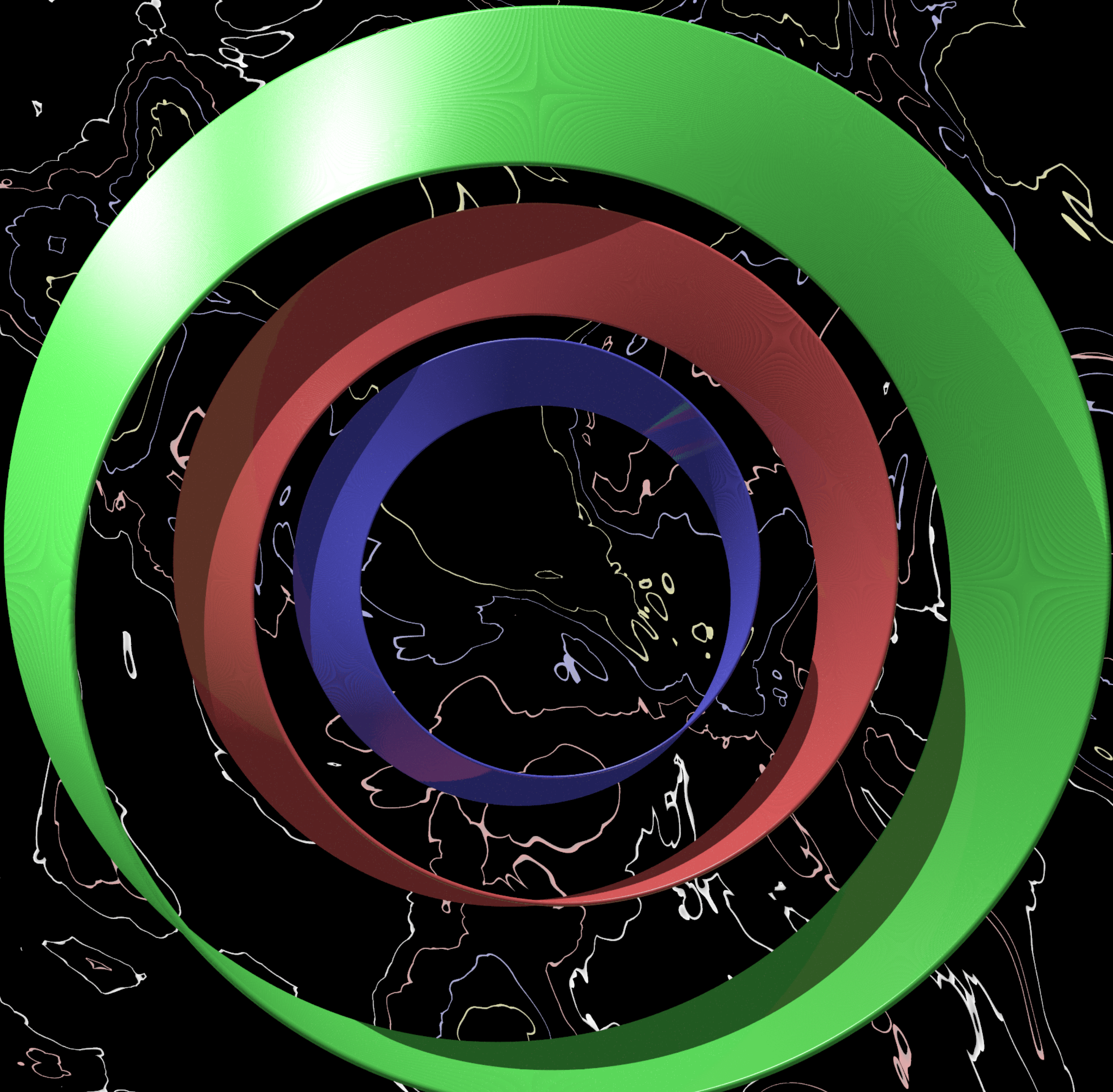
Spinors & U(1) Symmetries



N. Booker

To my parents

Contents

Chapter 1

Instead of a foreword

1.1 How to use this book

> **Quote 1.1** Stop trying to understand the axioms of math. Just try to work with them.
>
> *Stefan Fredenhagen, allegedly*

This book condenses semi-related topics in symmetries, Lie theory and spinors into a compact book that is meant to be sufficient for physics students. The book consists of three parts:

- Part I and Part II condenses a 'Symmetries' or 'Lie Theory' course in most universities.
- Part III is an introduction to various topics related to spinors that one may use in physics.

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1.2 Acknowledgements

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References

- *Symmetries in Physics* by Niklas Beisert (ETH Zürich)
- *Foundations of Mathematical Physics* by Paul P. Cook (King's College London) and Neil Lambert (King's College London)
- *QCD and Hadron Physics* by Gernot Eichmann (Technische Universität Graz)
- *Symmetries in Physics* by Matthias R. Gaberdiel (ETH Zürich)
- *Foundations of Mathematical Physics* by Eleni-Alexandra Kontou (King's College London)
- *Quantum Field Theory I* by Axel Maas (Technische Universität Graz)
- *Naive Lie Theory* by John Stillwell (University of San Francisco)
- *A Course in Modern Mathematical Physics* by Peter Szekeres (University of Adelaide)

Legend

For chapters that are not fully completed, a filled square is placed in front of its title to denote its completion status:

- ■ - mostly complete
- ■ - in progress
- ■ - empty

Part I

Symmetries

Chapter 2

Preliminaries

Quote 2.1 Sometimes, the background is more important.

Felix Halbwedl, 25 January 2025

2.1 Groups

Definition 2.1 (Group) A non-empty set G with a binary operation $\circ$ is a (finite) group if it satisfies the following properties:

- Closure:
$$a, b \in G \rightarrow a \circ b \in G \tag{2.1}$$

- Associativity:
$$(a \circ b) \circ c = a \circ (b \circ c) \tag{2.2}$$

- Identity:
$$\exists e \in G \quad \text{such that} \quad a \circ e = e \circ a = a \tag{2.3}$$

- Inverse:
$$\forall a \in G,\, \exists a^{-1} \in G \quad \text{such that} \quad a \circ a^{-1} = a^{-1} \circ a = e \tag{2.4}$$

The *group action* of some group G on some set X describes how the elements of G 'act' on elements of X in a way that respects the group structure:

Definition 2.2 (Group action) The action $\alpha(g, x)$ for $g \in G$ on $x \in X$ is a map $\alpha : G \times X \rightarrow X$ with the following properties:

- Identity:
$$\alpha(e, x) = x \quad \forall x \in X \tag{2.5}$$

- Compatibility:
$$\alpha(g_1 \circ g_2, x) = \alpha(g_1, \alpha(g_2, x)) \quad \forall g_1, g_2 \in G \quad \text{and} \quad \forall x \in X \tag{2.6}$$

Three types of important group actions are named:

- **Transitive action:** A group action α is *transitive* if
$$\forall x_1, x_2 \in X \quad \exists g \in G \quad \text{such that} \quad \alpha(g, x_1) = x^2 \tag{2.7}$$

- **Free action:** A group action α is *free* if, for *some* $x \in X$
$$\alpha(g, x) = x \quad \text{for some} \quad x \in X \implies g = e \tag{2.8}$$
i.e. if no non-identity element fixes any individual point in X.

- **Faithful action:** A group action α is *faithful* if

$$\alpha(g,x) = x \quad \forall x \in X \implies g = e \tag{2.9}$$

i.e. if no non-identity element fixes *all* points in X (but it might fix *some* points).

A general example of this is the so-called *transformation groups*:

Definition 2.3 (Transformation group) For some set W, the transformation group G is the set of one-to-one transformations

$$f : W \to W \tag{2.10}$$

G then defines a group whose operation $\circ$ is composition:

$$(f \circ g)(x) = f(g(x)) \tag{2.11}$$

The identity element is the trivial transformation $e(x) = x$. As the transformations are one-to-one and onto, inverses always exist.

A specific category of transformation groups is the so-called *symmetry groups*:

Definition 2.4 (Symmetry group) A symmetry group S_n is the transformation group whose elements are the permutations of some set $W_n = \{1, \cdots, n\}$. i.e. the invertible maps from $\{1, \cdots, n\}$ to itself.

Remark 2.1 For example, for $n = 7$, one symmetry group is $(1324)(5)(57)$.

Definition 2.5 (Abelian group) A group G is *abelian* if, for $A, B \in G$

$$a \circ b = b \circ a \tag{2.12}$$

Definition 2.6 (Cyclic group) A group C_p is *cyclic* if it consists only powers of some element a:

$$C_p = \{a^0, a, \cdots a^p\} \tag{2.13}$$

where a^0 is the identity.

Definition 2.7 (Dihedral group) A group D_n is *dihedral* if it is the symmetry group of an n-sided polygon in the plane. For clockwise rotations d and reflection around the axis s:

$$D_n = \{d, d^2, \cdots d^n, s, sd, \cdots sd^{n-1}\} \tag{2.14}$$

where $s^2 = d^n$ is the identity and the following relation is satisfied for some k:

$$d^{-k}s = sd^k \tag{2.15}$$

Definition 2.8 (Centre) The *centre* of a group $Z(g)$ is a set of elements that commute with all other elements of that group:

$$Z(G) = \{g \in G | g \circ g_1 = g_1 \circ 1 \forall g_1 \in G\} \tag{2.16}$$

Theorem 2.1 (Centre properties)

- The identity e is always part of the centre. i.e. $e \in Z(G)$.
- The centre of an abelian group is the group itself.

Definition 2.9 (Subgroup) A *subgroup* H of some group G is a non-empty subset of G that is a group itself. However, the verification can be simplified. For some $H \subset G$, H is a subgroup of G if

$$\forall g_1, g_2 \in H \quad g_1 \circ g_2^{-1} \in H \tag{2.17}$$

Remark 2.2 A group G always has the identity e and itself G as its subgroups. These two subgroups are known as its *trivial subgroups*. Any other subgroup H is known as a *proper subgroup*, denoted by $H < G$.

Definition 2.10 (Coset) For some (proper) subgroup H of G, $g \in G$ and $h \in H$, the *left coset* is

$$gH = \{g \circ h \in G\} \tag{2.18}$$

while the *right coset* is

$$Hg = \{h \circ g \in G\} \tag{2.19}$$

Definition 2.11 (Normal subgroup) A subgroup H of G is said to be *normal* or a *normal subgroup* of G if its left coset gH is equal to the right coset Hg. This is denoted by $H \triangleleft G$.

Definition 2.12 (Simple group) A group G is called a *simple group* if it does not have any non-trivial (i.e. e and G itself do not count) normal subgroups.

Definition 2.13 (Quotient group) For the normal subgroup $H \triangleleft G$, the set of all left cosets of H in G is denoted by G/H:

$$\frac{G}{H} = \{gH|g \in G\} \tag{2.20}$$

2.2 Operations between groups

One can, loosely speaking, 'multiply' two groups:

Definition 2.14 (Direct product) For the groups A and B with elements $a \in A$ and $b \in B$ and operations $\circ$ and $\diamond$, the *direct product* $A \times B$ is the set of ordered pairs $\{(a, b)\}$ with an operation called *product of pairs* defined by

$$(a_1, b_1) \times (a_2, b_2) = (a_1 \circ a_2, b_1 \diamond b_2) \tag{2.21}$$

Remark 2.3 This new group has normal subgroups of the form $\{(a, 1)\}$ and $\{(1, b)\}$ isomorphic to G and H respectively.
Let us proceed with some examples:

- The group $\mathbb{R}^n$ is actually the n-fold direct product $\underbrace{\mathbb{R}^1 \times \cdot \times \mathbb{R}^1}_{n \text{ times}}$, with the product of pairs being vector addition.
- For some groups A and B representing $n \times n$ matrices and elements $a \in A$ and $b \in B$, a group isomorphic to $A \times B$ are matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where 0 is the zero matrix of dimension $n \times n$. The product of pairs is simply the *block multiplication* of matrices:

$$\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix} \begin{pmatrix} a_2 & 0 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & 0 \\ 0 & b_1 b_2 \end{pmatrix} \tag{2.22}$$

In the finite case, the direct product is identical to the direct sum. However, there is a discrepancy in the infinite case. The direct product allows elements with an infinity of non-identity components, while the direct sum requires that elements have only finitely many non-identity components. Because of this, the result of the direct sum tends to be 'larger'.
More rigorously:

- For infinite direct products, given a collection of groups $\{G_i\}_{i \in I}$, the direct product is:

$$\prod_{i \in I} G_i = \{(g_i)_{i \in I} \mid g_i \in G_i \text{ for all } i \in I\} \tag{2.23}$$

- For infinite collections of groups, the direct sum consists of all elements $(g_i)_{i \in I}$ such that only finitely many coordinates g_i are non-identity elements:

$$\bigoplus_{i \in I} G_i = \{(g_i)_{i \in I} \mid g_i \neq e \text{ for only a finite } i\} \tag{2.24}$$

where e is the identity in each G_i and I is an *indexing set* (i.e. a set of indices). A finite I is of the form $\{1, 2, 3, \cdots\}$ while an infinite I is simply $I = \mathbb{Z}$.

Note 2.1 (Direct sum and tensor product shorthands) One will often see notations like $\bigoplus_i$ and $\bigotimes_i$. These notations simply aggregatate structures, much like $\sum_i$ and $\prod_i$. For example:

$$\bigoplus_i V_i := V_1 \oplus \cdots \oplus V_i \qquad \bigotimes_i V_i := V_1 \otimes \cdots \otimes V_i \tag{2.25}$$

Even more annoyingly, convention has it that $\oplus$ and $\otimes$ can also occupy upper indices:

$$V^{\oplus n} := \underbrace{V \oplus \cdots \oplus V}_{n \text{ direct sums among the same } V} \qquad V^{\otimes n} := \underbrace{V \otimes \cdots \otimes V}_{n \text{ tensor products among the same } V} \tag{2.26}$$

Lastly, the usage of $\prod$ indicates an aggregatation of direct products:

$$\prod_i G_i := G_1 \times \cdots \times G_i \tag{2.27}$$

2.3 Maps

Definition 2.15 (Homomorphism) The map $F : G \to G'$ between two groups $(G, \circ)$ and $(G', \circ')$ is a *homomorphism* or *group homomorphism* if

$$F(g_1 \circ g_2) = F(g_1) \circ' F(g_2) \tag{2.28}$$

Derivation 2.1 (Homomorphism) Consider the following two groups:

- Real positive numbers with multiplication $(\mathbb{R}^+, \cdot)$
- Complex numbers with multiplication $(\mathbb{C}, \cdot)$

and the map

$$F_u : \mathbb{R}^+ \to \mathbb{C} \quad \text{where} \quad F_U(a) = a^u \tag{2.29}$$

For some $a, b \in \mathbb{R}^+$ and $u \in \mathbb{C}$, we have

$$F_u(a \cdot b) = (a \cdot b)^u = a^u \cdot b^u \tag{2.30}$$

Hence, F_u is a homomorphism.

Definition 2.16 (Bijective) A function is *bijective* if it is:

- Injective/one-to-one (self-explanatory)
- Surjective/onto (all gs have corresponding g's)

Definition 2.17 (Isomorphism) If a homomorphism F is also bijective, then it is an *isomorphism* or *group isomorphism*. Two isomorphic groups are denoted by $G \cong G'$.

Definition 2.18 (Endomorphism) A map $F : (G, \circ) \to (G, \circ)$ that sends the group and the operation back to itself is an *endomorphism* or *group endomorphism*.

Definition 2.19 (Automorphism) An endomorphism that is also bijective (invertible) is an *automorphism* or *group automorphism*.

Definition 2.20 (Kernel) The *kernel* $\mathrm{Ker}(F)$ of a map $F : G \to G'$ is a normal subgroup of G that includes all the elements of G that is mapped to the identity e' of G':

$$\mathrm{Ker}(F) = \{g \in G | F(g) = e'\} \tag{2.31}$$

Definition 2.21 (Image and preimage) For the map $F : X \to Y$, $x \in X$ and $y \in Y$, we have (somewhat confusingly), the three usages of the terms *image* and *preimage*:

- **For elements:** y is the image of x, and x is the preimage of y.
- **For subsets:** For the subset $A \subset X$ and elements $a \in A$, the image of A under F is the set of all $F(a)$.
- **For functions:** Some subgroup of Y (possibly Y itself) is the image of F, and X is the preimage of F[a].

[a]Remember that while there may be elements in Y that do not correspond to elements in X, all elements in X should correspond to an element in Y.

We are now in a position to connect several of the definitions we have made so far:

Theorem 2.2 (1st isomorphism theorem) For groups G and G' and the homomorphism $F : G \to G'$, the image of F, which is a subgroup of G', is isomorphic to the quotient group $G/\operatorname{Ker}(F)$:

$$\operatorname{Img}(F) \cong \frac{G}{\operatorname{Ker}(F)} \tag{2.32}$$

If F is onto, its image will simply be G', and the above statement reduces to

$$G' \cong \frac{G}{\operatorname{Ker}(F)} \tag{2.33}$$

2.4 Algebras

Quote 2.2 In this context, we're discussing "an algebra" which refers to something like a number system - a mathematic structure that includes some sort of number and a multiplication operation between them. This is different from the general term "algebra" which describes math involving variables.

Ian Dunn and Zoë Wood, in the Graphics Programming Compendium

Definition 2.22 (Bilinear) For a vector space V over a field F[a], vectors $v, w_1, w_2 \in V$ and parameters $c, d \in F$, a *bilinear*, *bilinear product* or a *bilinear form* is a map $B : V \times V \to F$ that satisfies:

- **Bilinearity:**
$$B((cw_1 + dw_2), v) = cB(w_1, v) + dB(w_2, v) \tag{2.34}$$
$$B(v, (cw_1 + dw_2)) = cB(v, w_1) + dB(v, w_2) \tag{2.35}$$
- **Closure:**
$$B(v, w) \in V \quad \forall v, w \in V \tag{2.36}$$

[a]e.g. real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$.

Remark 2.4 Dot products, matrix multiplications and inner products in vector spaces are all bilinear products.

Definition 2.23 (Algebra) An *algebra* is a vector space V over a field F with a bilinear operation defined on it.

Derivation 2.2 (Algebra of a group) As an example, we consider a group G and the algebra $\mathbb{C}G$, which is defined as the group algebra of G over $\mathbb{C}$.

- $\mathbb{C}G$ is the vector space over $\mathbb{C}$ with a basis given by the elements of G, which we denote as $\{e_a | a \in G\}$.

- The product $\circ$ defined on $\mathbb{C}G$ is the bilinear extension of the group operation in G. For the basis vectors $\{e_a\}$, the product is given by

$$e_a \circ e_b = e_{ab}, \quad \text{for} \quad a, b \in G \tag{2.37}$$

The structure of the group depends on ab, the group operation in G.

2.5 Representations

So far, much has been discussed about the groups and algebra. We know that both consist of a series of elements, but are yet to discuss the elements themselves in depth. This is because the elements are so-called *abstract elements*, which are nothing but abstract objects that reflect/observe the nature of the group symmetry (e.g. $\mathrm{SO}(n)$ groups are rotation groups, and so on).

In physics, we are not terribly interested in the abstract elements themselves with the exception of their relevance in group symmetries. This is intuitive in that we cannot perform interesting transformations with abstract elements. To do this, we will need to consult *representations* of the group/algebra, which are the 'descriptions' of the its elements in terms of matrices and algebraic operations. In a more rigorous definition:

Definition 2.24 (Representation) A *representation* ρ of a (finite) group G on a vector space V is a homomorphism

$$\rho : G \to \mathrm{Aut}(V) \quad \text{or} \quad \rho : G \to \mathrm{GL}(V) \tag{2.38}$$

where $\mathrm{Aut}(V)$ is the group of automorphisms of V, which is actually the general linear group $\mathrm{GL}(V)$.

Remark 2.5 An equivalent statement is as follows: a representation is some map ρ that gives, from a n-dimensional group G, an invertible $n \times n$ matrix.

For the corresponding algebra, the representation is

$$\rho : \mathfrak{g} \to \mathrm{End}(V) \tag{2.39}$$

The following properties of ρ are observed:

- Preservation of algebraic operations $\circ$:

$$\forall a, b \in G \quad \rho(a \circ b) = \rho(a)\rho(b) \tag{2.40}$$

- Identity mapped to identity matrix $\mathbb{I}$:

$$\rho(e) = \mathbb{I} \tag{2.41}$$

- Equivalence of inverses:

$$\rho(a^{-1}) = (\rho(a))^{-1} \tag{2.42}$$

Note 2.2 *Sometimes*, we will call V the representation of G, and thus n as the dimension of the representation[a].

[a]This is a convention we will use out of convenience, as we will soon see.

The description of a group's representations and their relations is known as the *representation theory* of the group.

Definition 2.25 (Subrepresentation) A *subrepresentation* W of a representation V is a vector subspace $W \subseteq V$ that is invariant under G:

$$\rho(a)w \in W \quad \forall w \in W, \, a \in G \tag{2.43}$$

Definition 2.26 (Faithful representation) A representation on a group $\rho(G)$ *or* that on an algebra $\rho(\mathfrak{g})$ is *faithful* if

$$\mathrm{Ker}(\rho(G)) = e \quad \text{or} \quad \mathrm{Ker}(\rho(\mathfrak{g})) = 0 \quad \text{respectively} \tag{2.44}$$

Definition 2.27 (Invariant subspace) One can define *invariant subspaces* for both groups and algebras:

- For the representation ρ a group G defined on a vector space V. A subspace $W \subset V$ is invariant if
$$\forall g \in G \quad \text{and} \quad \forall w \in W \quad \rho(g)w \in W \tag{2.45}$$
- For the representation ρ an algebra $\mathfrak{g}$ defined on a vector space V. A subspace $W \subset V$ is invariant if
$$\forall X \in \mathfrak{g} \quad \text{and} \quad \forall w \in W \quad \rho(X)w \in W \tag{2.46}$$

Definition 2.28 (Irreducible representation) A representation V is *irreducible* if its only subrepresentations are itself and $\{0\}$. i.e. if it has no proper invariant subspaces. Otherwise, V is *reducible*. An irreducible representation is also called an *irrep* in short.

There are three standard tools we can use to derive further representations from known representations:

- For representations V and W of G, their direct sum $V \oplus W$ is also a representation of G.
- For representations V and W of G, their tensor product $V \otimes W$ is also a representation of G.
- For some representation V of G, its dual vector space V^* is also a representation of G.

Definition 2.29 (Indecomposable representations) A representation that is not derived via such algebraic operations is said to be *indecomposable*. Otherwise, it is *decomposable*.

Theorem 2.3 (Schur's lemma) The so-called *Schur's lemma* has two forms:

- Suppose we have a intertwining map $T : V \to W$ and ρ_1 and ρ_2, two irreducible representations of G
$$\rho_1 : G \to \mathrm{GL}(V) \quad \rho_2 : G \to \mathrm{GL}(W) \tag{2.47}$$
where $\mathrm{GL}(V)$ and $\mathrm{GL}(W)$ are general linear groups we will see later. One of the two must be true:
  - T is trivial/a zero map, i.e. $T = 0$.
  - T is an isomorphism.
- Now consider the case of $V = W$, where irreducible representations are
$$\rho_1 : G \to \mathrm{GL}(V) \quad \rho_2 : G \to \mathrm{GL}(V) \tag{2.48}$$
In this case, the only possible T is the identity and its scalar multiples:
$$T = \lambda \mathbb{I} \quad \lambda \in \mathbb{C} \tag{2.49}$$

Definition 2.30 (Equivalent representations) Two representation $\rho_1(G) \in \mathrm{GL}(V)$ and $\rho_2(G) \in \mathrm{GL}(W)$ are *equivalent* if there exists a invertible linear map $T : V \to W$ for which
$$T\rho_1(g) = \rho_2(g)T \quad \forall g \in G \tag{2.50}$$
This T is then called the *intertwiner* or *intertwining map* of ρ_1 and ρ_2.

Definition 2.31 (Unitary representations) Often used in quantum mechanics, *unitary representations* can be defined for both groups and algebras:

- The representation ρ a group G defined on a vector space V is unitary if
$$\forall g \in G \quad \rho(g)\rho(g)^\dagger = e \tag{2.51}$$
- The representation ρ an algebra $\mathfrak{g}$ defined on a vector space V is unitary if
$$\forall X \in \mathfrak{g} \quad \rho(X) = \rho(X)^\dagger \tag{2.52}$$

Definition 2.32 (Trivial representations) A representation is *trivial* if it maps all elements to the identity:

$$\forall g \in G \quad \rho(g) = e \quad \text{for Lie groups and} \quad \forall X \in \mathfrak{g} \quad \rho(X) = 0 \quad \text{for Lie algebras} \tag{2.53}$$

Definition 2.33 (Fundamental representations) A representation is *fundamental* if it maps all elements to themselves:

$$\forall g \in G \quad \rho(g) = g \quad \text{for Lie groups and} \quad \forall X \in \mathfrak{g} \quad \rho(X) = X \quad \text{for Lie algebras} \tag{2.54}$$

Note 2.3 (Representations in disguise) In physics literature, very little distinction is given between the abstract elements themselves and their representations. Often, physicists speak of 'elements' of a group when they are, in actuality, representations that are well-defined and take the form of matrices. So what is the takeaway of all this?

- In pure mathematics, an element of a group is simply an abstract object satisfying group multiplication rules.
- In physics, we care about how these elements act on physical states (e.g. spinors, wavefunctions), which naturally leads us to work with representations.
- In some literature, so-called 'elements' (almost always without the term 'abstract' before it) with well-defined forms are actually *representations in disguise*.

2.6 Characters

Definition 2.34 (Character) For an element $a \in G$ of some group G with representation ρ, its character $\chi_\rho(a)$ of some element $a \in G$ is the trace of a on the representation V:

$$\chi_\rho(a) = \mathrm{Tr}(\rho a) \tag{2.55}$$

Remark 2.6 Put simply, characters are traces of representations. We also note that:

- A representation is a matrix.
- The trace of a matrix is the sum of its eigenvalues.

Hence, character theory is a good way to keep track of the eigenvalues of group element actions[1]. Even better is the fact that, since traces are invariant under similarity transformations, the eigenvalues of a representation do not change under a change of basis.

Definition 2.35 (Conjugate) Two elements $a, b \in G$ are *conjugates* to each other if there exists a third element $g \in G$ which satisfies

$$b = g^{-1}ag \tag{2.56}$$

Definition 2.36 (Conjugacy class) The *conjugacy class* of a is then the collection

$$\mathrm{Cl}(a) = \{g^{-1}ag : g \in G\} \tag{2.57}$$

Theorem 2.4 (Character properties) For two representations V and W of G with eigenvalues $\{\lambda_i\}$ and $\{\mu_j\}$, we have

- Invariance on the conjugacy classes of G:

$$\chi_\rho(g^{-1}ag) = \chi_\rho(a) \tag{2.58}$$

In other words, the character $\chi_\rho(a)$ of the representation $\rho(a)$ of a group G is a function that depends only on the conjugacy class $\mathrm{Cl}(a)$ of a in G, meaning that it can be considered as a function on the set of conjugacy classes of G rather than G itself.

[1] Note that while the group representation describes how the element acts, it is not the action itself: a representation must be a linear action (i.e. matrix), while the action itself can be more general (e.g. permutation of a set, affine transformations).

- Direct sum:
$$\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2} \quad \text{whose eigenvalues are} \quad \lambda_i, \mu_j \tag{2.59}$$
- Tensor product:
$$\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1} \circ \chi_{\rho_2} \quad \text{whose eigenvalues are} \quad \lambda_i \circ \mu_j \tag{2.60}$$
- Dual:
$$\chi_{\rho^*} = \bar{\chi}_\rho \quad \text{whose eigenvalues are} \quad \{\lambda_i^{-1}\} = \{\bar{\lambda}_i\} \tag{2.61}$$

Definition 2.37 (Conjugation action) A group G has a natural action on itself $\mathrm{Ad}_G : G \times G \to G$ is the *conjugation action*[a]. It satisfies, for $g, h \in G$:

$$\mathrm{Ad}_G(g, h) = ghg^{-1} \tag{2.62}$$

This is a left action of G on itself which is smooth.

[a] Also called the *adjoint action* like its counterpart in algebras.

Definition 2.38 (Adjoint action) The extension of the conjugation action to algebras is known as the *adjoint action*, which is derived by differentiating the conjugation action at the identity. For $X, Y \in \mathfrak{g}$, the adjoint action for this algebra $\mathfrak{g}$ is

$$\mathrm{ad}_{\mathfrak{g}}(X, Y) = [X, Y] \tag{2.63}$$

where [] is the Lie bracket which we will soon see.

Chapter 3

Naive Lie theory

Quote 3.1 Yeah those... Not boring :)

Francisco Silva, on Lie theory and spinors, 19 January 2025

3.1 Lie derivatives

Definition 3.1 (Lie derivative) The *Lie derivative* evaluates the change of a tensor field along the flow defined by another vector field. For two vectors U and V, the Lie derivative of U^i with respect to (i.e. along) V^i is

$$\mathcal{L}_V U = V^j \partial_j U^i - U^j \partial_j V^i \tag{3.1}$$

where:

- The first term represents the directional derivative of U along V. i.e. how U changes along the flow of U.
- The second term is associated with the change in the vector field V as it moves along U.

Remark 3.1 By comparing this to the directional derivative in vector calculus, it would be intuitive that the Lie derivative likewise transforms as a vector. Conceptually, the Lie derivative is the derivative of U along the flow generated by V.

Remark 3.2 Much like the covariant derivative, the Lie derivative illustrated how tensor fields change when 'dragged' along the flow generated by a vector field. Unlike the covariant derivative, however, it does not consider the underlying connection or curvature.

Definition 3.2 (Lie bracket) In some literature, the Lie derivative is written as the so-called *Lie bracket* instead:

$$\underbrace{[V,U]}_{\text{Lie bracket}} := \underbrace{\mathcal{L}_V U}_{\text{Lie derivative}} \tag{3.2}$$

Theorem 3.1 (Lie derivative properties) For vectors U, V and W in a Lie algebra[a]:

- **Linearity:**
$$[V, U + W] = [V, U] + [V, W] \tag{3.3}$$
$$[V, UW] = [V, U]W + U[V, W] \tag{3.4}$$
- **Alternativity:**
$$[V, V] = 0 \tag{3.5}$$
- **Anticommutativity[b]:**
$$[V, U] = -[U, V] \tag{3.6}$$
- **Jacobi identity:**
$$[V, [U, W]] + [U, [W, V]] + [W, [V, U]] = 0 \tag{3.7}$$

[a]You will soon see what this means.
[b]This is implied by linearity and alternativity.

Remark 3.3 The vectors V and W can likewise be replaced by arbitrary functions f and g, and the Lie derivative stay the same by definition.

Derivation 3.1 (Higher-rank Lie derivatives) We can also derive the Lie derivative of a rank-2 tensor:

$$\mathcal{L}_V W_i^j = V^k \partial_k W_i^j + (\partial_i V^k) W_k^j - (\partial_k V^j) W_i^k \tag{3.8}$$

Remark 3.4 Here we see the tendency of the operator $\partial_i V^j$ to sacrifice one of its indices for the sake of the partial derivative as well as the target tensor, much like the poor Christoffel symbol in *Metrics and Cosmos and Trilobites*. However, we find that unlike the covariant derivative, terms which have arbitrary indices assuming covariant positions are *positive*.

3.2 Lie groups and Lie albegras

Definition 3.3 (Lie algebra) A *Lie algebra* is a specific form of algebra whose bilinear operation is the Lie bracket. In other words, a Lie algebra is a vector space V with a Lie bracket $[x, y]$ for $x, y \in V$ defined on it.

Definition 3.4 (Lie group) The manifold on which the Lie algebra rests is called a *Lie group*. It is both a group and a differentiable manifold.

Remark 3.5 Physically, one can regard a Lie algebra as the *tangent space* of a Lie group. If the Lie group is 3D, the Lie algebra is then a tangent plane.
Remark 3.6 The corresponding Lie algebra of some Lie group G is usually represented in small letter *Fraktur* as $\mathfrak{g}$.

Derivation 3.2 (Generator) Now we will see how this works in practice. First, we recall what we have so far:

- A Lie group is made up of a series of abstract elements $g(\theta)$.
- θ^a is a set of (real) parameters that corresponds to the 'coordinates' on the group manifold. For a n-dimensional group, its index runs over $a = 1, \cdots, n$.
- By definition, the representation of the abstract elements $g(\theta)$ are a series of linear operators (i.e. transformation matrices) $\rho(g(\theta))$. In physics literature, they are often loosely said to be the elements themselves.

Outrageously, we have not derived any representations in practice up to this point. Any *smol* (or infinitesimal) transformation ϵ in the a Lie group G can take the form of the representation $\rho(g(\theta))$[a] can be written in terms of the representation $\rho(X)$ (again, often simply called *elements*) of $\mathfrak{g}$, where $X \in \mathfrak{g}$:

$$\rho(g(\theta)) = I + \epsilon\rho(X) + \mathcal{O}(\epsilon^2) \tag{3.9}$$

where ϵ is another *smol* parameter. For a finite transformation, we obtain:

$$\rho(g(\theta)) = e^{\rho(X)} \tag{3.10}$$

Generally, one can write $\rho(X)$ in the form of $i\theta^a D_a$, which gives us:

Theorem 3.2 (Generation of Lie group representations from Lie algebra representations)

$$\rho(g(\theta)) = e^{i\theta^a D_a} \tag{3.11}$$

Due to this peculiar relationship, D_a are said to be the *generators* of $g(\theta)$.

Let us now set out to find an expression for these so-called generators. Differentiating Equation 3.11 by θ yields

$$\partial_\theta \rho(g(\theta)) = iD_a e^{i\theta^a D_a} \tag{3.12}$$

Physically, θ is a rotation angle and thus merely a phase. This allows us to set $\theta = 0$ for convenience and find that

$$\left.\frac{\partial \rho(g(\theta)}{\partial \theta}\right|_{\theta=0} = iD_a e^0 \tag{3.13}$$

Hence the generator can be so-created:

Definition 3.5 (Generator) The abstract elements of some Lie algebra $\mathfrak{g}$ are known as the generators of the elements of the corresponding Lie group G.

$$D_a = -i \left.\frac{\partial \rho(g(\theta))}{\partial \theta^a}\right|_{\theta=0} \tag{3.14}$$

[a]Often written the alternative symbol $U(\epsilon)$, especially when it is loosely designated as an 'element' (instead of a representation) of the Lie group G.

Some comments are in order:

- The generators we see here are merely the generators *independent from each other*[1]. This is important in that the generators *do not* make up the entirety of their corresponding Lie algebra, *but merely its bases.*
- Equation 3.11 has a complex phase $i\theta$ which ultimately makes the generators Hermitian. This is the convention in physics. In pure mathematics, this is not needed, and one only has a real phase. i.e.

$$\rho(g(\theta)) = e^{\theta^a D_a} \tag{3.15}$$

therefore giving rise to anti-Hermitian generators which observe

$$D_a = \left.\frac{\partial \rho(g(\theta))}{\partial \theta^a}\right|_{\theta=0} \tag{3.16}$$

where we note the distinct lack of the $-i$ factor.
- Generators are not restricted to representations, and one can also say that the elements of a Lie algebra $X \in \mathfrak{g}$ are the generators of the elements of its corresponding Lie group $g \in G$. However, this is not very interesting as again, both X and g are abstract elements.

Definition 3.6 (Structure constant) The *structure constants* f^c_{ab} of a Lie algebra $\mathfrak{g}$ in a vector space V are defined with respect to the following Lie bracket:

$$[t_a, t_b] = f^c_{ab} t_c \tag{3.17}$$

where t_i are the bases of the vector space V and a, b and c are coordinate indices. Structure constants are invariant under Lie algebra isomorphisms[a] which nonetheless change under a basis change.

[a]We will define this almost immediately.

Note 3.1 (A not-so-plot twist) As it turns out, the bases t_i are actually our good friends, the generators D_i in disguise. The usage of the different notation t_i seems to be merely a result of convention.

Definition 3.7 (Killing form) The so-called *Killing form* B_{ij} or $B(t_i, t_j)$ is a symmetric bilinear form emerging from the bases/generators of its corresponding Lie group:

$$B(t_i, t_j) = \mathrm{Tr}(\rho(t_i) \circ \rho(t_j)) \tag{3.18}$$

where Tr is the trace, ρ is the representation and $\circ$ is the group operation.
The Killing form can be represented in terms of structure constants:

$$B(t_i, t_j) = f^n_{im} \circ f^m_{jn} \tag{3.19}$$

[1]This is like how a 4D Ricci tensor has 16 components but only 10 independent components due to symmetry.

Quote 3.2 Now, in the autumnal serenity of semi-retirement, having finally looked at some of Wilhelm Killing's writings, without any doubt or hesitation I choose his paper dated 'Braunsberg, 2 Februar, 1888' as the most significant mathematical paper I have read or heard about in fifty years.

A. John Coleman, in The Mathematical Intelligencer Vol. 11, No. 3, *1980*

Definition 3.8 (Lie algebra isomorphism) A *Lie algebra isomorphism* $\varphi : \mathfrak{g} \to \mathfrak{g}'$ satisfies

- φ is a linear map:
$$\varphi(aX + bY) = a\varphi(X) + b\varphi(Y) \tag{3.20}$$
- φ preserves the Lie bracket:
$$\varphi([X, Y]) = [\varphi(X), \varphi(Y)] \tag{3.21}$$
- φ is bijective.

Now we will go through a few significant classifications of Lie algebras.

Definition 3.9 (Abelian Lie algebra) A Lie algebra is *abelian* if all of its structure constants vanish.

Definition 3.10 (Lie subalgebra) For two Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$, $\mathfrak{h}$ is a *Lie subalgebra* of $\mathfrak{g}$ (i.e. $\mathfrak{h} \subset \mathfrak{g}$) if:

- $\mathfrak{h}$ is a subspace of $\mathfrak{g}$.
- The Lie bracket of any two elements/generators in $\mathfrak{h}$ are also in $\mathfrak{h}$:
$$[\mathfrak{h}_a, \mathfrak{h}_b] \subseteq \mathfrak{h} \tag{3.22}$$

Definition 3.11 (Invariant Lie subalgebra) $\mathfrak{h}$ is called an *invariant Lie subalgebra* of $\mathfrak{g}$ if $\mathfrak{h} \subset \mathfrak{g}$ and any Lie bracket between elements of $\mathfrak{h}$ and $\mathfrak{g}$ remain within $\mathfrak{h}$:
$$[\mathfrak{h}_a, \mathfrak{g}_b] \subseteq \mathfrak{h} \tag{3.23}$$

Definition 3.12 (Simple Lie subalgebra) A *simple Lie algebra* is a Lie algebra that does not contain any invariant Lie subalgebra.

Definition 3.13 (Semisimple Lie algebra) A Lie algebra $\mathfrak{g}$ is *semisimple* if it contains no abelian and invariant Lie subalgebra[a].

[a]Although it might contain non-abelian and invariant Lie subalgebras.

Finally:

Definition 3.14 (Compact Lie algebra) A *compact Lie algebra* is associated with a *compact Lie group*, which is (loosely speaking) a fancy way of saying that a Lie group is finite.

3.3 Special Lie groups

Of paramount interest is the so-called *special Lie groups*. In practice, we will see many variants of special Lie groups, which are all 'special groups'. One important caveat is the fact that a *special group* is not formally well-defined. For physicists, however, it is useful to use simply the following working definition:

Definition 3.15 (Special group) For an n-dimensional space, we can denote the group of all square matrices with determinant 1 (i.e. $\det A = 1$) by $\mathrm{S}(n)$. These matrices are known as *unimodular matrices*.

Later on[2], we will combine the properties of this group with those of the three groups we will introduce.

[2]Read: very soon

We begin with the most general of the three groups:

Definition 3.16 (General linear group) For an n-dimensional space, the group of all invertible $n \times n$ matrices is known as the *general linear group* $\mathrm{GL}(n)$.

Now let us impose some more constraints. One can recall, perhaps from their quantum mechanics course:

Definition 3.17 (Unitary operator) An operator is *unitary* if its Hermitian conjugate is its inverse:

$$\hat{A}^\dagger = \hat{A}^{-1} \quad \text{or} \quad \hat{A}^\dagger \hat{A} = 1 \tag{3.24}$$

We denote a group of such matrices as a *unitary group*, which is a Lie group.

Definition 3.18 (Unitary group) Here, we can loosely equivalate a *matrix* and an operator. For an n-dimensional space, we can denote the group of all unitary $n \times n$ matrices by $\mathrm{U}(n)$.

We can make even more constraints:

Definition 3.19 (Orthogonal operator) An operator is *orthogonal* if its Hermitian conjugate is its inverse:

$$\hat{A}^T = \hat{A}^{-1} \quad \text{or} \quad \hat{A}^T \hat{A} = 1 \tag{3.25}$$

We can rephrase this in a more satisfying way: an orthogonal operator is an operator for which the columns and rows forming its matrix are orthonormal vectors (i.e. the columns form an orthonormal basis).

Remark 3.7 All orthogonal matrices are unitary. That is to say, orthogonal matrices are a specific case of unitary matrices. The significance of this will soon be apparent.

Definition 3.20 (Orthogonal group) For an n-dimensional space, the group of norm- (i.e. distance-) and angle-preserving transformations is the *orthogonal group* $\mathrm{O}(n)$, which is the group of $n \times n$ real matrices.

We can now combine these three groups with the special group, which yields three more sophisticated Lie groups:

Definition 3.21 ($\mathrm{SL}(n)$, $\mathrm{SU}(n)$ and $\mathrm{SO}(n)$ groups) By considering unimodular matrices only, we can define the following:

- As $\mathrm{SL}(n)$ group whose elements are unimodular is called a *special linear group* or a $\mathrm{SL}(n)$ group.
- An $\mathrm{U}(n)$ group whose elements are unimodular is called a *special unitary group* or a $\mathrm{SU}(n)$ group.
- An $\mathrm{O}(n)$ group whose elements are unimodular is called a *special orthogonal group* or a $\mathrm{SO}(n)$ group.

Their mathematical and physical meanings are as follows:

- $\mathrm{SL}(n)$:
  - Mathematically, the elements of $\mathrm{SL}(n)$ represent volume-preserving transformations in $\mathbb{R}^n$. They preserve the volume of objects in space, but not necessarily angles or lengths.
  - Physically, $\mathrm{SL}(n)$ appears in physics in the study of phase space transformations and Hamiltonian mechanics.
- $\mathrm{SU}(n)$: Its elements preserve the Hermitian inner product in $\mathbb{C}^n$, making it naturally the symmetry group of quantum mechanical systems, which take place in complex vector spaces. We will discuss this in greater detail later on.
- $\mathrm{SO}(n)$:
  - Mathematically, the elements of $\mathrm{SO}(n)$ are linear transformations of $\mathbb{R}^n$ that preserve the Euclidean inner product (hence distances and angles) and have determinant $+1$ (ensuring proper orientation is preserved, i.e., no reflections).

  - Physically, the elements of $\mathrm{SO}(n)$ correspond to the set of all possible rigid body rotations in n-dimensional Euclidean space. Hence, $\mathrm{SO}(n)$ is famously said to be the group of rotations of $\mathbb{R}^n$ space.

Finally, we note down some general ideas which will be reflected in later chapters, where we discuss specific Lie groups:

- Addition or subtraction between two generators always yields an antisymmetric (and traceless if $\mathrm{SU}(n)$) matrix which is then another generator.
- Multiplication between two generators does not always yield another generator. The result is not necessarily antisymmetric (but always traceless for $\mathrm{SU}(n)$).
- We do not consider division as algebras only need to have addition, subtraction and multiplication defined.

3.4 Topological properties

So far, we have only defined Lie groups and Lie algebras with respect to matrices. As it turns out, one can give a more generalised definition using topology we have seen in *Metrics and Cosmos and Trilobites*:

- A Lie group $(G, \circ)$ a group that has the structure of a smooth manifold, where the two group operations
  - Multiplication: $G \times G \to G$
  - Inversion: $g \to g^{-1}$

  are smooth maps.
- A Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space $T_e G$ of the identity e of G.

A Lie group can hold a number of topological properties. We will discuss the three main properties. Note that some examples will make more sense if one previews specific sections in the next chapters on specific Lie groups.

Definition 3.22 (Compactness) A Lie group is *compact* if its underlying manifold is compact (i.e., closed and bounded).

Remark 3.8 Compactness guarantees finite-dimensional representations decompose into irreducibles. It often simplifies the analysis and representation theory of the group.

- An example of a compact Lie group is $\mathrm{SO}(2)$, which runs over real numbers. It corresponds to a one-dimensional manifold that is a circle, which is closed and bounded. Thus, $\mathrm{SO}(2)$ is compact.
- A counter-example of a compact Lie group is $\mathrm{SL}(2)$, which corresponds to infinite lines on the hyperbolic planes. Thus, $\mathrm{SL}(2)$ is not compact.

These examples can be generalised:

Theorem 3.3 (Compact Lie groups) Lie groups of the form $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are *always* compact, while Lie groups of the form $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ are *never* compact.

Definition 3.23 (Path-connectedness) A Lie group is *path-connected* if any two points on its corresponding manifold can be joined by a continuous path *on the manifold.* i.e.

$$\forall x, y \in G \quad \exists \text{ a map } c : [0,1] \quad \text{such that} \quad c(0) = x \quad \text{and} \quad c(1) = y \tag{3.26}$$

Remark 3.9 Path-connectedness ensures that the group acts as a single entity and that group is connected in a 'strong' sense, in that every connected component of the group contains a path.

Theorem 3.4 (Path-connected Lie groups) Lie groups of the form $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{SO}(n)$ and $\mathrm{SL}(n)$ are *always* path-connected, while Lie groups of the form $\mathrm{O}(n)$ and $\mathrm{gL}(n)$ are *never* compact, as both are partitioned into two disjoint subsets. However, each subset itself is path-connected.

Definition 3.24 (Fundamental group) Consider a point x_0 in a topological space X, which we shall call the *base point*. A *loop* in X based at x_0 is a continuous function

$$\gamma : [0, 1] \to X \quad \text{such that} \quad \gamma(0) = \gamma(1) = x_0 \tag{3.27}$$

Two loops γ_1 and γ_2 are considered *homotopic* if one can be continuously deformed into the other without breaking. The set of equivalence classes of loops under this homotopy relation forms a group, called the *fundamental group* $\pi_1(X, x_0)$, with the operation given by concatenation of loops.

Definition 3.25 (Simple connectedness) A Lie group is *simply connected* if every closed loop on its corresponding manifold can be continuously deformed (to a point) and remain on the manifold. In other words, if its fundamental group is trivial

$$\pi_1(G) = 0 \tag{3.28}$$

For example:

- A sphere is simply connected as any loop can reduce to a point.
- A torus is not simply connected as loops around the toroidal and poloidal directions cannot reduce to a point.

More sophisticatedly:

- The special unitary group $\mathrm{SU}(n)$ is simply connected, so $\pi_1(\mathrm{SU}(n)) = 0$ for $n \geq 2$.
- The general linear group $\mathrm{GL}(n, \mathbb{R})$ is *not* simply connected, and its fundamental group is nontrivial.

Remark 3.10 Simple connectedness ensures that all topological ambiguities are resolved (e.g., universal covering spaces are trivial). Non-simple connectedness implies nontrivial loops that cannot be shrunk to a point, indicating the presence of *holes* in the space.

Chapter 4

$\mathrm{SU}(n)$ and $\mathrm{SO}(n)$ groups

Quote 4.1 I'm just a tiny gear in this giant clockwork and I don't know how late it is.

Felix Halbwedl, 19 January 2025

4.1 Overview of $\mathrm{SU}(n)$ groups

As mentioned before, SU(n) groups are often used in high energy physics:

- SU(2) encodes spin and isospin (i.e. QED).
- SU(3) encodes QCD.
- SU(2) $\times$ SU(2) encodes the (Euclidian) Lorentz groups.

Each SU(n) group has $n^2 - 1$ real group parameters (i.e. generators M_i), which are Hermitian and traceless. They form the Lie algebra $\mathfrak{su}(n)$. To study SU(n), it is useful to begin by looking at the U(n) groups. Unitary transformations stem from U(n) where n is the dimensionality/a constant associated with the degrees of freedom. Each set of transformations is then said to be generated[1] by our good friends, the generators:

$$R_i = e^{iM_i\theta} \quad M_i \in \mathfrak{u}(n) \tag{4.1}$$

Remark 4.1 The U(n) group has n^2 generators M_i which are $n \times n$ matrices.
For some dimension n, infinitesimal spin rotation (or equivalents) can be approximated as a phase:

$$\lim_{n\to\infty}\left(1 + \frac{i\alpha}{n}M_i\right)^n \approx e^{iM_i\theta} = R_i \tag{4.2}$$

This leads to the generalised phase change $\phi \to \phi' = \phi e^{iM_i\cdot\theta}$, which allows for us to represent these spin rotations as a Lie group.

4.2 $\mathrm{SU}(1)$ group

A general phase change $e^{i\theta}$ is governed by U(1), also called the *circle group*. In other words, the elements of U(1) are $e^{i\theta}$ for all possible θs:

$$U(1) = \{e^{i\theta} | \theta \in [0, 2\pi)\} \tag{4.3}$$

This group only has one generator: the 1D unit matrix $\mathbb{I}_1$. This makes up the Lie algebra $\mathfrak{u}(1)$.

$$\mathfrak{u}(1) = \mathbb{I}_1 = \begin{pmatrix}1\end{pmatrix} \tag{4.4}$$

Remark 4.2 U(1) is isomorphic to SO(2), which is also called the circle group.
Now we look at the SU(1) group. We impose the condition $\det R = 1$, after which the only remaining element is the number 1. This makes SU(1) quite a trivial group as no transformation happen at all. Even more so is its Lie algebra $\mathfrak{su}(1)$, which consists only of the number 0 - the generator of 1.

[1] To put it unimaginatively...

4.3 SU(2) group

These are the groups that govern spin-$\frac{1}{2}$. If we recall quantum mechanics, we will see that the wavefunction is transformed by $\psi \to \psi' = \psi e^{iS\theta}$:

$$U(2) = \{e^{iS\theta} | \theta \in [0, 2\pi)\} \tag{4.5}$$

Hence for U(2), one has the set of 4 generators $\vec{S} = \frac{1}{2}\vec{\sigma}$, where σ_i are the infamous Pauli matrices:

Definition 4.1 (Pauli matrices)

$$\sigma_0 = \mathbb{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.6}$$

After halving, these matrices make up the Lie algebra $\mathfrak{u}(2)$. σ_0 corresponds to a general phase change while the rest correspond to rotations on the yz, xz and xy planes.
One might recall that in many sources σ_0 is omitted, as it is merely the 2D identity matrix in a new coat of paint. There is, however, another reason. For SU(2), we have only 3 generators: the universally accepted Pauli matrices σ_1, σ_2 and σ_3. This is because $e^{i\sigma_0\theta}$ fails the conditon $\det R = 1$, which is required by SU(2). Finally, the Lie algebra $\mathfrak{su}(2)$ are made up of Σ_1, Σ_2 and Σ_3, which are the Pauli matrices σ_1, σ_2 and σ_3 with a prefactor of 1/2. They satisfy

$$[\Sigma_j, \Sigma_k] = i\epsilon_{jlk}\Sigma_l \tag{4.7}$$

Remark 4.3 Here we make a significant observation: In 2D and above, the generators of U(n) is always the combination of those of SU(n) and U(1). One then says that U(n) is isomorphic to the direct product of SU(n) and U(1). i.e.

Theorem 4.1 (U(n)-SU(n) relation)

$$\mathrm{U}(n) = \mathrm{SU}(n) \times \mathrm{U}(1) \tag{4.8}$$

Owing to its simplicity, the $\mathfrak{su}(2)$ Lie algebra can be used as a vehicle for some new concepts:

Derivation 4.1 (Complexification of Lie algebras) We note that (aside from in this derivation box) every Lie algebra in this section has been defined over real space. Ambitious as we are, we are then compelled to extend our lovely Lie algebras, from $\mathbb{R}$ to $\mathbb{C}$. The process is known as *complexification*:

Definition 4.2 (Complexification) The complefixation of a real Lie algebra $\mathfrak{g}$ is a *complex Lie algebra* $\mathfrak{g}^{\mathbb{C}}$ obtained by considering complex-linear combinations of the elements of $\mathfrak{g}$

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}. \tag{4.9}$$

Formally, if $\{x_i\}$ is a basis of $\mathfrak{g}$, then $\mathfrak{g}^{\mathbb{C}}$ consists of all expressions of the form

$$Z = \sum_i (a_i + b_i i)X_i, \quad a_i, b_i \in \mathbb{R} \tag{4.10}$$

Remark 4.4 There are several motivations for complexifying Lie algebras, although the most important of them is probably the fact that a complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ *always* contains its real counterpart $\mathfrak{g}$ as a Lie subalgebra.

Theorem 4.2 (Bilinearity of complex Lie algebras) For $a, b, c, d \in \mathbb{C}$ and $X, Y, Z, W \in \mathfrak{g}$, the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$ observes *complex bilinearity*:

$$[aX + bY, cZ + dW] = ac[X, Z] + ad[X, W] + bc[Y, Z] + bd[Y, W] \tag{4.11}$$

Now let us consider the complexification of $\mathfrak{su}(2)$:

$$\mathfrak{su}(2)^{\mathbb{C}} = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C} \tag{4.12}$$

where $\mathfrak{su}(2)^{\mathbb{C}}$ consists of all linear combinations of σ_i with complex coefficients and is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, which we remind ourselves to be the Lie algebra of traceless 2×2 complex matrices:

$$\mathfrak{sl}(2, \mathbb{C}) = \{X \in M_2(\mathbb{C}) | \mathrm{Tr}(X) = 0\} \tag{4.13}$$

The isomorphism arises because the basis σ_i (with complex coefficients) can be reinterpreted as spanning $\mathfrak{sl}(2,\mathbb{C})$ under the same commutation relations. A common basis for $\mathfrak{sl}(2,\mathbb{C})$ is given by the generators:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.14}$$

which satisfy the commutation relations

$$[H,E] = 2E \quad [H,F] = -2F \quad [E,F] = H \tag{4.15}$$

Significantly, we note that the complication is nothing but combinations like $\sigma_1 + i\sigma_2$ and $\sigma_1 - i\sigma_2$, which can be mapped to E and F in $\mathfrak{sl}(2,\mathbb{C})$.
Remark 4.5 Importantly, as $\mathfrak{so}(3)$ is isomorphic to $\mathfrak{su}(2)$ (as we will see later), the complexification of $\mathfrak{so}(3)$ is also $\mathfrak{sl}(2,\mathbb{C})$.

Derivation 4.2 (Irreducible representations of $\mathfrak{su}(2)$) Consider an irreducible representation $\rho(\mathfrak{su}(2)) = \mathrm{End}(V)$ where $\mathrm{End}(V)$ is the group of endomorphisms on V, a finite dimensional vector space.
$\rho(\mathfrak{su}(2))$ is indexed by $j \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$, where j is a non-negative half-integer or integer. The dimension of the irreducible representation with label j is $2j+1$. For example:

- For $j = 0$, the representation is 1D (i.e. trivial).
- For $j = \frac{1}{2}$, the representation is 2D (i.e. the fundamental representation).
- For $j = 1$, the representation is 3D.

The representation space for a given j is spanned by $2j+1$ basis vectors labeled by the eigenvalue m of the J_z operator, where $m \in \{-j, -j+1, \ldots, j\}$. Thus, for a given j, the basis vectors are:

$$|j,m\rangle \quad \text{where } m = -j, -j+1, \ldots, j \tag{4.16}$$

The generators of $\mathfrak{su}(2)$, denoted as J_x, J_y, J_z, satisfy the commutation relations

$$[J_x, J_y] = iJ_z \quad [J_y, J_z] = iJ_x \quad [J_z, J_x] = iJ_y \tag{4.17}$$

We now define the *raising and lowering operators*

$$J_\pm = J_x \pm iJ_y \tag{4.18}$$

under which the commutators are

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z \tag{4.19}$$

The action of these operators on the basis vectors is

$$J_z|j,m\rangle = m|j,m\rangle \tag{4.20}$$
$$J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle \tag{4.21}$$
$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m-1\rangle \tag{4.22}$$

where we have recognised them to be the good ol' raising and lowering operators in quantum mechanics that describe angular momentum states:

- $j = 0$: Scalar representation (trivial representation).
- $j = \frac{1}{2}$: Spin-$\frac{1}{2}$ particles like electrons.
- $j = 1$: Vector representation (e.g., describing certain gauge bosons).

Derivation 4.3 (SU(2) × SU(2) → SO(4)) The homomorphism $\mathrm{SU}(2)\times\mathrm{SU}(2)\to\mathrm{SO}(4)$ is another interesting example emerging from SU(2). Recall that SO(4) is the group of 4D space rotations while SU(2) is topologically equivalent to the 3-sphere S^3.

$$A \to g_L A g_R^\dagger, \quad \text{for } (g_L, g_R) \in \mathrm{SU}(2)\times\mathrm{SU}(2). \tag{4.23}$$

Since a real four-dimensional vector space can be identified with the space of Hermitian 2×2 matrices (which have four real degrees of freedom), this action induces a map

$$\phi : \mathrm{SU}(2)\times\mathrm{SU}(2) \to \mathrm{SO}(4) \tag{4.24}$$

which is a double cover, meaning that its kernel consists of $\{(\pm\mathbb{I}, \pm\mathbb{I})\}$. Thus, we have the isomorphism:

$$\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)}{\mathbb{Z}_2} \cong \mathrm{SO}(4) \tag{4.25}$$

4.4 SU(3) group

For QCD we have one more dimension and hence 9 dimensions. The transformations are $R_i = e^{i\frac{1}{2}\lambda_i\theta}$:

$$U(3) = \{e^{i\frac{1}{2}\lambda\theta} | \theta \in [0, 2\pi)\} \tag{4.26}$$

$\frac{1}{2}\lambda_i$ are the generators for U(3) called the *Gell-Mann matrices*:

Definition 4.3 (Gell-Mann matrices)

$$\begin{aligned}
&\lambda_0 = \mathbb{I}_3 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \quad \lambda_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} \quad \lambda_2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}\\
&\lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \quad \lambda_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix} \quad \lambda_5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}\\
&\lambda_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \lambda_7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}
\end{aligned} \tag{4.27}$$

Again, all $\frac{1}{2}\lambda_i$ matrices make up the Lie algebra $\mathfrak{u}(3)$.
To preserve $\det R = 1$, the identity matrix in disguise λ_0 is eliminated for $\mathfrak{su}(3)$. Hence SU(3) has only 8 generators λ_1 to λ_8 - the universally accepted Gell-Mann matrices.
The Gell-Mann matrices are the SU(3) equivalent of the SU(2) Pauli matrices. While the SU(2) eigenvectors correspond to a physical quantity (spin-up and spin-down states in spin-1/2 systems), the Gell-Mann matrices' eigenstates do not directly correspond to physical quantities.
Instead, one can think of them as different 'basis states' in the *colour space*, and linear combinations of these states describe physical colour configurations of quarks and gluons.

Definition 4.4 (Colour basis) The eigenvectors of the Gell-Mann matrices are known as the *colour basis*:

$$|\mathrm{red}\rangle = c_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad |\mathrm{green}\rangle = c_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad |\mathrm{blue}\rangle = c_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \tag{4.28}$$

From the 3 eigenstates, we have 3 colours: **red**, **green** and **blue**. They are complemented by the three antiquark colours: **antired**, **antigreen** and **antiblue**. A general *colour state* can therefore be represented as

$$c = \alpha c_1 + \beta c_2 + \gamma c_3 \tag{4.29}$$

where α, β and γ are complex numbers. The complex numbers represent probabilities and phase relationships in quantum states, same as complex spin states.
Just like how Pauli matrices can be combined to form the so-called *ladder operators* to move between spin

states, we can also combine the Gell-Mann matrices to form ladder operators to move between colour states:

Definition 4.5 (QCD ladder operators)

$$\underbrace{T_{\pm} = \frac{1}{2}(\lambda_1 \pm i\lambda_2)}_{\text{red} \to \text{green}} \quad \underbrace{V_{\pm} = \frac{1}{2}(\lambda_4 \pm i\lambda_5)}_{\text{red} \to \text{blue}} \quad \underbrace{U_{\pm} = \frac{1}{2}(\lambda_6 \pm i\lambda_7)}_{\text{green} \to \text{blue}} \tag{4.30}$$

4.5 Overview of $\mathrm{SO}(n)$ groups

Before considering the SO(3) group (or indeed, any generic special orthogonal group SO(n), it is expedient to look at a general case, which is the orthogonal group O(n).
O(n) consists of all $n \times n$ matrices R such that

$$R^T R = \mathbb{I}_n, \quad \det(R) = \pm 1 \tag{4.31}$$

In terms of its generators M, R is predictably represented by

$$R_i = e^{iM_i\theta} \quad M_i \in \mathfrak{o}(n) \tag{4.32}$$

where M are the collection of $n \times n$ skew-symmetric matrices M (i.e., $M^T = -M$) and make up the Lie algebra $\mathfrak{o}(n)$.
SO(n) is identical to O(n) with one exception: its elements R can only have determinant $+1$ instead of ± 1, as it is special:

$$R^T R = \mathbb{I}_n, \quad \det(R) = 1 \tag{4.33}$$

again, its generators M are members of the Lie algebra $\mathfrak{so}(n)$.
Remark 4.6 Importantly, the physical implication of SO(n)'s extra rescriction is that SO(n) includes only the rotation components of O(n), excluding reflections.

4.6 $\mathrm{SO}(1)$ group

The group O(1) consists of all 1×1 orthogonal matrices, which implies that $R^2 = 1$ and $\det R = \pm 1$. Possible Rs are hence

$$R = \pm 1 \to \mathrm{O}(1) = \{1, -1\} \tag{4.34}$$

where -1 and 1 correspond to reflection and identity respectively.
Now we consider SO(1), which consists of all 1×1 orthogonal matrices with determinant 1. As it turns out, there is only one candidate that fits the description:

$$R = 1 \to \mathrm{SO}(1) = \{1\} \tag{4.35}$$

Thus, the SO(1) group is trivial, containing only the identity element.
It can then be calculated that both $\mathfrak{o}(1)$ and $\mathfrak{so}(1)$ are zero Lie groups:

$$\mathfrak{o}(1) = \mathfrak{so}(1) = \{0\} \tag{4.36}$$

4.7 $\mathrm{SO}(2)$ group

The O(2) group or the *group of planar isometries* consists of all 2×2 orthogonal matrices R which then satisfy $R^T R = \mathbb{I}_2$ and $\det R = \pm 1$, which ensures that R preserves lengths and angles. In 2 dimensions, O(2) can be described as the group of rotations and reflections in the plane:

- Rotations:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{4.37}$$

- Reflections[2]:

$$M(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \tag{4.38}$$

where $\theta \in [0, 2\pi)$. Thus, O(2) can be expressed as

$$\mathrm{O}(2) = \{R(\theta) \mid \theta \in [0, 2\pi)\} \cup \{M(\theta) \mid \theta \in [0, 2\pi)\} \tag{4.39}$$

Now we consider the SO(2) group or the *circle group*, which consists of all 2×2 orthogonal matrices with determinant 1. This condition means that reflections are excluded, and the only transformations remaining are (pure) rotations. Thus, SO(2) can be expressed as:

$$\mathrm{SO}(2) = \{R(\theta) \mid \theta \in [0, 2\pi)\} \tag{4.40}$$

Remark 4.7 As mentioned when we discussed U(1), SO(2) is a 1D manifold, specifically on a circle S^1 By calculation, it can be found that bothO(2) and SO(2) both have only one generator, which is the entirety of the (one-dimensional Lie algebras) $\mathfrak{o}(2)$ and $\mathfrak{so}(2)$[3]:

Definition 4.6 ($\mathfrak{o}(2)$ and $\mathfrak{so}(2)$ elements)

$$A = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4.41}$$

4.8 SO(3) group

The group O(3) or the *group of 3D isometries* consists of all 3×3 orthogonal matrices, which then satisfy $R^T R = \mathbb{I}_3$ and $\det R = \pm 1$, where $\det(R) = 1$ represent (proper) rotations while $\det(R) = -1$ represent improper rotations (including reflections). Thus, O(3) can be written as:

$$\mathrm{O}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det(R) = \pm 1\} \tag{4.42}$$

The SO(3) group or the *rotation group* has only Rs with determinant $+1$, which are proper rotations. As such, we have

$$\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I, \det(R) = 1\} \tag{4.43}$$

SO(3) can be parameterised in terms of a rotation axis and angle:

$$R_{xy}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_{yz}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} \quad R_{zx}(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix} \tag{4.44}$$

For the same reason as $\mathfrak{o}(2)$ and $\mathfrak{so}(2)$, the elements of $\mathfrak{o}(3)$ and $\mathfrak{so}(3)$ are identical, only differing in physical interpretations:

Definition 4.7 ($\mathfrak{o}(3)$ and $\mathfrak{so}(3)$ elements)

$$M_{xy} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad M_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad M_{zx} = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \tag{4.45}$$

By explicit calculation, it is possible to find that

$$[M_i, M_j] = i\epsilon_{ijk} M_k \tag{4.46}$$

Derivation 4.4 (Double cover of SU(2) on SO(3)) Amazingly, this commutation is identical to that

[2] Also known as *improper rotations*.

[3] The Lie algebra $\mathfrak{so}(2)$ is identical to $\mathfrak{o}(2)$ because SO(2) is a connected Lie group, and the Lie algebra of a connected Lie group coincides with the Lie algebra of its associated orthogonal group.

of $\mathfrak{su}(2)$ in Equation 4.6, which suggests that $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$:

$$\mathfrak{su}(2) \cong \mathfrak{so}(3) \tag{4.47}$$

which suggests that a relationship also exists between SU(2) and SO(3). We begin by analysing how elements of SU(2) are related to rotations. Consider the following arbitrary element of SU(2), which is parameterised by the rotation axis $\hat{n}_i$ and the angle $|\theta|$:

$$U = \exp\left(\frac{i|\theta|}{2}\hat{n}_i \cdot \sigma_i\right) \tag{4.48}$$

where $\hat{n}_i$ is a unit vector along the axis of rotation and $|\theta|$ is the rotation angle. Expanding the exponential using properties of the Pauli matrices yields:

$$U = \cos\left(\frac{|\theta|}{2}\right)\mathbb{I} + i(\hat{n}_i \cdot \sigma_i)\sin\left(\frac{|\theta|}{2}\right) \tag{4.49}$$

where the parameter $|\theta|/2$ ranges over $0 \leq |\theta|/2 < 2\pi$, which corresponds to $0 \leq |\theta| < 4\pi$ for SU(2). Geometrically, SU(2) can be visualised as the 3-sphere S^3, a compact, simply connected Lie group. As such, we can also define the negation $-U$, which is the antipodal point of U on the SU(2) manifold (i.e. the 3-sphere S^3):

$$-U = -\left(\cos\left(\frac{|\theta|}{2}\right)\mathbb{I} + i(\hat{n}_i \cdot \sigma_i)\sin\left(\frac{|\theta|}{2}\right)\right) = \cos\left(\frac{|\theta|}{2}\right)(-\mathbb{I}) - i(\hat{n}_i \cdot \sigma_i)\sin\left(\frac{|\theta|}{2}\right) \tag{4.50}$$

Now, define a surjective homomorphism $\Phi : \mathrm{SU}(2) \to \mathrm{SO}(3)$, such that

$$\Phi\left(\exp\left(\frac{i|\theta|}{2}\hat{n}_i \cdot \sigma_i\right)\right) = \exp(i|\theta|\hat{n}_i \cdot M_i) \tag{4.51}$$

where M_i are the generators of $\mathfrak{so}(3)$, and the RHS corresponds to a rotation in 3D space about the axis $\hat{n}_i$ by an angle $|\theta|$. This mapping is well-defined because:

- **Preservation of structure:** Φ respects the group multiplication law. Specifically, if $U_1, U_2 \in$ SU(2), then $\Phi(U_1 U_2) = \Phi(U_1)\Phi(U_2)$.
- **Surjectivity:** Every element of SO(3) can be expressed as a rotation about some axis $\hat{n}_i$ by an angle $|\theta|$, and these rotations are fully covered by the image of Φ.

Importantly, we note that under Ψ, the rotation which has previously covered SU(2) under $0 \leq |\theta| < 4\pi$ now covers SO(3) under the very same $0 \leq |\theta| < 4\pi$. This has two equivalent statements:

- Ψ maps U, the elements of SU(2) to SO(3) in such a way that $\Phi(U)$ and $\Phi(-U)$ correspond to the very same rotation in SO(3).
- SU(2) is a *double cover* of SO(3), meaning that each element of SO(3) has exactly two preimages in SU(2):

$$\Phi^{-1}(R) = \{U, -U\}, \quad \text{for } R \in \mathrm{SO}(3) \tag{4.52}$$

Remark 4.8 The double covering is so-called as SU(2) spans twice the range of rotations, i.e., $0 \leq |\theta| < 4\pi$, while SO(3) only spans $0 \leq |\theta| < 2\pi$.

Hence, SO(3) is isomorphic to $S^3/\mathbb{Z}_2$ or rather $\mathrm{SU}(2)/\mathbb{Z}_2$, giving rise to the almighty conclusion

Theorem 4.3 (Isomorphism between SO(3) and SU(2))

$$\mathrm{SO}(3) \cong \mathrm{SU}(2)/\mathbb{Z}_2 \tag{4.53}$$

where $\mathbb{Z}_2 = \{\pm 1\}$ acts by identifying antipodal points on S^3.

Thus, SO(3) is not simply connected, and SU(2) serves as its *covering group*[a].

[a]SU(2) is also the *covering space* of SO(3). All covering groups are covering spaces, but a covering space can only be a group if it is a topological group.

4.9 $\mathrm{SO}(1,3)$ group

Shockingly, SR has once again risen from the grave. We now consider the $\mathrm{SO}(1,3)$ group or the so-called *Lorentz group*, which includes all Lorentz transformations (i.e. rotations and boosts) in Minkowski spacetime[4].

Definition 4.8 (Isometry) A coordinate transformation $x^\mu \to x'^\nu$ is called an *isometry* or a *local isometry* if the metric (i.e. the line element) is invariant.

The Lorentz group encodes all Minkowski spacetime isometries *under which the origin is invariant*. A generalisation of the Lorentz group (or in this context, the *homogeneous Lorentz group*) that includes all isometries is hence the *inhomogeneous Lorentz group* or the *inhomogeneous special orthogonal group* or the *Poincaré group*.

Remark 4.9 The only difference between the two is that the Poincaré group includes isometries that change the position of the origin. Now what is another name for this?
If you have translations in mind, then you would be right.

Definition 4.9 (Poincaré group) A Poincaré group $\mathrm{ISO}(1,3)$[a] is the full symmetry group of Minkowski spacetime (i.e. SR).

[a] As you may have guessed, ISO stands for *inhomogeneous special orthogonal* group.

It is then intuitive that an element of the Poincaré group would be called the *Poincaré transformation*. A generalised example would be a combination of a Lorentz transformation $\Lambda_{\mu\nu}$ and a translation vector a_μ:

$$x_\mu \to x'_\mu = \Lambda_{\mu\nu} x^\nu + a_\mu \tag{4.54}$$

or in wavefunction form

$$\phi(x) \to \phi(\Lambda x + a_\mu) \tag{4.55}$$

Remark 4.10 In 3D, the Poincaré group reduces to an *(inhomogeneous) Galiean group* $\mathrm{ISO}(3)$, also known as a *Euclidian group* $\mathrm{E}(3)$.
To construct the $\mathfrak{so}(1,3)$ algebra, one first parameterise the generators of the $\mathrm{SO}(1,3)$ group. Instead of the usual notation J^a for generators, we use $\omega_{\rho\sigma} J^{\rho\sigma}$ where $\omega_{\rho\sigma}$ are a series of parameters. It then follows that, for some metric trace g

$$(e^{i\omega_{\rho\sigma} J^{\rho\sigma}})^T g e^{i\omega_{\rho\sigma} J^{\rho\sigma}} = g \tag{4.56}$$

which gives

$$i\omega_{\rho\sigma} (J^{\rho\sigma})^T g J^{\rho\sigma} = 0 \tag{4.57}$$

Removing the prefactor and unpacking the metric trace yields

$$J^{\rho\sigma}_{\alpha\nu} g^{\alpha\mu} + g_{\nu\alpha} J^{\rho\sigma\alpha\mu} = J^{\rho\sigma\mu}_{\nu} + J^{\rho\sigma\nu}_{\mu} = 0 \tag{4.58}$$

which means that this rank-4 J must be antisymmetric in μ and ν. It then makes sense for us to choose it to be also antisymmetric in ρ and σ. This yields the expression

$$J^{\rho\sigma\mu}_{\nu} = i(g^{\rho\mu}\delta^\sigma_\nu - g^{\sigma\mu}\delta^\rho_\nu) \tag{4.59}$$

To recover the rank-2 $J^{\rho\sigma}$, we set $\mu = \nu$ and contract them. Solving the resultant expression gives the list of generators:

[4] Sometimes one includes reflections, and the Lorentz group simply becomes $\mathrm{O}(3)$. Its identity element, $\mathrm{SO}(1,3)$, is then called the *rescricted Lorentz group*.

Definition 4.10 ($\mathfrak{so}(1,3)$ generators)

$$
\begin{aligned}
J^{10} &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^{20} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^{30} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\
J^{12} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \quad J^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}
\end{aligned}
\tag{4.60}
$$

Theorem 4.4 ($\mathfrak{so}(1,3)$ generator commutation relations) By recovering the $\mathfrak{su}(2)$ generators as $\Sigma_j = \frac{1}{2}\epsilon_{jkl}J^{kl}$, we find the $\mathfrak{so}(1,3)$ generator commutation relations:

$$
[J^{j0}, J^{k0}] = -i\epsilon_{jkl}J^{l0} \quad [\Sigma_j, j^{k0}] = i\epsilon_{jkm}\Sigma_l \tag{4.61}
$$

where the first and second item is simply the $\mathfrak{so}(3)$ and $\mathfrak{su}(2)$ generator commutation relations respectively.

In fact, we can note that the generators in Equation 4.60 include the generators in Equation 4.45 in the form of J^{12}, J^{13} and J^{23}, where we have expanded into the time coordinate by adding one row on top and one column to the left. The purposes of the generators are hence clear:

- *Rotations* are generated by J^{12}, J^{13} and J^{23}.
- *Boosts* are generated by J^{10}, J^{20} and J^{30}.

Remark 4.11 If one considers the Minkowski space (t, x, y, z) as a hyperbolic plane, boosts are in fact rotations on this hyperbolic plane.

> **Quote 4.2** the boosts *are* rotations (...) well, hyperbolic rotations
>
> *Paulina Schlachter, 2 February 2025*

Part II

Graphics

Chapter 5

Young diagrams

> **Quote 5.1** In other words, the lower-left boundary of the collection of boxes takes the form of a staircase[a].
>
> *Niklas Beisert, in his lecture notes, 2020*
>
> [a]...defying the laws of gravity.

5.1 Young diagrams, or the rise of eye candy

A good way to visualise representations is the so-called *Young diagram*, which takes the form of an inverse staircase. The filling (😋) of a Young diagram is known as a *Young tableau*[1].

Definition 5.1 (Standard filling) A so-called *standard filling* obeys the following:

- All entries have a unique number $1, \cdots, m$
- The number in each entry increases...
  - ...from left to right.
  - ...from top to bottom.

For example, a standard filling might look like

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 & 7 \\ \hline 3 \\ \cline{1-1}\end{array} \tag{5.1}$$

where the squares make up the Young diagrams.

Definition 5.2 (Hook length) The *hook length* for one box in a Young diagram is the number of the boxes below it plus the number of boxes to its right plus one (i.e. itself).

As an example, the hook length of each box in the following Young diagram is written as the filling:

$$\begin{array}{|c|c|c|c|c|}\hline 8 & 7 & 5 & 4 & 1 \\ \hline 6 & 5 & 3 & 2 \\ \cline{1-4} 5 & 4 & 2 & 1 \\ \cline{1-4} 2 & 1 \\ \cline{1-2}\end{array} \tag{5.2}$$

Hook lengths can be used to calculate the number of possible standard fillings a certain Young diagram can have. The number of standard fillings $|\mu|$, also known as the *dimension* of the Young diagram, is given by

[1]Plural: tableaux

Theorem 5.1 (Hook length formula)

$$|\mu| = \frac{m!}{\prod \text{hook lengths}} \tag{5.3}$$

where m is the number of boxes in the Young diagram.

For example, for the Young diagram $\ydiagram{2,2,1}$, one has the hook lengths $\begin{ytableau} 4 & 2 \\ 3 & 1 \\ 1 \end{ytableau}$ and the number of standard fillings

$$|\mu| = \frac{m!}{4 \times 2 \times 3 \times 1 \times 1} = 5 \tag{5.4}$$

corresponding to the following fillings:

$$\begin{ytableau} 1 & 2 \\ 3 & 4 \\ 5 \end{ytableau} \qquad \begin{ytableau} 1 & 2 \\ 3 & 5 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 3 \\ 2 & 5 \\ 4 \end{ytableau} \qquad \begin{ytableau} 1 & 4 \\ 2 & 5 \\ 3 \end{ytableau} \tag{5.5}$$

Now we connect Young diagrams to representation theory.

5.2 Groups revisited

Now let us connect Young diagrams to actual groups. A permutation group S_n corresponds to a series of Young diagrams with n boxes:

- For S_2:

$$\ydiagram{2} \quad \ydiagram{1,1} \tag{5.6}$$

- For S_3:

$$\ydiagram{3} \quad \ydiagram{2,1} \quad \ydiagram{2,1} \quad \ydiagram{1,1,1} \tag{5.7}$$

- For S_4:

$$\ydiagram{4} \quad \ydiagram{3,1} \quad \ydiagram{2,2} \quad \ydiagram{3,1} \quad \ydiagram{1,1,1,1} \tag{5.8}$$

where:

- A diagram that is a single row only is totally symmetric.
- A diagram that is a single column is totally antisymmetric.
- A diagram with both more than one column and more than one row has mixed symmetry.

This gives rise to another super-duper secret way to determine the dimension of a Young diagram. We make a connection between the group S_n, perfectly diagonal Young diagrams of n rows and n columns and the smaller 'permutation diagrams'. we have seen before. For example, for S_3:

$$S_3 \rightarrow \ydiagram{3,2,1} \rightarrow \ydiagram{3} \quad \ydiagram{2,1} \quad \ydiagram{2,1} \quad \ydiagram{1,1,1} \tag{5.9}$$

Amazingly, there is a relationship between $n!$ and the dimensions of smaller diagrams (of indices labelled $1, \cdots, i, \cdots$):

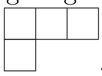
$$n! = \sum_i d_i^2 \tag{5.10}$$

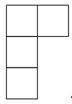
- For S_2:
$$2! = 2 = 1^2 + 1^2 \tag{5.11}$$

- For S_3:
$$3! = 6 = 1^2 + 2^2 + 1^2 \tag{5.12}$$

- For S_4:
$$4! = 24 = 1^2 + 3^2 + 2^2 + 3^2 + 1^2 \tag{5.13}$$

5.3 Young symmetriser

Consider a (finite) symmetry group S_n on n elements, which consists of all permutations of $\{1, \cdots, n\}$ under composition. A *partition* of n is a sequence of non-increasing positive integers $\lambda = (\lambda_1, \cdots, \lambda_k)$ which satisfies
$$\lambda_1 + \cdots + \lambda_k = n \quad \lambda_1 \geq \cdots \geq \lambda_k \tag{5.14}$$
We can describe partitions in terms of Young diagrams. Under this scheme, each partition λ_i corresponds to the number of squares on the i^{th} row of the Young diagram that the whole partition λ makes up. For example, for $\lambda_1 = 3$, $\lambda_2 = 1$, one has $\lambda = (3, 1)$ and  .

The *conjugate partition* is then defined as $\lambda' = (2, 1, 1)$ and corresponds to the Young diagram  . A more mathematically rigorous definition is as follows:

Definition 5.3 (Conjugate partition) For a partition $\lambda = (\lambda_j)$ one has the conjugate partition $\lambda' = (\lambda_i)$, where each term λ'_i is the number of λ_js that satisfy $\lambda_j \leq i$.

Each partition λ corresponds to a Young diagram, which is filled by a Young tableau consisting of the elements $\{1, \cdots, n\}$ (i.e. elements of S_n). A Young tableau whose filling is of this form is known to be *standard*.
Now consider, for a specific partition λ, two subgroups of S_n:

- The *row stabiliser* P_λ is generated by all permutations within each row.
$$P_\lambda = \{g \in S_n : g \text{ preserves each row of } \lambda\} \tag{5.15}$$

- The *column stabiliser* Q_λ is generated by all permutations within each column.
$$Q_\lambda = \{g \in S_n : g \text{ preserves each column of } \lambda\} \tag{5.16}$$

From these, we define:

- The *row symmetriser* a_λ sums over all row permutations:
$$a_\lambda = \sum_{g \in P_\lambda} e_g \tag{5.17}$$
where e_g is the unit vector corresponding to g.

- The *column symmetriser* Q_λ takes the alternating sum over all column permutations:
$$b_\lambda = \sum_{h \in Q_\lambda} \text{sign}(h) e_h \tag{5.18}$$
where $\text{sign}(h)$ is the sign of the permutation h:
  - If h swaps an *even number* of transpositions, $\text{sign}(h) = +1$.
  - If h swaps an *odd number* of transpositions, $\text{sign}(h) = -1$.

Remark 5.1 For example:

- Identity permutation e (does nothing), $\text{sign}(e) = +1$.
- Swapping 1 and 2 (in column 1): $(1\ 2)$, $\text{sign}(1\ 2) = -1$.
- Swapping 3 and 4 (in column 2): $(3\ 4)$, $\text{sign}(3\ 4) = -1$.
- Swapping both pairs simultaneously: $(1\ 2)(3\ 4)$, $\text{sign}((1\ 2)(3\ 4)) = (-1) \cdot (-1) = +1$.

Definition 5.4 (Young symmetriser)

$$c_\lambda := a_\lambda b_\lambda \in \mathbb{C}S_n \tag{5.19}$$

Theorem 5.2 (Young symmetriser properties)

- Young symmetrisers are projectors[a]:

$$c_\lambda c_\lambda = c_\lambda \tag{5.20}$$

- Two symmetrisers for different Young diagrams have zero product:

$$c_\lambda c_{\lambda'} = 0 \tag{5.21}$$

Although the product between two symmetrisers of different rows of the same diagram might not be zero.

[a] i.e. they behave as idempotent elements in the group algebra.

Now we are in a position to discuss the significance of Young symmetrisers. First, we note that Young symmetrisers act on the algebra $\mathbb{C}S_n$ of the group S_n. Using Young symmetrisers, we can construct the Specht modules S^λ, which form a complete set of irreps of S_n.

Definition 5.5 (Specht module) A Specht module S^λ is the representation of S_n associated with the partition λ

$$S^\lambda = c_\lambda \mathbb{C}S_n \tag{5.22}$$

Theorem 5.3 (Specht module properties)

- Action on the group algebra: c_λ acts as a projection operator in $\mathbb{C}S_n$, isolating a specific subspace that carries the irreducible representation.
- Irreducibility: The space S^λ is isomorphic to V^λ, an irreducible representation of S_n.
- Dimension Formula: The dimension of S^λ is given by the hook-length formula:

$$\dim S^\lambda = \frac{n!}{\prod_{i,j\in\lambda} h_{i,j}} \tag{5.23}$$

where $h_{i,j}$ is the hook length of the cell (i, j).

$\mathbb{C}S_n$ can be decomposed into a direct sum of matrix algebras associated with irreducible representations V^λ:

$$\mathbb{C}S_n = \bigoplus_\lambda \text{End}(V^\lambda), \tag{5.24}$$

However, as the Specht modules S^λ are isomorphic to the irreducible representations V^λ of S_n corresponding to the partitions λ, one can also write

$$\mathbb{C}S_n = \bigoplus_\lambda \text{End}(S^\lambda), \tag{5.25}$$

and go even as far as to say that the Specht modules themselves make up the irreps!

5.4 Frobenius formula

To determine the character of the irreducible representation associated with a Young diagram λ, we make use of the infamous *Frobenius formula*. Let S_n be the symmetric group on n elements, with:

- An irreducible representation whose character is χ_λ.
- A partition λ of n, satisfying $\lambda_1 + \cdots + \lambda_k = n$.
- A conjugacy class $\mathrm{Cl}(\mu)$ in S_n, corresponding to a partition λ_μ of n.
- i_j, the number of times where j appears in μ, such that $\sum_j i_j j = n$.

Now, define for $1 \leq j \leq n$:

- Power sum:
$$P_j(x) = x_1^j + x_2^j + \cdots + x_k^j \tag{5.26}$$
where $x = (x_1, \ldots, x_k)$ is a set of independent variables, j is an exponent (not an index), and k is at most the number of rows in λ.
- Vandermonde determinant:
$$\Delta(x) = \prod_{1 \leq i < j \leq k} (x_i - x_j) \tag{5.27}$$

This leads to the Frobenius formula for the character:

Theorem 5.4 (Frobenius formula)

$$\chi_\lambda(\mathrm{Cl}(\mu)) = \left[\Delta(x) \cdot \prod_{j=1}^{n} P_j(x)^{i_j} \right]_{(l_1, \ldots, l_k)} \tag{5.28}$$

where we define $l_m = \lambda_m + k - m$, and the notation $[\cdot]_{(l_1,\ldots,l_k)}$ extracts the coefficient of $x_i^{l_1}, \cdots x_k^{l_k}$ within $[\cdot]$.

So far, this section might feel quite self-contained[2]. However, we remind ourselves that the irrep of a partition λ_μ of a symmetric group S_n is isomorphic to the Specht module S^{λ_μ}. Hence, we can simply say that the Frobenius formula calculates the characters of the Specht modules.

[2] Which is not good, as it is about to end!

Chapter 6

Root systems

6.1 Cartan-Weyl basis

Definition 6.1 (Cartan subalgebra) A *maximal Cartan subalgebra* (usually simply called a *Cartan subalgebra*) $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is the maximal abelian subalgebra of $\mathfrak{g}$ that is self-normalising. It satisfies the following properties:

- **Nilpotency:** $\mathfrak{h}$ is nilpotent. i.e. the Lie brackets are trivial:

$$\mathrm{Ad}_{X_1} \mathrm{Ad}_{X_2} \dots \mathrm{Ad}_{X_k}(Y) = 0 \tag{6.1}$$

  for all $X_1, X_2, \dots, X_k \in \mathfrak{h}$ and all $Y \in \mathfrak{g}$.

- **Self-normalisation:** The normaliser of $\mathfrak{h}$ in $\mathfrak{g}$ is precisely $\mathfrak{h}$ itself:

$$\{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subseteq \mathfrak{h}\} = \mathfrak{h}. \tag{6.2}$$

  This means that $\mathfrak{h}$ is as large as possible while maintaining the nilpotency property.

- **Maximality:** $\mathfrak{h}$ is a maximal abelian subalgebra.

Some examples are in order:

- For the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, the set of diagonal matrices forms a Cartan subalgebra.
- For the classical semisimple Lie algebras like $\mathfrak{sl}_n(\mathbb{C})$, a Cartan subalgebra consists of the diagonal matrices with trace zero.

Using elements of a Cartan subalgebra $\mathfrak{h}$, its corresponding Lie algebra $\mathfrak{g}$ can be decomposed into a direct sum eigenspaces called *weight spaces* or *root spaces* that are determined by eigenvalues corresponding to the adjoint actions of elements in $\mathfrak{h}$.
In the case of a semisimple Lie algebra, this is mathematically

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \tag{6.3}$$

where Δ is the set of *roots* that are nonzero eigenvalues.

Definition 6.2 (Root space) Each so-called root space $\mathfrak{g}_\alpha$ consists of elements X satisfying:

$$\mathrm{ad}(H, X) = [H, X] = \alpha(H, X) \quad \forall H \in \mathfrak{h} \tag{6.4}$$

where $\alpha : \mathfrak{h} \to \mathbb{C}$ is a linear function that results in a root, and X is an eigenvector under the adjoint action.

Derivation 6.1 (Cartan-Weyl basis) Decomposing the $\mathfrak{sl}(3)$ Lie algebra results in the so-called *Cartan-Weyl basis*. Consider a Lie algebra $\mathfrak{g}$ and its Cartan subalgebra $\mathfrak{h}$. The decomposition then takes the form

$$\mathfrak{sl}(3) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha} \mathfrak{g}_{\alpha} \right) \tag{6.5}$$

We set Cartan elements $B \in \mathfrak{g}$ and raising and lowering operators $H \in \mathfrak{h}$ to be of the form

$$H \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \tag{6.6}$$

which satisfy

$$\mathrm{Tr}(H) + \mathrm{Tr}(B) = 0 \tag{6.7}$$

Imposing the adjoint action gives

$$\mathrm{ad}(H, B) = [H, B] = \begin{pmatrix} 0 & (a_1 - a_2)\, b_{12} & (a_1 - a_3)\, b_{13} \\ (a_2 - a_1)\, b_{21} & 0 & (a_2 - a_3)\, b_{23} \\ (a_3 - a_1)\, b_{31} & (a_3 - a_2)\, b_{32} & 0 \end{pmatrix} \tag{6.8}$$

The eigenvectors are the matrices B_{ij} with $i \neq j$. The Cartan-Weyl basis is

$$\{B_{11} - B_{22}, B_{22} - B_{33}, B_{12}, B_{13}, B_{21}, B_{23}, B_{31}, B_{32}\} \tag{6.9}$$

Part III

Spinors

Chapter 7

Grassmann mathematics

7.1 Grassmann algebra

Before Clifford algebras, we first need to understand the so-called *Grassmann algebras*.

Definition 7.1 (Multivector & wedge product) For any vectors u, v, w, etc., we can define the *wedge product*

$$u \wedge v \wedge w \wedge \cdots \tag{7.1}$$

The result of a wedge product of k vectors is called a *k-vector* or a *k-blade*. k is called the *grade* which is analogous to the tensorial rank.

Remark 7.1 A sum of k-vectors of different grades is called a *multivector*. To get an intuitive understanding, we look at the 2-vector (also called the *bivector* or the *antivector*) and the 3-vector.

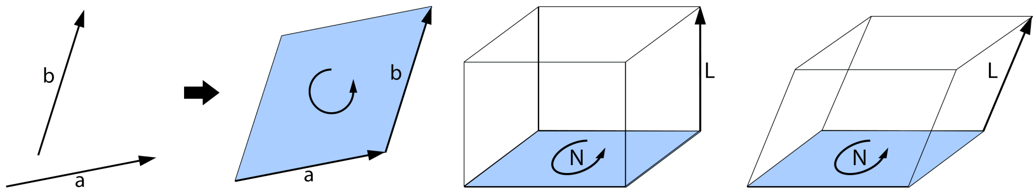


Figure 7.1: Windings of a 2-vector (left) and two similar 3-vectors (right).

Aside from the 2D area in blue[1], the bivector also contains an orientation, or rather a *winding*, which is the direction as shown by the anticlockwise arrow. The inclusion of the winding distinguishes a bivector from simply an area.

Derivation 7.1 (2-vectors and 3-vectors) Suppose we have two vectors $u = u_1e_1 + u_2e_2 + u_3e_3$ and $v = v_1e_1 + v_2e_2 + v_3e_3$. The wedge product is then

$$\begin{aligned} u \wedge v &= (u_1e_1 + u_2e_2 + u_3e_3) \wedge (v_1e_1 + v_2e_2 + v_3e_3) \\ &= (u_2v_3 - u_3v_2)(e_2 \wedge e_3) + (u_3v_1 - u_1v_3)(e_3 \wedge e_1) + (u_1v_2 - u_2v_1)(e_1 \wedge e_2) \end{aligned} \tag{7.2}$$

To simplify the expression, the following shorthand exists:

$$e_{12} = e_1 \wedge e_2 \quad e_{23} = e_2 \wedge e_3 \quad e_{31} = e_3 \wedge e_1 \quad e_{123} = e_1 \wedge e_2 \wedge e_3 \tag{7.3}$$

Which gives

$$u \wedge v = (u_2v_3 - u_3v_2)e_{23} + (u_3v_1 - u_1v_3)e_{31} + (u_1v_2 - u_2v_1)e_{12} \tag{7.4}$$

While this might look like an outer product, the wedge product is associative, while the outer product is not.
We now consider a 3-vector:

$$u \wedge v \wedge w = (u_1v_2w_3 + u_2v_3w_1 + u_3v_1w_2 - u_1v_3w_2 - u_2v_1w_3 - u_3v_2w_1)e_{123} \tag{7.5}$$

[1] The equivalent of this for 3-vectors is the 3D volume seen on the right.

There is only one component e_{123}. A 3-vector changes sign under a mirror reflection due to the presence of the winding. As such, it is also called an *antiscalar* or *pseudoscalar* when we work in 3D space[a].

[a]The implication being that a n-vector is only an antiscalar in n-dimensional space.

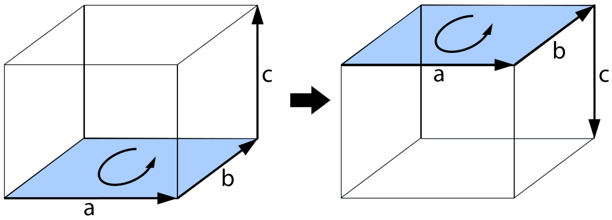


Figure 7.2: Change of sign of the antiscalar due to opposite windings.

Remark 7.2 To reverse the winding of a trivector, simply reverse the windings of all its bivector faces.

Definition 7.2 (Exterior product) The generalisation of the wedge product to multivectors of any grade is called the *exterior product*. Consider two differential forms α and β. In index notation, their exterior product $\alpha \wedge \beta$ can be represented as

$$(\alpha \wedge \beta)_{i_1 \dots i_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[i_1 \dots i_p} \beta_{i_{p+1} \dots i_{p+q}]} \tag{7.6}$$

where $_{[i_1 \dots i_p} \beta_{i_{p+1} \dots i_{p+q}]}$ denotes the antisymmetrisation of the indices[a]. The winding is shown in the signs of each term. Consider the wedge product of a 1-form α_i and a 2-form β_{jk}. The resultant 3-form is

$$(\alpha \wedge \beta)_{ijk} = 3\alpha_{[i}\beta_{jk]} = \alpha_i \beta_{jk} - \alpha_j \beta_{ik} + \alpha_k \beta_{ij} \tag{7.7}$$

[a]You have hopefully seen this in *Metrics and Cosmos and Trilobites*.

Theorem 7.1 (Exterior product properties) The exterior product observes the following properties:

- If one term is a scalar, the wedge product involving that term reduces to scalar multiplication. For scalar a and some n-form B:
$$a \wedge B = B \wedge a = aB \tag{7.8}$$
- Linearity. For scalars a and b:
$$u \wedge (av + bw) = au \wedge v + bu \wedge w \tag{7.9}$$
- The exterior product of a vector against itself is meaningless:
$$u \wedge u = -u \wedge u = 0 \tag{7.10}$$

Definition 7.3 (Grassmann algebra) As the exterior product is linear, the resultant vector space of the resultant n-form $\Lambda^n(V)$ is an algebra. This algebra is known as the *exterior algebra* or the *Grassmann*[a] *algebra*.

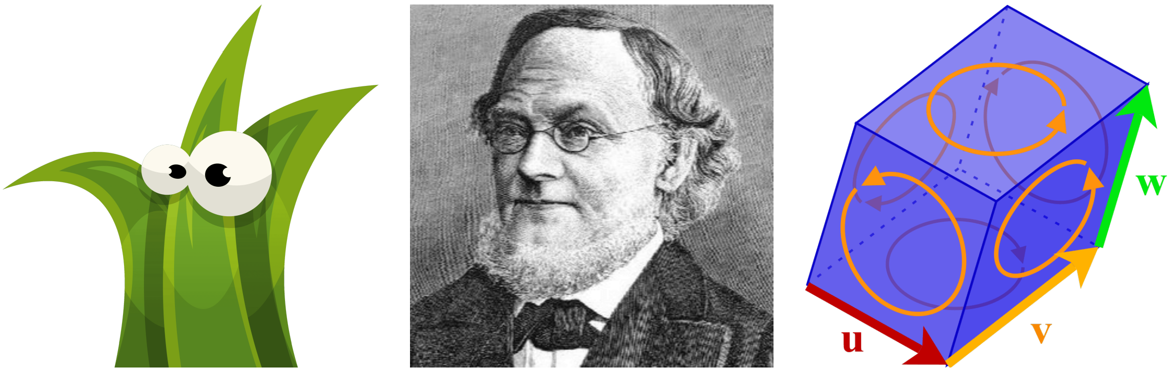
[a]This is the international spelling. The German spelling is *Graßmann*.

Quote 7.1 I think it uses the ß (or ss in international spelling)

Paul Kothgasser, on a different name, 29 September 2024

Quote 7.2 the ß is my favourite letter
i even managed to weasel it into my bsc thesis even tho that was in english

Paulina Schlachter, 29 September 2024



The Generic Grass Man | Hermann Grassmann | The Grassmann Algebra

Figure 7.3: Three types of Grassmann.

Remark 7.3 Get it? *Grass*mann? *Blade*?

Note 7.1 The fact that $u \wedge u = 0$ is what makes the wedge product different from the tensor product. Otherwise, the corresponding Grassmann algebra simply reduces to the so-called *tensor algebra* which defines the tensor product.

7.2 Clifford algebra

Definition 7.4 (Clifford algebra) The Clifford algebra $\mathrm{Cl}(m, n)$ is a collection of matrices (which we denote as *symbols*) with m elements squaring to 1 and n elements squaring to -1. The symbols anti-commute:

$$s_i s_j = -s_i s_j \quad \text{for} \quad i \neq j \tag{7.11}$$

Note 7.2 (Metric signature) Importantly, the Clifford algebra $\mathrm{Cl}(m, n)$ also corresponds to a metric of signature (m, n). For example, $\mathrm{Cl}(3, 0)$ (often simply denoted $\mathrm{Cl}(3)$) corresponds to a 3D Euclidian space - i.e. a metric of signature $(+, +, +)$ while $\mathrm{Cl}(1, 3)$ corresponds to Minkowski spacetime, which has the metric signature $(-, +, +, +)$.

Remark 7.4 For example, we can interpret the number 1 and -1 as the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the negative identity $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. The imaginary number i, interpreted as the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is then a Clifford algebra denoted as $\mathrm{Cl}(0, 1)$.

Definition 7.5 (Clifford product) The rather annoyingly notationless *Clifford product* between vectors $\vec{u}$ and $\vec{v}$ is defined as

$$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v} \tag{7.12}$$

The object resulting from a Clifford product of vectors is called a *versor*.

Remark 7.5 For orthogonal vectors, $\vec{u}\vec{v}$ reduces to $\vec{u} \wedge \vec{v}$, and for parallel vectors, $\vec{u}\vec{v}$ reduces to $\vec{u} \cdot \vec{v}$.

7.3 Grassmann numbers

The so-called *Grassmann numbers* define the notion of anti-commuting numbers.

Definition 7.6 (Grassmann number) The Grassmann numbers θ_i are numbers defined by their anti-commutation

$$\{\theta_i, \theta_j\} = 0 \tag{7.13}$$

or

$$\theta_i \theta_j = -\theta_j \theta_i \tag{7.14}$$

Theorem 7.2 (Grassmann number properties)

- **Nilpotency:**
$$\theta_i^2 = 0 \tag{7.15}$$
- **Conjugation:** This is defined as
$$(\theta^1 \cdots \theta^n)^* = (\theta^n)^* \cdots (\theta^1)^* \tag{7.16}$$

Hence, a set of *independent* Grassmann numbers that go up to some index N is

$$\{1, \theta^1, \theta^1\theta^2, \cdots, \theta^1 \cdots \theta^n\} \tag{7.17}$$

Grassmann numbers are related to Clifford algebra elements by Previously, we have seen that the complex number system can be expanded by our good friends, the quaternions. A similar extension of the complex number system can be accomplished by Grassmann numbers. A generalised example might look like

$$c_0 + c_1\theta^1 + \frac{1}{2!}c_{12}\theta^1\theta^2 + \cdots + \frac{1}{N!}\theta^1 \cdots \theta^n \tag{7.18}$$

where:

- The factorials are included for simplicity later on.
- The coefficients c are matrices, which are antisymmetric in all indices[2].

While conceptually similar to quaternions, the Grassmann number system is different in that it is anti-commutative and nilpotent[3].

7.4 Grassmann analysis

Definition 7.7 (Grassmann-even and Grassmann-odd) In practice, it is useful to split a Grassmann number z into its *Grassmann-even* and *Grassmann-odd* parts b and f:

$$z = b + f \tag{7.19}$$

where:

- The Grassmann-odd part f is purely anticommuting, meaning it contains at least one Grassmann variable. It is nilpotent and satisfies Equation 7.14.
- The Grassmann-even b commuting, meaning it can be made up of ordinary numbers (i.e. real or complex numbers) and products of an even number of Grassmann variables.

The simplest analytical function of z can then be written as a series

$$F(z) = F(b) + \frac{dF(b)}{db}f \tag{7.20}$$

where terms with higher orders of f have vanished due to the nilpotency of f. We sometimes call functions of this form a *Grassmann function*.
For two variables, this becomes

$$F(z_1, z_2) = F(b_1, b_2) + \frac{\partial F(b_1, b_2)}{\partial b_1}f_1 + \frac{\partial F(b_1, b_2)}{\partial b_2}f_2 + \frac{1}{2}\frac{\partial^2 F(b_1, b_2)}{\partial b_1 \partial b_2}f_1 f_2 \tag{7.21}$$

For non-analytical functions, we can write

$$F(z) = F_0(b) + F_1(b)f \tag{7.22}$$

[2] Since the products of the Grassmann numbers are also antisymmetric.

[3] Clifford algebra elements also obey anticommutation relations. However, it has a quadratic norm structure instead of a nilpotent one.

and for two variables

$$F(z_1, z_2) = F_0(b_1, b_2) + F_1(b_1, b_2)f_1 + F_2(b_1, b_2)f_2 + F_{12}(b_1, b_2)f_1 f_2 \tag{7.23}$$

We now discuss how calculus can be done in Grassmann numbers.

Theorem 7.3 (Differentiation of Grassmann numbers)

- Real numbers continue to follow the well-known

$$\frac{\partial}{\partial f_i} 1 = 0 \tag{7.24}$$

- Alternatively, Grassmann numbers obey

$$\frac{\partial}{\partial f_i} f_j = \delta_{ij} \tag{7.25}$$

  where δ_{ij} is the all-too-familiar delta function.

- They anti-commute with respect to Grassmann numbers:

$$\frac{\partial}{\partial f_1} f_2 f_1 = -f_2 \frac{\partial}{\partial f_1} f_1 = -f_2 = \frac{\partial}{\partial f_1} (-f_1 f_2) = \frac{\partial}{\partial f_1} f_2 f_1 \tag{7.26}$$

We also note that derivatives follow the Leibniz rule, which is a good point for us to generalise them to some z:

$$\frac{\partial}{\partial z_i} (z_j z_k) = \left(\frac{\partial}{\partial z_i} z_j \right) z_k + (-1)^{\pi(a)\pi(\partial/\partial z_i)} z_j \frac{\partial}{\partial z_i} z_k \tag{7.27}$$

where π is the Grassmann parity.

Definition 7.8 (Grassmann parity)

$$\pi(x) = \begin{cases} 0 & x \text{ is Grassmann-even} \\ 1 & x \text{ is Grassmann-odd} \end{cases} \tag{7.28}$$

Theorem 7.4 (Integration of Grassmann numbers) To preserve translational invariance, the integration must follow

$$\int df = 0 \quad \int (f) df = 1 \tag{7.29}$$

Chapter 8

Spinors

8.1 Quaternions

Previously, we have seen that the imaginary number i can be represented by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This idea can in fact be generalised into 3 symbols i, j and k which (implicitly along with the identity $\mathbb{I}$ or 1) are called the *quarternions*, which can be seen as an extension of the concept of complex numbers.

Definition 8.1 (Quaternion) The *quaternion* is number system a number system that is an extension of complex numbers. It is of the form

$$q = a + bi + cj + dk \tag{8.1}$$

where a, b, c and d are coefficients. Its members, i, j and k (known as *quaternions*) satisfy:

$$i^2 = -1 \quad j^2 = -1 \quad k^2 = -1 \quad ijk = -1 \tag{8.2}$$

Two algebraic operations are defined on quaternions. For the zero matrix 0 and the identity matrix $\mathbb{I}$:

- **Addition:**
  - **Commutativity:**
  $$q_1 + q_2 = q_2 + q_1 \tag{8.3}$$
  - **Associativity:**
  $$q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3 \tag{8.4}$$
  - **Inverse:**
  $$q + (-q) = 0 \tag{8.5}$$
  - **Identity:**
  $$q + 0 = q \tag{8.6}$$
- **Multiplication:** Given two quaternions
  $$q_1 = a_1 + b_1 i + c_1 j + d_1 k \quad q_2 = a_2 + b_2 i + c_2 j + d_2 k \tag{8.7}$$
  their product is
  $$\begin{aligned} q_1 q_2 =& (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + b_1 a_2 + c_1 d_2 - d_1 c_2)i + \\ & (a_1 c_2 - b_1 d_2 + c_1 a_2 + d_1 b_2)j + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2)k \end{aligned} \tag{8.8}$$
  which satisfies the following properties:
  - **Associativity:**
  $$q_1(q_2 q_3) = (q_1 q_2) q_3 \tag{8.9}$$

- **Inverse:**

$$qq^{-1} = \mathbb{I} \tag{8.10}$$

- **Identity:**

$$q\mathbb{I} = q \tag{8.11}$$

- **Left distribution:**

$$q_1(q_2 + q_3) = q_1q_2 + q_1q_3 \tag{8.12}$$

Remark 8.1 Importantly, as expected for de facto matrices, the quaternion product does *not* observe commutativity.

Derivation 8.1 (Quaternion properties) We can multiply both sides of $ijk = -1$ by k:

$$ijkk = -k \to ij(-1) = -k \to ij = k \tag{8.13}$$

All other relationships then follow:

Theorem 8.1 (Quarternion properties) i, j and k are related by

$$ij = k \quad jk = i \quad ki = j \tag{8.14}$$

or equivalently

$$ji = -k \quad kj = -i \quad ik = -j \tag{8.15}$$

Remark 8.2 Any pair out of i, j and k are hence anticommutative.

Definition 8.2 (Quaternion conjugate) The *quaternion conjugate* q^* is likewise an extension of the complex conjugate. For a quaternion q, it is

$$q^* = a - bi - cj - dk \tag{8.16}$$

where a, b, c and d are coefficients. Its members, the quaternions i, j and k satisfy:

$$i^2 = -1 \quad j^2 = -1 \quad k^2 = -1 \quad ijk = -1 \tag{8.17}$$

Definition 8.3 (Quaternion norm) Quite similar to complex numbers, the *norm* of a quaternion is defined as

$$||q|| = \sqrt{a^2 + b^2 + c^2 + d^2} \tag{8.18}$$

Definition 8.4 (Quaternion inverse) The *quaternion inverse* is

$$q^{-1} = \frac{q^*}{||q||^2} \tag{8.19}$$

Definition 8.5 (Unit quaternion) A so-called *unit quaternion* is a quaternion whose conjugate is simply its reverse. It satisfies

$$||q|| = 1 \tag{8.20}$$

or equivalently

$$a^2 + b^2 + c^2 + d^2 = 1 \tag{8.21}$$

We recall that our good ol' complex numbers define the so-called 1-sphere $\mathbb{S}^1$. Similarily, the unit quaternions are significant in that, from Equation 8.21, they form the *3-sphere* $\mathbb{S}^3$, the analogue of the sphere, in the space $\mathbb{R}^4$ of (a, b, c, d).

Theorem 8.2 (Quarternion-Pauli matrices equivalence) Each quaternion corresponds to a matrix that can be represented by Pauli matrices:

$$i \leftrightarrow -\sigma_y\sigma_z \quad j \leftrightarrow -\sigma_z\sigma_x \quad k \leftrightarrow -\sigma_x\sigma_y \tag{8.22}$$

Remark 8.3 Each quaternion and each of the RHS Pauli matrix products are isomorphic. In fact, from the RHS equivalents, we can think of i, j and k as the x, y and z axes.

8.2 Quaternions and rotations

Let us define a *pure imaginary quaternion* which takes the form

$$q = bi + cj + dk \tag{8.23}$$

and a *real quaternion* of the form

$$q = a\mathbb{I} \tag{8.24}$$

From this notion, we can split a quaternion into two parts: the *scalar part* or *real part* s and the *vector part* or *imaginary part* v:

$$q = a + bi + cj + dk = (s, v) \quad \text{where} \quad s = a\mathbb{I} = a \quad v = bi + cj + dk = (b, c, d) \tag{8.25}$$

Theorem 8.3 (Reflections) The reflection in some direction i is governed by

$$V' = -UVU^{-1} \tag{8.26}$$

U is the versor corresponding to the dimension. In 3D this is σ_i. In 4D this is γ_i where we have the extended Pauli matrices or the γ *matrices* $\gamma_t = \gamma_0 = \mathbb{I}$ and $\gamma_i = \sigma_i$ for $i = 1, 2, 3$.

Remark 8.4 A 'negative reflection' yields the same result as the negative signs for $-U$ and $-U^{-1}$ cancel out.

Theorem 8.4 (Rotations in 3D space and 4D spacetime) A rotation in a 3D space $\mathrm{Cl}(3, 0)$ in the ij plane for some indices i and j by an angle of θ can be represented by

$$V' = e^{-\sigma_i\sigma_j\frac{\theta}{2}}Ve^{\sigma_i\sigma_j\frac{\theta}{2}} \tag{8.27}$$

In a 4D spacetime $\mathrm{Cl}(1, 3)$, this is

$$V' = e^{-\gamma_i\gamma_j\frac{\theta}{2}}Ve^{\gamma_i\gamma_j\frac{\theta}{2}} \tag{8.28}$$

A Lorentz boost in some non-temporal direction k is

$$V' = e^{-\gamma_t\gamma_k\frac{\theta}{2}}Ve^{\gamma_t\gamma_k\frac{\theta}{2}} \tag{8.29}$$

where t is the time direction.

To generalise, a rotation in some space $\mathrm{Cl}(p, q)$ can be written as

$$V' = e^{-B\frac{\theta}{2}}Ve^{B\frac{\theta}{2}} \tag{8.30}$$

where B is the corresponding versor. We then say that $e^{-B\frac{\theta}{2}}$ and $e^{B\frac{\theta}{2}}$ are members of the $\mathbf{Spin}(p, q)$ group.

Remark 8.5 In the meantime, we note that a rotation U_1U_2 is simply two reflections.

8.3 $\mathrm{Spin}(n)$ groups

Again we generalise this to transformations of

$$V' = (U_1 \cdots U_k)V(U_1 \cdots U_k)^{-1} \tag{8.31}$$

Definition 8.6 (Pin group in space) Members of $\mathrm{Cl}(n)$ that are *normalised* versors make up the $\mathrm{Pin}(n)$ group. They represent all rotations/reflected rotations in n-dimensional space.

Definition 8.7 (Spin group in space) Members of $\mathrm{Cl}(n)$ that are *normalised* versors *of even length* (i.e. k is even) make up the $\mathrm{Spin}(n)$ group. They represent all rotations[a] in n-dimensional space.

[a]But not reflected rotations as any odd-length elements are eliminated.

Remark 8.6 Here we observe two double covers:

- The $\mathrm{Pin}(n)$ group is a double cover of the $\mathrm{O}(n)$ group.
- The $\mathrm{Spin}(n)$ group is a double cover of the $\mathrm{SO}(n)$ group.

We now incorporate time dimensions

Definition 8.8 (Spin group in space) Members of $\mathrm{Cl}(p,q)$ that are versors *of even length* (i.e. k is even) and observe $U_i^2 = \pm 1$[a] make up the $\mathrm{Spin}(p,q)$ group. They represent all rotations[b] in a spacetime with p temporal dimensions and q spatial dimensions.

[a]U_i is spacelike if $U_i^2 = -1$ and timelike if $U_i^2 = 1$
[b]But not reflected rotations as any odd-length elements are eliminated.

Previously we have seen how 2×2 matrices that square to $\mathbb{I}$ can represent i. This can be expanded to matrices of any size. Such matrices are known as *representations* of i. From this concept of representation we consider analogues for $\mathrm{SO}(3)$ and $\mathfrak{so}(3)$.
As it turns out the previous $\mathrm{SO}(3)$ rotation matrices and $\mathfrak{so}(3)$ generators we have derived are those of spin-1. This is the *spin-1 representation* of $\mathrm{SO}(3)$ and $\mathfrak{so}(3)$. If we recall an undergrad course:

Spin	Corresponding object	Example particle
0	scalar	Higgs boson
1/2	spinor	quarks & leptons
1	vector	gluon, photon & W and Z bosons
2	matrix	graviton
n	rank-n tensor	N/A

In spin-0, each generator is simply a number, that being 0. Each rotation matrix is likewise a number, this time 1. This is known as the *spin-0 representation* or the *trivial representation* due to how utterly simple it is.
Remark 8.7 Using these rotation 'matrices', we recover the fact that scalars undergo no change under rotation.
However, for spin-$\frac{1}{2}$ particles, we have a problem as no 2×2 matrices satisfy the conditions for generators. That is, no *spin-$\frac{1}{2}$ representation* of $\mathrm{SO}(3)$ and $\mathfrak{so}(3)$ exist. There is nonetheless a workaround: An equivalent representation exists in the $\mathrm{SU}(2)$ Lie group, which double-covers $\mathrm{SO}(3)$.In the same vein, while we cannot acquire spin-$\frac{1}{2}$ representation of $\mathrm{SO}^+(1,3)$, we can find its equivalent in $\mathrm{SL}(2,\mathbb{C})$ which double covers to $\mathrm{SO}^+(1,3)$.

8.4 Emergence of spinors

But how do we use Lie theory in practice? We can, for example, consider the innocent 2D space. Both $\mathrm{SO}(2)$ and $\mathrm{SU}(2)$ reduce to the form of

$$U = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{8.32}$$

There is, however, one crucial difference: for $\mathrm{SO}(2)$, both a and b are real, while for $\mathrm{SU}(2)$ they are complex. In fact, we can make a further simplification here. In the case of $\mathrm{SO}(2)$, the group of matrices reduce further, and contains only 2D rotation matrices which will presumably look quite familiar:

$$U_{\mathrm{SO}(2)} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{8.33}$$

Remark 8.8 This is rotation around a point. The $\mathrm{SO}(3)$ group is not much worse, consisting only of rotations on a line. For this reason, $\mathrm{SO}(n)$ groups are also called *rotation groups*.

We remain in 2D space for a bit more and investigate how the elements of SO(2) and SU(2) operate. Quite intuitively, SO(2) operates on real vectors:

$$\vec{v}' = U_{\mathrm{SO}(2)}\vec{v} \tag{8.34}$$

For SU(2), the transformation matrix is a bit more tricky. Consider again a rotation by θ:

$$U_{\mathrm{SU}(2)} = \exp\left(-\frac{i\theta}{2}\hat{n}_i \cdot \sigma_i\right) \tag{8.35}$$

where $\hat{n}_i = (x, y, z)$ are again the unit vectors along the rotation matrices. To observe this rotation, we employ the same strategem, but with the *complex* vectors:

$$\psi' = U_{\mathrm{SU}(2)}\psi \tag{8.36}$$

This ψ is called a *spinor*. Here, we might be tempted to commit the same old mistake we did in the very beginning and presume that spinors are just tensors with complex components.
However, one can immediately see that this is not the case even easier than how we realised that tensors are more than glorified matrices. This is because tensors, on their own, can be complex. Instead, spinors are defined with respect to orthogonal rotation groups. Spinors are not something 'physical' because they are not invariant under coordinate transforms.

Definition 8.9 (Spinor) For the transformation matrix $U \in \mathrm{SO}(n)$ in n dimensions, a spinor[a] S transforms as

$$\psi' = U\psi \tag{8.37}$$

[a]Pronounced like 'spinner'.

Spinors in 3D space transform invariantly under SO(3), and spinors in 4D spacetime transform invariantly under SO(1, 3). We will see this in detail in the Pauli and Weyl spinors.
Remark 8.9 Here we note that SO(1, 3) is distinct from SO(4). The former is defined with respect to the 4D Lorentian spacetime metric (i.e. Minkowski metric), while the latter is defined with respect to a metric with 4 spatial dimensions and signature $(+, +, +, +)$.

8.5 Pauli spinor

Derivation 8.1 (Factoring of matrices) We first revise a bit of mathematics: We can *factor* a range of matrices into a vector-dual vector pair. For example:

$$\begin{pmatrix} 1 & 100 \\ 4 & 400 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 100 \end{pmatrix} \tag{8.38}$$

where $\otimes$ is the *tensor product*. As a reminder, we can see how the components of the pair are determined by writing the tensor product more intuitively:

$$\begin{array}{cc} & \begin{pmatrix} 1 & 100 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 4 \end{pmatrix} & \begin{pmatrix} 1 & 100 \\ 4 & 400 \end{pmatrix} \end{array} \tag{8.39}$$

Note that not all matrices can be factored directly: One example is $\begin{pmatrix} 1 & 100 \\ 4 & 500 \end{pmatrix}$. For direct factoring, the matrix must satisfy:

- Columns must be multiples of each other.
- Rows must be multiples of each other.
- The determinant of the matrix must be 0.

Still we can factor other matrices. We can, for example, 'break down' the matrix into components:

$$\begin{pmatrix} 1 & 100 \\ 4 & 500 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 100 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 500 \end{pmatrix} \tag{8.40}$$

Now we can calculate vector-dual vector pair for each matrix. The end result is the sum of a series of pairs:

$$\begin{pmatrix} 1 & 100 \\ 4 & 500 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} + 100 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} + 500 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} \tag{8.41}$$

Remark 8.10 We can multiply one vector/dual vector by some number m and divide the other by m. The resulting pair will still be a solution. However, for all solutions $\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c & d \end{pmatrix}$, the ration $\frac{a}{b}$ must be the same.

Normally, from a set of x, y and z coordinates, one represents a vector by $\vec{v} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$. While this is indeed the most intuitive way to write down a vector, one should note that $\vec{v} = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$ is merely a formalism. We use it because it is the simplest formalism that successfully encodes information about x, y and z. For example, one can always encode 3D coordinates using a system of three equations with three unknowns (and only one solution for each), but we are not motivated to do this as it is very ineffective.

However, for reasons which will be apparent later on, we often want to encode a 3-vector with Pauli matrices instead. We first recall the Pauli matrices:

Definition 8.10 (Pauli matrices)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8.42}$$

As it turns out, a combination of all three Pauli matrices, or rather a *Pauli vector*, can encode any 3-vector. Consider a set of coordinates x, y and z, we can represent them with a so-called *Pauli vector*:

Definition 8.11 (Pauli vector)

$$V = x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{8.43}$$

Remark 8.11 Even though it is mathematically a 2×2 matrix, it still physically represents the same 3 coordinates as its 3-vector counterpart. In fact, the determinant of the Pauli vector is the negative of the magnitude of the vector squared:

$$\det V = -||\vec{v}||^2 \tag{8.44}$$

Theorem 8.5 (Conjugation of Pauli vectors) We can flip a Pauli vector by the i direction by using *conjugation*:

$$V' = -\sigma_i V \sigma_i \tag{8.45}$$

Noting that the Pauli matrices are unitary, we are somewhat inspired to find a connection between them and the aforementioned SU groups. If we consider a rotation by π on the xy plane, we simply do that by using

$$V' = -\sigma_y(-\sigma_x V \sigma_x)\sigma_y = \sigma_y\sigma_x V(\sigma_y\sigma_x)^\dagger \tag{8.46}$$

We see that this is not quite the same as how spinors rotate - this is because we are so far one step away from spinors. But we can formulate how Pauli vectors rotate.

Theorem 8.6 (Rotation of Pauli vectors) Assuming a rotation matrix A:

$$V' = AVA^\dagger \tag{8.47}$$

where A and $A^\dagger$ are each a so-called *half-rotation*.

A is 2D, has determinant 1 and is unitary (i.e. $V' = V$). Thus, it must belong to the SU(2) group. Every two elements in SU(2) correspond to (i.e. map to) one element in SO(3). Here, we again see double cover SU(2) $\rightarrow$ SO(3).

Theorem 8.7 (Double cover invariance) In n dimensions, a spinor is invariant under any transformation group that is a double cover of the SO(n) group.

Definition 8.12 (Pauli spinor) A Pauli vector can be decomposed into two *Pauli spinors* or a spinor-dual spinor pair. Due to its unique structure, the spinors simplify:

$$\begin{pmatrix} z & x-iy \\ x+iy & z \end{pmatrix} \to \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \begin{pmatrix} \zeta_1 & \zeta_2 \end{pmatrix} \tag{8.48}$$

where

$$\xi^1 = \zeta_2 = \sqrt{x-yi} \quad \xi^2 = -\zeta_1 = -i\sqrt{x+yi} \tag{8.49}$$

Remark 8.12 Again, ξ^1 and ξ^2 are not related to Killing vectors.
We can plug the pair form back into the rotation of Pauli vectors:

$$V' = A \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \begin{pmatrix} \zeta_1 & \zeta_2 \end{pmatrix} A^\dagger \tag{8.50}$$

Here, it can be seen that A operates on the spinor $\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}$, while $A^\dagger$ operates on the dual spinor $\begin{pmatrix} \zeta_1 & \zeta_2 \end{pmatrix}$.
The nominal significance of half-rotations is thus seen. Now consider the equivalence of a 3D vector v_i and a Pauli vector V^a_b. We can expand this to higher ranks, and thus represent all tensors with spinors. We can do so by the Pauli matrices, which, as a matter of fact, each have three indices: σ^a_{ib}, where $_i$ is the *tensor index* ^a_b is the *spinor index*.

$$g_{ij}\sigma^a_{ib}\sigma^c_{jd} = g^{ac}_{bd} \tag{8.51}$$

Remark 8.13 Every two spinor indices correspond to one tensor index. Thus, a spinor is often informally called a rank $\frac{1}{2}$ tensor. This is, however, a very misleading statement as a spinor of dimension n does *not* have $\sqrt{n}$ components. As we have just seen, a 3D spinor[1] has 2 components.
We can further sit on this by considering the following: If we take both vector and spinor indices into account, σ^c_{jd} is effectively a *map* which takes us from the 3D vector space to the 4D spinor space, which is the tensor product of the 2D spaces of the covariant and contravariant spinors.

8.6 Weyl spinor

As the whole idea of Pauli spinors is based on the Pauli matrices in x, y and z directions, we can comfortably conclude that Pauli spinors are associated with 3D space. In GR, where we consider 4D spacetime, *Weyl spinors*, which also have 2 components, are used instead.
We can consider an analogue of the Pauli vector in 4D spacetime. This is the *Weyl vector*.

Definition 8.13 (Weyl vector)

$$W = t\sigma_t + x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \tag{8.52}$$

where σ_t is simply I, the identity matrix.

Remark 8.14 The determinant of the Weyl vector is the proper distance squared. i.e. the spacetime interval:

$$\det V = s^2 \tag{8.53}$$

Derivation 8.2 (Lorentz transformation) One notable example is the Lorentz transformation or the *Lorentz boost*. In $\mathrm{SO}(1,3)$, one has 6 transformations in total:

- Rotation in the xy, yz and zx planes
- Boosts in the x, y and z directions

For a general Lorentz transformation $\Lambda \in \mathrm{SO}(1,3)$, we have the corresponding *special linear group* $\mathrm{SL}(2,\mathbb{C})$[a] transformation L which acts on Weyl spinors. Every two elements in SL(2) correspond to (i.e. map to) one element in $\mathrm{SO}(1,3)$. Again, this is a double cover $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)$.
A Weyl spinor ψ transforms under a Lorentz transformation as $\psi' = L\psi$, where the precise form of L

[1] Even though a Pauli spinor is itself 2D, its corresponding vector Lies in 3D space.

is yet to be determined. Again as $L \in \mathrm{SL}(2,\mathbb{C})$, we have $\psi' = \psi$.
We define the concept of *rapidity*:

Definition 8.14 (Rapidity)

$$w = \tanh^{-1}(v/c) \tag{8.54}$$

For a boost in some direction i by rapidity w:

$$S_i = \exp\left(\frac{w}{2}\sigma_i\right) \tag{8.55}$$

[a]$\mathbb{C}$ reminds us that we are dealing with complex numbers.

Definition 8.15 (Weyl spinor) A Weyl vector can be decomposed into two *Weyl spinors* or a spinor-dual spinor pair:

$$\begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \to \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \begin{pmatrix} \psi^{1*} & \psi^{2*} \end{pmatrix} \tag{8.56}$$

where

$$|\psi^1| = \sqrt{ct+z} \quad |\psi^2| = \sqrt{ct-z} \tag{8.57}$$

Now we consider inner products. We want them to be invariant under rotations. This is simple for Pauli spinors as their rotations are unitary. Thus the inner product of two Pauli spinors ξ and χ are simply $\xi^\dagger\chi$. For two Weyl spinors, however, the linear transformations are not always unitary. So we introduce a 'correction' matrix to ensure that the inner product is preserved under transformations.

Theorem 8.8 (Weyl spinor inner products) The inner product between two Weyl spinors ψ and ϕ is $\psi^T\epsilon\phi$ where we have the *spinor metric*

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{8.58}$$

Note 8.1 (Dual spinor form of a Weyl spinor) The dual spinor form of a spinor is hence given by $\psi^T\epsilon$.

Remark 8.15 Note that this leads to $L^T\epsilon = \epsilon L^{-1}$.
Due to this lack of direct correspondence there is an extra element called *chariality* which essentially denotes the *handedness* of a Weyl spinor. Each physical transformation of Weyl spinors corresponds to two matrices in the $\mathrm{SL}(2,\mathbb{C})$ group, if one adjusts the axes of the first matrix, they find the second matrix, making both physically equivalent. One is then said to be *left-handed* and the other is said to be *right-handed*. Mathematically, the two matrices are complex conjugates of each other and give rise to left-handed and right-handed Weyl spinors.
Remark 8.16 By convention, we take the previously seen Weyl spinors as left-handed.

Definition 8.16 (Right-handed Weyl spinors) We can derive the right-handed Weyl spinors from their left-handed counterparts:

- The left dual spinor $\psi^{\dot a}$ is the complex conjugate of the right spinor ψ_a.
- The right dual spinor $\psi_{\dot a}$ is the complex conjugate of the left spinor ψ^a.

We can thus summarise the Weyl and dual Weyl spinors:

Weyl and dual Weyl spinors

Type	Lorentz transformation	Notation	In terms of left spinor components
left	$\psi \to L\psi$	$\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$	self
left dual	$\psi^T\epsilon \to \psi^T\epsilon L^{-1}$	$\begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix}$	$\begin{pmatrix} -\psi^2 & \psi^1 \end{pmatrix}$
right	$\psi^\dagger\epsilon \to \psi^\dagger\epsilon(L^{-1})^*$	$\begin{pmatrix} \psi_{\dot 1} & \psi_{\dot 2} \end{pmatrix}$	$\begin{pmatrix} -\psi^{2*} & \psi^{1*} \end{pmatrix}$
right dual	$\psi^* \to L^*\psi^*$	$\begin{pmatrix} \psi^{\dot 1} \\ \psi^{\dot 2} \end{pmatrix}$	$\begin{pmatrix} \psi^{1*} \\ \psi^{2*} \end{pmatrix}$

Figure 8.1: Hermann and Helene Weyl (Konrad Jacobs, March 1913)

8.7 Dirac spinor

When dealing with spin-$\frac{1}{2}$ particles in SR, we need to keep track of both charalities. As such we employ the *Dirac spinor*.

Definition 8.17 (Dirac spinor) The Dirac spinor is simply a left-chiral Weyl spinor stacked on top of a right-chiral Weyl spinor which is switched from row representation to column representation. As such, it has 4 components.

Definition 8.18 (Double cover of $\mathrm{SO}(3)$ by $\mathrm{SU}(2)$) e recall that a Pauli vector V transforms as

$$V' = AVA^\dagger \tag{8.59}$$

where $R \in \mathrm{SO}(3)$. For the equivalent 3D vector $\vec{v}$, the equivalent is

$$\vec{v}' = R\vec{v} \tag{8.60}$$

where $R \in \mathrm{SO}(3)$. As we have to use both A and $A^\dagger$ to accomplish what R did,

In real life, we are primarily concerned with Pauli, Weyl and Dirac spinors due to the dimensionality of our own spacetime. However, we ultimately want to find a mechanism to generate spinors in any number of dimensions. This mechanism is the *Clifford algebra*, which we will strive to arrive at in the next few sections.

8.8 Motivating examples in physics

Remark 8.17 A famous motivating example is usually the comparison of physical space and state space. In physical space, the spin-up and spin-down states $|\uparrow\rangle$ and $|\downarrow\rangle$ are π apart. In state space, as they must be orthogonal, they are $\pi/2$ apart. In state space, a state returns itself after one spin (2π). A similar relation happens in polarisation with respect to physical space and polarisation space. After half a rotation in physical space or a whole rotation in polarised space, the wave is phase-shifted by π and has to do another rotation in polarised space to return to its original phase.
Fun fact 8.1 In fact, spinors came from the idea of the Dirac equation and the theory of complex potentials.
Remark 8.18 In particle physics:

- Spin-0 particles (e.g. Higgs bosons) are represented by scalars.
- Spin-$\frac{1}{2}$ particles (e.g. quarks, electrons, neutrinos) are represented by spinors.
- Spin-1 particles (e.g. photons, gluons, W and Z bosons) are represented by vectors.

- Spin-2 particles (e.g. gravitons) are represented by matrices (in the case of the graviton, the stress-energy tensor).