In fact, second quantisation is a slight misnomer as quantising operators is also possible (although unnecessary) in QM.

- **Jacobi identity:** The cyclic sum of nested commutators vanishes

$$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0 \tag{4.18}$$

- **Product rules:** These are Leibniz-type identities

$$[A,BC]=[A,B]C+B[A,C] \quad [AB,C]=A[B,C]+[A,C]B \tag{4.19}$$

Anticommutators:

- **Bilinearity:** The anticommutator is linear in both the first and second arguments

$$\{\alpha A+\beta C,B\}=\alpha\{A,B\}+\beta\{C,B\} \quad \{A,\alpha B+\beta C\}=\alpha\{A,B\}+\beta\{A,C\} \tag{4.20}$$

- **Product rules:** They look a bit different than for commutators

$$\{A,BC\}=\{A,B\}C-B[A,C] \quad \{AB,C\}=A\{B,C\}-[A,C]B \tag{4.21}$$

So far we have been working in position space, which is, informally speaking, the collection of all possible positional vectors. However, it is often convenient to employ *momentum space* in canonical quantisation. This is because, as we will see:

- The free field Hamiltonian becomes diagonal.
- Expressions for propagators are simplified.

A momentum space is a generalisation of the *reciprocal space* or *wavevector space* you may have seen before in crystallography. The momentum and position (or physical) spaces are Fourier transforms of each other.

Definition 4.4 (Fourier transform)

$$\tilde{f}(p)=\int d^4x e^{ipx}f(x) \tag{4.22}$$

At this point, one should familiarise themselves with three implications. The first two stem from the so-called sifting property of the Dirac delta:

Theorem 4.5 (Sifting property of the Dirac delta)

$$\int d^nx\delta^n(x-y)f(x)=f(y) \tag{4.23}$$

This is very useful because we then have the first implication

Theorem 4.6 (Elimination of integral via the Fourier transformation)

$$\int d^my\int d^nx\delta^n(x-y)f(x)=\int d^myf(y) \tag{4.24}$$

Now let us set $y=0$. We then have

$$\int d^nxf(x)\delta^n(x)=f(0) \tag{4.25}$$

which gives the second implication

Theorem 4.7 (Fourier transform and exponential function)

$$\int d^nxe^{\pm ikx}=(2\pi)^n\delta^nk \tag{4.26}$$

The final implication concerns the 4-derivative ∂_μ. Under a Fourier transform, we have

$$\partial_\mu f(x)\to\int d^4xe^{ip\cdot x}\partial_\mu f(x)=-ip_\mu\tilde{f}(p) \tag{4.27}$$

if one removes the functions themselves, one finds

Theorem 4.8 (Fourier transform of 4-derivative)

$$\partial_\mu \to -ip_\mu \tag{4.28}$$

For example, the momentum space Klein-Gordon equation is

$$(p^2 + m^2)\phi = 0 \tag{4.29}$$

In the following derivation, even though the field and canonical momentum operators are still in position space, their expressions are in momentum space, which we Fourier-transform back to position space.

4.3 Quantisation of the Klein-Gordon field

Derivation 4.2 (Field operator) The Klein-Gordon equation general solution in Equation 4.3, can be rewritten to account for negative energy solutions:

$$\phi(x,t) = \int d^3pN_p \left(f_p e^{-i(E_p t - p\cdot x)} + f_p^* e^{i(E_p t - p\cdot x)}\right) \tag{4.30}$$

where N_p, a real function of p, is the previously seen normalisation factor and f_p is a complex function of p (and hence based in momentum space). Importantly, as the field operator $\phi(x,t)$ is based in position space, we must perform a Fourier transform $\int d^3pN_p$ to convert f_p from momentum space to position space.
To ensure that the resultant quantised Hamiltonian will evolve with time in the same way, we replace f_p and f_p^* with the annihilation and creation operators a_p and $a_p^\dagger$. As the Klein-Gordon field is a scalar field (spin-0), it is a bosonic field and its components commute under field quantisation:

Theorem 4.9 (Bosonic creation and annigilation operator commutations) For two arbitrary momenta p and q in bosonic fields, their creation and annihilation operators a_p, $a_p^\dagger$, a_q and $a_q^\dagger$ observe:

$$[a_p, a_q^\dagger] = (2\pi)^3\delta^3(p-q) \tag{4.31}$$

$$[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0 \tag{4.32}$$

Recalling that special relativity must be observed, we must choose an N_p that makes $\phi(x)$ Lorentz-invariant. The Lorentz-invariant phase space volume element for a single particle is given by

Definition 4.5 (Lorentz-invariant phase space volume element)

$$dV = \frac{dp^3}{(2\pi)^3 2E_p} \tag{4.33}$$

For the field operator, we can remove a factor of $1/\sqrt{2E_p}$ to this volume element so that no factors of E_p emerge in the field and momentum operator commutation relations[a]. Taking $N_p = 1/((2\pi)^3\sqrt{2E_p})$ the Klein-Gordon field operator is then written as

Definition 4.6 (Klein-Gordon field operator)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right) \tag{4.34}$$

[a] Lorentz invariance is preserved as the factor $1/\sqrt{2E_p}$ is not really gone, but rather absorbed into the field amplitude.

Note 4.1 (Two formalisms of $\phi(p)$) Intuitively, the field operator in momentum space is the Fourier transform of its position space counterpart

$$\phi(p) = \int d^3x e^{-ip\cdot x}\phi(x) \tag{4.35}$$

However, the notation $\phi(p)$ is overloaded: a second formalism exists in some literature where $\phi(p)$ denotes the *momentum space contribution* to the field operator seen in Equation 4.34:

$$\phi(p)_{\text{alt}} = \frac{1}{\sqrt{2E_p}}\left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right) \tag{4.36}$$

which gives rise the (equally correct) formula for the position space field operator

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\phi(p)_{\text{alt}} \tag{4.37}$$

In this book, we use the first formalism.

The momentum operator can then be written by adding a factor of iE_p and changing the sign on the first term

Definition 4.7 (Klein-Gordon momentum operator)

$$\pi(x) = -\int \frac{d^3p}{(2\pi)^3\sqrt{2}}\sqrt{E_p}\left(-a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right) \tag{4.38}$$

The commutation relations for bosonic fields are then

Theorem 4.10 (Bosonic field and momentum operator commutations) For some arbitrary spacetime coordinates x and y

$$[\phi(x), \pi(y)] = i\delta^3(p-q) \tag{4.39}$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = [\phi^*(x), \phi(y)] = [\pi^*(x), \pi(y)] = 0 \tag{4.40}$$

Remark 4.6 The vanishing of the commutator between ϕ and π with their complex conjugates is a result of the Grassmann-evenness of bosonic fields[3]. The implication of this is that ϕ and π are treated as independent from their complex conjugates.

By recalling the definition of the Hamiltonian, we can derive the Klein-Gordon Hamiltonian density from the Klein-Gordon Lagrangian density as

$$\mathcal{H} = \frac{1}{2}(\pi^2 + |\nabla\phi|^2 + m^2\phi^2) \tag{4.41}$$

We then find the (time-independent) Hamiltonian operator:

$$H = \int \frac{d^3p}{(2\pi)^3 2E_p}E_p^2\left(a_p^\dagger a_p + a_p a_p^\dagger\right) = \underbrace{\int \frac{d^3p}{(2\pi)^3}E_p a_p^\dagger a_p}_{①} + \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{E_p}{2}\left[a_p, a_p^\dagger\right]}_{②} \tag{4.42}$$

However, as $\left[a_p, a_p^\dagger\right]$ is proportional to the identity, so is the term ②, and we can neglect it for convenience. Due to the ordering of its creation and annihilation operators, the remaining term ① is known as the *normal-ordered* or *Wick-ordered* Hamiltonian, named after Gian Carlo Wick:

Definition 4.8 (Normal-ordered Klein-Gordon Hamiltonian)

$$:H: = \int \frac{d^3p}{(2\pi)^3}E_p a_p^\dagger a_p \tag{4.43}$$

Here we have introduced the concept of *normal ordering*:

Definition 4.9 (Normal ordering) For some operator O that can be expressed as a polynomial of creation and annihilation operators, its normal ordering or *Wick ordering* $:O:$ is the same polynomial, but with all creation operators to the left of all annihilation operators. This is often necessary in that it eliminates unphysicality created by vacuum fluctuations like $a_p^\dagger a_p$.

[3] In other words, they are *enforced* by the Grassmann-evenness of the fields.

Remark 4.7 For example, given some $O = a_p a_q a_p^\dagger$, we have $:O: = a_p^\dagger a_p a_q$.
By considering Equation 4.43 as the entire Hamiltonian[4], we are able to easily diagonalise it. Therefore, we can find that $:H:$ admits the eigenvectors $|n\rangle$ or $a_p|n\rangle$, with the eigenvalues being

$$:H:a_p|n\rangle = (-n - E_p)a_p|n\rangle \tag{4.44}$$

Remark 4.8 Conversely, *antinormal ordering* places creation operators to the right instead.

Theorem 4.11 (Vacuum expectation value) The vacuum expectation value of any normal-ordered expression yields zero.

We are now in a position to consider the physicality of our free field, Klein-Gordon theory. This involves distinguishing between free field theories, of which our Klein-Gordon theory is one, and interacting theories, in which every physical theory in the standard model lies.

Definition 4.10 (Free and interacting field theories)

- *Free terms* in the Lagrangian are terms which, upon insertion into the Euler-Lagrange equations, yield a linear equation of motion (like the Klein-Gordon equation we have seen and the Dirac equation we will see).
- *Interacting terms* in the Lagrangian are terms which, upon insertion into the Euler-Lagrange equations, yield a nonlinear equation of motion (like the full ϕ^4 theory and QED equations of motion, which are not very interesting for us except for completeness).

We will elaborate on these two statements in the next chapter. For now, it suffices to understand that the Klein-Gordon theory Lagrangian consists purely of free terms due to the linearity of the Klein-Gordon equation, which makes it clearly unphysical:

- As you will see in the next chapter, Feynman diagrams include propagators, vertices and loops. A free field theory can only have propagators: As there are no interactions, vertices do not exist.
- For this reason, loops do not exist either, except for standalone ones.
- The propagator goes on forever in time in the absence of interactions, giving rise to a scattering matrix that is always trivial (simply the unit matrix).
- The cross-section for any scattering process (where next to nothing happens) is then always zero. It should then be obivous that this field theory is not very interesting.

Before we get into any physical theories (which has the problem of gauge fixing), we want to look a toy model that is interacting to get the full grasp of the building block of QFT while not having to deal with those pesky gauges. In most literatire, this toy model is ϕ^4 theory. This is nothing but Klein-Gordon theory with an added quartic interacting term which gives rise to the vertex. As the free terms are identical, the propagator stays the same. For this reason, we will derive the free field propagator not in this chapter, but rather the next one where, in ϕ^4 theory, its significance can be fully apperciated.

[4] An advantage of considering a quantity's normal-ordered counterpart as itself is that we eliminate any uninteresting constant terms. This often simplifies calculations.

Chapter 5

Interacting fields I: ϕ^4 theory

> **Quote 5.1** Tbh, hand-drawn sketches have their own Charme.
>
> *Felix Halbwedl, 27 August 2024*

We now investigate particle interactions as well as the two matrices S_{fi} and $\mathcal{M}_{fi}$, the first of which physically governs the probability a certain interaction will take place. The Feynman rules, which we use to build the formula for calculating elements of $\mathcal{M}_{fi}$, are then derived for ϕ^4 theory, a simple toy model. Some concluding discussions on their use in high energy physics are then made.

5.1 Self-interaction

We now consider an interacting field theory where a field interacts with its environment. This comes into play by introducing interacting terms in the Lagrangian. One type of interaction is *self-interaction*, which is the interaction between a particle and its own field. A good toy model is the so-called ϕ^4 *theory*[1], which gets rid of the problematic[2] 3$^{\text{rd}}$-order term, adding only a quartic interaction term $-\frac{\lambda}{4!}\phi^4$ to Equation 2.4:

Definition 5.1 (ϕ^4 theory Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{5.1}$$

where λ is some dimensionless coupling constant that determines the strength of the interaction, and the $1/4!$ is a combinatorial normalisation factor that compensates for the fact that there are 4! ways to permute the (4 identical) fields in ϕ^4 theory.

ϕ^4 theory is among a family of theories known as ϕ^n theories. It is superior to all other ϕ^n theories for two reasons:

- **Energetic stability:** ϕ^4 *theory* has the second-simplest interaction term that respects the symmetry $\phi \to -\phi$[3]:
  - The general form for the potential energy in ϕ^3 theory is $V(\phi) = m^2\phi^2 + g\phi^3$. Due to the odd exponential in $g\phi^3$, one can get a negative Hamiltonian expectation for a large, negative ϕ, even if the coupling constant g is positive.
  - In contrast, the $\lambda\phi^4$ term ϕ^4 theory is positive as long as λ is positive, ensuring a stable minimum, which is crucial for physical systems.
- **Renormalisability:** In certain dimensions, ϕ^3 theory is non-renormalisable, unlike ϕ^4 theory.

[1] Even though its only physical application is describing the self-interaction term of the Higgs field, it has become a staple of QFT books.

[2] In that it makes the theory non-renormalisable. You will see what this means in Part IV.

[3] ϕ^2 theory has the simplest interaction term. But this term is simply the previously seen mass term $\frac{1}{2}m^2\phi^2$.

Note that this is still a bosonic field theory that satisfies the Klein-Gordon equation.
Before moving onto scattering, we need introduce one last mathematical tool, which is the *Wick contraction*.

Definition 5.2 (Wick contraction) For operators A and B, the Wick contraction[a] or simply *contraction* is simply its vacuum expectation value

$$\overbrace{AB} = \langle 0|AB|0\rangle \tag{5.2}$$

[a] So-called as just like the contraction of indices in GR (which, in the case of a rank-2 tensor, starts with two indices and ends with a scalar), it starts with two operators and ends with a number.

Derivation 5.1 (Alternative forms of the Wick contraction) From the definition of Wick contractions, we can final several equivalences:

- The product of two operators can always be split into a part that contributes to the vacuum expectation value and a part that does not, which is actually the normal ordering[a] $:AB:$.

  Theorem 5.1 (Wick's first theorem)

  $$AB = \langle 0|:AB:|0\rangle + :AB: = \overbrace{AB} + :AB: \tag{5.3}$$

  In some literature, a rearranged version of *Wick's first theorem* is actually used as a less intuitive definition of Wick contractions.

  $$\overbrace{AB} = AB - :AB: \tag{5.4}$$

- The vacuum expectation value is inherently time-ordered, so if A and B are the fields $\phi(x)$ and $\phi(y)$, we can even make the equivalence

  $$\overbrace{\phi(x)\phi(y)} = \langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|T[\phi(x)\phi(y)]|0\rangle = D_F(x-y) \tag{5.5}$$

  where $D_F(x-y)$ is the Feynman propagator.

[a] As normal-ordered operators always have creation operators before annihilation operators, their vacuum expectation value $\langle 0|:AB:|0\rangle$ is always zero.

Remark 5.1 In other words, the Wick contraction of operators returns their 'nontrivial' part. This part, which represents quantum fluctuations and interactions, contributes to the vacuum expectation value, whereas the normal-ordered part does not.
Often, it is more convenient to use a very similar operation called the *time-ordered pairing* instead of Wick contractions.

Definition 5.3 (Time-ordered pairing)

$$\underbrace{A(x)B(y)} = \begin{cases} \overbrace{A(x)B(y)} & x^0 > y^0 \\ (-1)^p \overbrace{B(x)A(y)} & y^0 > x^0 \end{cases} \tag{5.6}$$

where p, last seen in Definition 3.8, is our good friend, the parity.

Quote 5.2 In more accurate books like Bogoliubov's, the time dependent contractions are written down with bottom brackets.

Felix Halbwedl, 22 December 2024

Derivation 5.2 (Wick's second theorem for two fields) A field can be split into creation and annihilation parts:

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{5.7}$$

where

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ip\cdot x} \quad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p^\dagger e^{+ip\cdot x} \tag{5.8}$$

Let us evaluate $T[\phi(x)\phi(y)]$ under this decomposition. When $x^0 > y^0$, we have, by time-ordering

$$T[\phi(x)\phi(y)] = \phi(x)\phi(y)$$

Now substitute Equation 5.7

$$\phi(x)\phi(y) = [\phi^+(x) + \phi^-(x)][\phi^+(y) + \phi^-(y)] \tag{5.9}$$

We then rewrite this sum by grouping into the normal-ordered piece plus the piece that arises from swapping one creation past one annihilation:

$$\phi(x)\phi(y) = :\phi(x)\phi(y): + [\phi^+(x), \phi^-(y)] \tag{5.10}$$

But using the mode expansions, one can directly see that

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} = D_F(x-y) \tag{5.11}$$

where the Feynman propagator is exactly the contraction of two fields. Thus for $x^0 > y^0$, we have

$$T[\phi(x)\phi(y)] = :\phi(x)\phi(y): + D_F(x-y) \tag{5.12}$$

By the same argument, this is also true for $y^0 > x^0$. Equation 5.12 is known as Wick's second theorem for two fields.

Now we generalise this to an arbitrary number of fields.

Theorem 5.2 (Wick's second theorem) The time-ordering $T[A_1A_2A_3A_4A_5A_6\cdots]$, where all operators are made up of creation and annihilation operators like $A_i = A_i^+ + A_i^-$, can be expressed in terms of time-ordered pairings:

$$T[A_1A_2A_3A_4A_5A_6\cdots] = :A_1A_2A_3A_4A_5A_6\cdots: + \underbrace{\sum_{\text{single}} :\underbrace{A_1}\,A_2A_3A_4A_5A_6\cdots:}_{①} + \underbrace{\sum_{\text{double}} :\underbrace{A_1}\,A_2\underbrace{A_3}\,A_4A_5A_6\cdots:}_{②} + \cdots \tag{5.13}$$

where:

- ① denotes the sum of all the possible results of $A_1A_2A_3A_4A_5A_6\cdots$ undergoing one Wick contraction somewhere in the expression:

$$\sum_{\text{single}} :\underbrace{A_1}\,A_2A_3A_4A_5A_6\cdots: = :\underbrace{A_1}\,A_2A_3A_4A_5A_6\cdots: + :\underbrace{A_1}\,A_3A_2A_4A_5A_6\cdots: + \cdots \tag{5.14}$$

- ② denotes the sum of all the possible results of $A_1A_2A_3A_4A_5A_6\cdots$ undergoing two Wick contractions somewhere in the expression:

$$\sum_{\text{double}} :\underbrace{A_1}\,A_2\underbrace{A_3}\,A_4A_5A_6\cdots: = :\underbrace{A_1}\,A_2\underbrace{A_3}\,A_4A_5A_6\cdots: + :\underbrace{A_1}\,A_3\underbrace{A_2}\,A_4A_5A_6\cdots: + \cdots \tag{5.15}$$

- ...and so on.

An alternative, non-time-ordered version of *Wick's second theorem* is

$$A_1A_2A_3A_4A_5A_6\cdots = :A_1A_2A_3A_4A_5A_6\cdots: + \sum_{\text{single}} :\overbrace{A_1}\,A_2A_3A_4A_5A_6\cdots: + \sum_{\text{double}} :\overbrace{A_1}\,A_2\overbrace{A_3}\,A_4A_5A_6\cdots: + \cdots \tag{5.16}$$

where we simply use the normal contraction.

Finally, a nice trick is the so-called *Wick's third theorem* or *Wick's theorem for vacuum expectation values*:

Theorem 5.3 (Wick's third theorem) For operators A, B_1, $\cdots$, B_n, the following is observed:

$$\langle 0|T[AB_1\cdots B_n]|0\rangle = \sum_i \langle 0|T[\overbrace{AB_1\cdots B_i}\cdots B_n]|0\rangle \tag{5.17}$$

5.2 Scattering matrix

An interacting field theory gives rise to *scattering*, where we have the *S-matrix* or the *scattering matrix* that encodes all the information about the probabilities of different scattering processes. It can be derived by taking limits of $U(t,t_0)$, where time evolution covers the entire history of the system, from the infinite past to the infinite future:

$$S = \lim_{t\to\infty}\lim_{t_0\to 0\infty} U(t,t_0) \tag{5.18}$$

Definition 5.4 (S-matrix operator) For the initial/incoming state $|\psi,\text{in}\rangle$ and the final/outgoing state $\langle\alpha,\text{out}|$, the S-matrix element S_{fi} represents the probability amplitude that $|\psi,\text{in}\rangle$ evolves into $\langle\alpha,\text{out}|$. It can be represented via the S-matrix operator S:

$$S_{fi} = \langle\alpha,\text{out}|S|\psi,\text{in}\rangle \tag{5.19}$$

An important property is the so-called *cluster decomposition*, which states that an experiment cannot influence another experiment carried out far away from it, and vice versa. Its implication in QFT is as follows:

Theorem 5.4 (Cluser decomposition) Consider two experiments (or rather *clusters*) 1 and 2. An initial state α which includes parts in both clusters α_1 and α_2 evolves into a final state β which likewise includes parts in both clusters β_1 and β_2. The scattering matrix can always be decomposed to

$$S_{\beta\alpha} = S_{\beta_1\alpha_1}S_{\beta_2\alpha_2} \tag{5.20}$$

where $S_{\beta_1\alpha_1}$ and $S_{\beta_2\alpha_2}$ are 'parts' of $S_{\beta\alpha}$ in 1 and 2.

Now consider a highly idealised system of n particles which are prepared with momenta $\{p_i, i=1,\cdots,n\}$ at time $t_0\to-\infty$. They interact with (i.e. scatter in) a perturbed Hamiltonian $H = H_0 + H_{\text{int}}$, reaching a set of final momenta $\{q_i, i=1,\cdots,n\}$ time $t\to\infty$. One can represent the initial and final states with

$$|p,\text{in}\rangle = \prod_{i=1}^{n} a^\dagger_{p_i}|0\rangle \quad \langle q,\text{out}| = \langle 0|\prod_{i=1}^{n} a^\dagger_{q_i} \tag{5.21}$$

where $|0\rangle$ is the ground state. In this scenario, the S-matrix elements are

$$S_{qp} = \langle q,\text{out}|e^{-iH(t-t_0)}|p,\text{in}\rangle \tag{5.22}$$

To evaluate the elements of S_{qp}, we need to diagonalise the full Hamiltonian H. As this is typically impossible, we must use a perturbative approach to deal with H_{int}, where we assume that the interaction Hamiltonian $H_{\text{int}}(t)$ is zero at $t\to-\infty$ and $t\to\infty$.

Derivation 5.3 (LSZ reduction formula) Let us now consider scattering in the Heisenberg picture. We apply the Heisenberg equation to a_p:

$$\dot{a}_p = i[:H:,a_p] \to \dot{a}_p = iE_p a_p \tag{5.23}$$

Plugging this result into the field operator yields

$$\phi(x)_H = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a^\dagger_p e^{ip\cdot x}\right) \tag{5.24}$$

From this, one can represent the creation and annihilation operators in terms of the wavefunctions.

$$ia^\dagger_{p,H} = \int \frac{d^3\mathbf{x}}{\sqrt{2E_p}}\left(\mathrm{e}^{-ip\cdot x}\left(\partial_0\varphi_H(x)\right) - \varphi_H(x)\left(\partial_0\mathrm{e}^{-ip\cdot x}\right)\right) \tag{5.25}$$

In the Heisenberg picture, the S-matrix element is

Definition 5.5 (Heisenberg picture S-matrix element)

$$S_{qp,H} = \langle 0|a_{q_1,H}\cdots a_{q_m,H}a^\dagger_{p_1,H}\cdot a^\dagger_{p_n,H}|0\rangle \tag{5.26}$$

We can repeatedly insert Equation 5.25, which yields the so-called *LSZ formula* or the *LSZ reduction formula* for S-matrix elements, named after Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann. By defining a so-called *vacuum expectation value*

Definition 5.6 (Vacuum expectation value) The expectation value of time-ordered field operators evaluated at $m+n$ spacetime points is labelled G_{m+n} as the time ordering symbol is a $(m+n)$-point *Green's function*:

$$G_{m+n} = \langle 0|T[\phi_H(x_1)\cdots\phi_H(x_{m+n})]|0\rangle \tag{5.27}$$

We can write the LSZ formula as

Theorem 5.5 (LSZ reduction formula for S-matrix elements) For a series of spacetime coordinates x_i, the scattering matrix elements can be represented by

$$S_{qp,H} = \int \frac{d^4x_1}{\sqrt{2E_{p_1}}}\cdots\int \frac{d^4x_{m+n}}{\sqrt{2E_{q_m}}}e^{-i\sum_{j=1}^n p_j\cdot x_j + i\sum_{j=1}^m q_j\cdot x_{n+j}}\prod_{j=1}^{m+n}(\partial^2_{x_j}+m^2)G_{m+n} \tag{5.28}$$

where the Klein-Gordon operators $(\partial^2_{x_j}+m^2)$ enforce that the external particles are on-shell.

Remark 5.2 Here, we see that all our field quantisation in Fock space and derivation of (anti)commutation relations have paid off. Exactly because of canonical quantisation, we were able to construct the LSZ formula, which is essential for constructing the S-matrix. We thus see the significance of canonical quantisation, even though, as we will discover, our end result actually does *not* make use of field operators whatsoever.

Now we would like to migrate to the interaction picture (denoted by the subscript $_I$) due to its simplicity[4]. Our vacuum expectation value/Green's function then becomes the *interacting time-ordered propagator*:

Theorem 5.6 (Interacting time-ordered propagator)

$$G_{m+n,I} = \frac{\langle 0|T[\phi_I(x_1)\cdots\phi_I(x_{m+n})]S|0\rangle}{\langle 0|S|0\rangle} \tag{5.29}$$

It is also called the *interacting*[a] *Green's function*, the *$(m+n)$-point Green's function* or the *correlation function*[b].

[a] The 'interacting' does not refer to the dynamic picture. Instead, it is used to describe the presence of an interaction term in the Hamiltonian.

[b] So-called as it is used to study correlations between field operators at different spacetime points in the interacting vacuum.

As is easily seen, the LSZ formula is very cumbersome. As such, while itis theoretically significant, we usually do not actually use it as-is when deriving S_{fi}. Instead, we utilise the *Feynman rules*, which are the practical implementation of what the LSZ formula theoretically justifies. The next two sections will be dedicated to deriving the Feynman rules and elucidating their physical and practical meaning.

5.3 Feynman diagrams

Now we will begin our long journey to the Feynman rules which we use in lieu of the LSZ reduction formula. One can construct $\mathcal{M}_{fi}$ (and by that, S_{fi}) of a given field theory through its Feynman rules. These rules make use of the so-called *Feynman diagrams*, which is effectively the graphical representation of a specific process:

[4] Also because we have not derived an expression for the Heisenberg picture fields ϕ_H anyway.

Definition 5.7 (Feynman diagram elements) A Feynman diagram includes propagators, vertices, and loops[a]:

- **Propagators:** They correspond to moving free particles and correspond to the free part of the Lagrangian.
  - The propagators representing the initial and final particle states are represented by *external propagators*, *external legs* or *external lines*, whose ends are *external points*:
    - An *incoming external leg* is an initial state (i.e. incoming) particle, typically on the left side.
    - An *outgoing external leg* is a final state (i.e. outgoing) particle, typically on the right side.

    For a total number of n such external points, one has an n-point Feynman diagram.
  - The propagators that are intermediate lines between vertices and represent virtual particles are *internal propagators* or *internal lines*.

  Remark 5.3 External and internal propagators have the have identical mathematical formulation.However, external propagators do not contribute to the scattering matrix or the transition amplitude as are a part of the *initial & final states*, while internal propagators contribute to both as they are a part of the transition/scattering process.

  Vertices: Denoted by visible round dots. The number of vertices, known as the *order*, corresponds to the order of the coupling constant of the field theory. Vertices correspond to the interacting part of the Lagrangian. Importantly, as we will soon demonstrate:
  - The number of particles/fields/lines that connect to the propagator of a field theory is strictly fixed.
  - As mentioned previously, vertices cannot be derived exactly. Rather, we must employ perturbation theory in both the canonical quantisation and path integral formulations.

  Internal loops: Internal loops can be loosely thought of as in internal line coming back to itself. Mathematically, we write it as a loop integral of the (internal) line propagator. We speak of a *tree-level diagram* if the diagram in question has no loops. Loops do not correspond to any part of the Lagrangian. Instead, they arise from combining interaction and free parts in higher-order (loop) perturbation theory[b].

[a] As ϕ^4 theory is a purely self-interacting toy model and does not involve any specific particles, *which is not the case in real life*. We will use plain lines for all particles and propagators.

[b] As we will soon see, internal loops integrate over internal momenta. This often creates divergences, which is a hallmark of loop-order QFT that much be treated by *renormalisation*.

A Feynman diagram of the 0^{th} order has no vertices:

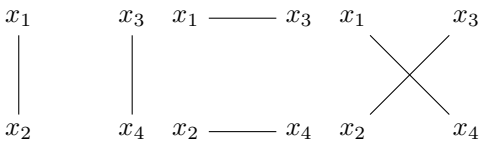


Figure 5.1: 0^{th}-order Feynman diagrams

A Feynman diagram of the 1^{st} order has a single vertex:

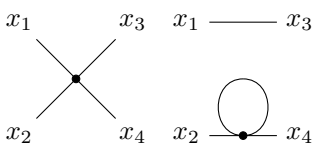


Figure 5.2: 1^{st}-order Feynman diagrams

Note the existence of the loop in the diagram on the right. Functionally, this diagram is identical to the middle entry in the 0^{th}-order diagrams, with the only difference being the removal of the loop. Hence, it is not the simplest expression this particular interaction can take. The middle 0^{th}-order diagram is then known as the *leading order* Feynman diagram with respect to the interaction it represents as it is the most reduced form of the interaction.
A Feynman diagram of the 2^{nd} order has two vertices[5]:

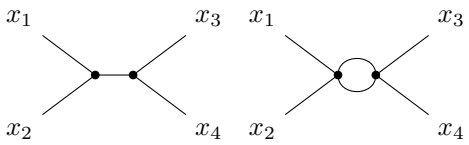


Figure 5.3: 2^{nd}-order Feynman diagrams

All of the diagrams above are 4-point Feynman diagrams.

> **Quote 5.3** You can still insert some hand drawings. Hand drawings are better than no drawings, and if you don't do it now, it eventually never happens.
>
> *Felix Halbwedl, encouraging the author to finish the Feynman diagram illustrations, 22 November 2024*

One importance of Feynman diagrams is that it gives more intuitive physical meanings to G_{m+n} and S_{fi}:

- G_{m+n} sums over all possible Feynman diagrams (both connected and disconnected) with $m+n$ external points. That is to say, it encodes the *sum* of probability *amplitudes* over all possible processes allowed by the theory of interest for fields to be created or annihilated at those points, including all quantum fluctuations consistent with the field dynamics.
- S_{fi} is the probability *itself*, and encodes a specific the scattering process (i.e. only a specific Feynman diagram). So in a way, G_{m+n} is the full 'storyboards' of diagrams, and S_{fi} is the final edited 'scene' of a specific storyboard that makes it to the screen.

5.4 ϕ^4 theory Feynman rules

The first derivation we shall make is that of the propagator which we, in our infinite wisdom, have deferred up to this point. At this point, we can already evaluate the multi-point Green's function by using Wick's second theorem. This may look like a tedious process. Luckily for us, in doing so, many terms cancel out, and Equation 5.29 reduces to a series of two-point Green's functions or the *Feynman propagators* of some spacetime coordinates x and y, labelled $D_F(x-y)$:

$$D_F(x-y) = \langle 0|T[\phi_I(x)\phi_I(y)]|0\rangle \tag{5.30}$$

Derivation 5.4 (Feynman propagator) Now we want to solve for the exact form of this propagator. The central idea which we shall utilise is the fact that $D_F(x-y)$ is a solution of the Klein-Gordin equation. Before we mindlessly intert this into Equation 4.2, however, let us perform a mathematical trick by performing a Fourier transform:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \tilde{D}_F(p) e^{-ip\cdot(x-y)} \tag{5.31}$$

Substituting this into the equation gives

$$\int \frac{d^4p}{(2\pi)^4} \tilde{D}_F(p) e^{-ip\cdot(x-y)} (p^2 - m^2) = \delta^4(x-y) \tag{5.32}$$

[5] At this point, it should be clear that the Feynman diagrams we have drawn are nothing but 'toy' Feynman diagrams. Recall that a 3-point vertex is impossible in ϕ^4 theory.

This implies that $\tilde{D}_F(p)$ must satisfy, for the momentum space Klein-Gordon operator $(p^2 - m^2)$

$$(p^2 - m^2)\tilde{D}_F(p) = i \tag{5.33}$$

Thus, the propagator in momentum space is

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2} \tag{5.34}$$

There is one problem with this propagator. Two zero denominator singularities emerge at so-called *poles*, positions where the on-shell condition is enforced:

$$p^0 = \pm E_p = \pm\sqrt{p^2 + m^2} \tag{5.35}$$

As such a *smol* $i\epsilon$ is included to avoid singularities at the poles. This ensures that our integral is well-defined and preserves causality:

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{5.36}$$

Now, we return to position space by computing the inverse Fourier transform:

Definition 5.8 (Feynman propagator) The Feynman propagator is the probability amplitude for a scalar particle to propagate from the spacetime point x to y, taking into account quantum fluctuations:

$$D_F(x - y) = \lim_{\epsilon \to 0+} i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{5.37}$$

Equation 5.33 can likewise be moved to position space. After generalising, we have

$$(\Box + m^2)D_F(x - y) = -i\delta^4(x - y) \tag{5.38}$$

where, importantly, we see that the Feynman propagator is a Green's function related to the Klein-Gordon operator. It represents the probability amplitude for a particle to propagate from spacetime point y to x. The equation above simply shows that the Klein-Gordon operator governs this propagation.

Note 5.1 For convenience, we often suppress (i.e. conveniently forget writing) $\lim_{\epsilon\to 0+}$ or both $\lim_{\epsilon\to 0+}$ and $i\epsilon$ for brievity, in which case their existence is assumed.

This is the easy part. Next, we will see how a perturbative expansion canallow us to read off the propagator. First, we show that Equation 5.29 is a Taylor expansion of λ in the ϕ^4 theory. A Feynman diagram with m incoming external legs and n outgoing external legs is represented a $(m+n)$-point Green's function, which itself is made up of Feynman propagators. As the Feynman propagator has a S_{fi} term, it can be written as a perturbative expansion like S_{fi} in Equation **??**:

$$G_{m+n} = \sum_k G^{(k)}_{m+n} \tag{5.39}$$

Each $G^{(k)}_{m+n}$ corresponds to a class of (various possible) Feynman diagrams with k vertices[6], where the coupling constant λ is of the order k. For example:

- The 0^{th} term represents the so-called *free propagator* where no interactions happen. It has only external leg propagators.
- For $k > 0$, the k^{th} term represents the Feynman propagator where k interaction happens (represented by a Feynman diagram of order k). It has both external and internal propagators.

Derivation 5.5 ($2 \to 2$ processes)

[6]That is to say, each interaction vertex arises from an insertion of the interaction term from the Lagrangian.

> **Quote 5.4** It's like an electric board with four sockets, and each field is a plug.
>
> *Alessio Serafini, on $2 \to 2$ Feynman diagrams, 27 February 2025*

Putting it all together, we now look at the example of a $(2+2)$-point Green's function with incoming external legs x_1 and x_2, outgoing external legs x_3 and x_4 and no interaction in the middle of the Feynman diagram. From Wick's third theorem in Equation 5.17, we see that the Green's function can be represented entirely in terms of Feynman propagators as

$$G^{(0)}_{2+2} = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3) \tag{5.40}$$

If 1 interaction happens at point y, we have the extra term

$$G^{(1)}_{2+2} = -i\frac{\lambda}{4!}\frac{4!}{1!}\int d^4y D_F(x_1 - y)D_F(x_2 - y)D_F(x_3 - y)D_F(x_4 - y) \tag{5.41}$$

where λ and $1/4!$ are the coupling constant and the normalisation term seen in Equation 5.1. The factor 4! that follows accounts for the fact that each vertex has four ϕ-fields, and there are 4! ways to contract these (four) fields with external legs or propagators[a]. The factor 1! accounts for the first-order perturbation.
If 2 interactions happen at points y_1 and y_2 on the left and right sides respectively, we have the extra term, where the factors λ, $1/4!$ and 4! are squared:

$$G^{(2)}_{2+2} = -\left(\frac{\lambda}{4!}\right)^2 \frac{4!^2}{2!}\int d^4y_1 \int d^4y_2 D_F(x_1 - y1)D_F(x_2 - y_1)D_F(x_3 - y_2)D_F(x_4 - y_2) \tag{5.42}$$

where the factor 1! accounts for the second-order perturbation, and so on in higher orders.

[a] This is the multiplicity C which we will see later.

We also note that the same Feynman diagram can be yielded from different ways of contracting operators, which may arise multiple times in the series expansion. This is reflected in the Feynman diagram by the multiplicity C. We thus define:

Definition 5.9 (Multiplicity) The *multiplicity* or *multiplicity factor* C with respect to a specific Feynman diagram is the number of possible Wick contractions that result in the specific diagram.

This is a natural result from Wick's second theorem, where we have summed over all possible contractions of the field operators, many of which are actually identical.

> **Quote 5.5** It is not trivial.
>
> *Paulina Schlachter, on calculating C, 25 February 2025*

Finally, we are in a position to discuss the *Feynman rules*. The Feynman rules do not produce S_{fi} itself, but rather a quantity called the *scattering amplitude* or *transition amplitude* $\mathcal{M}_{fi}$, a matrix related to the S-matrix. The advantage of the Feynman rules is that they are set up specifically to make constructing $\mathcal{M}_{fi}$ fast and intuitive. But first, we need to provide a definition for $\mathcal{M}_{fi}$:

Derivation 5.6 (Scattering amplitude) Let us begin with Equation 5.28, where we have m incoming external legs and n outgoing external legs. At this point, two simplifications can be made. The first is *amputating* our poor propagators (also known as *truncating* our poor external legs):

Definition 5.10 (Amputated propagator) It is convenient to write all 'internal' or 'interior' parts of the propagator as a so-called *amputated propagator* $\bar{G}(y_1, \cdots, y_l)$:

$$G_{m+n} = D_F(x_1 - y_1)\cdots D_F(x_{n+m} - y_l) \times \bar{G}(y_1, \cdots y_l) \tag{5.43}$$

where we have $l \leq m + n$ since more than one leg may couple to the same vertex.

Remark 5.4 As the amputated propagator is effectively the whole propagator with all external leg

propagators removed[a], it is known to be 'amputated'.

> **Quote 5.6** Arm raised, but not too high.
>
> *Felix Halbwedl, 30 March 2025*

Now the question arises: Why do we do this? Let us inspect the RHS of Equation 5.28. On the far right, we have the two factors $\prod_{j=1}^{m+n}(\partial_{x_j}^2 + m^2)G_{m+n}$. Importantly, we see that $\prod_{j=1}^{m+n}(\partial_{x_j}^2 + m^2)$ are effectively the inverses of the external leg Feynman propagators, which cancel out with the external leg Feynman propagators themselves in G_{m+n}. As such, the two factors together is effectively the amputated propagator we saw in Equation 5.43.
The second simplification is the Fourier transformation, through which we eliminate the rather pesky 4-integrals. We can then define the scattering amplitude $\mathcal{M}_{fi}$ as

Definition 5.11 (Transition amplitude)

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(p_f - p_i)\mathcal{M}_{fi} \tag{5.44}$$

where m and n are indices for initial and final particles, $p_i = \sum_{j=1}^{n} p_j$ and $p_f = \sum_{j=1}^{n} q_j$, and $\delta^4(p_i - p_f)$ enforces momentum conservation.

Remark 5.5 The physical significance of $\mathcal{M}_{fi}$ can be quickly found: The Kronecker delta δ_{fi} is essentially the identity that accounts for the case where there is no interaction. Hence, $\mathcal{M}_{fi}$ is a rescaled version of the interaction-dependent part of S_{fi}. Crudely speaking, we can reduce the relationship to

$$\mathcal{M}_{fi} = \langle f|S_{fi} - 1|i\rangle \tag{5.45}$$

[a] Note however that external *points* which represent initial and final states are preserved. This is trivial in ϕ^4 theory as all external points are 1, but become important in QED where external points represent states, which are 4-spinors or 4-versors.

The Feynman rules of ϕ^4 theory are thus as follows:

Theorem 5.7 (ϕ^4 theory Feynman rules) For a given Feynman diagram in ϕ^4 theory, the scattering amplitude matrix elements $\mathcal{M}_{fi}$ is constructed as follows:

ϕ^4 theory Feynman rules (partial)

For each	Add to expression
Incoming/outgoing scalar particle	1[a]
Internal line	$\frac{i}{k_j^2 - m^2}$
Internal loop	$\int \frac{d^4k_j}{(2\pi)^4} \frac{i}{k_j^2 - m^2}$
Vertex	$-i\frac{\lambda}{4!}(2\pi)^4\delta^4 \sum_j p_j$[b]

where k_j is the propagator momenta, p_j is the incoming momenta and q_j is the outgoing momenta. We also perform the following steps:

- Introduce the multiplicity C for the number of contractions leading to the same diagram.
- Include the $1/k!$ factor from the Taylor expansion, where k is the perturbative order.
- Remove a factor of $(2\pi)^4\delta^3(p - q)$, where p and q are the total initial and detected momenta, as it is usually already accounted for in the definition of the cross-section or decay rate in terms of the matrix elements.

[a] Unlike spinor and vector fields in QED, scalar fields in our good ol' harmless ϕ^4 theory manifests in a mere number (i.e. scalar) which, in our case, just so happens to be unity.
[b] The sum is over all lines exiting the vertex and forces four-momentum conservation at the vertex.

Remark 5.6 For $2 \to 2$ interactions ($m = n = 2$), $\mathcal{M}_{fi}$ reduces very nicely to $-i\lambda$. Here the nature of ϕ^4 theory as a useful toy model becomes clear.

Note 5.2 (Sneak peek of renormalisation) It should be obvious that, to make sure that our results are physical, the scattering amplitude must observe

$$0 \leq |\mathcal{M}|^2 \leq 1 \tag{5.46}$$

At the tree level, this is always observed. However, the introduction of loops will often bring about divergences (often into infinity) that violate this condition. These divergences suggest that our theory is not entirely physical beyond the tree level. To kill the divergences, we must *renormalise* our field theory of interest. How we do this will be at the heart of Part IV.

Happily, we have now constructed $\mathcal{M}_{fi}$ from the Feynman rules. By inserting it into Equation 5.44, we can solve for S_{fi}, which is much easier than using the LSZ formula. Finally, we note that:

- External propagators/legs/lines have the same expression as internal lines, which is $i(k_j^2 - m^2)^{-1}$. However, they are not included in the Feynman rules as the external legs are already truncated in $\mathcal{M}_{fi}$.
- In the same vein, external legs are truncated in S_{fi}. Both are a result of the cancelling out of external legs in Equation 5.28. However, this does not mean that external legs are physically meaningless, as they still appear in Feynman diagrams.

5.5 Beginnings of HEP

> **Quote 5.7** At least my garden is not big enough to build it there.
>
> *Felix Halbwedl, on the Tevratron, 5 January 2025*

We continue our discussion in momentum space and investigate a few experimental HEP-adjacent concepts: the *transition rate*, the *decay rate* and the *cross-section*.

Derivation 5.7 (Probability density) We first adopt the shorthand notation

$$A_p = \prod_{j=1}^{n} a_{p_j} \quad B_q = \prod_{j=1}^{n} a_{q_j,H} \tag{5.47}$$

From the Born rule, the probability density of scattering involving m particles is

$$dP(q_1, \cdots, q_m) = \mathrm{Tr}[\rho d \prod_{q_1} \cdots d\pi_{q_m}] \tag{5.48}$$

In momentum space, we have

$$d\prod_q = \frac{d^3q}{(2\pi)^3} a^\dagger|0\rangle\langle 0|a_q \tag{5.49}$$

where $1/(2\pi)^3$ is a normalisation factor resulting from normalising the previously unnormalised plane wave solution ϕ

$$\langle 0|a_p a_p^\dagger|0\rangle = (2\pi)^3\delta(0) \tag{5.50}$$

In any case, by inserting Equation 5.49, Equation 5.48 becomes

$$\begin{aligned} dP(q_1, \cdots, q_m) &= V^{-n} \int \frac{d^{3n}k}{(2\pi)^{3n}} \frac{d^{3m}q}{(2\pi)^{3m}} \langle 0|A_k B_q^\dagger|0\rangle\langle 0|B_q A_p^\dagger|0\rangle\langle 0|A_p A_k^\dagger|0\rangle \\ &= V^{-n} \frac{d^{3m}q}{(2\pi)^{3m}} \langle 0|A_k B_q^\dagger|0\rangle\langle 0|B_q A_p^\dagger|0\rangle \\ &= \frac{1}{V^n} \frac{d^{3m}q}{(2\pi)^{3m}} |S_{qp}|^2 \end{aligned} \tag{5.51}$$

which is the simplified form of the probability density.

A problem in Equation 5.44 that we have left unaddressed is the momentum conservation-enforcing term $\delta^4(p_i - p_f)$. In it, the momentum states exist throughout the entirety of the space-time. In a real experiment, however, incoming and outgoing states are localised. To deal with this, we assume the interaction happens over a time of T in a system localised in some volume V[7]. First, we can perform the rewrite

$$(2\pi)^4\delta^4(p_i - p_f) = \int_{VT} d^4x e^{i(P_f - P_i)x} \tag{5.52}$$

Taking the square modulus gives

$$|(2\pi)^4\delta^4(p_i - p_f)|^2 \approx (2\pi)^4\delta^4(p_i - p_f)| \int_{VT} d^4x e^{i(P_f - P_i)x}| = VT(2\pi)^4\delta^4(p_i - p_f) \tag{5.53}$$

The transition rate, which is the probability per unit time, is denoted by W.

Definition 5.12 (Transition rate differential) The differential of the transition rate is given by

$$dW = |\mathcal{M}_{fi}^2|V \prod_{j=1}^{n} \frac{1}{2E_{p_j}V} \,\mathrm{LIPS}(m) \tag{5.54}$$

In this expression, we have crammed all Lorentz-invariant terms together as the so-called *Lorentz-invariant phase space*, which is defined with respect to m particles in the final state:

Definition 5.13 (Lorentz-invariant phase space)

$$\mathrm{LIPS}(m) \equiv (2\pi)^4\delta^4(p_i - p_f) \prod_{k=1}^{m} \frac{d^3q_k}{(2\pi)^3 2E_k} \tag{5.55}$$

Now we consider decays. Unlike other interactions, there is only one initial particle in decays. As such, the decay rate Γ_{if}, which is the transition rate for decays, is

Definition 5.14 (Decay rate)

$$\Gamma_{if} = \frac{1}{2m} \int |\mathcal{M}_{fi}|^2 \,\mathrm{LIPS}(m) \tag{5.56}$$

In the case where the end product consists of 2 particles, the decay rate reduces to

$$\Gamma_{if} = \frac{1}{2m} \int |\mathcal{M}_{fi}|^2 q_f d\Omega \tag{5.57}$$

where Ω is the so-called *solid angle*.
A quantity ultimately related to the transition rate is the cross-section. We begin with the *particle flux*:

Definition 5.15 (Particle flux) The particle flux for a beam with velocity v_1 and a density of particles of $1/V$[a] and a target with velovity v_2 is

$$F = \frac{|v_1 - v_2|}{V} \tag{5.58}$$

[a] i.e. 1 particle in a volume of V.

This is simply the number of particles per unit area which run past each other per unit time.
As the *cross section* is the transition rate for a single particle *per unit beam flux*, we can find the differential cross section by dividing the transition rate by the flux:

$$d\sigma = \frac{dW}{F} = \frac{1}{|v_1 - v_2|} \frac{1}{4E_1E_2} |\mathcal{M}_{fi}|^2 \,\mathrm{LIPS}(m) \tag{5.59}$$

This expression is Lorentz-invariant. We then integrate and find

[7] This is not a problem, as both T and V ultimately disappear.

Definition 5.16 (Cross section)

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}_{fi}|^2 \, \mathrm{LIPS}(m) \tag{5.60}$$

Remark 5.7 If the final particles are identical, LIPS(m) is divided by $m!$.

Chapter 6

Interacting fields II: QED

A generalised and actually physical version of the Klein-Gordon equation is the Dirac equation. The Dirac field, which represents fermions (i.e. electron and positron), then accounts for one-half of QED. The other half is the photon field, which is governed by Maxwell's equations and is the half of QED that makes it a gauge theory. Using canonical quantisation, we will quantise the Dirac and photon (Maxwell) fields and develop a Lagrangian for quantum electrodynamics, which accounts for fields generated by both electrons/positrons and photons. Both fields are a departure from our innocent ϕ^4 fields: the Dirac field is a fermion field instead of a boson field, and the photon field, while a boson field, is a gauge field that requires gauge fixing. field

6.1 Dirac equation

The ill-fated Klein-Gordon equation, which we have found to be *kaputt*, is a Lorentz-invariant 2^{nd}-order DE. We now propose a better candidate in the form of a Lorentz-invariant 1^{st}-order DE, whose most general form would be known as the *Dirac equation*

Theorem 6.1 (Dirac equation)

$$(i\gamma^\mu\partial_\mu - m)\phi = (i\not{\partial} - m)\psi = 0 \tag{6.1}$$

γ^μ is a yet undetermined 4-vector, and we have defined the so-called *Feynman slash notation* for some four-vector a

$$\not{a} = \gamma^\mu a_\mu \tag{6.2}$$

Quote 6.1 The equation was more intelligent than its author.

Paul Dirac, on his equation[a] *(disputed)*

[a] The rationale behind the quote, as per Victor Weisskopf, was that 'A great deal more was hidden in the Dirac equation than the author had expected when he wrote it down in 1928'.

Remark 6.1 Again, this formulation stems from intuition. The modern standard derivation involves inserting the Dirac Lagrangian in Equation 6.18 (which we ultimately construct by intuition) into the Euler-Lagrange equations.

Derivation 6.1 (γ^μ matrices) The Dirac equation does not directly conflict with the Klein-Gordon equation:

- As the Klein-Gordon equation must still be satisfied to fulfil the SR energy-momentum relation, every solution to the Dirac equation is also a solution to the Klein-Gordon equation.
- The reverse is not true, as the spinorial nature of the solution means that the probability $\rho = \psi^\dagger\psi$ will always be non-negative, and *all negative probability states are excluded.*

The first observation then inspires us to investigate the nature of the γ^μ matrices by considering what happens in the Klein-Gordon limit. We impose the differential operator $i\gamma^\mu\partial_\mu + m$ to the Dirac

equation[a] and equating it with the Klein-Gordon equation

$$\underbrace{-(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi}_{\text{application of } i\gamma^\mu\partial_\mu + m} = \underbrace{-(\partial^\mu\partial_\mu - m^2)\psi}_{\text{Klein-Gordon equation}} = 0 \tag{6.3}$$

Immediately, we see that the term $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu$ must be equal to $\partial^\mu\partial_\mu$. Effectively, through $\gamma^\mu\gamma^\nu$, one index was shifted up. Hence, a sensible guess of $\gamma^\mu\gamma^\nu$ would be the metric:

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} \tag{6.4}$$

However, this is wrong for the reason that the off-diagonal components of the 4-metric are zero, thus implying

$$\gamma^0\gamma^1 = 0 \quad \text{and} \quad (\gamma^0)^2 = -(\gamma^1)^2 = 1 \tag{6.5}$$

at the same time. In fact, such conditions can never be satisfied as long as the components of γ^μ are mere numbers. However, if one switches the indices on the LHS of Equation 6.3 and adds this otherwise identical expression to Equation 6.3, they will find

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \gamma^\nu\gamma^\mu\partial_\nu\partial_\mu = 2\partial^\mu\partial_\mu \tag{6.6}$$

This becomes an anticommutation relation in the form of

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{6.7}$$

One can see from Equation that γ^μ are elements of a *Clifford algebra*[b], from the definition of which it is clear that elements of γ^μ must be a set of 4 matrices.

Definition 6.1 (γ^μ matrices in the Dirac basis) In the *Dirac basis*, the γ^μ matrices are

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \tag{6.8}$$

where $j \in \{1, 2, 3\}$ (sometimes also labelled as i), and σ_j are the Pauli matrices we know and love.

[a] Note the + sign of the second term of this operator!

[b] This poses no conflict with Grassmann algebras that define the fields because no single algebra can be said to formulate QED. Rather, both do so simultaneously.

Note 6.1 (Alterjjnative formulation of the γ^μ matrices) It is often convenient to redefine γ^0 and γ^i. We define β and α^i:

$$\beta := \gamma^0 \quad \gamma^i = \beta\alpha^i \tag{6.9}$$

While β is nothing but a relabelling of γ^0 meant to confuse people, α^i actually has its merits, as we will see later.

A second basis is the so-called *Weyl basis* or *chiral basis*, where the γ^0 is slightly changed, while the others stay the same

Definition 6.2 (Weyl basis)

$$\gamma^0_{\text{ch}} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \quad \gamma^j_{\text{ch}} = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \tag{6.10}$$

Remark 6.2 Like the majority of the literature, we follow the Dirac basis in this book. Interestingly, the Pauli matrices satisfy the following relation:

Theorem 6.2 (Pauli matrices properties)

$$\sigma_j\sigma_k = i\epsilon_{jkl}\sigma_l + \delta_{jk}\mathbb{I}_2 \tag{6.11}$$

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l \tag{6.12}$$

where we have once again encountered our good friends, the Kronecker delta δ_{jk} and the Levi-Civita symbol ϵ_{jkl}.

Remark 6.3 As the γ^μ matrices are 4×4, the wave solution of the Dirac equation ϕ has 4 components. However, it is important to note that this ψ is *not* a vector due to it not transforming under general coordinate transformations. Rather, it is a *spinor* which one can better understand by reading the companion book *Spinors & Symmetries*. While it is possible to simply regard ψ as a 4-vector in some regards, doing so would be quite morally questionable.
Before we recover the Dirac Lagrangian, there remains one loose end to take care of. As we will soon see, this innocent Dirac field is not alone. Rather, it is coupled to a (scalar) photon field A^μ that we will see later. Here, a problem has arisen in the partial derivative field $\partial_\mu A^\mu$, which does not transform covariantly. To compensate for this failure, we augment the partial derivative by introducing a correction term. This new derivative is called the *covariant derivative*, which, in a general gauge theory with a vector field A^μ and coupling constant g, is given by

Definition 6.3 (Abelian covariant derivative)

$$D^\mu = \partial^\mu + igA^\mu \tag{6.13}$$

From this, we can construct the field strength tensor for some vector field A^μ, which has the form

Definition 6.4 (Field strength tensor)

$$F^{\mu\nu} = i[D^\mu, D^\nu] = \partial^\mu A^\nu - \partial^\nu A^\mu - i[A_\mu, A_\nu] \tag{6.14}$$

Remark 6.4 But this feels a bit familiar, doesn't it?
As it turns out, we can make an analogy between gauge theories and good ol' GR:

- The general coordinate transformations we see in GR are analogous to the gauge transformations in gauge theories.
- The Christoffel symbols $\Gamma^\sigma_{\mu\nu}$ we use in GR covariant derivatives are analogous to the gauge field A^μ, and both represent a 'correction' to their respective transformation.
- The Ricci tensor $R_{\mu\nu}$ is analogous to the the field strength $F^{\mu\nu}$.
- ...and so on.

The implication of this for QED is then intuitive: In the presence of this scalar field, Our innocent partial derivative ∂ must be rewritten as D. Its second term is modified from that of the generalised covariant derivative by the charge e, which is the coupling constant of QED:

$$D_\mu = \partial_\mu + ieA_\mu \tag{6.15}$$

We can now write down the Lagrangian and Hamiltonian densities for the (free field) Dirac equation for abelian gauge theories:

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi \tag{6.16}$$

which, by noting the Dirac equation itself, is always zero. The bar on top denotes the *Dirac adjoint*:

Definition 6.5 (Dirac adjoint) The Dirac adjoint $\bar{\psi}$ of some ψ is

$$\bar{\psi} = \psi^\dagger \gamma^0 \tag{6.17}$$

$\bar{\psi}$ is not a new field or, indeed, anything physical. It is a mathematically altered ψ that ensures the Lorentz invariance of the Dirac Lagrangian.

Let us write out the covariant derivative explicitly. The Lagrangian is then

Definition 6.6 (Dirac Lagrangian)

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi = \underbrace{\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi}_{\text{Dirac field}} \underbrace{- e\bar{\psi}\gamma^\mu\psi A_\mu}_{\text{interaction}} \tag{6.18}$$

The interaction term denotes interaction with the photon field A^μ. It replaces the interaction $-J^\mu A_\mu$ in the classical Maxwell Lagrangian we will see in Equation 6.91. If further fields (e.g. proton fields) are added, this changes to

$$\mathcal{L} = \sum_f \bar{\psi}(i\gamma^\mu(\partial_\mu + ie_f A_\mu) - m_f)\psi \tag{6.19}$$

where f is an index representing the range of fermions we are concerned with.
Sticking to the simpler case, it then follows that the canonical momentum reads

Definition 6.7 (Dirac canonical momentum)

$$\pi = i\bar{\psi}\gamma^0 = i\psi^\dagger\gamma^0\gamma^0 = i\psi^\dagger \tag{6.20}$$

The Hamiltonian is then

Definition 6.8 (Dirac Hamiltonian)

$$\mathcal{H} = i\psi^\dagger\partial_0\psi - \mathcal{L} = i\psi^\dagger\partial_0\psi = \bar{\psi}(-i\gamma^j\partial_j + m)\psi \tag{6.21}$$

Remark 6.5 The canonical momentum is $\pi = i\psi^\dagger$ by dint of $\gamma^0\gamma^0 = \mathbb{I}_4$.

Derivation 6.2 (Dirac equation general solution) Despite ψ being a spinor[a], a general plane wave solution is again of the form $\psi = ue^{-ip\cdot x}$. From this and the Klein-Gordon on-shell condition, the eigenvalue equation is

$$(\not{p} - m)u = 0 \tag{6.22}$$

The slashed momentum matrix is of the form

$$\not{p} = \begin{pmatrix} p^0\mathbb{I}_2 & p\cdot\sigma \\ p\cdot\sigma & -p^0\mathbb{I}_2 \end{pmatrix} \tag{6.23}$$

where p is the 3-vector and σ is a 3D vector whose elements are the Pauli matrices.
Remark 6.6 Note that we are still living in momentum space.
The eigenspinors, called *Dirac spinors*, are

$$u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{p\cdot\sigma}{E_p+m}\chi_s \end{pmatrix} \quad v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \frac{p\cdot\sigma}{E_p+m}\chi_s \\ \chi_s \end{pmatrix} \tag{6.24}$$

for $s = 1, 2$. χ_s are 2-component spinors (or so-called *Weyl spinors*), and are defined by

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.25}$$

> **Quote 6.2** This is a four-dimensional beast.
>
> *Alessio Serafini, on the Dirac spinor, 13 March 2025*

The physical meaning of the Weyl spinors can be investigated:

- χ_s is the so-called *large component*. It lives in the rest frame and describes spin eigenstates.
- $\frac{p\cdot\sigma}{E_p+m}\chi_s$ is the so-called *small component*. It describes relativistic corrections to the spin eigenstates due to the particle's motion and encodes spin-orbit coupling via $p\cdot\sigma$.

By noting that

$$(p\cdot\sigma)^2 = |p|^2 = E_p^2 - m^2 \tag{6.26}$$

it can be seen that the eigenvectors are normalised as

$$u_r^\dagger(p)u_s(p) = v_r^\dagger(p)v_s(p) = \delta_{rs}2E_p \tag{6.27}$$

As $\not{p}$ is not Hermitian, the eigenvectors are not orthogonal. However, we do have a metric relation that is quite similar to orthogonality

$$u_r^\dagger(p)v_s(-p) = 0 \tag{6.28}$$

Equation 6.27 and Equation 6.28 can be represented in Lorentz-invariant form:

$$\bar{u}_r(p)u_s(p) = -\bar{v}_r(p)\bar{v}_s(p) = \sigma_{rs}2m \quad \bar{u}_r(p)v_s(p) = 0 \tag{6.29}$$

Finally, we are in a position to write down the full general solution of the Dirac equation, which represents fermions like electrons and positions:

Theorem 6.3 (Dirac equation general solution) The general solution of the Dirac equation is a spinor field known as the *Dirac field*:

$$\psi = \int d^3p \sum_{s=1}^{2} (b_s(p)u_s(p)e^{-ip\cdot x} + d_s(p)v_s(p)e^{ip\cdot x}) \tag{6.30}$$

where $b_s(p)$ and $d_s(p)$ are 4 constants.

The Dirac field is a so-called *Grassmann-valued field*, which is a field whose components take values in a Grassmann algebra. Recalling Grassmann mathematics from *Spinors & Symmetries*, one can realise that the components of Grassmann fields anticommute rather than commute.
From their Dirac equation, spinors $u_s(p)$ and $v_s(p)$ further satisfy

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \quad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{6.31}$$

The on-shell condition is enforced by $p^0 = E_p$. The $u_s(p)$ term accounts for positive energy solutions which represent matter, while the $v_s(p)$ term accounts for negative energy solutions which represent antimatter.

[a] The horror!

Let us continue with two physical discussions. The first concerns how the negative energy solutions gave rise to antimatter:

- Like the infamous Klein-Gordon equation, the Dirac equation also have negative energy solutions. However, rather than dismiss them as unphysical, Dirac proposed that all negative energy states are filled in a kind of vacuum state, which is now known as the *Dirac sea*.
- A 'hole' in this sea - i.e. a missing negative-energy electron - would appear as a positive-energy, positive-charge particle. This is the positron which was discovered by Carl Anderson in 1932.
- The formulation of the positron was later expanded, which gave rise to antimatter as antiparticles of matter. The emergence of antiparticles have thus justified the negative energy solutions as physical.

The second is to do with the nature of the Dirac equation. The Dirac equation fails to describe a single particle and requires a many-particle treatment like QFT for two reasons:

- For a single particle, the Dirac equation implies unbounded negative energy. This is solved by the Dirac sea and antimatter, which inherently implies many particles.
- Even if we remove this historical convenience, the Dirac equation still implies particle creation and annihilation, which is impossible for a single particle.

6.2 Quantisation of the Dirac field

Having acquired the the Dirac field as the general solution of the Dirac equation in Equation 6.30, we now attempt to quantise it, which allows us to quantise several significant quantities using it. In quantising the Klein-Gordon equation, we replaced f_p and f_p^* with creation and annihilation operators (see Derivation 4.2). Here we start less ambitiously. We order that $b_s(p)$ become an operator. For $d_s(p)$, we do the same but replace it with $d_s^\dagger(p)$. The general solution then reads

$$\psi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} (b_s(p)u_s(p)e^{-ip\cdot x} + d_s^\dagger(p)v_s(p)e^{ip\cdot x}) \tag{6.32}$$

where again, for the sake of convenient normalisations, we have added a normalisation factor of $\frac{1}{(2\pi)^3\sqrt{2E_p}}$, previously seen in Equation 4.33. The canonical momentum can likewise be found:

$$\pi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} (b_s^\dagger(p)u_s^\dagger(p)e^{ip\cdot x} + d_s(p)v_s^\dagger(p)e^{-ip\cdot x}) \tag{6.33}$$

One can then represent the fermionic field creation and annihilation operators $b_s(p)$ and $d_s^\dagger(p)$ in terms of wavefunctions, much like the interacting field creation and annihilation operators in Equation 5.25:

$$-id_s^\dagger(p) = \int \frac{d^3x}{\sqrt{2E_p}} \frac{\bar{v}_s}{2m} (e^{-ip\cdot x}\partial_0\psi - \psi\partial_0 e^{-ip\cdot x}) \quad ib_s^\dagger(p) = \int \frac{d^3x}{\sqrt{2E_p}} (e^{-ip\cdot x}\partial_0\bar{\psi} - \bar{\psi}\partial_0 e^{-ip\cdot x}) \frac{u_s}{2m} \tag{6.34}$$

where the factor of $1/2m$ arises from the normalisation condition in Equation 6.29.
Before mindlessly assigning the bosonic commutation relations to $b_s(p)$ and $d_s(p)$, we stop for a moment and realise that the Dirac field is not actually a bosonic field. The solution to the Dirac equation is not a scalar, but a spinor. Hence, it describes spin-$\frac{1}{2}$ particles[1] and is a fermionic field instead. As such, we instead impose the following *fermionic anticommutation relations* for some indices i and j:

Theorem 6.4 (Fermionic creation and annihilation operator anticommutations)

$$\{b_i(p), b_j^\dagger(q)\} = \{d_i(p), d_j^\dagger(q)\} = \delta_{ij}(2\pi)^2\delta^3(p-q) \tag{6.35}$$

$$\{b_i(p), b_j(q)\} = \{d_i(p), d_j(q)\} = 0 \tag{6.36}$$

Operators for different particle species (electrons and positrons) anticommute to zero:

$$\{b_i(p), d_j(q)\} = \{b_i^\dagger(p), d_j(q)\} = \{b_i(p), d_j^\dagger(q)\} = \{b_i^\dagger(p), d_j^\dagger(q)\} = 0 \tag{6.37}$$

Suddenly recalling the useful relation in Equation 6.31 for no reason whatsoever, we can rewrite it using the definition of the Dirac adjoint in Equation 6.17 as

$$\sum_s u_s(p)_\alpha u_s^\dagger(p)_\beta = (E_p + m)\delta_{\alpha\beta} - (p\cdot\gamma\gamma^0)_{\alpha\beta} \quad \sum_s v_s(p)_\alpha v_s^\dagger(p)_\beta = (E_p - m)\delta_{\alpha\beta} - (p\cdot\gamma\gamma^0)_{\alpha\beta} \tag{6.38}$$

Putting Equation 6.33, Equation 6.35, Equation 6.36 and Equation 6.38 all together, we can solve for the commutator $[\phi_\alpha(x), \pi_\beta(y)]$ and find the following anticommutation relation

Theorem 6.5 (Fermionic field and momentum operator anticommutations)

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\pi_\alpha(x), \pi_\beta(y)\} = \{\pi_\alpha(x), \pi_\beta^\dagger(y)\} = 0 \tag{6.39}$$

$$\{\psi_\alpha^\dagger(x), \psi_\beta(y)\} = \delta_{\alpha\beta}\delta^3(x-y) \tag{6.40}$$

Importantly, this anticommutator, like its boson counterpart $[\phi, \phi^*]$, is not so much derived as it is enforced, in this case by the fact that fermionic fields are Grassmann-odd. The implication is that $\psi^\dagger$, and by extension, $\bar{\psi}$[a], are not independent from ψ.
Recalling the result from Equation 6.20, we have

$$\{\psi_\alpha(x), \pi_\beta(y)\} = i\delta_{\alpha\beta}\delta^3(x-y) \tag{6.41}$$

[a] This justifies our comment following Equation 6.17.

By integrating Equation 6.21 and then using Equation 6.27, we can find the normal-ordered Hamiltonian

Definition 6.9 (Dirac equation normal-ordered Hamiltonian)

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=1}^{2} (b_s^\dagger(p)b_s(p) + d_s^\dagger(p)d_s(p)) \tag{6.42}$$

We can likewise find the charge by integrating the 0^{th} component 4-current $J^0 = \psi^\dagger\psi$:

[1] Here we see why a spinor is often called a 'rank-half tensor'.

Definition 6.10 (Electric charge)

$$Q = e\int d^3x{:}\psi^\dagger\psi{:} = e\int \frac{d^3p}{(2\pi)^3}\sum_{s=1}^{2}(b_s^\dagger(p)b_s(p) - d_s^\dagger(p)d_s(p)) \tag{6.43}$$

where the normal ordering is used to ensure the result is physical. This is simply the electric charge, from which one can verify a particle and its antiparticle carry equal but opposite charges.

6.3 Story of a spinor

Previously, we have said that spinors, unlike tensors, do not undergo general coordinate transformations. Instead, they undergo rotation-like transformations defined by Lie groups. With respect to Dirac spinors, we consider several Lie groups. The starting point is the *homoneneous Lorentz group* $\mathrm{SO}^+(1,3)$, which is a *connected* group that governs all (continuous) spacetime boosts and rotations[2]:

- If we add three more components (parity P, time reversal T and PT), we get the *full lorentz group* or the *inhomogeneous Lorentz group* $\mathrm{O}(1,3)$, a *disconnected* group with four *smoothly separated* components:

$$\mathrm{O}(1,3) \cong \mathrm{SO}^+(1,3) \cup P\cdot\mathrm{SO}^+(1,3) \cup T\cdot\mathrm{SO}^+(1,3) \cup PT\cdot\mathrm{SO}^+(1,3) \tag{6.44}$$

- $\mathrm{SO}^+(1,3)$ is a double cover of $\mathrm{SL}(2,\mathbb{C})$. This is how we go from the 4D Dirac spinor to the 2D Weyl spinor[3].
- The rotation subgroup of $\mathrm{SO}^+(1,3)$ is $\mathrm{SO}(3)$, which itself is famously a double cover of $\mathrm{SU}(2)$.
- $\mathrm{SU}(2)$ is fundamental for the concept of spin and all that follows.

Definition 6.11 (Proper and improper Lorentz transformations) There exists two kinds of Lorentz transformations:

- *Proper Lorentz transformations* have matrices with determinant $+1$. These are rotations and boosts that make up $\mathrm{SO}^+(1,3)$
- *Improper Lorentz transformations* have matrices with determinant -1. These are the P, T and PT transformations that make up the rest of $\mathrm{O}(1,3)$.

Let us first consider $\mathrm{SO}^+(1,3)$.

Derivation 6.3 (Lorentz group generators) Given a Lorentz transformation $x \to x' = \Lambda x$, we expect the Dirac equation to be Lorentz-invariant:

$$(i\gamma^\mu\partial_\mu - m)\psi(x) \to (i\gamma^\mu\partial'_\mu - m)\psi'(x') \tag{6.45}$$

where the wavefunction transforms according to the (internal) spinor transformation $T(\Lambda)$, defined by

$$\psi'(x') = T(\Lambda)\psi(x) = T(\Lambda)\psi(\Lambda^{-1}x') \tag{6.46}$$

Now we try to determine $T(\Lambda)$. One can write out the transformation in terms of indices:

$$x'^\mu = \Lambda^\nu_\mu x^\mu \quad \partial_\mu\Lambda^\nu_\mu\partial'_\nu \tag{6.47}$$

The original, untransformed and final, transformed equations can then be written as

$$(i\gamma^\nu\Lambda^\mu_\nu\partial'_\nu - m)\psi(\Lambda^{-1}x') = 0 \quad T\Lambda^{-1}(i\gamma^\mu\partial'_\mu - m)T(\Lambda)\psi(\Lambda^{-1}x') = 0 \tag{6.48}$$

By equating the two[a], one finds

$$T(\Lambda)^{-1}\gamma^\mu T(\Lambda) = \lambda^\mu_\nu\gamma^\nu \tag{6.49}$$

[2]This is a spacetime symmetry that transforms the spinors and is unrelated to the U(1) internal symmetry we will see later.

[3]Recall that the transformations of spinors are ultimately governed by Lie groups.

Using the parameterisation of generators in *Spinors & Symmetries*, an infinitesimal Lorentz transformation may be parameterised as

$$\Lambda^\mu_\nu = \delta^\mu_\nu - \omega_{\rho\sigma}(g^{\rho\nu}\delta^\sigma_\nu - g^{\sigma\mu}\delta^\rho_\nu) + O(\omega^2) \tag{6.50}$$

and a transformation may be given as

$$T(\Lambda) = e^{i\omega_{\rho\sigma}s^{\rho\sigma}} = \mathbb{I}_4 + i\omega_{\rho\sigma}s^{\rho\sigma} + O(\omega^2) \tag{6.51}$$

where $s^{\rho\sigma}$ are the all-too-familiar generators that we have parameterised via $\omega_{\rho\sigma}$ in *Spinors & Symmetries*. Plugging the two expressions into Equation 6.49 gives

$$i[s^{\rho\sigma}, \gamma^\mu] = g^{\rho\mu}\gamma^\sigma - g^{\sigma\mu}\gamma^\rho \tag{6.52}$$

which surprisingly reduces, via (anti)commutation relations to the simple expression

Definition 6.12 (Lorentz group generators) The following generators define the spinor representation of the (proper) Lorentz group

$$s^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma] \tag{6.53}$$

We conclude by saying that the transformations *are* the representations $T(\Lambda) = e^{i\omega_{\rho\sigma}s^{\rho\sigma}}$.

[a] One can do so as both equations apply for all ψs.

6.4 CPT theorem

This entire mathematical toolkit has led us to a famous theorem, which is the so-called *CPT* (*charge conjugation*, *parity*, *time reversal*) *theorem*, also known with an alternate initial ordering as the *PCT theorem*. All three are symmetries of the Dirac field.

> **Quote 6.3** Yes, it's the oldschool order.
>
> *Felix Halbwedl, on the alternate initial order 'PCT', 3 January 2025*

Transformations form a connected Lie groups are continuous, while the as-of-yet-undefined transformations P and T are discrete transformations, which involve specific finite changes that cannot be continuously connected to the identity transformation. They are not elements of actually $\mathrm{O}(1,3)$, but rather representations of the elements Λ_P and Λ_T (which are Lorentz transformations) of the group $\mathrm{O}(1,3)$:

$$\Lambda_P = \mathrm{diag}(1,-1,-1,-1) \quad \Lambda_T = \mathrm{diag}(-1,1,1,1) \tag{6.54}$$

A third discrete symmetry we are interested in is the charge conjugation C, which is an outer automorphism[4] of the U(1) group, even though it is not continuous like U(1) symmetry.
Now, let us define C, P and T explicitly:

Definition 6.13 (Parity operation) P stands for the *parity operation* P, which changes the 3-position to its inverse:

$$P : (t, \mathbf{x}) \to (t, -\mathbf{x}) \tag{6.55}$$

Theorem 6.6 (Transformations under P)

[4] An automorphism is a symmetry of the symmetry... oh well.

Transformations under P

Object	Behaviour under P[a]	Intrinsic parity P(object)
Scalar S	Invariant	$+1$
Pseudoscalar S_p	Sign flip	-1
4-spinor ψ in Dirac representation	$P\psi = \gamma^0\psi$	$+1$ (fermions), -1 (antifermions)
Vector V^μ	Sign flip on V^i, invariant otherwise	-1[b]
Pseudovector A^μ	Sign flip on A^0, invariant otherwise	$+1$
Metric $g^{\mu\nu}$	Invariant[c]	$+1$
Stress-energy tensor $T^{\mu\nu}$	Sign flip on T^{0j}, T^{i0} and some of T^{ij}, invariant otherwise	$+1$
Faraday tensor $F^{\mu\nu}$	Sign flip on F^{0j}, invariant otherwise	None (mixed)

[a] $i, j \in [1, 2, 3]$
[b] If it describes an interacting particle like a photon.
[c] Components undergo general coordinate transformations.

Remark 6.7 As $P^2 = \mathbb{I}$ for a Dirac spinor, it is a Z_2 symmetry labelled Z_2^P.

Definition 6.14 (Time reversal) T stands for the so-called *time reversal* T, which flips the time coordinate:

$$T : (t, \mathbf{x}) \to (-t, \mathbf{x}) \tag{6.56}$$

Theorem 6.7 (Transformations under T)

Transformations under T

Object	Behaviour under P
Scalar S	Invariant
Pseudoscalar S_p	Invariant
4-spinor ψ in Dirac representation	$T = i\gamma^1\gamma^3$[a]
Vector V^μ	Sign flip on V^i, invariant otherwise
Pseudovector A^μ	Sign flip on A^0, invariant otherwise
Metric $g^{\mu\nu}$	Invariant[b]
Stress-energy tensor $T^{\mu\nu}$	Sign flip on T^{0j} and T^{i0}, invariant otherwise
Faraday tensor $F^{\mu\nu}$	Sign flip on F^{ij}, invariant otherwise

[a] This is a mathematical tool. The actual physical operator has an extra complex conjugation operator K and is thus anti-unitary. It has the form $\mathcal{T} = TK = i\gamma^1\gamma^3 K$.
[b] Components undergo general coordinate transformations.

Definition 6.15 (Charge conjugation) C stands for *charge conjugation* C, which is the sign-flip of all charges:

$$C : Q \to -Q \tag{6.57}$$

Theorem 6.8 (Transformations under C) The action of C on other quantities are simpler, and we summarise them below:

Transformations under C

Object	Behaviour under C
Scalar S	Sign flip
Pseudoscalar S_p	Invariant
4-spinor ψ in Dirac representation	$C = i\gamma^2\gamma^0$
Vector V^μ	Sign flip
Pseudovector A^μ	Invariant
Metric $g^{\mu\nu}$	Invariant[a]
Stress-energy tensor $T^{\mu\nu}$	Invariant
Faraday tensor $F^{\mu\nu}$	Sign flip

[a] Applies even for charged fields. This is because the stress-energy tensor depends quadratically on fields, and the sign changes cancel out.

From the actions of C, P, T on Dirac spinors, it is then easy to infer their actions on Weyl spinors. This brings about some funny effects. For example, Theorem 6.8 implies, for the left- and right-handed Weyl spinors χ_L and χ_R that make up the Dirac spinor:

$$C\chi_L = -i\sigma_2\chi_R^* \quad C\chi_R = -i\bar{\sigma}_2\chi_L^* \tag{6.58}$$

Hence, for a theory to be C invariant, it must contain:

- Left-handed Weyl spinors
- Right-handed complex conjugate of Weyl spinors

However, it is possible to get away with only one handedness. This is the so-called *Majorana*[5] *spinor*, which is technically a very specific subcategory of the Dirac spinor:

Definition 6.16 (Majorana spinor) A Majorana spinor is a Dirac spinor with the following construction

$$\psi = \begin{pmatrix} \chi_L \\ -i\sigma_2\chi_L^* \end{pmatrix} \tag{6.59}$$

Due to being a Dirac spinor, it transforms as one, with the key distinction of undergoing charge conjugation as

$$C\psi = \psi \tag{6.60}$$

meaning that a *Majorana fermion* is its own antiparticle. As a result, a Majorana particle is always charge neutral.

Derivation 6.4 (γ^5 matrix) Interestingly, if one consults the parts of Theorem 6.8, Theorem 6.6 and Theorem 6.7 on Dirac spinors, one sees that CPT, put together, acts on a Dirac spinor as

$$CPT\psi = \gamma^5\psi \tag{6.61}$$

Where we have defined the so-called γ^5 matrix as

Definition 6.17 (γ^5 matrix)

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{6.62}$$

This is a γ^μ matrix in name only which has nothing to do with the Lorentz group generators, and whose notation γ^5 is a historical artefact. That being said, it is not entirely a mathematical convenience either, as we will see in the next section.

[5] To this day, his demise remains a historical mystery. For more, see here.

Theorem 6.9 (γ^5 matrix properties)

- γ^5 is unitary:

$$\gamma^{5\dagger} = \gamma^5 \quad (\gamma^5)^2 = 1 \tag{6.63}$$

- γ^5 anticommutes with γ^μ matrices:

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{6.64}$$

As seen in all three transformations, symmetry can be broken, which we call *symmetry breaking*.

Definition 6.18 (Symmetry breaking) Two types of symmetry breaking exist:

- In *spontaneous symmetry breaking*, the equations of motion are invariant, but the ground state (vacuum) of the system is not.
- In *explicit symmetry breaking*, the equations of motion are not invariant.

The breaking of individual or two (e.g. C, P, T) symmetries is not prohibited in QFT. However, due to Lorentz invariance, the breaking of CPT symmetry is disallowed in QFT. This is illustrated by the CPT theorem:

Theorem 6.10 (CPT theorem) Any quantum field theory that is Lorentz-invariant and has a well-defined local interaction must respect CPT symmetry. i.e. for some quantity H, the combination of charge conjugation, parity, and time reversal is always a symmetry:

$$(CPT)H(CPT)^{-1} = H \tag{6.65}$$

As mentioned, the CPT theorem ultimately results from Lorentz invariance. Hence, it cannot be spontaneously broken like gauge symmetries (e.g. electroweak symmetry). At the time of writing (2025), experimentalists have yet to observe CPT symmetry breaking, showing how well QFT has withstood the tests of time.
There is, of course, a more terrifying implication if we turn the first statement above backwards: If CPT were spontaneously broken, it would suggest a violation of Lorentz invariance, a common feature of physics beyond the standard model (BSM).

6.5 Spin, helicity and charality

All this hard work has finally put us in a position to return to quantum mechanics. Spin concerns the rotation of Dirac equation solutions, which are Dirac spinors. We already know that the entirety of Lorentz transformations - translations and rotations - are governed by $\mathrm{SO}^+(1,3)$, which is where we will start:

- If we remove translations, we get the 3D rotation group SO(3).
- Now consider which double cover this group forms (i.e. go from Dirac spinors to Weyl spinors), and we get SU(2), which is intuitively the Lie group that governs spin.
- The generators of SU(2) are the Pauli matrices.

However, at this point, we have to go back from Weyl spinors to Dirac spinors. This results in the so-called *spin operator* $\mathbf{S}$, which is actually a 'vector of matrices'. To see what this means, we write down the spin operator in its individual components S_i as well as its entirety:

Definition 6.19 (Spin angular momentum operator)

$$\hat{S}_i = \frac{1}{2}\Sigma_i = \frac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix} \quad \mathbf{S} = \frac{1}{2}\mathbf{\Sigma} = \frac{1}{2}\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} \tag{6.66}$$

where we have defined

$$\Sigma_i = \frac{i}{2}\epsilon_{ijk}\gamma^j\gamma^k = \frac{i}{2}\epsilon_{ijk}\alpha^j\alpha^k \tag{6.67}$$

Essentially, $\mathbf{S}$ is a 3-vector whose components are operators.

From quantum mechanics, one recalls that spin has no classical analogue. It contributes, along with the *orbital angular momentum* which is carried over from classical mechanics, to the total angular momentum, which is ultimately conserved. We will prove this now.

Derivation 6.5 (Conservation of angular momentum) The total angular momentum operator $\mathbf{J}$ is the sum of the orbit and spin angular momentum operators. This is known as *spin-orbit coupling*:

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \tag{6.68}$$

To prove that it is a conserved quantity, all we have to do is to show that it satisfies Equation 3.38. This can be done by evaluating $[\hat{L}_i, H]$ and $[\hat{S}_i, H]$ individually. Let us first recall the definition of $\hat{L}_i$:

Definition 6.20 (Orbital angular momentum operator)

$$\hat{L}_i = \epsilon_{ijk} x^j p^k \quad \mathbf{L} = \mathbf{x} \times \mathbf{p} \tag{6.69}$$

We now calculate the Dirac Hamiltonian (density) by inserting *the free part of* Equation 6.18 into Equation 2.7 and then both into Equation 2.28:

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta m \tag{6.70}$$

Now take the commutator of Equation 6.69 and this. We get

$$[\hat{L}_i, H] = i\epsilon_{ijk}\gamma^j p^k \quad [\mathbf{L}, H] = i\boldsymbol{\gamma} \times \mathbf{p} \tag{6.71}$$

Doing the same for Equation 6.68 gives

$$[\hat{S}_i, H] = -i\epsilon_{ijk}\gamma^j p^k \quad [\mathbf{S}, H] = -i\boldsymbol{\gamma} \times \mathbf{p} \tag{6.72}$$

Indeed, we are left with

$$[\hat{L}_i + \hat{S}_i, H] = 0 \tag{6.73}$$

from which we recover the conservation of the total angular momentum of $\mathbf{J}$. We then know that $\mathbf{J}$ is also Lorentz-invariant.

Having finished our little recap, we can discuss some concepts relevant in HEP.

Definition 6.21 (Helicity operator) The *helicity operator* is the spin operator projected in the direction of the momentum. In index and vector notations, it is written as[a]

$$\hat{h} = \frac{\Sigma_i p^i}{|p_i|} = \frac{\gamma^5\gamma^0\gamma_\mu p^\mu}{|p_\mu|} \quad \hat{h} = \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{|\mathbf{p}|} = \frac{\gamma^5\gamma^0\boldsymbol{\gamma} \cdot \mathbf{p}}{|\mathbf{p}|} \tag{6.74}$$

Dirac spinors in the helicity basis are eigenspinors of the helicity operator with eigenvalues $h = \pm 1$:

$$\hat{h}\psi = \lambda\psi = \pm\psi \tag{6.75}$$

[a] Some alternative definitions exist up to a rescaling factor.

Interestingly, we see that γ^5 has reemerged. That is to say, the ghost of γ^5 past has yet again come back to haunt us. To see what this means, we begin by noting that $\gamma^5\psi$ is a solution of the Dirac equation.

Derivation 6.6 ($\gamma^5\psi$ as massless solution) We consider the Dirac equation of a massless particle *in momentum space*, where $m = 0$ and $i\partial_\mu$ is replaced with p_μ:

$$\gamma^\mu p_\mu \psi = (\gamma^0 p_0 + \gamma^i p_i)\psi = 0 \tag{6.76}$$

Multiply both sides by γ^5 yields

$$\gamma^5(\gamma^0 p_0 + \gamma^i p_i)\psi = 0 \tag{6.77}$$

Using the anticommutation property in Equation 6.64, we can move γ^5 past the bracket and then multiply both sides by -1. This conveniently lands us at

$$(\gamma^0 p_0 + \gamma^i p_i)(\gamma^5\psi) = 0 \tag{6.78}$$

Hence, $\gamma^5\psi$ is the solution of the massless Dirac equation.

As such, the eigenstates of $\hat{h}$ should also be solutions to the Dirac equation, each representing the particle's solution in a different helicity.

Derivation 6.7 ($\gamma^5\psi$ flips the helicity) Let us multiply γ^5 on both sides of Equation6.75:

$$\gamma^5\hat{h}\psi = \gamma^5\lambda\psi = \lambda\gamma^5\psi = \pm\gamma^5\psi \tag{6.79}$$

where we have intuitively moved γ^5 on the RHS. On the LHS, we recall the composition of $\hat{h}$ and the anticommutation property in Equation 6.64. This gives

$$-\hat{h}\gamma^5\psi = \lambda\gamma^5\psi = \pm\gamma^5\psi \quad \hat{h}\gamma^5\psi = -\lambda\gamma^5\psi = \mp\gamma^5\psi \tag{6.80}$$

Hence, γ^5, which is the Dirac basis CPT action, acts as a helicity-flip operator.

This then connects to the concept of *chirality* (from 'hand' in Greek), which denotes the 'handedness' of a particle.

- When $h = 1$, helicity is positive as the particle's spin direction is the same as its direction of motion. Chirality is *left-handed*. This orientation is so-called as it observes the right-hand rule: Align the right thumb in the momentum's direction, and the curled fingers should align with the spin direction.
- When $h = -1$, helicity is negative as the particle's spin direction is opposite from its direction of motion. Chirality is *right-handed*. This orientation is so-called as it observes the left-hand rule: Align the left thumb in the momentum's direction, and the curled fingers should align with the spin direction.

Ultimately, this intuitive definition does not find much use in practice, and we yet again find ourselves terrorised by γ^5, which is known as the *chirality operator*.

Definition 6.22 (Chirality) Chirality is eigenvalue of γ^5, which takes the value of ± 1.

This is defined in conjunction with two, so-called, *chiral projectors*:

Definition 6.23 (Chiral projectors) The left-handed and right-handed chiral projectors are defined as

$$P_L = \frac{\mathbb{I}_4 - \gamma^5}{2} \quad P_R = \frac{\mathbb{I}_4 + \gamma^5}{2} \tag{6.81}$$

Through them, we acquire the left-chiral and right-chiral spinors:

$$\psi_L = P_L\psi = \frac{\mathbb{I}_4 - \gamma^5}{2}\psi \quad \psi_R = P_R\psi = \frac{\mathbb{I}_4 + \gamma^5}{2}\psi \tag{6.82}$$

which are also flipped by γ^5:

$$\gamma^5\psi_L = \psi_R \quad \gamma^5\psi_R = \psi_L \tag{6.83}$$

These spinors are 4D on paper, and we formally have

$$\psi = \psi_L + \psi_R \tag{6.84}$$

However, half of their components are zero, and only 2 independent degrees of freedom are left. As such, one can effectively discard the zero components. Then, ψ_L and ψ_R reduce to 2D and, hence, Weyl spinors:

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} \quad \psi_L = \begin{pmatrix}\xi \\ 0\end{pmatrix} \quad \psi_R = \begin{pmatrix}0 \\ \eta\end{pmatrix} \tag{6.85}$$

where ξ and η are 2-spinors.

Derivation 6.8 (Massless limit) We can split a Dirac spinor into its left- and right-chiral parts. This involves treating ψ_L and ψ_R as Weyl spinors instead of half-zero Dirac spinors. Inserting Equation

6.85 into the Dirac equation gives

$$i\gamma^\mu\partial_\mu\psi_L - m\psi_R = 0 \quad i\gamma^\mu\partial_\mu\psi_R - m\psi_L = 0 \tag{6.86}$$

The mass term couples left and right chiralities, and one cannot have a mass term for a fermion unless both ψ_L and ψ_R are present. In the ultra-relativistic limit ($E \gg m$), and $m \to 0$. We then have

Theorem 6.11 (Weyl equations)

$$i\gamma^\mu\partial_\mu\psi_L = 0 \quad i\bar{\gamma}^\mu\partial_\mu\psi_R = 0 \tag{6.87}$$

Now, since the particle is massless, its speed is c in all reference frames. Unlike the massive particle case, it is no longer possible to 'overtake' it by a Lorentz boost. That is to say, helicity becomes Lorentz-invariant like chirality, and the two become identical.

Finally, let us wrap up this section, which is already too long, by tying up one last loose end with respect to the almighty γ^5 matrix. We add to the linear σ Lagrangian in Equation 8.59 a new term that describes the interaction between a scalar field ϕ and a Dirac fermion field ψ. This results in the so-called *Yukawa theory* which has the Lagrangian

Definition 6.24 (Yukawa interaction Lagrangian)

$$\mathcal{L} = \frac{1}{2}\bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m_\phi^2\phi^2 - \frac{\xi\phi^3}{3!} - \frac{\lambda\phi^4}{4!} - g\phi\bar{\psi}_i\Gamma_{ij}\psi_j \tag{6.88}$$

where the final term describes the coupling between ϕ and ψ.

We now want to solve for Γ_{ij}, which is indeterminate. Recall that all quantum field theories within the standard model must be Lorentz-invariant. This can only be satisfied if this final interaction term (or effectively, the part $\bar{\psi}_i\Gamma_{ij}\psi_j$) transforms as a scalar. Hence, two candidates exist for Γ_{ij}:

- The unit matrix $\mathbb{I}_4$, under which $\bar{\psi}_i\Gamma_{ij}\psi_j$ transforms as a scalar.
- $i\gamma^5$, under which $\bar{\psi}_i\Gamma_{ij}\psi_j$ transforms as a pseudoscalar.

> **Quote 6.4** The γ^5 matrix is the trilobites of high energy physics!
>
> *The Author, 21 April 2025*

Taking the latter case gives rise to a funny implication. As it turns out, the Higgs field gives mass to fermions by coupling left- and right-chiral fields through a term $y\bar{\psi}_L H\psi_R$. Once the Higgs gets a vacuum expectation value, this becomes the Yukawa coupling term $m\bar{\psi}_L\psi_R$ which ties chirality and mass together.

6.6 Maxwell's equations

It is well-known that QED is a gauge theory governed by U(1) symmetry. This gauge symmetry is so-called as it is defined with respect to the Lie group U(1)[6], which is abelian. In *Spinors & Symmetries*, it is seen that representations of abstract elements of an abelian Lie group commute.

$$[A_\mu, A_\nu] = 0 \tag{6.89}$$

In our case, this is the gauge (photon) field A^μ, which is the 4-potential, consisting of the electric scalar potential and the magnetic vector potential[7]. Inserting Equation 6.89 into Equation 6.14 gives us the QED field strength tensor, which is the well-known Faraday tensor:

Definition 6.25 (Faraday tensor)

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{6.90}$$

[6] We will investigate its gauge transformations very soon.

[7] We can also use an alternative 4-potential, comprising of the magnetic scalar potential and the electric vector potential. However, this is rarely used due to the absence of observed magnetic monopoles.

Unsurprisingly, the equations of motion that govern the photon half of QED are Maxwell's equations, which we can derive using the standard method of constructing the Lagrangian and insert it into the Euler-Lagrange equations.

Derivation 6.9 (Recovery of Maxwell's equations) We set the photon field Lagrangian (density) as

Definition 6.26 (Photon field Lagrangian)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.91}$$

We can write down the full form of Equation **??** by inserting Equation 6.90:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= -\frac{1}{4}(\partial^\mu A^\nu \partial_\mu A_\nu - \partial^\nu A^\mu \partial_\mu A_\nu - \partial^\mu A^\nu \partial_\nu A_\mu + \partial^\nu A^\mu \partial_\nu A_\mu)\end{aligned} \tag{6.92}$$

Noting that all indices are free, we relabel some of them and find that

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) \tag{6.93}$$

Now let us take a *smol* detour and consider a 'total derivative' term of the form $\partial_\mu(A_\nu \partial^\mu A^\nu)$. As per the product rule, we immediately see that one can expand it as

$$\partial_\mu(A_\nu \partial^\mu A^\nu) = (\partial_\mu A_\nu)(\partial^\mu A^\nu) + A_\nu(\partial_\mu \partial^\mu A^\nu) \tag{6.94}$$

Using this relation, one can rewrite the two terms in Equation 6.93 as

$$\frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) = \frac{1}{2}\partial_\mu(A_\nu \partial^\nu A^\mu) - \frac{1}{2}A_\nu \partial^\nu \partial^\mu A_\mu \tag{6.95}$$

$$-\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) = \frac{1}{2}A_\nu \Box A^\nu - \frac{1}{2}\partial_\mu(A_\nu \partial^\mu A^\nu) \tag{6.96}$$

But at the same time, the term $\partial_\mu(A_\nu \partial^\mu A^\nu)$ looks a bit familiar, doesn't it? One can see that this so-called term is nothing but our good friend, the boundary term in Equation 2.12. Recalling that zero terms and the boundary terms can be added or removed from the Lagrangian to our liking, Equation 6.93 is then

$$\mathcal{L} = \frac{1}{2}A_\mu(\Box A^\mu - \partial^\mu \partial^\nu)A_\nu \tag{6.97}$$

By applying the Euler-Lagrange equations, we find that

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0 \tag{6.98}$$

where A^ν, which we ultimately recognise as a field variable, replaces ψ.
We then recover Maxwell's equations in index notation

Theorem 6.12 (Maxwell's equations)

$$\partial_\mu F^{\nu\mu} = J^\nu \tag{6.99}$$

The electric and magnetic 3-fields then follow intuitively:

$$E_i = F^{0i} = -\partial_i A^0 + \partial_0 A^i \quad B^i = -\frac{1}{2}\epsilon^{ijk}F_{jk} = \epsilon^{ijk}\partial_j A_k \tag{6.100}$$

Now we investigate the gauge transformation brought about by our so-called U(1) symmetry. In gauge theory, there exist physical observables that are invariant under certain transformations of the potentials. In the case of electromagnetism, this manifests in the invariance of the Faraday tensor $F^{\nu\mu}$ (and hence, the EM fields E and B) under a set of 4-potential gauge transformations that is unsurprisingly called the U(1) transformations:

Definition 6.27 (U(1) transformations)

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{6.101}$$

where $\Lambda(x)$ is a scalar field and satisfies the wave equation.

One can verify the invariance of the Faraday tensor this by noting that the following conditions simultaneously hold

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0} \quad \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = 0 \tag{6.102}$$

where $F^{\mu 0}$ are the canonically conjugate EM fields.
Another way to see this is that, assuming zero 4-current, Maxwell's equations can be written as

$$\partial_\nu F^{\nu\mu} = \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = 0 \tag{6.103}$$

Immediately, we note that the equation does not depend on A^μ itself. Rather, the only dependence lies in $\partial_\nu A^\nu$.
Either way, we see that we can shift A_μ by any gradient $\partial_\mu \Lambda$, as we just did in our U(1) transformation, without affecting the Faraday tensor $F_{\mu\nu}$. As it turns out, this is expected from the so-called *Noether's second theorem*.

Theorem 6.13 (Noether's second theorem) If a Lagrangian is invariant under an infinite-dimensional local symmetry group (like U(1) symmetry), there exist differential identities that the Euler-Lagrange equations are subject to, known as Noether identities, that make the equations to be not independent from each other. These Noether identities then reflect the redundancy in the equations of motion.

We can verify that such an identity exists for Maxwell's equations.

Derivation 6.10 (Noether identity in Maxwell's equations) The Lagrangian is invariant under our U(1) transformation, and it is possible to write the variation of the action as

$$\delta S = \int d^4x \partial_\mu \Lambda \cdot \left(\frac{\delta \mathcal{L}}{\delta A_\mu}\right) = \int d^4x \Lambda \cdot \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta A_\mu}\right) = 0 \tag{6.104}$$

But since $\Lambda(x)$ is arbitrary, we know that, identically:

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta A_\mu}\right) = 0 \tag{6.105}$$

Substituting gives

$$\frac{\delta \mathcal{L}}{\delta A_\mu} = \partial_\nu F^{\nu\mu} \tag{6.106}$$

This yields our Noether identity

$$\partial_\mu \partial_\nu F^{\nu\mu} = 0 \tag{6.107}$$

which holds identically due to the antisymmetry of $F^{\nu\mu}$.

The takeaway from this gauge invariance of $F^{\nu\mu}$ is that not all degrees of freedom in A_μ are physical. The unphysical degrees of freedom correspond to pure gauge and are thus called *gauge freedoms* which, if not eliminated, will leas us to mistakenly count multiple configurations of A^μs as distinct and give rise in erroneous results.

Derivation 6.11 (Fixing the gauge) The standard procedure to eliminate gauge freedoms is *gauge fixing*. In classical EM, we attempt to eliminate this gauge freedom in Equation 6.101 by the Lorentz-invariant Lorenz gauge.

> **Quote 6.5** Amazingly, the missing "t" is not a typo here.
>
> *Alessio Serafini*

$$\partial_\mu A^\mu = 0 \tag{6.108}$$

whence Maxwell's equations reduce to the d'Alembertian (wave) equation

$$\Box A^\mu = 0 \tag{6.109}$$

The mathematical justification of this gauge fixing is known as the *Gupta-Bleuler formalism*[a]. We will not justify it fully, but the main ideas are as follows:

- So far our Fock space has been positive-definite. However, when $A^\mu(x)$ is quantised, we have one timelike component which leads to negative norm states. We then construct a so-called *Krein space*, which is a Fock space that includes these negative norms.
- The negative norm is clearly unphysical. So we impose the Lorenz gauge. In doing so, we have reduced our Krein space into a physical subspace.

The quantum analogue of the Lorenz gauge is the R_ξ gauge. We begin with the photon field Lagrangian in Equation 6.91. The R_ξ gauge adds a gauge-fixing term to the Lagrangian:

Theorem 6.14 (R_ξ gauge)

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \tag{6.110}$$

where ξ is a parameter.

Like in Equation 9.2, we can rewrite the gauge-fixing term and recognise the vanishing of the 'total derivative' boundary term:

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\frac{1}{2\xi}(\partial^\mu A_\mu \partial^\nu A_\nu) \\ &= -\frac{1}{2\xi}\partial_\mu(A^\mu \partial^\nu A_\nu) + \frac{1}{2\xi}A^\mu \partial_\mu \partial^\nu A_\nu \\ &= \frac{1}{2\xi}A^\mu \partial_\mu \partial_\nu A^\nu\end{aligned} \tag{6.111}$$

The addition of this gauge-fixing term is justified because it is the simplest local, renormalisable term one can introduce to fix the gauge. We will prove this by using the Faddeev-Popov method in Part III.

[a] In non-abelian gauge theories, it is superceded by the BRST formalism.

The simplest R_ξ gauge is the Feynman-'t Hooft gauge, which is used in most QFT calculations. In this gauge, one has $\xi = 1$, and the Lagrangian becomes

Theorem 6.15 (Feynman-'t Hooft gauge)

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{2}(\partial_\mu A^\mu)^2 \tag{6.112}$$

We can now quantise the 4-potential, which is a real, massless Klein-Gordon (scalar) field. The classical wave solution is

$$A^\mu(x) = \int d^3p \sum_{\lambda=0}^{3} (\epsilon^\mu_\lambda(p) f_\lambda(p) e^{-ip\cdot x} + \epsilon^{\mu *}_\lambda(p) f^*_\lambda(p) e^{ip\cdot x}) \tag{6.113}$$

where ϵ^μ is a *polarisation vector*, a 4-versor[8]. Significantly, it holds the following summation property:

Theorem 6.16 (Polarisation vector property)

$$\sum_{\lambda=0}^{d} \epsilon^{\nu *}_\lambda(p) \epsilon^\mu_\lambda(p) = -g^{\mu\nu} \tag{6.114}$$

where the index λ labels the polarisation states of the photon.

[8] One can recall from *Spinors & Symmetries* that a versor is simply a unit quaternion (i.e. it has norm 1). In simpler terms, a versor is simply a vector whose magnitude is unity (i.e. 1).

One notes this to quite resemble the *tetrad fields* in general relativity.

Derivation 6.12 (Polarisation and the Lorentz gauge) For a photon, there are four possible indices, but not all are physical:

- $\lambda = 0$: This is a *longitudinal polarisation* that is often unphysical.
- $\lambda = 1, 2$: They are the two physical *transverse polarisations*[a] of the photon.
- $\lambda = 3$: This is a *scalar polarisation* that is also often unphysical.

As photons in QED are gauge bosons, the choice of polarisation vectors is not unique. This gauge freedom can be removed by applying Lorenz gauge:

$$p_\mu \epsilon^\mu_\lambda = 0 \tag{6.115}$$

where p^μ is the photon's 4-momentum.
In doing so, the two unphysical components of the 4-versor have been eliminated due to them being unphysical under the gauge, and only the two transverse polarisation states remain physical for photons. This means that $k_\mu \varepsilon^\mu(\lambda, k) = 0$ for transverse polarisations $\lambda = 1, 2$, but not necessarily for $\lambda = 0, 3$.

[a] i.e. they are perpendicular to the direction of propagation and to each other.

6.7 Quantisation of the photon field

Using the same procedure we have done before, we insert the normalisation factor into Equation 6.116 and replace $f_\lambda(p)$ and $f^*_\lambda(p)$ with creation and annihilation operators. This gives the photon field, which is actually the 4-potential we see in classical electromagnetism.

Definition 6.28 (Photon field)

$$A^\mu(x) = \int \frac{d^3p}{(2\pi^3)\sqrt{2E_p}} \sum_{\lambda=0}^{3} (\epsilon^\mu_\lambda(p) a_\lambda(p) e^{-ip\cdot x} + \epsilon^{\mu *}_\lambda(p) a^\dagger_\lambda(p) e^{ip\cdot x}) \tag{6.116}$$

where the creation and annihilation operators have the form

$$i a^\dagger_\lambda(p) = \int \frac{d^3x}{\sqrt{2E_p}} g_{\lambda\lambda} \epsilon^\mu_\lambda(p) (e^{-ip\cdot x} \partial_0 A_\mu - A_\mu \partial_0 e^{-ip\cdot x}) \tag{6.117}$$

Note 6.2 (Gauge field normalisation) So far, we have been using *relativistic normalisation*, which is given by Equation 4.31. For gauge fields, a different normalisation convention is used instead. The key difference is in how the one-particle states are defined:

$$[a_p, a^\dagger_q] = 2E(2\pi)^3 \delta^3(p - q) \tag{6.118}$$

To account for (or rather *absorb*) this extra factor of $2E$, Equation **??** becomes

$$S_{fi} = \mathcal{M}_{fi} i \prod_{j=1}^{n} \frac{1}{2E_{f_j}} \prod_{j=1}^{m} \frac{1}{2E_{i_j}} (2\pi)^4 \delta^4(p_i - p_f) \tag{6.119}$$

In this gauge theory normalisation convention, Equation 5.55 takes the form

$$\text{LIPS}(m) \equiv (2\pi)^4 \delta^4(p_i - p_f) \prod_{k=1}^{m} \frac{d^3 q_k}{(2\pi)^3} \tag{6.120}$$

They observe the commutation relation

$$[a_\lambda(p), a^\dagger_{\lambda'}(q)] = -g_{\lambda\lambda'} (2\pi)^3 2E \delta^3(p - q) \tag{6.121}$$

We then derive the canonical momentum of a photon field by inserting Equation 6.91:

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = \partial_0 A^\mu \tag{6.122}$$

Plugging in Equation 6.116 gives

Definition 6.29 (Photon field canonical momentum)

$$\pi^\mu = -\int \frac{d^3p}{(2\pi^3)\sqrt{2E_p}} \sum_{\lambda=0}^{3} (\epsilon_\lambda^{\mu*}(p) a_\lambda^\dagger(p) e^{ip\cdot x} - \epsilon_\lambda^\mu(p) a_\lambda(p) e^{-ip\cdot x}) \tag{6.123}$$

Again we consider the nature of the photon. It has spin 1, and is thus a boson. The standard bosonic commutations thus apply:

$$[A^\mu(x), \pi^\nu(u)] = -ig^{\mu\nu}\delta^3(x-y) \tag{6.124}$$

$$[A^\mu(x), A^\nu(u)] = [\pi^\mu(x), \pi^\nu(u)] = 0 \tag{6.125}$$

One final loose end is the Hamiltonian. By inserting Equation 6.122 into Equation 2.28 (where, notably, the field is A^μ instead of ϕ), the photon field Hamiltonian reads

Definition 6.30 (Photon field Hamiltonian)

$$\mathcal{H} = \frac{1}{2}\dot{A}^\nu \dot{A}_\nu + \frac{1}{2} DA^\nu DA_\nu \tag{6.126}$$

6.8 QED Feynman rules

The final term we need to add to the QED Lagrangian is the interaction term that represents the vertex, which is the coupling of two fermion fields to a photon field. This allows us to write the QED Lagrangian as

Definition 6.31 (QED Lagrangian)

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{6.127}$$

We now include the gauge-fixing term and write down the full QED Lagrangian by consulting Equation 6.15, Equation 9.2 and Equation 6.111. The fully simplified, gauge-fixed QED Lagrangian is then

$$\mathcal{L}_{\text{QED}} = \underbrace{\bar{\psi}(i\gamma^\mu\partial - m)\psi}_{\text{free fermion}} - \underbrace{e\bar{\psi}\gamma^\mu A_\mu \psi}_{\text{vertex}} - \underbrace{\frac{1}{2}A^\mu(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu}_{\text{free photon}} + \underbrace{\frac{1}{2\xi}A^\mu\partial_\mu\partial_\nu A^\nu}_{R_\xi \text{ gauge}} \tag{6.128}$$

We now calculate the QED propagators.

Derivation 6.13 (Fermionic propagator) As the fermionic propagator is fermionic[a], we denote it distinctly as S_F:

$$S_F(x-y) = \langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle \tag{6.129}$$

This is the only useful fermionic propagator, as anticommutation implies that

$$\langle 0|T[\psi_\alpha(x)\psi_\beta(y)]|0\rangle = \langle 0|T[\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)]|0\rangle = 0 \tag{6.130}$$

By definition, it is a Green's function of the Dirac operator:

$$(i\not{\partial}_x - m)S_F(x-y) = \delta^4(x-y) \tag{6.131}$$

Now we want to convert this expression to momentum space. We begin by Fourier-transforming Equation 6.1 into momentum space. The field transformations are

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4}\psi(p)e^{-ip\cdot x} \quad \bar{\psi}(x) = \int \frac{d^4p}{(2\pi)^4}\bar{\psi}(p)e^{ip\cdot x} \tag{6.132}$$

Using this and Equation 4.28, we have

$$(\not{p} - m)\tilde{\psi}(p) = 0 \tag{6.133}$$

The second element to be Fourier-transformed is the propagator itself:

$$S_F(p) = \int d^4x e^{ip\cdot x} S_F(x) \tag{6.134}$$

Putting it all together, we find

$$(\not{p} - m)S_F(p) = i\mathbb{I} \rightarrow S_F(p) = \frac{i}{\not{p} - m} = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \tag{6.135}$$

where we added a *smol*, pole-eliminating term in the last step. Returning to position space gives us

Definition 6.32 (Fermionic propagator)

$$S_F(x - y) = -i \lim_{\epsilon \to 0+} \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p} + m)e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{6.136}$$

[a] Who could've guessed?

Derivation 6.14 (Photon field propagator) Now we attack the photon field propagator. As it is bosonic, we denote it as $D_F^{\mu\nu}$, identical to the Feynman propagator save for the tensorial indices:

$$D_F^{\mu\nu}(y - y) = \langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle \tag{6.137}$$

The derivation is conceptually similar to that of the Feynman and fermion propagators. While we have not derived the (gauge) photon field equations of motion before, we can do so now by inserting the gauge EM Lagrangian in Equation 6.112 into Equation 2.26.

$$\partial_\mu F^{\mu\nu} + \frac{1}{\xi}\partial^\nu(\partial_\mu A^\mu) = 0 \tag{6.138}$$

Expanding $F^{\mu\nu}$ and simplifying yields

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) + \frac{1}{\xi}\partial^\nu(\partial_\mu A^\mu) = 0 \tag{6.139}$$

Now we move this to momentum space. The the Fourier transform of the photon field is

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\tilde{A}_\mu(k) \tag{6.140}$$

Using this and Equation 4.28, the equations of motion become

$$-k^2\tilde{A}^\nu(k) + \left(1 - \frac{1}{\xi}\right)k^\nu(k_\mu\tilde{A}^\mu(k)) = 0 \tag{6.141}$$

where k^μ is the 4-momentum. We define a so-called *kernel* that is inverse of the photon propagator, which satisfies

$$D^{-1}_{\mu\nu}(k)\tilde{A}^\nu(k) = 0 \rightarrow D^{-1}_{\mu\nu}(k) = -k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right)k_\mu k_\nu \tag{6.142}$$

It should not be surprising that the product of the propagator and the kernel must be the Kronecker tensor[a] δ^ρ_μ:

$$D^{-1}_{\mu\nu}(k)D^{\nu\rho}(k) = \delta^\rho_\mu \tag{6.143}$$

We conveniently assume the (tensorial) propagator to be of the form

$$D_{\mu\nu}(k) = A(k^2)g_{\mu\nu} + B(k^2)k_\mu k_\nu \tag{6.144}$$

Inserting this into Equation 6.143 yields

$$\left[-k^2 g_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) k_\mu k_\nu\right] \left[A(k^2) g^{\nu\rho} + B(k^2) k^\nu k^\rho\right] = \delta_\mu^\rho \tag{6.145}$$

This is a set of tensorial equations which have the solution

$$A(k^2) = -\frac{1}{k^2} \quad B(k^2) = -\frac{\xi - 1}{\xi} \frac{1}{k^4} \tag{6.146}$$

This gives the photon field propagator as

$$D_{\mu\nu}(k) = -\frac{1}{k^2}\left[g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right] \tag{6.147}$$

Do not forget that we still need to fix the gauge fully. We use the Feynman gauge, which is $\xi = 1$:

$$D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2} \tag{6.148}$$

> **Quote 6.6** Do not forghetti, die Kirche ruft uns arme Seelen zur Führung! Morgen um 15 Uhr bei da Hans Jesus Kirche
>
> *Julia Neumeister, 4 April 2025*

Transforming back into position space, we find

Definition 6.33 (Photon field propagator)

$$D_F^{\mu\nu}(y-y) = -i \lim_{\epsilon \to 0+} \int \frac{d^4x}{(2\pi)^4} \frac{g^{\mu\nu} e^{-ip\cdot(x-y)}}{p^2 + i\epsilon} \tag{6.149}$$

[a] How rude!

Finally, using Wick's second theorem and taking the Grassmann parity of Fermions into account, we can formulate the QED Feynman rules:

Theorem 6.17 (QED Feynman rules) For a given Feynman diagram in QED, the scattering amplitude matrix elements $\mathcal{M}_{fi}$ is constructed as follows:

QED Feynman rules (partial)

For each	Add to expression
Incoming and outgoing electron	$\bar{u}_\alpha(s,p)$ and $u_\alpha(s,p)$
Incoming and outgoing positron	$v_\alpha(s,p)$ and $\bar{v}_\alpha(s,p)$
Incoming and outgoing photon	$\epsilon^{*\mu}(\lambda,p)$ and $\epsilon^\mu(\lambda,p)$
Internal photon line	$\frac{-ig^{\mu\nu}}{p^2}$
Internal fermion line	$\frac{i(\not{p}+m)e^{-ip\cdot(x-y)}}{p^2-m^2}$
Internal photon loop	$\int d^4 l_n \frac{1}{(2\pi)^4} \frac{-ig^{\mu\nu}}{p^2}$
Internal femion loop	$\int d^4 l_n \frac{1}{(2\pi)^4} \frac{i(\not{p}+m)e^{-ip\cdot(x-y)}}{p^2-m^2}$
Vertex	$-ie\gamma^n_{\alpha\beta}$
Vertex	$(2\pi)^4 \delta^3(k_i - k_f)$[a]

In the order along the direction of the external leg arrows, the incoming and outgoing photon indices are μ and ν, and the incoming and outgoing fermion indices are α and β. The photon/fermion propagator index is n, and the incoming and outgoing 4-momenta are k_i and k_f. Each internal loop has a so-called *internal momenta* l_n.
Finally, before taking a well-deserved break, remove a factor of $(2\pi)^4\delta^3(p-q)$, where p and q are the *total* initial and detected momenta.

[a] This term enforces 4-momentum conservation.

One should note the following points:

- The symbols s and λ seen in the electrons, positrons and photons are actually indices, which are put into the bracket purely for ease of viewing.
- In QED, the previously seen factor C observes $C = k!$, and C and $1/k!$ cancel out.
- When the scattering involves more than one diagram, Wick's theorem will be used, in which the exchange of two fermion operators changes the sign of the expression as per the Grassman parity.

We can now draw QED Feynman diagrams. Unlike ϕ^4 theory, QED is not a toy model and concerns itself with real particles - in specific, fermions and photons. One can represent them in a Feynman diagram as follows:

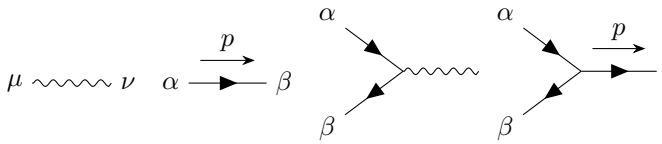


Figure 6.1: QED Feynman diagram elements. L-R: photon propagator, fermion propagator, vertex leading to photon propagator, vertex leading to fermion propagator

Thus, we have finished the canonical quantisation derivation of QED.

Part III

Path integrals

Chapter 7

Free fields

Quote 7.1 Juice, juice, orange juice...

Richard Feynman, playing the bongos, September 1981

While canonical quantisation is the more intuitive approach to developing field theories, it suffers from certain drawbacks, especially with respect to extra degrees of gauge freedom (as we have seen in deriving the QED Feynman rules). In developing more complicated field theories like QCD, we will make use of the other formalism: path integrals.

7.1 Path integrals in non-relativistic quantum mechanics

Unsurprisingly, the end goal in path integrals is the same as that in canonical quantisation, which is to derive the S-matrix S_{fi}, the transition amplitude $\mathcal{M}_{fi}$ and the Feynman rules of a given field theory that constructs $\mathcal{M}_{fi}$.
It is useful to recognise the relationship between S_{fi} and the unitary time evolution operator $U(t, t_0)$. When t is the final time t_f of some process and t_0 is the initial time t_i, one recovers the scattering matrix:

$$S_{fi} = U(t_f, t_i) \tag{7.1}$$

We can see why this reasoning is useful by investigating path integrals in innocent non-relativistic quantum mechanics. The central idea is that a particle in motion can and *will* take every possible trajectory or *path*. We postulate that each path contributes a factor of e^{iS} to the $U(t, t_0)$:

$$U(t, t_0) = \sum_{\text{all paths}} e^{iS} \tag{7.2}$$

where S is the action.
As is well known, in non-relativistic quantum mechanics, $U(t, t_0)$ is given by

$$U(t, t_0) = \langle f | e^{-iHT} | 1 \rangle \tag{7.3}$$

where 1 is the initial state[1] and $T = t - t_0$ is the time interval.

Derivation 7.1 (Time-slicing) One can solve the contribution to $U(t, t_0)$ of a certain path by slicing the path's time interval into *smol* time steps of ϵ. We use the *Lie product formula*:

Theorem 7.1 (Lie product formula) For any operators or square matrices $\hat{A}$ and $\hat{B}$, one has

$$e^{\hat{A}+\hat{B}} = \lim_{N\to\infty} \left(e^{\hat{A}/N} e^{\hat{B}/N} \right)^N = \lim_{N\to\infty} \left(e^{\hat{B}/N} e^{\hat{A}/N} \right)^N \tag{7.4}$$

where N is the so-called *Trotter number*.

[1] We have avoided writing i to prevent confusion with indices that will appear later.

Practically, this has an alternative formulation. Suppose that, instead of solving directly for $e^{\hat{A}+\hat{B}}$, we solve it segment by segment, solving first a $e^{\epsilon(\hat{A}+\hat{B})}$ for a *smol* ϵ and then calculate $(e^{\epsilon(\hat{A}+\hat{B})})^{1/\epsilon}$, where the segment $e^{\epsilon(\hat{A}+\hat{B})}$ has the form

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} + O(\epsilon^2) \tag{7.5}$$

Our good friend, the unitary time evolution operator, can then be approximated as

$$e^{-iHT} = \left(e^{-iH\epsilon}\right)^N \tag{7.6}$$

where $N = T/\epsilon$ is again the Trotter number.
From this, we are in a position to construct the generic time evolution operator $U(t, t_0)$. We write over a series of N steps:

$$\begin{aligned} U(t,t_0) =& \langle f, t_N | e^{-iH\epsilon} | N-1, t_{N-1}\rangle \cdots \langle 1, t_1 | e^{-iH\epsilon} | i, t_0\rangle \\ =& \int dx_{N-1}\cdots dx_1 \langle x_f, t_N | e^{-iH\epsilon} | x_{N-1}, t_{N-1}\rangle \cdots \langle x_1, t_1 | e^{-iH\epsilon} | x_i, t_0\rangle \end{aligned} \tag{7.7}$$

where we have integrated over all intermediate positions $dx_1 \cdots dx_{N-1}$[a] and $\epsilon = T/N$ is again a single time step.
Remark 7.1 Note that the second line is not a single integral but $N-1$ integrals.

[a] The initial and final positons x_0 and x_N are not integrated as they are fixed - remember that the path integral is integrating over a range of *possible* positions!

The Hamiltonian can be decomposed as

$$H = \frac{1}{2}p_i^2 + V(q_i) \tag{7.8}$$

where we recall p_i and q_i to be generalised momenta and coordinates.

Derivation 7.2 (One slice in phase space) We now want to evaluate one slice, represented by $\langle q_{i+1}, t_{i+1}| e^{-iH\epsilon} |q_i, t_i\rangle$. Importantly, as neither p_i nor q_i are scalars, we *cannot* simply say that $e^{-iH\epsilon} = e^{-i\epsilon(p_i^2/2+V(q_i))} = e^{-i\epsilon p_i^2/2}e^{-i\epsilon V(q_i))}$. A trick must be used to solve for the decomposed result:

Theorem 7.2 (Baker-Campbell-Hausdorff formula) Suppose one has the known matrices X and Y and the unknown matrix Z which satisfy $e^X e^Y = e^Z$. Z can be solved by

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots \tag{7.9}$$

where square brackets are commutators.

Amazingly, however, as ϵ is *smol*, Equation 7.9 yields convenient approximation

$$e^{-iH\epsilon} \approx e^{-i\epsilon p_i^2/2}e^{-i\epsilon V(q_i))} \tag{7.10}$$

which is *almost* identical to the result if p_i and q_i *were* scalars. We can then say that, for some arbitrary step i, that

$$\langle q_{i+1}, t_{i+1}| e^{-iH\epsilon} |q_i, t_i\rangle \approx \langle q_{i+1}, t_{i+1}| e^{-i\epsilon p_i^2/2}e^{-i\epsilon V(q_i))} |q_i, t_i\rangle \tag{7.11}$$

We exploit the following identity, which is a complete set of momentum eigenstates:

$$\int \frac{dp}{2\pi} |p\rangle\langle p| = 1 \tag{7.12}$$

Insertion gives

$$\langle q_{i+1}| e^{-i\epsilon\frac{p^2}{2}} e^{-i\epsilon V(q)} |q_i\rangle = \int \frac{dp}{2\pi} \langle q_{i+1}| e^{-i\epsilon\frac{p^2}{2}} |p\rangle\langle p| e^{-i\epsilon V(q)} |q_i\rangle \tag{7.13}$$

Both $V(q)$ and p^2 act diagonally in position space, and we find

$$\langle p|e^{-i\epsilon V(q)}|q_i\rangle = e^{-i\epsilon V(q_i)}\langle p|q_i\rangle = e^{-i\epsilon V(q_i)}\frac{e^{-ipq_i}}{\sqrt{2\pi}} \quad \langle q_{i+1}|e^{-i\epsilon\frac{p^2}{2}}|p\rangle = e^{-i\epsilon\frac{p^2}{2}}\langle q_{i+1}|p\rangle = e^{-i\epsilon\frac{p^2}{2}}\frac{e^{ipq_{i+1}}}{\sqrt{2\pi}} \tag{7.14}$$

Putting it all together, and considering multiple degrees of freedom labeled by $j = 1, \ldots, M$, we have

$$\langle q_{i+1}, t_{i+1}|\, e^{-iH\epsilon}\, |q_i, t_i\rangle = e^{-\epsilon V(q_i)} \int \prod_j^M \frac{dp_j^i}{2\pi} \exp\left[i\epsilon\left(p_j\frac{q_j^{i+1}-q_j^i}{\epsilon} - \frac{p_j^{i2}}{2}\right)\right] \tag{7.15}$$

where M is the degree of freedom (i.e. the number of dimensions/coordinates).

This is as far as we can go in phase space, and we are forced to move to momentum space.

Derivation 7.3 (Many slices in configuration space) Let us now go back to many slices so that we can recover the time evolution operator. Say that we have N slices. The integrals are then defined be over all intermediate positions and momenta:

$$\langle q_f, t_f|q_0, t_0\rangle = \int \prod_{i=1}^{N-1} dq_i \prod_{i=0}^{N-1}\left(\frac{dp_i}{2\pi}\right) \exp\left[i\epsilon\sum_{i=0}^{N-1}\left(p_i\frac{q_{i+1}-q_i}{\epsilon} - \frac{p_i^2}{2} - V(q_i)\right)\right] \tag{7.16}$$

The momenta p_i can now be integrated out. This is, up to phase, standard Gaussian integral, which can be evaluated as

$$\int_{-\infty}^{\infty} dp e^{-ap^2/2+bp+c} = \sqrt{2\pi/a}e^{b^2/2a+c} \tag{7.17}$$

In our case, this is

$$\langle q_f, t_f|q_0, t_0\rangle = \int \prod_{i=1}^{N-1} dq_i i\sum^{N}\epsilon\left(\frac{1}{2}\dot{q}_i^2 - V(q_i)\right) = \int \prod_{i=1}^{N-1} dq_i i\sum^{N}\epsilon L(q_i, \dot{q}_i) \tag{7.18}$$

We now make a shorthand that represents the integration over *all possible intermediate configurations* of the path $q(t)$ between the endpoints q_0 and q_f - i.e. the 'sum over all paths'. This is sometimes called the *integration measure*:

Definition 7.1 (Integration measure)

$$\mathcal{D}q = \prod_{i=1}^{N-1} dq_i \tag{7.19}$$

where N is once again the number of steps.

Thus, the transition amplitude becomes

$$\langle q_f, t_f|q_0, t_0\rangle = \int \mathcal{D}q e^{i\sum_{i=0}^{N-1}\epsilon L(q_i,\dot{q}_i)} \tag{7.20}$$

In the continuum limit, we send N to infinity and find

$$\langle q_f, t_f|q_0, t_0\rangle = \int \mathcal{D}q e^{iS} = \int \mathcal{D}q e^{i\int_{t_0}^{t_f} dtL(q,\dot{q})} \tag{7.21}$$

where S is the action. This verifies our postulate in Equation 7.2[a].

[a] Importantly, time ordering is not a concern as the integration the Lagrangian naturally preserves the time order. The same can be seen in the time steps in Equation 7.7.

Theorem 7.3 (Integration measure properties)

$$\int \mathcal{D}\phi = \phi(x) \tag{7.22}$$

$$\int \mathcal{D}\phi\phi = \frac{\phi^2(x)}{2} \tag{7.23}$$

> **Quote 7.2** The strict built-in ordering of times makes commutation relations between the operators irrelevant.
>
> *Niklas Beisert, 2017*

Now we turn this from quantum mechanics to a field theory:

- The Lagrangian L is replaced by the Lagrangian density $\mathcal{L}$.
- The coordinates q are replaced with the fields ϕ.
- From the last point, $\mathcal{D}q$ becomes $\mathcal{D}\phi$.

which gives, after introducing sensible limits

Theorem 7.4 (Time evolution of fields)

$$U(t,t_0) = \int_{\phi(t_0)}^{\phi(t)} \mathcal{D}\phi e^{i\int d^4x\mathcal{L}} \tag{7.24}$$

7.2 Generating functional

> **Quote 7.3** As any reader of Dirac knows, it is sometimes convenient to speak of a distribution as if it were a function.
>
> *Sidney Coleman and Jeffrey Mandula, in 'All Possible Symmetries of the S Matrix', 16 March 1967*

In the path integral formulation, we introduce, for every field $\phi(x)$, a classical external field $J(x)$ called a *source* that couples linearly to the field. This so-called source is more of a mathematical convenience than a physical entity, and it is significant in that it allows the creation of Green's functions.
Before introducing the main concepts of this section in formulae, we summarise the main ideas in text:

- The Lagrangian density (and by this, the action) is modified by adding a source term $J(x)\phi(x)$.
- The time evolution operator, which is an exponential of the Hamiltonian, is generalised as the almighty $Z(J(x))$ *generating functional* which, as we will see later, is so-called because it is used to generate Green's functions.
- Loosely speaking, the Green's function can be derived by differentiating the *generating functional* by the source and setting $J(x) = 0$ afterwards[2].

Definition 7.2 (Generating functional) The generating function is an exponential of the now-modified action:

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}(\phi(x))+J(x)\phi(x))} \tag{7.25}$$

Remark 7.2 But this looks a bit familiar, doesn't it?
If you share this opinion, you'd be right. As $Z(J(x))$ integrates over all possible field configurations, it is the quantum analogue of the well-known *partition function* in statistical physics. In fact, $Z(J(x))$ is simply called the *partition function* in some literature.
So far, this generating functional diverges into infinity. We thus introduce the *normalised generating functional* $Z_0[J]$:

[2] This is where one can appreciate the nature of the source as a mathematical convenience. Ultimately, QFT describes free or interacting fields instead of external influences. Hence, the source to zero as this external influence would have remained otherwise.

Definition 7.3 (Normalised generating functional)

$$Z_0[J] = \frac{Z[J]}{Z[0]} \tag{7.26}$$

Derivation 7.4 (Klein-Gordon field) We are now in a position to derive the (renormalised) generating functional of the Klein-Gordon field. Consider the Klein-Gordon Lagrangian we have seen in Equation 2.4:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

The generating functional is hence

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + i\int d^4x J(x)\phi(x)} \tag{7.27}$$

Let us analyse the exponential

$$i\int d^4x \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + i\int d^4x J(x)\phi(x) \tag{7.28}$$

Assuming fields vanish at infinite, we can employ integration by parts. This gives

$$\frac{1}{2}i\int d^4x\phi(-\partial^2 - m^2)\phi + i\int d^4x J(x)\phi(x) \tag{7.29}$$

Importantly, the field $\phi(x)$ can be decomposed into two parts, a classical (background) part and a quantum fluctuation part:

$$\phi(x) = \phi_c(x) + \phi_q(x) \tag{7.30}$$

where:

- $\phi_c(x)$ is chosen to satisfy the classical equation of motion involving the source

$$(\partial^2 + m^2)\phi_c(x) = -J(x) \tag{7.31}$$

- $\phi_q(x)$ is the quantum fluctuation around the classical solution.

This is the essence of the path integral method: quantum effects come from summing over fluctuations around classical paths. From Equation 7.31, the exponential reduces to

$$-\frac{1}{2}i\int d^4x\phi_q(\partial^2 + m^2)\phi_q + i\int d^4x J(x)\phi_c(x) \tag{7.32}$$

At the same time, it is easy to rewrite Equation 7.31 as

$$\phi_c(x) = \int d^4y D_F(x-y)J(y) \tag{7.33}$$

where, in our utter ignorance, we have accidentally rederived the Feynman propagator $D_F(x-y)$ in the path integral formulation.
Hence:

$$i\int d^4x J(x)\phi_c(x) = i\int d^4x d^4y J(x)D_F(x-y)J(y) \tag{7.34}$$

Turning back to the generating functional, we now have

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x\phi_q(\partial^2+m^2)\phi_q + i\int d^4x d^4y J(x)D_F(x-y)J(y)} \tag{7.35}$$

However, we are still integrating over the whole field ϕ. As elucidated earlier, ϕ_c is nothing but a measly classical background, and the real object of interest is ϕ_q. That is to say, it is ϕ_q instead of the

whole ϕ that we now want to integrate over. This is a linear change of variables, so the integration measure reduces simply:

$$\mathcal{D}\phi = \mathcal{D}(\phi_c + \phi_q) = \mathcal{D}\phi_q \tag{7.36}$$

Quote 7.4 Ha!

Nikos Stavros, in Red Alert 1

The generating functional is then

$$Z[J] = \int \mathcal{D}\phi_q e^{i\int d^4x \phi_q(\partial^2+m^2)\phi_q + i\int d^4x d^4y J(x) D_F(x-y) J(y)} \tag{7.37}$$

Now consider the case where the source is zero. As the source is the cause of the classical field ϕ_c, $\phi_c = 0$ when $J = 0$, and we can effectively rewrite $\phi(x)$ as $\phi_q(x)$. Hence, we can write

$$Z[J] = Z[0] e^{i\int d^4x d^4y J(x) D_F(x-y) J(y)} \tag{7.38}$$

We now derive, for the Klein-Gordon field, the expression of the normalised generating function $Z_0[J]$ in Equation 7.26. Amazingly, this is already implied in Equation 7.38:

$$Z_0[J(x)] = e^{i\int d^4x d^4y J(x) D_F(x-y) J(y)} \tag{7.39}$$

By the definition of the exponential function, we can easily see that $Z_0[0]$ is always 1. This verifies that our result is indeed normalised.

7.3 Green's function

We are now almost in a position to derive the full Green's function from the generating function we have just derived. This involves differentiating $Z_0[J]$, which is a *functional*, by sources $J(x_n)$, which are *functions* themselves. Hence, standard derivatives fail, and we need to introduce the so-called *functional derivative* δ, which is the analogue of a normal derivative for a function.

Derivation 7.5 (Motivation of the functional derivative) Suppose we have a functional

$$J[f] = \int_a^b L(x, f(x), f'(x))dx \tag{7.40}$$

where we add a *smol* δf to $f(x)$. L then becomes

$$L(x, f(x), f'(x)) \to L(x, f + \delta f, f' + \delta f) \tag{7.41}$$

where $\delta f'$ is the *variation* of the *derivative*. The change in J, due to this variation, is then

$$\begin{aligned} \delta J &= \int_a^b \left(\frac{\partial L}{\partial f}\delta f(x) + \frac{\partial L}{\partial f'}\frac{d}{dx}\delta f(x) \right) dx \\ &= \int_a^b \left(\frac{\partial L}{\partial f} - \frac{d}{dx}\frac{\partial L}{\partial f'} \right) \delta f(x) dx + \frac{\partial L}{\partial f'}(b)\delta f(b) - \frac{\partial L}{\partial f'}(a)\delta f(a) \end{aligned} \tag{7.42}$$

where, importantly, $(\delta f)'$ is the *derivative* of the *variation*.

The functional derivative satisfies the following properties:

Theorem 7.5 (Functional derivative properties)

$$\frac{\delta\phi(y)}{\delta\phi(x)} = \delta(x-y) \tag{7.43}$$

$$\frac{\delta}{\delta\phi(x)}(\alpha(y)\beta(z)) = \frac{\delta\alpha(y)}{\delta\phi(x)}\beta(z) + \alpha(x)\frac{\delta\beta(z)}{\delta\phi(x)} \tag{7.44}$$

Derivation 7.6 (Green's function) We again begin with the (unnormalised) generating functional for a (free or interacting) scalar field theory:

$$Z(J) = \int \mathcal{D}\phi \exp\left(iS[\phi] + i\int d^4x J(x)\phi(x)\right) \tag{7.45}$$

Taking the functional derivative of $Z(J)$ with respect to the source $J(x)$ brings down factors of $\phi(x)$ from the exponential:

$$\frac{\delta Z(J)}{\delta J(x)} = i\int \mathcal{D}\phi\phi(x) \exp\left(iS[\phi] + i\int d^4x J(x)\phi(x)\right) \tag{7.46}$$

Now, for a system of n 4-positions, we have an n-point function, and hence n sources. Repeating the functional derivative for every source gives us

$$\frac{\delta^n Z(J)}{\delta J(x_1)\cdots\delta J(x_n)} = i^n \int \mathcal{D}\phi\phi(x_1)\cdots\phi(x_n) \exp\left(iS[\phi] + i\int d^4x J(x)\phi(x)\right) \tag{7.47}$$

Recall that it is the Green's function we want. This is the *vacuum* expectation value of the time-ordered product. We hence set the source as zero:

$$\left.\frac{\delta^n Z(J)}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} = i^n \int \mathcal{D}\phi\phi(x_1)\cdots\phi(x_n)e^{iS[\phi]} \tag{7.48}$$

Normalising the generating functional gives

$$\frac{1}{i^n}\left.\frac{\delta^n Z_0(J)}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} = \frac{1}{Z[0]}\int \mathcal{D}\phi\phi(x_1)\cdots\phi(x_n)e^{iS[\phi]} \tag{7.49}$$

Now let us connect this to the vacuum expectation value. In (classical) probability theory, the definition of the expectation value is well-known as

$$\langle f\rangle = \frac{1}{Z}\int f(x)e^{-S(x)}dx \tag{7.50}$$

From Equation 7.24, we see that time evolution and expectation values are calculated as weighted sums over all field configurations, with weight $e^{iS[\phi]}$. The VEV of a time-ordered product of field operators is a probabilistic average over all possible quantum fluctuations of the vacuum state fields. Hence, like Equation 7.50, we can construct the expression

$$\langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle = \frac{\int \mathcal{D}\phi\phi(x_1)\cdots\phi(x_n)e^{iS[\phi]}}{\int \mathcal{D}\phi e^{iS[\phi]}} = \frac{1}{Z[0]}\int \mathcal{D}\phi\phi(x_1)\cdots\phi(x_n)e^{iS[\phi]} \tag{7.51}$$

where, again, $Z[0]$ normalises the expression.
Hence, we finally see that Equation 7.49 is exactly the Green's function in disguise:

Theorem 7.6 (Propagator-generating functional relation)

$$G_0^{(n)} = \langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle = \frac{1}{i^n}\left.\frac{\delta^n Z_0[J]}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} \tag{7.52}$$

This then allows us to reconstruct the generating functional in terms of the propagator:

$$Z[J] = \sum_{n=0}^{\infty}\int d^dx_1\ldots d^dx_n\langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle J(x_1)\cdots J(x_n) \tag{7.53}$$

Chapter 8

Interacting fields I: ϕ^4 theory

In the last chapter, we left off from Equation 7.52, which can be written explicitly as[1]:

$$\langle T[\phi_1 \cdots \phi_n] \rangle = \left. \frac{\int \mathcal{D}\phi \phi_1 \cdots \phi_n e^{iS[\phi,J]}}{\int \mathcal{D}\phi e^{iS[\phi,J]}} \right|_{J=0} \tag{8.1}$$

This chapter will see us tackling this expression.

8.1 Generating functional

While innocent-looking, the expression of our so-called generating functional is actually quite unwieldy. Similar to what we did in canonical quantisation, we split the Lagrangian in the numerator (i.e. the generating functional $Z[J]$) into two parts, the (quadratic) free part $\mathcal{L}_F$ which we can single out from the integration measure and the interacting part $\mathcal{L}_I$ with a dependence on ϕ.

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^dx(\mathcal{L}_F+\mathcal{L}_I+J\phi)} = \int \mathcal{D}\phi e^{i\int d^dx(\mathcal{L}_F+J\phi)} e^{i\int d^dx\mathcal{L}_I} \tag{8.2}$$

where we have a third implicit term $\mathcal{L}_S = J(x)\phi(x)$, which is the source part.
Now we evaluate this expression. The free part of the Lagrangian is nothing but the Klein-Gordon Lagrangian, whose generating functional we have already derived in Equation 7.39.

Derivation 8.1 (Interaction part) For the interaction term, we rewrite it in terms of functional derivatives,

$$e^{iS_I[\phi]} = e^{i\int d^dx\mathcal{L}_I[\phi]} \tag{8.3}$$

Here a cute trick can be used. Consider the expression $\frac{\delta}{\delta J(x)} e^{i\int J\phi}$. By evaluating the partial derivative, we can write

$$\frac{\delta}{\delta J(x)} e^{i\int J\phi} = i\phi(x) e^{i\int J\phi} \tag{8.4}$$

If we eventually set the source to zero (which, as we have seen, we will), the exponential term is nothing but unity. Hence, assuming that the source is ultimately set to zero, we can always make the substitution

$$\phi(x) = -\frac{\delta}{\delta J(x)} \tag{8.5}$$

Thus, the interaction part becomes

$$e^{i\int d^dx\mathcal{L}_I} = e^{i\int d^dx\mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} \tag{8.6}$$

which we can single out from the integral due to the lack of dependence on ϕ.

[1] At first glance this might seem slightly confusing. Note that the indexless ϕ is not standalone but is to be read as a part of the integration measure $\mathcal{D}\phi$.

Hence, the generating functional is

$$Z[J] = e^{i\int d^dx \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} e^{-\frac{i}{2}\int d^dx d^dy J(x) D_F(x-y) J(y)} \tag{8.7}$$

The free part of the ϕ^4 Lagrangian is identical to that of the Klein-Gordon Lagrangian, and this is why we have already re-derived our good friend, the Feynman propagator in the last chapter. However, it remains for us to derive the vertex which arises from the new ϕ^4 quartic interacting term. One strength of the path integral formulation lies in the fact that one can directly read off the vertex term from the Lagrangian. We will do so now:

Derivation 8.2 (Lazy vertex) Let us take ϕ^4 theory as an example, where $\mathcal{L}_I$ is actually $-\lambda\phi^4/4!$. Immediately, we can see that two terms have nothing to do with the vertex:

- ϕ^4 denote field self-interactions.
- 4! is the combinatorial factor.

We then have $-\lambda$ left. However, there is also a prefactor i brought about by the generating functional. If we combine the two, the resulting expression $-i\lambda$ is exactly the vertex factor in the Feynman rules.

However, this is not very rigorous, and we will proceed with the full derivation in the next section.

8.2 Perturbative expansion

Let us again consider a $2 \to 2$ process in ϕ^4 theory. The compact form of the Green's function reads

$$\langle T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle = \left.\frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)e^{iS[\phi,J]}}{\int \mathcal{D}\phi e^{iS[\phi,J]}}\right|_{J=0} \tag{8.8}$$

The denominator is merely the generating functional $Z[J]$ (in this case $Z[0]$), while the numerator can be realised as the generating functional differentiated with respect to $x_1, \cdots, x_4$.

$$\left\langle T\left[\prod_i^4 \phi(x_i)\right]\right\rangle = \frac{1}{Z[0]}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\int \mathcal{D}\phi e^{iS[\phi,J]}\Big|_{J=0} \tag{8.9}$$

Note that while $\int \mathcal{D}\phi e^{iS[\phi,J]}$ is essentially $Z[J]$, we cannot cancel it with $1/Z[0]$ due to the existence of the differential operators $\delta^4/\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)$. Instead, we decompose the action as per Equation 8.2.
Substituting the integration results from Equation 8.7:

$$\left\langle T\left[\prod_i^4 \phi(x_i)\right]\right\rangle = \frac{1}{Z[0]}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} e^{i\int d^dx \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} e^{-\frac{i}{2}\int d^dx d^dy J(x) D_F(x-y) J(y)}\Big|_{J=0} \tag{8.10}$$

where d is the number of dimensions in the spacetime.

Derivation 8.3 ($2 \to 2$ processes) We cannot calculate the Green's function with this result alone. Rather, we perform a perturbative expansion of the exponential. We start with $e^{i\int d^dx \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} e^{-\frac{i}{2}\int d^dx d^dy J(x) D_F(x-y) J(y)}$, which is the evaluated form of $Z[J]$.
Recall from Part II that the interacting Lagrangian in ϕ^4 theory is

$$\mathcal{L}_I = -\frac{\lambda}{4!}\phi^4 \tag{8.11}$$

where, importantly, λ is the almighty coupling constant. The two exponentials then each expand to a series of polynomials:

$$\left\langle T\left[\prod_i^4 \phi(x_i)\right]\right\rangle = \frac{1}{Z[0]}\frac{\delta^4}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)\,\delta J(x_4)}\left(\sum_n \frac{1}{n!}\left(\frac{\lambda}{4!}i\int d^dy \frac{\delta^4}{i\delta J(y)^4}\right)^n\right)\times \left(\sum_m \frac{1}{m!}\left(-\frac{i}{2}\int d^dx d^dy J(x)D_F(x-y)J(y)\right)^m\right)\Bigg|_{J=0} \tag{8.12}$$

where m and n are the powers.
Both indices m and n go up to infinity. However, we are actually not too interested in them on their own. Rather, we inspect the order of λ which, like in canonical quantisation, denotes the number of vertices. This puts us in a position to return to Equation 5.39.
Merely at first order, we have the terrible-looking expression

$$\left\langle T\left[\prod_i^4 \phi(x_i)\right]\right\rangle = \frac{Z_2[0]}{Z[0]}\frac{\delta^4}{\delta J(x_1)\,\delta J(x_2)\,\delta J(x_3)\,\delta J(x_4)}\left(\frac{1}{2!}\left(-\frac{i}{2}\int d^dx d^dy J(x)D_F(x-y)J(y)\right)^2 - \frac{i\lambda}{4!}\int d^dz \frac{1}{4!}\frac{\delta^4}{\delta J(z)^4}\left(-\frac{i}{2}\int d^dx d^dy J(z)D_F(x-y)J(y)\right)^4 + \mathcal{O}\left(\lambda^2\right)\right) \tag{8.13}$$

This will produce a series of derivatives, many of which are identical. In the interest of brevity, we will jump through this hard part and arrive at the conclusion that one can write

$$\frac{1}{Z[0]} = \frac{1}{1+\lambda D} \quad \text{the rest} = A + \lambda(B + C + AD) + O(\lambda^2) \tag{8.14}$$

With some effort, A, B, C and D can be solved. We first look at A:

$$A = G^{(0)}_{2+2} = D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3) \tag{8.15}$$

This is *exactly* what we got in Part I using Wick's third theorem! Now let us investigate the other terms:

$$\lambda B = -i\lambda\int d^dx D_F(x-x_1)\,D_F(x-x_2)\,D_F(x-x_3)\,D_F(x-x_4) \tag{8.16}$$

$$\lambda C = -\frac{i\lambda}{2}\sum_{P(ijkl)} D_F(x_i-x_j)\int d^dx D_F(x-x)D_F(x-x_k)\,D_F(x-x_l) \tag{8.17}$$

where $P(ijkl)$ permutes over all possible indices i, j, k and l (i.e. 1 and 2).

$$\lambda AD = -\frac{i\lambda}{8}\int d^dx D_F(x-x)D_F(x-x)\sum_{P(ijkl)} D_F(x_i-x_j)\,D_F(x_k-x_l) \tag{8.18}$$

Again, these results agree with what we got from canonical quantisation. Putting it all together, we have

$$\langle T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle = \frac{A+\lambda(B+C+AD)}{1+\lambda D} \tag{8.19}$$

One can expand the RHS factorial, yielding

$$\langle T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle = (A+\lambda(B+C+AD))(1-\lambda D)+O(\lambda^2) = A+\lambda(B+C)+O(\lambda^2) \tag{8.20}$$

We hence recover

$$G^{(1)}_{2+2} = \lambda(B+C) \tag{8.21}$$

If one is masochistic, it is possible to write out the full form of the propagator via this expansion and set the sources to zero. In principle, this recovers the very same ϕ^4 Feynman rules as we have seen before.

Already, we could give some comments comparing path integrals and canonical quantisation:

- So far, we have categorically avoided using Wick's theorem, and we will not use it even if we write out the full form of the propagator. As we have seen in the $2 \to 2$ process, all possible contractions are 'automatically' generated (i.e. arise naturally) through differentiating the (Gaussian[2]) path integral. Here we see the advantage of the path integral formulation.
- We note that the expansion is very unwieldy in position space, and the expansion is again better carried out in momentum space. Here we see the disadvantage of the path integral formulation.

8.3 Effective action

Our previous encounter with amputated propagators was brief. Now, with path integrals in our hands, we can investigate it and its related concepts more sophisticatedly. Starting from the full propagator or the *dressed propagator*, one can increasingly simplify it:

- If we preserve the connected parts of the Feynman diagram only, the Green's functions reduce to *connected Green's functions*, which is generated by the so-called *free energy* $W[J]$.
- If we further remove (i.e. amputate) the external (leg) propagators[3], we are left with the previously seen amputated propagators. This extracts the core interaction structure.
- If we remove even the reducible parts of the Feynman diagram, we are left with the *one-particle-irreducible* (*1PI* or *OPI* in short) *Green's functions*, which describe fundamental interaction vertices beyond the classical action (e.g. loops).
- Conversely, if we remove all OPI Green's functions from the full propagator, we find the *bare propagator* G_0.

> **Quote 8.1** Help me, OPI-Wan. You're my only hope.
>
> *Star Wars, 1977*

Dear reader (yes, you), chapters ago, you learned about amputated propagators in Equation 5.43. Now they beg you to help them in their struggle against path integrals.
We first define the *free energy*, which is simply the generating functional of connected Green's functions.

Definition 8.1 (Free energy)

$$W[J] = -i \ln(Z[J]) \tag{8.22}$$

Expanding $W[J]$ in terms of $J(x)$ provides the connected n-point Green's functions G_c

Definition 8.2 (Connected Green's function)

$$G_c = \langle T[\phi(x_1) \cdots \phi(x_n)] \rangle_c = \left. \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} \tag{8.23}$$

Thus, $W[J]$ generates the connected Green's functions. To cement this in our memories, we note that they are distinct from normal Green's functions, which, for n points, are given by

$$G = \left. \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} \tag{8.24}$$

from which we can amputate our poor propagators.
The so-called *classical field* φ is defined as the expectation value of the quantum field, or essentially the mean field over all configurations, in the presence of the source. It is given by taking the functional derivative of W with respect to J:

[2] Gaussian integrals inherently sum over all possible contractions.
[3] Whether one does so in the full Green's function or the connected Green's function makes no difference, and both result in the amputated propagator.

Definition 8.3 (Classical field)

$$\varphi = \langle \phi(x) \rangle = \frac{\delta W[J]}{\delta J(x)} \tag{8.25}$$

This so-called classical field is distinct from the classical field ϕ_c from the last section, and the two converge only when there are no quantum corrections.

Before we proceed, a few comments are in order:

- Our classica field contains all loop corrections as well as the source.
- If we set the source to zero, our classical field φ reduces to the vacuum expectation value $\langle \phi \rangle$.
- At tree level, our classical field becomes the classical part of the field ϕ_c that we get when we decompose the field like in Equation 7.30.
- At both zero source and tree level, the classical field is zero[4], with the exception of spontaneously symmetry-breaking theories we will see by the end of this chapter, where it could take values of equal magnitude and opposite signs.

The *effective action* is then the Legendre transform of $W[J]$:

Definition 8.4 (Effective action)

$$\Gamma[\varphi] = W[J] - \int d^d x J(x)\varphi(x) \tag{8.26}$$

where $J(x)$ is understood as a functional of $\varphi(x)$ through the inversion of $\varphi(x) = \delta W/\delta J$.

From the effective action, one can derive a galaxy of useful quantities. For example, if one is to assume a constant classical field φ_{const}, one gets the sign-flipped *effective potential* V_{eff} at that constant field, multiplied by the spacetime volume:

Definition 8.5 (Effective potential) The effective potential is simply the renormalised potential at a constant field, given by

$$\Gamma[\varphi_{\text{const}}] = -V_{\text{eff}} \int d^d x \tag{8.27}$$

where the superscript d is the spacetime dimension.

Now consider derivatives against the classical field φ. The first functional derivative of $\Gamma[\varphi]$ gives the source. This is nothing but the equation of motion for our so-called classical field[5]:

$$\frac{\delta \Gamma}{\delta \varphi} = -J \tag{8.28}$$

The second functional derivative of $\Gamma[\varphi]$ gives the *inverse propagator*:

$$\frac{\delta^2 \Gamma[\varphi]}{\delta\varphi(x)\delta\varphi(y)} = (G_{\text{full}})^{-1}(x, y) \tag{8.29}$$

where $G_{\text{full}}(x, y)$ is the full propagator or the *dressed propagator* including quantum corrections.
Any higher-order derivatives of $\Gamma[\varphi]$ yield the OPI Green's functions G_{OPI}, sometimes also denoted as γ:

Definition 8.6 (OPI Green's function)

$$G_{\text{OPI}} = \left. \frac{\delta^n \Gamma[\varphi]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} \tag{8.30}$$

In other words, $\Gamma[\varphi]$ is the generating functional of OPI Green's functions. For example, $\Gamma^{(4)}$ represents the four-point interaction vertex.

[4] Recall that when we *now* set the source to zero, we have $\phi_c = 0$.

[5] This also justified why our so-called classical field is so-named. Its behaviour is governed by an action principle due to lacking any quantum corrections.

Remark 8.1 The amputated Green's function essentially represents the sum of OPI diagrams that make up the full vertex function.
Remark 8.2 Importantly, the term 'one-particle-irreducible' does not describe interactions involving only one particle. Rather, a Feynman diagram is called OPI if it cannot be split into two separate diagrams by cutting a single internal propagator. This means:

- The diagram remains connected if any single internal propagator is removed.
- It does not factorise into two separate diagrams through a single-particle exchange.

This is in contrast to one-particle-reducible (OPR) diagrams, which can be split by cutting a single propagator, meaning they contain a disconnected propagator that acts as a bridge.
Let us summarise what we have so far:

- We have 4 main quantities of which we can take functional derivatives - the classical action and the 3 we have seen in this section.
- They are functionals of either the source $J(x)$ or the field $\phi(x)$.
- In an n-point QFT process, one has a series of positions $x_1 \cdots x_n$.

We go over them one by one:

- **Classical action $S[\phi]$:** Functional derivatives are taken w.r.t. $\phi(x)$.
  - The first derivative simply generates the classical equations of motion.
  - Higher derivatives against the n points generate vertices in the classical theory.
- **Generating functional $Z[J]$:** Functional derivatives are taken w.r.t. $J(x)$ and generate vacuum expectation values (i.e. correlation functions or Green's functions).
- **Free energy $W[J]$:** Functional derivatives are taken w.r.t. $J(x)$ and generate connected Green's functions.
- **Effective action $\Gamma[\phi]$:** Functional derivatives are taken w.r.t. $\phi(x)$.
  - The first derivative generates the master Dyson-Schwinger equation in Equation 8.38 which incorporate loop corrections[6].
  - The second derivative generates the inverse propagator.
  - Higher derivatives against the n points generate the OPI Green's functions.

One final, albeit important, remark concerns the so-called *self-energy* Σ, which we have seen to be instrumental in the Dyson equation we will use in renormalisation.

Definition 8.7 (Self-energy) We define the self-energy of a particle as the energy that a particle has due to its interaction and its environment[a]. Mathematically, it is the sum of all its OPI *two-point* diagrams:

$$\Sigma(p) := \sum G_{\text{OPI}}(p) \tag{8.31}$$

[a] i.e. the part of the total energy that relates the particle back to itself

Remark 8.3 It is worth noting that self-energy is not actually energy. Rather, it represents corrections to the full propagator arising from the interaction between a particle and its environment. This is analogous to the original concept of self-energy in electromagnetism, which refers to the energy required to assemble a charged particle due to its own electromagnetic field.
So far, we have seen quite a few gimmicks that we can exploit from the effective action formalism. However, this is not the end of the story. Rather, as we will see in the rest of this chapter, the effective action gives rise to two fundamental results.

[6] Hence the name 'effective action' is justified.

8.4 Dyson-Schwinger equations

An interesting analogy exists between the action S and the generating functional Z. Consider a *smol* variation of the field (i.e. a gauge transformation). As this is nothing a change of variables in the path integral, the generating functional in Equation 7.25 is invariant:

$$\delta Z = \int \mathcal{D}\phi \frac{\delta}{\delta\phi(x)} e^{i\int d^4x(\mathcal{L}(\phi(x))+J(x)\phi(x))} = 0 \tag{8.32}$$

This is analogous to the action principle, which states that the action is invariant under a *smol* variation of coordinates[7]. Also by analogy to the action, we can, using the gauge invariance of the generating functional, derive a series of equation that look like equations of motion.

Derivation 8.4 (Dyson-Schwinger equation) We first evaluate the functional derivative inside the integral

$$\frac{\delta}{\delta\phi(x)} e^{i\int d^4y(\mathcal{L}(\phi(y))+J(y)\phi(y))} = \left(i\frac{\delta\mathcal{L}}{\delta\phi(x)} + iJ(x)\right) e^{i\int d^4y(\mathcal{L}+J\phi)} \tag{8.33}$$

This gives

$$\delta Z = \int \mathcal{D}\phi \left(i\frac{\delta\mathcal{L}}{\delta\phi(x)} + iJ(x)\right) e^{i\int d^4y(\mathcal{L}+J\phi)} = 0 \tag{8.34}$$

By realising that the action is nothing but the 4-integral of the Lagrangian (density), we can write

$$\frac{\delta\mathcal{L}}{\delta\phi(x)} = \frac{\delta S}{\delta\phi(x)} \tag{8.35}$$

Thus

$$\delta Z = i\int \mathcal{D}\phi \frac{\delta S}{\delta\phi(x)} e^{iS[\phi]+i\int J\phi} + iJ(x)\int \mathcal{D}\phi e^{iS[\phi]+i\int J\phi} = 0 \tag{8.36}$$

Factoring off i on both sides and recognising that $Z[J] = \int \mathcal{D}\phi e^{iS+i\int J\phi}$, this becomes

$$\frac{\delta S}{\delta\phi(x)}[\phi]\, Z[J] = -J(x)Z[J] \tag{8.37}$$

Recalling the cute trick that was Equation 8.5, we replace ϕ by functional derivatives of J:

Theorem 8.1 (Dyson-Schwinger equation)

$$\frac{\delta S}{\delta\phi}\left(-i\frac{\delta}{\delta J}\right) Z[J] = -J(x)Z[J] \tag{8.38}$$

This is the master equation for the rest of the Dyson-Schwinger equations, which are infinite. The Dyson-Schwinger equations for propagators are derived by performing functional derivatives of the source J.

An alternate form of this master equation exists, which involves the quantities we derived in the last section. Recall from Equation 8.25 that the classical field is the expectation value of a field and is related to the free energy W in Equation 8.22. By inserting the field expectation/classical field it into Equation 8.38, we can rewrite it in terms of the free energy W:

$$\frac{\delta S}{\delta\phi}\left(\frac{\delta W}{\delta J} + \frac{\delta}{\delta J}\right) = -J(x) \tag{8.39}$$

Using a Legendre transformation and inserting Equation 8.28, a dependence on the effective action Γ in Equation 8.26 can also be acquired:

$$\frac{\delta\Gamma}{\delta\varphi(x)} + \frac{\delta S}{\delta\varphi(x)}\left(\varphi(x) + \frac{\delta^2 W}{\delta J(x)\delta J(y)} + \frac{\delta}{\delta J(y)}\right) = 0 \tag{8.40}$$

from which the Dyson-Schwinger equations for propagators are derived by taking derivatives of the classical field φ.

[7] Remember that in QFT, we have replaced 4-coordinates with 4-fields.

Derivation 8.5 (Dyson equation) We will now derive the Dyson-Schwinger equation of propagators. We perform a single functional derivative with respect to the source:

$$\begin{aligned} -i\frac{\delta}{\delta J(y)}\left[\frac{\delta S}{\delta\phi(x)}\left[-i\frac{\delta}{\delta J}\right]Z[J]\right] &= -i\frac{\delta}{\delta J(y)}\left[-J(x)Z[J]\right] \\ &= -i\left[-\delta(x-y)Z[J] - J(x)(-i)\frac{\delta Z}{\delta J(y)}\right] \end{aligned} \tag{8.41}$$

Now we set the source to zero:

- The LHS becomes a time-ordered two-point function.
- The second LHS term vanishes.

Hence:

$$\langle T\phi(x)\phi(y)\rangle = -i\frac{\delta}{\delta J(y)}\left[-J(x)Z\right]\Big|_{J=0} \tag{8.42}$$

$$\frac{1}{Z[0]}\left(-i\frac{\delta}{\delta J(x)}\right)\left(-i\frac{\delta}{\delta J(y)}\right)Z[J]\Big|_{J=0} = -i\left[-\delta(x-y)Z[0]\right] \tag{8.43}$$

$$Z[0]\left\langle\frac{\delta S}{\delta\phi(x)}\phi(y)\right\rangle = i\delta(x-y)Z[0] \tag{8.44}$$

Finally, we see that the two-point Dyson-Schwinger equation or the Dyson-Schwinger equation for propagators in

Theorem 8.2 (Dyson-Schwinger equation for propagators)

$$\left\langle\frac{\delta S}{\delta\phi(x)}\phi(y)\right\rangle = i\delta(x-y) \tag{8.45}$$

As an example, let us see what this implies in ϕ^4 theory. Let us insert the ϕ^4 Lagrangian in Equation 5.1 and recognise that $\delta(x-y)$ becomes $\delta^4(x-y)$ due to dimensionality:

$$\left\langle\left(-\partial_x^2 - m^2\right)\phi(x)\phi(y)\right\rangle - \frac{\lambda}{3!}\left\langle\phi(x)^3\phi(y)\right\rangle = i\delta^4(x-y) \tag{8.46}$$

We now rewrite this expression by introducing two objects:

- The (now full) propagator $G(x,y) \equiv \langle T[\phi(x)\phi(y)]\rangle$
- The OPI insertion Σ, given by

$$\langle\phi(x)^3\phi(y)\rangle_{\text{connected}} \to 3\int d^4z d^4w G(x,z)\Sigma(z,w)G(w,y) \tag{8.47}$$

  where the factor of 3 comes from choosing one of the three $\phi(x)$ lines to hook into the OPI two-point subdiagram.

The equation then reads

$$\left(-\partial_x^2 - m^2\right)G(x,y) - \int d^4z\Sigma(x,z)G(z,y) = i\delta^4(x-y) \tag{8.48}$$

By recognising that the bare propagator $G_0(x,y)$, which is the Feynman propagator we have seen in Equation 5.37, satisfies

$$G_0^{-1}(x,y) = \left(-\partial_x^2 - m^2\right)\delta(x-y) \tag{8.49}$$

We can again rewrite the equation as

$$\left(G_0^{-1} - \Sigma\right)G = i \tag{8.50}$$

which, after a Fourier transformation to momentum space, is

$$\left(G_0^{-1}(p) - \Sigma(p)\right)G(p) = i \tag{8.51}$$

Rearranging gives us the so-called *Dyson equation*:

Theorem 8.3 (Dyson equation)

$$G_{\text{full}} = G_0 + \Sigma = \frac{i}{p^2 - m^2 - \Sigma(p) + i\epsilon} \tag{8.52}$$

where we have clearly labelled G as the full propagator G_{full}.

If we are to interpret this equation a geometric series (which we will), it essentially decomposes the full propagator into the bare and dressed propagators. Equivalently, one can see this as summing all insertions of the OPI self-energy. The expansion reads:

$$\begin{aligned} D_F(p) = &\underbrace{\frac{i}{p^2 - m_0^2 + i\epsilon}}_{0 \text{ loops}} + \underbrace{\frac{i}{p^2 - m_0^2 + i\epsilon}\left(-i\Sigma\left(p^2\right)\right)\frac{i}{p^2 - m_0^2 + i\epsilon}}_{1 \text{ loop}} + \\ &\underbrace{\frac{i}{p^2 - m_0^2 + i\epsilon}\left(-i\Sigma\left(p^2\right)\right)\frac{i}{p^2 - m_0^2 + i\epsilon}\left(-i\Sigma\left(p^2\right)\right)\frac{i}{p^2 - m_0^2 + i\epsilon}}_{2 \text{ loops}} + \cdots \end{aligned} \tag{8.53}$$

where the terms go up to infinity.
But what does this physically mean? The key to interpreting this result lies in the realisation that our assumption in Part I that a propagator is nothing but a single line has been a lie all along. The single-line propagator we have grown to know and love is nothing but the *bare propagator* - that is to say, the simplest possible version of it. In contrast, when we go to higher loop orders, we have so-called *dressed propagators*, which is the result when one attaches internal loops to a bare propagator. That is to say, like the vertex, the propagator *cannot* be exactly solved either - it, too, must be derived perturbatively.
With this realisation, we can make sense of our result:

- The n^{th} term has n loops sandwiched among (i.e. multiplied by) $n+1$ internal lines.
- The LHS is the dressed Feynman propagator, and the first RHS term is the bare Feynman propagator.
- The rest of the RHS are OPI Feynman propagators, collectively known as self-energy, relate the dressed and bare Feynman propagators.

The usefulness of this section and the previous one will not be immediately obvious. However, they will prove essential in Part IV.

8.5 Spontaneous symmetry breaking

> **Quote 8.2** Flatfish evolved to survive on ocean bed. And basically, they moved their eyes to one side of the body. (...) Halibuts have their eyes on the right side of the body, while flounders have their eyes on the left side of the body. Why is that? We don't know. But they are symmetric.
>
> *Emilia Szymańska, 13 September 2024*

Outside of the convenient tricks we introduced in the last section, another important implication of the effective action formalism is the spontaneous symmetry breaking. We introduce spontaneous symmetry breaking via the motivating example of a modified ϕ^4 theory. If one replaces the mass coupling m^2 in the Lagrangian with some negative parameter $-\mu^2$, we find

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{8.54}$$

Unlike standard ϕ^4 theory, this model exhibits a funny effect upon closer inspection. As we have learned

in Part I, a Lagrangian can be split into kinetic terms and the potential, which is

$$V = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{8.55}$$

It is intuitive to see that this potential has two minima $\pm v$, where v is the vacuum expectation value of ϕ:

$$v = \pm\sqrt{\frac{6}{\lambda}}\mu \tag{8.56}$$

This minima is completely fixed in that it has no dependence on the 4-coordinates. It is then possible to decompose the field as

$$\phi(x) = v + \sigma(x) \tag{8.57}$$

Let us consider the case where the field lives near the VEV. Inserting this decomposition then gives

$$\mathcal{L} = \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - 2\mu^2\sigma^2 - \sqrt{\frac{6}{\lambda}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4 \tag{8.58}$$

We can immediately see that:

- The field perturbation $\sigma(x)$ is massive as we have a mass-like quadratic term $-2\mu^2\sigma^2$.
- Symmetry is no longer preserved when we flip the sign of ϕ. This is due to the sign also flipping in the third-order term $-\sqrt{\frac{6}{\lambda}}\mu\sigma^3$.

What is the physical implication of the second point? Well, nothing happens when the energy is well above the VEV, where the ϕ-V function still looks like a single trough. However, as we approach the VEV, the shape of the function begins to resemble a so-called *sombrero*. That is to say, instead of having one minimum, we have two minima along with a third unstable stationary point in between. Importantly, this means that a minimum can be reached at two different points of ϕ.

Figure 8.1: Various sombreros at the Museo de Arte Popular, CDMX. Credit: Alejandro Linares García

The breaking of this spontaneous symmetry is discrete. However, just like how we have discrete and continuous symmetries, there exist discrete and continuous spontaneous symmetry breakings, the latter of which gives rise to the so-called *Goldstone's theorem*.
Consider the so-called *linear sigma model* or the *linear σ model*. Superficially, it seems identical to ϕ^4 theory, of which it is a generalised version. The key lies in the fact that instead of a *single* scalar field, the linear σ model has a series of N scalar fields with identical couplings, denoted (i.e. summed over) by index $i = 1, \cdots, N$.

Definition 8.8 (Linear σ Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i + \frac{1}{2}m^2\phi_i^2 + \frac{\lambda}{4!}\phi_i^4 \tag{8.59}$$

The important realisation here is to imagine the collection of scalar fields as a vector, albeit not in the same way as a vector field. This requires us to realise the fact that the number of our scalar fields N can be different from the spacetime dimensions:

- The components of this so-called vector are full-fledged scalar fields determined by the 4-coordinates in Minkowski space instead of components of a vector field.
- Rather, the vector components together make up a 'field space' of N dimensions, distinct from our Minkowski spacetime.

Now consider a good ol' orthogonal group of rotations denoted by $\mathrm{O}(N)$. As we know, the representations of this group encompass all $N \times N$ orthogonal matrices R_{ij}. The linear σ Lagrangian is then invariant under the transformation

$$\phi_j = R^i_j\phi^i \tag{8.60}$$

This transformation, which is *continuous*, is analogous to our previous *discrete* sign-flip.
Again we solve for our minima, which reads

$$(\phi_{i,0})^2 = \frac{\mu^2}{\lambda} \tag{8.61}$$

This only poses a constraint on the magnitude of $\phi_{i,0}$. For our convenience, we can set our so-called 'field space' coordinates such that the only non-zero component of $\phi_{i,0}$ lies in the N^{th} direction:

$$\phi_{i,0} = (0, \cdots, 0, v) \quad v = \frac{\mu}{\sqrt{\lambda}} \tag{8.62}$$

The shifted fields then read, for some index $k = 1, \cdots, N-1$:

$$\phi_i(x) = (\pi_k(x), v + \sigma(x)) \tag{8.63}$$

where $\pi_k(x)$ is the shift of the (previously zero) fields of index $1, \cdots, N-1$, and $\sigma(x)$ is the shift of our (previously non-zero) field of index N.
Our good friend, the Lagrangian, is then

$$\mathcal{L} = \frac{1}{2}\partial_\mu\pi_k\partial^\mu\pi_k + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \mu^2\sigma^2 - \sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\pi_k^2\sigma - \frac{\lambda}{4}\pi_k^2\sigma^2 - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{4}\pi_k^4 \tag{8.64}$$

where Greek indices indicate spacetime components, and Latin indices indicate 'field space' components.

Quote 8.3 Because it is a Griechenlandflagge.

Felix Halbwedl, erroneously asserting that the Scottish flag in the BaGru room is a Greek flag, 26 September 2024

Three physical comments are in order:

- Again, we see that σ is massive due to the existence of the quadratic mass-like term, but the fields π_k are massless.
- There exists a broken continuous symmetry due to the existence of the terms $-\sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu\pi_k^2\sigma$.
- While we no longer have $\mathrm{O}(N)$ symmetry, a $\mathrm{O}(N-1)$ symmetry corresponding to the index k is preserved.

So what does the example tell us?

Derivation 8.1 (Goldstone's theorem) Let us generalise our linear σ result to an arbitrary field theory with several fields $\lambda_i(x)$. Like before, we can split the Lagrangian into the kinetic terms and the

potential:

$$\mathcal{L} = T + V \tag{8.65}$$

where, for a minimum in V, the fields find themselves at $\lambda_{i,0}$.
Expanding the potential about $\lambda_{i,0}$ gives

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi - \phi_0)_i (\phi - \phi_0)_j \left(\frac{\partial^2}{\partial\phi_i \partial\phi_j} V\right)_{\phi_0} + \cdots \tag{8.66}$$

As before, we recognise that the coefficient of the quadratic (mass-like) term is nothing but the mass coupling, or rather the *mass couplings*, which are now a matrix:

$$(m^2)_{ab} = \left(\frac{\partial^2}{\partial\phi_i \partial\phi_j} V\right)_{\phi_0} \tag{8.67}$$

Let us now impose a continuous transformation, which looks like

$$\phi_i \to \phi_i + \alpha\chi_i(\phi_i) \tag{8.68}$$

Now suppose that V is invariant under this *smol* transformation, we then have

$$V(\phi_i + \alpha\chi_i(\phi_i)) = V(\phi_i) \tag{8.69}$$

We can rephrase this as

$$\frac{\partial}{\partial\alpha} V(\phi + \alpha\chi(\phi))\Big|_{\alpha=0} = 0 \tag{8.70}$$

By the chain rule,

$$\frac{\partial V}{\partial\phi_i}\chi_i(\phi) = 0 \tag{8.71}$$

At the minimum $\phi = \phi_0$, we have $\partial_i V(\phi_0) = 0$ by definition of a stationary point, and this expression is trivially satisfied. We must then impose a second partial derivative with respect to some ϕ_j and then evaluate at ϕ_0:

$$0 = \frac{\partial}{\partial\phi_j}\left(\frac{\partial V}{\partial\phi_i}\chi_i(\phi)\right)\Big|_{\phi_0} = \frac{\partial^2 V}{\partial\phi_j\partial\phi_i}\Big|_{\phi_0}\chi_i(\phi_0) + \frac{\partial V}{\partial\phi_i}\Big|_{\phi_0}\frac{\partial\chi_i}{\partial\phi_j}\Big|_{\phi_0} \tag{8.72}$$

But $\partial_i V(\phi_0) = 0$, so the second term vanishes and we get

$$\frac{\partial^2 V}{\partial\phi_j\partial\phi_i}\Big|_{\phi_0}\chi_i(\phi_0) = 0 \tag{8.73}$$

$$(m^2)_{ji}\chi_i(\phi_0) = 0 \tag{8.74}$$

Again we have two cases:

- $\chi_i(\phi_0) = 0$: This means that symmetry is preserved at the VEV (ground state), which makes this case uninteresting to look at.
- $\chi_i(\phi_0) = 0$: This is the interesting case as symmetry is broken by the non-zero transformation. However, as the product is zero, we now must have $m^2 = 0$.

The second point yields Goldstone's theorem:

> **Theorem 8.4 (Goldstone's theorem)** If a global continuous symmetry is spontaneously broken, the field theory must contain massless scalar particles, or so-called *Goldstone bosons*, which corresponds to the broken symmetry generators.

However, the reverse is not true. Massless scalar particles do not always give rise to the spontaneous breaking of global symmetries.

But what about renormalisation? This is actually a good question, and using the effective action formal-

ism, we can see that Goldstone's theorem is satisfied in all loop orders[8].

Derivation 8.2 (Goldstone's theorem for effective potentials) Suppose $V_{\text{eff}}(\varphi)$ has a non-trivial minimum φ_0. The stationary point condition then reads

$$\left.\frac{\partial V_{\text{eff}}}{\partial \varphi}\right|_{\varphi_0} = 0 \tag{8.75}$$

The mass-squared matrix of *smol* fluctuations $\varphi_i = \varphi_0 + \eta_i$ is

$$(m^2)_{ij} = \left.\frac{\partial^2 V_{\text{eff}}}{\partial\varphi\partial\varphi}\right|_{\varphi_0} \tag{8.76}$$

We can see that the variation of the effective action is zero:

$$\delta\Gamma[\varphi] = \int d^4x \frac{\delta\Gamma}{\delta\varphi_i(x)}(T^a)_{ij}\varphi_j(x) = 0 \tag{8.77}$$

Hence yielding the condition

$$(T^a)_{ij}\varphi\frac{\partial V_{\text{eff}}}{\partial\varphi} = 0 \tag{8.78}$$

Again, this relationship is at the first derivative order and is trivial. Taking a second derivative gives

$$(T^a)_{ik}\delta_{ji}\varphi\frac{\partial V_{\text{eff}}}{\partial\varphi} + (T^a)_{ik}\varphi\frac{\partial^2 V_{\text{eff}}}{\partial\varphi\partial\varphi} = 0 \tag{8.79}$$

Now we set the field to the minimum φ_0. The first term vanishes by Equation 8.75, and one has

$$(m^2)_{ij}(T^a)_{jk}\varphi_0 = 0 \tag{8.80}$$

Hence, each broken generator T^a with $T^a\varphi \neq 0$ is an eigenvector of m^2 with eigenvalue zero. That is to say, our (massless) Goldstone bosons remain massless after renormalisation.

[8] You may be tempted to skip the rest of the section and only return to it after finishing Chapter 11.

Chapter 9

Interacting fields II: QED

As we already saw, QED is a gauge theory, which means that we need gauge fixing to quantise it. In this chapter, we will see that gauge fixing in path integrals is slightly different from canonical quantisation due to the structure of the integral and the presence of functionals.

9.1 Fermion propagator

As seen in Equation 6.18, the Lagrangian for the Dirac fermion (without any further fermions) is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu$$

The generating functional is then

Definition 9.1 (Dirac equation generating functional)

$$Z[J] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int d^d x(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta)} \tag{9.1}$$

where η is the (Grassmann) source for the Dirac field.

From this, we can derive the fermion propagator.

Derivation 9.1 (Fermion propagator) Once again, we can decompose the field into two parts:

$$\psi = \psi_c + \psi_q \tag{9.2}$$

where the classical-like ψ_c and satisfies the sourceful Dirac equation

$$(i\partial\!\!\!/ - m)\psi_c = -\eta \tag{9.3}$$

and is given by

$$\psi_c = -D^{-1}\eta \tag{9.4}$$

where we have used a shorthand by defining the so-called *Dirac operator*, which is, in some sense, an extended derivative[a]:

Definition 9.2 (Dirac operator)

$$D = i\partial\!\!\!/ - m \tag{9.5}$$

Essentially, we have shifted fields to absorb the linear source terms:

$$\psi = \psi_q - D^{-1}\eta \quad \bar{\psi} = \bar{\psi}_q - \bar{\eta}D^{-1} \tag{9.6}$$

such that our Lagrangian reads

$$\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta = \bar{\psi}D\psi + \bar{\eta}\psi + \bar{\psi}\eta = \bar{\psi}_q D\psi_q - \bar{\eta}D^{-1}\eta \tag{9.7}$$

Again, as we have nothing but a linear change of variables, the integration measure shifts as

$$\mathcal{D}\phi = \mathcal{D}(\psi_c + \psi_q) = \mathcal{D}\psi_q \tag{9.8}$$

and the generating functional reads

$$Z[\eta,\bar{\eta}] = \int \mathcal{D}\psi_q \mathcal{D}\bar{\psi}_q e^{i\int \bar{\psi}_q D\psi_q} e^{-i\int d^d x d^d y \bar{\eta}(x) D^{-1}(x,y)\eta(y)} \tag{9.9}$$

Like in the case of the Feynman propagator, if one sets the source to zero, they find

$$Z[0,0] = \int \mathcal{D}\psi_q \mathcal{D}\bar{\psi}_q e^{i\int \bar{\psi}_q D\psi_q} \tag{9.10}$$

As such, Equation 9.9 is normalised via the relation

$$Z_0[\eta,\bar{\eta}] = \frac{Z[\eta,\bar{\eta}]}{Z[0,0]} = e^{-i\int d^d x d^d y \bar{\eta}(x) D^{-1}(x,y)\eta(y)} \tag{9.11}$$

By consulting Equation 7.52, we know that the fermion two-point function is simply the generating functional after taking the functional derivative twice by the source. That is to say:

$$\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle = -\frac{\delta Z_0[\eta,\bar{\eta}]}{\delta\bar{\eta}(x)\delta\eta(y)}\bigg|_{\eta=\bar{\eta}=0} \tag{9.12}$$

Differentiating the exponent $-i\int \bar{\eta} D^{-1}\eta$ twice gives exactly our fermion propagator:

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = D^{-1}(x,y) = S_F(x-y) \tag{9.13}$$

By consulting the standard Fourier transform result Equation 4.28, it is easy to see that the momentum space Dirac operator is $D = \not{p} - m$. Hence, its inverse is

$$S_F(p) = \frac{i}{\not{p} - m + i\epsilon} = i\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \tag{9.14}$$

Making the Fourier transform back to position space, we finally rederive the (tree-level) fermion propagator as

$$S_F(x-y) = \int \frac{d^d p}{(2\pi)^d} i\frac{\not{p} + m}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \tag{9.15}$$

[a] But *not* a covariant derivative as no gauges are involved.

This concludes the easy half.

9.2 Photon field propagator

Now we arrive at the hard part. Recall the photon field Lagrangian from Equation . Adding a source term yields the generating functional

Definition 9.3 (Photon field generating functional)

$$Z[J_\mu] = \int \mathcal{D}A_\mu e^{i\int d^d x \left(\frac{1}{2}A^\mu(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu + J^\mu A_\mu\right)} \tag{9.16}$$

Note 9.1 Here, J_μ remains an arbitrary source instead of the 4-current. This is because we have not included an interaction term (where the 4-current J^μ is present) in the (free) photon field Lagrangian.

Again, gauge freedom complicates many things. We show this by first mindlessly evaluating the path integral without considering them:

- We evaluate $\Box g_{\mu\nu} - \partial_\mu\partial_\nu$ in momentum space. Using the property of derivatives in Fourier transforms, its equivalent in momentum space is $p^2 g_{\mu\nu} - p_\mu p_\nu$.
- If we contract this with k^ν, we obtain:

$$(p^2 g_{\mu\nu} - p_\mu p_\nu)p^\nu = g_{\mu\nu}p^2 p^\nu - p_\mu(p\cdot p) = p^2 p_\mu - p^2 p_\mu = 0 \tag{9.17}$$

which means that our expression maps any vector proportional to p_μ to zero. It is hence not invertible.

- Let us now recall the gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda(x)$. In momentum space, it becomes

$$A_\mu(k) \to A_\mu(k) + ip_\mu \Lambda(k) \tag{9.18}$$

Since $p^2 g_{\mu\nu} - p_\mu p_\nu$ annihilates any vector proportional to p_μ, it cannot distinguish between different gauge-equivalent configurations of A_μ.

Like in canonical quantisation, extra degrees of freedom must be eliminated by gauges.

Derivation 9.2 (Gauge fixing, take 2) In the case of electromagnetism (or indeed non-abelian gauge theories in general), the gauge fields A_μ are not uniquely defined because they transform under a gauge transformation as:

$$A_\mu \to A^g_\mu = A_\mu + \partial_\mu g \tag{9.19}$$

where A^g_μ is the field after it is gauge-transformed by some arbitrary function $g(x)$.
Now we want to remove the gauge freedom (i.e. invariance of A_μ). One can always write a gauge condition as a functional $C[A_\mu, x_\mu]$ satisfying

$$C[A_\mu, x_\mu] = 0 \tag{9.20}$$

which selects a unique representative from each equivalence class of gauge-related fields. However, this is not done by imposing restrictions directly on C[a]. Instead, we use the gauge function $g(x)$ to actively transform the gauge field to satisfy the condition.
In other words, given any A_μ, we can always find a suitable gauge function $g(x)$ such that the transformed field A^g_μ satisfies the gauge condition in Equation 9.20.
In the case of the Lorenz gauge, seen in Equation 6.108, the condition we want to impose is

$$\partial^\mu A^g_\mu = 0 \tag{9.21}$$

which we can expand as

$$\partial^\mu (A_\mu + \partial_\mu g) = \partial^\mu A_\mu + \partial^\mu \partial_\mu g = 0 \tag{9.22}$$

This is a differential equation for $g(x)$, which can always be solved, ensuring that we can always reach the Lorenz gauge by an appropriate choice of $g(x)$. Let us now make use of this in the photon field. This involves the functional $\Delta[A_\mu]$, which is essentially a delta functional. It is the generalisation of the Dirac delta function, which is related to it by

Definition 9.4 (Delta functional)

$$\Delta[A_\mu]^{-1} = \int \mathcal{D}g \delta(C[A_\mu, x_\mu]) \tag{9.23}$$

This functional is gauge-invariant. For a gauge transformation $g \to g + g'$:

$$\Delta[A_\mu]^{-1} = \Delta[A^{g'}_\mu]^{-1} \tag{9.24}$$

Interestingly, if we invert the expression above, we get

$$1 = \Delta[A_\mu] \int \mathcal{D}g \delta(C[A_\mu, x_\mu]) \tag{9.25}$$

which allows us to semi-cheatingly insert this expression as we please. Let us do so in Equation 9.16:

$$Z[J^\mu] = \int \mathcal{D}A_\mu \Delta[A_\mu] \int \mathcal{D}g \delta\left(C\left[A^g_\mu\right]\right) e^{iS[A_\mu]} \tag{9.26}$$

where we have used the shorthand

$$S[A_\mu] = i \int d^d x \left(\frac{1}{2} A^\mu \left(\Box g_{\mu\nu} - \partial_\mu \partial_\nu \right) A^\nu + J^\mu A_\mu \right) \tag{9.27}$$

We now implement the gauge transformation $g \to g + g'$:

$$Z[J^\mu] = \int \mathcal{D}g \int \mathcal{D}A_\mu^{g'} \Delta\left[A_\mu^{g'}\right] \delta\left(C\left[A_\mu^{g+g'}\right]\right) e^{iS[A_\mu^{g'}]} \tag{9.28}$$

and change the variables of integration from $A_\mu^{g'}$ to $A_\mu^{-g-g'}$

$$Z[J^\mu] = \int \mathcal{D}g \int \mathcal{D}A_\mu \Delta[A_\mu] \delta\left(C[A_\mu]\right) e^{iS[A_\mu]} \tag{9.29}$$

To proceed from here, we must employ a nice trick on the term $\Delta[A_\mu]$ in Equation 9.23, which we will otherwise struggle to integrate. Let us change the variable of integration from G to C:

$$\Delta[A_\mu]^{-1} = \int \mathcal{D}C \left(\det \frac{\delta C}{\delta g}\right)^{-1} \delta(C) = \left(\det \frac{\delta C\left[A_\mu, x\right]}{\delta g}\right)^{-1}_{C=0} \tag{9.30}$$

which easily transforms to

$$\Delta[A_\mu] = \left(\det \frac{\delta C\left[A_\mu, x\right]}{\delta g}\right)_{C=0} \tag{9.31}$$

Let us now define the so-called *Faddeev-Popov operator* by

$$\det M(x,y) = \left(\det \frac{\delta C\left[A_\mu, x\right]}{\delta g}\right)_{C=0} \tag{9.32}$$

Using the chain rule, one can derive

Definition 9.5 (Faddeev-Popov operator)

$$M(x,y) = -\partial_\mu^y \frac{\delta C\left[A_\mu, x\right]}{\delta g} \tag{9.33}$$

Using the Lorenz gauge $C\left[A_\mu, x\right] = \partial^\mu A_\mu = 0$, we have

$$M(x,y) = -\partial^2 \delta(x-y) \tag{9.34}$$

which is nothing but a field-independent functional determinant that *does not* introduce interactions. This allows us to replace $\Delta[A_\mu]$ in Equation 9.29 with $\det M$:

$$Z[J^\mu] = \int \mathcal{D}g \int \mathcal{D}A_\mu \det M \delta\left(C[A_\mu]\right) e^{iS[A_\mu]} \tag{9.35}$$

Recall from Equation 6.101 that one more gauge freedom exists due to U(1) symmetry transformations. We then implement the R_ξ Landau gauge in Equation 6.110, which, in the form of the functional C, is

$$C = D[A_\mu, x] + \Lambda(x) \tag{9.36}$$

where we have relabelled the original Lorenz gauge as $D[A_\mu, x]$, and Λ is an arbitrary function.
By using the standard trick of integrating over λ with a Gaussian weighting function, Z becomes

$$Z[J^\mu] = \int \mathcal{D}\Lambda e^{\frac{i}{2\xi}\int d^dx \Lambda^2} \tag{9.37}$$

which allows us to rewrite the delta function in an equivalent integral form:

$$\delta C = \delta(D[A_\mu, x] + \Lambda) = \int \mathcal{D}\Lambda e^{-\frac{i}{2\xi}\int d^dx \Lambda^2} \delta(D[A_\mu, x]) \tag{9.38}$$

Plugging this into the generating functional, we obtain:

$$Z[J^\mu] = \int \mathcal{D}\Lambda \mathcal{D}A_\mu \det M \delta C e^{iS[A_\mu] + \frac{i}{2\xi}\int d^dx \Lambda^2} \tag{9.39}$$

integrating yields

$$Z[J^\mu] = \int \mathcal{D}A_\mu \det M e^{i\int d^dx \left(S[A_\mu] + \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)} \tag{9.40}$$

Finally, we reinsert the full form of the shorthand in Equation 9.27, consult Equation 6.111 and acknowledge that, as the photon field is abelian, the Faddeev-Popov operator term evolves into its fairly harmless form in Equation 9.34. This gives our final, well-defined expression

Definition 9.6 (Gauge-fixed photon field generating functional)

$$Z[J^\mu] = \int \mathcal{D}A_\mu e^{i\int d^dx \frac{1}{2}A^\mu \left(\Box g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu +\right)A^\nu + \int d^dx J^\mu A_\mu} \tag{9.41}$$

[a] e.g. defining it in a way that inherently limits A_μ.

Note 9.2 (Faddeev-Popov ghosts) We append this discussion with a taste of things to come. In QED, which is a *non-abelian* gauge theory, Equation 9.34 applies. In abelian gauge theories like QCD, however, the Faddeev-Popov operator term in the generating functional

$$\det M = \int \mathcal{D}c\mathcal{D}\bar{c}e^{iS_{\text{ghost}}} \tag{9.42}$$

which contributes to the Lagrangian. c and $\bar{c}$ are unphysical fields known as the *Faddeev-Popov ghost fields*[a] or simply *Faddeev-Popov ghosts* which obey Grassmann anticommutations. We will discuss the treatment of Faddeev-Popov Ghosts much later.

[a] The general notion of a *ghost field* denote unphysical fields emerging in the Lagrangian. Specifically, ghost fields in QFT assume the name 'Faddeev-Popov ghost fields'.

After gauge fixing, we can now derive the photon propagator.

Derivation 9.3 (Photon propagator) We again begin by renormalising our generating function, which is seen in Equation 9.41. When the source is zero, one has

$$Z[0] = \int \mathcal{D}A_\mu e^{i\int d^dx \frac{1}{2}A^\mu \left(\Box g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu +\right)A^\nu} \tag{9.43}$$

We go momentum space via a Fourier transform:

$$Z[0] = \int \mathcal{D}A_\mu e^{\frac{1}{2}\int \frac{d^4p}{(2\pi)^4} A^\mu(-p)\left(-p^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right)p_\mu p_\nu\right)A^\nu(p)} \tag{9.44}$$

Performing the Gaussian integral yields us the normalised generating functional, which reads

$$Z[J] = Z[0]e^{\frac{i}{2}\int \frac{d^dk}{(2\pi)^d} J^\mu(-k)D_{\mu\nu}(k)J^\nu(k)} \tag{9.45}$$

with the propagator $D_{\mu\nu}(k)$ defined by

$$\left(-p^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right)p_\mu p_\nu\right) D^\alpha{}_\nu(k) = i\delta_{\mu\nu} \tag{9.46}$$

The rest of the derivation is identical to what we did in Part II from Equation 6.143 on.

9.3 Vertex

We once again find ourselves on the doorsteps of greatness as we formulate QED for a second time. Putting it all together, the QED generating functional is

Definition 9.7 (QED generating functional)

$$Z[J^\mu] = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int d^dx \left(A^\mu \left(\Box g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu +\right) A^\nu + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu A_\mu\right)} \tag{9.47}$$

where $\bar{\eta}\psi + \bar{\psi}\eta$ is the fermion field source term, and $J^\mu A_\mu$ is the photon (gauge) field source term.

Remark 9.1 Strictly speaking, one can even introduce the previously seen ghost field terms

$$Z[J] = \int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}c\mathcal{D}\bar{c}e^{i\int d^4x \left(A^\mu \left(\Box g_{\mu\nu} - \left(1-\frac{1}{\xi}\right)\partial_\mu\partial_\nu +\right) A^\nu + \mathcal{L}_{\text{ghost}} + J^\mu A_\mu + \bar{J}\psi + \bar{\psi}J\right)} \tag{9.48}$$

which are trivial (i.e. vanish) in QED anyway.
From the interaction term, we can then derive the 3-point function, which is the vertex.

Part IV

Renormalisation and regularisation

Chapter 10

Regularisation

> **Quote 10.1** The sun set in the west on a notion where no man had dared to venture. And beyond that - infinity.
>
> *1492: Conquest of Paradise*

What?... You thought that was it? Oh no. We still have to treat infinities. Even though we noted loops in the Feynman rules, everything we have done so far has been restricted to the tree level. This is obviously problematic because loop interactions emerge in real life.

10.1 Motivation

Previously in Part II, we introduced the complete set of Feynman rules and the concept of the scattering amplitude $\mathcal{M}$. As the objective now is to go beyond the tree level, it is a good idea to review the full form of $\mathcal{M}$ up to the infinite order:

Theorem 10.1 (Full ϕ^4 theory scattering amplitude)

$$i\mathcal{M} = -i\lambda + \frac{(-i\lambda)^2}{2}\sum I_2 + O(\lambda^3) \tag{10.1}$$

where $\sum I_n$ is the sum of all possible *Feynman integrals* of that order n.

This systematic treatment makes use of the so-called Feynman integrals, which are representations of loop contributions to the scattering amplitude. The Feynman integral of a diagram arises naturally from the Feynman rules:

Definition 10.1 (Feynman integral)

$$I = \lim_{\epsilon\to 0}\int \prod_{k=1}^{l}\frac{d^d p_k}{i\pi^{d/2}}\prod_{k=1}^{n}\frac{1}{(p_k^2 - m_1^2 + i\epsilon)} \tag{10.2}$$

where:

- d is the physical dimension of the manifold.
- l is the number of loops.
- n is the number of the internal legs/lines.

Note 10.1 One can think of Feynman integrals I_n as a generalised version of Feynman propagators D_F. Indeed, a Feynman integral is ultimately a combination of Feynman propagators integrated over internal momenta. It corresponds to the probability amplitude for a particle to propagate from a point *back to the same point* some l times and hence represents l loops.

Previously, the reader has been led to believe that this is the end of the story. Beyond the tree level,

this is unfortunately a lie. Uniquely among all Feynman diagram elements, loops are mathematically represented by integrating over internal loop momenta. To be exact, loop integrals.
It is well-known that loop integrals often diverge into infinity, which is indeed the case for our Feynman integrals. Unsurprisingly, physical systems are not infinite, which suggests that our current formulation of QFT fails when we go beyond the tree level. In this chapter, we restrict our discussion to ϕ^4 theory.

Derivation 10.1 (Tadpole diagram) We illustrate this with the simplest possible example, that being a 1-loop, 1-point/2-point function known as the so-called *tadpole diagram*:

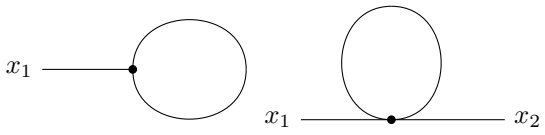


Figure 10.1: Archetypical tadpole diagrams.

The self-energy of a Feynman diagram is directly related to its Feynman integral by a coupling constant (and arbitrary constants). In ϕ^4 theory, the self-energy of an is simply the Feynman integral multiplied by a factor of $\frac{\lambda^n}{2}$ where n is the *number of vertices*. In our case, this is $\frac{\lambda}{2}$. Take the one-point tadpole as an example:

$$\Sigma_{\text{tadpole}} = \frac{\lambda}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 - m^2 - i\epsilon} \tag{10.3}$$

where, notably, we assume an arbitrary number of dimensions d for the time being. As we will prove later on, this integral evaluates as

$$\Sigma_{\text{tadpole}} = \frac{\lambda}{2}\frac{\pi^{d/2}}{(2\pi)^d}\Gamma\left(1 - \frac{d}{2}\right) m^{d-2} \tag{10.4}$$

where $\Gamma(n)$ is a so-called *gamma function* defined by

Definition 10.2 (Gamma function)

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt \tag{10.5}$$

If n is a positive integer, $\Gamma(n)$ reduces to

$$\Gamma(n) = (n-1)! \tag{10.6}$$

Theorem 10.2 (Gamma function properties)

- $\Gamma(n)$ has poles at $n = 0, -1, -2, \cdots$
- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma'(1) = -\gamma \approx -0.5772$ where γ is the Euler-Mascheroni constant.

We now make the arbitrary choice of $d = 2$. The gamma function becomes

$$\Gamma\left(1 - \frac{2}{2}\right) = \int_0^\infty t^{-1}e^{-t}dt = \infty \tag{10.7}$$

Hence, we have $\Sigma_{\text{tadpole}} = \infty$ for $d = 2$.
What is the physical meaning of this? Essentially, when p is *large*, the original integral approaches

$$\Sigma_{\text{tadpole}} = \frac{\lambda}{2(2\pi)^d}\int d^d p \frac{1}{p^2} \tag{10.8}$$

which diverges to infinity for $d \geq 2$. This result is clearly unphysical and is known as the so-called *ultraviolet (UV) singularity*.

Remark 10.1 Another singularity is the so-called *infrared (IR) singularity* which takes occur in low energies/*smol* momenta.

10.2 Mathematical tooklit

The tadpole diagram is a good example of infinities as its integral is already simplified and can be evaluated as-is. This is, regrettably, not the case for almost all other diagrams, for which the elephant in the room remains our inability to integrate over the four-momentum p_μ. Rather, a mathematical trick known as *Feynman parameterisation* is required.

Derivation 10.2 (Feynman parameterisation)

Definition 10.3 (Feynman parameters) In his 1949 paper 'Space-Time Approach to Quantum Electrodynamics', Feynman noted the relation

$$\begin{aligned}\frac{1}{ab} &= -\frac{1}{a-b}\left(\frac{1}{a}-\frac{1}{b}\right) = -\frac{1}{a-b}\left[\frac{1}{xa+(1-x)b}\right]_0^1 \\ &= \int_0^1 dx \frac{1}{(xa+(1-x)b)^2} = \int_0^1 dx \int_0^1 dy \delta(x+y-1)\frac{1}{(xa+yb)^2}\end{aligned} \tag{10.9}$$

where $x, y \in [0,1]$ are the so-called *Feynman parameters*. One can generalise this relation as

$$\frac{1}{a_1 \cdots a_n} = \int dx_1 \cdots dx_n \delta\left(\sum_{i=1}^n x_i - 1\right)\frac{(n-1)!}{[\sum_{i=1}^n x_i a_i]^n} \tag{10.10}$$

Now let us apply this in the context of Feynman integrals. Assume the loop momentum k, external momenta p_i, mass m_i and the singularity-avoiding term $i\epsilon$, we define our a_i as

$$a_i = (k+p_i)^2 - m^2 + i\epsilon \tag{10.11}$$

The delta function in Equation 10.10 essentially enforces the relation

$$\sum_{i=1}^n x_i - 1 = 0 \to \sum_{i=1}^n x_i = 1 \tag{10.12}$$

We can thus write, using Equation 10.11 and Equation 10.12:

$$\sum_{i=1}^n x_i a_i = k^2 \sum_{i=1}^n x_i (2kp_i + P_i^2 - m_i^2) + i\epsilon \tag{10.13}$$

We define a rescaled loop momentum l, which is

$$l = k + \sum_{i=1}^n x_i p_i \tag{10.14}$$

This gives rise to the expression

$$\sum_{i=1}^n x_i a_i = l^2 - \left(\sum_{i=1}^n x_i p_i\right)^2 - \sum_{i=1}^n x_i (p_i^2 - m_i^2) + i\epsilon = l^2 - \mu^2 + i\epsilon \tag{10.15}$$

where we have defined a (rescaled) mass parameter μ that is independent of l, satisfying

$$\mu^2 = \left(\sum_{i=1}^n x_i p_i\right)^2 - \sum_{i=1}^n x_i (p_i^2 - m_i^2) \tag{10.16}$$

This is nice, as we have now eliminated the Feynman parameter x from the final expression completely! Here we see why Feynman parameterisation is so-called. Essentially, the new loop momenta and energy are rescaled (and parameterised) using our new friends, the Feynman parameters.

Another nice standard trick one can employ to simplify Feynman integrals is the so-called *Wick rotation*:

Definition 10.4 (Wick rotation) Consider the 4D Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{10.17}$$

The Wick rotation involves a coordinate rescaling

$$t = -i\tau \tag{10.18}$$

after which the metric reads

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2 \tag{10.19}$$

which is simply the 4D Euclidian metric. This simplifies calculations, and the Minkowski space version of our quantity of interest can be recovered by reversing our rescaling in Equation 10.18.

Note 10.2 In this book, some sacrifices are made for our convenience:

- We will relabel the rescaled l and μ back to p and m after the Feynman parameterisation is complete. This ensures that we see symbols we are familiar with in the final result.
- The same thing will happen w.r.t. Wick rotation. We will announce a Wick rotation has been made, at which point it should be assumed that the physical quantities, which retain their pre-Wick rotation notation, are those after Wick rotation.

Derivation 10.3 (Bubble diagram) We are now in a position to observe the emergence of infinities in a 1-loop, 2-point/4-point function, whose Feynman diagrams are called *bubble diagrams*:

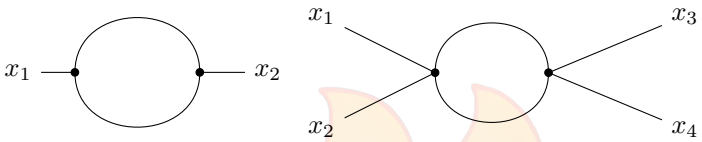


Figure 10.2: Archetypical bubble diagrams.

Let us take the 4-point diagram as an example. We recall that the self-energy of a diagram with n vertices is related to the Feynman integral by $\frac{\lambda^n}{2}$. For the bubble diagram, this is $\frac{\lambda^2}{2}$:

$$\Sigma_{\text{bubble}} = \frac{\lambda^2}{6} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2)((q-p)^2 + m^2)} \tag{10.20}$$

where $q = p_1 + p_2 = -p_3 - p_4$. Performing Feynman parameterisation and a Wick rotation yields

$$\Sigma_{\text{bubble}} = -\frac{\lambda^2}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 - m^2 + i\epsilon)^2} \tag{10.21}$$

where it should be clear that p and m are nothing but the rescaled l and μ which we have relabelled. This simplified integral evaluates as

$$\Sigma_{\text{bubble}} = \frac{\lambda^2}{2} \frac{\pi^{d/2}}{(2\pi)^d} \Gamma\left(2 - \frac{d}{2}\right) \int_0^1 x(1-x)q^2 + m^{2^{d-2-2}} dx \tag{10.22}$$

Here, the UV singularity emerges at $d = 4$. The physical implication is likewise slightly altered: When p is *large*, the original integral approaches

$$\Sigma_{\text{bubble}} = \frac{\lambda^2}{2(2\pi)^d} \int d^d p \frac{1}{p^4} \tag{10.23}$$

which again is a UV singularity where the integral diverges to infinity for $d \geq 4$.

Exercise 10.1 Using Feynman parameterisation, show that a UV singularity emerges in a *sunset diagram*, which looks like

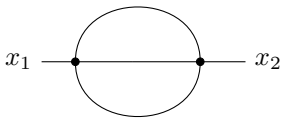


Figure 10.3: Archetypical sunset diagram.

and has the self-energy

$$\Sigma_{\text{sunset}} = \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\epsilon)((p-k-q)^2 - m^2 + i\epsilon)(q^2 - m^2 + i\epsilon)} \tag{10.24}$$

where p is the external momentum, k and a are the loop momenta and m is the mass of the internal propagators.

Quote 10.2 There's potential for a lot of upheval.

Alessio Serafini, on the sunset diagram, 6 March 2025

In fact, for a sufficiently large d, most Feynman diagrams display UV singularities. They are hence said to be *UV divergent*. One way to treat these infinities is *regularisation*, which, as we will see, is a very hand-wavy way to deal with infinities.

10.3 Cutoff regularisation

Derivation 10.4 (Cutoff regularisation) We return to the tadpole diagram self-energy in Equation 10.3. We have written it with an indefinite integral, but it actually spans over positive and negative infinities:

$$\Sigma_{\text{tadpole}} = \frac{\lambda}{2} \int_{-\infty}^{\infty} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2 - i\epsilon} \tag{10.25}$$

It has already been established that integrating over the positive infinity is what causes infinities to emerge. So what if we replace the upper limit with a *large* but finite Λ? The integral becomes

$$\Sigma_{\text{tadpole}} = \frac{i\lambda}{4\pi^2} \left(\Lambda^2 \sqrt{1 + \frac{m^2}{\Lambda^2}} - m^2 \ln \left(\frac{\Lambda + \Lambda\sqrt{1 + \frac{m^2}{\Lambda^2}}}{m} \right) \right) \tag{10.26}$$

where Λ is known as the *regulator*.
This is known as *cutoff regularisation* or the *UV momentum cutoff*, one of the many ways to regularise Feynman integrals. Its physical meaning lies in the fact that we essentially introduce a ceiling of the biggest momentum the system we have, which allows us to avoid the singularity in Equation 10.8.
The original singularity is then recovered when $\Lambda \to \infty$

The same happens to a bubble diagram, although in this case we use Feynman parameterisation and Wick rotation, given its complexity.

10.4 Pauli-Villars regularisation

The so-called *Pauli-Villars regularisation* or *P-V regularisation* is named after Wolfgang Pauli and Felix Villars, and involves rewriting the first propagator using Feynman parameterisation. In the general case, the P-V regularisation is accomplished by

$$\frac{1}{k^2 - m^2 + i\epsilon} \to \frac{1}{k^2 - m^2 + i\epsilon} - \sum_i \frac{a_i}{k^2 - \Lambda_i^2 + i\epsilon} \tag{10.27}$$

for a series of mass parameters Λ_i with indices i.

Derivation 10.5 (Pauli-Villars regularisation) Now consider the bubble diagram. Again, writing out the integration explicitly gives

$$\Sigma_{\text{bubble}} = \frac{(-i\lambda)^2}{2}\int_{-\infty}^{\infty}\frac{d^dp}{(2\pi)^d}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(p-k)^2+m^2+i\epsilon} \tag{10.28}$$

where the factor $-i$ accounts for the Wick rotation. We then perform the P-V regularisation:

$$\frac{1}{k^2-m^2+i\epsilon}\to\frac{1}{k^2-m^2+i\epsilon}-\underbrace{\frac{1}{k^2-\Lambda^2+i\epsilon}}_{①}=\frac{m^2-\Lambda^2}{(k^2-m^2+i\epsilon)(k^2-\Lambda^2+i\epsilon)} \tag{10.29}$$

where λ is again *large* (i.e. $\Lambda \gg m$) but finite, and ① is a pseudo-mass term that represents a fictitious photon mass. The self-energy then reads

$$\Sigma_{\text{bubble}} = -\frac{\lambda^2\Lambda^2}{2}\int_{-\infty}^{\infty}\frac{d^dp}{(2\pi)^d}\frac{1}{(k^2-m^2+i\epsilon)(k^2-\Lambda^2+i\epsilon)((p-k)^2+m^2+i\epsilon)} \tag{10.30}$$

In regularisation, it is often useful to split the self-energy into finite and infinite (i.e. convergent and divergent) parts so that their physical significance can be better understood:

$$\Sigma(p^2) = \Sigma(0) + \tilde{\Sigma}(p^2) \tag{10.31}$$

In our example, they are

$$\Sigma_{\text{bubble}}(0) = -\frac{\lambda^2\Lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\epsilon)^2(k^2-\Lambda^2+i\epsilon)} \tag{10.32}$$

$$\tilde{\Sigma}_{\text{bubble}}\left(p^2\right) = -\frac{\lambda^2\Lambda^2}{2}\int\frac{d^4k}{(2\pi)^4}\frac{2pk-p^2}{(k^2-m^2)^2(k^2-\Lambda^2)((p-k)^2-m^2)} \tag{10.33}$$

10.5 Dimensional regularisation

The process known as *dimensional regularisation* is arguably the most important method of regularisation in perturbation theory. As we will see later, it can be used in conjunction with the minimal subtraction scheme to perform renormalisation.

Derivation 10.6 (Tadpole diagram) Let us tie up our previous loose end by evaluating the tadpole diagram step by step. The Feynman (loop) integral reads

$$I_2 = \int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2-m_0^2+i\epsilon} \tag{10.34}$$

In this case, we label the regularisation scale M instead of Λ and make the rescaling

$$I_2 = \frac{1}{M^{d-4}}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2-m_0^2+i\epsilon} \tag{10.35}$$

We define another parameter $\varepsilon = 4-d$ (not to be confused with the ϵ in the term $i\epsilon$) and transition to Euclidean space by performing a Wick rotation $k^0 \to ik^0$, giving:

$$I_2 = iM^{\varepsilon}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m_0^2} \tag{10.36}$$

which integrates as

$$I_2 = \frac{iM^{\varepsilon}}{(2\pi)^d}V_S\int_0^{\infty}d^dp\frac{p^{d-1}}{p^2+m_0^2} \tag{10.37}$$

where V_S is the surface volume of a unit sphere in d dimensions that has the standard formula $V_S = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.
By making the substitution $x = m_0^2/(p^2 + m_0^2)$, the integral evaluates as

$$I_2 = V_S \frac{m_0^2}{2(2\pi)^d} \left(\frac{M}{m_0}\right)^{\varepsilon} \int_0^1 (1-x)^{d/2-1} x^{-d/2} \tag{10.38}$$

Very sneakily, the integral $\int_0^1 (1-x)^{d/2-1} x^{-d/2}$ is actually three beta functions in disguise. We know this as it is a standard *Euler beta function*, which has the form

Definition 10.5 (Euler beta function)

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1} dx \tag{10.39}$$

We can hence rewrite

$$I_2 = V_S \frac{m_0^2}{2(2\pi)^d} \left(\frac{M}{m_0}\right)^{\varepsilon} \frac{\Gamma(d/2)\Gamma(1-d/2)}{\Gamma(1)} \tag{10.40}$$

where $\Gamma(1)$ is simply 1. Inserting the so-called surface volume $V_S = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, we get

$$I_2 = \frac{m_0^2}{(2\pi)^{2-\varepsilon/2}} \left(\frac{M}{m_0}\right)^{\varepsilon} \Gamma(1-d/2) \tag{10.41}$$

We can perform the following expansions

$$\Gamma(1-d/2) = \Gamma(-\varepsilon/2) \approx -\frac{2}{\varepsilon} - \gamma - 1 + O(\varepsilon) \tag{10.42}$$

where γ is again the Euler-Mascheroni constant.

$$\left(\frac{M}{m_0}\right)^{4-d} = 1 + \epsilon \ln\left(\frac{M}{m_0}\right) + \mathcal{O}(\epsilon^2) \tag{10.43}$$

$$(2\pi)^{d/2} = (2\pi)^2 \left(1 - \frac{\epsilon}{2}\ln(2\pi) + \mathcal{O}(\epsilon^2)\right) \tag{10.44}$$

Putting it all together, and treating ε as the parameter of interest, we find

$$I_2 = -\frac{m_0^2}{(4\pi)^{d/2}} \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m_0^2}\right) \tag{10.45}$$

which diverges as $\epsilon \to 0$ or $d \to 4$.
As the tadpole diagram has only one measly vertex, its self-energy contribution is related to the Feynman integral by $\frac{\lambda^1}{2}$. Hence, the self-energy is

$$\Sigma_{\text{tadpole}} = -\frac{m_0^2 \lambda}{2(4\pi)^{d/2}} \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m_0^2}\right) \tag{10.46}$$

Derivation 10.7 (Bubble diagram) We know that the loop integral of a bubble diagram is

$$I_4 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \tag{10.47}$$

where $p = p_1 + p + 2 = p_3 + p_4$ is the external momentum and k is the internal loop momentum. Again, we employ the regularisation scale M and make the rescaling:

$$I_4 = \frac{1}{M^{d-4}} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\epsilon)^2} \frac{1}{((p-k)^2 - m^2 + i\epsilon)^2} \tag{10.48}$$

After Feynman parameterisation, we get

$$\begin{aligned} I_4 =& \frac{1}{M^{d-4}} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{1}{(x((p-k)^2 - m^2) + (1-x)(k^2 - m^2))^2} \\ =& M^\varepsilon \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{1}{(k - xp)^2 + x(1+x)p^2 - m^2} \end{aligned} \tag{10.49}$$

By defining $l^\mu = k^\mu - xp^\mu$ and performing a Wick rotation, we can rewrite

$$I_4 = iM^\varepsilon \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{1}{(l^2 + m^2 - x(1-x)p^2 + i\epsilon)^2} \tag{10.50}$$

Using the properties of the Gaussian function, we have

$$I_4 = \frac{i\pi^{d/2}}{(2\pi)^d} \Gamma(\varepsilon/2) \int_0^1 dx \frac{1}{(m^2 - x(1-x)p^2)^{\varepsilon/2}} \tag{10.51}$$

Let us perform expansions of these terms. Labelling $a^2 = m^2 - x(1-x)p^2$:

$$\Gamma(\varepsilon/2) = \frac{2}{\varepsilon} - \gamma + O(\epsilon) \quad \frac{1}{(a^2)^{\varepsilon/2}} = 1 - \frac{\varepsilon}{2} \ln a^2 + O(\varepsilon^2) \quad M^\varepsilon = (M^2)^{\varepsilon/2} = 1 + \frac{2}{\varepsilon} \ln M^2 + \cdots \tag{10.52}$$

Putting it all together, and treating ε as the parameter of interest, we find

$$I_4 = \frac{i\pi^{d/2}}{(2\pi)^d} \int_0^1 dx \left(\frac{2}{\varepsilon} - \gamma - \ln \frac{m^2 - x(1-x)p^2}{M^2} + O(\varepsilon) \right) \tag{10.53}$$

Using the standard integration techniques, we have

$$I_4 = -\frac{m_0^2}{(4\pi)^{d/2}} \left(\frac{2}{\varepsilon} - \gamma + \ln \frac{m_0^2}{\mu^2} \right) \tag{10.54}$$

which also diverges as $\epsilon \to 0$ or $d \to 4$.
As we have 2 vertices, the one-loop contribution in the Green's function, or rather the self-energy is related to the Feynman integral by $\frac{\lambda^2}{2}$. Hence

$$\Sigma_{\text{bubble}} = \frac{i\lambda^2 \pi^{d/2}}{2(2\pi)^d} \left(\frac{2}{\varepsilon} - \gamma + \ln \frac{m_0^2}{\mu^2} \right) \tag{10.55}$$

where the factor of λ^2 denotes 2 vertices in the bubble diagram.

Chapter 11

Renormalisation I: ϕ^4 theory

> **Quote 11.1** The infinities are hidden behind the cloud?
>
> *Felix Halbwedl, on the watermark of Part IV, 3 March 2025*

Fortunately, there are no infinities in real-world physics. Unfortunately, we have not recovered this via regularisation, which only serves as an intermediate step to control divergences by introducing a regulator. To obtain finite, physically meaningful predictions, we must employ *renormalisation*.

11.1 Emergence of renormalisation

The key point lies in realising that quantities depending on λ, including e, m, ψ and A_μ, are actually merely the *tree-level* versions of the terms they *claim* to represent:

- So far, we have mistakenly treated the coupling constant λ as a fixed quantity. This is, in fact, not true: Quantum corrections introduce scale dependence, and λ evolves with energy scales as a result.
- Due to this, quantities we previously employed like the mass m and the charge e, also change in higher orders. This extends to the scalar field ψ, the vector field A_μ and the Green's function G, all of which are dependent on e and m. In higher orders, they are augmented by a series of correction terms, which can be written as a power series of the coupling constant.
- We will now denote the previously seen λ, m, e, ϕ, A and G as λ_0, m_0, e_0, ψ_0, A_0 and G_0, which we call the *bare* (read: unrenormalised) quantities.
- The renormalised (read: actual) quantities are then denoted simply as m, e, ψ, A and G, since they are the true quantities.

Remark 11.1 As infinities only emerge in loop integrals, the tree-level part of a propagator is unaffected by renormalisation. At each order, the renormalised counterpart of a bare quantity is different.

> **Quote 11.2** In practice, we can never measure bare charge or bare mass, because nature always includes the higher order corrections in real world interactions. But the concept of such bare quantities will serve us well in our analyses.
>
> *Robert D. Klauber*

Importantly, renormalisation does not render regularisation redundant. In fact, the standard way of renormalising a field theory involves regularisation:

- Label the bare quantities with the subscript $_0$.
- Identify the divergences (i.e. infinities).
- Regularise the theory[1].

[1] We need to do this so that there is actually a way to evaluate our integral

- Replace all instances of the bare coupling constant and mass λ_0 and m_0 with their renormalised counterparts λ and m.
- Remove infinities from the bare quantities using the counterterms. This must be accomplished via specific normalisation schemes.
- Remove the regulator to 'deregularise' the theory, e.g.
  - In general, set d to the actual number of physical dimensions.
  - In cutoff regularisation, send $\Lambda \to \infty$.
  - In dimensional regularisation, send $\varepsilon \to 0$.

Theorem 11.1 (Renormalisability) If all divergences can be absorbed into a *finite* number of redefined physical parameters[a], the theory is known to be *renormalisable*. If an *infinite* number of counterterms is required, the theory is known to be *non-renormalisable*.

[a] Even renormalisable theories may still have infinities at intermediate stages, but these are systematically removed.

11.2 Renormalisability

In addition to our previous loose definition, we want to systematically determine whether a field theory is renormalisable. This can be done in more than one way.

Definition 11.1 (Mass dimension) The so-called *mass dimension* $[Q]$ of some quantity Q is the dimension of mass (or, assuming natural units, energy as well) units in physical units of Q.

The definition above is quite a mouthful, so let us clarify it with a few examples. For a theory in d dimensions:

- The mass dimension $[S]$ of an action S is always zero. As the action is dimensionless, it cannot possibly have units of mass. It can be decomposed to
$$[S] = [\mathcal{L}] + [d^d x] = 0 \tag{11.1}$$
where $[d^d x] = -d$ as $\int d^d x$ *integrates* over d independent coordinates.
- The mass dimension $[\mathcal{L}]$ of the Lagrangian density $\mathcal{L}$ is always d. This is derived by inserting $[d^d x] = -d$ into Equation 11.1.
- Consider a Lagrangian with only the kinetic term $\mathcal{L} = \frac{\partial^2 \phi^2}{2}$, which gives the expression
$$[\mathcal{L}] = [\partial^2] + [\phi^2] = 2[\partial] + 2[\phi] = 2 + 2[\phi] = d \tag{11.2}$$
The mass dimension $[\phi]$ of a field ϕ is thus
$$[\phi] = \frac{d-2}{2} \tag{11.3}$$
- Now consider a Lagrangian with a series of interacting terms $\mathcal{L} = \lambda_i \phi^i$[2], which gives the expression
$$[\mathcal{L}] = [\lambda_i] + [\phi^i] = [\lambda_i] + \frac{i(d-2)}{2} = d \tag{11.4}$$
For a specific term in the Lagrangian with i fields, the mass dimension of the term's corresponding coupling constant λ_i[3] is thus
$$[\lambda] = d - \frac{i(d-2)}{2} \tag{11.5}$$
Take ϕ^4 theory for example. We have $i = 4$ fields. As such, in 4 dimensions:
$$[\lambda] = d - \frac{4(d-2)}{2} = 4 - d = \varepsilon \tag{11.6}$$
where we recall the definition of ε from dimensional regularisation.

[2] Here, the subscript is an index while the superscript is an exponential.
[3] This denotes the renormalised coupling constant. As we know, the bare coupling constant is dimensionless. We will return to this later.

Theorem 11.2 (Renormalisability from $[\lambda_i]$) A coupling i in a field theory is renormalisable if $[\lambda_i] \geq 0$. If $[\lambda_i] < 0$, it is non-renormalisable. This means it needs a suppression factor and is treated in an *effective field theory* (EFT) framework.

A second framework is *power counting*, which is a very sophisticated way of saying dimensional analysis:

Definition 11.2 (Superficial degree of divergence) The *superficial degree of divergence* D is a convenient way to determine if a Feynman diagram diverges into infinity. It is the power of the momentum p in the Feynman diagram[a]:

$$D = dL - \sum_i (d_i - d)V_i \tag{11.7}$$

where L is the number of (independent) loops (i.e. momentum integrations) in the diagram, V_i is the number of vertices of the vertex type corresponding to the interaction term with i fields, and d_i is the mass dimension of the interaction term associated with vertex type i.

- If $D \geq 0$, the diagram is known to be *superficially divergent*:
  - If $D > 0$, the diagram leads to *logarithmic divergence*.
  - If $D = 0$, the diagram leads to *power law divergence* and requires counterterms.
- If $D < 0$, the diagram is convergent, and no renormalisation is needed.

[a] i.e. the momentum power in the numerator minus that in the denominator.

Theorem 11.3 (Renormalisability from power counting)

- A field theory is renormalisable if $d_i \leq d$. Through redefinitions, only a finite number of counterterms are needed. Examples are QED and QCD in $d = 4$.
- A field theory is super-renormalisable if $d_i < d$. This is a subset of renormalisable theories that have only a finite number of divergent Feynman diagrams (usually low-order loops). An example is ϕ^3 theory in $d = 6$.
- A field theory is non-normalisable if $d_i > d$. An infinite number of counterterms are needed. As this is not accomplishable, the theory becomes meaningless in high energies (but is acceptable in low energies). An example is GR in $d = 4$.

Let us consider this in the context of ϕ^4 theory in 4D, our simplest toy model. In the one-loop order, the 1-point loop (tadpole), the 2-point loop (bubble) and the 4-point loop are all divergent. Conversely, n-point loops with $n > 4$ do not contribute, as they have a negative superficial degree of divergence $D < 0$.

11.3 Counterterms

The bare quantities are related to their renormalised counterparts by the so-called *renormalisation factors* Z_λ, Z_m and Z_ϕ:

Definition 11.3 (ϕ^4 renormalisation factors)

$$\lambda_0 = Z_\lambda \lambda \quad \phi_0 = \sqrt{Z_\phi}\phi \quad m_0^2 = Z_m m^2 \tag{11.8}$$

The ϕ^4 theory Lagrangian in Equation 5.1 becomes

$$\mathcal{L} = \frac{Z_\phi}{2}\partial_\mu\phi\partial^\mu\phi + \frac{Z_m Z_\phi}{2}m^2\phi^2 + \frac{Z_\lambda \lambda Z_\phi^2}{4!}\phi^4 \tag{11.9}$$

The renormalised Feynman propagator, n-point vertex and n-point Green's function are then

$$D_F = \frac{D_{F,0}}{Z_\phi} = \frac{i}{Z_\phi(p^2 - Z_m m^2 + i\epsilon)} \quad \Gamma^{(n)} = Z_\phi^{-n/2}\Gamma_0^{(n)} \quad G^{(n)} = Z_\phi^{-n/2}G_0^{(n)} \tag{11.10}$$

The propagator and vertex can be decomposed. For example, the 2-point propagator function and 4-point vertex function decomposes as

$$\Gamma^{(2)}(p^2) = p^2 - m^2 + \Sigma(p^2) \quad \Gamma^{(4)}(s,t,u) = -i\lambda + \Gamma^{(4)}_{\text{1-loop}}(s,t,u) + \Gamma^{(4)}_{\text{2-loop}}(s,t,u) + \cdots \tag{11.11}$$

This decomposition is quite intuitive, as ϕ^4 theory is merely a scalar theory. Its QED counterpart will be much scarier.
Here we acknowledge a certain degree of hand-waviness of the renomalisation parameters:

- All physical quantities are finite.
- All physical quantities are independent of the renormalisation constants Z_i and ϵ (of the $i\epsilon$ fame).

One notable exception to this is the (bare and renormalised) Lagrangian, which is not an observable. With our new friends, the renormalisation factors, the Lagrangian, which we now acknowledge to be actually nothing but the *bare Lagrangian*, can be written as

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4}_{\text{renormalised Lagrangian}} + \underbrace{\frac{\delta Z_\phi}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\delta m^2\phi^2 + \frac{\delta\lambda}{4!}\phi^4}_{\text{counterterms}} \tag{11.12}$$

where the δZ_ϕ, δm^2 and $\delta\lambda$ are the so-called *counterterms*[4]:

Definition 11.4 (ϕ^4 counterterms)

$$Z_\phi = 1 + \delta Z_\phi \quad m_0^2 Z_\phi = m^2 + \delta m^2 \quad \lambda_0 Z_\phi^2 = \mu^{-\varepsilon}(\lambda + \delta\lambda) \tag{11.13}$$

Let us justify the counterterms one by one:

- Both Z_ϕ and δZ_ϕ are dimensionless.
- Z_ϕ is dimensionless, while m_0^2, m^2 and δm^2 have mass dimensions of 2.
- λ_0 is dimensionless, while from Equation 11.6, λ has mass dimension ϵ. To preserve the dimensionlessness of λ_0, we introduce the term μ^ε, where μ is the so-called *renormalisation scale* and has mass dimension 1.

Thus, we have explicitly established the ultimate goal of renormalisation. To solve for the renormalised quantities or the renormalisation constants, we must calculate the counterterms δ.

Note 11.1 (Running coupling) Importantly, we observe that the renormalised mass and field are independent of the renormalisation scale μ while the renormalised coupling constant is not. The *renormalised* coupling λ is well-known as the *running coupling* as the dependence shows that λ 'runs' with μ.

But what does μ physically mean? As it turns out, it is a part of *renormalisation conditions*, which are constraints imposed on the Z_is. This is actually an umbrella term for many concepts:

- We choose an arbitrary *renormalisation scale* μ[5] which the counterterms δ are dependent on.
- A corresponding (and likewise arbitrary) *renormalisation point*, which is essentially an artificial constraint we place on μ, is defined.
- Different choices of renormalisation conditions correspond to different *renormalisation schemes*, each of which fixes the counterterms in a specific way.

Some comments on renormalisation schemes should be made:

- Renormalisation schemes are merely mathematical conveniences, and physical variables should not differ among schemes.

[4] We can think of the divergence as rubbish that leaks out during a calculation. We don't simply throw the rubbish onto the bare parameter directly. Instead, we introduce the counterterm, a trash bin, whose only job is to collect and *cancel* out the garbage. Once that is done, the renormalised result is 'clean'.

[5] This is meaningless without the next bullet point.

- The lack of any technical restrictions on the choice of μ can be exploited: It is often practically convenient to choose μ to be of the same order as the characteristic energy scale of the physical process being studied.
- As different renormalisation schemes are different only by the choice of μ, the different incarnations of a renormalised quantity in different schemes are related to each other by finite constants.
- However, the renormalised Green's function is always the same regardless of the renormalisation scheme. This invariance (i.e. symmetry) gives rise to the *renormalisation group* which we will soon discuss. The renormalisation group also ties up our other loose end, which is the running of the coupling constants.

11.4 Renormalisation schemes

After many, many pages, we are now in a position to actually carry out what we set out to do at the beginning of this chapter - removing the divergence. Now that we have established that our arbitrary choice of μ does not affect the renormalised quantities, we can discuss specific *renormalisation schemes*. In this section, we shall introduce three schemes, which are by design almost always used with dimensional regularisation:

- The *minimal subtraction (MS) scheme*.
- The *modified minimal subtraction ($\overline{MS}$) scheme*, which is a slightly modified form of the MS scheme.
- The *on-shell (OS) scheme* or the *physical scheme*.

The first question that arises is why dimensional regularisation is the 'golden boy' of all three schemes. Previously, we introduced the renormalisation scale μ as a bookkeeping device to make the coupling constants dimensionally correct. The same can be said about the regulator M in dimensional regularisation. As such, we recognise that M is nothing but μ. For a tadpole diagram, this means:

$$I_2 = -\frac{m_0^2}{(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) + \frac{\mu^2}{m_0^2}\right) \tag{11.14}$$

From here on, the three schemes are rather intuitive. The idea is to modify the Feynman integral according to our renormalisation conditions, which ultimately leave the divergence (e.g. $\frac{2}{\varepsilon}$) is intact by design. The infinities in the bare term and the counterterm cancel out.

> **Quote 11.3** -The infinities might cancel each other out.
> -Grüß Gott! They just might.
>
> *Barclay and Einstein, in 'The N^{th} Degree'*

Let us first introduce the MS and $\overline{\text{MS}}$ schemes, which are highly similar.

Definition 11.5 (MS renormalisation conditions) In the MS scheme, only the divergent part $\frac{2}{\varepsilon}$ (i.e. the infinite pole) is preserved in the counterterm. It is so-called because the minimal possible subtraction is made: *only* the divergent pole $\frac{2}{\varepsilon}$ is deleted from the final renormalised mass.

Definition 11.6 ($\overline{\text{MS}}$ renormalisation conditions) The $\overline{\text{MS}}$ scheme is almost identical to the MS scheme. However, instead of only preserving the infinity in the counterterm, we also preserve the finite constants, removing only the renormalisation scale term $\frac{\mu^2}{m_0^2}$. The $\overline{\text{MS}}$ scheme is often preferred over the MS scheme as the nasty finite constants are also removed.

Now we go onto the OS scheme, which, terrifyingly, does not get rid of the arbitrary renormalisation scale μ from the final result. Instead, all quantities (mass, coupling/charge, etc.) are defined to be the physical observables themselves[6] through setting the renormalisation point to correspond exactly to the on-shell condition:

[6] Hence its alternate name.

Definition 11.7 (OS renormalisation point)

$$p^2 = m^2 \tag{11.15}$$

The implication is that any required scale dependence is effectively hidden inside the physical parameters. Instead of deleting certain terms, we instead define a series of renormalisation conditions around the renormalisation point $p^2 = m^2$:

Definition 11.8 (ϕ^4 theory OS renormalisation conditions)

- **Mass counterterm:** The 2-point propagator vanishes exactly at the physical mass shell

$$\Gamma^{(2)}(p^2 = m^2) = 0 \tag{11.16}$$

  This ensures the physical renormalised mass m is the pole of the full propagator, i.e., the physical particle has mass m.

- **Field counterterm:**

$$\left.\frac{d\Gamma^{(2)}(p^2)}{dp^2}\right|_{p^2=m^2} = 1 \tag{11.17}$$

  This ensures the propagator has unit residue at the pole, corresponding to a properly normalised one-particle state.

- **Coupling counterterm:** The renormalised 4-point vertex function $\Gamma^{(4)}(p_i)$ is set to equal the physical coupling λ at a particular kinematic configuration. In the on-shell scheme, this is usually:

$$\Gamma^{(4)}(p^2 = m^2) = -i\lambda \tag{11.18}$$

  This means that the coupling λ is defined as the physical 2-to-2 scattering amplitude at the symmetric point where all external particles are on-shell and the center-of-mass energy is just at threshold.

The final loose end we need to take care of in the OS scheme is the surviving μ. To tie it up, we once again consult dimensional analysis. For physical intuition, we hence set μ to be simply the mass m, which has the benefit of making logarithm corrections simple:

$$\mu = m \tag{11.19}$$

This brings about another subtlety. Curiously, we note that the renormalised coupling, which depends on μ, now becomes

$$\lambda_0 Z_\phi^2 = m^{-\varepsilon}(\lambda + \delta\lambda) \tag{11.20}$$

which is no longer dependent on μ. The implication is that the coupling $\lambda(\mu)$, which runs with μ in the general case, is no longer running. Instead, it is fixed to m. This is exactly the point of the OS scheme. By making the fixture $\mu = m$, we have now defined the otherwise running coupling to be the actual, physically observed coupling.

11.5 1-loop renormalisation of ϕ^4 theory

Let us get a first taste of what renormalised variables look like:

Derivation 11.1 (Renormalised field) By applying the renormalisation condition in Equation 11.16 to Equation 11.11, the field counterterm can be derived as

$$\delta Z_\phi = -\left.\frac{\partial\Sigma(p^2)}{\partial p^2}\right|_{p^2=m^2} \tag{11.21}$$

where the on-shell condition is accounted for. However, the self-energy is related to the Feynman integral, which is momentum-independent, by $\frac{\lambda}{2}$, and is as such also momentum-independent. This gives

$$\delta Z_\phi = 0 \to Z_\phi = 1 \tag{11.22}$$

in the tadpole and bubble diagrams. That is to say, in the 1-loop order, the renormalised field is identical to the bare field. This applies to all three schemes. In fact, the simplest correction emerges in the 2-loop order in a sunset diagram.

Derivation 11.2 (Renormalised mass) From Equation 11.22, the mass counterterm relationship for a tadpole diagram reduces nicely to

$$m_0^2 Z_\phi = m_0^2 = m^2 + \delta m^2 \tag{11.23}$$

From mass-energy equivalence (note that momentum is already out of the picture from the renormalised field), the mass counterterm is

$$\delta m^2 = \frac{\lambda}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 - m_0^2 + i\epsilon} + \underbrace{O(\lambda^n)}_{①} \tag{11.24}$$

where only the first-order counterterm is written explicitly, and ① are higher-order counterterms. Treating the first-order counterterm (i.e. tadpole) Feynman integral with dimensional regularisation, we find that it is

$$\delta m^2 = -\frac{m_0^2\lambda}{2(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) + \frac{\mu^2}{m_0^2}\right) \tag{11.25}$$

- **MS scheme:** The Feynman integral and the first-order counterterm are

$$I_{2,\mathrm{MS}} = \frac{m_0^2}{(4\pi)^{d/2}}\frac{2}{\varepsilon} \quad \delta m^2_{\mathrm{MS}} = -\frac{m_0^2\lambda}{(4\pi)^{d/2}\varepsilon} \tag{11.26}$$

  Thus, the divergence-free renormalised mass becomes

$$m^2 = m_0^2 + \delta m^2 = m_0^2 - \frac{m_0^2\lambda}{(4\pi)^{d/2}\varepsilon} \tag{11.27}$$

- **$\overline{\mathrm{MS}}$ scheme:** The Feynman integral and the first-order counterterm are

$$I_{2,\overline{\mathrm{MS}}} = \frac{m_0^2}{(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right) \quad \delta m^2_{\overline{\mathrm{MS}}} = -\frac{m_0^2\lambda}{2(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right) \tag{11.28}$$

  Thus, the renormalised mass is

$$m^2 = m_0^2 + \delta m^2 = m_0^2 - \frac{m_0^2\lambda}{2(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right) \tag{11.29}$$

- **OS scheme:** The mass counterterm is fixed to exactly remove the self-energy correction:

$$m^2_{\mathrm{phys}} = m^2 + \Sigma^{(1)}(m^2) - \delta m^2 = m^2 \tag{11.30}$$

  Thus, the Feynman integral and the renormalised mass are

$$I_{2,\mathrm{OS}} = \frac{m_0^2}{(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi + \frac{\mu^2}{m_0^2}\right) \quad m^2 = m_0^2 + \delta m^2 = m_0^2 - \frac{m_0^2\lambda}{2(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi + \frac{\mu^2}{m_0^2}\right) \tag{11.31}$$

Derivation 11.3 (Renormalised coupling) Only the 4-point 1-loop (bubble) diagram contributes to the coupling counterterm. We remember that using dimensional regularisation, its integral evaluates as Equation 10.54. The general form of counterterm reads

$$\delta\lambda = \sum_i^{\text{all diagrams}} C_i \mathrm{divergence}_i \tag{11.32}$$

where C is the multiplicity.
At one loop order, the only contributing diagram is the 4-point, 1-loop diagram, which has 3 possible Feynman diagrams corresponding to the s, t and u channels. The counterterm hence reads

$$\delta\lambda = 3I_4 \tag{11.33}$$

- **MS scheme:**

$$\delta\lambda = 3\frac{i\lambda^2\pi^{d/2}}{2(2\pi)^d}\frac{2}{\varepsilon} \tag{11.34}$$

- **$\overline{\text{MS}}$ scheme:**

$$\delta\lambda = 3\frac{i\lambda^2\pi^{d/2}}{2(2\pi)^d}\left(\frac{2}{\varepsilon} - \gamma\right) \tag{11.35}$$

- **OS scheme:** The conditon $\Gamma^{(4)}(p^2 = m^2) = -i\lambda$ makes the fixture $\lambda_{\text{phys}} = \lambda$. Hence

$$\delta\lambda = 3\frac{i\lambda^2\pi^{d/2}}{2(2\pi)^d}\left(\frac{2}{\varepsilon} - \gamma + \ln\frac{m_0^2}{\mu^2}\right) \tag{11.36}$$

One problem arises here. We see that different renormalisation schemes give different counterterms:

Aphorism 11.1 (Felix Halbwedl, 22 March 2025) What matters most is the divergent part of the counterterms, they have to agree for all incarnations.
The finite part of the counterterms is not relevant at all, it can be anything finite.
The job of the counterterm is to hunt down and kill the divergency inside the bare mass. It cannot retrieve the physical mass held captive by the divergency. There we need to ask Mama nature for a helping hand, and measure the physical mass.
I know, it's mathematically horrible, but in the end it's Mama nature we want to describe.

In this sense, the dependence of the renormalised quantities on the renormalisation scale is also eliminated by measuring quantities physically. This is the central point of the next two sections.

11.6 Callan-Symanzik equation

The creation of our so-called renormalisation scale may not seem immediately satisfying, as we have merely transferred the arbitrariness of δ with an arbitrariness of the renormalisation scheme (i.e. of μ). This apparent contradiction is ultimately reconciled by the so-called *Callan-Symanzik equation*, which makes sure that the theory remains physically meaningful despite the arbitrariness of μ.

Derivation 11.4 (Callan-Symanzik equation)

Quote 11.4 Now we exploit the simple but powerful fact that the bare vertex functions[a] $\Gamma_0^{(n)}$ do not know anything about the renormalisation scale μ.

John Cardy

[a] For our example, we use a generic bare Green's function $G_{n,0}$ instead.

As the bare n-point Green's function $G_0^{(n)}$is independent from the renormalisation scale μ, it is safe to write

$$\mu\partial_\mu G_0^{(n)}(p, \lambda_0) = 0 \tag{11.37}$$

By consulting the part of Equation 11.10 that concerns the Green's function, we can write $G_0^{(n)}$ in terms of Z_ϕ and $G^{(n)}$. Hence, the equation above becomes

$$\mu\partial_\mu(Z_\phi^{n/2}(\lambda, \mu)G^{(n)}(p, \lambda, \mu)) = 0 \tag{11.38}$$

Finally, we use the chain rule:

Theorem 11.4 (Callan-Symanzik equation) To preserve the invariance of physical variables, any *direct* change of Green's function $G_n(p, \lambda, \mu)$ due to the change of μ is compensated by corresponding changes of G_n due to changes of the coupling constant λ and the field ϕ, and the *total* change of G_n arising from μ is zero:

$$\frac{\partial G_n}{\partial \mu} + \frac{\partial \lambda}{\partial \mu}\frac{\partial G_n}{\partial \lambda} + \frac{n}{Z_\phi}\frac{\partial Z_\phi}{\partial \mu}G_n = 0 \tag{11.39}$$

where n is the number of points of the n-point function.

Let us go through the terms one by one:

- The first term is the *direct* dependence of G_n on the renormalisation scale μ.
- The second term is the *implicit* dependence of G_n on μ, but through the running coupling $\lambda(\mu)$, which is ultimately dependent on (or 'runs' with) μ. Due to this dependence, $\lambda(\mu)$ is ultimately not a physical quantity[a].
- The third term is a rescaling that accounts for the fact that the field itself is renormalised, and represents the contribution of the field strength renormalisation to the dependence of G_n on μ.
- As the sum of the terms is zero, μ is made irrelevant w.r.t. the physical quantities. We often speak of this as the scale dependence being essentially 'absorbed' into physical quantities.

Remark 11.2 As noted in Quote 11.4, it should be immediately obivous that Equation 11.39 can also be derived from $\Gamma^{(n)}$. In fact, the same can be said for the field ϕ. The equations are

$$\frac{\partial \Gamma^{(n)}}{\partial \mu} + \frac{\partial \lambda}{\partial \mu}\frac{\partial \Gamma^{(n)}}{\partial \lambda} - \frac{n}{2Z_\phi}\frac{\partial Z_\phi}{\partial \mu}\Gamma^{(n)} = 0 \quad \frac{\partial \phi}{\partial \mu} + \frac{1}{\phi}\frac{\partial \phi}{\partial \mu}\phi = 0 \tag{11.40}$$

where the running coupling term is missing in the ϕ equation as the field has nothing to do with the coupling constant.

[a] Recall from experimental HEP that perturbation theory naturally fails for large coupling constants. Since different choices of μ effectively shift the way we split between 'low-energy' and 'high-energy' contributions, $\lambda(\mu)$ changes accordingly.

To better illustrate its physical meaning, we can now introduce two functions which are dimensionless and thus depend only on the equally dimensionless λ:

Definition 11.9 (Beta function) The *beta function* $\beta(\lambda)$, which describes how running coupling 'runs' with μ:

$$\beta(\lambda) = \frac{\delta \lambda}{\delta \ln \mu} = \mu \frac{\delta \lambda}{\delta \mu} \tag{11.41}$$

where δ is nothing but the functional derivative. This expression is also called the *renormalisation group equations*.

Definition 11.10 (Anomalous dimension) The *anomalous dimension* $\gamma(\lambda)$, which is technically a correction to the *scaling dimension*:

$$\gamma(\lambda) = -\frac{\mu}{Z_\phi}\frac{\delta Z_\phi}{\delta \mu} \tag{11.42}$$

We then have the massless version of Equation 11.39, which reads

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda)\frac{\partial}{\partial \lambda} + n\gamma(\lambda)\right) G_n = 0 \tag{11.43}$$

This establishes a relation between the dependence of G_n on μ and the dependence of G_n on λ.

Note 11.2 (Plot twist) At this point, we note that:

- The total dependence $\mu \frac{\partial}{\partial \mu} + \beta(\lambda)\frac{\partial}{\partial \lambda}$ of the Green's function G_n or the vertex $\Gamma^{(n)}$, explicit *and* implicit, is in fact non-zero and is equal to the field renormalisation term $n\gamma(\lambda)$.

- The presence of the field renormalisation term thus reveals a sinister plot twist. Neither G_n nor $\Gamma^{(n)}$ are physical quantities.

11.7 Renormalisation group

Now that we have eliminated $\lambda(\mu)$ and G_n as unphysical variables, we can finally turn to physical variables.

Derivation 11.5 (Renormalisation group invariance) A physical observable O observes

$$O \sim Z_\phi^{-n} G_n \tag{11.44}$$

where Z_ϕ^{-n} cancels out with the field renormalisation dependence of the Green's function. We thus have

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda}\right) O = 0 \tag{11.45}$$

which implies that the total explicit and implicit dependence of O on μ is zero.
This can be written more concisely. Inserting the full form of the beta function gives

$$\mu\left(\frac{\partial}{\partial\mu} + \frac{d\lambda}{d\mu}\frac{\partial}{\partial\lambda}\right) O = 0 \tag{11.46}$$

Contracting the chain rule, and we have

$$\mu\frac{d}{d\mu}O = 0 \tag{11.47}$$

Finally, we see that a physical observable O is μ-independent.

The μ-invariance illustrated by Equation 11.47 is interesting as it reminds us of symmetries under Lie groups we saw in *Spinors & Symmetries*. For this reason, we often speak of a *renormalisation group*. This is not actually a group, but rather a semigroup s transformations are not necessarily invertible.

Derivation 11.6 (Running coupling) We also want to establish a relation between the dependence of G_n on p and the dependence of G_n on λ. Such a relation can be found in a second alternative form of Equation 11.39, which involves a rescaling of the momentum. Let us begin with the *classical scaling equation*:

$$\left(p\frac{\partial}{\partial p} - n[\phi] + \mu\frac{\partial}{\partial\mu}\right) G_n = 0 \tag{11.48}$$

where we recall that $[\phi]$ is the mass dimension of ϕ.
Subtracting Equation 11.43 from Equation 11.48 gives

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda} - n([\phi] + \gamma(\lambda))\right) G_n = 0 \tag{11.49}$$

This equation can be fitted for our good friend, the running coupling:

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial\lambda}\right) \lambda(p/\mu) = 0 \tag{11.50}$$

Another note is that this running coupling must be a solution to the beta function equation (Equation 11.41), and an initial condition exists in the form of

$$\lambda(1) = \lambda_0 \tag{11.51}$$

One can thus rewrite the equation as

$$\int_{\lambda_0}^{\lambda(p/\mu)} \frac{\delta\lambda'(p/\mu)}{\beta(\lambda')} = \ln(p/\mu) \tag{11.52}$$

To sum it up, this equation tells us that a change in p inevitably induces a rescaling of λ and a rescaling

of ϕ by proxy of $\gamma(\lambda)$:

- For $\beta(\lambda) > 0$, p rises and falls as λ does.
- For $\beta(\lambda) < 0$, p rises as λ falls and falls as λ rises.
- For $\beta(\lambda) = 0$, we label $\lambda|_{\beta(\lambda)=0} := \lambda^*$.

The idea of the *renormalisation group flow* lies in the change of the coupling constant λ due to the change in the beta function $\beta(\lambda)$[7]. In other words, it describes how $\beta(\lambda)$ 'flows' with λ. This is captured in Equation 11.43, the massless Callan-Symanzik equation. Disregarding the anomalous dimension $\gamma(\lambda)$, we see that

- For $\beta(\lambda) > 0$, the RG flow is towards larger values of λ:
  - The IR singularity happens when $\lambda \to 0$ and $\beta(\lambda) \to 0$. Due to the weak λ, perturbation theory is well-suited for the IR region. This is known as an *IR stable fixed point*.
  - The UV singularity happens when λ is *large* and positive. $\beta(\lambda)$ is thus *large* and positive. Due to the weak λ, perturbation theory fails in the UV region.
- For $\beta(\lambda) < 0$, the RG flow is towards smaller values of λ:
  - The UV singularity happens when $\lambda \to 0$ and $\beta(\lambda) \to 0$. Due to the weak λ, perturbation theory is well-suited for the UV region. This is known as an *UV stable fixed point*.
  - The IR singularity happens when λ is *large* and positive. $\beta(\lambda)$ is thus *large* and negative. Due to the weak λ, perturbation theory fails in the IR region.
- For $\beta(\lambda) = 0$, we consult Equation 11.49:
  - If $\beta(\lambda)$ goes from negative to positive through λ^*, we will see the momentum approaching zero as $\lambda \to \lambda^*$:
    $$p \to 0 \tag{11.53}$$
    This is an *IR stable zero*.
  - If $\beta(\lambda)$ goes from positive to negative through λ^*, we will see the momentum approaching infinity as $\lambda \to \lambda^*$:
    $$p \to \infty \tag{11.54}$$
    This is an *UV stable zero*.

  This is an interesting case, as we can extract the behaviour of the Green's functions by setting $\lambda \to \lambda^*$ in Equation 11.49.

11.8 Källén-Lehmann spectral representation

Finally, it is useful to reflect upon what we have gone through in a more physical way. The default form of a renormalised tree-level Feynman propagator with mass m in Equation 11.10 should be well-known to the reader at this point. However, we have yet to expand this to higher orders.
As it turns out, there is a default way to represent a propagator of arbitrary order called the *Källén-Lehmann spectral representation*, where the full propagator *in momentum space* is solved via integrating over the mass parameter/renormalisation scale μ:

Definition 11.11 (Källén-Lehmann spectral representation)

$$G = i \int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{p^2 - m^2 + i\epsilon} \tag{11.55}$$

Note 11.3 The Källén-Lehmann spectral representation does not actually assist with calculating counterterms. Rather, it shows that interacting propagators can always be expressed as a weighted sum of

[7] Which itself dictates how λ evolves with the renormalisation scale μ.

free propagators, which provides a non-perturbative insight into the structure of the theory.

The central point of this formalism is as follows:

- Effectively, the tree-level propagator can be regarded as a free particle, regardless of whether the theory itself is free or interacting.
- Higher-order propagators, which are effectively quantum corrections, represent the interacting parts of an (interacting) theory.

We can show this formulaically. For a free particle with mass m, the spectral function is

$$\rho(\mu^2) = Z_\phi^{-1}\delta(\mu^2 - m^2) \tag{11.56}$$

This ρ fixes μ to m, and Equation 11.55 reduces to Equation 11.10.
For interacting fields, our propagator under the Källén-Lehmann spectral representation can be rewritten as

$$G = \underbrace{\frac{i}{Z_\phi(p^2 - m^2 + i\epsilon)}}_{\text{free particle term}} + \underbrace{i\int_0^\infty d\mu^2 \frac{\sigma(\mu^2)}{p^2 - m^2 + i\epsilon}}_{\text{continuum terms}} \tag{11.57}$$

where the continuum terms are interacting field contributions from multi-particle states. Hence, the function $\sigma(\mu^2)$ is defined to be explicitly the interacting contribution:

$$\sigma(\mu^2) = \rho(\mu^2) - Z_\phi^{-1}\delta(\mu^2 - m^2) \tag{11.58}$$

Remark 11.3 Hence, the physical meaning of Equation 11.55 and (more obivously) Equation is seen. It describes the full propagator, representing the high-order quantum corrections that are deviations from the free propagator in Equation 11.10.
We end with some mathematical discussions. Intuitively, both $\rho(\mu^2)$ and $\sigma(\mu^2)$ are positive, the first of which implies that, for some order n of differentiation:

$$\frac{\partial^n D\left(-p^2\right)}{\partial\left(p^2\right)^n} = (-1)^n i\int_{m_t^2}^\infty d\mu^2 \frac{\rho\left(\mu^2\right)}{\left(p^2+\mu^2\right)^n} \tag{11.59}$$

Inserting the decomposed Equation 11.3 into this gives

$$Z_\phi^{-1} + \int_{m_t^2}^\infty d\mu^2\sigma(\mu^2) = 1 \tag{11.60}$$

As $\sigma(\mu^2) > 0$, we have $0 \leq Z_\phi^{-1} < 1$.

Chapter 12

Renormalisation II: QED

> **Quote 12.1** Miew. Mowem Purrum.
>
> *Felix Halbwedl, 8 March 2025*

We are finally in a position to calculate the QED counterterms. Fortunately, the conceptual ideas of renormalisation have already been detailed in the last chapter. The MS, $\overline{\text{MS}}$ and OS schemes we have introduced also apply to QED. All we have to do is to develop a corresponding framework for vector fields.

12.1 Ward-Takahashi identity

With the renormalisation of QED, we are now again in the realm of gauge theories. As it turns out, gauge invariance affects renormalisation factors in some funny ways. This can be verified by going over the Slavnov-Taylor, Ward-Takahashi and Ward identities, which are the final development from applying Noether's theorem in QFT that we will see in a while.
It is expeidient to clarify the context that each identity live in, and how the identities relate to each other:

- The Slavnov-Taylor identity applies to the most general case, which is non-abelian gauge theories like QCD.
- The Ward-Takahashi identity is a reduction of the Slavnov-Taylor identity in the abelian limit, like QED.
- The Ward identity is a reduction of the Ward-Takahashi identity in the limit when the external fermions are on-shell and when momentu transfer is zero[1].

As we have not reached non-abelian gauge theories yet, we first derive the Ward-Takahashi identity.

Derivation 12.1 (Ward-Takahashi identity) Beginning with the QED generating functional

$$Z[\bar{J}, J, J^\mu] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp\left[i\int d^4x \left(\mathcal{L}_{\text{QED}} + \bar{J}\psi + \bar{\psi}J + J^\mu A_\mu\right)\right] \tag{12.1}$$

where we recall J and $\bar{(J)}$ to be the fermion source terms, J^μ to be the boson source term, and that there exist the shorthands

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad D_\mu = \partial_\mu + ieA_\mu$$

It is again the Dyson-Schwinger equations that save the day. We recall that the generating functional is invariant under local gauge transformations, which, in QED, is our good friend, the U(1) symmetry:

$$\psi \to (1 + i\alpha(x))\psi, \quad \bar{\psi} \to \bar{\psi}(1 - i\alpha(x)), \quad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$$

[1] This is also called the *soft photon limit*.

Let us now solve for the variation of the generating functional. Intuitively, the variations of the fields read

$$\delta\psi(x) = i\alpha(x)\psi(x) \quad \delta\bar{\psi}(x) = -i\alpha(x)\bar{\psi}(x) \tag{12.2}$$

Under this, the action varies as

$$\delta S = \int d^4x \alpha(x)\partial_\mu j^\mu(x) \tag{12.3}$$

and the source terms vary as

$$\delta(\bar{J}\psi + \bar{\psi}J) = i\alpha(x)\bar{J}(x)\psi(x) - i\alpha(x)\bar{\psi}(x)J(x) \tag{12.4}$$

Putting it all together, the total variation of the integrand on the exponential is, to first order in α:

$$\delta\left[S + \int d^4x(\bar{J}\psi + \bar{\psi}J)\right] = \int d^4x\alpha(x)\left[\partial_\mu j^\mu(x) + i\bar{J}(x)\psi(x) - i\bar{\psi}(x)J(x)\right] \tag{12.5}$$

Hence, the variation of the generating functional reads

$$\delta Z = i\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS+i\int \bar{J}\psi+\bar{\psi}J}\int d^4x\alpha(x)\left[\partial_\mu j^\mu(x) + i\bar{J}(x)\psi(x) - i\bar{\psi}(x)J(x)\right] \tag{12.6}$$

Again, consulting the Dyson-Schwinger equations, we know that $\delta Z = 0$. We can then divide both sides by $i\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS+i\int \bar{J}\psi+\bar{\psi}J}$: Factor out the $\alpha(x)$ and write in terms of expectation values:

$$\int d^4x\alpha(x)\left[\partial_\mu\langle j^\mu(x)\rangle + \bar{J}(x)\langle\psi(x)\rangle - \langle\bar{\psi}(x)\rangle J(x)\right] = 0 \tag{12.7}$$

Let us consider the simplest vertex, which is a three-point function with two fermions and one photon, where the fermion absorbs/emits a photon of momentum q[a]. According to Equation 8.24, taking functional derivatives of the generating functional with respect to J and $\bar{J}$ gives us the Green's function:

$$\partial^x_\mu G^\mu(x,y,z) = \partial_\mu\langle T[j^\mu(x)\psi(y)\bar{\psi}(z)]\rangle = \delta(x-y)\langle T[\psi(y)\bar{\psi}(z)]\rangle - \delta(x-z)\langle T[\psi(y)\bar{\psi}(z)]\rangle \tag{12.8}$$

where x, y and z are the 4-positions of the three legs.
We take Fourier transforms of both sides. For each component, this works as

$$\begin{aligned} j^\mu(q) &= \int d^4x e^{iq\cdot x} j^\mu(x) \quad \psi(p+q) = \int d^4y e^{i(p+q)\cdot y}\psi(y) \\ \bar{\psi}(p) &= \int d^4z e^{-ip\cdot z}\bar{\psi}(z) \quad \langle T[\psi(y)\bar{\psi}(z)]\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(y-z)} S_F(p) \end{aligned} \tag{12.9}$$

where p is the total (4-)momentum of the incoming fermion, and q is the momentum of the incoming/outgoing photon[b].
Putting it all together, we see that the delta functions are eliminated:

$$\int d^4x e^{iqx}\partial_\mu\langle T[j^\mu(x)\psi(y)\bar{\psi}(z)]\rangle = e^{iqy}S_F(y-z) - e^{iqz}S_F(y-z) \tag{12.10}$$

This reduces to

$$iq^\mu\langle j^\mu(q)\psi(p+q)\bar{\psi}(p)\rangle = S_F(p) - S_F(p+q) \tag{12.11}$$

where From the OPI decomposition, we can rewrite the LHS in terms of the vertex function $\Gamma^\mu(p+q,p)$:

$$\langle j^\mu(q)\psi(p+q)\bar{\psi}(p)\rangle = S_F(p+q)\Gamma^\mu(p+q,p)S_F(p) \tag{12.12}$$

Inserting this gives

$$iq^\mu S_F(p+q)\Gamma_\mu(p+q,p)S_F(p) = S_F(p) - S_F(p+q) \tag{12.13}$$

Dividing both sides by $S_F^{-1}(p+q)S_F^{-1}(p)$, we find the common form of the Ward-Takahashi identities:

Theorem 12.1 (Ward-Takahashi identity)

$$iq^\mu \Gamma_\mu(p+q,p) = S_F^{-1}(p+q) - S_F^{-1}(p) \tag{12.14}$$

where $\Gamma_\mu(p+q,p)$ is the full (ampurared) vertex function, $S_F(p+q)$ is the full (interacting, renormalised) fermion propagator and $S_F(p)$ is the same propagator but lacking one external photon.

The Ward-Takahashi identity is also called the *rigid identity*, as *rigid symmetries* is an alternative name for global symmetries.

[a]Note that this is effectively a momentum transfer by q on the fermion.
[b]Note that the sign is automatically taken care of if we know whether it is incoming or outgoing.

Quote 12.2 Because the Ward-Takahashi identity relates the $n+1$-point function with the n-point function, which is useful if you want to construct the most general 3-points function.

Felix Halbwedl, on why Feynman diagrams are employed in the Ward-Takahashi identity, 5 February 2025

Some physical comments can be made:

- The identity is a non-perturbative statement as no perturbation theory is used. It holds beyond just tree level.
- While we used the simplest case of the fermion-photon propagator in our derivation, the final identity in Equation 12.14 actually applies to all vertices in QED. That is to say, we can use this identity for any vertex in QED that has at least one external photon involved (we will not prove this).
- In this case, we only consider (or 'zoom into') the fermion that the photon operates on, which has the incoming and outgoing full propagators $S_F(p)$ and $S_F(p+q)$, and all other fermions or photons in the vertex are ignored. $\Gamma_\mu(p+q,p)$ is still the full vertex, which can now be better written as $\Gamma_\mu(\cdots, p+q, p)$ where $\cdots$ are the momenta of all other particles we have ignored.

In conclusion, the key takeaway from this identity is twofold:

- Following directly from current conservation and Noether's theorem, there is, when a fermion emits or absorbs one photon, a correlation between the pre- and post-emission/absorption full fermion propagators with the product of the photon momentum and the full vertex function.
- The longitudinal part of the (3-point) vertex function (i.e. the part proportional to q^μ) is completely determined by the difference in the fermion self-energies (which are 2-point functions).

Note 12.1 (Sneak peek of gauge theory) The Ward-Takahashi identity is actually a specific form of the Slavnov-Taylor identity, which is used in the (most general case of) non-abelian gauge theories we will soon discuss, like QCD. The Slavnov-Taylor identity is derived from the BRST symmetry of the gauge-fixed Lagrangian.

$$q^\mu \Gamma_\mu(p+q,p) = S_F^{-1}(p+q)G(p+q,p) - G(p+q,p)S_F^{-1}(p) \tag{12.15}$$

where $G(p+q,p)$ is the ghost-fermion scattering kernel. This term disappears in the abelian limit, and we are left with the Ward-Takahashi identity.

12.2 Ward identity

Let us prove that the Ward-Takahashi identity reduces to the Ward identity in the long wavelength (IR singularity) case, where the vertex function is on-shell and when momentum transfer is zero ($q \to 0$).

Derivation 12.2 (Ward idnetity) As we are concerned with the $q \to 0$ limit, it is legal to perform a Taylor expansion of the RHS around $q = 0$

$$S_F^{-1}(p+q) - S_F^{-1}(p) \approx q^\mu \frac{\partial S_F^{-1}(p)}{\partial p^\mu} \tag{12.16}$$

Substituting this into the Ward-Takahashi identity, we find

$$q^\mu \Gamma_\mu(p,p) = q^\mu \frac{\partial S_F^{-1}(p)}{\partial p^\mu} \tag{12.17}$$

Eliminating the common q^μ:

Theorem 12.2 (Ward identity)

$$\Gamma_\mu(p,p) = \frac{\partial S_F^{-1}(p)}{\partial p^\mu} \tag{12.18}$$

This is the generalised form of the Ward identity.

We can derive a few useful variants of the Ward identity:

Derivation 12.3 (Differential form of the Ward idnetity) From the known relation

$$S_F(p) = \left[S_F^{-1}(p)\right]^{-1} \tag{12.19}$$

We can use the product rule and find

$$\frac{\partial S_F(p)}{\partial p^\mu} = -S_F(p)\left(\frac{\partial S_F^{-1}(p)}{\partial p^\mu}\right) S_F(p) \tag{12.20}$$

Inserting this into Equation 12.18, we find

$$\frac{\partial S_F(p)}{\partial p^\mu} = -S_F(p)\Gamma_\mu(p) S_F(p) \tag{12.21}$$

which is the differential form of the Ward identity. This is the incarnation of the Ward identity that Yasushi Takahashi started with when he derived the Ward-Takahashi identity in 1957.

Derivation 12.4 (Scattering amplitude form of the Ward idnetity) Consider a process where an *external* photon with momentum q^μ is emitted from an external charged fermion line. The scattering amplitude reads

$$\mathcal{M}^\mu(q) = \bar{u}(p')\Gamma^\mu(p',p)u(p) \tag{12.22}$$

Let us apply the external photon momentum q_μ on both sides:

$$q_\mu \mathcal{M}^\mu(k) = q_\mu \bar{u}(p')\Gamma^\mu(p',p)u(p) \tag{12.23}$$

Using Equation 12.14, we see that

$$q_\mu \mathcal{M}^\mu(k) = \bar{u}(p')[S_F^{-1}(p') - S_F^{-1}(p)]u(p) \tag{12.24}$$

When the external fermions are on-shell, we know that the inverse propagator annihilates on-shell spinors. As such

$$S_F^{-1}(p)u(p) = 0 \quad \bar{u}(p')S_F^{-1}(p') = 0 \tag{12.25}$$

We hence find the all-too-familiar form of the Ward identity

$$q_\mu \mathcal{M}^\mu(q) = 0 \tag{12.26}$$

This essentially reflects current conservation in QED, or equivalently, gauge invariance of the S-matrix. Let us now investigate this external photon. As it is external, there are only two transverse polarisations, whose 4-versors are, by convention:

$$\epsilon^1_\mu = (0,1,0,0) \quad \epsilon^2_\mu = (0,0,1,0) \tag{12.27}$$

As the matrix elements are those of the external photon, they project onto the polarisation versor states. As such, $\mathcal{M}^\mu(q)$ also has only two non-zero components[a] - $\mathcal{M}^1(q)$ and $\mathcal{M}^2(q)$. From Equation 12.18, the momentum hence has

$$q_\mu = (k_0, 0, 0, -k_3) \quad q^\mu = (k_0, 0, 0, k_3) \tag{12.28}$$

where the transverseness of the two polarisations is seen.

[a] Yes! $\mathcal{M}^\mu(q)$ is a vector! Bet you didn't catch that, did you?

12.3 Renormalisation constants

Let us begin with our good friend, the gauge-fixed Lagrangian in Equation 6.128 and make a cosmetic replacement of the gauge 'coupling' ξ in the gauge-fixing term

$$\frac{1}{\xi} = \lambda \quad \frac{1}{2\xi} A^\mu \partial_\mu \partial_\nu A^\nu = \frac{\lambda}{2} A^\mu \partial_\mu \partial_\nu A^\nu \tag{12.29}$$

Again, we acknowledge the fact that the quantities here are actually bare quantities:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}_0 (i\gamma^\mu \partial - m_0)\psi_0 - \frac{1}{2} A_0^\mu (\Box g_{\mu\nu} - \partial_\mu \partial_\nu) A_0^\nu - e_0 \bar{\psi}_0 \gamma^\mu A_{\mu,0} \psi_0 + \frac{\lambda_0}{2} A_0^\mu \partial_\mu \partial_\nu A_0^\nu \tag{12.30}$$

We can now write down the conventional renormalisation factors, some of which are defined slightly differently from those in ϕ^4 theory. To begin with, we start with the provisional set of five renormalisation factors

$$\psi_0 = \sqrt{Z_\psi}\psi \quad A_0 = \sqrt{Z_A} A \quad m_0 = Z_m m \quad e_0 = Z_e e \quad \lambda_0 = Z_\lambda \lambda \tag{12.31}$$

Let us define another three provisional, so-called, renormalisation factors:

$$S_{F,0} = Z_S S_F \quad D^{\mu\nu}_{F,0} F = Z_D D^{\mu\nu}_F \quad \Gamma_0 = Z_\Gamma^{-1} \Gamma \tag{12.32}$$

It should be intuitively obvious that we can represent Z_S, Z_D and Z_Γ[2] via the original five renormalisation factors, which we will prove now.

Derivation 12.5 (Propagators) The fermion and photon propagators are defined as the following two-point functions

$$S_F(x-y) = \langle 0 | T\{\psi(x)\bar{\psi}(y)\} | 0 \rangle \tag{12.33}$$

$$D^{\mu\nu}_F(x-y) = \langle 0 | T\{A^\mu(x) A^\nu(y)\} | 0 \rangle \tag{12.34}$$

Expressing the renormalised quantities in terms of their bare counterparts and renormalisation factors:

$$S_{F,0}(x-y) = \langle 0 | T\{\psi_0(x)\bar{\psi}_0(y)\} | 0 \rangle = Z_\psi \langle 0 | T\{\psi(x)\bar{\psi}(y)\} | 0 \rangle = Z_\psi S_F(x-y) \tag{12.35}$$

$$D^{\mu\nu}_{F,0}(x-y) = Z_A \langle 0 | T\{A^\mu(x) A^\nu(y)\} | 0 \rangle = Z_A D^{\mu\nu}_F(x-y) \tag{12.36}$$

Hence, we find that

$$Z_S = Z_\psi \quad Z_D = Z_A \tag{12.37}$$

Derivation 12.6 (Vertex) Z_Γ is a bit trickier and requires us to use the Ward-Takahashi identity. Let us begin with the Ward-Takahashi identity with renormalised quantities

$$q_\mu \Gamma^\mu(p+q, p) = S^{-1}(p+q) - S^{-1}(p)$$

Writing the renormalised quantities in their bare counterparts and renormalised factors give

$$q_\mu Z_\Gamma \Gamma^\mu_0(p+q, p) = Z_\psi \left[S_0^{-1}(p+q) - S_0^{-1}(p) \right] \tag{12.38}$$

However, we note that the Ward-Takahashi identity also holds for bare quantities:

$$q_\mu \Gamma^\mu_0(p+q, p) = S_0^{-1}(p+q) - S_0^{-1}(p) \tag{12.39}$$

[2] Which, as seen above, is defined differently from the other two renormalisation factors. This becomes convenient in calculating the counterterms.

Dividing Equation 12.39 by Equation 12.38 gives us

$$Z_\Gamma = Z_\psi \tag{12.40}$$

Remark 12.1 Often, Z_Γ, Z_ψ and Z_A are written as Z_1, Z_2 and Z_3 due to having been so-denoted in the original 1950s literature.

This is not the end of our troubles, because as it turns out, even the original five renormalisation factors relate to each other.

Derivation 12.7 ($Z_\lambda Z_A = 1$) This can be proven by looking at the gauge fixing term. Writing all the terms explicitly gives

$$\frac{\lambda_0}{2}\left(\partial_\mu A_0^\mu\right)^2 = \frac{Z_\lambda \lambda}{2}\left(\partial_\mu\left(Z_A^{1/2}A^\mu\right)\right)^2 \tag{12.41}$$

This is simply

$$\frac{\lambda_0}{2}(\partial_\mu A_0^\mu)^2 = Z_\lambda Z_A \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 \tag{12.42}$$

At this point, we note that the gauge fixing term has a unique property: its bare and renormalised forms are *always* identical. Hence

$$Z_\lambda Z_A \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 = \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 \tag{12.43}$$

from which we can immediately read off

$$Z_\lambda Z_A = 1 \tag{12.44}$$

Derivation 12.8 ($Z_e Z_A^{1/2} = 1$) This can be proven by inspecting the interaction term. Again, we write all terms explicitly

$$ie_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0,\mu} = i(Z_e e)(Z_\psi^{1/2}\psi)^\dagger\gamma^\mu(Z_\psi^{1/2}\psi)(Z_A^{1/2}A_\mu) = Z_e Z_\psi Z_A^{1/2}(ie\bar{\psi}\gamma^\mu\psi A_\mu) \tag{12.45}$$

Recall that the interaction term represents the vertex function. As such, this actually describes vertex renormalisation

$$ie_0\bar{\psi}_0\gamma^\mu\psi_0 A_{0,\mu} = Z_\Gamma(ie\bar{\psi}\gamma^\mu\psi A_\mu) = Z_e Z_\psi Z_A^{1/2}(ie\bar{\psi}\gamma^\mu\psi A_\mu) \to Z_\phi = Z_e Z_\psi Z_A^{1/2} \tag{12.46}$$

from which we can immediately read off

$$Z_e Z_A^{1/2} = 1 \tag{12.47}$$

We are now finally in a position to state all the QED renormalisation factors:

Definition 12.1 (QED renormalisation factors)

- **Variables:**

$$\psi_0 = \sqrt{Z_\psi}\psi \quad A_0 = \sqrt{Z_A}A \quad m_0 = Z_m m \quad e_0 = Z_A^{-1/2}e \quad \lambda_0 = Z_A^{-1}\lambda \tag{12.48}$$

- **Propagators and vertex:**

$$S_{F,0} = Z_\psi S_F \quad D_{F,0}^{\mu\nu}F = Z_A D_F^{\mu\nu} \quad \Gamma_0 = Z_\psi\Gamma \tag{12.49}$$

12.4 Tensor decomposition

In QED, the full forms of the fermion propagator, photon propagator and vertex can still be found by summing the loop contributions. This reads

$$iS^{-1}(p) = iS_0^{-1}(p) + \Sigma(p) \quad i\left(D^{-1}\right)^{\mu\nu}(q) = i\left(D_0^{-1}\right)^{\mu\nu}(q) + \Pi^{\mu\nu}(q) \quad \Gamma^\mu(p,q) = \Gamma_0^\mu(p,q) + \Lambda^\mu(p,q) \tag{12.50}$$

where $\Sigma(p)$ is the fermion self-energy, $\Pi^{\mu\nu}(q)$ is the photon vacuum polarisation and $\Lambda^\mu(p,q)$ is the vertex correction. We are now interested in formulating an expression for these three terms, which will involve

decomposing them. Unlike in Equation 11.11, the fermion and photon propagators and the vertex in QED all have tensorial structures, which forces us to make use of the so-called *tensor decomposition*. Let us go through the terms one by one.

Derivation 12.9 (Fermion propagator) We have two terms, the free (bare) propagator and the self-energy. The free propagator, which reads $S_{F,0}(p) = \frac{i}{\not{p}-m+i\epsilon}$, decomposes as

$$iS_{F,0}^{-1}(p) = \not{p} - m \tag{12.51}$$

Now we investigate self-energy. To preserve Lorentz covariance, the only possible tensor structures are $\not{p}$ and $\mathbb{I}$. The independent variable becomes p^2, which is Lorentz-invariant:

$$\Sigma(p) = \left[1 - Z_2^{-1}(p^2)\right] \not{p} - \left[Z_2^{-1}(p^2) Z_m(p^2) - 1\right] m \tag{12.52}$$

For ease of reading, we often conventionally define the *fermion wavefunction renormalisaton function* $A(p^2)$ *fermion mass function* $M(p^2)$, which are nothing but shorthands:

$$\Sigma(p) = A(p^2)\not{p} + B(p^2)m \tag{12.53}$$

Substituting them into Equation 12.50 the full inverse propagator decomposition:

$$iS_F^{-1}(p) = \not{p} - m + A(p^2)\not{p} + B(p^2)m = \left[1 + A(p^2)\right] \not{p} - \left[1 - B(p^2)\right] m \tag{12.54}$$

$$iS_F^{-1}(p) = A(p^2)(\not{p} - \Sigma_M(p^2)\mathbb{I}) = A(p^2)(\not{p} - \Sigma_M(p^2)) \tag{12.55}$$

Inverting this expression returns us

$$S_F(p) = \frac{i}{\left[1 + A(p^2)\right] \not{p} - \left[1 - B(p^2)\right] m} \tag{12.56}$$

Conversely, we can define the *wavefunction renormalisation function* $Z(p^2)$ and the *momentum-dependent effective mass* $M(p^2)$:

$$Z(p^2) = \frac{1}{1 + A(p^2)} \quad M(p^2) = \frac{1 - B(p^2)}{1 + A(p^2)} m \tag{12.57}$$

This gives us a form similar to the bare propagator

$$S_F(p) = \frac{iZ(p^2)}{\not{p} - M(p^2)} \tag{12.58}$$

Derivation 12.10 (Photon propagator) The photon propagator has rank 2, and the only possible structures are the metric $g^{\mu\nu}$ and the momenta product $q^\mu q^\nu$.
Let us again investigate the terms one by one. In the Feynman gauge, the free photon propagator is

$$D_0^{\mu\nu}(q) = \frac{-ig^{\mu\nu}}{q^2 + i\epsilon} \tag{12.59}$$

Its inverse with a factor of i is then

$$i\left(D_0^{-1}\right)^{\mu\nu}(q) = q^2 g^{\mu\nu} \tag{12.60}$$

Previously in the Ward identity, we have seen that $\Pi^{\mu\nu}(q)$ must be transverse. As such, it takes the form

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \Pi(q^2) \tag{12.61}$$

where $\Pi(q^2)$ is a scalar function known as the *vacuum polarisation scalar*.
Substituting into the inverse propagator, we find

$$i(D^{-1})^{\mu\nu}(q) = q^2 g^{\mu\nu} + \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right) \Pi(q^2) = q^2 \left[g^{\mu\nu} + \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) \Pi(q^2)\right] \tag{12.62}$$

Inverting this again in the Feynman gauge gives

$$D^{\mu\nu}(q) = \frac{-i}{q^2\left[1+\Pi(q^2)\right]}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \lambda^{-1}\frac{-iq^\mu q^\nu}{(q^2)^2} \tag{12.63}$$

We can write a shorthand of this by defining the transverse part of the inverse photon propagator:

$$T^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \tag{12.64}$$

which gives the decomposition as

$$D^{\mu\nu}(q) = \frac{-i}{q^2\left[1+\Pi(q^2)\right]}T^{\mu\nu} + \lambda^{-1}\frac{-iq^\mu q^\nu}{(q^2)^2} \tag{12.65}$$

Derivation 12.11 (Vertex) As per the Feynman rules, the tree-level vertex factor is $-ie\Gamma^\mu = -ie\gamma^\mu$, making the tree-level contribution to the vertex simply the gamma matrices γ^μ. Denoting loop corrections as $\Lambda^\mu(p', p)$, we then have the full, corrected vertex

$$\Gamma^\mu(p', p) = \gamma^\mu + \Lambda^\mu_{\text{1-loop}}(p', p) + \Lambda^\mu_{\text{2-loop}}(p', p) + \cdots \tag{12.66}$$

where the 1-loop correction reads, for free indices α and β

$$\Lambda^\mu(p', p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\gamma^\alpha\frac{i}{\not{k}-m}\gamma^\mu\frac{i}{\not{k}+\not{q}-m}\gamma^\beta\left(\frac{-ig_{\alpha\beta}}{(k-p)^2}\right) \tag{12.67}$$

and so on.
This decomposition is often given a more systematic treatment. Ultimately, we construct Γ^μ from available vectors and matrices: vectors p^μ, $p'^\mu = p^\mu + q^\mu$ and q^μ; Dirac matrices: γ^μ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Finally, we have scalar functions $F_i(q^2)$ known as *form factors*, which depend only on the Lorentz scalar q^2.

$$\Gamma^\mu(p, q) = F_1(q^2)\gamma^\mu + F_2(q^2)i\sigma^{\mu\nu}q_\nu + F_3(q^2)q^\mu \tag{12.68}$$

where:

- $F_1(q^2)$ is the *Dirac form factor* concerned with charge and normalisation.
- $F_2(q^2)$ is the *Pauli form factor* concerned with anomalous magnetic moment.
- $F_3(q^2)$ which vanishes due to the Ward identity in Equation 12.18.

Due to the vanishing of $F_3(q^2)$, the physically relevant vertex decomposition is

$$\Gamma^\mu(p, q) = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i}{2m}\sigma^{\mu\nu}q_\nu \tag{12.69}$$

This is identical to Equation 12.66. To see this, we consider the tree-level case, where $F_1(q^2) = 1$ and $F_2(q^2) = 0$. This recovers the tree-level vertex:

$$\Gamma^\mu(p, q)\big|_{\text{tree}} = \gamma^\mu = \Gamma^\mu_0(p, q) \tag{12.70}$$

The same idea follows at higher orders. The equivalence lies in the fact that the correction $\Lambda^\mu(p', p)$ for each order can be recovered by setting $F_1(q^2)$ and $F_2(q^2)$ to their specific incarnation in that order.

12.5 1-loop renormalisation of QED

Like in ϕ^4 theory, we define the field counterterms w.r.t. the *renormalisation constants*, and the mass and charge counterterms w.r.t. the *physical quantities themselves*. We also define a vertex counterterm δZ_Γ so that the Lagrangian can be written fully in terms of bare and counterterms:

Definition 12.2 (QED counterterms)

$$Z_\Gamma = 1 + \delta Z_\Gamma \quad Z_\psi = 1 + \delta Z_\psi \quad Z_A = 1 + \delta Z_A \quad m_0 Z_\psi = m^2 + \delta m^2 \tag{12.71}$$

From the Ward identity, we know that $Z_\Gamma = Z_\psi$. It is hence easy to see that

$$\delta Z_\Gamma = \delta Z_\psi \tag{12.72}$$

We get two counterterms for the price of one!

The bare Lagrangian then reads

$$\mathcal{L}_{\text{bare}} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu}_{\text{renormalised Lagrangian}} \underbrace{-\frac{1}{4}\delta Z_A F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left(i\delta Z_\phi \not{D} - \delta m\right)\psi - e\delta Z_\Gamma \bar{\psi}\gamma^\mu\psi A_\mu}_{\text{counterterms}} \tag{12.73}$$

A convenient renormalisation point is

$$\not{p} = m \tag{12.74}$$

for which we define the following renormalisation conditions:

Definition 12.3 (QED OS renormalisation conditions)

- **Mass counterterm:** The electron propagator has a pole at the physical (measured) mass m

$$\Sigma(\not{p})|_{\not{p}=m} = 0 \tag{12.75}$$

  where $\Sigma(\not{p})$ is the electron self-energy.

- **Fermion field counterterm:** The residue of the electron propagator at the physical pole is unity

$$\frac{d}{d\not{p}}\left[\not{p} - m - \Sigma(\not{p})\right]\Big|_{\not{p}=m} = 1 \rightarrow \frac{d\Sigma(\not{p})}{d\not{p}}\Big|_{\not{p}=m} = 0 \tag{12.76}$$

- **Photon field counterterm:** The photon remains massless, i.e., its propagator has a pole at $p^2 = 0$

$$\Pi^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi(q^2) \quad \text{with} \quad \Pi(0) = 0 \tag{12.77}$$

  where $\Pi^{\mu\nu}(q)$ is the photon self-energy (vacuum polarisation).

- **Coupling (i.e. charge) counterterm:**

$$Z_1 \Gamma^\mu(p', p)\big|_{p=p', q=0} = \gamma^\mu \tag{12.78}$$

  This is known as the *Thomson limit* and ensures that the vertex function reduces to the bare interaction in the limit of zero momentum exchange.

From here on, we are in a position to calculate the counterterms.

Derivation 12.12 (Fermion field, vertex and mass) The electron self-energy is the sum of the one-loop self-energy and the counterterms, with the standard formula of

$$-i\Sigma(\not{p}) = -i\Sigma_{\text{1-loop}}(\not{p}) + i\Sigma(\delta Z_\phi \not{p} - \delta m) \tag{12.79}$$

Now we impose the renormalisation conditions. By applying Equation 12.75, we have

$$\Sigma_{\text{1-loop}}(m) = m\delta Z_\psi - \delta m \tag{12.80}$$

Consulting Equation 12.76, we find that

$$\frac{d\Sigma_{\text{1-loop}}(\not{p})}{d\not{p}}\Big|_{\not{p}=m} = \delta Z_\psi \tag{12.81}$$

The one-loop self-energy is given by

$$-i\Sigma_{\text{1-loop}}(\not{p}) = (-ie)^2 \int \frac{d^4k}{(2\pi)^2}\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}\gamma^\nu \frac{-ig_{\mu\nu}}{(p-k)^2+i\varepsilon} \tag{12.82}$$

Using Feynman parameterisation and dimensional regularisation, this evaluates as

$$-i\Sigma_{\text{1-loop}}(\not{p}) = -ie^2\frac{\mu^\varepsilon}{(4\pi)^{d/2}}\int_0^1 dx\Gamma(\varepsilon/2)\frac{(4-\varepsilon)m-(2-\varepsilon)x\not{p}}{[(1-x)m^2-x(1-x)p^2]^{\varepsilon/2}} \tag{12.83}$$

We insert this into Equation 12.81, which gives the rather cumbersome result

$$\delta Z_\phi = \frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx\left(\frac{2}{\varepsilon}-\gamma_E-\ln\left(\frac{(1-x)m^2}{\mu^2}\right)+O(\varepsilon)\right)\times\left(-(2-\varepsilon)x+\left(\frac{\varepsilon}{2}\right)x\frac{[4-2x-\varepsilon(1-x)]}{(1-x)}\right) \tag{12.84}$$

Substituting this result back to Equation 12.79 yields

$$\delta m = m\frac{e^2}{16\pi^2}\int_0^1 dx\left(\frac{2}{\varepsilon}-\gamma_E-\ln\left(\frac{(1-x)m^2}{\mu^2}\right)+\mathcal{O}(\varepsilon)\right)\times\frac{(\varepsilon-2)x((1+\varepsilon)-2)[4-2x-\varepsilon(1-x)]}{2(1-x)} \tag{12.85}$$

This suggests that the counterterm δm must be directly proportional to m.

Derivation 12.13 (Photon field) Again, we first acknowledge that the photon self-energy is the sum of the one-loop self-energy and the counterterms

$$i\Pi^{\mu\nu}(q^2) = i\Pi^{\mu\nu}_{\text{1-loop}}(q^2) - i(g^{\mu\nu}q^2q^\mu q^\nu)\delta Z_A \tag{12.86}$$

where the photon one-loop self-energy is given as

$$i\Pi^{\mu\nu}_{\text{loop}} = (-1)(ie)^2\int\frac{d^4k}{(2\pi)^2}\,\text{Tr}\left[\gamma^\mu\frac{i}{\not{k}-m}\gamma^\nu\frac{i}{\not{k}+\not{q}-m}\right] \tag{12.87}$$

where the factor of -1 is due to the presence of one closed fermion loop.
Using Feynman paramterisation and exploiting the γ matrix formulae in Theorem **??**, This becomes

$$i\Pi^{\mu\nu}_{\text{loop}}\left(q^2\right) = -4e^2\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}\frac{N^{\mu\nu}}{\left[x\left((k+q)^2-m^2\right)+(1-x)\left(k^2-m^2\right)\right]^2} \tag{12.88}$$

where we have defined the shorthand

$$N^{\mu\nu} = k^\mu(k+q)^\nu + k^\nu(k+q)^\mu - g^{\mu\nu}[k\cdot(k+q)-m^2] \tag{12.89}$$

By redefining variables and dimensional regularisation, this becomes

$$i\Pi^{\mu\nu}_{\text{loop}} = -i(g^{\mu\nu}q^2-q^\mu q^\nu)\frac{8e^2}{(}(4\pi)^{d/2})\mu^\varepsilon\int_0^1 x(1-x)dx\frac{\Gamma(\varepsilon/2)}{(a^2)^{\varepsilon/2}} \tag{12.90}$$

Consulting Equation 12.61, we actially recover the vaccum polarisation scalar $\Pi(q^2)$ in the 1-loop order:

$$\Pi_{\text{loop}} = -\frac{8e^2}{(4\pi)^{d/2}}\mu^\varepsilon\int_0^1 x(1-x)dx\frac{\Gamma(\varepsilon/2)}{(a^2)^{\varepsilon/2}} \tag{12.91}$$

Inserting this into Equation 12.86 and imposing the renormalisation condition in Equation 12.77, we can derive the counterterm as

$$\delta Z_A = \Pi_{\text{loop}}(0) = -\frac{8e^2}{16\pi^2}\int_0^1 x(1-x)dx\left(\frac{2}{\varepsilon}-\gamma_E-\ln\frac{m^2}{\mu^2}+\cdots\right) \tag{12.92}$$

Derivation 12.14 (Running charge) Let us consider the true form of a full photon propagator. It will be the sum of the tree-level photon propagator, the 1-loop correction (i.e. OPI diagram), the 2-loop correction, etc. Using the Feynman rules, it is not hard to see the expression

$$\underbrace{\frac{-ig_{\mu\nu}}{q^2}F\left(q^2\right)}_{\text{full photon propagator}} = \underbrace{\frac{-ig_{\mu\nu}}{q^2}}_{\text{tree-level contribution}} + \underbrace{\frac{-ig_{\mu\rho}}{q^2}\left[i\left(g^{\rho\sigma}q^2 - q^\rho q^\sigma\right)\Pi\left(q^2\right)\right]\frac{-ig_{\sigma\nu}}{q^2}}_{\text{1-loop contribution}} + \cdots \tag{12.93}$$

where we have conveniently defined the *form factor* $F(q^2)$ that represents the 'summation' of the contributions at all levels.
Through a series of manipulations of the tensorial objects in this equation, as well as utilising the relationship

$$\Pi(q) + \Pi^2(q^2) + \cdots = \frac{1}{1-\Pi(q^2)} - 1 \tag{12.94}$$

we find that

$$\frac{-ig_{\mu\nu}}{q^2}F\left(q^2\right)e^2 = \frac{-ig_{\mu\rho}}{q^2}e^2\frac{1}{1-\Pi(q^2)} \tag{12.95}$$

Recognising that the LHS $F\left(q^2\right)e^2$ and the RHS e^2 can be rewritten as the effective coupling constant seen in interactions (which is essentially the (renormalised) running charge) e^2 and the bare charge e_0^2, we find the relationship

$$e^2(q^2) = \frac{e_0^2}{1-\Pi(q^2)} \tag{12.96}$$

where $\Pi(q^2)$ is the photon self-energy *scalar*. From a similar train of thought to that of the last derivation, we can see that $\Pi(q^2)$ is rather intuitively given by

$$\Pi(q^2) = \Pi_{\text{loop}}(q^2) - \delta Z_A = -\frac{2e^2}{4\pi^2}\int_0^1 dx x(1-x)\ln\left(\frac{m^2}{m^2 - x(1-x)q^2}\right) \tag{12.97}$$

Two comments are in order:

- $\Pi(q^2)$ is clearly a physical quantity independent of μ.
- e increases with the energy scale due to the log function. This is known as the *screening effect*, a general feature of abelian gauge theories.

We end with a note on HEP. Equation 12.96 can be recast by defining a so-called *fine structure constant*:

Definition 12.4 (Fine structure constant) We define the renormalised and bare fine structure constants as α and α_0 respectively:

$$\alpha_0 = \frac{e_0^2}{4\pi} \tag{12.98}$$

The previous relationship seen in Equation 12.96 then becomes

$$\alpha(q^2) = \frac{\alpha_0}{1-\Pi(q^2)} \tag{12.99}$$

From meticulous experiments, we know that α takes the infamous value of $1/137$.

Quote 12.3 Weesa free!

Unknown Gungan, in Return of the Jedi

As of the time of writing (2025), QED has been renormalised up to the fifth loop order in key quantities, particularly the anomalous magnetic moment of the electron. Calculations for fourth and fifth loop order renormalisations, pioneered by the much-celebrated Toichiro Kinoshita, were only developed from the 1990s on.

12.6 Beyond perturbation theory

> **Quote 12.4** Chromodynamics is weird, with couplings that grow with distances and...
>
> *Alessio Serafini, 27 March 2025*

While powerful, perturbation theory has limitations. It fails for large coupling constants ($\lambda \ll 1$). In cases where the coupling constant is not strictly *smol* or new physics appears at different energy scales, perturbation theory can still be applied in a restricted sense. *Effective field theories* (EFTs) use perturbative methods to focus on low-energy phenomena by integrating out high-energy degrees of freedom. We conclude with some philosophical remarks:

- As we proceeded through Part IV, we saw how everything became increasingly arbitrary and handwavy. There is first a breakdown of tree-level QFT at the loop level. However, instead of overthrowing this theory, we introduced renormalisation, which is effectively an 'extension' of tree-level QFT that becomes increasingly unwieldy as the diagrams increase in their complexity.
- This is because QFT is actually an EFT[3]. Unlike fundamental theories, EFTs are valid up to a certain energy scale but are expected to break down beyond that scale, giving rise to new physics. Higher-energy effects then manifest as *suppressed corrections* in the form of higher-dimensional operators. In this sense, all classical physics is nothing but effective theories w.r.t. modern physics. In the same vein, GR and modified theories of gravity are likewise EFTs of a future grand unified theory.
- As the standard model is effectively a collection of QFTs, it, too, is an EFT. Most of its extensions, like supersymmetry (SUSY), are likewise EFTs. In the 1970s, physicists initially thought that supergravity (SUGRA), a gravitational extension of SUSY, might provide a complete quantum theory of gravity by itself. However, it was later realised that supergravity alone is not sufficient to quantise gravity at all energy scales. Instead, string theory emerged as a more complete framework, with supergravity appearing as a low-energy effective theory of string theory.

Ultimately, the goal of physics is then to construct a single fundamental theory, the candidates of which include string theory as well as non-perturbative quantum gravity theories[4] asymptotically safe quantum gravity and loop quantum gravity.

> **Quote 12.5** Our two statements actually describe the same, but are written down on two different sides of the same medal.
> As they said, it's not the difficulty of the territory. You can avoid the most dangerous cliffs if you take the right route. It's more the sheer size which bends your knee, as you have to cross long distances in the realm of QFT.
>
> *Felix Halbwedl, on Quote 1.1 and Quote 1.2, 17 November 2024*

[3] This idea, considered revolutionary at the time of its inception, was formulated by Ken Wilson, whose general ideas are often known as *Wilsonian renormalisation*.

[4] This is motivated by the fact that perturbation theory makes GR non-renormalisable.