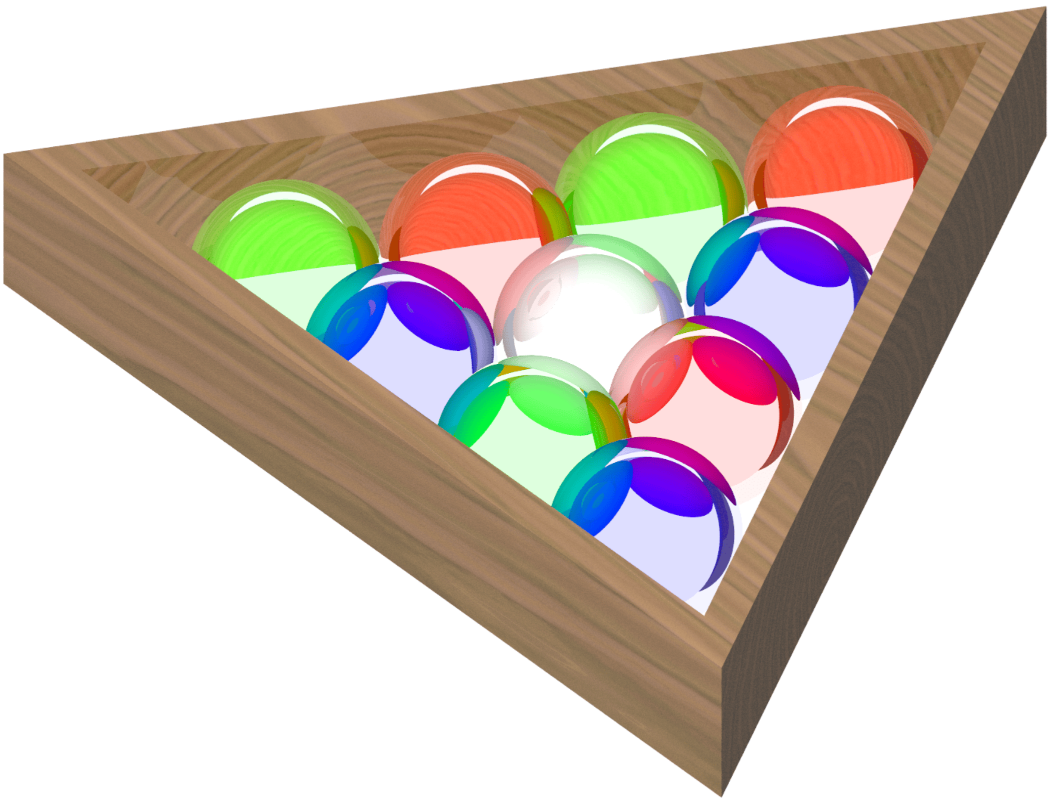
Particles & Wizards

Introduction to Quantum Field Theory



A Very Short Proto-book by

N. Booker

To my parents

Contents

Chapter 1

Instead of a foreword

1.1 How to use this book

> **Quote 1.1** But QFT is not conceptually difficult. It's just classical field theory with commutators, at least if one stays clear of more esoteric things like supersymmetry or string theory. If you already have experience in CFT, QFT should pose no difficulties to you at all.
>
> *Paulina Schlachter, 29 September 2024*

> **Quote 1.2** When conquering QFT you'll find yourself in a similar position as Napoleon with Russia.
>
> *Felix Halbwedl, 20 October 2024*

Quantum field theory is the unification of quantum mechanics and special relativity. It is not a theory of quantum gravity because it still operates within the completely flat Minkowski space. While the development of high-energy physics preceded that of QFT historically, QFT actually forms the theoretical basis of HEP.

Our ultimate objective in QFT is thus to calculate the final state from some initial state. This is accomplished by applying the scattering matrix S_{fi} to the initial state. The interaction-relevant part of S_{fi} is another matrix $\mathcal{M}_{fi}$ known as the transition amplitude, which is significant in its own right and can be derived from the Feynman rules. As it turns out, this can be done in two ways:

- The first approach is the canonical quantisation formulation, in which fields are quantised. Historically, it is also called 'second quantisation' as it builds on 'first quantisation' in quantum mechanics, where physical quantities are quantised. For this reason, it is the more intuitive approach, and is the staple of a standard 'Quantum Field Theory I' course in most universities. We will cover canonical quantisation in Part I.

- The second approach is the path integral formulation, in which, from the action S of a system, we derive a generating functional $Z[J]$ that takes the form of a path integral. Propagators are then derived by taking functional derivatives of $Z[J]$. This approach is somewhat easier but less intuitive. As such, it is usually at the intersection between 'Quantum Field Theory I' and 'Quantum Field Theory II' course in most universities. We will cover path integrals in Part II.

- As it turns out, both formulation, which yield the same Feynman rules, fail in structures beyond the tree level of Feynman diagrams due to the emergence of infinities. The elimination of these unphysical infinities requires the theory to be renormalised, which can only be done by first performing the nice mathematical trick of regularisation. We will cover regularisation and renormalisation in Part III.

We assume knowledge of classical field theory, special relativity and quantum mechanics in bra-ket notation, although a brief review is given in Chapter 2. Many parts of the book require knowledge of Lie theory. This will not be covered in this book. Rather, the reader is encouraged to consult the companion

book *Spinors & Symmetries* or other standard Lie theory texts.
For any comments, suggestions or typos, please e-mail the following address:

`neil(dot)booker(dot)21(at)ucl(dot)ac(dot)uk`

> **Quote 1.3** 🐈, aber sicher doch. Ich mache auch Fehler.
>
> ---
> *Felix Halbwedl, 8 February 2025*

1.2 Acknowledgements

> **Quote 1.4** Credit for those who work hard, I just happen to know some things.
>
> ---
> *Felix Halbwedl, in his infinite humility, 22 December 2024*

Work on *Particles & Wizards: Introduction to Quantum Field theory* started shortly before the 2024-25 quantum field theory course at University College London lectured by Prof. Alessio Serafini[1]. I would like to thank him for answering the many questions on QFT I had throughout my master's year and for his consistent support.
I want to extend my gratitude to Felix Halbwedl, who stimulated many physical discussions on various topics in QFT and offered much advice on the contents and the formatting of this book. I also thank him for the many quotes he has contributed to the book as well as his moral support. I am also thankful to Abhijeet Vats, under whose guidance I was able to develop my LaTeX skills to a satisfactory level. Without them, this book would undoubtedly not have been in its current form.

1.3 References

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- *Notes on Quantum Field Theory I* by Marco Serone (SISSA)
- *Quantenfeldtheorie* by Matthias Gaberdiel (ETH Zürich)
- *Quantum Field Theory I* by Niklas Beisert (ETH Zürich)
- *Quantum Field Theory II* by Niklas Beisert (ETH Zürich)
- *Quantum Field Theory II* by Matthias Gaberdiel (ETH Zürich) and Aude Gehrmann-De Ridder (ETH Zürich)
- *Quantum Fields* by Nikolay Bogoliubov (JINR) and Dmitry Shirkov (JINR)
- *Quantum Field Theory* by Gernot Eichmann (Technische Universität Graz)
- *Quantum Field Theory I* by Axel Maas (Technische Universität Graz)
- *Quantum Field Theory* by Alessio Serafini (University College London)

[1]Known lovingly as the 'Wizard' due to his character appearing as a wizard in the UCL PandA Day plays.

Chapter 2

Preliminaries

2.1 Quantum mechanics of mixed states

Quote 2.1 Fortunately, quantum mechanics is easy and can be summarised in a few lines.

Alessio Serafini

A quantum state can always be represented by a Hermitian, positive semi-definite operator with trace 1 ϱ[1] (i.e. all eigenvalues of ϱ are positive semi-definite and add up to 1).

Definition 2.1 (Positive definiteness and positive semi-definiteness) A positive definite operator ϱ always yields a positive expectation value

$$\langle\psi|\varrho|\psi\rangle > 0 \tag{2.1}$$

A positive semi-definite operator ϱ always yields a non-negative expectation value

$$\langle\psi|\varrho|\psi\rangle \leq 0 \tag{2.2}$$

We now introduce the so-called *Sylvester's criterion*.

Definition 2.2 (Minor) A *minor* of some matrix is the determinant of the resultant matrix after deleting an arbitrary number of rows and columns from the initial matrix. For a square matrix, a minor is called a *principal minor* when the indices of the deleted rows and those of the deleted columns are *identical*[a].

[a]This is significant in that if one deletes, say, the 3$^{\text{rd}}$ row and the 4$^{\text{th}}$ column, the resulting matrix is a minor but not a principal minor.

Theorem 2.1 (Sylvester's criterion) One can use minors of a matrix to test positive definiteness and positive semi-definiteness. For a Hermitian $n \times n$ matrix:

- Positive definiteness holds if all the *leading* principal minors are positive. i.e. if the determinants of the top-left $1 \times 1, \cdots, n \times n$ sub-matrices are positive.
- Positive semi-definiteness holds if *all* principal minors are non-negative.

While a pure quantum state is simply a bra or a ket, the bra-ket notation is insufficient for a *mixed state*, which is described by a *density matrix* or a *density operator*. Unlike a state vector, which is an element *of* the Hilbert space, the density matrix is an operator *on* the Hilbert space.
Physically, a mixed state is a statistical mixture (i.e. ensemble) of i different pure states $|\Psi_i\rangle$ with probabilities p_i, such that:

$$\varrho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i| \tag{2.3}$$

[1]Note that this might not be the *density operator* ρ.

In this way, the density matrix extends the concept of quantum state to mixed states, systems where we do not have complete knowledge.

Derivation 2.1 (Von Neumann equation) We recall the *theorem of Liouville*[a] in classical field theory, which states that the phase space distribution function $\rho(p,q)$ is constant along the trajectories of the system:

Theorem 2.2 (Theorem of Liouville)

$$\partial_t \rho = \{H, \rho\} \tag{2.4}$$

where H is the Hamiltonian and $\{\}$ is the *Poission bracket*, which for functions f and g and phase space coordinates (q_i, p_i) satisfy

$$\{f, g\} = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right) \tag{2.5}$$

In QFT, the density operator ρ is functionally equivalent to the phase space distribution function. Hence we have an equivalent for the theorem of Liouville

Theorem 2.3 (Von Neumann equation)

$$\dot{\rho} = -i[H, \rho] \tag{2.6}$$

where we have assumed that $\hbar = 1$.

Due to the aforementioned equivalence, the equation is also called the *quantum Liouville equation* or the *Liouville-von Neumann equation.* It is also the mixed state equivalent of the Schrödinger equation, which deals with pure states.

[a] Or more commonly in the English-speaking world, *Liouville's theorem.*

Derivation 2.2 (Reduction to pure state) To prove the last statement, we consider a pure state, where we only have one possible i. The density matrix is

$$\rho = |\Psi\rangle\langle\Psi| \tag{2.7}$$

Substituting this into the LHS of the von Neumann equation gives

$$\partial_t \rho = (\partial_t |\Psi\rangle)\langle\Psi| + |\Psi\rangle(\partial_t \langle\Psi|) \tag{2.8}$$

Substituting into the RHS yields

$$-i[H, \rho] = -iH|\Psi\rangle\langle\Psi| + i|\Psi\rangle\langle\Psi|H \tag{2.9}$$

We now equate the two sides:

$$(\partial_t |\Psi\rangle)\langle\Psi| + |\Psi\rangle(\partial_t \langle\Psi|) = -iH|\Psi\rangle\langle\Psi| + i|\Psi\rangle\langle\Psi|H \tag{2.10}$$

where we have recovered the Schrödinger equation and its adjoint form

$$\partial_t |\Psi\rangle = -iH|\Psi\rangle \quad \partial_t \langle\Psi| = i\langle\Psi|H \tag{2.11}$$

Now we remind ourselves of how measurements work. We have already seen the so-called *projection-valued measures* or *PVMs.* Previously, they have been known to us as *projectors.*

Definition 2.3 (Projection-valued measure) A projection-valued measure P_i is a linear and positive semi-definite operator that satisfies, for the density operator ρ:

- Normalisation condition:

$$\sum_i P_i = \mathbb{I} \quad \text{or} \quad \sum_i \mathrm{Tr}(\rho P_i) = 1 \tag{2.12}$$

  where $\mathbb{I}$ is the unit matrix of appropriate dimension.

- Orthogonality condition:

$$P_iP_j = \delta_{ij}P_i \quad \text{or} \quad P_i^2 = P_i \tag{2.13}$$

Functionally, it maps a quantity in a vector space V into a subspace $W \subset V$. Each P_i corresponds to an eigenvalue of an observable (e.g. position or spin).

Theorem 2.4 (Born rule) The probability of obtaining the outcome i is, for a mixed state:

$$p_j = \mathrm{Tr}(\rho P_j) = \mathrm{Tr}(|j\rangle\langle\phi|k\rangle\langle j|) = \langle j|\phi\rangle\langle\phi|j\rangle = |\langle j|\phi\rangle|^2 \tag{2.14}$$

For a pure state, this simplifies to

$$p_i = \langle\Psi|P_i|\Psi\rangle \tag{2.15}$$

Remark 2.1 Here we see the significance of Equation 2.12, which is that all probabilities expectedly sum up to 1.
So far, we have been working with PVMs, which are utterly perfect, innocent and idealised measurements. In real life, measurement devices are not ideal, giving rise to noise. These imperfect (or rather *generalised*) measurements are described by *positive operator-valued measures* or *POVMs*.

> **Quote 2.2** 'Positive Operator Valued Measure', an acronym fabricated by mathematical physicists to scare all others away.
>
> *Alessio Serafini*

Definition 2.4 (Positive operator-valued measure) A positive operator-valued measure $\prod_i$ is a linear and positive semi-definite operator that satisfies the normalisation condition only:

$$\sum_i \prod_i = \mathbb{I} \quad \text{or} \quad \sum_i \mathrm{Tr}(\rho \prod_i) = 1 \tag{2.16}$$

By imposing the condition

$$\mathrm{Tr}[\prod_i \prod_j] = \delta_{ij} \tag{2.17}$$

POVMs reduce to PVMs/projectors.
Remark 2.2 The Born rule is the same as for PVMs, save for the nominal replacement of P_is by $\prod_i$s.

2.2 Action principle

The two elements leading to the action principle are the action itself and the concept of symmetries. We begin by working slowly towards action.

Note 2.1 (Metric signature) Unlike GR, convention dictates that the Minkowski 4-metric in HEP has the signature $(+,-,-,-)$. That is, the line element has the form

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{2.18}$$

We now briefly discuss the *variational formalism*. Two equivalent formulations of the variational formalism exist - Lagrangian and Hamiltonian mechanics. In classical mechanics, the central quantities are the 4-position x (or often q) and momentum p.

Note 2.2 (Reference frames) We can choose certain frames that simplify calculations:

- For a spacelike separation $(x-y)^2 < 0$, one can always, without loss of generality, choose a frame to set $(x^0 - y^0) = 0$.
- For a timelike separation $(x-y)^2 > 0$, one can always, without loss of generality, choose a frame to set $(\vec{x} - \vec{y}) = 0$.

In field theories, the 4-position x is replaced with a 4-field $\phi(x) = (\phi_0, \phi_1, \phi_2, \phi_3)$. ϕ_1, ϕ_2 and ϕ_3 are simply the spatial components of the corresponding 3-field, while ϕ_0 is a *scalar* or *time-like* component of the 4-field[2].

Definition 2.5 (Action) For a set of fields ϕ_i with the 4-position x^i, the *action* is defined as

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi) d^4x = \int L dx^0 = \int L dt \tag{2.19}$$

where L is the Lagrangian and $\mathcal{L}$ is the *Lagrangian density*[a].

[a]Often also simply called the *Lagrangian*, although you will be able to tell the difference by looking at the notation.

Now we look at what a field Lagrangian actually looks like. The simplest Lagrangian is that of a free massless scalar field, which can be used to model particles like massless scalar bosons[3]. It is given by

Definition 2.6 (Free massless scalar field Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi \tag{2.20}$$

The sole term is the *kinetic energy density*, which arises from the variation of the field ϕ over the 4-coordinates.

Remark 2.3 A *free field* is so-called as it has no interactions, which manifests in extra terms in the Lagrangian.
One can introduce mass to the free field. Note that this does not mean that the field itself is massive (which makes little sense) but rather that the particle that generates the field is massive. With the addition of a mass term, the Lagrangian becomes

Definition 2.7 (Free massive scalar field Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2}m^2\phi^2 \tag{2.21}$$

where the second term is the *potential energy density*.

Remark 2.4 This Lagrangian is actually a reduced form of the free massive complex scalar field Lagrangian[4], which is

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi^* \partial_\mu \phi - \frac{1}{2}m^2|\phi|^2 \tag{2.22}$$

We are now in a position to discuss symmetries. Physicists use the word 'symmetry' as a more sophisticated way of saying that a physical quantity we are interested in stays invariant under the change of some other 'background' quantity. If this physical quantity of interest instead changes with the background quantity, we then sadly say that symmetry is 'broken'. Broadly speaking, we are interested in the following symmetries:

- **External and internal symmetries:**
  - *External symmetries* are transformations that involve changes to the spacetime coordinates themselves. One example is Poincaré symmetry.
  - *Internal symmetries* are transformations that act on internal degrees of freedom of fields (e.g. charge, spin, etc.), leaving spacetime coordinates unchanged.
- **Discrete and continuous symmetries:**
  - *Continuous symmetries* are governed by transformation parameters that can admit a continuous range of values. One example is Poincaré symmetry.
  - *Discrete symmetries* involve transformations that take on only specific values. Examples are the C, P and T symmetries.

[2]For example, in the electromagnetic 4-potential, ϕ_0 is the electric scalar potential.
[3]We see it more often in approximate models as fundamental massless scalar particles with zero mass are rare.
[4]Boy, is that a mouthful!

So far, we have heard about the layman's version of Noether's theorem 'All symmetries lead to conservation laws'. We have made a conceptual overview of symmetries, but what are their mathematical implications? As it turns out, symmetries are defined with respect to an action principle. Consider an *infinitesimal coordinate transformation*

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu \tag{2.23}$$

Up to the first order expansion, a generic field in x^μ undergoes the corresponding transformation:

$$\phi(x) \to \phi'(x') = \phi(x) - \epsilon^\mu \partial_\mu \phi(x) \tag{2.24}$$

The variation of the field can then be written as

$$\delta\phi(x) = \phi'(x') - \phi(x) = -\epsilon^\mu \partial_\mu \phi(x) \tag{2.25}$$

An important point of note concerns the Lagrangian (density). While it has rank 0, the Lagrangian $\mathcal{L}$ in some theories might not transform as a scalar. Rather, its variation takes the general form of the total derivative of a current-like vector field K^μ:

$$\delta\mathcal{L} = \partial_\mu K^\mu \tag{2.26}$$

where physically, K^μ is the measure of the failure of $\mathcal{L}$ to transform as a scalar.

Derivation 2.3 (Action principle) We can solve for the variation of the action by integrating the variation of the Lagrangian in Equation 2.41, which gives:

$$\delta S = \int d^4x \delta\mathcal{L} = \int d^4x \partial_\mu K^\mu \tag{2.27}$$

Using the divergence theorem, this integral can be converted into a surface integral over the boundary of the spacetime region ∂V:

$$\delta S = \int_{\partial V} d^3x K^\mu n_\mu \tag{2.28}$$

where n_μ is the normal vector to the boundary. The bulk term has vanished, leaving us the boundary term that is exactly $K^\mu n_\mu$ (effectively K^μ).
Importantly, we now assume that this boundary term K^μ vanishes[a]:

$$\int_S d^3x K^\mu n_\mu = 0 \tag{2.29}$$

It can then be concluded that a symmetry inevitavly implies *an invariant action*:

Theorem 2.5 (Action principle)

$$\delta S = 0 \tag{2.30}$$

This is the almighty *action principle*.

[a]This is typically justified in physical field theories where fields and their variations vanish at spatial or temporal infinity. We will not show this here.

Remark 2.5 The *action principle* is simply another name for the *principle of stationary action*, which is itself often erroneously known as the *principle of least action*[5]. This simply means that the time derivative of the action of an isolated system is zero. As the principle can be used for action generated by *any* field, it is often mentioned as '*an* action principle' instead of '*the* action principle'.

2.3 Classical field theory

The action principle is important as it allows us to derive the equations of motion. Let us show this with the general example in classical mechanics.

[5]This is because the principle states that instead of at a minimum, action tends to stay *stationary*, be it a maximum, a minimum or a saddle point.

Derivation 2.4 (Euler-Lagrange equations and the boundary term) We begin by varying the action S given in Equation 2.19 with respect to ϕ, which involves integration by parts:

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right] \tag{2.31}$$

Noting that $\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi)$, we can write

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta \phi) \right] \tag{2.32}$$

We can apply integration by parts to the term involving $\partial\mu(\delta\phi)$. The variation of the action is thus

$$\delta S = \int d^4x \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi}_{①} + \int d^4x \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial\phi \right)}_{②} \tag{2.33}$$

Through this process, we have exposed the so-called *boundary term* ②, which is a total derivative and does not contribute to the equations of motion. This is because it can be converted into a surface integral over the boundary of the integration region using Gauss's law[a] In contrast, ① is known as the *bulk term*[b].
There are two scenarios in which the boundary term can be ignored:

- The field and its derivative vanishes at the boundary.
- The boundary extends into infinity.

Assuming the first point and applying the action principle leads to

Theorem 2.6 (Euler-Lagrange equations)

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \tag{2.34}$$

By recognising that ∂_μ is just the derivative over 4-coordinates or the *4-derivative*, we can see that this is the Euler-Lagrange equations that we have previously seen.

[a] As such, the boundary term is also called the *surface term*.
[b] The designations 'bulk term' and 'boundary term' are more common in general relativity with respect to the Einstein-Hilbert action, but are nice to remember nonetheless.

Before proceeding, we make a few comments on the bulk and boundary terms:

- The bulk term is so-called because it integrates over the entire volume of spacetime. When an action principle is imposed, the integrand of the bulk term vanishes, as seen in Equation 2.34.
- Hence, the bulk term reflects how the action responds to changes in the field ϕ in the 'bulk' or the 'interior' (i.e. everything minus the boundary) of the spacetime we consider. Under an action principle, the bulk term defines the allowed configurations for ϕ via the Euler-Lagrange equations.
- The boundary term reflects the influence of boundary conditions for the action. By imposing an action principle, we have also assumed that $\delta\phi = 0$ on the boundary - a boundary condition.
- This is why boundary conditions (like Dirichlet or Neumann conditions) are usually imposed in variational problems to ensure well-defined dynamics in the bulk term.

The *Hamiltonian* is essentially a Legendre transformation of the Lagrangian:

Definition 2.8 (Hamiltonian and Hamiltonian density) The *Hamiltonian* H is

$$H = \int \mathcal{H}(\phi, \pi, \partial_\mu\phi) d^3x \tag{2.35}$$

which is the volume integral of the *Hamilton density* $\mathcal{H}$. Also simply called the *Hamiltonian*, it is given by

$$\mathcal{H}(\phi_i, \pi_i, t) = \sum_i \pi_i \dot{\phi}_i(\phi_j, \pi_j) - \mathcal{L}(\dot{x}_k(x_j, p_j), x_k, t) \tag{2.36}$$

where ϕ is the field and π is the and *canonical momentum*, the equivalent of momentum in field theory.

Remark 2.6 As it turns out, the quantity which we have been led to believe to be the Hamiltonian as undergrads is actually the Hamiltonian density $\mathcal{H}$.

Derivation 2.5 (Hamilton's equations) By taking the variation of Equation 2.36, one finds

$$\delta\mathcal{H} = \sum_i \delta\pi_i \dot{\phi}^i(\phi_j, \pi_j) - \delta\pi^i \frac{\partial\mathcal{L}}{\partial\pi^i} = \sum_i \delta\pi_i \dot{\phi}^i(\phi_j, \pi_j) - \sum_i \delta\phi^i(\phi_j, \pi_j)\dot{\pi}_i \tag{2.37}$$

Now compare this against the general variation:

$$\delta\mathcal{H} = \delta\phi^i(\phi_j, \pi_j)\frac{\partial\mathcal{H}}{\partial\phi^i(\phi_j, \pi_j)} + \delta\pi_i \frac{\partial\mathcal{H}}{\partial\pi_i} \tag{2.38}$$

By equating the two expressions for $\delta\mathcal{H}$, we recover the so-called *Hamilton's equations*:

Theorem 2.7 (Hamilton's equations)

$$\frac{\partial\mathcal{H}}{\partial\phi_i} = -\dot{\pi}_i \quad \frac{\partial\mathcal{H}}{\partial\pi_i} = \dot{\phi}_i \tag{2.39}$$

Remark 2.7 From this, we can rewrite Hamilton's equations in terms of Poisson brackets:

$$\dot{x}_i = \{x_i, H\} \quad \dot{p}_i = \{p_i, H\} \tag{2.40}$$

The quantum version of the first equation is simply the Schrödinger equation in the Heisenberg picture.

2.4 Noether's theorem

We can now finally derive Noether's theorem. The variation of the (field-dependent) Lagrangian can be written as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \tag{2.41}$$

where $\delta\phi$ is again the variation of the field ϕ under the symmetry transformation.
We can rewrite the first term, yielding

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \tag{2.42}$$

By inspection, this is actually the product rule expansion of Using the Euler-Lagrange equations, this becomes

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) \tag{2.43}$$

By substituting Equation 2.41, we find

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu K^\mu \to \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \partial_\mu K^\mu = 0 \tag{2.44}$$

Now let us define another current-like quantity called the *Noether current*:

Definition 2.9 (Noether current) The 4-vector equivalent of the probability density ϱ is the so-called Noether current, *probability 4-current* or *conserved current* J^μ. Its zeroth component is simply the good ol' probability density, and its 3 other components are the *probability (3-)current* $\mathbf{J}^\alpha$.

From the Noether current, one can derive a charge-like quantity representing the total probability called the *Noether charge*, the *probability charge* or the *conserved charge* as it is conserved with respect to time:

Definition 2.10 (Probability charge)

$$Q = \int d^3x J^0 \tag{2.45}$$

As probability is conserved, J^μ is Lorentz-invariant and satisfies the *continuity equation*:

Theorem 2.8 (Continuity equation)

$$\partial_\mu J^\mu = 0 \tag{2.46}$$

Rather cheatingly, we can now equate Equation 2.44 and Equation 2.46:

$$\partial_\mu J^\mu = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \partial_\mu K^\mu \tag{2.47}$$

By removing the partial derivatives, we recover the expression for the Noether current, known as Noether's theorem:

Theorem 2.9 (Noether's theorem)

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - K^\mu \tag{2.48}$$

Quote 2.3 Yes, but it is the same sound

David Steiner, comparing the 'oe' in 'Noether' with 'ö', 21 November 2024

Note 2.3 Noether's theorem implies the conservation of the charge associated with the probability current:

$$\frac{dQ}{dt} = \int d^3x \partial_0 J^0 = -\int d^3x \nabla \cdot \mathbf{J} = 0 \tag{2.49}$$

Finally, we can directly relate the Noether current and the action, from Equation 2.33 and Equation 2.34, one can see that

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\delta S}{\delta \phi} \tag{2.50}$$

which, by insertion into Equation 2.47, gives

$$\partial_\mu J^\mu = \delta\phi \frac{\delta S}{\delta \phi} = \frac{\delta \phi}{\delta \theta} \frac{\delta S}{\delta \phi} \tag{2.51}$$

where θ is the parameter for an infinitesimal transformation satisfying

$$x \to x' = x + \theta \delta x \tag{2.52}$$

Exercise 2.1 Let ϕ be a free scalar field obeying the Klein-Gordon equation (Equation 3.2), and let J^μ be the associated density and current 4-vector. Derive the continuity equation (Equation 2.46).

Remark 2.8 One essential type of symmetry in QFT is the so-called *gauge symmetries*, which is are both internal symmetries and continuous symmetries. Gauge symmetries are governed by Lie groups we have seen in *Spinors & Symmetry*.

2.5 Poincaré transformations

As an example, we now derive the Noether current under Poincaré transformations, which, as seen in *Spinors & Symmetries*, includes translations and Lorentz transformations.

Derivation 2.6 (Translation) Let us assume the same transformations as in Equation 2.23 and Equation . The field variation is then shown in Equation 2.25.
We know that $\mathcal{L}$ transforms as Equation 2.41. Substituting Equation 2.25, we obtain

$$\delta \mathcal{L} = -\epsilon^\nu \left(\frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \partial_\nu \phi \right) \tag{2.53}$$

Now substitute this into Equation 2.27:

$$\delta S = -\epsilon^\nu \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \partial_\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu \partial_\nu \phi \right) \tag{2.54}$$

By inserting the equivalence from the Euler-Lagrange equations, we can rewrite the integral as

$$\delta S = -\epsilon^\nu \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \right) \tag{2.55}$$

where the terms within the bracket is the canonical stress-energy tensor. A simple shift of indices gives its contravariant form:

Definition 2.11 (Stress-energy tensor)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \tag{2.56}$$

Importantly, by comparing Equation 2.27 and Equation 2.55, one can identify

$$\delta \mathcal{L} = \partial_\mu (\epsilon_\nu T^{\mu\nu}) = \partial_\mu K^\mu \tag{2.57}$$

In other words, our new friend $\epsilon_\nu T^{\mu\nu}$ resembles the previously seen boundary/surface term K^μ. Finally, if one takes a partial derivative of the stress-energy tensor and applies the Euler-Lagrange equations, they will find that

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.58}$$

This shows the well-known conservation of matter-energy content, and follows directly from Noether's theorem applied to spacetime translations.

Note 2.4 Equation 2.58 merely states that $T^{\mu\nu}$ is invariant when differentiated over *4-coordinates*. As such, it describes the conservation of energy-momentum *density* as opposed to energy and momentum themselves. In cosmology, where inflation exists, the total energy of the universe is *not* conserved. Rather, as inflation gives rise to the expansion of spacetime, the energy *increases* to preserve the invariance of energy-momentum density.

Derivation 2.7 (Lorentz transformation) A Lorentz transformation can be either a rotation or a Lorentz boost. Assuming the parameterisation

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu \quad \omega^{\mu\nu} = -\omega^{\nu\mu} \tag{2.59}$$

where $\omega^{\nu\mu}$ is some parameter, the coordinate and field transformations are

$$x^\mu \to x'^\mu = x^\mu + \Lambda^\mu_\nu x^\nu \quad \phi(x) \to \phi'(x) = \phi(x) + \frac{1}{2}\omega^{\rho\sigma}\Sigma_{\rho\sigma}\phi \tag{2.60}$$

where $\Sigma_{\rho\sigma}$ are the generators of the representations of the Lorentz group abstract elements corresponding to ϕ (e.g., for scalars $\Sigma_{\rho\sigma} = 0$, for vectors $\Sigma_{\rho\sigma}$ corresponds to antisymmetric tensors, and so on). Again, by using Noether's theorem, we can find that the contribution to the Noether current is the total angular momentum, which includes the orbital angular momentum and spin:

$$M^{\mu\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Sigma^{\rho\sigma} \phi \tag{2.61}$$

For Lorentz transformations, $K^\mu = \partial_\nu(x^\nu J^\mu - x^\mu J^\nu)$, and the Noether current becomes:

$$J^\mu = \omega_{\rho\sigma} M^{\mu\rho\sigma} \tag{2.62}$$

where the nature of $\omega_{\rho\sigma} M^{\mu\rho\sigma}$ as a boundary term is easily seen.

Hence, combining both types of symmetries, the most general form of the Noether current under Poincaré transformations can be written as:

$$J^\mu = \epsilon_\nu T^{\mu\nu} + \frac{1}{2}\omega_{\rho\sigma} M^{\mu\rho\sigma} \tag{2.63}$$

where $T^{\mu\nu}$ represents energy-momentum contributions, and $M^{\mu\rho\sigma}$ represents both orbital and intrinsic angular momentum contributions.
We will now discuss an important point that will pop up over and over again in the rest of the book. As should be well understood at this point, QFT is a quantum theory that incorporates SR, which is based on Lorentz transforms:

- A quantity is *Lorentz-covariant* if it transforms under the Lorentz group representation corresponding to its type (scalar, vector, axial vector, spinor, rank-2 tensor, etc.).
- A quantity is *Lorentz-invariant* if it is invariant under Lorentz transformations.
- A equation is Lorentz-covariant/invariant if all its quantities are Lorentz-covariant/invariant.

While Lorentz invariance is intuitive, we find it prudent to define Lorentz covariance rigorously:

Theorem 2.10 (Lorentz covariance) A field $\phi(x)$ transforms under a Lorentz transformation Λ as:

$$\phi'(x') = \rho(\Lambda)\phi(x)\rho(\Lambda)^{-1} \tag{2.64}$$

where the dash represents the new field/coordinate and $\rho(\Lambda)$ is a *representation* of the Lorentz group ($\mathrm{SO}(1,3)$) acting on the field.
As seen in *Spinors & Symmetries*, this representation differs:

- A scalar field $\phi(x)$ transforms trivially:

$$\phi'(x') = \phi(x) \tag{2.65}$$

- A vector field $A^\mu(x)$, transforms as:

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \tag{2.66}$$

- A spinor field $\psi(x)$, transforms as

$$\psi'(x') = T(\Lambda)\psi(x) \tag{2.67}$$

  where $T(\Lambda)$ is a spinorial representation we will derive in Derivation 5.3.

We end with some physical comments. Consider a quantum field theory *within the standard model*:

- Fields are Lorentz-covariant because they transform under representations of the Lorentz group.
- Equations of motion and physical laws are Lorentz-covariant as they involve only Lorentz-covariant quantities. This Lorentz covariance is effectively enforced due to SR.
- The Lagrangian is Lorentz-invariant, ensuring that the theory as a whole respects Lorentz symmetry.

A more complete overview of the representation theory of the Lorentz group, can be seen in *Spinors & Symmetries*.

Part I

Canonical quantisation

Chapter 3

Free fields

Quote 3.1 What could possible go wrong?

Alessio Serafini, 16 January 2024

In this chapter, we make an overview of the so-called Klein-Gordon equation, an intuitive attempt at developing a special relativistic quantum theory, as well as its failures. By quantising the Klein-Gordon field, we will get a first taste of canonical quantisation.

3.1 Klein-Gordon equation and its demise

In quantum mechanics, there is only one dimension, which is time. This becomes a problem in high-energy physics, where we take into account the movement of the particles. Also, we must note that particle interactions in high-energy physics often take place in the relativistic limit. As such, any candidates for a theory of HEP must satisfy the following:

- The theory must be in 4D (3 spatial dimensions and 1 temporal dimension).
- The theory must incorporate special relativity.

An earlier attempt at constructing such a theory is the infamous *Klein-Gordon equation*. At that point, QFT was still not developed, and it was naively assumed that the Schrödinger equation could be modified to be Lorentz-invariant by simply applying the wavefunction ϕ to both sides of the equivalence

$$E_p^2 = m^2 + |p|^2 \to -\partial_t^2 = m^2 - \nabla^2 \tag{3.1}$$

where, for convenience, we have set $c = \hbar = 1$. $E_p^2 = m^2 + |p|^2$ is known as the *on-shell condition*[1]. This gives

Theorem 3.1 (Klein-Gordon equation) The Klein-Gordon equation describes scalar (spin-0) particles in a relativistic framework:

$$(\Box + m^2)\phi = 0 \tag{3.2}$$

where $(\Box + m^2)$ is known as the *Klein-Gordon operator*.

Remark 3.1 As free fields are solutions to the Klein-Gordon equation, a Klein-Gordon operator acting on a free field always gives 0.

Exercise 3.1 Show that the Klein-Gordon equation can be recovered by using an action principle on the free massive scalar field Lagrangian in Equation 2.21.

The Klein-Gordon equation has a plane wave general solution

$$\phi(x) = Ne^{-iE_pt - p\cdot x} \tag{3.3}$$

where N is a normalisation constant.

[1] In reference to the so-called *mass shell* in momentum space, a surface where the energy and momentum satisfy the on-shell condition.

Derivation 3.1 (Demise) Consider a simple 1D potential barrier of the form

$$V(x) = \begin{cases} 0 & x < 0 \\ V & x \geq 0 \end{cases} \tag{3.4}$$

According to the Klein-Gordon equation, the simplest solution would be

$$\phi(t, x) = \begin{cases} e^{-i(E_p t - px)} + ae^{-i(E_p t + px)} & x < 0 \\ be^{-i(E_p t + kx)} & x \geq 0 \end{cases} \tag{3.5}$$

where p is the momentum, $k = \sqrt{(E_p - V_0)^2 - m^2}$, and

- $e^{-i(E_p t - px)}$ is the part of the field travelling at the +ve x-direction that has not yet reached the potential barrier.
- $ae^{-i(E_p t + px)}$ is the part of the field reflected at the barrier travelling at the $-$ve x-direction.
- $be^{-i(E_p t + kx)}$ is the part of the field transmitted through the barrier travelling at the +ve x-direction.

Intuitively, both $\phi(x)$ and $\partial_x \phi(x)$ are continuous at $x = 0$, from which we find the parameters

$$a = \frac{p - k}{p + k} \quad b = \frac{2p}{p + k} \tag{3.6}$$

By inserting the $x < 0$ solution into the Klein-Gordon equation, we find a dispersion relation

$$p = \pm\sqrt{E_p^2 + m^2} \tag{3.7}$$

To reflect the forward-travelling nature of $e^{-i(E_p t - px)}$m the group velocity $v_g = \partial_p E_p$ must be positive. This forces us to adopt the positive solution.

Definition 3.1 (Covariant derivative) The *covariant derivative* is the extension of the 4-derivative in the presence of a 4-vector field A^μ. It is

$$D^\mu = \partial^\mu + iA^\mu \tag{3.8}$$

Now we insert the $x \geq 0$ solution into the Klein-Gordon equation. Due to the non-zero potential V, we replace the partial derivatives with covariant derivatives:

$$i\partial_t \to i\partial_t - V \quad \partial_t \to \partial_t + iV \tag{3.9}$$

which gives

$$k = \mp\sqrt{(E_p - V)^2 - m^2} \tag{3.10}$$

Again, to reflect the forward-travelling nature of $be^{-i(E_p t + kx)}$, the group velocity or its inverse $\frac{1}{v_g} = \frac{\partial k}{\partial E_p} = \mp\frac{E_p - V}{|k|}$ must be positive. Now consider the case $V > E_p$. From the group velocity condition, we are forced to adopt the negative solution.

One can find a negative energy solution for each positive energy solution. However, this can be handwaved, as we will see much, much later, as antimatter. The real problem lies with the probability density, which we recall to be the 0^{th} component of the conserved current:

$$\varrho = i(\phi^*(\partial_t + iV)\phi - \phi(i\partial_t - V)\phi^*) \tag{3.11}$$

which, in this case, is simply

$$\varrho = 2b^2(E_p - V) \tag{3.12}$$

For $E < V$, this probability density is always negative.

Remark 3.2 Wait, what?

A negative probability density is always unphysical, which can be resolved by turning the (classical) Klein-Gordon field into an operator. This is the beginning of quantum field theory.
Historically, the quantisation of fields/operators as the so-called *field operator* is known as *second quantisation*, in contrast to quantised particles, which was known as the *first quantisation*[2]. Today, we call both *canonical quantisation* as canonical commutation relations are utilised in both quantisation processes.

3.2 Fock space

A generic state in QFT is essentially a linear combination of k particle states for some arbitrary k. This is significant in that k is not fixed - particles might be created and annihilated. Hilbert spaces, which have a fixed number of particles, fail to describe QFT. Rather, the vector space QFT lies in is known as a *Fock space*:

Definition 3.2 (Fock space) The Fock[a] space $\mathcal{F}(\mathcal{H}_1)$ is the direct sum of all n-particle Hilbert spaces:

$$\mathcal{F}(\mathcal{H}_1) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \quad n \in \mathbb{Z} \tag{3.13}$$

where $\mathcal{H}_0 = \mathbb{C}$ is the vacuum state space (zero particles), and $\mathcal{H}_n = \mathcal{H}_1^{\otimes n}$ is the n-particle Hilbert space.

[a]Named after Vladimir Fock, or *Fok* in scientific transliteration.

Remark 3.3 In mathematics, this decomposition is known as a $\mathbb{Z}$-grading as $n \in \mathbb{Z}$.

Definition 3.3 (Creation and annihilation operators) Fock spaces are equipped with creation and annihilation operators $a^\dagger$ and a that adds a particle to the final state and removes a particle from the initial state respectively:

$$a^\dagger|n\rangle = |n+1\rangle \quad a|n\rangle = |n-1\rangle \tag{3.14}$$

Remark 3.4 Bosonic creation and annihilation operators are near-identical the good ol' ladder operators in QM as they obey the same commutation relations. Fermionic creation and annihilation operators observe anticommutation relations, which are powered by Grassmann mathematics covered in *Spinors & Symmetries*.

Theorem 3.2 (Bosonic and fermionic operator commutations)

- Bosonic operators commute:

$$[B_i, B_j] = B_i B_j - B_j B_i = 0 \tag{3.15}$$

- Fermionic operators anticommute:

$$\{F_i, F_j\} = F_i F_j + F_j F_i = 0 \tag{3.16}$$

- A bosonic operator commutes with a fermionic operator:

$$[B, F] = BF - FB = 0 \tag{3.17}$$

3.3 Quantisation of the Klein-Gordon field

A momentum space is a generalisation of the *reciprocal space* or *wavevector space* you may have seen before in crystallography. The momentum and position (or physical) spaces are Fourier transforms of each other.

[2]In fact, second quantisation is a slight misnomer as quantising operators is also possible (although unnecessary) in QM.

Theorem 3.3 (Fourier transform identity) From the delta function property $\int_{-\infty}^{\infty} dk\, f(k)\delta(k) = f(0)$, one has

$$\int_{-\infty}^{\infty} e^{-ikx} dx = 2\pi\delta(k) \tag{3.18}$$

So far we have been working in position space, which is, informally speaking, the collection of all possible positional vectors. In the following derivation, even though the field and canonical momentum operators are still in position space, their expressions are in momentum space, which we Fourier-transform back to position space.

Derivation 3.2 (Field operator) The Klein-Gordon equation general solution in Equation 3.3, can be rewritten to account for negative energy solutions:

$$\phi(x,t) = \int d^3p N_p \left(f_p e^{-i(E_p t - p\cdot x)} + f_p^* e^{i(E_p t - p\cdot x)} \right) \tag{3.19}$$

where N_p, a real function of p, is the previously seen normalisation factor and f_p is a complex function of p (and hence based in momentum space). Importantly, as the field operator $\phi(x,t)$ is based in position space, we must perform a Fourier transform $\int d^3pN_p$ to convert f_p from momentum space to position space.
To ensure that the resultant quantised Hamiltonian will evolve with time in the same way, we replace f_p and f_p^* with the annihilation and creation operators a_p and $a_p^\dagger$. As the Klein-Gordon field is a scalar field (spin-0), it is a bosonic field and its components commute under field quantisation:

Theorem 3.4 (Bosonic creation and annigilation operator commutations) For two arbitrary momenta p and q in bosonic fields, their creation and annihilation operators a_p, $a_p^\dagger$, a_q and $a_q^\dagger$ observe:

$$[a_p, a_q^\dagger] = (2\pi)^3\delta^3(p-q) \tag{3.20}$$

$$[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0 \tag{3.21}$$

Recalling that special relativity must be observed, we must choose an N_p that makes $\phi(x)$ Lorentz-invariant. The Lorentz-invariant phase space volume element for a single particle is given by

Definition 3.4 (Lorentz-invariant phase space volume element)

$$dV = \frac{dp^3}{(2\pi)^3 2E_p} \tag{3.22}$$

For the field operator, we can remove a factor of $1/\sqrt{2E_p}$ to this volume element so that no factors of E_p emerge in the field and momentum operator commutation relations[a]. Taking $N_p = 1/((2\pi)^3\sqrt{2E_p})$ the Klein-Gordon field operator is then written as

Definition 3.5 (Klein-Gordon field operator)

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right) \tag{3.23}$$

[a]Lorentz invariance is preserved as the factor $1/\sqrt{2E_p}$ is not really gone, but rather absorbed into the field amplitude.

Note 3.1 (Two formalisms of $\phi(p)$) Intuitively, the field operator in momentum space is the Fourier transform of its position space counterpart

$$\phi(p) = \int d^3x e^{-ip\cdot x}\phi(x) \tag{3.24}$$

However, the notation $\phi(p)$ is overloaded: a second formalism exists in some literature where $\phi(p)$ denotes the *momentum space contribution* to the field operator seen in Equation 3.23:

$$\phi(p)_{\text{alt}} = \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right) \tag{3.25}$$

which gives rise the (equally correct) formula for the position space field operator

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \phi(p)_{\text{alt}} \tag{3.26}$$

In this book, we use the first formalism.

The momentum operator can then be written by adding a factor of iE_p and changing the sign on the first term

Definition 3.6 (Klein-Gordon momentum operator)

$$\pi(x) = -\int \frac{d^3p}{(2\pi)^3\sqrt{2}} \sqrt{E_p} \left(-a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right) \tag{3.27}$$

The commutation relations for bosonic fields are then

Theorem 3.5 (Bosonic field and momentum operator commutations) For some arbitrary spacetime coordinates x and y

$$[\phi(x), \pi(y)] = i\delta^3(p-q) \tag{3.28}$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0 \tag{3.29}$$

By recalling the definition of the Hamiltonian, we can derive the Klein-Gordon Hamiltonian density from the Klein-Gordon Lagrangian density as

$$\mathcal{H} = \frac{1}{2}(\pi^2 + |\nabla\phi|^2 + m^2\phi^2) \tag{3.30}$$

We then find the (time-independent) Hamiltonian operator:

$$H = \int \frac{d^3p}{(2\pi)^3 2E_p} E_p^2 \left(a_p^\dagger a_p + a_p a_p^\dagger\right) = \underbrace{\int \frac{d^3p}{(2\pi)^3} E_p a_p^\dagger a_p}_{①} + \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{E_p}{2} \left[a_p, a_p^\dagger\right]}_{②} \tag{3.31}$$

However, as $\left[a_p, a_p^\dagger\right]$ is proportional to the identity, so is the term ②, and we can neglect it for convenience. Due to the ordering of its creation and annihilation operators, the remaining term ① is known as the *normal-ordered* or *Wick-ordered*[3] Hamiltonian:

Definition 3.7 (Normal-ordered Klein-Gordon Hamiltonian)

$$:H: = \int \frac{d^3p}{(2\pi)^3} E_p a_p^\dagger a_p \tag{3.32}$$

Here we have introduced the concept of *normal ordering*:

Definition 3.8 (Normal ordering) For some operator O that can be expressed as a polynomial of creation and annihilation operators, its normal ordering or *Wick ordering* $:O:$ is the same polynomial, but with all creation operators to the left of all annihilation operators. This is often necessary in that it eliminates unphysicality created by vacuum fluctuations like $a_p^\dagger a_p$.

Remark 3.5 For example, given some $O = a_p a_q a_p^\dagger$, we have $:O: = a_p^\dagger a_p a_q$.
By considering Equation 3.32 as the entire Hamiltonian[4], we are able to easily diagonalise it. Therefore, we can find that $:H:$ admits the eigenvectors $|n\rangle$ or $a_p|n\rangle$, with the eigenvalues being

$$:H: a_p|n\rangle = (-n - E_p) a_p |n\rangle \tag{3.33}$$

Remark 3.6 Conversely, *antinormal ordering* places creation operators to the right instead.

[3]Named after Gian Carlo Wick.
[4]An advantage of considering a quantity's normal-ordered counterpart as itself is that we eliminate any uninteresting constant terms. This often simplifies calculations.

Theorem 3.6 (Vacuum expectation value) The vacuum expectation value of any normal-ordered expression yields zero.

Chapter 4

Interacting fields I: Preliminaries

We now investigate particle interactions as well as the two matrices S_{fi} and $\mathcal{M}_{fi}$, the first of which physically governs the probability a certain interaction will take place. The Feynman rules, which we use to build the formula for calculating elements of $\mathcal{M}_{fi}$, are then derived for ϕ^4 theory, a simple toy model. Some concluding discussions on their use in high energy physics are then made.

4.1 Dynamic pictures and time evolution

We are now in a position to expand our field theory beyond free particles and into particle interactions. To this end, we modify our Hamiltonian to include an interaction term known as the *interaction Hamiltonian* H_{int}:

$$H = H_0 + H_{\text{int}} \tag{4.1}$$

where H_0 is the *free Hamiltonian*.
In QM, there are three *dynamical pictures* or *representations*. Effectively, they are different formalisms through which one can represent time evolution. Now we revisit them in the context of scattering, where quantum states are slightly different. We have 'in' states $|\psi, \text{in}\rangle$ which denote *prepared* or *incoming* particles and 'out' states $\langle\alpha, \text{out}|$ which denote *detected* or *outgoing* particles.
Remark 4.1 Significantly, 'out' states are treated as half of the density operator. i.e. they are effectively regarded as operators, not states:

$$P_\alpha = |\alpha, \text{out}\rangle\langle\alpha, \text{out}| \tag{4.2}$$

Definition 4.1 (Schrödinger picture) The *Schrödinger picture* is the representation we have encountered in undergrad QM. Time evolution is represented as follows:

- Operators are time-invariant.
- 'in' states evolve under the *total Hamiltonian* H:

$$|\psi, t\rangle = e^{-iH(t-t_0)}|\psi, \text{in}\rangle \tag{4.3}$$

- 'out' states are time-invariant.

Definition 4.2 (Heisenberg picture) The *Heisenberg picture* is the opposite of the Schrödinger picture:

- Operators evolve under the *total Hamiltonian* H:

$$O_H = e^{iH(t-t_0)} O e^{-iH(t-t_0)} \tag{4.4}$$

- 'in' states are time-invariant.
- 'out' states evolve under the *total Hamiltonian* H:

$${}_H\langle\alpha, t| = \langle\alpha, \text{out}|e^{-iH(t-t_0)} \tag{4.5}$$

Definition 4.3 (Interaction picture) The *interaction picture* lies between the Schrödinger and Heisenberg pictures:

- Operators evolve under the *free Hamiltonian* H_0:

$$O_H = e^{iH_0(t-t_0)} O e^{-iH_0(t-t_0)} \tag{4.6}$$

- 'in' states evolve under both the *total* and *free Hamiltonians*:

$$|\psi, t\rangle = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} |\psi, \text{in}\rangle \tag{4.7}$$

  where we often label *interaction picture evolution operator* $U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$.

- 'out' states evolve under the *free Hamiltonian* H_0:

$${}_I\langle \alpha, t| = \langle \alpha, \text{out}| e^{-iH_0(t-t_0)} \tag{4.8}$$

Note 4.1 A few points of interest should be mentioned:

- There is a 'conservation of time evolution terms': Multiplying all 3 time-evolved terms should leave only $e^{-iH(t-t_0)}$, where H is expectedly the *total Hamiltonian*.
- t_0 refers to the time when the state is prepared as $|\psi, \text{in}\rangle$. At this time, all dynamical pictures are identical.
- While the Heisenberg picture generally simplifies calculations, the interaction picture is advantageous when the Hamiltonian includes an interaction term.

The evolution operator in the interaction picture is

$$U(t, t_0) = e^{iH_0(t-t_0)} e^{-iH(t-t_0)} \tag{4.9}$$

Theorem 4.1 (Interaction picture evolution operator properties) The interaction picture evolution operator has several properties:

- Under zero time evolution, it returns the identity:

$$U(t_0, t_0) = \mathbb{I} \tag{4.10}$$

- Inverse:

$$U^{-1}(t_1, t_2) = U(t_2, t_1) \tag{4.11}$$

- Stacking:

$$U(t_3, t_1) = U(t_3, t_2) U(t_2, t_1) \tag{4.12}$$

Derivation 4.1 (Time ordering) By differentiating Equation 4.9 against t and using the initial condition in Equation 4.10, one can derive an alternate expression of $U(t, t_0)$ in terms of itself at a different point in time:

$$U(t, t_0) = \mathbb{I} - i \int_{t_0}^{t} dt_1 H_{\text{int},I}(t_1) U(t_1, t_0) \tag{4.13}$$

By inserting $U(t, t_0)$ into $U(t_1, t_0)$ over and over and over again, one yields the so-called *Dyson series*:

$$U(t, t_0) = \mathbb{I} + \sum_{j=1}^{\infty} (-i)^j \int_{t_0}^{t} dt_j \cdots \int_{t_0}^{t} dt_1 H_{\text{int},I}(t_j) \cdots H_{\text{int},I}(t_1) \tag{4.14}$$

To ensure that the intergrations are performed at the correct temporal order, we introduce the so-called *time ordering symbol*:

Definition 4.4 (Time ordering symbol) Consider a series operators $A_1(x_1)\cdots A_n(x_n)$, each of which can be represented in the form of creation and annihilation operators like $A_n(x_n) = A_n^+(x_n) + A_n^-(x_n)$. The time ordering symbol is a loosely defined convenience which reorders the operators according to their 4-position:

$$T[A_1(t_1)\cdots A_n(t_n)] = (-1)^p A_{i_1}(t_{i_1})\cdots A_{i_n}(t_{i_n}) \quad \text{for} \quad x_{i_1} \leq \cdots \leq x_{i_n} \tag{4.15}$$

where p, the parity, can be represented by Grassmann parity[a]:

$$p = \sum_{i<j} \pi(A_i)\pi(A_j) \mod 2 \tag{4.16}$$

where the Grassmann parity $\pi(A_i)$ observes

- $\pi(A_i) = 0$ for bosonic operators, which commute as they are Grassmann-even.
- $\pi(A_i) = 1$ for fermionic operators, which anti-commute as they are Grassmann-odd.

[a] See *Spinors & Symmetries*.

Remark 4.2 One can also define this without Grassmann mathematics, albeit less satisfyingly:

- Bosonic-bosonic swaps do not contribute to p. No sign change occurs as bosonic operators commute.
- Bosonic-fermionic swaps do not contribute to p. No sign change occurs as bosonic and fermionic operators commute.
- Due to anticommutation, fermionic-fermionic swaps contribute to p by the following:

$$p = \text{num. of fermionic-fermionic swaps} \mod 2 \tag{4.17}$$

Remark 4.3 When a system is entirely comprised of bosons or fermions, this simplifies:

- For bosonic operators, $p = 0$.
- For fermionic operators, $p = 0$ if the number of swaps is even and $p = 1$ if the number of swaps is odd.

Noting that the Dyson series can be represented compactly via an exponential, we represent $U(t, t_0)$ as a time-ordered exponential:

$$U(t, t_0) = T\left[\exp\left(-i\int_{t_0}^{t} dt' H_{\text{int},I}(t')\right)\right] \tag{4.18}$$

In simplified sceanrios, T can also be represented mathematically with the *Heaviside step function* $\theta(t)$:

$$T[A_1(t_1)A_2(t_2)] = \theta(t_1 - t_2)A_1(t_1)A_2(t_2) \pm \theta(t_2 - t_1)A_2(t_2)A_1(t_1) \tag{4.19}$$

where:

Definition 4.5 (Heaviside step function)

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{4.20}$$

Theorem 4.2 (Heaviside step function properties)

$$\partial_t\theta(t) = \delta(t) \tag{4.21}$$

$$(\partial_t\theta(t))\phi(t) = -\delta(t)(\partial_t\phi(t)) \tag{4.22}$$

4.2 Scattering matrix

One way a field can interact with its environment is *scattering*. In scattering, we have the S*-matrix* or the *scattering matrix*, which encodes all the information about the probabilities of different scattering processes. It can be derived by taking limits of $U(t, t_0)$, where time evolution covers the entire history of the system, from the infinite past to the infinite future:

$$S = \lim_{t \to \infty} \lim_{t_0 \to 0\infty} U(t, t_0) \tag{4.23}$$

Definition 4.6 (S-matrix operator) For the initial/incoming state $|\psi, \text{in}\rangle$ and the final/outgoing state $\langle \alpha, \text{out}|$, one can find the S-matrix element S_{fi} via the S-matrix operator S:

$$S_{fi} = \langle \alpha, \text{out}|S|\psi, \text{in}\rangle \tag{4.24}$$

Like in HEP, S_{fi} represents the probability amplitude that $|\psi, \text{in}\rangle$ evolves into $\langle \alpha, \text{out}|$.

An important property is the so-called *cluster decomposition*, which states that an experiment cannot influence another experiment carried out far away from it, and vice versa. Its implication in QFT is as follows:

Theorem 4.3 (Cluser decomposition) Consider two experiments (or rather *clusters*) 1 and 2. An initial state α which includes parts in both clusters α_1 and α_2 evolves into a final state β which likewise includes parts in both clusters β_1 and β_2. The scattering matrix can always be decomposed to

$$S_{\beta\alpha} = S_{\beta_1\alpha_1} S_{\beta_2\alpha_2} \tag{4.25}$$

where $S_{\beta_1\alpha_1}$ and $S_{\beta_2\alpha_2}$ are 'parts' of $S_{\beta\alpha}$ in 1 and 2.

Now consider a highly idealised system of n particles which are prepared with momenta $\{p_i, i = 1, \cdots, n\}$ at time $t_0 \to -\infty$. They interact with (i.e. scatter in) a perturbed Hamiltonian $H = H_0 + H_{\text{int}}$, reaching a set of final momenta $\{q_i, i = 1, \cdots, n\}$ time $t \to \infty$. One can represent the initial and final states with

$$|p, \text{in}\rangle = \prod_{i=1}^{n} a_{p_i}^\dagger |0\rangle \quad \langle q, \text{out}| = \langle 0| \prod_{i=1}^{n} a_{q_i}^\dagger \tag{4.26}$$

where $|0\rangle$ is the ground state. In this scenario, the S-matrix elements are

$$S_{qp} = \langle q, \text{out}|e^{-iH(t-t_0)}|p, \text{in}\rangle \tag{4.27}$$

To evaluate the elements of S_{qp}, we need to diagonalise the full Hamiltonian H. As this is typically impossible, we must use a perturbative approach to deal with H_{int}, where we assume that the interaction Hamiltonian $H_{\text{int}}(t)$ is zero at $t \to -\infty$ and $t \to \infty$.

Derivation 4.2 (Scattering in the Heisenberg picture) One can recall the so-called *Heisenberg equation*, which is the Schrödinger equation in the Heisenberg picture. Here, the classical Poisson bracket is replaced with a commutator:

Theorem 4.4 (Heisenberg equation)

$$\dot{\phi} = i[:H:, \phi] \tag{4.28}$$

Remark 4.4 At $t = 0$, the Heisenberg equation is identical to the Schrödinger equation.

The same equation can be applied to a_p:

$$\dot{a}_p = i[:H:, a_p] \to \dot{a}_p = iE_p a_p \tag{4.29}$$

Plugging this result into the field operator yields

$$\phi(x)_H = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right) \tag{4.30}$$

where one has the additional on-shell condition

$$p^0 = \sqrt{|p|^2 + m^2} \tag{4.31}$$

From this, one can represent the creation and annihilation operators in terms of the wavefunctions.

$$ia^{\dagger}_{p,H} = \int \frac{\mathrm{d}^3\mathbf{x}}{\sqrt{2E_p}} \left(\mathrm{e}^{-ip\cdot x} \left(\partial_0 \varphi_H(x)\right) - \varphi_H(x) \left(\partial_0 \mathrm{e}^{-ip\cdot x}\right)\right) \tag{4.32}$$

By noting that the Heisenberg picture S-matrix element is

Definition 4.7 (Heisenberg picture S-matrix element)

$$S_{qp,H} = \langle 0| a_{q_1,H} \cdots a_{q_m,H} a^{\dagger}_{p_1,H} \cdot a^{\dagger}_{p_n,H} |0\rangle \tag{4.33}$$

We can plug in Equation 4.32, which yields the so-called *LSZ formula* or the *LSZ reduction formula* for S-matrix elements, named after Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann:

Theorem 4.5 (LSZ reduction formula for S-matrix elements) For a series of spacetime coordinates x_i, the scattering matrix elements can be represented by

$$\begin{aligned} S_{qp,H} = &\int \frac{d^4x_1}{\sqrt{2E_{p_1}}} \cdots \int \frac{d^4x_{m+n}}{\sqrt{2E_{q_m}}} e^{-i\sum_{j=1}^{n} p_j\cdot x_j + i\sum_{j=1}^{m} q_j\cdot x_{n+j}} \\ &\prod_{j=1}^{m+n} (\partial^2_{x_j} + m^2) \langle 0| T\left[\phi_H(x_1)\cdots\phi_H(x_{m+n})\right]|0\rangle \end{aligned} \tag{4.34}$$

We investigate the physical significance of each term:

- The integral terms $\int \frac{d^4x_j}{\sqrt{2E_{p_j}}}$. The energies E_{p_j} associated with momenta p_j are integrated over the spacetime coordinates x_j for each external particle. As the LSZ formula is normalised with respect to single-particle states, one has the normalisation factors $\sqrt{2E_{p_j}}$
- The exponential phases $e^{-i\sum_{j=1}^{n} p_j\cdot x_j + i\sum_{j=1}^{m} q_j\cdot x_{n+j}}$ correspond to plane waves representing the incoming and outgoing particles with momenta p_j and q_j respectively.
- The Klein-Gordon operators $(\partial^2_{x_j} + m^2)$ enforce that the external particles are on-shell.
- The time-ordered expectation value is labelled G as the time ordering symbol is a $(m+n)$-point Green's function:
$$G_{m+n} = \langle 0|T[\phi_H(x_1)\cdots\phi_H(x_{m+n})]|0\rangle \tag{4.35}$$
This is effectively a vacuum expectation value.

As of now, we cannot yet solve the LSZ reduction formula. This is because we do not have an expression for the Heisenberg picture fields ϕ_H. We can switch to the interaction picture (denoted by the subscript $_I$), which yields the expression

Theorem 4.6 (Interacting time-ordered propagator)

$$G_{m+n,I} = \frac{\langle 0|T[\phi_I(x_1)\cdots\phi_I(x_{m+n})]S|0\rangle}{\langle 0|S|0\rangle} \tag{4.36}$$

It is also[a] called the *interacting Green's function*, the $(m+n)$*-point Green's function* or the *correlation function*[b].

[a] The reason behind the name 'propagator' will be seen when we arrive at Feynman diagrams

[b] So-called as it is used to study correlations between field operators at different spacetime points in the interacting vacuum.

We note the following for this propagator:

- The *interacting time-ordered propagator* is so-called as it represents the probability amplitude for scattering processes involving $m+n$ field insertions (i.e. $m+n$ points in spacetime where the fields are evaluated). i.e. the probability amplitude for a particle (or more generally, a field excitation) to travel or 'propagate' from one point to another in spacetime.

- The 'interacting' does not refer to the dynamic picture. Instead, it is used to describe the presence of an interaction term in the Hamiltonian.
- Generally, one can use this expression without complications as it covers the whole term G_{m+n} and no operators are lost.

4.3 Self-interaction: ϕ^4 theory

We now introduce the concept of *self-interaction*, which is the interaction between a particle and its own field. A good toy model is the so-called ϕ^4 *theory*[1], which adds a *quartic interaction term* $-\frac{\lambda}{4!}\phi^4$ to Equation 2.21:

Definition 4.8 (ϕ^4 theory Lagrangian)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{4.37}$$

where λ is some dimensionless coupling constant that determines the strength of the interaction, and the $1/4!$ is a combinatorial normalisation factor that compensates for the fact that there are $4!$ ways to permute the (4 identical) fields in ϕ^4 theory.

ϕ^4 theory which is among a family of theories known as ϕ^n theory. It is superior to all other ϕ^n theories for two reasons:

- **Energetic stability:** ϕ^4 *theory* has the second-simplest interaction term that respects the symmetry $\phi \to -\phi$[2]:
  - The general form for the potential energy in ϕ^3 theory is $V(\phi) = m^2\phi^2 + g\phi^3$. Due to the odd exponential in $g\phi^3$, one can get a negative Hamiltonian expectation for a large, negative ϕ, even if the coupling constant g is positive.
  - In contrast, the $\lambda\phi^4$ term ϕ^4 theory is positive as long as λ is positive, ensuring a stable minimum, which is crucial for physical systems.
- **Renormalisability[3]:** In certain dimensions, ϕ^3 theory is non-renormalisable, unlike ϕ^4 theory.

By integrating the normal-ordered interaction term $:\phi^4:$[4], we find the so-called *interaction Hamiltonian density*:

$$\mathcal{H}_{\text{int}} = -\frac{\lambda}{4!}:\phi^4: \tag{4.38}$$

The Hamiltonian is thus

$$H_{\text{int}} = \int d^3x \mathcal{H}_{\text{int}} = \frac{\lambda}{4!}\int d^3x :\phi^4: \tag{4.39}$$

which yields the S-matrix operator

$$S = T\left[\sum_{j=0}^{\infty}\frac{(-i)^j}{j!}\left(\frac{\lambda}{4!}\int d^4 :\phi(x)^4:\right)\right] \tag{4.40}$$

Next, we will see how Equation 4.36 is a Taylor expansion of λ in the ϕ^4 theory. But before this, we introduce one last bit of formalism.

Definition 4.9 (Wick contraction) For operators A and B, the *Wick contraction*[a] or simply *contraction* is simply its vacuum expectation value

$$\langle AB \rangle = \langle 0|AB|0\rangle \tag{4.41}$$

[1] Even though its only physical application is describing the self-interaction term of the Higgs field, it has become a staple of QFT books.
[2] ϕ^2 theory has the simplest interaction term. But this term is simply the previously seen the mass term $\frac{1}{2}m^2\phi^2$.
[3] You will see what this means in Part II
[4] Recall that this is necessary to remove vacuum divergences like $\langle 0|aa^\dagger aa^\dagger|0\rangle$, which are not normal-ordered.

[a]So-called as just like the contraction of indices in GR (which, in the case of a rank-2 tensor, starts with two indices and ends with a scalar), it starts with two operators and ends with a number.

Derivation 4.3 (Alternative forms of the Wick contraction) From the definition of Wick contractions, we can final several equivalences:

- The product of two operators can always be split into a part that contributes to the vacuum expectation value and a part that does not, which is actually the normal ordering[a] $:AB:$.

  Theorem 4.7 (Wick's first theorem)

  $$AB = \langle 0|{:}AB{:}|0\rangle + {:}AB{:} = \overbracket{AB} + {:}AB{:} \tag{4.42}$$

  In some literature, a rearranged version of *Wick's first theorem* is actually used as a less intuitive definition of Wick contractions.

  $$\overbracket{AB} = AB - {:}AB{:} \tag{4.43}$$

- If A and B are the fields $\phi(x)$ and $\phi(y)$, the RHS becomes the Feynman propagator $D_F(x-y)$:

  $$\overbracket{\phi(x)\phi(y)} = D_F(x-y) \tag{4.44}$$

- The vacuum expectation value is inherently time-ordered, so we can even make the equivalence

  $$\overbracket{AB} = \langle 0|AB|0\rangle = \langle 0|T[AB]|0\rangle \tag{4.45}$$

[a]As normal-ordered operators always have creation operators before annihilation operators, their vacuum expectation value $\langle 0|{:}AB{:}|0\rangle$ is always zero.

Remark 4.5 In other words, the Wick contraction of operators returns their 'nontrivial' part. This part, which represents quantum fluctuations and interactions, contributes to the vacuum expectation value, whereas the normal-ordered part does not.
Often, it is more convenient to use a very similar operation called the *time-ordered pairing* instead of Wick contractions.

Definition 4.10 (Time-ordered pairing)

$$\underbracket{A(x)B(y)} = \begin{cases} \overbracket{A(x)B(y)} & x^0 > y^0 \\ (-1)^p \overbracket{B(x)A(y)} & y^0 > x^0 \end{cases} \tag{4.46}$$

where p, last seen in Definition 4.4, is our good friend, the parity.

> **Quote 4.1** In more accurate books like Bogoliubov's, the time dependent contractions are written down with bottom brackets.
>
> *Felix Halbwedl, 22 December 2024*

Theorem 4.8 (Wick's second theorem) The time-ordering $T[A_1A_2A_3A_4A_5A_6\cdots]$, where all operators are made up of creation and annihilation operators like $A_i = A_i^+ + A_i^-$, can be expressed in terms of time-ordered pairings:

$$T[A_1A_2A_3A_4A_5A_6\cdots] = {:}A_1A_2A_3A_4A_5A_6\cdots{:} + \underbrace{\sum_{\text{single}} {:}\underbracket{A_1A_2}A_3A_4A_5A_6\cdots{:}}_{①} + \underbrace{\sum_{\text{double}} {:}\underbracket{A_1A_2}\underbracket{A_3A_4}A_5A_6\cdots{:}}_{②} + \cdots \tag{4.47}$$

where:

- ① denotes the sum of all the possible results of $A_1A_2A_3A_4A_5A_6\cdots$ undergoing one Wick contraction somewhere in the expression:

$$\sum_{\text{single}} :\underbracket{A_1 A_2} A_3 A_4 A_5 A_6 \cdots: = :\underbracket{A_1 A_2} A_3 A_4 A_5 A_6 \cdots: + :\underbracket{A_1 A_3} A_2 A_4 A_5 A_6 \cdots: + \cdots \tag{4.48}$$

- ② denotes the sum of all the possible results of $A_1A_2A_3A_4A_5A_6\cdots$ undergoing two Wick contractions somewhere in the expression:

$$\sum_{\text{double}} :\underbracket{A_1 A_2}\underbracket{A_3 A_4} A_5 A_6 \cdots: = :\underbracket{A_1 A_2}\underbracket{A_3 A_4} A_5 A_6 \cdots: + :\underbracket{A_1 A_3}\underbracket{A_2 A_4} A_5 A_6 \cdots: + \cdots \tag{4.49}$$

- ...and so on.

An alternative, non-time-ordered version of *Wick's second theorem* is

$$A_1A_2A_3A_4A_5A_6\cdots = :A_1A_2A_3A_4A_5A_6\cdots: + \sum_{\text{single}} :\overbracket{A_1 A_2} A_3 A_4 A_5 A_6 \cdots: + \sum_{\text{double}} :\overbracket{A_1 A_2}\overbracket{A_3 A_4} A_5 A_6 \cdots: + \cdots \tag{4.50}$$

where we simply use the normal contraction.

Finally, a nice trick is the so-called *Wick's third theorem* or *Wick's theorem for vacuum expectation values*:

Theorem 4.9 (Wick's third theorem) For operators A, B_1, $\cdots$, B_n, the following is observed:

$$\langle 0|T[AB_1\cdots B_n]|0\rangle = \sum_i \langle 0|T[\overbracket{AB_1\cdots B_i}\cdots B_n]|0\rangle \tag{4.51}$$

4.4 Feynman diagrams

At this point, we can already evaluate the multi-point Green's function by using Wick's second theorem. This may look like a tedious process. Luckily for us, in doing so, many terms cancel out, and Equation 4.36 reduces to a two-point Green's function or the *Feynman propagator* of some spacetime coordinates x and y, labelled $D_F(x-y)$:

$$D_F(x-y) = \langle 0|T[\phi_I(x)\phi_I(y)]|0\rangle \tag{4.52}$$

Derivation 4.4 (Feynman propagator) Now we want to solve for the exact form of this propagator. The central idea which we shall utilise is the fact that $D_F(x-y)$ is a solution of the Klein-Gordin equation. Before we mindlessly intert this into Equation 3.2, however, let us perform a mathematical trick by performing a Fourier transform:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\tilde{D}_F(p)e^{-ip\cdot(x-y)} \tag{4.53}$$

Substituting this into the equation gives

$$\int \frac{d^4p}{(2\pi)^4}\tilde{D}_F(p)e^{-ip\cdot(x-y)}(p^2-m^2) = \delta^4(x-y) \tag{4.54}$$

This implies that $\tilde{D}_F(p)$ must satisfy

$$(p^2-m^2)\tilde{D}_F(p) = i \tag{4.55}$$

Thus, the propagator in momentum space is

$$\tilde{D}_F(p) = \frac{i}{p^2-m^2} \tag{4.56}$$

There is one problem with this propagator. Two zero denominator singularities emerge at so-called *poles*, positions where the on-shell condition is enforced:

$$p^0 = \pm E_p = \pm\sqrt{p^2+m^2} \tag{4.57}$$

As such a *small* $i\epsilon$ is included to avoid singularities at the poles. This ensures that our integral is well-defined and preserves causality:

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{4.58}$$

Now, we return to position space by computing the inverse Fourier transform:

Definition 4.11 (Feynman propagator) The Feynman propagator is the probability amplitude for a scalar particle to propagate from the spacetime point x to y, taking into account quantum fluctuations:

$$D_F(x-y) = \lim_{\epsilon \to 0+} i \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{4.59}$$

Note 4.2 For convenience, we often suppress (i.e. conveniently forget writing) $\lim_{\epsilon\to 0+}$ or both $\lim_{\epsilon\to 0+}$ and $i\epsilon$ for brievity, in which case their existence is assumed.

Now we introduce the *transition amplitude* or *scattering amplitude* $\mathcal{M}_{fi}$, a matrix related to the S-matrix whose physical significance we will see later.

Definition 4.12 (Transition amplitude)

$$S_{fi} = \delta_{fi} + i(2\pi)^4\delta^4(p_f - p_i)\mathcal{M}_{fi} \tag{4.60}$$

where $\delta^4(P_i - P_f)$ enforces momentum conservation.

Remark 4.6 The physical significance of $\mathcal{M}_{fi}$ can be quickly found: The Kronecker delta δ_{fi} is essentially the identity that accounts for the case where there is no interaction. Hence, $\mathcal{M}_{fi}$ is a rescaled version of the interaction-dependent part of S_{fi}. Crudely speaking, we can reduce the relationship to

$$\mathcal{M}_{fi} = \langle f|S_{fi} - 1|i\rangle \tag{4.61}$$

One can construct $\mathcal{M}_{fi}$ (and by that, S_{fi}) of a given field theory through its Feynman rules. These rules make use of the so-called *Feynman diagrams*, which is effectively the graphical representation of a S-matrix:

- **External legs**: The initial and final particle states, known as *external legs*, are the starting and end points on the left and right sides. For a total number of n such points, one has an n-point Feynman diagram.

- **Vertices:** Denoted by visible round dots. The number of vertices, known as the *order*, corresponds to the order of the coupling constant of the field theory.

- **Propagators**: The intermediate lines and loops represent abstract 'paths' the particles take or virtual particles whose terms and Feynman diagram representations are known as *propagators*[5].

A Feynman diagram of the 0^{th} order has no vertices:

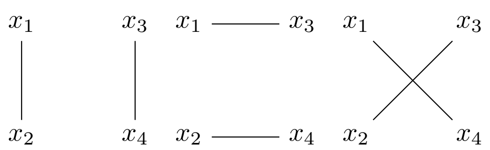

Figure 4.1: 0^{th}-order Feynman diagrams

A Feynman diagram of the 1^{st} order has a single vertex:

[5]As ϕ^4 theory is purely self-interacting and does not involve any specific particles, we will use plain lines for all particles and propagators, which is almost always not the case in real life.

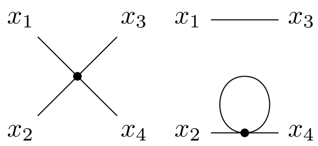

Figure 4.2: 1^{st}-order Feynman diagrams

Note the existence of the loop in the diagram on the right. Functionally, this diagram is identical to the middle entry in the 0^{th}-order diagrams, with the only difference being the removal of the loop. Hence, it is not the simplest expression this particular interaction can take. The middle 0^{th}-order diagram is then known as the *leading order* Feynman diagram with respect to the interaction it represents as it is the most reduced form of the interaction.
A Feynman diagram of the 2^{nd} order has two vertices:

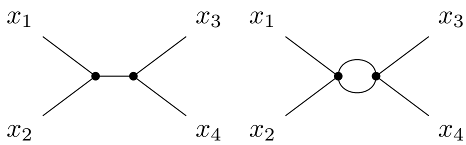

Figure 4.3: 2^{nd}-order Feynman diagrams

All of the diagrams above are 4-point Feynman diagrams.

> **Quote 4.2** You can still insert some hand drawings. Hand drawings are better than no drawings, and if you don't do it now, it eventually never happens.
>
> *Felix Halbwedl, encouraging the author to finish the Feynman diagram illustrations, 22 November 2024*

Definition 4.13 (Tree level) We speak of a *tree-level diagram* if the diagram in question has no loops. Mathematically, this means that the diagram does not integrate over internal momenta.

4.5 ϕ^4 theory Feynman rules

Aside from the previously seen initial and final states, the two important elements in a Feynman diagram are external legs and (internal) virtual particles:

Definition 4.14 (External leg) In a Feynman diagram, an *external leg* or more boringly an *external point* is an incoming or outgoing particle. Specifically:

- An *incoming external leg* is an initial state (i.e. incoming) particle, typically on the left side.
- An *outgoing external leg* is a final state (i.e. outgoing) particle, typically on the right side.

Definition 4.15 (External and internal propagators) In a Feynman diagram, two types of propagators exist:

- *External propagators* or *external leg propagators* represent *probability amplitudes* of incoming or outgoing particles in a scattering process.
- *Internal propagators* represent *probability amplitudes* of virtual particles created at some time and then annihilated at a later time.

A propagator is also sometimes called a *line* or an *edge*.

A Feynman diagram with m incoming external legs and n outgoing external legs is represented a $(m+n)$-point Green's function, which itself is made up of Feynman propagators. As the Feynman propagator has a S_{fi} term, it can be written as a perturbative expansion like S_{fi} in Equation 4.40:

$$G_{m+n} = \sum_k G^{(k)}_{m+n} \tag{4.62}$$

Each $G^{(k)}_{m+n}$ corresponds to a class of (various possible) Feynman diagrams with k vertices[6], where the coupling constant λ is of the order k. For example:

- The 0^{th} term represents the so-called *free propagator* where no interactions happen. It has only external leg propagators.
- For $k > 0$, the k^{th} term represents the Feynman propagator where k interaction happens (represented by a Feynman diagram of order k). It has both external and internal propagators.

Derivation 4.5 ($2 \to 2$ processes)

> **Quote 4.3** It's like an electric board with four sockets, and each field is a plug.
>
> *Alessio Serafini, on $2 \to 2$ Feynman diagrams, 27 February 2025*

Putting it all together, we now look at the example of a $(2+2)$-point Green's function with incoming external legs x_1 and x_2, outgoing external legs x_3 and x_4 and no interaction in the middle of the Feynman diagram can be represented in terms of Feynman propagators as

$$G^{(0)}_{2+2} = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3) \tag{4.63}$$

If 1 interaction happens at point y, we have the extra term

$$G^{(1)}_{2+2} = -i\frac{\lambda}{4!}\frac{4!}{1!}\int d^4y D_F(x_1 - y)D_F(x_2 - y)D_F(x_3 - y)D_F(x_4 - y) \tag{4.64}$$

where λ and $1/4!$ are the coupling constant and the normalisation term seen in Equation 4.37. The factor 4! that follows accounts for the fact that each vertex has four ϕ-fields, and there are 4! ways to contract these (four) fields with external legs or propagators[a]. The factor 1! accounts for the first-order perturbation.

If 2 interactions happen at points y_1 and y_2 on the left and right sides respectively, we have the extra term, where the factors λ, $1/4!$ and 4! are squared:

$$G^{(2)}_{2+2} = -\left(\frac{\lambda}{4!}\right)^2 \frac{4!^2}{2!}\int d^4y_1 \int d^4y_2 D_F(x_1 - y1)D_F(x_2 - y_1)D_F(x_3 - y_2)D_F(x_4 - y_2) \tag{4.65}$$

where the factor 1! accounts for the second-order perturbation, and so on in higher orders.

[a]This is the multiplicity C which we will see later.

We also note that the same Feynman diagram can be yielded from different ways of contracting operators, which may arise multiple times in the series expansion. This is reflected in the Feynman diagram by the multiplicity C. We thus define:

Definition 4.16 (Multiplicity) The *multiplicity* or *multiplicity factor* C with respect to a specific Feynman diagram is the number of possible Wick contractions that result in the specific diagram.

This is a natural result from Wick's second theorem, where we have summed over all possible contractions of the field operators, many of which are actually identical.

Derivation 4.6 ($1 \to 1$ process) Let us now consider the 'simple' example of a $1 \to 1$ process with a single vertex. This is expressed by the Green's function

$$G_{1+1} = \langle 0|T\left[\phi(x)\phi(y)(-i)\int dt \int d^3z \frac{\lambda}{4!}\phi^4\right]|0\rangle \tag{4.66}$$

[6]That is to say, each interaction vertex arises from an insertion of the interaction term from the Lagrangian.

which this is actually a disguised version of its full form

$$G_{1+1} = \langle 0|T\left[\phi(x)\phi(y)\left(\frac{-i\lambda}{4!}\right)\int d^4z\phi(z)\phi(z)\phi(z)\phi(z)\right]|0\rangle \tag{4.67}$$

Now we apply Wick's second theorem. There are a total of 6 fields, which gives 15 possible contractions, although one can catergorise them into two types:

- $\phi(x)$ and $\phi(y)$ each contract with a $\phi(z)$, and the remaining two $\phi(z)$s contract with each other. This has 12 possible contractions.
- $\phi(x)$ contracts with $\phi(y)$, and the four $\phi(z)$s contract with each other. This has 3 possible contractions.

Effectively, we then have

$$G_{1+1} = 3\left(\frac{-i\lambda}{4!}\right)D_F(x-y)\int d^4zD_F(z-z)D_F(z-z)+12\left(\frac{-i\lambda}{4!}\right)\int d^4zD_F(x-z)D_F(y-z)D_F(z-z) \tag{4.68}$$

Quote 4.4 It is not trivial.

Paulina Schlachter, on calculating C, 25 February 2025

Already, for such a simple process, our derivation is somewhat tedious. A (only slightly) more intuitive way of determining C comes from inspecting Feynman diagrams.

Note 4.3 (Multiplicity from pre-diagrams) To begin with, let us draw the external legs as points (called *external points*) and all the internal propagators as-is between the initial and final external points. This pseudo-Feynman diagram is often called a *pre-diagram*.
Now formulate the number of all possible configurations one can connect the external points to the propagator (i.e. end points) of each pre-diagram. This number is the corresponding C to the pre-diagram's corresponding diagram.
Importantly, when doing so, one should note that any loops *which have vertices with internal lines* further complicate matters as they can appear on more than one possible internal line. The C for a diagram with loop(s) that has/have a vertex/vertices is then the C of its corresponding loopless diagram multiplied by the number of possible loop configurations.
If the loop is completely isolated from any internal line, then we may treat its C as that of its loopless counterpart. The isolated loops are known as *vacuum bubbles*, and such diagrams are known as *vacuum diagrams*. They are unphysical and are not generated by the properly normalised generating functional we will see in Part II.

Already, we can see a pattern emerging. This is generalised by the so-called ϕ^4 theory Feynman rules (in momentum space).

Theorem 4.10 (ϕ^4 theory Feynman rules) For a given Feynman diagram in ϕ^4 theory, the transition amplitude matrix elements $\mathcal{M}_{fi}$ is constructed as follows:

ϕ^4 theory Feynman rules (partial)

For each	Add to expression
Incoming/outgoing scalar particle	1[a]
Internal line	$\frac{i}{k_j^2-m^2}$
Internal loop	$\int \frac{d^4k_j}{(2\pi)^4}$
Vertex	$-i\frac{\lambda}{4!}(2\pi)^4\delta^4\sum_j p_j$[b]

where k_j is the propagator momenta, p_j is the incoming momenta and q_j is the outgoing momenta. We also perform the following steps:

- Introduce the multiplicity C for the number of contractions leading to the same diagram.
- Include the $1/k!$ factor from the Taylor expansion, where k is the perturbative order.

- Remove a factor of $(2\pi)^4\delta^3(p-q)$, where p and q are the total initial and detected momenta, as it is usually already accounted for in the definition of the cross-section or decay rate in terms of the matrix elements.

[a]Unlike spinor and vector fields in QED, scalar fields in our good ol' harmless ϕ^4 theory manifests in a mere number (i.e. scalar).
[b]The sum is over all lines exiting the vertex and forces four-momentum conservation at the vertex.

Remark 4.7 For $2 \to 2$ interactions ($m = n = 2$), $\mathcal{M}_{fi}$ reduces very nicely to $-i\lambda$. Here the nature of ϕ^4 theory as a useful toy model becomes clear.

Note 4.4 (Sneak peek of renormalisation) It should be obvious that, to make sure that our results are physical, the scattering amplitude must observe

$$0 \leq |\mathcal{M}|^2 \leq 1 \tag{4.69}$$

At the tree level, this is always observed. However, the introduction of loops will often bring about divergences (often into infinity) that violate this condition. These divergences suggest that our theory is not entirely physical beyond the tree level. To produce non-divergent results, we must apply *renormalisation* to our quantum field theory of interest. How we do this will be at the heart of Part II.

4.6 Beginnings of HEP

Happily, the Feynman rules now allow us to calculate the transition amplitude $\mathcal{M}_{fi}$. However, our final goal remains calculating S_{fi}. The means to do so is the previously seen LSZ formula, which has finally become useful now that we have developed the complete toolkit to use it.

Derivation 4.7 (Scattering matrix) Let us begin with Equation 4.34, where we have m incoming external legs and n outgoing external legs. At this point, two simplifications can be made. The first is amputating our poor propagators:

Definition 4.17 (Amputated propagator) It is convenient to write all 'internal' or 'interior' parts of the propagator as a so-called *amputated propagator* $\bar{G}(y_1, \cdots, y_l)$, giving the whole propagator as

$$G_{m+n} = D_F(x_1 - y_1) \cdots D_F(x_{n+m} - y_l) \times \bar{G}(y_1, \cdots y_l) \tag{4.70}$$

where, $l \leq m+n$, given that more than one leg may couple to the same vertex.

Remark 4.8 As the amputated propagator is effectively the whole propagator with all external leg propagators removed, it is known to be 'amputated.' Conversely, one can construct the whole propagator G_{m+n} by multiplying all external legs, and finally the poor amputated propagator. A more rigourous treatment is given in Part II.

The second is the Fourier transformation: At this point, we are still living in position space. Here it is again convenient to perform a Fourier transform to momentum space. We can relate the position space field $\phi(x)$ and its momentum space counterpart $\tilde{\phi}(p)$ with the following Fourier transforms:

$$\tilde{\phi}(p) = \int d^4x e^{ip\cdot x}\phi(x) \quad \phi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\tilde{\phi}(p) \tag{4.71}$$

where $p \cdot x = p^\mu x_\mu = Et - \vec{p}\cdot\vec{x}$.

Now let us perform the amputation and Fourier transforms. In our case, the propagator is amputated by simply applying the Klein-Gordon operator:

$$S_{fi} = \int \frac{\mathrm{d}^4 x_1}{\sqrt{2E_{i_1}}} \cdots \int \frac{\mathrm{d}^4 x_{m+n}}{\sqrt{2E_{f_m}}} e^{-i\sum_{j=1}^{n} p_j\cdot x_j + i\sum_{j=1}^{m} q_j \cdot x_{n+j}} \prod_{j=1}^{m+n} \left(\partial^2_{x_j} + m^2\right) G_{m+n} \tag{4.72}$$

where, by switching to momentum space, we have introduced normalisation factors $1/\sqrt{2E}$.

Finally, recall that:

- The Klein-Gordon operator acting on the Green's function simply gives $(2\pi)^4\delta^4(P_i - P_f)$ enforcing energy-momentum conservation.

- $\mathcal{M}_{fi}$ can be singled out from its definition in Equation 4.61.

This gives the relation between S_{fi} and $\mathcal{M}_{fi}$ in ϕ^4 theory:

$$S_{fi} = \mathcal{M}_{fi} i \prod_{j=1}^{n} \frac{1}{\sqrt{2E_{f_j}}} \prod_{j=1}^{m} \frac{1}{\sqrt{2E_{i_j}}} (2\pi)^4 \delta^4(P_i - P_f) \tag{4.73}$$

where m and n are indices for initial and final particles, $P_i = \sum_{j=1}^{n} p_j$ and $P_f = \sum_{j=1}^{n} q_j$.

Note 4.5 (Alternative normalisation) In this book, we use *relativistic normalisation*, which is given by Equation 3.20. In some literature, a different normalisation convention is used instead. The key difference is in how the one-particle states are defined:

$$[a_p, a_q^\dagger] = 2E(2\pi)^3\delta^3(p - q) \tag{4.74}$$

To account for (or rather *absorb*) this extra factor of $2E$, Equation 4.73 becomes

$$S_{fi} = \mathcal{M}_{fi} i \prod_{j=1}^{n} \frac{1}{2E_{f_j}} \prod_{j=1}^{m} \frac{1}{2E_{i_j}} (2\pi)^4 \delta^4(P_i - P_f) \tag{4.75}$$

As we will soon see, in the alternative normalisation convention, Equation 4.85 takes the form

$$\text{LIPS}(m) \equiv (2\pi)^4 \delta^4(P_i - P_f) \prod_{k=1}^{m} \frac{d^3 q_k}{(2\pi)^3} \tag{4.76}$$

We continue our discussion in momentum space and investigate a few experimental HEP-adjacent concepts: the *transition rate*, the *decay rate* and the *cross-section*.

Derivation 4.8 (Probability density) We first adopt the shorthand notation

$$A_p = \prod_{j=1}^{n} a_{p_j} \quad B_q = \prod_{j=1}^{n} a_{q_j, H} \tag{4.77}$$

From the Born rule, the probability density of scattering involving m particles is

$$dP(q_1, \cdots, q_m) = \text{Tr}[\rho d \prod_{q_1} \cdots d\pi_{q_m}] \tag{4.78}$$

In momentum space, we have

$$d \prod_{q} = \frac{d^3 q}{(2\pi)^3} a^\dagger |0\rangle\langle 0| a_q \tag{4.79}$$

where $1/(2\pi)^3$ is a normalisation factor resulting from normalising the previously unnormalised plane wave solution ϕ

$$\langle 0| a_p a_p^\dagger |0\rangle = (2\pi)^3 \delta(0) \tag{4.80}$$

In any case, by inserting Equation 4.79, Equation 4.78 becomes

$$\begin{aligned} dP(q_1, \cdots, q_m) &= V^{-n} \int \frac{d^{3n} k}{(2\pi)^{3n}} \frac{d^{3m} q}{(2\pi)^{3m}} \langle 0| A_k B_q^\dagger |0\rangle\langle 0| B_q A_p^\dagger |0\rangle\langle 0| A_p A_k^\dagger |0\rangle \\ &= V^{-n} \frac{d^{3m} q}{(2\pi)^{3m}} \langle 0| A_k B_q^\dagger |0\rangle\langle 0| B_q A_p^\dagger |0\rangle \\ &= \frac{1}{V^n} \frac{d^{3m} q}{(2\pi)^{3m}} |S_{qp}|^2 \end{aligned} \tag{4.81}$$

which is the simplified form of the probability density.

A problem in Equation 4.73 that we have left unaddressed is the momentum conservation-enforcing term

$\delta^4(P_i - P_f)$. In it, the momentum states exist throughout the entirety of the space-time. In a real experiment, however, incoming and outgoing states are localised. To deal with this, we assume the interaction happens over a time of T in a system localised in some volume V[7]. First, we can perform the rewrite

$$(2\pi)^4\delta^4(P_i - P_f) = \int_{VT} d^4x e^{i(P_f - P_i)x} \tag{4.82}$$

Taking the square modulus gives

$$|(2\pi)^4\delta^4(P_i - P_f)|^2 \approx (2\pi)^4\delta^4(P_i - P_f)|\int_{VT} d^4x e^{i(P_f - P_i)x}| = VT(2\pi)^4\delta^4(P_i - P_f) \tag{4.83}$$

The transition rate, which is the probability per unit time, is denoted by W.

Definition 4.18 (Transition rate differential) The differential of the transition rate is given by

$$dW = |\mathcal{M}_{fi}^2|V \prod_{j=1}^{n} \frac{1}{2E_{p_j}V} \,\mathrm{LIPS}(m) \tag{4.84}$$

In this expression, we have crammed all Lorentz-invariant terms together as the so-called *Lorentz-invariant phase space*, which is defined with respect to m particles in the final state:

Definition 4.19 (Lorentz-invariant phase space)

$$\mathrm{LIPS}(m) \equiv (2\pi)^4\delta^4(P_i - P_f) \prod_{k=1}^{m} \frac{d^3q_k}{(2\pi)^3 2E_k} \tag{4.85}$$

Now we consider decays. Unlike other interactions, there is only one initial particle in decays. As such, the decay rate Γ_{if}, which is the transition rate for decays, is

Definition 4.20 (Decay rate)

$$\Gamma_{if} = \frac{1}{2m} \int |\mathcal{M}_{fi}|^2 \,\mathrm{LIPS}(m) \tag{4.86}$$

In the case where the end product consists of 2 particles, the decay rate reduces to

$$\Gamma_{if} = \frac{1}{2m} \int |\mathcal{M}_{fi}|^2 q_f d\Omega \tag{4.87}$$

where Ω is the so-called *solid angle*.
A quantity ultimately related to the transition rate is the cross-section. We begin with the *particle flux*:

Definition 4.21 (Particle flux) The particle flux for a beam with velocity v_1 and a density of particles of $1/V$[a] and a target with velovity v_2 is

$$F = \frac{|v_1 - v_2|}{V} \tag{4.88}$$

[a] i.e. 1 particle in a volume of V.

This is simply the number of particles per unit area which run past each other per unit time.
As the *cross section* is the transition rate for a single particle *per unit beam flux*, we can find the differential cross section by dividing the transition rate by the flux:

$$d\sigma = \frac{dW}{F} = \frac{1}{|v_1 - v_2|} \frac{1}{4E_1E_2} |\mathcal{M}_{fi}|^2 \,\mathrm{LIPS}(m) \tag{4.89}$$

This expression is Lorentz-invariant. We then integrate and find

[7] This is not a problem, as both T and V ultimately disappear.

Definition 4.22 (Cross section)

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |\mathcal{M}_{fi}|^2 \, \mathrm{LIPS}(m) \tag{4.90}$$

Remark 4.9 If the final particles are identical, LIPS(m) is divided by $m!$.
A useful shorthand when considering any $2 \to 2$ process (e.g. annihilation, scattering) is the Mandelstam variables:

Definition 4.23 (Mandelstam variables) The Mandelstam variables s, t and u correspond to the s-, t- and u- channels respectively.

$$s = (p_1 + p_2)^2 \quad t = (p_1 - p_3)^2 \quad u = (p_1 - p_4)^2 \tag{4.91}$$

s, t and u are equal to the four-momentum exchange q^2 in their own channels. They are Lorentz-invariant and satisfy

$$s > 0 \quad t < 0 \quad u < 0 \quad s + t + u = m_1^2 + m_1^2 + m_3^2 + m_4^2 \tag{4.92}$$

Chapter 5

Interacting fields II: QED

A generalised and actually physical version of the Klein-Gordon equation is the Dirac equation. Using canonical quantisation, we will quantise the Dirac field and develop a Lagrangian for quantum electrodynamics, which accounts for fields generated by both electrons/positrons and photons.

5.1 Dirac equation

The ill-fated Klein-Gordon equation, which we have found to be *kaputt*, is a Lorentz-invariant 2^{nd}-order DE. We now propose a better candidate in the form of a Lorentz-invariant 1^{st}-order DE, whose most general form would be known as the *Dirac equation*

Theorem 5.1 (Dirac equation)

$$(i\gamma^\mu \partial_\mu - m)\phi = (i\partial\!\!\!/ - m)\psi = 0 \tag{5.1}$$

γ^μ is a yet undetermined 4-vector, and we have defined the so-called *Feynman slash notation* for some four-vector a

$$a\!\!\!/ = \gamma^\mu a_\mu \tag{5.2}$$

Quote 5.1 The equation was more intelligent than its author.

Paul Dirac, on his equation[a] *(disputed)*

[a]The rationale behind the quote, as per Victor Weisskopf, was that 'A great deal more was hidden in the Dirac equation than the author had expected when he wrote it down in 1928'.

Remark 5.1 The Dirac equation does not directly conflict with the Klein-Gordon equation. In fact, every solution to the Dirac equation is also a solution to the Klein-Gordon equation[1]. However, the reverse is not true: the spinorial nature of the solution means that the probability $\rho = \psi^\dagger\psi$ will always be non-negative, and the Dirac equation excludes negative probability states that can be admitted as solutions to the Klein-Gordon equation.

Derivation 5.1 (γ matrices) The previous remark allows us to determine the γ matrices by applying the differential operator $i\gamma^\mu\partial_\mu + m$ to the Dirac equation[a] and equating it with the Klein-Gordon equation

$$\underbrace{-(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi}_{\text{Application of } i\gamma^\mu\partial_\mu + m} = \underbrace{-(\partial^\mu\partial_\mu - m^2)\psi}_{\text{Klein-Gordon equation}} = 0 \tag{5.3}$$

By equating the two, we have seen that the term $\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu$ must be equal to $\partial^\mu\partial_\mu$. Effectively, through $\gamma^\mu\gamma^\nu$, one index was shifted up. Hence, a sensible guess of $\gamma^\mu\gamma^\nu$ would be the metric:

$$\gamma^\mu\gamma^\nu = g^{\mu\nu} \tag{5.4}$$

However, this is wrong for the reason that the off-diagonal components of the 4-metric are zero, thus

[1]This is because the Klein-Gordon equation must still be satisfied to fulfil the SR energy-momentum relation.

implying

$$\gamma^0\gamma^1 = 0 \quad \text{and} \quad (\gamma^0)^2 = -(\gamma^1)^2 = 1 \tag{5.5}$$

at the same time. In fact, such conditions can never be satisfied as long as the components of γ^μ are mere numbers. However, if one switches the indices on the LHS of Equation 5.3 and adds this otherwise identical expression to Equation 5.3, they will find

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + \gamma^\nu\gamma^\mu\partial_\nu\partial_\mu = 2\partial^\mu\partial_\mu \tag{5.6}$$

This becomes

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{5.7}$$

where one has the anticommutator

Definition 5.1 (Anticommutator)

$$\{a, b\} = ab + ba \tag{5.8}$$

One can see from Equation that γ^μ are elements of a *Clifford algebra*[b], from the definition of which it is clear that elements of γ^μ must be matrices.

Definition 5.2 (γ matrices in the Dirac basis) There exist 4[a] γ matrices. In the *Dirac basis*, they are

$$\gamma^0 = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix} \quad \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \tag{5.9}$$

for $j = 1, 2, 3$. σ_j are the Pauli matrices we know and love.

[a] There is a 5$^{\text{th}}$ γ matrix, but it is defined purely for our convenience and is actually not a part of the γ matrices.

[a] Note the $+$ sign of the second term of this operator!

[b] This is covered in *Spinors & Symmetries*.

Interestingly, the Pauli matrices satisfy the following relation:

Theorem 5.2 (Pauli matrices property)

$$\sigma_j\sigma_k = i\epsilon_{jkl}\sigma_l + \delta_{jk}\mathbb{I}_2 \tag{5.10}$$

where we have once again encountered our good friends, the Kronecker delta δ_{jk} and the Levi-Civita symbol ϵ_{jkl}.

Finally, we conclude with a list of useful formulae for γ matrices:

Theorem 5.3 (Commonly used γ matrix formulae)

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu} \tag{5.11}$$

$$\mathrm{Tr}\left(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n}}\right) = g^{\mu_1\mu_2}\,\mathrm{Tr}\left(\gamma^{\mu_3}\ldots\gamma^{\mu_{2n}}\right) - g^{\mu_1\mu_3}\,\mathrm{Tr}\left(\gamma^{\mu_2}\gamma^{\mu_4}\ldots\gamma^{\mu_{2n}}\right) + \cdots + g^{\mu_1\mu_n}\,\mathrm{Tr}\left(\gamma^{\mu_2}\ldots\gamma^{\mu_{2n-1}}\right) \tag{5.12}$$

$$\mathrm{Tr}\left(\gamma^{\mu_1}\ldots\gamma^{\mu_{2n+1}}\right) = 0 \tag{5.13}$$

$$\mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b \tag{5.14}$$

$$\mathrm{Tr}(\not{a}\not{b}\not{c}\not{d}) = 4(a \cdot bc \cdot d - a \cdot cb \cdot d + a \cdot db \cdot c) \tag{5.15}$$

$$\gamma^\alpha\gamma^\mu\gamma_\alpha = -2\gamma^\mu \tag{5.16}$$

$$\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha = 4g^{\mu\nu} \tag{5.17}$$

$$\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha = -2\gamma^\rho\gamma^\nu\gamma^\mu \tag{5.18}$$

Remark 5.2 As the γ matrices are 4×4, the wave solution of the Dirac equation ϕ has 4 components. However, it is important to note that this ψ is *not* a vector due to it not transforming under general coordinate transformations. Rather, it is a *spinor* which one can better understand by reading the companion book *Spinors & Symmetries*. While it is possible to simply regard ψ as a 4-vector in some regards, doing so would be quite morally questionable.

Derivation 5.2 (Dirac equation general solution) Despite ψ being a spinor[a], a general plane wave solution is again of the form $\psi = ue^{-ip\cdot x}$. From this and the Klein-Gordon on-shell condition, the eigenvalue equation is

$$(\not{p} - m)u = 0 \tag{5.19}$$

The slashed momentum matrix is of the form

$$\not{p} = \begin{pmatrix} p^0\mathbb{I}_2 & p\cdot\sigma \\ p\cdot\sigma & -p^0\mathbb{I}_2 \end{pmatrix} \tag{5.20}$$

where p is the 3-vector and σ is a 3D vector whose elements are the Pauli matrices.
Remark 5.3 Note that we are still living in momentum space.
The eigenspinors, called *Dirac spinors*, are

$$u_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{p\cdot\sigma}{E_p+m}\chi_s \end{pmatrix} \quad v_s(p) = \sqrt{E_p + m}\begin{pmatrix} \chi_s \\ \frac{p\cdot\sigma}{E_p+m}\chi_s \end{pmatrix} \tag{5.21}$$

for $s = 1, 2$. χ_s are 2-component spinors (or so-called *Weyl spinors*), and are defined by

$$\chi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.22}$$

By noting that

$$(p\cdot\sigma)^2 = |p|^2 = E_p^2 - m^2 \tag{5.23}$$

it can be seen that the eigenvectors are normalised as

$$u_r^\dagger(p)u_s(p) = v_r^\dagger(p)v_s(p) = \delta_{rs}2E_p \tag{5.24}$$

As $\not{p}$ is not Hermitian, the eigenvectors are not orthogonal. However, we do have a metric relation that is quite similar to orthogonality

$$u_r^\dagger(p)v_s(-p) = 0 \tag{5.25}$$

Equation 5.24 and Equation 5.25 can be represented in Lorentz-invariant form:

$$\bar{u}_r(p)u_s(p) = -\bar{v}_r(p)\bar{v}_s(p) = \sigma_{rs}2m \quad \bar{u}_r(p)v_s(p) = 0 \tag{5.26}$$

where the bar on top denotes the *Dirac adjoint*:

Definition 5.3 (Dirac adjoint) The Dirac adjoint $\bar{\psi}$ of some ψ is

$$\bar{\psi} = \psi^\dagger\gamma^0 \tag{5.27}$$

Finally, we are in a position to write down the full general solution of the Dirac equation, which represents fermions like electrons and positions:

Theorem 5.4 (Dirac equation general solution) The general solution of the Dirac equation is a spinor field known as the *Dirac field*:

$$\psi = \int d^3p \sum_{s=1}^{2}(b_s(p)u_s(p)e^{-ip\cdot x} + d_s(p)v_s(p)e^{ip\cdot x}) \tag{5.28}$$

where $b_s(p)$ and $d_s(p)$ are 4 constants.

The Dirac field is a so-called *Grassmann-valued field*, which is a field whose components take values in a Grassmann algebra. Recalling Grassmann mathematics from *Spinors & Symmetries*, one can realise that the components of Grassmann fields anticommute rather than commute.
From their Dirac equation, spinors $u_s(p)$ and $v_s(p)$ further satisfy

$$\sum_s u_s(p)\bar{u}_s(p) = \not{p} + m \quad \sum_s v_s(p)\bar{v}_s(p) = \not{p} - m \tag{5.29}$$

Remark 5.4 The on-shell condition is enforced by $p^0 = E_p$. The $u_s(p)$ term accounts for positive energy solutions which represent particles, while the $v_s(p)$ term accounts for negative energy solutions which represent antiparticles[b].

[a] The horror!
[b] Here, the emergence of antiparticles have justified the negative energy solutions as physical.

We can now write down the Lagrangian and Hamiltonian densities for the (free field) Dirac equation:

Definition 5.4 (Dirac Lagrangian)

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \tag{5.30}$$

which, by noting the Dirac equation itself, is always zero.

As we will soon see, this innocent Dirac field is not alone. Rather, it is coupled to a (scalar) photon field A^μ that we will see later. In the presence of this scalar field, Our slashed partial derivative $\not{\partial}$ must be rewritten as $\not{D}$. Its second term is modified from that of the generalised covariant derivative by the charge e, effectively the coupling constant:

$$D_\mu = \partial_\mu + ieA_\mu \tag{5.31}$$

Let us write out the covariant derivative explicitly. The Lagrangian is then

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi = \underbrace{\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi}_{\text{Dirac field}} - \underbrace{e\bar{\psi}\gamma^\mu\psi A_\mu}_{\text{interaction}} \tag{5.32}$$

where the interaction term denotes interaction with the photon field A^μ. It replaces the interaction $= J^\mu A_\mu$ in the classical Maxwell Lagrangian we will see in Equation 5.108. If further fields (e.g. proton fields) are added, this changes to

$$\mathcal{L} = \sum_f \bar{\psi}(i\gamma^\mu(\partial_\mu + ie_f A_\mu) - m_f)\psi \tag{5.33}$$

where f is an index representing the range of fermions we are concerned with.
Sticking to the simpler case, it then follows that

Definition 5.5 (Dirac Hamiltonian)

$$\mathcal{H} = i\psi^\dagger\partial_0\psi - \mathcal{L} = i\psi^\dagger\partial_0\psi = \bar{\psi}(-i\gamma^j\partial_j + m)\psi \tag{5.34}$$

Remark 5.5 The canonical momentum is $\pi = i\psi^\dagger$ by dint of $\gamma^0\gamma^0 = \mathbb{I}_4$.

5.2 Story of a spinor

Previously, we have said that spinors, unlike tensors, do not undergo general coordinate transformations. Instead, they undergo rotation-like transformations defined by Lie groups[2]. In the case of the Dirac equation solutions, the rotation transformations are defined by the SU(2) group, whose representation is

$$\hat{S}_i = \frac{1}{2}\Sigma_i = \frac{1}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix} \tag{5.35}$$

which is actually a 3-vector whose components are operators. We have the following commutation relations

$$[\hat{S}_i, H] = i\epsilon_{ijk}p^i\gamma_j \tag{5.36}$$

Such a commutation relation does not preserve Lorentz invariance. However, if we recall LS coupling, we will remember that the *total* angular momentum is both the orbital angular momentum $\hat{L}$ and the spin $\hat{S}$, and it just so happens that there is the following commutation relation

$$[\hat{L} + \hat{S}_i, H] = 0 \tag{5.37}$$

from which we recover the Lorentz invariance of the total angular momentum.

Derivation 5.3 (Lorentz group generators) Given a Lorentz transformation $x \to x' = \Lambda x$, we expect the Dirac equation to be Lorentz-invariant:

$$(i\gamma^\mu\partial_\mu - m)\psi(x) \to (i\gamma^\mu\partial'_\mu - m)\psi'(x') \tag{5.38}$$

[2]Again see *Spinors & Symmetries*.

where the wavefunction transforms according to the (internal) spinor transformation $T(\Lambda)$, defined by

$$\psi'(x') = T(\Lambda)\psi(x) = T(\Lambda)\psi(\Lambda^{-1}x') \tag{5.39}$$

Now we try to determine $T(\Lambda)$. One can write out the transformation in terms of indices:

$$x'^{\mu} = \Lambda^{\nu}_{\mu} x^{\mu} \quad \partial_{\mu} \Lambda^{\nu}_{\mu} \partial'_{\nu} \tag{5.40}$$

The original, untransformed and final, transformed equations can then be written as

$$(i\gamma^{\nu}\Lambda^{\mu}_{\nu}\partial'_{\nu} - m)\psi(\Lambda^{-1}x') = 0 \quad T\Lambda^{-1}(i\gamma^{\mu}\partial'_{\mu} - m)T(\Lambda)\psi(\Lambda^{-1}x') = 0 \tag{5.41}$$

By equating the two[a], one finds

$$T(\Lambda)^{-1}\gamma^{\mu}T(\Lambda) = \lambda^{\mu}_{\nu}\gamma^{\nu} \tag{5.42}$$

Using the parameterisation of generators in *Spinors & Symmetries*, an infinitesimal Lorentz transformation may be parameterised as

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} - \omega_{\rho\sigma}(g^{\rho\nu}\delta^{\sigma}_{\nu} - g^{\sigma\mu}\delta^{\rho}_{\nu}) + o(\omega) \tag{5.43}$$

and a transformation may be given as

$$T(\Lambda) = e^{i\omega_{\rho\sigma}s^{\rho\sigma}} = \mathbb{I}_4 + i\omega_{\rho\sigma}s^{\rho\sigma} + o(\omega) \tag{5.44}$$

where $s^{\rho\sigma}$ are the all-too-familiar generators that we have parameterised via $\omega_{\rho\sigma}$ in *Spinors & Symmetries*. Plugging the two expressions into Equation 5.42 gives

$$i[s^{\rho\sigma}, \gamma^{\mu}] = g^{\rho\mu}\gamma^{\sigma} - g^{\sigma\mu}\gamma^{\rho} \tag{5.45}$$

which surprisingly reduces, via (anti)commutation relations to the simple expression

Definition 5.6 (Lorentz group generators) The following generators define the spinor representation of the Lorentz group

$$s^{\rho\sigma} = \frac{i}{4}[\gamma^{\rho}, \gamma^{\sigma}] \tag{5.46}$$

[a] One can do so as both equations apply for all ψs.

Remark 5.6 We conclude by saying that the transformations are the representations $T(\Lambda) = e^{i\omega_{\rho\sigma}s^{\rho\sigma}}$.

Definition 5.7 (Proper and improper Lorentz transformations) There exists two kinds of Lorentz transformations:

- *Proper Lorentz transformations* have matrices with determinant 1.
- *Improper Lorentz transformations* have matrices with determinant -1.

Remark 5.7 For example, the parity and time reversal Lorentz transformations Λ_P and Λ_T are improper:

$$\Lambda_P = \mathrm{diag}(1, -1, -1, -1) \quad \Lambda_T = \mathrm{diag}(-1, 1, 1, 1) \tag{5.47}$$

One can find more improper transformations by multiplying proper ones with them. Their spinorial representations are

$$T(\Lambda_P) = \gamma^0 \quad T(\Lambda_T)\psi(x) = -\gamma^1\gamma^3\psi(\Lambda_T x)^* \tag{5.48}$$

Some generalised 'not-so-physical' models can now be discussed. They are less useful in QED than they are in QCD[3], so we will see them again.
If one considers only self-interaction, the most generalised model will be, up to the 4$^{\text{th}}$ order, the so-called *linear sigma model* or the *linear σ model*.
The linear σ model Lagrangian is

[3] The linear σ model is a simple toy model for chiral symmetry breaking in QCD, while Yukawa theory serves as a simple analogue for strong interactions.

Definition 5.8 (Linear σ Lagrangian)

$$\mathcal{L} = \underbrace{\frac{1}{2}\bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m_\phi^2\phi^2}_{①} - \underbrace{\frac{\xi\phi^3}{3!}}_{②} - \underbrace{\frac{\lambda\phi^4}{4!}}_{③} \tag{5.49}$$

Noting that:

- ① are the free terms or the so-called *kinetic terms.*
- ② is the ϕ^3 theory self-interaction term.
- ③ is the ϕ^4 theory self-interaction term.

Remark 5.8 The linear σ model is parity-invariant if and only if ϕ is a scalar. If ϕ is a pseudoscalar, parity is broken by the cubic term unless its coupling constant ξ is zero. The fermion transformation under parity follows $P\psi = \eta\gamma^0\psi$, where η is a phase factor.
We now add a new term that describes the interaction between a scalar field ϕ and a Dirac fermion field ψ. This results in a Lagrangian used in the so-called *Yukawa interaction* or *Yukawa coupling*, which takes the form

Definition 5.9 (Yukawa interaction Lagrangian)

$$\mathcal{L} = \frac{1}{2}\bar{\psi}(i\partial\!\!\!/ - m_\psi)\psi + \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m_\phi^2\phi^2 - \frac{\xi\phi^3}{3!} - \frac{\lambda\phi^4}{4!} - g\phi\bar{\psi}_i\Gamma_{ij}\psi_j \tag{5.50}$$

where the final term describes the coupling between ϕ and ψ.

We now want to determine Γ_{ij}. Recall that all quantum field theories within the standard model must be Lorentz-invariant. This can only be satisfied if this final interaction term (or effectively, the part $\bar{\psi}_i\Gamma_{ij}\psi_j$) transforms as a scalar. Hence, two candidates exist for Γ_{ij}:

- The unit matrix $\mathbb{I}$, under which $\bar{\psi}_i\Gamma_{ij}\psi_j$ transforms as a scalar.
- $i\gamma^5$, where γ^5 is the previously teased γ^5 matrix. Under it, $\bar{\psi}_i\Gamma_{ij}\psi_j$ transforms as a pseudoscalar.

Definition 5.10 (γ^5 matrix)

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{5.51}$$

Theorem 5.5 (γ^5 matrix properties) The γ^5 matrix satisfies

$$\gamma^{5\dagger} = \gamma^5 \quad \{\gamma^5, \gamma^\mu\} = 0 \quad (\gamma^5)^2 = 1 \tag{5.52}$$

Finally, γ^5 changes sign for improper Lorentz transformations. e.g.

$$\gamma^5\gamma^0 = -\gamma^0\gamma^5 \tag{5.53}$$

With the γ^5 finally set up, we can define the basis for all possible γ matrix products, which consists of a lofty 16 matrices:

$$\{1, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, \sigma^{\mu\nu}\} \tag{5.54}$$

Here $\sigma^{\mu\nu}$ is defined as

$$\sigma^{\mu\nu} = 2s^{\mu\nu} \tag{5.55}$$

in which $s^{\mu\nu}$ is the Lorentz group generators defined in Equation 5.46.

5.3 Discrete symmetries and the CPT theorem

This puts us in an intuitive position to discuss the so-called *bilinears*, which are useful for defining quantities with particular properties under Lorentz transformations and appear in Lagrangians for fermion field theories.

Derivation 5.4 (Bilinears) Recall from *Spinors & Symmetries* that mathematically, a bilinear is a map that is linear in both its arguments.
For Lorentz groups, these 2 arguments are ψ and $\bar{\psi}$, and a Lorentz group bilinear is essentially a map which sandwiches some quantity between $\bar{\psi}$ and ψ and returns a result that transforms in a certain way.
There are a total of 16 Lorentz group bilinears. Depending on the quantity we sandwich with $\bar{\psi}$ and ψ, we can categorise them according to their transformation properties:

- **Scalar:** This quantity remains unchanged under any Lorentz transformation. An example is the mass m.
$$\bar{\psi}\psi \to \bar{\psi}\psi \tag{5.56}$$
- **Pseudoscalar**[a]**:** This quantity accounts for parity by changing signs under spatial inversion (parity transformation). It is associated with chirality.
$$\bar{\psi}\gamma^5\psi \to \det(\Lambda)\bar{\psi}\gamma^5\psi \tag{5.57}$$
- **Vector:** This quantity transforms as a 4-vector under Lorentz transformations.
$$\bar{\psi}\gamma^\mu\psi \to \Lambda^\mu_\nu\bar{\psi}\gamma^\mu\psi \tag{5.58}$$
It can represent conserved currents[b] and obeys the continuity equation to enforce charge conservation:
$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \tag{5.59}$$
- **Pseudovector:** This quantity transforms identically to a 4-vector, except that it switches sign under parity. It appears in theories involving axial currents[c] and chiral symmetry.
$$\bar{\psi}\gamma^\mu\gamma^5\psi \to \det(\Lambda)\Lambda^\mu_\nu\bar{\psi}\gamma^\mu\gamma^5\psi \tag{5.60}$$
- **Rank-2 tensor:** This quantity transforms as a rank-2 antisymmetric tensor.
$$\bar{\psi}\sigma^{\mu\nu}\psi \to \Lambda^\nu_\lambda\Lambda^\nu_\sigma\bar{\psi}\sigma^{\lambda\sigma}\psi \tag{5.61}$$

We can verify that all quantities are linear in both arguments, which is skippable if taken for granted:

- **Scalar:**
$$\bar{\psi}(a\psi_1 + b\psi_2) = a\bar{\psi}\psi_1 + b\bar{\psi}\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\psi = a\bar{\psi}_1\gamma^5\psi + b\bar{\psi}_2\psi \tag{5.62}$$
- **Pseudoscalar:**
$$\bar{\psi}(a\psi_1 + b\psi_2)\gamma^5 = a\bar{\psi}\gamma^5\psi_1 + b\bar{\psi}\gamma^5\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\gamma^5\psi = a\bar{\psi}_1\gamma^5\psi + b\bar{\psi}_2\gamma^5\psi \tag{5.63}$$
- **Vector:**
$$\bar{\psi}(a\psi_1 + b\psi_2)\gamma^\mu = a\bar{\psi}\gamma^\mu\psi_1 + b\bar{\psi}\gamma^\mu\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\gamma^\mu\psi = a\bar{\psi}_1\gamma^\mu\psi + b\bar{\psi}_2\gamma^\mu\psi \tag{5.64}$$
- **Pseudovector:**
$$\bar{\psi}(a\psi_1 + b\psi_2)\gamma^\mu\gamma^5 = a\bar{\psi}\gamma^\mu\gamma^5\psi_1 + b\bar{\psi}\gamma^\mu\gamma^5\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\gamma^\mu\gamma^5\psi = a\bar{\psi}_1\gamma^\mu\gamma^5\psi + b\bar{\psi}_2\gamma^\mu\gamma^5\psi \tag{5.65}$$
- **Rank-2 tensor:**
$$\bar{\psi}(a\psi_1 + b\psi_2)\sigma^{\mu\nu} = a\bar{\psi}\sigma^{\mu\nu}\psi_1 + b\bar{\psi}\sigma^{\mu\nu}\psi_2 \quad (a\bar{\psi}_1 + b\bar{\psi}_2)\sigma^{\mu\nu}\psi = a\bar{\psi}_1\sigma^{\mu\nu}\psi + b\bar{\psi}_2\sigma^{\mu\nu}\psi \tag{5.66}$$

[a] So-called as it switches sign under parity, *not* because it has zero rank (which it doesn't).
[b] In fact, the electromagnetic 4-current can be written as $J^\mu = \bar{\psi}\gamma^\mu\psi$.
[c] As such, it is also called a *axial vector*. For example, the axial current in weak interactions is $\bar{\psi}\gamma^\mu\gamma^5\psi$ and plays a role in describing the *handedness* of particles.

A famous theorem is then the so-called *CPT* (charge, parity, time) *theorem*, also known with an alternate initial ordering as the *PCT theorem*. All three are symmetries of the Dirac field.

> **Quote 5.2** Yes, it's the oldschool order.
>
> *Felix Halbwedl, on the alternate initial order 'PCT', 3 January 2025*

Previously, we have seen transformations like Lorentz transformations, which include rotations and boosts. These transformations (or indeed, any transformation that form a Lie group) is known as a *continuous transformation*. Meanwhile, all three CPT transformations are discrete symmetries, which involve specific finite changes that cannot be continuously connected to the identity transformation.

Definition 5.11 (Parity operation) P stands for the *parity operation* P, which changes the 3-position to its inverse:

$$P : (t, \mathbf{x}) \to (t, -\mathbf{x}) \tag{5.67}$$

which leads to

$$\mathbf{p} \to -\mathbf{p} \quad \mathbf{l} \to -\mathbf{l} \tag{5.68}$$

where $\mathbf{p}$ and $\mathbf{l}$ are the *spatial components* of the momentum and angular momentum.

Theorem 5.6 (Transformations under P)

- Scalars ψ are invariant under P:

  $$P\psi P^{-1} = \psi \tag{5.69}$$

  Thus, the intrinsic parity of a scalar field is usually defined as $P(S) = +1$.

- Pseudoscalars ψ_p change sign under P:

  $$P\psi_p P^{-1} = -\psi_p \tag{5.70}$$

  Thus, the intrinsic parity of a pseudoscalar is $P(\mathcal{P}) = -1$.

- Vectors T^μ transform like spacetime coordinates. Under P:

  $$PV^0P^{-1} = V^0 \quad PV^iP^{-1} = -V^i \tag{5.71}$$

  As the spatial components flip sign, the intrinsic parity of a vector field is typically $P(V) = -1$ if it describes an interacting particle like a photon.

- Pseudovectors A^μ do not transform like normal vectors under P but instead gain an additional sign:

  $$PA^0P^{-1} = -A^0 \quad PA^iP^{-1} = A^i \tag{5.72}$$

  As the temporal component flips sign, the intrinsic parity of a pseudovector field is $P(A) = +1$.

- Rank-2 tensors $T^{\mu\nu}$ transform under P depending on its indices:

  $$PT^{00}P^{-1} = T^{00} \quad PT^{0i}P^{-1} = -T^{0i} \quad PT^{ij}P^{-1} = T^{ij} \tag{5.73}$$

  The intrinsic parity depends on the context.

Definition 5.12 (Time reversal) T stands for the so-called *time reversal* T, which flips the time coordinate:

$$T : (t, \mathbf{x}) \to (-t, \mathbf{x}) \tag{5.74}$$

Theorem 5.7 (Transformations under T)

- Scalars ψ are invariant under T:

  $$T\psi T^{-1} = \psi \tag{5.75}$$

- Pseudoscalars ψ_p are also invariant under T:

$$T\phi_p T^{-1} = \phi_p \tag{5.76}$$

- Vectors V^μ transform like spacetime coordinates. Under T:

$$TV^0T^{-1} = V^0 \quad TV^iT^{-1} = -V^i \tag{5.77}$$

- Pseudovectors A^μ transform in the opposite way to vectors:

$$TA^0T^{-1} = -A^0 \quad TA^iT^{-1} = A^i \tag{5.78}$$

- Rank-2 tensor transformations vary by tensor under T:
  - The metric is invariant.
  - The energy-momentum tensor follows

$$TT^{00}T^{-1} = T^{00} \quad TT^{0i}T^{-1} = -T^{0i} \quad TT^{ij}T^{-1} = T^{ij} \tag{5.79}$$

  - The Faraday tensor follows

$$TE^iT^{-1} = E^i \quad TB^iT^{-1} = -B^i \tag{5.80}$$

  Only the magnetic field flips sign, as magnetic fields originate from moving charges, and motion reverses under T.

Definition 5.13 (Charge conjugation) C stands for *charge conjugation* C, which is the sign-flip of all charges:

$$C : Q \to -Q \tag{5.81}$$

Theorem 5.8 (Transformations under C)

- Scalars ψ change sign under C:

$$C\phi C^{-1} = \phi \tag{5.82}$$

- Pseudoscalars ψ_p are invariant under C:

$$C\phi_p C^{-1} = -\phi_p \tag{5.83}$$

- Vectors V^μ change sign under C:

$$CV^\mu C^{-1} = -V^\mu \tag{5.84}$$

- Pseudovectors A^μ are invariant under C:

$$CA^\mu C^{-1} = A^\mu \tag{5.85}$$

- Rank-2 tensors $T^{\mu\nu}$ change sign under C:

$$CF^{\mu\nu}C^{-1} = -F^{\mu\nu} \tag{5.86}$$

- Rank-2 pseudotensors $\tilde{F}^{\mu\nu}$ are invariant under C:

$$C\tilde{F}^{\mu\nu}C^{-1} = \tilde{F}^{\mu\nu} \tag{5.87}$$

Charge conjugation also has some interesting effects on spinors. Consider our good friend, the Dirac spinor ψ:

$$C\psi = -i\gamma_2\psi^* \to \bar{\psi} = (C\psi)^T\gamma_0 \tag{5.88}$$

This implies, for the left- and right-handed Weyl spinors ψ_{Weyl} and χ that make up the Dirac spinor:

$$C\psi_{\text{Weyl}} = -i\sigma_2\chi^* \quad C\chi = -i\bar{\sigma}_2\psi^*_{\text{Weyl}} \tag{5.89}$$

The second implication is that for a theory to be C invariant, it must contain:

- Left-handed Weyl spinors
- Right-handed complex conjugate of Weyl spinors

However, it is possible to get away with only one handedness. This is the so-called *Majorana*[4] *spinor*, which is technically a very specific subcategory of the Dirac spinor:

Definition 5.14 (Majorana spinor) A Majorana spinor is a Dirac spinor with the following construction

$$\psi = \begin{pmatrix} \psi_{\text{Weyl}} \\ -i\sigma_2\psi^*_{\text{Weyl}} \end{pmatrix} \tag{5.90}$$

Due to being a Dirac spinor, it transforms as one, with the key distinction of undergoing charge conjugation as

$$C\psi = \psi \tag{5.91}$$

meaning that a *Majorana fermion* is its own antiparticle. As a result, a Majorana particle is always charge neutral.

As seen in all three transformations, symmetry can be broken, which we call *symmetry breaking*. Two types of symmetry breaking exist:

- In *spontaneous symmetry breaking*, the equations of motion are invariant, but the ground state (vacuum) of the system is not.
- In *explicit symmetry breaking*, the equations of motion are not invariant.

The breaking of individual or two (e.g. CP, P, T) symmetries is not prohibited in QFT. However, due to Lorentz invariance, the breaking of CPT symmetry is disallowed in QFT. This is illustrated by the CPT theorem:

Theorem 5.9 (CPT theorem) Any quantum field theory that is Lorentz-invariant and has a well-defined local interaction must respect CPT symmetry. i.e. for some quantity H, the combination of charge conjugation, parity, and time reversal is always a symmetry:

$$(CPT)H(CPT)^{-1} = H \tag{5.92}$$

As mentioned, the CPT theorem ultimately results from Lorentz invariance. Hence, it cannot be spontaneously broken like gauge symmetries (e.g. electroweak symmetry). At the time of writing (2025), experimentalists have yet to observe CPT symmetry breaking, showing how well QFT has withstood the tests of time.
There is, of course, a more terrifying implication if we turn the first statement above backwards: If CPT were spontaneously broken, it would suggest a violation of Lorentz invariance, a common feature of physics beyond the standard model (BSM).

5.4 Helicity and charality

Definition 5.15 (Weyl basis) In the so-called *Weyl basis* or *chiral basis*, the γ^0 is slightly changed, while the others stay the same

$$\gamma^0_{\text{ch}} = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix} \quad \gamma^j_{\text{ch}} = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \tag{5.93}$$

[4] To this day, his demise remains a historical mystery. For more, see here.

Definition 5.16 (Helicity operator) The *helicity operator* is the spin operator projected in the direction of the momentum:

$$\hat{h} = \frac{\sigma_\mu \cdot p^\mu}{|p_\mu|} \tag{5.94}$$

Remark 5.9 For a Dirac spinor, the eigenvalues of the helicity operator are $h = \pm 1$.
Helicity is compatible with the symmetries of the Dirac equation, as such, the eigenstates of $\hat{h}$ should also be solutions to the Dirac equation, each representing the particle's solution in a different helicity.

Theorem 5.10 (Helicity eigenstates) For a massive spin-$\frac{1}{2}$ fermions which are free particles, the helicity eigenstates of matter propagating in the (θ, ϕ) direction are

$$u_\uparrow = \sqrt{E+m}\begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \\ \frac{|\vec{p}|}{E+m}\cos(\theta/2) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin(\theta/2) \end{pmatrix} \quad u_\downarrow = \sqrt{E+m}\begin{pmatrix} -\sin(\theta/2) \\ e^{i\phi}\cos(\theta/2) \\ \frac{|\vec{p}|}{E+m}\sin(\theta/2) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos(\theta/2) \end{pmatrix} \tag{5.95}$$

The states observe

$$\hat{h}u_\uparrow = +u_\uparrow \quad \hat{h}u_\downarrow = -u_\downarrow \tag{5.96}$$

For antimatter, the helicity operator $\hat{h}^v = -\hat{h}$ is the negative of its matter counterpart, and one has the eigenstates

$$v_\uparrow = \sqrt{E+m}\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin(\theta/2) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos(\theta/2) \\ -\sin(\theta/2) \\ e^{i\phi}\cos(\theta/2) \end{pmatrix} \quad v_\downarrow = \sqrt{E+m}\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos(\theta/2) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin(\theta/2) \\ \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix} \tag{5.97}$$

which observe

$$\hat{h}^v v_\uparrow = +v_\uparrow \quad \hat{h}^v v_\downarrow = -v_\downarrow \tag{5.98}$$

Definition 5.17 (Chirality) Chirality (from 'hand' in Greek) denotes the *handedness* of a particle:

- When $h = 1$, helicity is positive as the particle's spin direction is the same as its direction of motion. Chirality is *left-handed.* This orientation is so-called as it observes the right-hand rule: Align the right thumb in the momentum's direction, and the curled fingers should align with the spin direction.
- When $h = -1$, helicity is negative as the particle's spin direction is opposite from its direction of motion. Chirality is *right-handed.* This orientation is so-called as it observes the left-hand rule: Align the left thumb in the momentum's direction, and the curled fingers should align with the spin direction.

Chirality is conserved by $\bar{u}\gamma^\mu u$. In the massless limit, chirality is identical to helicity.

5.5 Quantisation of the Dirac field

Having acquired the the Dirac field as the general solution of the Dirac equation in Equation 5.28, we now attempt to quantise it, which allows us to quantise several significant quantities using it. In quantising the Klein-Gordon equation, we replaced f_p and f_p^* with creation and annihilation operators (see Derivation 3.2). Here we start less ambitiously. We order that $b_s(p)$ become an operator. For $d_s(p)$, we do the same but replace it with $d_s^\dagger(p)$. The general solution then reads

$$\psi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} (b_s(p)u_s(p)e^{-ip\cdot x} + d_s^\dagger(p)v_s(p)e^{ip\cdot x}) \tag{5.99}$$

where again, for the sake of convenient normalisations, we have added a normalisation factor of $\frac{1}{(2\pi)^3\sqrt{2E_p}}$, previously seen in Equation 3.22. The canonical momentum can likewise be found:

Definition 5.18 (Dirac equation canonical momentum)

$$\pi = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} (b_s^\dagger(p) u_s^\dagger(p) e^{ip\cdot x} + d_s(p) v_s^\dagger(p) e^{-ip\cdot x}) \tag{5.100}$$

Before mindlessly assigning the bosonic commutation relations to $b_s(p)$ and $d_s(p)$, we stop for a moment and realise that the Dirac field is not actually a bosonic field. The solution to the Dirac equation is not a scalar, but a spinor. Hence, it describes spin-$\frac{1}{2}$ particles[5] and is a fermionic field instead. As such, we instead impose the following *fermionic anticommutation relations* for some indices i and j:

Theorem 5.11 (Fermionic creation and annihilation operator anticommutations)

$$\{b_i(p), b_j^\dagger(q)\} = \{d_i(p), d_j^\dagger(q)\} = \delta_{ij}(2\pi)^2\delta^3(p-q) \tag{5.101}$$

$$\{b_i(p), b_j(q)\} = \{d_i(p), d_j(q)\} = \{b_i(p), d_j(q)\} = 0 \tag{5.102}$$

Suddenly recalling the useful relation in Equation 5.29 for no reason whatsoever, we can rewrite it using the definition of the Dirac adjoint in Equation 5.27 as

$$\sum_s u_s(p)_\alpha u_s^\dagger(p)_\beta = (E_p + m)\delta_{\alpha\beta} - (p\cdot\gamma\gamma^0)_{\alpha\beta} \qquad \sum_s v_s(p)_\alpha v_s^\dagger(p)_\beta = (E_p - m)\delta_{\alpha\beta} - (p\cdot\gamma\gamma^0)_{\alpha\beta} \tag{5.103}$$

Putting Equation 5.100, Equation 5.101, Equation 5.102 and Equation 5.103 all together, we can solve for the commutator $[\phi_\alpha(x), \pi_\beta(y)]$ and find the following anticommutation relation

Theorem 5.12 (Fermionic field and momentum operator anticommutations)

$$\{\psi_\alpha(x), \pi_\beta(y)\} = i\delta_{\alpha\beta}\delta^3(x-y) \tag{5.104}$$

$$\{\psi_\alpha(x), \psi_\beta(y)\} = \{\pi_\alpha(x), \pi_\beta(y)\} = 0 \tag{5.105}$$

By integrating Equation 5.34 and then using Equation 5.24, we can find the normal-ordered Hamiltonian

Definition 5.19 (Dirac equation normal-ordered Hamiltonian)

$$H = \int \frac{d^3p}{(2\pi)^3} E_p \sum_{s=1}^{2} (b_s^\dagger(p) b_s(p) + d_s^\dagger(p) d_s(p)) \tag{5.106}$$

We can likewise find the charge by integrating the 0^{th} component 4-current J^0, given by the vector bilinear in Equation 5.58 as $\psi^\dagger\psi$:

$$Q = e\int d^3x {:}\psi^\dagger\psi{:} = e\int \frac{d^3p}{(2\pi)^3} \sum_{s=1}^{2} (b_s^\dagger(p) b_s(p) - d_s^\dagger(p) d_s(p)) \tag{5.107}$$

where the normal ordering is used to ensure the result is physical.
Remark 5.10 From this, one can verify a particle and its antiparticle carry equal but opposite charges.

5.6 Quantisation of the electromagnetic field

Derivation 5.5 (Recovery of Maxwell's equations) By setting the (ultimately zero) Lagrangian density as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu \tag{5.108}$$

where $-J^\mu A_\mu$ is an interaction term.
The Faraday tensor in terms of the 4-potential is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{5.109}$$

[5]Here we see why a spinor is often called a 'rank-half tensor'.

Inserting this, we can apply the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0 \tag{5.110}$$

where A^ν, which we ultimately recognise as a field variable, replaces ψ.
We then recover Maxwell's equations in index notation

Theorem 5.13 (Maxwell's equations)

$$\partial_\mu F^{\nu\mu} = J^\nu \tag{5.111}$$

In gauge theory, there exist physical observables that are invariant under certain transformations of the potentials. In the case of electromagnetism, this manifests in the invariance of the Faraday tensor $F^{\nu\mu}$ (and hence, the EM fields E and B) under the following gauge transformation of the 4-potential[6]. This is brought about by the U(1) transformations:

Definition 5.20 (U(1) transformations)

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{5.112}$$

where $\Lambda(x)$ is a scalar field and satisfies the wave equation $\partial_\mu \partial^\mu = 0$.

This gauge invariance of $F^{\nu\mu}$ leads to one redundant degree of freedom, called a *gauge freedom*: an infinity of A^μs corresponds to the same set of E and B. If such redundancies are not eliminated, calculations involving A^μ will mistakenly count multiple configurations of A^μs as distinct and result in erroneous results.

Derivation 5.6 (Fixing the gauge) The standard procedure to eliminate gauge freedoms is *gauge fixing*. In classical EM, we attempt to eliminate this gauge freedom in Equation 5.112 by the Lorentz-invariant Lorenz gauge.

> **Quote 5.3** Amazingly, the missing "t" is not a typo here.
>
> *Alessio Serafini*

$$\partial_\mu A^\mu = 0 \tag{5.113}$$

The quantum analogue of the Lorenz gauge is the R_ξ Landau gauge. We begin with the photon field Lagrangian, which is

Definition 5.21 (Photon field Lagrangian)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{5.114}$$

The R_ξ Landau gauge adds a gauge-fixing term to the Lagrangian. This is also known as the *Gupta-Bleuler formalism*:

Theorem 5.14 (R_ξ Landau gauge)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \tag{5.115}$$

where ξ is a parameter.

The addition of this term is justified by the fact that we can add or subtract terms that are zero or boundary terms to/from a Lagrangian arbitrarily. In the Lorenz gauge, the gauge fixing term in the R_ξ Landau gauge is zero, and satisfies this condition.
The simplest R_ξ Landau gauge is the Feynman-'t Hooft gauge, which is used in most QFT calculations. In this gauge, one has $\xi = 1$, and the Lagrangian becomes

[6] One can verify this by noting that the conditions $\frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0}$ (where $F^{\mu 0}$ are the canonically conjugate EM fields) and $\frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = 0$ simultaneously hold.

Theorem 5.15 (Feynman-'t Hooft gauge)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{5.116}$$

We can write the Faraday tensors in the first term in terms of the 4-potential:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu \tag{5.117}$$

The term $-\frac{1}{2}(\partial_\mu A^\mu)^2$ can be written as $-\frac{1}{2}\partial_\nu A^\mu \partial_\mu A^\nu$. From differential geometry, we know that it is equivalent to $-\frac{1}{2}\partial_\mu A_\nu \partial^\nu A^\mu$ from a trivial manipulation of indices. The photon field Lagrangian, arising from electromagnetism, hence reduces to

Definition 5.22 (Photon field Lagrangian under the Feynman-'t Hooft gauge)

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A^\nu \partial^\mu A^\nu \tag{5.118}$$

where we see that the Lagrangian is now diagonal in terms of the field derivatives, indicating that the gauge has removed any redundant degrees of freedom.

Fun fact 5.1 We can also use an alternative 4-potential, comprising of the magnetic scalar potential and the electric vector potential. However, this is rarely used due to the absence of observed magnetic monopoles. We can now quantise the 4-potential, which is a real, massless Klein-Gordon (scalar) field. The classical wave solution is

$$A^\mu(x) = \int d^3p \sum_{\lambda=0}^{3} (\epsilon^\mu_\lambda(p) f_\lambda(p) e^{-ip\cdot x} + \epsilon^{\mu *}_\lambda(p) f^*_\lambda(p) e^{ip\cdot x}) \tag{5.119}$$

where ϵ^μ is a *polarisation vector*, a 4-versor[7]. Significantly, it holds the following summation property:

Theorem 5.16 (Polarisation vector property)

$$\sum_{\lambda=0}^{d} \epsilon^{\nu *}_\lambda(p)\epsilon^\mu_\lambda(p) = -g^{\mu\nu} \tag{5.120}$$

where the index λ labels the polarisation states of the photon.

One notes this to quite resemble the *tetrad fields* in general relativity.

Derivation 5.7 (Polarisation and the Lorentz gauge) For a photon, there are four possible indices, but not all are physical:

- $\lambda = 0$: This is a *longitudinal polarisation* that is often unphysical.
- $\lambda = 1, 2$: They are the two physical *transverse polarisations*[a] of the photon.
- $\lambda = 3$: This is a *scalar polarisation* that is also often unphysical.

As photons in QED are gauge bosons, the choice of polarisation vectors is not unique. This gauge freedom can be removed by applying Lorenz gauge:

$$p_\mu \epsilon^\mu_\lambda = 0 \tag{5.121}$$

where p^μ is the photon's 4-momentum.
In doing so, the two unphysical components of the 4-versor have been eliminated due to them being unphysical under the gauge, and only the two transverse polarisation states remain physical for photons.

[a] i.e. they are perpendicular to the direction of propagation and to each other.

[7] One can recall from *Spinors & Symmetries* that a versor is simply a unit quaternion (i.e. it has norm 1). In simpler terms, a versor is simply a vector whose magnitude is unity (i.e. 1).

Using the same procedure we have done before, we insert the normalisation factor into Equation 5.122 and replace $f_\lambda(p)$ and $f^*_\lambda(p)$ with creation and annihilation operators

Definition 5.23 (Photon field 4-potential)

$$A^\mu(x) = \int \frac{d^3p}{(2\pi^3)\sqrt{2E_p}} \sum_{\lambda=0}^{3} (\epsilon^\mu_\lambda(p) a_\lambda(p) e^{-ip\cdot x} + \epsilon^{\mu *}_\lambda(p) a^\dagger_\lambda(p) e^{ip\cdot x}) \tag{5.122}$$

where the creation and annihilation operators observe the commutation relation

$$[a_\lambda(p), a^\dagger_{\lambda'}(q)] = -g_{\lambda\lambda'}(2\pi)^3\delta^3(p-q) \tag{5.123}$$

We then derive the canonical momentum of a photon field. Noting from previously that it satisfies

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} \tag{5.124}$$

we can further say that, by inserting Equation 5.114

$$\pi^\mu = \partial_0 A^\mu \tag{5.125}$$

Finally, plugging in Equation 5.122 gives

Definition 5.24 (Photon field canonical momentum)

$$\pi^\mu = -\int \frac{d^3p}{(2\pi^3)\sqrt{2E_p}} \sum_{\lambda=0}^{3} (\epsilon^{\mu *}_\lambda(p) a^\dagger_\lambda(p) e^{ip\cdot x} - \epsilon^\mu_\lambda(p) a_\lambda(p) e^{-ip\cdot x}) \tag{5.126}$$

Again we consider the nature of the photon. It has spin 1, and is thus a boson. The standard bosonic commutations thus apply:

$$[A^\mu(x), \pi^\nu(u)] = -ig^{\mu\nu}\delta^3(x-y) \tag{5.127}$$

$$[A^\mu(x), A^\nu(u)] = [\pi^\mu(x), \pi^\nu(u)] = 0 \tag{5.128}$$

One final loose end is the Hamiltonian. By inserting Equation 5.125 into Equation 2.36 (where, notably, the field is A^μ instead of ϕ), the photon field Hamiltonian reads

Definition 5.25 (Photon field Hamiltonian)

$$\mathcal{H} = \frac{1}{2}\dot{A}^\nu \dot{A}_\nu + \frac{1}{2} D A^\nu D A_\nu \tag{5.129}$$

5.7 QED Feynman rules

We now begin constructing the QED Lagrangian by coupling the field Lagrangian from the Dirac equation general solution to the photon field Lagrangian, seen in Equation 5.32 and Equation 5.114 respectively[8]. This allows us to write the QED Lagrangian as

Definition 5.26 (QED Lagrangian)

$$\mathcal{L} = \bar{\psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.130}$$

We now include the gauge-fixing term and write down the full QED Lagrangian by consulting Equation 5.31, with each term labelled:

$$\mathcal{L}_{\text{QED}} = \underbrace{\bar{\psi}(i\gamma^\mu\partial - m)\psi}_{\text{free fermion}} - \underbrace{e\bar{\psi}\gamma^\mu A_\mu \psi}_{\text{interaction vertex}} \underbrace{- \frac{1}{2}A^\mu(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)A^\nu}_{\text{free photon}} + \underbrace{\frac{1}{2\xi}A^\mu\partial_\mu\partial_\nu A^\nu}_{\text{Landau gauge}} \tag{5.131}$$

[8]Note that they are not the free and interaction Lagrangians, as we will see almost immediately.

Remark 5.11 Those who are observant will see that we have used the pre-Feynman-'t Hooft gauge term $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$. This is because the term's incarnation after gauge fixing $-\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$ is not appropriate *before* fixing the gauge. We quantised the Dirac and photon fields separately merely so that we could insert the results. That is to say, we have not yet quantised QED at this point.

Derivation 5.8 (Fermionic field creation and annihilation operators) One can also represent the fermionic field creation and annihilation operators $b_s(p)$ and $d_s^\dagger(p)$ in terms of wavefunctions, much like the interacting field creation and annihilation operators in Equation 4.32:

$$-id_s^\dagger(p) = \int \frac{d^3x}{\sqrt{2E_p}} \frac{\bar{v}_s}{2m}(e^{-ip\cdot x}\partial_0\psi - \psi\partial_0 e^{-ip\cdot x}) \tag{5.132}$$

$$ib_s^\dagger(p) = \int \frac{d^3x}{\sqrt{2E_p}} (e^{-ip\cdot x}\partial_0\bar{\psi} - \bar{\psi}\partial_0 e^{-ip\cdot x})\frac{u_s}{2m} \tag{5.133}$$

where the factor of $1/2m$ arises from the normalisation condition in Equation 5.26. The photon field creation and annihilation operators can likewise be represented by

$$ia_\lambda^\dagger(p) = \int \frac{d^3x}{\sqrt{2E_p}} g_{\lambda\lambda}\epsilon_\lambda^\mu(p)(e^{-ip\cdot x}\partial_0 A_\mu - A_\mu\partial_0 e^{-ip\cdot x}) \tag{5.134}$$

Derivation 5.9 (Propagators) We can now calculate the QED propagators. Using Equation 5.120 which eliminates those pesky 4-versors, the photon field propagator is

Definition 5.27 (Photon field propagator) As the photon field propagator is bosonic, we denote it as D_F, identical to the Feynman propagator:

$$D_F^{\mu\nu}(p) = \langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle = -i\lim_{\epsilon\to 0+}\int \frac{d^4x}{(2\pi)^4}\frac{g^{\mu\nu}e^{-ip\cdot(x-y)}}{p^2+i\epsilon} \tag{5.135}$$

and the fermionic propagator is

Definition 5.28 (Fermionic propagator) As the fermionic propagator is fermionic[a], we denote it distinctly as S_F:

$$S_F(p) = \langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle = -i\lim_{\epsilon\to 0+}\int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \tag{5.136}$$

[a] Who could've guessed?

$S_F(p)$ is a Green's function of the Dirac operator:

$$(\not{p}-m)\lim_{\epsilon\to 0+}\int \frac{d^4p}{(2\pi)^4}\frac{i(\not{p}+m)e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} = i\delta^4(z-y) \tag{5.137}$$

Remark 5.12 The only useful fermionic propagator is $\langle 0|T[\psi(x)\bar{\psi}(y)]|0\rangle$. This is because anti-commutation implies that

$$\langle 0|T[\psi_\alpha(x)\psi_\beta(y)]|0\rangle = \langle 0|T[\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y)]|0\rangle = 0 \tag{5.138}$$

Finally, using Wick's second theorem and taking the Grassmann parity of Fermions into account, we can formulate the QED Feynman rules:

Theorem 5.17 (QED Feynman rules) For a given Feynman diagram in QED, the transition amplitude matrix elements $\mathcal{M}_{fi}$ is constructed as follows:

QED Feynman rules (partial)

For each	Assign
Incoming and outgoing electron	$\bar{u}_\alpha(s,p)$ and $u_\alpha(s,p)$
Incoming and outgoing positron	$v_\alpha(s,p)$ and $\bar{v}_\alpha(s,p)$
Incoming and outgoing photon	$\epsilon^{*\mu}(\lambda,p)$ and $\epsilon^\mu(\lambda,p)$
Internal photon line	$\frac{-ig^{\mu\nu}}{p^2}$
Internal fermion line	$\frac{i(\not{p}+m)e^{-ip\cdot(x-y)}}{p^2-m^2}$
Internal femion loop[a]	$\int d^4k_n/(2\pi)^4$
Vertex	$-ie\gamma^n_{\alpha\beta}$ [b]
Vertex	$(2\pi)^4\delta^3(k_i-k_f)$ [c]

where the incoming and outgoing photon indices are μ and ν, the incoming and outgoing fermion indices are α and β, and the incoming and outgoing 4-momenta are k_i and k_f. Each internal loop has a so-called *internal momenta* k_n.

Finally, before taking a well-deserved break, remove a factor of $(2\pi)^4\delta^3(p-q)$, where p and q are the total initial and detected momenta.

[a] There are no internal photon loops.

[b] The external leg indices, in the order along the direction of the external leg arrows, are α and β while the propagator index (photon *or* fermion) is n.

[c] This term enforces 4-momentum conservation.

Note 5.1 One should note the following points:

- $\bar{u}_\alpha(s,p)$ and $u_\alpha(s,p)$ are electron 4-spinors, $v_\alpha(s,p)$ and $\bar{v}_\alpha(s,p)$ are position 4-spinors, and $\epsilon^{*\mu}(\lambda,p)$ are photon 4-versors.
- The symbols s and λ seen in the electrons, positrons and photons are actually indices, which are put into the bracket purely for ease of viewing.
- In QED, the previously seen factor C observes $C=k!$, and C and $1/k!$ cancel out.
- When the scattering involves more than one diagram, Wick's theorem will be used, in which the exchange of two fermion operators changes the sign of the expression as per the Grassman parity.

Unlike ϕ^4 theory, QED is not a toy model and concerns itself with real particles - in specific, fermions and photons. One can represent them in a Feynman diagram as follows:

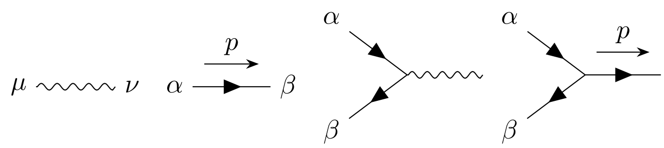

Figure 5.1: QED Feynman diagram elements. L-R: photon propagator, fermion propagator, vertex leading to photon propagator, vertex leading to fermion propagator

Thus, we have finished the canonical quantisation derivation of QED.

Part II

Path integrals

Chapter 6

Free fields

Quote 6.1 Juice, juice, orange juice...

Richard Feynman, playing the bongos, September 1981

While canonical quantisation is the more intuitive approach to developing field theories, it suffers from certain drawbacks, especially with respect to extra degrees of gauge freedom (as we have seen in deriving the QED Feynman rules). In developing more complicated field theories like QCD, we will make use of the other formalism: path integrals.

6.1 Path integrals in non-relativistic quantum mechanics

Unsurprisingly, the end goal in path integrals is the same as that in canonical quantisation, which is to derive the S-matrix S_{fi}, the transition amplitude $\mathcal{M}_{fi}$ and the Feynman rules of a given field theory that constructs $\mathcal{M}_{fi}$.
It is useful to recognise the relationship between S_{fi} and the unitary time evolution operator $U(t, t_0)$. When t is the final time t_f of some process and t_0 is the initial time t_i, one recovers the scattering matrix:

$$S_{fi} = U(t_f, t_i) \tag{6.1}$$

We can see why this reasoning is useful by investigating path integrals in innocent non-relativistic quantum mechanics. The central idea is that a particle in motion can and *will* take every possible trajectory or *path*. We postulate that each path contributes a factor of e^{iS} to the $U(t, t_0)$:

$$U(t, t_0) = \sum_{\text{all paths}} e^{iS} \tag{6.2}$$

where S is the action.
As is well known, in non-relativistic quantum mechanics, $U(t, t_0)$ is given by

$$U(t, t_0) = \langle f|e^{-iHT}|1\rangle \tag{6.3}$$

where 1 is the initial state[1] and $T = t - t_0$ is the time interval.

Derivation 6.1 (Time-slicing) One can solve the contribution to $U(t, t_0)$ of a certain path by slicing the path's time interval into *smol* time steps of ϵ. Recall the *Lie product formula* we have seen in *Electron's Destiny*. For any operators or square matrices $\hat{A}$ and $\hat{B}$, one has

$$e^{\hat{A}+\hat{B}} = \lim_{N\to\infty} \left(e^{\hat{A}/N} e^{\hat{B}/N}\right)^N = \lim_{N\to\infty} \left(e^{\hat{B}/N} e^{\hat{A}/N}\right)^N \tag{6.4}$$

where N is the so-called *Trotter number*.
Practically, this has an alternative formulation. Suppose that, instead of solving directly for $e^{\hat{A}+\hat{B}}$, we solve it segment by segment, solving first a $e^{\epsilon(\hat{A}+\hat{B})}$ and then calculate $(e^{\epsilon(\hat{A}+\hat{B})})^{1/\epsilon}$, where the segment $e^{\epsilon(\hat{A}+\hat{B})}$ has the form

[1]We have avoided writing i to prevent confusion with indices that will appear later.

Theorem 6.1 (Lie product formula)

$$e^{\epsilon(\hat{A}+\hat{B})} = e^{\epsilon\hat{A}}e^{\epsilon\hat{B}} + O(\epsilon^2) \tag{6.5}$$

where, as $\epsilon \to 0$, $O(\epsilon^2)$ vanishes.

Our good friend, the unitary time evolution operator, can then be approximated as

$$e^{-iHT} = \left(e^{-iH\epsilon}\right)^N \tag{6.6}$$

where $N = T/\epsilon$ is again the Trotter number.
From this, we are in a position to construct the generic time evolution operator $U(t, t_0)$. We write over a series of N steps:

$$\begin{aligned} U(t,t_0) =& \langle f, t_N|e^{-iH\epsilon}|N-1, t_{N-1}\rangle \cdots \langle 1, t_1|e^{-iH\epsilon}|i, t_0\rangle \\ =& \int dx_{N-1}\cdots dx_1 \langle x_f, t_N|e^{-iH\epsilon}|x_{N-1}, t_{N-1}\rangle \cdots \langle x_1, t_1|e^{-iH\epsilon}|x_i, t_0\rangle \end{aligned} \tag{6.7}$$

where we have integrated over all intermediate positions $dx_1 \cdots dx_{N-1}$[a] and $\epsilon = T/N$ is again a single time step.
Remark 6.1 Note that the second line is not a single integral but $N-1$ integrals.

[a]The initial and final positons x_0 and x_N are not integrated as they are fixed - remember that the path integral is integrating over a range of *possible* positions!

The Hamiltonian can be decomposed as

$$H = \frac{1}{2}p_i^2 + V(q_i) \tag{6.8}$$

where we recall p_i and q_i to be generalised momenta and coordinates.

Derivation 6.2 (One slice in phase space) We now want to evaluate one slice, represented by $\langle q_{i+1}, t_{i+1}|\, e^{-iH\epsilon}\,|q_i, t_i\rangle$. Importantly, as neither p_i nor q_i are scalars, we *cannot* simply say that $e^{-iH\epsilon} = e^{-i\epsilon(p_i^2/2+V(q_i))} = e^{-i\epsilon p_i^2/2}e^{-i\epsilon V(q_i))}$. A trick must be used to solve for the decomposed result:

Theorem 6.2 (Baker-Campbell-Hausdorff formula) Suppose one has the known matrices X and Y and the unknown matrix Z which satisfy $e^X e^Y = e^Z$. Z can be solved by

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots \tag{6.9}$$

where square brackets are commutators.

Amazingly, however, as ϵ is *small*, Equation 6.9 yields convenient approximation

$$e^{-iH\epsilon} \approx e^{-i\epsilon p_i^2/2}e^{-i\epsilon V(q_i))} \tag{6.10}$$

which is *almost* identical to the result if p_i and q_i *were* scalars. We can then say that, for some arbitrary step i, that

$$\langle q_{i+1}, t_{i+1}|\, e^{-iH\epsilon}\,|q_i, t_i\rangle \approx \langle q_{i+1}, t_{i+1}|\, e^{-i\epsilon p_i^2/2}e^{-i\epsilon V(q_i))}\,|q_i, t_i\rangle \tag{6.11}$$

We exploit the following identity, which is a complete set of momentum eigenstates:

$$\int \frac{dp}{2\pi}|p\rangle\langle p| = 1 \tag{6.12}$$

Insertion gives

$$\langle q_{i+1}|e^{-i\epsilon\frac{p^2}{2}}e^{-i\epsilon V(q)}|q_i\rangle = \int \frac{dp}{2\pi}\langle q_{i+1}|e^{-i\epsilon\frac{p^2}{2}}|p\rangle\langle p|e^{-i\epsilon V(q)}|q_i\rangle \tag{6.13}$$

Both $V(q)$ and p^2 act diagonally in position space, and we find

$$\langle p|e^{-i\epsilon V(q)}|q_i\rangle = e^{-i\epsilon V(q_i)}\langle p|q_i\rangle = e^{-i\epsilon V(q_i)}\frac{e^{-ipq_i}}{\sqrt{2\pi}} \quad \langle q_{i+1}|e^{-i\epsilon\frac{p^2}{2}}|p\rangle = e^{-i\epsilon\frac{p^2}{2}}\langle q_{i+1}|p\rangle = e^{-i\epsilon\frac{p^2}{2}}\frac{e^{ipq_{i+1}}}{\sqrt{2\pi}} \tag{6.14}$$

Putting it all together, and considering multiple degrees of freedom labeled by $j = 1, \dots, M$, we have

$$\langle q_{i+1}, t_{i+1}| \, e^{-iH\epsilon} \, |q_i, t_i\rangle = e^{-\epsilon V(q_i)} \int \prod_j^M \frac{dp_j^i}{2\pi} \exp\left[i\epsilon \left(p_j \frac{q_j^{i+1} - q_j^i}{\epsilon} - \frac{p_j^{i2}}{2} \right) \right] \tag{6.15}$$

where M is the degree of freedom (i.e. the number of dimensions/coordinates).

This is as far as we can go in phase space, and we are forced to move to momentum space.

Derivation 6.3 (Many slices in configuration space) Let us now go back to many slices so that we can recover the time evolution operator. Say that we have N slices. The integrals are then defined be over all intermediate positions and momenta:

$$\langle q_f, t_f | q_0, t_0\rangle = \int \prod_{i=1}^{N-1} dq_i \prod_{i=0}^{N-1} \left(\frac{dp_i}{2\pi} \right) \exp\left[i\epsilon \sum_{i=0}^{N-1} \left(p_i \frac{q_{i+1} - q_i}{\epsilon} - \frac{p_i^2}{2} - V(q_i) \right) \right] \tag{6.16}$$

The momenta p_i can now be integrated out. This is, up to phase, standard Gaussian integral, which can be evaluated as

$$\int_{-\infty}^{\infty} dp e^{-ap^2/2 + bp + c} = \sqrt{2\pi/a} e^{b^2/2a + c} \tag{6.17}$$

In our case, this is

$$\langle q_f, t_f | q_0, t_0\rangle = \int \prod_{i=1}^{N-1} dq_i i \sum^N \epsilon \left(\frac{1}{2} \dot{q}_i^2 - V(q_i) \right) = \int \prod_{i=1}^{N-1} dq_i i \sum^N \epsilon L(q_i, \dot{q}_i) \tag{6.18}$$

We now make a shorthand that represents the integration over *all possible intermediate configurations* of the path $q(t)$ between the endpoints q_0 and q_f - i.e. the 'sum over all paths'. This is sometimes called the *integration measure*:

Definition 6.1 (Integration measure)

$$\mathcal{D}q = \prod_{i=1}^{N-1} dq_i \tag{6.19}$$

where N is once again the number of steps.

Thus, the transition amplitude becomes

$$\langle q_f, t_f | q_0, t_0\rangle = \int \mathcal{D}q e^{i \sum_{i=0}^{N-1} \epsilon L(q_i, \dot{q}_i)} \tag{6.20}$$

In the continuum limit, we send N to infinity and find

$$\langle q_f, t_f | q_0, t_0\rangle = \int \mathcal{D}q e^{iS} = \int \mathcal{D}q e^{i \int_{t_0}^{t_f} dt L(q, \dot{q})} \tag{6.21}$$

where S is the action. This verifies our postulate in Equation 6.2[a].

[a]Importantly, time ordering is not a concern as the integration the Lagrangian naturally preserves the time order. The same can be seen in the time steps in Equation 6.7.

Theorem 6.3 (Integration measure properties)

$$\int \mathcal{D}\phi = \phi(x) \tag{6.22}$$

$$\int \mathcal{D}\phi\phi = \frac{\phi^2(x)}{2} \tag{6.23}$$

> **Quote 6.2** The strict built-in ordering of times makes commutation relations between the operators irrelevant.
>
> *Niklas Beisert, 2017*

Now we turn this from quantum mechanics to a field theory:

- The Lagrangian L is replaced by the Lagrangian density $\mathcal{L}$.
- The coordinates q are replaced with the fields ϕ.
- From the last point, $\mathcal{D}q$ becomes $\mathcal{D}\phi$.

which gives, after introducing sensible limits

$$U(t, t_0) = \int_{\phi(t_0)}^{\phi(t)} \mathcal{D}\phi e^{i \int d^4x \mathcal{L}} \tag{6.24}$$

6.2 Sources

> **Quote 6.3** As any reader of Dirac knows, it is sometimes convenient to speak of a distribution as if it were a function.
>
> *Sidney Coleman and Jeffrey Mandula, in 'All Possible Symmetries of the S Matrix', 16 March 1967*

In the path integral formulation, we introduce, for every field $\phi(x)$, a classical external field $J(x)$ called a *source* that couples linearly to the field. This so-called source is more of a mathematical convenience than a physical entity[2] , and it is significant in that it allows the creation of Green's functions.
Before doing this derivation, we will summarise it in text:

- The Lagrangian density is modified by adding a source term $J(x)\phi(x)$. The action is modified by proxy.
- The time evolution operator, which is an exponential of the Hamiltonian, is generalised as the almighty $Z(J(x))$ *generating functional*[3], which is an exponential of the now-modified action.
- Loosely speaking, the Green's function can be derived by differentiating the *generating functional* by the source and setting $J(x) = 0$ afterwards[4].

Definition 6.2 (Generating functional)

$$Z[J] = \int \mathcal{D}\phi e^{i \int d^4x (\mathcal{L}(\phi(x)) + J(x)\phi(x))} \tag{6.25}$$

Remark 6.2 But this looks a bit familiar, doesn't it?
If you share this opinion, you'd be right. As $Z(J(x))$ integrates over all possible field configurations, it is the quantum analogue of the well-known *partition function* in statistical physics[5].

Derivation 6.4 (Free massive scalar field) We are now in a position to derive the generating functional of some free field. Consider the following Lagrangian of a free massive scalar field[a]

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{6.26}$$

[2]For this reason, it does not appear in canonical quantisation.
[3]It is so-called as it is used in path integrals to generate Green's functions.
[4]This is where one can appreciate the nature of the source as a mathematical convenience. Ultimately, QFT describes free or interacting fields instead of external influences. Hence, the source to zero as this external influence would have remained otherwise.
[5]In fact, it is simply called the *partition function* in some literature.

The generating functional is hence

$$Z[J] = \int \mathcal{D}\phi e^{i \int d^4x \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2}m^2\phi^2 + i\int d^4x J(x)\phi(x)} \tag{6.27}$$

Let us first analyse the exponential

$$i\int d^4x \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2}m^2\phi^2 + i\int d^4x J(x)\phi(x) \tag{6.28}$$

Now we can introduce the two-point Green's function or the propogator, which, as seen before, satisfies

$$(\partial^2 + m^2)D_F(x-y) = \delta^4(x-y) \tag{6.29}$$

For our convenience, the field $\phi(x)$ can be decomposed

$$\phi(x) = \varphi(x) + \phi_q(x) \tag{6.30}$$

where:

- $\varphi(x)$ satisfies the classical equation of motion involving the source

$$(\partial^2 + m^2)\varphi(x) = J(x) \tag{6.31}$$

- $\phi_q(x)$ is the quantum fluctuation around the classical solution.

The exponential term is hence

$$-\frac{1}{2}i\int d^4x \phi_q(\partial^2 + m^2)\phi_q + i\int d^4x d^4y J(x)D_F(x-y)J(y) \tag{6.32}$$

So far, this generating functional remains unnormalised and diverges into infinity. We thus introduce the so-called *normalised generating functional* $Z_0[J]$, which has the form

Definition 6.3 (Normalised generating functional)

$$Z_0[J] = \frac{Z[J]}{Z[J=0]} \tag{6.33}$$

In our case, this is

Definition 6.4 (Normalised free massive scalar field generating functional)

$$Z_0[J(x)] = \frac{\int \mathcal{D}\phi e^{-\frac{1}{2}i\int d^4x(\partial^2+m^2)\phi + i\int d^4x J(x)\phi(x)}}{\int \mathcal{D}\phi e^{-\frac{1}{2}i\int d^4x(\partial^2+m^2)\phi}} \tag{6.34}$$

The numerator integrates as

$$e^{-\frac{1}{2}\int d^4x d^4y J(x)D_F(x-y)J(y)} \underbrace{\int \mathcal{D}\phi_q e^{-\frac{1}{2}i\int d^4x \phi_q(\partial^2+m^2)\phi_q}}_{1} \tag{6.35}$$

The denominator is a Gaussian normalisation integral where, by definition, $J = 0$ (see Equation 6.33). As such, the $\varphi(x)$ has no effect, and we can effectively rewrite $\phi(x)$ as $\phi_q(x)$. This makes the denominator equivalent to 1, and we are left with

$$Z_0[J(x)] = e^{-\frac{1}{2}\int d^4x d^4y J(x)D_F(x-y)J(y)} \tag{6.36}$$

where again, $D_F(x-y)$ is the all-too-familiar two-point Green's function or the Feynman propagator:

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \tag{6.37}$$

[a]This is identical to Equation 2.21 save for the minus sign.

From this, we are almost capable of establishing a relationship between the propogatoe and the generating functional. The only missing piece is the so-called *functional derivative* δ, which is the analogue of a normal derivative for a function.

Theorem 6.4 (Functional derivative properties)

$$\frac{\delta 1}{\delta\phi(x)} = 0 \tag{6.38}$$

$$\frac{\delta\phi(y)}{\delta\phi(x)} = \delta(x-y) \tag{6.39}$$

$$\frac{\delta}{\delta\phi(x)}(\alpha(y)\beta(z)) = \frac{\delta\alpha(y)}{\delta\phi(x)}\beta(z) + \alpha(x)\frac{\delta\beta(z)}{\delta\phi(x)} \tag{6.40}$$

A general relation between propagators and the normalised generating functional in free fields can now be made. For a system of n 4-positions, we have

Theorem 6.5 (Propagator-generating functional relation)

$$G_0^{(n)} = \langle 0|T[\phi\left(x_1\right)\cdots\phi\left(x_n\right)]|0\rangle = \frac{1}{i^n}\left.\frac{\delta^n Z_0(J)}{\delta J(x_1)\cdots\delta J(x_n)}\right|_{J=0} \tag{6.41}$$

This then allows us to reconstruct the generating functional in terms of the propagator:

$$Z[J] = \sum_{n=0}^{\infty}\int d^d x_1 \ldots d^d x_n \langle 0|T[\phi\left(x_1\right)\cdots\phi\left(x_n\right)]|0\rangle J\left(x_1\right)\cdots J\left(x_n\right) \tag{6.42}$$

Chapter 7

Interacting fields I: Preliminaries

In the last chapter, we left off from Equation 6.34, which can be generalised to many fields[1]:

$$\langle T[\phi_1 \cdots \phi_n] \rangle = \left. \frac{\int \mathcal{D}\phi \phi_1 \cdots \phi_n e^{iS[\phi,J]}}{\int \mathcal{D}\phi e^{iS[\phi,J]}} \right|_{J=0} \tag{7.1}$$

This chapter will see us tackling this expression.

7.1 Generating functional

While innocent-looking, the expression of the generating functional is actually quite unwieldy. Similar to what we did in canonical quantisation, we split the Lagrangian in the numerator (i.e. the generating functional $Z[J]$) into two parts, the (quadratic) free part $\mathcal{L}_F$ which we can single out from the integration measure and the interacting part $\mathcal{L}_I$ with a dependence on ϕ.

$$Z[J] = \int \mathcal{D}\phi e^{i \int d^d x (\mathcal{L}_F + \mathcal{L}_I + J\phi)} = \int \mathcal{D}\phi e^{i \int d^d x (\mathcal{L}_F + J\phi)} e^{i \int d^d x \mathcal{L}_I} \tag{7.2}$$

In fact, there is a third implicit term $\mathcal{L}_S = J(x)\phi(x)$, which is the source part.
Now we evaluate this expression:

Derivation 7.1 (Interaction part) For the interaction term, we rewrite it in terms of functional derivatives,

$$e^{iS_I[\phi]} = e^{i \int d^d x \mathcal{L}_I[\phi]} \tag{7.3}$$

Since we will later set $J = 0$, we replace $\phi(x)$ with a functional derivative:

$$\phi(x) = \frac{\delta}{i\delta J(x)} \tag{7.4}$$

Thus, the interaction part becomes

$$e^{i \int d^d x \mathcal{L}_I} = e^{i \int d^d x \mathcal{L}_I \left[\frac{\delta}{i\delta J}\right]} \tag{7.5}$$

which we can single out from the integral due to the lack of dependence on ϕ.

Derivation 7.2 (Free part) The removal of the interaction term from the integral allows us to integrate the free term by itself. As it is Gaussian, we can perform the path integral to obtain

$$\int \mathcal{D}\phi e^{i \int d^d x (\mathcal{L}_F + J\phi)} = e^{-\frac{i}{2} \int d^d x d^d y J(x) D_F(x-y) J(y)} \tag{7.6}$$

where $D_F(x-y)$ is our good friend, the two-point Feynman propagator.

[1]At first glance this might seem slightly confusing. Note that the indexless ϕ is not standalone but is to be read as a part of the integration measure $\mathcal{D}\phi$.

Hence, the generating functional is

$$Z[J] = e^{i\int d^dx \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} e^{-\frac{i}{2}\int d^dx d^dy J(x) D_F(x-y) J(y)} \tag{7.7}$$

7.2 Perturbative expansion: ϕ^4 theory again

Let us again consider a $2 \to 2$ process in ϕ^4 theory. The compact form of the Green's function reads

$$\langle T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle = \left.\frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)e^{iS[\phi,J]}}{\int \mathcal{D}\phi e^{iS[\phi,J]}}\right|_{J=0} \tag{7.8}$$

The denominator is merely the generating functional $Z[J]$ (in this case $Z[0]$), while the numerator can be realised as the generating functional differentiated with respect to $x_1, \cdots, x_4$.

$$\left\langle T\left[\prod_i^4 \phi(x_i)\right]\right\rangle = \frac{1}{Z[0]}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\int \mathcal{D}\phi e^{iS[\phi,J]}\bigg|_{J=0} \tag{7.9}$$

Note that while $\int \mathcal{D}\phi e^{iS[\phi,J]}$ is essentially $Z[J]$, we cannot cancel it with $1/Z[0]$ due to the existence of the differential operators $\delta^4/\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)$. Instead, we decompose the action as per Equation 7.2.
Substituting the integration results from Equation 7.7:

$$\left\langle T\left[\prod_i^4 \phi(x_i)\right]\right\rangle = \frac{1}{Z[0]}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)} e^{i\int d^dx \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} e^{-\frac{i}{2}\int d^dx d^dy J(x) D_F(x-y) J(y)}\bigg|_{J=0} \tag{7.10}$$

where d is the number of dimensions in the spacetime.

Derivation 7.3 (2 $\to$ 2 processes) We cannot calculate the Green's function with this result alone. Rather, we perform a perturbative expansion of the exponential. We start with $e^{i\int d^dx \mathcal{L}_I\left[\frac{\delta}{i\delta J}\right]} e^{-\frac{i}{2}\int d^dx d^dy J(x) D_F(x-y) J(y)}$, which is the evaluated form of $Z[J]$.
Recall from Part I that the interacting Lagrangian in ϕ^4 theory is

$$\mathcal{L}_I = \frac{\lambda}{4!}\phi^4 \tag{7.11}$$

where, importantly, λ is the almighty coupling constant. The two exponentials then each expand to a series of polynomials:

$$\begin{aligned}\left\langle T\left[\prod_i^4 \phi(x_i)\right]\right\rangle =& \frac{1}{Z[0]}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\left(\sum_n \frac{1}{n!}\left(\frac{\lambda}{4!}i\int d^dy \frac{\delta^4}{i\delta J(y)^4}\right)^n\right)\times \\ &\left(\sum_m \frac{1}{m!}\left(-\frac{i}{2}\int d^dx d^dy J(x) D_F(x-y) J(y)\right)^m\right)\bigg|_{J=0}\end{aligned} \tag{7.12}$$

where m and n are the powers.
Both indices m and n go up to infinity. However, we are actually not too interested in them on their own. Rather, we inspect the order of λ which, like in canonical quantisation, denotes the number of vertices. This puts us in a position to return to Equation 4.62.
Merely at first order, we have the terrible-looking expression

$$\begin{aligned}\left\langle T\left[\prod_i^4 \phi(x_i)\right]\right\rangle =& \frac{Z_2[0]}{Z[0]}\frac{\delta^4}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\left(\frac{1}{2!}\left(-\frac{i}{2}\int d^dx d^dy J(x) D_F(x-y) J(y)\right)^2\right. \\ &\left. -\frac{i\lambda}{4!}\int d^dz \frac{1}{4!}\frac{\delta^4}{\delta J(z)^4}\left(-\frac{i}{2}\int d^dx d^dy J(z) D_F(x-y) J(y)\right)^4 + \mathcal{O}\left(\lambda^2\right)\right)\end{aligned} \tag{7.13}$$

This will produce a series of derivatives, many of which are identical. In the interest of brevity, we will jump through this hard part and arrive at the conclusion that one can write

$$\frac{1}{Z[0]} = \frac{1}{1+\lambda D} \quad \text{the rest} = A + \lambda(B + C + AD) + O(\lambda^2) \tag{7.14}$$

where:

$$A = G^{(0)}_{2+2} = D_F(x_1 - x_2)D_F(x_3 - x_4) + D_F(x_1 - x_3)D_F(x_2 - x_4) + D_F(x_1 - x_4)D_F(x_2 - x_3) \tag{7.15}$$

$$\lambda B = -i\lambda \int d^d x D_F\left(x - x_1\right) D_F\left(x - x_2\right) D_F\left(x - x_3\right) D_F\left(x - x_4\right) \tag{7.16}$$

$$\lambda C = -\frac{i\lambda}{2} \sum_{P(ijkl)} D_F\left(x_i - x_j\right) \int d^d x D_F(x - x) D_F\left(x - x_k\right) D_F\left(x - x_l\right) \tag{7.17}$$

$$\lambda AD = -\frac{i\lambda}{8} \int d^d x D_F(x - x) D_F(x - x) \sum_{P(ijkl)} D_F\left(x_i - x_j\right) D_F\left(x_k - x_l\right) \tag{7.18}$$

where $P(ijkl)$ permutes over all possible indices i, j, k and l (i.e. 1 and 2). This gives

$$\langle T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle = \frac{A + \lambda(B + C + AD)}{1 + \lambda D} \tag{7.19}$$

One can expand the RHS factorial, yielding

$$\langle T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]\rangle = (A + \lambda(B + C + AD))(1 - \lambda D) + O(\lambda^2) = A + \lambda(B + C) + O(\lambda^2) \tag{7.20}$$

We hence recover

$$G^{(1)}_{2+2} = \lambda(B + C) \tag{7.21}$$

If one is masochistic, it is possible to write out the full form of the propagator via this expansion and set the sources to zero. In principle, this recovers the very same ϕ^4 Feynman rules as we have seen before.

Already, we could give some comments comparing path integrals and canonical quantisation:

- So far, we have categorically avoided using Wick's theorem, and we will not use it even if we write out the full form of the propagator. As we have seen in the $2 \to 2$ process, all possible contractions are 'automatically' generated (i.e. arise naturally) through differentiating the (gaussian[2]) path integral. Here we see the advantage of the path integral formulation.
- We note that the expansion is very unwieldy in position space, and the expansion is again better carried out in momentum space. Here we see the disadvantage of the path integral formulation.

7.3 Further generating functionals

Our previous encounter with amputated propogators was brief. Now, with path integrals in our hands, we can investigate it and its related concepts more sophisticatedly. Starting from the full propagator or the *dressed propagator*, one can increasingly simplify it:

- If we preserve the connected parts of the Feynman diagram only, the Green's functions reduce to *connected Green's functions*, which is generated by the so-called *free energy* $W[J]$.
- If we further remove (i.e. amputate) the external (leg) propagators[3], we are left with the previously seen amputated propagators. This extracts the core interaction structure.
- If we remove even the reducible parts of the Feynman diagram, we are left with the *one-particle-irreducible* (*1PI* or *OPI* in short) *Green's functions*, which describe fundamental interaction vertices beyond the classical action (e.g. loops).

[2]Gaussian integrals inherently sum over all possible contractions.

[3]Whether one does so in the full Green's function or the connected Green's function makes no difference, and both result in the amputated propagator.

- Conversely, if we remove all OPI Green's functions from the full propagator, we find the *bare propagator* G_0.

> **Quote 7.1** Help me, OPI-Wan. You're my only hope.
>
> *Star Wars, 1977*

Dear reader (yes, you), chapters ago, you learned about amputated propagators in Equation 4.70. Now they beg you to help them in their struggle against path integrals.
We first define the *free energy*, which is simply the generating functional of connected Green's functions.

Definition 7.1 (Free energy)

$$W[J] = -i \ln(Z[J]) \tag{7.22}$$

Expanding $W[J]$ in terms of $J(x)$ provides the connected n-point Green's functions G_c

Definition 7.2 (Connected Green's function)

$$G_c = \langle T[\phi(x_1) \cdots \phi(x_n)] \rangle_c = \left. \frac{\delta^n W[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} \tag{7.23}$$

Thus, $W[J]$ generates the connected Green's functions. To cement this in our memories, we note that they are distinct from normal Green's functions, which, for n points, are given by

$$G = \left. \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} \tag{7.24}$$

from which we can amputate our poor propagators.
The so-called *classical field* φ, which is defined as the expectation value of the quantum field in the presence of the source, is then given by taking the functional derivative of W with respect to J:

Definition 7.3 (Classical field)

$$\varphi = \langle \phi(x) \rangle = \frac{\delta W[J]}{\delta J(x)} \tag{7.25}$$

The *effective action* is then the Legendre transform of $W[J]$:

Definition 7.4 (Effective action)

$$\Gamma[\varphi] = W[J] - \int d^d x J(x) \varphi(x) \tag{7.26}$$

where $J(x)$ is understood as a functional of $\varphi(x)$ through the inversion of $\varphi(x) = \delta W / \delta J$.

The first functional derivative of $\Gamma[\varphi]$ gives the source:

$$\frac{\delta \Gamma}{\delta \varphi} = -J \tag{7.27}$$

The second functional derivative of $\Gamma[\varphi]$ gives the *inverse propagator*:

$$\frac{\delta^2 \Gamma[\varphi]}{\delta \varphi(x) \delta \varphi(y)} = (G_{\text{full}})^{-1}(x, y) \tag{7.28}$$

where $G_{\text{full}}(x, y)$ is the full propagator or the *dressed propagator* including quantum corrections.
Any higher-order derivatives of $\Gamma[\varphi]$ yield the OPI Green's functions G_{OPI}, sometimes also denoted as γ:

Definition 7.5 (OPI Green's function)

$$G_{\text{OPI}} = \left. \frac{\delta^n \Gamma[\varphi]}{\delta J(x_1) \cdots \delta J(x_n)} \right|_{J=0} \tag{7.29}$$

In other words, $\Gamma[\varphi]$ is the generating functional of OPI Green's functions. For example, $\Gamma^{(4)}$ represents the four-point interaction vertex.

Remark 7.1 The amputated Green's function essentially represents the sum of OPI diagrams that make up the full vertex function.
Remark 7.2 Importantly, the term 'one-particle-irreducible' does not describe interactions involving only one particle. Rather, a Feynman diagram is called OPI if it cannot be split into two separate diagrams by cutting a single internal propagator. This means:

- The diagram remains connected if any single internal propagator is removed.
- It does not factorise into two separate diagrams through a single-particle exchange.

This is in contrast to one-particle-reducible (OPR) diagrams, which can be split by cutting a single propagator, meaning they contain a disconnected propagator that acts as a bridge.
Let us summarise what we have so far:

- We have 4 main quantities of which we can take functional derivatives - the classical action and the 3 we have seen in this section.
- They are functionals of either the source $J(x)$ or the field $\phi(x)$.
- In an n-point QFT process, one has a series of positions $x_1 \cdots x_n$.

We go over them one by one:

- **Classical action $S[\phi]$:** Functional derivatives are taken w.r.t. $\phi(x)$.
  - The first derivative simply generates the classical equations of motion.
  - Higher derivatives against the n points generate vertices in the classical theory.
- **Generating functional $Z[J]$:** Functional derivatives are taken w.r.t. $J(x)$ and generate vacuum expectation values (i.e. correlation functions or Green's functions).
- **Free energy $W[J]$:** Functional derivatives are taken w.r.t. $J(x)$ and generate connected Green's functions.
- **Effective action $\Gamma[\phi]$:** Functional derivatives are taken w.r.t. $\phi(x)$.
  - The first derivative generates the master Dyson-Schwinger equation in Equation 7.34 which incorporate loop corrections[4].
  - The second derivative generates the inverse propagator.
  - Higher derivatives against the n points generate the OPI Green's functions.

One final, albeit important, remark concerns the so-called *self-energy* Σ, which is instrumental in the Dyson equation we will use in renormalisation.

Definition 7.6 (Self-energy) We define the self-energy of a particle as the energy that a particle has due to its interaction and its environment [a]. Mathematically, it is the sum of all its OPI *two-point* diagrams:

$$\Sigma(p) := \sum G_{\mathrm{OPI}}(p) \tag{7.30}$$

[a] i.e. the part of the total energy that relates the particle back to itself

Remark 7.3 It is worth noting that self-energy is not actually energy. Rather, it represents corrections to the full propagator arising from the interaction between a particle and its environment. This is analogous to the original concept of self-energy in electromagnetism, which refers to the energy required to assemble a charged particle due to its own electromagnetic field.

[4] Hence the name 'effective action' is justified.

7.4 Dyson-Schwinger equations

An interesting analogy exists between the action S and the generating functional Z. Consider a *smol* variation of the field (i.e. a gauge transformation). As this is nothing a change of variables in the path integral, the generating functional in Equation 6.25 is invariant:

$$\delta Z = \int \mathcal{D}\phi \frac{\delta}{\delta\phi(x)} e^{i\int d^4x(\mathcal{L}(\phi(x))+J(x)\phi(x))} = 0 \tag{7.31}$$

This is analogous to the action principle, which states that the action is invariant under a *smol* variation of coordinates[5].
Integrating by parts, one finds the first incarnation of the Dyson-Schwinger equation:

Theorem 7.1 (Dyson-Schwinger equation)

$$\frac{\delta S}{\delta\phi}\left(-i\frac{\delta}{\delta J}\right) Z[J] = -J(x)Z[J] \tag{7.32}$$

Recall from Equation 7.25 that the classical field is the expectation value of a field and is related to the free energy W in Equation 7.22. By inserting the field expectation/classical field it into Equation 7.32, we can rewrite it in terms of the free energy W:

$$\frac{\delta S}{\delta\phi}\left(\frac{\delta W}{\delta J} + \frac{\delta}{\delta J}\right) = -J(x) \tag{7.33}$$

Using a Legendre transformation and inserting Equation 7.27, a dependence on the effective action Γ in Equation 7.26 can also be acquired:

$$\frac{\delta\Gamma}{\delta\varphi(x)} + \frac{\delta S}{\delta\varphi(x)}\left(\varphi(x) + \frac{\delta^2 W}{\delta J(x)\delta J(y)} + \frac{\delta}{\delta J(y)}\right) = 0 \tag{7.34}$$

This is the master equation for the rest of the Dyson-Schwinger equations, which are infinite. These further iterations of the first Dyson-Schwinger equation are derived by performing further derivatives w.r.t. the classical field φ.

Derivation 7.4 (Dyson equation) As an example, let us perform a single φ-functional derivative on the Equation 7.34. The first term is then the second φ-derivative of Γ, which, according to Equation 7.28, is the inverse propagator from which we can solve the actual propagator.
If we consider the 2-point case, the propagator is then a Feynman propagator, which can be seen to observe

$$\begin{aligned} D_F(p) = &\underbrace{\frac{i}{p^2-m_0^2+i\epsilon}}_{\text{0 loops}} + \underbrace{\frac{i}{p^2-m_0^2+i\epsilon}\left(-i\Sigma\left(p^2\right)\right)\frac{i}{p^2-m_0^2+i\epsilon}}_{\text{1 loop}} + \\ &\underbrace{\frac{i}{p^2-m_0^2+i\epsilon}\left(-i\Sigma\left(p^2\right)\right)\frac{i}{p^2-m_0^2+i\epsilon}\left(-i\Sigma\left(p^2\right)\right)\frac{i}{p^2-m_0^2+i\epsilon}}_{\text{2 loops}} + \cdots \end{aligned} \tag{7.35}$$

where the terms go up to infinity. The physical interpretation is as follows:

- The n^{th} term has n loops sandwiched among (i.e. multiplied by) $n+1$ internal lines.
- The LHS is the dressed Feynman propagator, and the first RHS term is the bare Feynman propagator.
- The rest of the RHS are OPI Feynman propagators, collectively known as self-energy, relate the dressed and bare Feynman propagators.

The third point can be realised by recognising that Equation 7.35 is actually really an expansion:

[5]Remember that in QFT, we have replaced 4-coordinates with 4-fields.

Theorem 7.2 (Dyson equation) The Dyson equation is a specific of the Dyson-Schwinger equation:

$$G_{\text{full}} = G_0 + \Sigma \tag{7.36}$$

which can be rewritten as

$$G_{\text{full}} = \frac{i}{p^2 - m^2 - \Sigma(p) + i\epsilon} \tag{7.37}$$

The usefulness of this section and the previous one will not be immediately obvious. However, they will prove essential in Part III.

Chapter 8

Interacting fields II: QED

QED is an abelian[1] gauge theory, which means that, as seen before, we need gauge fixing to quantise it. In this chapter, we will see that gauge fixing in path integrals is slightly different from canonical quantisation due to the structure of the integral and the presence of functionals.

8.1 Quantisation of the Dirac field

As seen in Equation 5.32, the Lagrangian for the Dirac fermion (without any further fermions) is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu$$

The generating functional is then

Definition 8.1 (Dirac equation generating functional)

$$Z[J] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{i\int d^dx(\bar{\psi}(i\partial\!\!\!/ - m)\psi + \bar{J}\psi + \bar{\psi}J)} \tag{8.1}$$

from which we can derive the fermion propagator. This concludes the easy half.

8.2 Quantisation of the electromagnetic field

Now we arrive at the hard part. Recall from Equation 5.114 that the photon field Lagrangian is of the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Writing this with the full form of the Faraday tensor in Equation 5.109 yields the generating functional

Definition 8.2 (Photon field generating functional)

$$Z\left[J_\mu\right] = \int \mathcal{D}A_\mu e^{i\int d^dx\left(\frac{1}{2}A^\mu\left(g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)A^\nu + J^\mu A_\mu\right)} \tag{8.2}$$

Note 8.1 Here, J_μ remains an arbitrary source instead of the 4-current. This is because we have not included an interaction term (where the 4-current J^μ is present) in the (free) photon field Lagrangian.

Again, gauge freedom complicates many things. We show this by first mindlessly evaluating the path integral without considering them:

- We evaluate $g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu$ in momentum space. Using the property of derivatives in Fourier transforms, its equivalent in momentum space is $g_{\mu\nu}k^2 - k_\mu k_\nu$.

[1] We will find out what this means later on.

- If we contract this with k^ν, we obtain:

$$(g_{\mu\nu}k^2 - k_\mu k_\nu)k^\nu = g_{\mu\nu}k^2k^\nu - k_\mu(k\cdot k) = k^2k_\mu - k^2k_\mu = 0 \tag{8.3}$$

which means that our expression maps any vector proportional to k_μ to zero. It is hence not invertible.

- Let us now recall the gauge transformation $A_\mu \to A_\mu + \partial_\mu\Lambda(x)$. In momentum space, it becomes

$$\tilde{A}_\mu(k) \to \tilde{A}_\mu(k) + ik_\mu\tilde{\Lambda}(k) \tag{8.4}$$

Since $g_{\mu\nu}k^2 - k_\mu k_\nu$ annihilates any vector proportional to k_μ, it cannot distinguish between different gauge-equivalent configurations of A_μ.

Like in canonical quantisation, extra degrees of freedom must be eliminated by gauges.

Derivation 8.1 (Gauge fixing, take 2) In the case of electromagnetism (or indeed non-abelian gauge theories in general), the gauge fields A_μ are not uniquely defined because they transform under a gauge transformation as:

$$A_\mu \to A^g_\mu = A_\mu + \partial_\mu g \tag{8.5}$$

where A^g_μ is the field after it is gauge-transformed by some arbitrary function $g(x)$.
Now we want to remove the gauge freedom (i.e. invariance of A_μ). One can always write a gauge condition as a functional $C[A_\mu, x_\mu]$ satisfying

$$C[A_\mu, x_\mu] = 0 \tag{8.6}$$

which selects a unique representative from each equivalence class of gauge-related fields. However, this is not done by imposing restrictions directly on C[a]. Instead, we use the gauge function $g(x)$ to actively transform the gauge field to satisfy the condition.
In other words, given any A_μ, we can always find a suitable gauge function $g(x)$ such that the transformed field A^g_μ satisfies the gauge condition in Equation 8.6.
In the case of the Lorenz gauge, seen in Equation 5.113, the condition we want to impose is

$$\partial^\mu A^g_\mu = 0 \tag{8.7}$$

which we can expand as

$$\partial^\mu(A_\mu + \partial_\mu g) = \partial^\mu A_\mu + \partial^\mu\partial_\mu g = 0 \tag{8.8}$$

This is a differential equation for $g(x)$, which can always be solved, ensuring that we can always reach the Lorenz gauge by an appropriate choice of $g(x)$. Let us now make use of this in the photon field. This involves the functional $\Delta[A_\mu]$, which is essentially a delta functional. It is the generalisation of the Dirac delta function, which is related to it by

Definition 8.3 (Delta functional)

$$\Delta[A_\mu]^{-1} = \int \mathcal{D}g\delta(C[A_\mu, x_\mu]) \tag{8.9}$$

This functional is gauge-invariant. For a gauge transformation $g \to g + g'$:

$$\Delta[A_\mu]^{-1} = \Delta[A^{g'}_\mu]^{-1} \tag{8.10}$$

Interestingly, if we invert the expression above, we get

$$1 = \Delta[A_\mu]\int \mathcal{D}g\delta(C[A_\mu, x_\mu]) \tag{8.11}$$

which allows us to semi-cheatingly insert this expression as we please. Let us do so in Equation 8.2:

$$Z = \int \mathcal{D}A_\mu\Delta\left[A_\mu\right]\int \mathcal{D}g\delta\left(C\left[A^g_\mu\right]\right)\exp\left(iS\left[A_\mu\right]\right) \tag{8.12}$$

We now implement the gauge transformation $g \to g + g'$:

$$Z = \int \mathcal{D}g \int \mathcal{D}A_\mu^{g'} \Delta\left[A_\mu^{g'}\right] \delta\left(C\left[A_\mu^{g+g'}\right]\right) \exp\left(iS\left[A_\mu^{g'}\right]\right) \tag{8.13}$$

and change the variables of integration from $A_\mu^{g'}$ to $A_\mu^{-g-g'}$

$$Z = \int \mathcal{D}g \int \mathcal{D}A_\mu \Delta\left[A_\mu\right] \delta\left(C\left[A_\mu\right]\right) \exp\left(iS\left[A_\mu\right]\right) \tag{8.14}$$

To proceed from here, we must employ a nice trick on the term $\Delta\left[A_\mu\right]$ in Equation 8.9, which we will otherwise struggle to integrate. Let us change the variable of integration from G to C:

$$\Delta\left[A_\mu\right]^{-1} = \int \mathcal{D}C \left(\det \frac{\delta C}{\delta g}\right)^{-1} \delta(C) = \left(\det \frac{\delta C\left[A_\mu, x\right]}{\delta g}\right)^{-1}_{C=0} \tag{8.15}$$

which easily transforms to

$$\Delta\left[A_\mu\right] = \left(\det \frac{\delta C\left[A_\mu, x\right]}{\delta g}\right)_{C=0} \tag{8.16}$$

Let us now define the so-called *Faddeev-Popov operator* by

$$\det M(x,y) = \left(\det \frac{\delta C\left[A_\mu, x\right]}{\delta g}\right)_{C=0} \tag{8.17}$$

Using the chain rule, one can derive

Definition 8.4 (Faddeev-Popov operator)

$$M(x,y) = -\partial_\mu^y \frac{\delta C\left[A_\mu, x\right]}{\delta g} \tag{8.18}$$

Using the Lorenz gauge $C\left[A_\mu, x\right] = \partial^\mu A_\mu = 0$, we have

$$M(x,y) = -\partial^2 \delta(x-y) \tag{8.19}$$

which is nothing but a field-independent functional determinant that *does not* introduce interactions. This allows us to replace $\Delta\left[A_\mu\right]$ in Equation 8.14 with $\det M$:

$$Z = \int \mathcal{D}g \int \mathcal{D}A_\mu \det M \delta\left(C\left[A_\mu\right]\right) \exp\left(iS\left[A_\mu\right]\right) \tag{8.20}$$

Recall from Equation 5.112 that one more gauge freedom exists due to U(1) symmetry transformations. We then implement the R_ξ Landau gauge in Equation 5.115, which, in the form of the functional C, is

$$C = D[A_\mu, x] + \Lambda(x) \tag{8.21}$$

where we have relabelled the original Lorenz gauge as $D[A_\mu, x]$, and Λ is an arbitrary function. By using the standard trick of integrating over λ with a Gaussian weighting function, Z becomes

$$Z = \int \mathcal{D}\Lambda \exp\left(-\frac{i}{2\xi}\int d^dx \Lambda^2\right) \tag{8.22}$$

which allows us to rewrite the delta function in an equivalent integral form:

$$\delta[C] = \delta(D[A_\mu, x] + \Lambda) = \int \mathcal{D}\Lambda e^{-\frac{i}{2\xi}\int d^dx\Lambda^2} \delta(D[A_\mu, x]) \tag{8.23}$$

Plugging this into the generating functional, we obtain:

$$Z = \int \mathcal{D}\Lambda \mathcal{D}A_\mu \det M \exp\left(-\frac{i}{2\xi}\int d^dx\Lambda^2\right) \delta(C) \exp(iS) \tag{8.24}$$

integrating yields our final, well-defined expression

$$Z = \int \mathcal{D}A_\mu \det M \exp\left(iS - \frac{i}{2\xi}\int d^dx (\partial_\mu A^\mu)^2\right) \tag{8.25}$$

where $S = \int d^dx \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right)$ is the free field action and $-\frac{i}{2\xi}\int d^dx(\partial_\mu A^\mu)^2$ is the gauge fixing term.

[a] e.g. defining it in a way that inherently limits A_μ.

Note 8.2 (Faddeev-Popov ghosts) We append this discussion with a taste of things to come. In QED, which is a *non-abelian* gauge theory, the Faddeev-Popov operator evolves into its fairly harmless form in Equation 8.19, as we have seen in Equation 8.19. In abelian gauge theories like QCD, however, the Faddeev-Popov operator term in the generating functional

$$\det M = \int \mathcal{D}c\mathcal{D}\bar{c}e^{iS_{\text{ghost}}} \tag{8.26}$$

which contributes to the Lagrangian. c and $\bar{c}$ are unphysical fields known as the *Faddeev-Popov ghost fields*[a] or simply *Faddeev-Popov ghosts* which obey Grassmann anticommutations. We will discuss the treatment of Faddeev-Popov Ghosts much later.

[a]The general notion of a *ghost field* denote unphysical fields emerging in the Lagrangian. Specifically, ghost fields in QFT assume the name 'Faddeev-Popov ghost fields'.

8.3 Emergence of the QED Feynman rules

We once again find ourselves on the doorsteps of greatness as we formulate QED for a second time. Putting it all together, the QED generating functional is

Definition 8.5 (QED generating functional)

$$Z[J^\mu] = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(i\int d^dx(\mathcal{L}_{\text{QED}} + \bar{J}\psi + \bar{\psi}J + J^\mu A_\mu)\right) \tag{8.27}$$

where $\bar{J}\psi + \bar{\psi}J$ is the fermion field source term, $J^\mu A_\mu$ photon (gauge) field source term[a].

[a]No source term exists for $F^{\mu\nu}$ as it is not a variable of integration, although in some extensions of QED or effective field theories, one might introduce a source term for it.

Remark 8.1 Strictly speaking, one can even introduce the previously seen ghost field terms

$$Z[J] = \int \mathcal{D}A_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}c\mathcal{D}\bar{c} \exp\left[i\int d^4x \left(\mathcal{L}_{\text{QED}} + \mathcal{L}_{\text{ghost}} + J^\mu A_\mu + \bar{J}\psi + \bar{\psi}J\right)\right] \tag{8.28}$$

which are trivial (i.e. vanish) in QED anyway.
From here, it is then intuitive to derive the QED Feynman rules.
One final comment concerns quantum symmetries. The QED Dyson-Schwinger equations can be written as

Theorem 8.1 (QED Dyson-Schwinger equations)

- **Fermionic Dyson-Schwinger equations:**

$$\langle (i\gamma^\mu D_\mu - m)\psi(x)\rangle = -\bar{\eta}(x) \tag{8.29}$$

- **Photon Dyson-Schwinger equations:**

$$\left\langle \partial_\nu F^{\nu\mu}(x) - e\bar{\psi}(x)\gamma^\mu\psi(x)\right\rangle = -J^\mu(x) \tag{8.30}$$

Part III

Renormalisation and regularisation

Chapter 9

Regularisation

> **Quote 9.1** The sun set in the west on a notion where no man had dared to venture. And beyond that - infinity.
>
> *1492: Conquest of Paradise*

What?... You thought that was it? Oh no. We still have to treat infinities. Even though we noted loops in the Feynman rules, everything we have done so far has been restricted to the tree level. This is obviously problematic because loop interactions emerge in real life.

9.1 Motivation

Previously in Part I, we introduced the complete set of Feynman rules and the concept of the scattering amplitude $\mathcal{M}$. As the objective now is to go beyond the tree level, it is a good idea to review the full form of $\mathcal{M}$ up to the infinite order:

Theorem 9.1 (Full ϕ^4 theory scattering amplitude)

$$i\mathcal{M} = -i\lambda + \frac{(-i\lambda)^2}{2}\sum I_2 + O(\lambda^3) \tag{9.1}$$

where $\sum I_n$ is the sum of all possible *Feynman integrals* of that order n.

This systematic treatment makes use of the so-called Feynman integrals, which are representations of loop contributions to the scattering amplitude. The Feynman integral of a diagram arises naturally from the Feynman rules:

Definition 9.1 (Feynman integral)

$$I = \lim_{\epsilon\to 0}\int \prod_{k=1}^{l} \frac{d^d p_k}{i\pi^{d/2}} \prod_{k=1}^{n} \frac{1}{(p_k^2 - m_1^2 + i\epsilon)} \tag{9.2}$$

where:

- d is the physical dimension of the manifold.
- l is the number of loops.
- n is the number of the internal legs/lines.

Note 9.1 One can think of Feynman integrals I_n as a generalised version of Feynman propagators D_F. Indeed, a Feynman integral is ultimately a combination of Feynman propagators integrated over internal momenta. It corresponds to the probability amplitude for a particle to propagate from a point *back to the same point* some l times and hence represents l loops.

Previously, the reader has been led to believe that this is the end of the story. Beyond the tree level,

this is unfortunately a lie. Uniquely among all Feynman diagram elements, loops are mathematically represented by integrating over internal loop momenta. To be exact, loop integrals.
It is well-known that loop integrals often diverge into infinity, which is indeed the case for our Feynman integrals. Unsurprisingly, physical systems are not infinite, which suggests that our current formulation of QFT fails when we go beyond the tree level. In this chapter, we restrict our discussion to ϕ^4 theory.

Derivation 9.1 (Tadpole diagram) We illustrate this with the simplest possible example, that being a 1-loop, 1-point/2-point function known as the so-called *tadpole diagram*:

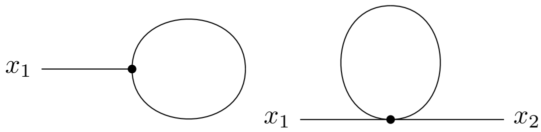

Figure 9.1: Archetypical tadpole diagrams.

The self-energy of a Feynman diagram is directly related to its Feynman integral by a coupling constant (and arbitrary constants). In ϕ^4 theory, the self-energy of an is simply the Feynman integral multiplied by a factor of $\frac{\lambda^n}{2}$ where n is the *number of vertices.* In our case, this is $\frac{\lambda}{2}$. Take the one-point tadpole as an example:

$$\Sigma_{\text{tadpole}} = \frac{\lambda}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 - m^2 - i\epsilon} \tag{9.3}$$

where, notably, we assume an arbitrary number of dimensions d for the time being. As we will prove later on, this integral evaluates as

$$\Sigma_{\text{tadpole}} = \frac{\lambda}{2}\frac{\pi^{d/2}}{(2\pi)^d}\Gamma\left(1-\frac{d}{2}\right)m^{d-2} \tag{9.4}$$

where $\Gamma(n)$ is a so-called *gamma function* defined by

Definition 9.2 (Gamma function)

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt \tag{9.5}$$

If n is a positive integer, $\Gamma(n)$ reduces to

$$\Gamma(n) = (n-1)! \tag{9.6}$$

Theorem 9.2 (Gamma function properties)

- $\Gamma(n)$ has poles at $n = 0, -1, -2, \cdots$
- $\Gamma(n+1) = n\Gamma(n)$
- $\Gamma'(1) = -\gamma \approx -0.5772$ where γ is the Euler-Mascheroni constant.

We now make the arbitrary choice of $d = 2$. The gamma function becomes

$$\Gamma\left(1-\frac{2}{2}\right) = \int_0^\infty t^{-1}e^{-t}dt = \infty \tag{9.7}$$

Hence, we have $\Sigma_{\text{tadpole}} = \infty$ for $d = 2$.
What is the physical meaning of this? Essentially, when p is *large*, the original integral approaches

$$\Sigma_{\text{tadpole}} = \frac{\lambda}{2(2\pi)^d}\int d^d p\frac{1}{p^2} \tag{9.8}$$

which diverges to infinity for $d \geq 2$. This result is clearly unphysical and is known as the so-called *ultraviolet (UV) singularity.*

Remark 9.1 Another singularity is the so-called *infrared (IR) singularity* which takes occur in low energies/*small* momentums.

9.2 Mathematical tooklit

The tadpole diagram is a good example of infinities as its integral is already simplified and can be evaluated as-is. This is, regrettably, not the case for almost all other diagrams, for which the elephant in the room remains our inability to integrate over the four-momentum p_μ. Rather, a mathematical trick known as *Feynman parameterisation* is required.

Derivation 9.2 (Feynman parameterisation)

Definition 9.3 (Feynman parameters) In his 1949 paper ‘Space-Time Approach to Quantum Electrodynamics’, Feynman noted the relation

$$\begin{aligned}\frac{1}{ab} &= -\frac{1}{a-b}\left(\frac{1}{a}-\frac{1}{b}\right) = -\frac{1}{a-b}\left[\frac{1}{xa+(1-x)b}\right]_0^1 \\ &= \int_0^1 dx \frac{1}{(xa+(1-x)b)^2} = \int_0^1 dx \int_0^1 dy \delta(x+y-1)\frac{1}{(xa+yb)^2}\end{aligned} \tag{9.9}$$

where $x, y \in [0,1]$ are the so-called *Feynman parameters*. One can generalise this relation as

$$\frac{1}{a_1 \cdots a_n} = \int dx_1 \cdots dx_n \delta\left(\sum_{i=1}^n x_i - 1\right) \frac{(n-1)!}{\left[\sum_{i=1}^n x_i a_i\right]^n} \tag{9.10}$$

Now let us apply this in the context of Feynman integrals. Assume the loop momentum k, external momenta p_i, mass m_i and the singularity-avoiding term $i\epsilon$, we define our a_i as

$$a_i = (k+p_i)^2 - m^2 + i\epsilon \tag{9.11}$$

The delta function in Equation 9.10 essentially enforces the relation

$$\sum_{i=1}^n x_i - 1 = 0 \to \sum_{i=1}^n x_i = 1 \tag{9.12}$$

We can thus write, using Equation 9.11 and Equation 9.12:

$$\sum_{i=1}^n x_i a_i = k^2 \sum_{i=1}^n x_i(2kp_i + P_i^2 - m_i^2) + i\epsilon \tag{9.13}$$

We define a rescaled loop momentum l, which is

$$l = k + \sum_{i=1}^n x_i p_i \tag{9.14}$$

This gives rise to the expression

$$\sum_{i=1}^n x_i a_i = l^2 - \left(\sum_{i=1}^n x_i p_i\right)^2 - \sum_{i=1}^n x_i(p_i^2 - m_i^2) + i\epsilon = l^2 - \mu^2 + i\epsilon \tag{9.15}$$

where we have defined a (rescaled) mass parameter μ that is independent of l, satisfying

$$\mu^2 = \left(\sum_{i=1}^n x_i p_i\right)^2 - \sum_{i=1}^n x_i(p_i^2 - m_i^2) \tag{9.16}$$

This is nice, as we have now eliminated the Feynman parameter x from the final expression completely! Here we see why Feynman parameterisation is so-called. Essentially, the new loop momenta and energy are rescaled (and parameterised) using our new friends, the Feynman parameters.

Another nice standard trick one can employ to simplify Feynman integrals is the so-called *Wick rotation*:

Definition 9.4 (Wick rotation) Consider the 4D Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{9.17}$$

The Wick rotation involves a coordinate rescaling

$$t = -i\tau \tag{9.18}$$

after which the metric reads

$$ds^2 = d\tau^2 + dx^2 + dy^2 + dz^2 \tag{9.19}$$

which is simply the 4D Euclidian metric. This simplifies calculations, and the Minkowski space version of our quantity of interest can be recovered by reversing our rescaling in Equation 9.18.

Note 9.2 In this book, some sacrifices are made for our convenience:

- We will relabel the rescaled l and μ back to p and m after the Feynman parameterisation is complete. This ensures that we see symbols we are familiar with in the final result.
- The same thing will happen w.r.t. Wick rotation. We will announce a Wick rotation has been made, at which point it should be assumed that the physical quantities, which retain their pre-Wick rotation notation, are those after Wick rotation.

Derivation 9.3 (Bubble diagram) We are now in a position to observe the emergence of infinities in a 1-loop, 2-point/4-point function, whose Feynman diagrams are called *bubble diagrams*:

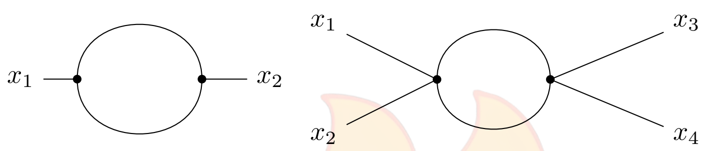

Figure 9.2: Archetypical bubble diagrams.

Let us take the 4-point diagram as an example. We recall that the self-energy of a diagram with n vertices is related to the Feynman integral by $\frac{\lambda^n}{2}$. For the bubble diagram, this is $\frac{\lambda^2}{2}$:

$$\Sigma_{\text{bubble}} = \frac{\lambda^2}{6}\int \frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-m^2)((q-p)^2+m^2)} \tag{9.20}$$

where $q = p_1 + p_2 = -p_3 - p_4$. Performing Feynman parameterisation and a Wick rotation yields

$$\Sigma_{\text{bubble}} = -\frac{\lambda^2}{2}\int \frac{d^dp}{(2\pi)^d}\frac{1}{(p^2-m^2+i\epsilon)^2} \tag{9.21}$$

where it should be clear that p and m are nothing but the rescaled l and μ which we have relabelled. This simplified integral evaluates as

$$\Sigma_{\text{bubble}} = \frac{\lambda^2}{2}\frac{\pi^{d/2}}{(2\pi)^d}\Gamma\left(2-\frac{d}{2}\right)\int_0^1 x(1-x)q^2 + m^{2}{}^{d-2-2}\,dx \tag{9.22}$$

Here, the UV singularity emerges at $d = 4$. The physical implication is likewise slightly altered: When p is *large*, the original integral approaches

$$\Sigma_{\text{bubble}} = \frac{\lambda^2}{2(2\pi)^d}\int d^dp\frac{1}{p^4} \tag{9.23}$$

which again is a UV singularity where the integral diverges to infinity for $d \geq 4$.

Exercise 9.1 Using Feynman parameterisation, show that a UV singularity emerges in a *sunset diagram*, which looks like

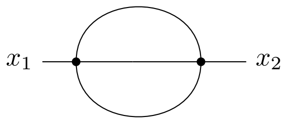

Figure 9.3: Archetypical sunset diagram.

and has the self-energy

$$\Sigma_{\text{sunset}} = \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\epsilon)((p-k-q)^2 - m^2 + i\epsilon)(q^2 - m^2 + i\epsilon)} \tag{9.24}$$

where p is the external momentum, k and a are the loop momenta and m is the mass of the internal propagators.

Quote 9.2 There's potential for a lot of upheaval.

Alessio Serafini, on the sunset diagram, 6 March 2025

In fact, for a sufficiently large d, most Feynman diagrams display UV singularities. They are hence said to be *UV divergent*. One way to treat these infinities is *regularisation*, which, as we will see, is a very hand-wavy way to deal with infinities.

9.3 Cutoff regularisation

Derivation 9.4 (Cutoff regularisation) We return to the tadpole diagram self-energy in Equation 9.3. We have written it with an indefinite integral, but it actually spans over positive and negative infinities:

$$\Sigma_{\text{tadpole}} = \frac{\lambda}{2} \int_{-\infty}^{\infty} \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 - m^2 - i\epsilon} \tag{9.25}$$

It has already been established that integrating over the positive infinity is what causes infinities to emerge. So what if we replace the upper limit with a *large* but finite Λ? The integral becomes

$$\Sigma_{\text{tadpole}} = \frac{i\lambda}{4\pi^2} \left(\Lambda^2 \sqrt{1 + \frac{m^2}{\Lambda^2}} - m^2 \ln \left(\frac{\Lambda + \Lambda\sqrt{1 + \frac{m^2}{\Lambda^2}}}{m} \right) \right) \tag{9.26}$$

where Λ is known as the *regulator*.
This is known as *cutoff regularisation* or the *UV momentum cutoff*, one of the many ways to regularise Feynman integrals. Its physical meaning lies in the fact that we essentially introduce a ceiling of the biggest momentum the system we have, which allows us to avoid the singularity in Equation 9.8.
The original singularity is then recovered when $\Lambda \to \infty$

The same happens to a bubble diagram, although in this case we use Feynman parameterisation and Wick rotation, given its complexity.

9.4 Pauli-Villars regularisation

The so-called *Pauli-Villars regularisation* or *P-V regularisation* is named after Wolfgang Pauli and Felix Villars, and involves rewriting the first propagator using Feynman parameterisation. In the general case, the P-V regularisation is accomplished by

$$\frac{1}{k^2-m^2+i\epsilon} \to \frac{1}{k^2-m^2+i\epsilon} - \sum_i \frac{a_i}{k^2-\Lambda_i^2+i\epsilon} \tag{9.27}$$

for a series of mass parameters Λ_i with indices i.

Derivation 9.5 (Pauli-Villars regularisation) Now consider the bubble diagram. Again, writing out the integration explicitly gives

$$\Sigma_{\text{bubble}} = \frac{(-i\lambda)^2}{2} \int_{-\infty}^{\infty} \frac{d^dp}{(2\pi)^d} \frac{i}{k^2-m^2+i\epsilon} \frac{i}{(p-k)^2+m^2+i\epsilon} \tag{9.28}$$

where the factor $-i$ accounts for the Wick rotation. We then perform the P-V regularisation:

$$\frac{1}{k^2-m^2+i\epsilon} \to \frac{1}{k^2-m^2+i\epsilon} - \underbrace{\frac{1}{k^2-\Lambda^2+i\epsilon}}_{1} = \frac{m^2-\Lambda^2}{(k^2-m^2+i\epsilon)(k^2-\Lambda^2+i\epsilon)} \tag{9.29}$$

where λ is again *large* (i.e. $\Lambda \gg m$) but finite, and 1 is a pseudo-mass term that represents a fictitious photon mass. The self-energy then reads

$$\Sigma_{\text{bubble}} = -\frac{\lambda^2\Lambda^2}{2} \int_{-\infty}^{\infty} \frac{d^dp}{(2\pi)^d} \frac{1}{(k^2-m^2+i\epsilon)(k^2-\Lambda^2+i\epsilon)((p-k)^2+m^2+i\epsilon)} \tag{9.30}$$

In regularisation, it is often useful to split the self-energy into finite and infinite (i.e. convergent and divergent) parts so that their physical significance can be better understood:

$$\Sigma(p^2) = \Sigma(0) + \tilde{\Sigma}(p^2) \tag{9.31}$$

In our example, they are

$$\Sigma_{\text{bubble}}(0) = -\frac{\lambda^2\Lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2+i\epsilon)^2\,(k^2-\Lambda^2+i\epsilon)} \tag{9.32}$$

$$\tilde{\Sigma}_{\text{bubble}}\left(p^2\right) = -\frac{\lambda^2\Lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{2pk-p^2}{(k^2-m^2)^2\,(k^2-\Lambda^2)\left((p-k)^2-m^2\right)} \tag{9.33}$$

9.5 Dimensional regularisation

The process known as *dimensional regularisation* is arguably the most important method of regularisation in perturbation theory. As we will see later, it can be used in conjunction with the minimal subtraction scheme to perform renormalisation.

Derivation 9.6 (Tadpole diagram) Let us tie up our previous loose end by evaluating the tadpole diagram step by step. The Feynman (loop) integral reads

$$I_2 = \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2-m_0^2+i\epsilon} \tag{9.34}$$

In this case, we label the regularisation scale M instead of Λ and make the rescaling

$$I_2 = \frac{1}{M^{d-4}} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2-m_0^2+i\epsilon} \tag{9.35}$$

We define another parameter $\varepsilon = 4-d$ (not to be confused with the ϵ in the term $i\epsilon$) and transition to Euclidean space by performing a Wick rotation $k^0 \to ik^0$, giving:

$$I_2 = iM^{\varepsilon} \int \frac{d^dp}{(2\pi)^d} \frac{1}{p^2+m_0^2} \tag{9.36}$$

which integrates as

$$I_2 = \frac{iM^\varepsilon}{(2\pi)^d} V_S \int_0^\infty d^d p \frac{p^{d-1}}{p^2 + m_0^2} \tag{9.37}$$

where V_S is the surface volume of a unit sphere in d dimensions that has the standard formula $V_S = \frac{2\pi^{d/2}}{\Gamma(d/2)}$.
By making the substitution $x = m_0^2/(p^2 + m_0^2)$, the integral evaluates as

$$I_2 = V_S \frac{m_0^2}{2(2\pi)^d} \left(\frac{M}{m_0}\right)^\varepsilon \int_0^1 (1-x)^{d/2-1} x^{-d/2} \tag{9.38}$$

Very sneakily, the integral $\int_0^1 (1-x)^{d/2-1} x^{-d/2}$ is actually three beta functions in disguise. We know this as it is a standard *Euler beta function*, which has the form

Definition 9.5 (Euler beta function)

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 x^{a-1}(1-x)^{b-1} dx \tag{9.39}$$

We can hence rewrite

$$I_2 = V_S \frac{m_0^2}{2(2\pi)^d} \left(\frac{M}{m_0}\right)^\varepsilon \frac{\Gamma(d/2)\Gamma(1-d/2)}{\Gamma(1)} \tag{9.40}$$

where $\Gamma(1)$ is simply 1. Inserting the so-called surface volume $V_S = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, we get

$$I_2 = \frac{m_0^2}{(2\pi)^{2-\varepsilon/2}} \left(\frac{M}{m_0}\right)^\varepsilon \Gamma(1-d/2) \tag{9.41}$$

We can perform the following expansions

$$\Gamma(1-d/2) = \Gamma(-\varepsilon/2) \approx -\frac{2}{\varepsilon} - \gamma - 1 + O(\varepsilon) \tag{9.42}$$

where γ is again the Euler-Mascheroni constant.

$$\left(\frac{M}{m_0}\right)^{4-d} = 1 + \epsilon \ln\left(\frac{M}{m_0}\right) + \mathcal{O}(\epsilon^2) \tag{9.43}$$

$$(2\pi)^{d/2} = (2\pi)^2 \left(1 - \frac{\epsilon}{2}\ln(2\pi) + \mathcal{O}(\epsilon^2)\right) \tag{9.44}$$

Putting it all together, and treating ε as the parameter of interest, we find

$$I_2 = -\frac{m_0^2}{(4\pi)^{d/2}} \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m_0^2}\right) \tag{9.45}$$

which diverges as $\epsilon \to 0$ or $d \to 4$.
As the tadpole diagram has only one measly vertex, its self-energy contribution is related to the Feynman integral by $\frac{\lambda^1}{2}$. Hence, the self-energy is

$$\Sigma_{\text{tadpole}} = -\frac{m_0^2 \lambda}{2(4\pi)^{d/2}} \left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) + \ln\frac{\mu^2}{m_0^2}\right) \tag{9.46}$$

Derivation 9.7 (Bubble diagram) We know that the loop integral of a bubble diagram is

$$I_4 = \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 - m^2 + i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon} \tag{9.47}$$

where $p = p_1 + p + 2 = p_3 + p_4$ is the external momentum and k is the internal loop momentum. Again, we employ the regularisation scale M and make the rescaling:

$$I_4 = \frac{1}{M^{d-4}} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2 + i\epsilon)^2} \frac{1}{((p-k)^2 - m^2 + i\epsilon)^2} \tag{9.48}$$

After Feynman parameterisation, we get

$$\begin{aligned} I_4 =& \frac{1}{M^{d-4}} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{1}{(x((p-k)^2 - m^2) + (1-x)(k^2 - m^2))^2} \\ =& M^{\varepsilon} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{1}{(k - xp)^2 + x(1+x)p^2 - m^2} \end{aligned} \tag{9.49}$$

By defining $l^\mu = k^\mu - xp^\mu$ and performing a Wick rotation, we can rewrite

$$I_4 = iM^{\varepsilon} \int \frac{d^d k}{(2\pi)^d} \int_0^1 dx \frac{1}{(l^2 + m^2 - x(1-x)p^2 + i\epsilon)^2} \tag{9.50}$$

Using the properties of the Gaussian function, we have

$$I_4 = \frac{i\pi^{d/2}}{(2\pi)^d} \Gamma(\varepsilon/2) \int_0^1 dx \frac{1}{(m^2 - x(1-x)p^2)^{\varepsilon/2}} \tag{9.51}$$

Let us perform expansions of these terms. Labelling $a^2 = m^2 - x(1-x)p^2$:

$$\Gamma(\varepsilon/2) = \frac{2}{\varepsilon} - \gamma + O(\epsilon) \quad \frac{1}{(a^2)^{\varepsilon/2}} = 1 - \frac{\varepsilon}{2} \ln a^2 + O(\varepsilon^2) \quad M^{\varepsilon} = (M^2)^{\varepsilon/2} = 1 + \frac{2}{\varepsilon} \ln M^2 + \cdots \tag{9.52}$$

Putting it all together, and treating ε as the parameter of interest, we find

$$I_4 = \frac{i\pi^{d/2}}{(2\pi)^d} \int_0^1 dx \left(\frac{2}{\varepsilon} - \gamma - \ln \frac{m^2 - x(1-x)p^2}{M^2} + O(\varepsilon) \right) \tag{9.53}$$

Using the standard integration techniques, we have

$$I_4 = -\frac{m_0^2}{(4\pi)^{d/2}} \left(\frac{2}{\varepsilon} - \gamma + \ln \frac{m_0^2}{\mu^2} \right) \tag{9.54}$$

which also diverges as $\epsilon \to 0$ or $d \to 4$.
As we have 2 vertices, the one-loop contribution in the Green's function, or rather the self-energy is related to the Feynman integral by $\frac{\lambda^2}{2}$. Hence

$$\Sigma_{\text{bubble}} = \frac{i\lambda^2 \pi^{d/2}}{2(2\pi)^d} \left(\frac{2}{\varepsilon} - \gamma + \ln \frac{m_0^2}{\mu^2} \right) \tag{9.55}$$

where the factor of λ^2 denotes 2 vertices in the bubble diagram.

Chapter 10

Renormalisation I: Preliminaries

Quote 10.1 The infinities are hidden behind the cloud?

Felix Halbwedl, on the watermark of Part III, 3 March 2025

Fortunately, there are no infinities in real-world physics. Unfortunately, we have not recovered this via regularisation, which only serves as an intermediate step to control divergences by introducing a regulator. To obtain finite, physically meaningful predictions, we must employ *renormalisation*.

10.1 Emergence of renormalisation

The key point lies in realising that quantities depending on λ, including e, m, ψ and A_μ, are actually merely the *tree-level* versions of the terms they *claim* to represent:

- So far, we have mistakenly treated the coupling constant λ as a fixed quantity. This is, in fact, not true: Quantum corrections introduce scale dependence, and λ evolves with energy scales as a result.
- Due to this, quantities we previously employed like the mass m and the charge e, also change in higher orders. This extends to the scalar field ψ, the vector field A_μ and the Green's function G, all of which are dependent on e and m. In higher orders, they are augmented by a series of correction terms, which can be written as a power series of the coupling constant.
- We will now denote the previously seen λ, m, e, ϕ, A and G as λ_0, m_0, e_0, ψ_0, A_0 and G_0, which we call the *bare* (read: unrenormalised) quantities.
- The renormalised (read: actual) quantities are then denoted simply as m, e, ψ, A and G, since they are the true quantities.

Remark 10.1 As infinities only emerge in loop integrals, the tree-level part of a propagator is unaffected by renormalisation. At each order, the renormalised counterpart of a bare quantity is different.

Quote 10.2 In practice, we can never measure bare charge or bare mass, because nature always includes the higher order corrections in real world interactions. But the concept of such bare quantities will serve us well in our analyses.

Robert D. Klauber

Importantly, renormalisation does not render regularisation redundant. In fact, the standard way of renormalising a field theory involves regularisation:

- Label the bare quantities with the subscript $_0$.
- Identify the divergences (i.e. infinities).
- Regularise the theory[1].

[1] We need to do this so that there is actually a way to evaluate our integral

- Replace all instances of the bare coupling constant and mass λ_0 and m_0 with their renormalised counterparts λ and m.
- Remove infinities from the bare quantities using the counterterms. This must be accomplished via specific normalisation schemes.
- Remove the regulator to 'deregularise' the theory, e.g.
  - In general, set d to the actual number of physical dimensions.
  - In cutoff regularisation, send $\Lambda \to \infty$.
  - In dimensional regularisation, send $\varepsilon \to 0$.

Theorem 10.1 (Renormalisability) If all divergences can be absorbed into a *finite* number of redefined physical parameters[a], the theory is known to be *renormalisable*. If an *infinite* number of counterterms is required, the theory is known to be *non-renormalisable*.

[a] Even renormalisable theories may still have infinities at intermediate stages, but these are systematically removed.

10.2 Renormalisability

In addition to our previous loose definition, we want to systematically determine whether a field theory is renormalisable. This can be done in more than one way.

Definition 10.1 (Mass dimension) The so-called *mass dimension* $[Q]$ of some quantity Q is the dimension of mass (or, assuming natural units, energy as well) units in physical units of Q.

The definition above is quite a mouthful, so let us clarify it with a few examples. For a theory in d dimensions:

- The mass dimension $[S]$ of an action S is always zero. As the action is dimensionless, it cannot possibly have units of mass. It can be decomposed to
$$[S] = [\mathcal{L}] + [d^d x] = 0 \tag{10.1}$$
where $[d^d x] = -d$ as $\int d^d x$ *integrates* over d independent coordinates.
- The mass dimension $[\mathcal{L}]$ of the Lagrangian density $\mathcal{L}$ is always d. This is derived by inserting $[d^d x] = -d$ into Equation 10.1.
- Consider a Lagrangian with only the kinetic term $\mathcal{L} = \frac{\partial^2 \phi^2}{2}$, which gives the expression
$$[\mathcal{L}] = [\partial^2] + [\phi^2] = 2[\partial] + 2[\phi] = 2 + 2[\phi] = d \tag{10.2}$$
The mass dimension $[\phi]$ of a field ϕ is thus
$$[\phi] = \frac{d-2}{2} \tag{10.3}$$
- Now consider a Lagrangian with a series of interacting terms $\mathcal{L} = \lambda_i \phi^i$[2], which gives the expression
$$[\mathcal{L}] = [\lambda_i] + [\phi^i] = [\lambda_i] + \frac{i(d-2)}{2} = d \tag{10.4}$$
For a specific term in the Lagrangian with i fields, the mass dimension of the term's corresponding coupling constant λ_i[3] is thus
$$[\lambda] = d - \frac{i(d-2)}{2} \tag{10.5}$$
Take ϕ^4 theory for example. We have $i = 4$ fields. As such, in 4 dimensions:
$$[\lambda] = d - \frac{4(d-2)}{2} = 4 - d = \varepsilon \tag{10.6}$$
where we recall the definition of ε from dimensional regularisation.

[2] Here, the subscript is an index while the superscript is an exponential.
[3] This denotes the renormalised coupling constant. As we know, the bare coupling constant is dimensionless. We will return to this later.

Theorem 10.2 (Renormalisability from $[\lambda_i]$) A coupling i in a field theory is renormalisable if $[\lambda_i] \geq 0$. If $[\lambda_i] < 0$, it is non-renormalisable. This means it needs a suppression factor and is treated in an *effective field theory* (EFT) framework.

A second framework is *power counting*, which is a very sophisticated way of saying dimensional analysis:

Definition 10.2 (Superficial degree of divergence) The *superficial degree of divergence* D is a convenient way to determine if a Feynman diagram diverges into infinity. It is the power of the momentum p in the Feynman diagram[a]:

$$D = dL - \sum_i (d_i - d)V_i \tag{10.7}$$

where L is the number of (independent) loops (i.e. momentum integrations) in the diagram, V_i is the number of vertices of the vertex type corresponding to the interaction term with i fields, and d_i is the mass dimension of the interaction term associated with vertex type i.

- If $D \geq 0$, the diagram is known to be *superficially divergent*:
  - If $D > 0$, the diagram leads to *logarithmic divergence*.
  - If $D = 0$, the diagram leads to *power law divergence* and requires counterterms.
- If $D < 0$, the diagram is convergent, and no renormalisation is needed.

[a] i.e. the momentum power in the numerator minus that in the denominator.

Theorem 10.3 (Renormalisability from power counting)

- A field theory is renormalisable if $d_i \leq d$. Through redefinitions, only a finite number of counterterms are needed. Examples are QED and QCD in $d = 4$.
- A field theory is super-renormalisable if $d_i < d$. This is a subset of renormalisable theories that have only a finite number of divergent Feynman diagrams (usually low-order loops). An example is ϕ^3 theory in $d = 6$.
- A field theory is non-normalisable if $d_i > d$. An infinite number of counterterms are needed. As this is not accomplishable, the theory becomes meaningless in high energies (but is acceptable in low energies). An example is GR in $d = 4$.

Let us consider this in the context of ϕ^4 theory in 4D, our simplest toy model. In the one-loop order, the 1-point loop (tadpole), the 2-point loop (bubble) and the 4-point loop are all divergent. Conversely, n-point loops with $n > 4$ do not contribute, as they have a negative superficial degree of divergence $D < 0$.

10.3 Counterterms

The bare quantities are related to their renormalised counterparts by the so-called *renormalisation factors* Z_λ, Z_m and Z_ϕ:

Definition 10.3 (ϕ^4 renormalisation factors)

$$\lambda_0 = Z_\lambda \lambda \quad \phi_0 = \sqrt{Z_\phi}\phi \quad m_0^2 = Z_m m^2 \tag{10.8}$$

The ϕ^4 theory Lagrangian in Equation 4.37 becomes

$$\mathcal{L} = \frac{Z_\phi}{2}\partial_\mu\phi\partial^\mu\phi + \frac{Z_m Z_\phi}{2}m^2\phi^2 + \frac{Z_\lambda \lambda Z_\phi^2}{4!}\phi^4 \tag{10.9}$$

The renormalised Feynman propagator, n-point vertex and n-point Green's function are then

$$D_F = \frac{D_{F,0}}{Z_\phi} = \frac{i}{Z_\phi(p^2 - Z_m m^2 + i\epsilon)} \quad \Gamma^{(n)} = Z_\phi^{-n/2}\Gamma_0^{(n)} \quad G^{(n)} = Z_\phi^{-n/2}G_0^{(n)} \tag{10.10}$$

Here we acknowledge a certain degree of hand-waviness of the renomalisation parameters:

- All physical quantities are finite.
- All physical quantities are independent of the renormalisation constants Z_i and ϵ (of the $i\epsilon$ fame).

One notable exception to this is the Lagrangian, which is not an observable. With our new friends, the renormalisation factors, it can be written as

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4}_{\text{renormalised Lagrangian}} + \underbrace{\frac{\delta Z_\phi}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\delta m^2\phi^2 + \frac{\delta\lambda}{4!}\phi^4}_{\text{counterterms}} \tag{10.11}$$

where the δZ_ϕ, δm^2 and $\delta\lambda$ are the so-called *counterterms*[4]:

Definition 10.4 (ϕ^4 counterterms)

$$Z_\phi = 1 + \delta Z_\phi \quad m_0^2 Z_\phi = m^2 + \delta m^2 \quad \lambda_0 Z_\phi^2 = \mu^{-\varepsilon}(\lambda + \delta\lambda) \tag{10.12}$$

Let us justify the counterterms one by one:

- Both Z_ϕ and δZ_ϕ are dimensionless.
- Z_ϕ is dimensionless, while m_0^2, m^2 and δm^2 have mass dimensions of 2.
- λ_0 is dimensionless, while from Equation 10.6, λ has mass dimension ϵ. To preserve the dimensionlessness of λ_0, we introduce the term μ^ε, where μ is the so-called *renormalisation scale* and has mass dimension 1.

Thus, we have explicitly established the ultimate goal of renormalisation. To solve for the renormalised quantities or the renormalisation constants, we must calculate the counterterms δ.

Note 10.1 (Running coupling) Importantly, we observe that the renormalised mass and field are independent of the renormalisation scale μ while the renormalised coupling constant is not. The renormalised coupling λ is well-known as the *running coupling* as the dependence shows that λ 'runs' with μ.

But what does μ physically mean? As it turns out, it is a part of *renormalisation conditions*, which are constraints imposed on the Z_is. This is actually an umbrella term for many concepts:

- We choose an arbitrary *renormalisation scale* μ[5] which the counterterms δ are dependent on.
- A corresponding (and likewise arbitrary) *renormalisation point*, which is essentially an artificial constraint we place on μ, is defined.
- Different choices of renormalisation conditions correspond to different *renormalisation schemes*, each of which fixes the counterterms in a specific way.

Some comments on renormalisation schemes should be made:

- Renormalisation schemes are merely mathematical conveniences, and physical variables should not differ among schemes.
- The lack of any technical restrictions on the choice of μ can be exploited: It is often practically convenient to choose μ to be of the same order as the characteristic energy scale of the physical process being studied.
- As different renormalisation schemes are different only by the choice of μ, the different incarnations of a renormalised quantity in different schemes are related to each other by finite constants.
- However, the renormalised Green's function is always the same regardless of the renormalisation scheme. This invariance (i.e. symmetry) gives rise to the *renormalisation group* which we will soon discuss. The renormalisation group also ties up our other loose end, which is the running of the coupling constants.

[4]We can think of the divergence as rubbish that leaks out during a calculation. We don't simply throw the rubbish onto the bare parameter directly. Instead, we introduce the counterterm, a trash bin, whose only job is to collect and *cancel* out the garbage. Once that is done, the renormalised result is 'clean'.

[5]This is meaningless without the next bullet point.

10.4 Renormalisation schemes

After many, many pages, we are now in a position to actually carry out what we set out to do at the beginning of this chapter - removing the divergence. Now that we have established that our arbitrary choice of μ does not affect the renormalised quantities, we can discuss specific *renormalisation schemes*. In this section, we shall introduce three schemes, which are by design almost always used with dimensional regularisation:

- The *minimal subtraction (MS) scheme*.
- The *modified minimal subtraction ($\overline{\text{MS}}$) scheme*, which is a slightly modified form of the MS scheme.
- The *on-shell (OS) scheme* or the *physical scheme*.

The first question that arises is why dimensional regularisation is the 'golden boy' of all three schemes. Previously, we introduced the renormalisation scale μ as a bookkeeping device to make the coupling constants dimensionally correct. The same can be said about the regulator M in dimensional regularisation. As such, we recognise that M is nothing but μ. For a tadpole diagram, this means:

$$I_2 = -\frac{m_0^2}{(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) + \frac{\mu^2}{m_0^2}\right) \tag{10.13}$$

From here on, the three schemes are rather intuitive. The idea is to modify the Feynman integral according to our renormalisation conditions, which ultimately leave the divergence (e.g. $\frac{2}{\varepsilon}$) is intact by design. The infinities in the bare term and the counterterm cancel out.

> **Quote 10.3** -The infinities might cancel each other out.
> -Grüß Gott! They just might.
>
> *Barclay and Einstein, in 'The Nth Degree'*

Let us first introduce the MS and $\overline{\text{MS}}$ schemes, which are highly similar.

Definition 10.5 (MS renormalisation conditions) In the MS scheme, only the divergent part $\frac{2}{\varepsilon}$ (i.e. the infinite pole) is preserved in the counterterm. It is so-called because the minimal possible subtraction is made: *only* the divergent pole $\frac{2}{\varepsilon}$ is deleted from the final renormalised mass.

Definition 10.6 ($\overline{\text{MS}}$ renormalisation conditions) The $\overline{\text{MS}}$ scheme is almost identical to the MS scheme. However, instead of only preserving the infinity in the counterterm, we also preserve the finite constants, removing only the renormalisation scale term $\frac{\mu^2}{m_0^2}$. The $\overline{\text{MS}}$ scheme is often preferred over the MS scheme as the nasty finite constants are also removed.

Now we go onto the OS scheme, which does not utilise dimensional regularisation. Terrifyingly, the renormalisation scale is not used either. Instead of introducing an arbitrary μ, everything is defined in terms of physical quantities[6], such as the physical mass, and any required scale dependence is effectively hidden inside the physical parameters.

Definition 10.7 (ϕ^4 theory OS renormalisation conditions)

- **Mass counterterm:** The 2-point propagator vanishes exactly at the physical mass shell
$$\Gamma^{(2)}(p^2 = m^2) = 0 \tag{10.14}$$
This ensures the physical renormalised mass m is the pole of the full propagator, i.e., the physical particle has mass m.
- **Field counterterm:**
$$\left.\frac{d\Gamma^{(2)}(p^2)}{dp^2}\right|_{p^2=m^2} = 1 \tag{10.15}$$
This ensures the propagator has unit residue at the pole, corresponding to a properly normalised one-particle state.

[6] Hence its alternate name.

- **Coupling counterterm:** The renormalised 4-point vertex function $\Gamma^{(4)}(p_i)$ is set to equal the physical coupling λ at a particular kinematic configuration. In the on-shell scheme, this is usually:

$$\Gamma^{(4)}(p^2 = m^2) = -i\lambda \tag{10.16}$$

This means that the coupling λ is defined as the physical 2-to-2 scattering amplitude at the symmetric point where all external particles are on-shell and the center-of-mass energy is just at threshold.

Remark 10.2 There is one *smol* caveat to the OS scheme. Even though we are not playing with dimensional regularisation or μ anymore, a vestige of μ remains in the form of the renormalised coupling definition $\lambda_0 Z_\phi^2 = \mu^{-\varepsilon}(\lambda + \delta\lambda)$. We note that the scale dependence is essentially 'absorbed' into physical quantities. Luckily, this means that the final result is μ-independent. For physical intuition, we hence often set μ to be simply the mass m. That is to say:

$$\lambda_0 Z_\phi^2 = m^{-\varepsilon}(\lambda + \delta\lambda) \tag{10.17}$$

10.5 1-loop renormalisation of ϕ^4 theory

Let us get a first taste of what renormalised variables look like:

Derivation 10.1 (Renormalised field) The field counterterm can be derived as

$$\delta Z_\phi = -\left.\frac{\partial \Sigma(p^2)}{\partial p^2}\right|_{p^2=m^2} \tag{10.18}$$

where the on-shell condition is accounted for. However, the self-energy is related to the Feynman integral, which is momentum-independent, by $\frac{\lambda}{2}$, and is as such also momentum-independent. This gives

$$\delta Z_\phi = 0 \to Z_\phi = 1 \tag{10.19}$$

in the tadpole and bubble diagrams. That is to say, in the 1-loop order, the renormalised field is identical to the bare field. This applies to all three schemes. In fact, the simplest correction emerges in the 2-loop order in a sunset diagram.

Derivation 10.2 (Renormalised mass) From Equation 10.19, the mass counterterm relationship for a tadpole diagram reduces nicely to

$$m_0^2 Z_\phi = m_0^2 = m^2 + \delta m^2 \tag{10.20}$$

From mass-energy equivalence (note that momentum is already out of the picture from the renormalised field), the mass counterterm is

$$\delta m^2 = \frac{\lambda}{2}\int \frac{d^d p}{(2\pi)^d}\frac{1}{p^2 - m_0^2 + i\epsilon} + \underbrace{O(\lambda^n)}_{①} \tag{10.21}$$

where only the first-order counterterm is written explicitly, and ① are higher-order counterterms. Treating the first-order counterterm (i.e. tadpole) Feynman integral with dimensional regularisation, we find that it is

$$\delta m^2 = -\frac{m_0^2 \lambda}{2(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln(4\pi) + \frac{\mu^2}{m_0^2}\right) \tag{10.22}$$

- **MS scheme:** The Feynman integral and the first-order counterterm are

$$I_{2,\mathrm{MS}} = \frac{m_0^2}{(4\pi)^{d/2}}\frac{2}{\varepsilon} \qquad \delta m^2_{\mathrm{MS}} = -\frac{m_0^2\lambda}{(4\pi)^{d/2}\varepsilon} \tag{10.23}$$

Thus, the divergence-free renormalised mass becomes

$$m^2 = m_0^2 + \delta m^2 = m_0^2 - \frac{m_0^2\lambda}{(4\pi)^{d/2}\varepsilon} \tag{10.24}$$

- **$\overline{\text{MS}}$ scheme:** The Feynman integral and the first-order counterterm are

$$I_{2,\overline{\text{MS}}} = \frac{m_0^2}{(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right) \qquad \delta m^2_{\overline{\text{MS}}} = -\frac{m_0^2\lambda}{2(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right) \tag{10.25}$$

  Thus, the renormalised mass is

$$m^2 = m_0^2 + \delta m^2 = m_0^2 - \frac{m_0^2\lambda}{2(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi\right) \tag{10.26}$$

- **OS scheme:** The mass counterterm is fixed to exactly remove the self-energy correction:

$$m^2_{\text{phys}} = m^2 + \Sigma^{(1)}(m^2) - \delta m^2 = m^2 \tag{10.27}$$

  Thus, the Feynman integral and the renormalised mass are

$$I_{2,\text{OS}} = \frac{m_0^2}{(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi + \frac{\mu^2}{m_0^2}\right) \quad m^2 = m_0^2 + \delta m^2 = m_0^2 - \frac{m_0^2\lambda}{2(4\pi)^{d/2}}\left(\frac{2}{\varepsilon} - \gamma + \ln 4\pi + \frac{\mu^2}{m_0^2}\right) \tag{10.28}$$

Derivation 10.3 (Renormalised coupling) Only the 4-point 1-loop (bubble) diagram contributes to the coupling counterterm. We remember that using dimensional regularisation, its integral evaluates as Equation 9.54. The general form of counterterm reads

$$\delta\lambda = \sum_i^{\text{all diagrams}} C_i \text{divergence}_i \tag{10.29}$$

where C is the multiplicity.
At one loop order, the only contributing diagram is the 4-point, 1-loop diagram, which has 3 possible Feynman diagrams corresponding to the s, t and u channels. The counterterm hence reads

$$\delta\lambda = 3I_4 \tag{10.30}$$

- **MS scheme:**

$$\delta\lambda = 3\frac{i\lambda^2\pi^{d/2}}{2(2\pi)^d}\frac{2}{\varepsilon} \tag{10.31}$$

- **$\overline{\text{MS}}$ scheme:**

$$\delta\lambda = 3\frac{i\lambda^2\pi^{d/2}}{2(2\pi)^d}\left(\frac{2}{\varepsilon} - \gamma\right) \tag{10.32}$$

- **OS scheme:** The conditon $\Gamma^{(4)}(p^2 = m^2) = -i\lambda$ makes the fixture $\lambda_{\text{phys}} = \lambda$. Hence

$$\delta\lambda = 3\frac{i\lambda^2\pi^{d/2}}{2(2\pi)^d}\left(\frac{2}{\varepsilon} - \gamma + \ln\frac{m_0^2}{\mu^2}\right) \tag{10.33}$$

One problem arises here. We see that different renormalisation schemes give different counterterms:

> **Aphorism 10.1 (Felix Halbwedl, 22 March 2025)** What matters most is the divergent part of the counterterms, they have to agree for all incarnations.
> The finite part of the counterterms is not relevant at all, it can be anything finite. The job of the counterterm is to hunt down and kill the divergency inside the bare mass. It cannot retrieve the physical mass held captive by the divergency. There we need to ask Mama nature for a helping hand, and measure the physical mass.
> I know, it's mathematically horrible, but in the end it's Mama nature we want to describe.

In this sense, the dependence of the renormalised quantities on the renormalisation scale is also eliminated by measuring quantities physically. This is the central point of the next two sections.

10.6 Callan-Symanzik equation

The creation of our so-called renormalisation scale may not seem immediately satisfying, as we have merely transferred the arbitrariness of δ with an arbitrariness of the renormalisation scheme (i.e. of μ). This apparent contradiction is ultimately reconciled by the so-called *Callan-Symanzik equation*, which makes sure that the theory remains physically meaningful despite the arbitrariness of μ.

Derivation 10.4 (Callan-Symanzik equation)

> **Quote 10.4** Now we exploit the simple but powerful fact that the bare vertex functions[a] $\Gamma_0^{(n)}$ do not know anything about the renormalisation scale μ.
>
> *John Cardy*
>
> [a] In our case, we use the bare Green's functions $G_{n,0}$ instead.

As the bare n-point Green's function $G_0^{(n)}$is independent from the renormalisation scale μ, it is safe to write

$$\mu\partial_\mu G_0^{(n)}(p,\lambda_0) = 0 \tag{10.34}$$

By consulting the part of Equation 10.10 that concerns the Green's function, we can write $G_0^{(n)}$ in terms of Z_ϕ and $G^{(n)}$. Hence, the equation above becomes

$$\mu\partial_\mu(Z_\phi^{n/2}(\lambda,\mu)G^{(n)}(p,\lambda,\mu) = 0 \tag{10.35}$$

Finally, we use the chain rule:

Theorem 10.4 (Callan-Symanzik equation) To preserve the invariance of physical variables, any *direct* change of Green's function $G_n(p,\lambda,\mu)$ due to the change of μ is compensated by corresponding changes of G_n due to changes of the coupling constant λ and the field ϕ, and the *total* change of G_n arising from μ is zero:

$$\frac{\partial G_n}{\partial\mu} + \frac{\partial\lambda}{\partial\mu}\frac{\partial G_n}{\partial\lambda} + \frac{n}{Z_\phi}\frac{\partial Z_\phi}{\partial\mu}G_n = 0 \tag{10.36}$$

where n is the number of points of the n-point function.

Let us go through the terms one by one:

- The first term is the *direct* dependence of G_n on the renormalisation scale μ.
- The second term is the *implicit* dependence of G_n on μ, but through the running coupling $\lambda(\mu)$, which is ultimately dependent on (or 'runs' with) μ. Due to this dependence, $\lambda(\mu)$ is ultimately not a physical quantity[a].
- The third term is a rescaling that accounts for the fact that the field itself is renormalised, and represents the contribution of the field strength renormalisation to the dependence of G_n on μ.
- As the sum of the terms is zero, μ is made irrelevant w.r.t. the physical quantities.

Remark 10.3 As noted in Quote 10.4, it should be immediately obivous that Equation 10.36 can also be derived from $\Gamma^{(n)}$. In fact, the same can be said for the field ϕ. The equations are

$$\frac{\partial\Gamma^{(n)}}{\partial\mu} + \frac{\partial\lambda}{\partial\mu}\frac{\partial\Gamma^{(n)}}{\partial\lambda} - \frac{n}{2Z_\phi}\frac{\partial Z_\phi}{\partial\mu}\Gamma^{(n)} = 0 \quad \frac{\partial\phi}{\partial\mu} + \frac{1}{\phi}\frac{\partial\phi}{\partial\mu}\phi = 0 \tag{10.37}$$

where the running coupling term is missing in the ϕ equation as the field has nothing to do with the coupling constant.

[a] Recall from experimental HEP that perturbation theory naturally fails for large coupling constants. Since different choices of μ effectively shift the way we split between 'low-energy' and 'high-energy' contributions, $\lambda(\mu)$ changes accordingly.

To better illustrate its physical meaning, we can now introduce two functions which are dimensionless and thus depend only on the equally dimensionless λ:

Definition 10.8 (Beta function) The *beta function* $\beta(\lambda)$, which describes how running coupling 'runs' with μ:

$$\beta(\lambda) = \frac{\delta\lambda}{\delta \ln \mu} = \mu \frac{\delta\lambda}{\delta\mu} \tag{10.38}$$

where δ is nothing but the functional derivative. This expression is also called the *renormalisation group equations.*

Definition 10.9 (Anomalous dimension) The *anomalous dimension* $\gamma(\lambda)$, which is technically a correction to the *scaling dimension*:

$$\gamma(\lambda) = -\frac{\mu}{Z_\phi}\frac{\delta Z_\phi}{\delta\mu} \tag{10.39}$$

We then have the massless version of Equation 10.36, which reads

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right) G_n = 0 \tag{10.40}$$

This establishes a relation between the dependence of G_n on μ and the dependence of G_n on λ.

Note 10.2 (Plot twist) At this point, we note that:

- The total dependence $\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda}$ of the Green's function G_n or the vertex $\Gamma^{(n)}$, explicit *and* implicit, is in fact non-zero and is equal to the field renormalisation term $n\gamma(\lambda)$.
- The presence of the field renormalisation term thus reveals a sinister plot twist. Neither G_n nor $\Gamma^{(n)}$ are physical quantities.

10.7 Renormalisation group

Now that we have eliminated $\lambda(\mu)$ and G_n as unphysical variables, we can finally turn to physical variables.

Derivation 10.5 (Renormalisation group invariance) A physical observable O observes

$$O \sim Z_\phi^{-n} G_n \tag{10.41}$$

where Z_ϕ^{-n} cancels out with the field renormalisation dependence of the Green's function. We thus have

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda}\right) O = 0 \tag{10.42}$$

which implies that the total explicit and implicit dependence of O on μ is zero.
This can be written more concisely. Inserting the full form of the beta function gives

$$\mu\left(\frac{\partial}{\partial\mu} + \frac{d\lambda}{d\mu}\frac{\partial}{\partial\lambda}\right) O = 0 \tag{10.43}$$

Contracting the chain rule, and we have

$$\mu\frac{d}{d\mu} O = 0 \tag{10.44}$$

Finally, we see that a physical observable O is μ-independent.

The μ-invariance illustrated by Equation 10.44 is interesting as it reminds us of symmetries under Lie groups we saw in *Spinors & Symmetries.* For this reason, we often speak of a *renormalisation group.* This is not actually a group, but rather a semigroup s transformations are not necessarily invertible.

Derivation 10.1 (Running coupling) We also want to establish a relation between the dependence of G_n on p and the dependence of G_n on λ. Such a relation can be found in a second alternative form of Equation 10.36, which involves a rescaling of the momentum. Let us begin with the *classical scaling* *equation*:

$$\left(p\frac{\partial}{\partial p} - n[\phi] + \mu\frac{\partial}{\partial \mu}\right) G_n = 0 \tag{10.45}$$

where we recall that $[\phi]$ is the mass dimension of ϕ.
Subtracting Equation 10.40 from Equation 10.45 gives

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} - n([\phi] + \gamma(\lambda))\right) G_n = 0 \tag{10.46}$$

This equation can be fitted for our good friend, the running coupling:

$$\left(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda}\right) \lambda(p/\mu) = 0 \tag{10.47}$$

Another note is that this running coupling must be a solution to the beta function equation (Equation 10.38), and an initial condition exists in the form of

$$\lambda(1) = \lambda_0 \tag{10.48}$$

One can thus rewrite the equation as

$$\int_{\lambda_0}^{\lambda(p/\mu)} \frac{\delta\lambda'(p/\mu)}{\beta(\lambda')} = \ln(p/\mu) \tag{10.49}$$

To sum it up, this equation tells us that a change in p inevitably induces a rescaling of λ and a rescaling of ϕ by proxy of $\gamma(\lambda)$:

- For $\beta(\lambda) > 0$, p rises and falls as λ does.
- For $\beta(\lambda) < 0$, p rises as λ falls and falls as λ rises.
- For $\beta(\lambda) = 0$, we label $\lambda|_{\beta(\lambda)=0} := \lambda^*$.

The idea of the *renormalisation group flow* lies in the change of the coupling constant λ due to the change in the beta function $\beta(\lambda)$[7]. In other words, it describes how $\beta(\lambda)$ 'flows' with λ. This is captured in Equation 10.40, the massless Callan-Symanzik equation. Disregarding the anomalous dimension $\gamma(\lambda)$, we see that

- For $\beta(\lambda) > 0$, the RG flow is towards larger values of λ:
  - The IR singularity happens when $\lambda \to 0$ and $\beta(\lambda) \to 0$. Due to the weak λ, perturbation theory is well-suited for the IR region. This is known as an *IR stable fixed point*.
  - The UV singularity happens when λ is *large* and positive. $\beta(\lambda)$ is thus *large* and positive. Due to the weak λ, perturbation theory fails in the UV region.
- For $\beta(\lambda) < 0$, the RG flow is towards smaller values of λ:
  - The UV singularity happens when $\lambda \to 0$ and $\beta(\lambda) \to 0$. Due to the weak λ, perturbation theory is well-suited for the UV region. This is known as an *UV stable fixed point*.
  - The IR singularity happens when λ is *large* and positive. $\beta(\lambda)$ is thus *large* and negative. Due to the weak λ, perturbation theory fails in the IR region.
- For $\beta(\lambda) = 0$, we consult Equation 10.46:
  - If $\beta(\lambda)$ goes from negative to positive through λ^*, we will see the momentum approaching zero as $\lambda \to \lambda^*$:

    $$p \to 0 \tag{10.50}$$

    This is an *IR stable zero*.

[7]Which itself dictates how λ evolves with the renormalisation scale μ.

– If $\beta(\lambda)$ goes from positive to negative through λ^*, we will see the momentum approaching infinity as $\lambda \to \lambda^*$:

$$p \to \infty \tag{10.51}$$

This is an *UV stable zero.*

This is an interesting case, as we can extract the behaviour of the Green's functions by setting $\lambda \to \lambda^*$ in Equation 10.46.

10.8 Källén-Lehmann spectral representation

Finally, it is useful to reflect upon what we have gone through in a more physical way. The default form of a renormalised tree-level Feynman propagator with mass m in Equation 10.10 should be well-known to the reader at this point. However, we have yet to expand this to higher orders.
As it turns out, there is a default way to represent a propagator of arbitrary order called the *Källén-Lehmann spectral representation*, where the full propagator *in momentum space* is solved via integrating over the mass parameter/renormalisation scale μ:

Definition 10.10 (Källén-Lehmann spectral representation)

$$G = i\int_0^\infty d\mu^2 \frac{\rho(\mu^2)}{p^2 - m^2 + i\epsilon} \tag{10.52}$$

Note 10.3 The Källén-Lehmann spectral representation does not actually assist with calculating counterterms. Rather, it shows that interacting propagators can always be expressed as a weighted sum of free propagators, which provides a non-perturbative insight into the structure of the theory.

The central point of this formalism is as follows:

- Effectively, the tree-level propagator can be regarded as a free particle, regardless of whether the theory itself is free or interacting.
- Higher-order propagators, which are effectively quantum corrections, represent the interacting parts of an (interacting) theory.

We can show this formulaically. For a free particle with mass m, the spectral function is

$$\rho(\mu^2) = Z_\phi^{-1}\delta(\mu^2 - m^2) \tag{10.53}$$

This ρ fixes μ to m, and Equation 10.52 reduces to Equation 10.10.
For interacting fields, our propagator under the Källén-Lehmann spectral representation can be rewritten as

$$G = \underbrace{\frac{i}{Z_\phi(p^2 - m^2 + i\epsilon)}}_{\text{free particle term}} + \underbrace{i\int_0^\infty d\mu^2 \frac{\sigma(\mu^2)}{p^2 - m^2 + i\epsilon}}_{\text{continuum terms}} \tag{10.54}$$

where the continuum terms are interacting field contributions from multi-particle states. Hence, the function $\sigma(\mu^2)$ is defined to be explicitly the interacting contribution:

$$\sigma(\mu^2) = \rho(\mu^2) - Z_\phi^{-1}\delta(\mu^2 - m^2) \tag{10.55}$$

Remark 10.4 Hence, the physical meaning of Equation 10.52 and (more obivously) Equation is seen. It describes the full propagator, representing the high-order quantum corrections that are deviations from the free propagator in Equation 10.10.
We end with some mathematical discussions. Intuitively, both $\rho(\mu^2)$ and $\sigma(\mu^2)$ are positive, the first of which implies that, for some order n of differentiation:

$$\frac{\partial^n D\left(-p^2\right)}{\partial\left(p^2\right)^n} = (-1)^n i\int_{m_t^2}^\infty d\mu^2 \frac{\rho\left(\mu^2\right)}{\left(p^2+\mu^2\right)^n} \tag{10.56}$$

Inserting the decomposed Equation 10.4 into this gives

$$Z_\phi^{-1} + \int_{m_t^2}^{\infty} d\mu^2 \sigma(\mu^2) = 1 \tag{10.57}$$

As $\sigma(\mu^2) > 0$, we have $0 \leq Z_\phi^{-1} < 1$.

Chapter 11

Renormalisation II: QED

> **Quote 11.1** Miew. Mowem Purrum.
>
> *Felix Halbwedl, 8 March 2025*

We are finally in a position to calculate the QED counterterms. Fortunately, the conceptual ideas of renormalisation have already been detailed in the last chapter. The MS, $\overline{\text{MS}}$ and OS schemes we have introduced also apply to QED. All we have to do is to develop a corresponding framework for vector fields.

11.1 Ward-Takahashi identity

With the renormalisation of QED, we are now again in the realm of gauge theories. As it turns out, gauge invariance affects renormalisation factors in some funny ways. This can be verified by going over the Slavnov-Taylor, Ward-Takahashi and Ward identities, which are the final development from applying Noether's theorem in QFT that we will see in a while.
It is expeidient to clarify the context that each identity live in, and how the identities relate to each other:

- The Slavnov-Taylor identity applies to the most general case, which is non-abelian gauge theories like QCD.
- The Ward-Takahashi identity is a reduction of the Slavnov-Taylor identity in the abelian limit, like QED.
- The Ward identity is a reduction of the Ward-Takahashi identity in the limit when the external fermions are on-shell and when momentu transfer is zero[1].

As we have not reached non-abelian gauge theories yet, we first derive the Ward-Takahashi identity.

Derivation 11.1 (Ward-Takahashi identity) Beginning with the QED generating functional

$$Z[\bar{J}, J, J^\mu] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu \exp\left[i\int d^4x \left(\mathcal{L}_{\text{QED}} + \bar{J}\psi + \bar{\psi}J + J^\mu A_\mu\right)\right] \tag{11.1}$$

where we recall J and $\bar(J)$ to be the fermion source terms, J^μ to be the boson source term, and that there exist the shorthands

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad D_\mu = \partial_\mu + ieA_\mu$$

It is again the Dyson-Schwinger equations that save the day. We recall that the generating functional is invariant under local gauge transformations, which, in QED, is our good friend, the U(1) symmetry:

$$\psi \to (1 + i\alpha(x))\psi, \quad \bar{\psi} \to \bar{\psi}(1 - i\alpha(x)), \quad A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\alpha(x)$$

[1] This is also called the *soft photon limit*.

Let us now solve for the variation of the generating functional. Intuitively, the variations of the fields read

$$\delta\psi(x) = i\alpha(x)\psi(x) \quad \delta\bar{\psi}(x) = -i\alpha(x)\bar{\psi}(x) \tag{11.2}$$

Under this, the action varies as

$$\delta S = \int d^4x\alpha(x)\partial_\mu j^\mu(x) \tag{11.3}$$

and the source terms vary as

$$\delta(\bar{J}\psi + \bar{\psi}J) = i\alpha(x)\bar{J}(x)\psi(x) - i\alpha(x)\bar{\psi}(x)J(x) \tag{11.4}$$

Putting it all together, the total variation of the integrand on the exponential is, to first order in α:

$$\delta\left[S + \int d^4x(\bar{J}\psi + \bar{\psi}J)\right] = \int d^4x\alpha(x)\left[\partial_\mu j^\mu(x) + i\bar{J}(x)\psi(x) - i\bar{\psi}(x)J(x)\right] \tag{11.5}$$

Hence, the variation of the generating functional reads

$$\delta Z = i\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS+i\int \bar{J}\psi+\bar{\psi}J}\int d^4x\alpha(x)\left[\partial_\mu j^\mu(x) + i\bar{J}(x)\psi(x) - i\bar{\psi}(x)J(x)\right] \tag{11.6}$$

Again, consulting the Dyson-Schwinger equations, we know that $\delta z = 0$. We can then divide both sides by $i\int\mathcal{D}\bar{\psi}\mathcal{D}\psi e^{iS+i\int\bar{J}\psi+\bar{\psi}J}$: Factor out the $\alpha(x)$ and write in terms of expectation values:

$$\int d^4x\alpha(x)\left[\partial_\mu\langle j^\mu(x)\rangle + \bar{J}(x)\langle\psi(x)\rangle - \langle\bar{\psi}(x)\rangle J(x)\right] = 0 \tag{11.7}$$

Let us consider the simplest vertex, which is a three-point function with two fermions and one photon, where the fermion absorbs/emits a photon of momentum q[a]. According to Equation 7.24, taking functional derivatives of the generating functional with respect to J and $\bar{J}$ gives us the Green's function:

$$\partial^x_\mu G^\mu(x,y,z) = \partial_\mu\langle T[j^\mu(x)\psi(y)\bar{\psi}(z)]\rangle = \delta(x-y)\langle T[\psi(y)\bar{\psi}(z)]\rangle - \delta(x-z)\langle T[\psi(y)\bar{\psi}(z)]\rangle \tag{11.8}$$

where x, y and z are the 4-positions of the three legs.
We take Fourier transforms of both sides. For each component, this works as

$$\begin{aligned} j^\mu(q) &= \int d^4xe^{iq\cdot x}j^\mu(x) \quad \psi(p+q) = \int d^4ye^{i(p+q)\cdot y}\psi(y) \\ \bar{\psi}(p) &= \int d^4ze^{-ip\cdot z}\bar{\psi}(z) \quad \langle T[\psi(y)\bar{\psi}(z)]\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(y-z)}S_F(p) \end{aligned} \tag{11.9}$$

where p is the total (4-)momentum of the incoming fermion, and q is the momentum of the incoming/outgoing photon[b].
Putting it all together, we see that the delta functions are eliminated:

$$\int d^4xe^{iqx}\partial_\mu\langle T[j^\mu(x)\psi(y)\bar{\psi}(z)]\rangle = e^{iqy}S_F(y-z) - e^{iqz}S_F(y-z) \tag{11.10}$$

This reduces to

$$iq^\mu\langle j^\mu(q)\psi(p+q)\bar{\psi}(p)\rangle = S_F(p) - S_F(p+q) \tag{11.11}$$

where From the OPI decomposition, we can rewrite the LHS in terms of the vertex function $\Gamma^\mu(p+q,p)$:

$$\langle j^\mu(q)\psi(p+q)\bar{\psi}(p)\rangle = S_F(p+q)\Gamma^\mu(p+q,p)S_F(p) \tag{11.12}$$

Inserting this gives

$$iq^\mu S_F(p+q)\Gamma_\mu(p+q,p)S_F(p) = S_F(p) - S_F(p+q) \tag{11.13}$$

Dividing both sides by $S_F^{-1}(p+q)S_F^{-1}(p)$, we find the common form of the Ward-Takahashi identities:

Theorem 11.1 (Ward-Takahashi identity)

$$iq^\mu\Gamma_\mu(p+q,p) = S_F^{-1}(p+q) - S_F^{-1}(p) \tag{11.14}$$

where $\Gamma_\mu(p+q,p)$ is the full (ampurared) vertex function, $S_F(p+q)$ is the full (interacting, renormalised) fermion propagator and $S_F(p)$ is the same propagator but lacking one external photon.

The Ward-Takahashi identity is also called the *rigid identity*, as *rigid symmetries* is an alternative name for global symmetries.

[a]Note that this is effectively a momentum transfer by q on the fermion.
[b]Note that the sign is automatically taken care of if we know whether it is incoming or outgoing.

Quote 11.2 Because the Ward-Takahashi identity relates the $n+1$-point function with the n-point function, which is useful if you want to construct the most general 3-points function.

Felix Halbwedl, on why Feynman diagrams are employed in the Ward-Takahashi identity, 5 February 2025

Some physical comments can be made:

- The identity is a non-perturbative statement as no perturbation theory is used. It holds beyond just tree level.
- While we used the simplest case of the fermion-photon propagator in our derivation, the final identity in Equation 11.14 actually applies to all vertices in QED. That is to say, we can use this identity for any vertex in QED that has at least one external photon involved (we will not prove this).
- In this case, we only consider (or 'zoom into') the fermion that the photon operates on, which has the incoming and outgoing full propagators $S_F(p)$ and $S_F(p+q)$, and all other fermions or photons in the vertex are ignored. $\Gamma_\mu(p+q,p)$ is still the full vertex, which can now be better written as $\Gamma_\mu(\cdots,p+q,p)$ where $\cdots$ are the momenta of all other particles we have ignored.

In conclusion, the key takeaway from this identity is twofold:

- Following directly from current conservation and Noether's theorem, there is, when a fermion emits or absorbs one photon, a correlation between the pre- and post-emission/absorption full fermion propagators with the product of the photon momentum and the full vertex function.
- The longitudinal part of the (3-point) vertex function (i.e. the part proportional to q^μ) is completely determined by the difference in the fermion self-energies (which are 2-point functions).

Note 11.1 (Sneak peek of gauge theory) The Ward-Takahashi identity is actually a specific form of the Slavnov-Taylor identity, which is used in the (most general case of) non-abelian gauge theories we will soon discuss, like QCD. The Slavnov-Taylor identity is derived from the BRST symmetry of the gauge-fixed Lagrangian.

$$q^\mu\Gamma_\mu(p+q,p) = S_F^{-1}(p+q)G(p+q,p) - G(p+q,p)S_F^{-1}(p) \tag{11.15}$$

where $G(p+q,p)$ is the ghost-fermion scattering kernel. This term disappears in the abelian limit, and we are left with the Ward-Takahashi identity.

11.2 Ward identity

Let us prove that the Ward-Takahashi identity reduces to the Ward identity in the long wavelength (IR singularity) case, where the vertex function is on-shell and when momentum transfer is zero ($q \to 0$).

Derivation 11.2 (Ward idnetity) As we are concerned with the $q \to 0$ limit, it is legal to perform a Taylor expansion of the RHS around $q = 0$

$$S_F^{-1}(p+q) - S_F^{-1}(p) \approx q^\mu \frac{\partial S_F^{-1}(p)}{\partial p^\mu} \tag{11.16}$$

Substituting this into the Ward-Takahashi identity, we find

$$q^\mu \Gamma_\mu(p,p) = q^\mu \frac{\partial S_F^{-1}(p)}{\partial p^\mu} \tag{11.17}$$

Eliminating the common q^μ:

Theorem 11.2 (Ward identity)

$$\Gamma_\mu(p,p) = \frac{\partial S_F^{-1}(p)}{\partial p^\mu} \tag{11.18}$$

This is the generalised form of the Ward identity.

We can derive a few useful variants of the Ward identity:

Derivation 11.3 (Differential form of the Ward idnetity) From the known relation

$$S_F(p) = \left[S_F^{-1}(p)\right]^{-1} \tag{11.19}$$

We can use the product rule and find

$$\frac{\partial S_F(p)}{\partial p^\mu} = -S_F(p)\left(\frac{\partial S_F^{-1}(p)}{\partial p^\mu}\right) S_F(p) \tag{11.20}$$

Inserting this into Equation 11.18, we find

$$\frac{\partial S_F(p)}{\partial p^\mu} = -S_F(p)\Gamma_\mu(p)S_F(p) \tag{11.21}$$

which is the differential form of the Ward identity. This is the incarnation of the Ward identity that Yasushi Takahashi started with when he derived the Ward-Takahashi identity in 1957.

Derivation 11.4 (Scattering amplitude form of the Ward idnetity) Consider a process where an *external* photon with momentum q^μ is emitted from an external charged fermion line. The scattering amplitude reads

$$\mathcal{M}^\mu(q) = \bar{u}(p')\Gamma^\mu(p',p)u(p) \tag{11.22}$$

Let us apply the external photon momentum q_μ on both sides:

$$q_\mu \mathcal{M}^\mu(k) = q_\mu \bar{u}(p')\Gamma^\mu(p',p)u(p) \tag{11.23}$$

Using Equation 11.14, we see that

$$q_\mu \mathcal{M}^\mu(k) = \bar{u}(p')[S_F^{-1}(p') - S_F^{-1}(p)]u(p) \tag{11.24}$$

When the external fermions are on-shell, we know that the inverse propagator annihilates on-shell spinors. As such

$$S_F^{-1}(p)u(p) = 0 \quad \bar{u}(p')S_F^{-1}(p') = 0 \tag{11.25}$$

We hence find the all-too-familiar form of the Ward identity

$$q_\mu \mathcal{M}^\mu(q) = 0 \tag{11.26}$$

This essentially reflects current conservation in QED, or equivalently, gauge invariance of the S-matrix. Let us now investigate this external photon. As it is external, there are only two transverse polarisations, whose 4-versors are, by convention:

$$\epsilon_\mu^1 = (0,1,0,0) \quad \epsilon_\mu^2 = (0,0,1,0) \tag{11.27}$$

As the matrix elements are those of the external photon, they project onto the polarisation versor states. As such, $\mathcal{M}^\mu(q)$ also has only two non-zero components[a] - $\mathcal{M}^1(q)$ and $\mathcal{M}^2(q)$. From Equation 11.18, the momentum hence has

$$q_\mu = (k_0, 0, 0, -k_3) \quad q^\mu = (k_0, 0, 0, k_3) \tag{11.28}$$

where the transverseness of the two polarisations is seen.

[a] Yes! $\mathcal{M}^\mu(q)$ is a vector! Bet you didn't catch that, did you?

11.3 Counterterms

Let us begin with our good friend, the gauge-fixed Lagrangian in Equation 5.131 and make a cosmetic replacement of the gauge 'coupling' ξ in the gauge-fixing term

$$\frac{1}{\xi} = \lambda \quad \frac{1}{2\xi}A^\mu\partial_\mu\partial_\nu A^\nu = \frac{\lambda}{2}A^\mu\partial_\mu\partial_\nu A^\nu \tag{11.29}$$

Again, we acknowledge the fact that the quantities here are actually bare quantities:

$$\mathcal{L}_{\text{QED}} = \bar{\psi}_0(i\gamma^\mu\partial - m_0)\psi_0 - \frac{1}{2}A_0^\mu(\Box g_{\mu\nu} - \partial_\mu\partial_\nu)A_0^\nu - e_0\bar{\psi}_0\gamma^\mu A_{\mu,0}\psi_0 + \frac{\lambda_0}{2}A_0^\mu\partial_\mu\partial_\nu A_0^\nu \tag{11.30}$$

We can now write down the conventional renormalisation factors, some of which are defined slightly differently from those in ϕ^4 theory. To begin with, we start with the provisional set of five renormalisation factors

$$\psi_0 = \sqrt{Z_\psi}\psi \quad A_0 = \sqrt{Z_A}A \quad m_0 = Z_m m \quad e_0 = Z_e e \quad \lambda_0 = Z_\lambda\lambda \tag{11.31}$$

Let us define another three provisional, so-called, renormalisation factors:

$$S_{F,0} = Z_S S_F \quad D_{F,0}^{\mu\nu}F = Z_D D_F^{\mu\nu} \quad \Gamma_0 = Z_\Gamma\Gamma \tag{11.32}$$

It should be intuitively obvious that we can represent Z_S, Z_D and Z_Γ via the original five renormalisation factors, which we will prove now.

Derivation 11.5 (Propagators) The fermion and photon propagators are defined as the following two-point functions

$$S_F(x-y) = \langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle \tag{11.33}$$
$$D_F^{\mu\nu}(x-y) = \langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle \tag{11.34}$$

Expressing the renormalised quantities in terms of their bare counterparts and renormalisation factors:

$$S_{F,0}(x-y) = \langle 0|T\{\psi_0(x)\bar{\psi}_0(y)\}|0\rangle = Z_\psi\langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle = Z_\psi S_F(x-y) \tag{11.35}$$
$$D_{F,0}^{\mu\nu}(x-y) = Z_A\langle 0|T\{A^\mu(x)A^\nu(y)\}|0\rangle = Z_A D_F^{\mu\nu}(x-y) \tag{11.36}$$

Hence, we find that

$$Z_S = Z_\psi \quad Z_D = Z_A \tag{11.37}$$

Derivation 11.6 (Vertex) Z_Γ is a bit trickier and requires us to use the Ward-Takahashi identity. Let us begin with the Ward-Takahashi identity with renormalised quantities

$$q_\mu\Gamma^\mu(p+q,p) = S^{-1}(p+q) - S^{-1}(p)$$

Writing the renormalised quantities in their bare counterparts and renormalised factors give

$$q_\mu Z_\Gamma^{-1}\Gamma_0^\mu(p+q,p) = Z_\psi\left[S_0^{-1}(p+q) - S_0^{-1}(p)\right] \tag{11.38}$$

However, we note that the Ward-Takahashi identity also holds for bare quantities:

$$q_\mu\Gamma_0^\mu(p+q,p) = S_0^{-1}(p+q) - S_0^{-1}(p) \tag{11.39}$$

Dividing Equation 11.39 by Equation 11.38 gives us

$$Z_\Gamma = \frac{1}{Z_\psi} \tag{11.40}$$

Remark 11.1 Often, Z_Γ, Z_ψ and Z_A are written as $1/Z_1$ (where $\Gamma = Z_1\Gamma_0$)[a], Z_2 and Z_3 due to having been so-denoted in the original 1950s literature.

[a] Usually Z_Γ is defined as $\Gamma = Z_\gamma\Gamma_0$, making it identical to Z_1. In this book, we define it as $\Gamma_0 = Z_\Gamma\Gamma$ (i.e. the inverse of Z_1) to keep it consistent with Z_S and Z_D.

This is not the end of our troubles, because as it turns out, even the original five renormalisation factors relate to each other.

Derivation 11.7 ($Z_\lambda Z_A = 1$) This can be proven by looking at the gauge fixing term. Writing all the terms explicitly gives

$$\frac{\lambda_0}{2}\left(\partial_\mu A_0^\mu\right)^2 = \frac{Z_\lambda\lambda}{2}\left(\partial_\mu\left(Z_A^{1/2}A^\mu\right)\right)^2 \tag{11.41}$$

This is simply

$$\frac{\lambda_0}{2}(\partial_\mu A_0^\mu)^2 = Z_\lambda Z_A\frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 \tag{11.42}$$

At this point, we note that the gauge fixing term has a unique property: its bare and renormalised forms are *always* identical. Hence

$$Z_\lambda Z_A\frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 = \frac{\lambda}{2}\left(\partial_\mu A^\mu\right)^2 \tag{11.43}$$

from which we can immediately read off

$$Z_\lambda Z_A = 1 \tag{11.44}$$

Derivation 11.8 ($Z_eZ_A^{1/2} = 1$) This can be proven by inspecting the interaction term. Again, we write all terms explicitly

$$ie_0\bar{\psi}_0\gamma^\mu\psi_0A_{0,\mu} = i(Z_ee)(Z_\psi^{1/2}\psi)^\dagger\gamma^\mu(Z_\psi^{1/2}\psi)(Z_A^{1/2}A_\mu) = Z_eZ_\psi Z_A^{1/2}(ie\bar{\psi}\gamma^\mu\psi A_\mu) \tag{11.45}$$

Recall that the interaction term represents the vertex function. As such, this actually describes vertex renormalisation

$$ie_0\bar{\psi}_0\gamma^\mu\psi_0A_{0,\mu} = Z_\Gamma(ie\bar{\psi}\gamma^\mu\psi A_\mu) = Z_\phi(ie\bar{\psi}\gamma^\mu\psi A_\mu) = Z_eZ_\psi Z_A^{1/2}(ie\bar{\psi}\gamma^\mu\psi A_\mu) \to Z_\phi = Z_eZ_\psi Z_A^{1/2} \tag{11.46}$$

from which we can immediately read off

$$Z_eZ_A^{1/2} = 1 \tag{11.47}$$

We are now finally in a position to state all the QED renormalisation factors:

Definition 11.1 (QED renormalisation factors)

- **Variables:**

$$\psi_0 = \sqrt{Z_\psi}\psi \quad A_0 = \sqrt{Z_A}A \quad m_0 = Z_mm \quad e_0 = Z_A^{-1/2}e \quad \lambda_0 = Z_A^{-1}\lambda \tag{11.48}$$

- **Propagators and vertex:**

$$S_{F,0} = Z_\psi S_F \quad D_{F,0}^{\mu\nu}F = Z_AD_F^{\mu\nu} \quad \Gamma_0 = Z_\psi\Gamma \tag{11.49}$$

The Lagrangian hence looks like

$$\mathcal{L}_{\text{QED}} = Z_\psi\bar{\psi}\left(i\partial\!\!\!/ - Z_mm\right)\psi + Z_A\frac{1}{2}A^\mu\left(\Box g_{\mu\nu} - \partial_\mu\partial_\nu\right)A^\nu + Z_\psi e\bar{\psi}A\!\!\!/\psi + \frac{\lambda}{2}A^\mu\partial_\mu\partial_\nu A^\nu \tag{11.50}$$

11.4 Tensor decomposition

In QED, the full forms of the fermion propagator, photon propagator and vertex can still be found by summing the loop contributions. This reads

$$iS^{-1}(p) = iS_0^{-1}(p) + \Sigma(p) \quad i\left(D^{-1}\right)^{\mu\nu}(q) = i\left(D_0^{-1}\right)^{\mu\nu}(q) + \Pi^{\mu\nu}(q) \quad \Gamma^{\mu}(p,q) = \Gamma_0^{\mu}(p,q) + \Lambda^{\mu}(p,q) \tag{11.51}$$

where $\Sigma(p)$ is the fermion self-energy, $\Pi^{\mu\nu}(q)$ is the photon vacuum polarisation and $\Lambda^{\mu}(p,q)$ is the vertex correction. We are now interested in formulating an expression for these three terms, which will involve decomposing them.
We recall that in ϕ^4 theory, the 2-point propagator function and 4-point vertex function decomposes as

$$\Gamma^{(2)}(p^2) = p^2 - m^2 + \Sigma(p^2) \quad \Gamma^{(4)}(s,t,u) = -i\lambda + \Gamma^{(4)}_{\text{1-loop}}(s,t,u) + \Gamma^{(4)}_{\text{2-loop}}(s,t,u) + \cdots \tag{11.52}$$

This decomposition is quite intuitive, as ϕ^4 theory is merely a scalar theory. QED is more complicated in that one of the propagators is tensorial. Hence, all three quantities have a tensorial structure, which forces us to make use of the so-called *tensor decomposition*. Let us go through the terms one by one.

Derivation 11.9 (Fermion propagator) We have two terms, the free (bare) propagator and the self-energy. The free propagator, which reads $S_{F,0}(p) = \frac{i}{\not{p} - m + i\epsilon}$, decomposes as

$$iS_{F,0}^{-1}(p) = \not{p} - m \tag{11.53}$$

Now we investigate self-energy. To preserve Lorentz covariance, the only possible tensor structures are $\not{p}$ and $\mathbb{I}$. The independent variable becomes p^2, which is Lorentz-invariant:

$$\S_F igma(p) = \left[1 - Z_2^{-1}(p^2)\right]\not{p} - \left[Z_2^{-1}(p^2)Z_m(p^2) - 1\right]m \tag{11.54}$$

For ease of reading, we often conventionally define the *fermion wavefunction renormalisaton function* $A(p^2)$ *fermion mass function* $M(p^2)$, which are nothing but shorthands:

$$\S_F igma(p) = A(p^2)\not{p} + B(p^2)m \tag{11.55}$$

Substituting them into Equation 11.51 the full inverse propagator decomposition:

$$iS_F^{-1}(p) = \not{p} - m + A(p^2)\not{p} + B(p^2)m = \left[1 + A(p^2)\right]\not{p} - \left[1 - B(p^2)\right]m \tag{11.56}$$

$$iS_F^{-1}(p) = A(p^2)(\not{p} - \S_F igma_M(p^2)\mathbb{I}) = A(p^2)(\not{p} - \S_F igma_M(p^2)) \tag{11.57}$$

Inverting this expression returns us

$$S_F(p) = \frac{i}{\left[1 + A(p^2)\right]\not{p} - \left[1 - B(p^2)\right]m} \tag{11.58}$$

Conversely, we can define the *wavefunction renormalisation function* $Z(p^2)$ and the *momentum-dependent effective mass* $M(p^2)$:

$$Z(p^2) = \frac{1}{1 + A(p^2)} \quad M(p^2) = \frac{1 - B(p^2)}{1 + A(p^2)}m \tag{11.59}$$

This gives us a form similar to the bare propagator

$$S_F(p) = \frac{iZ(p^2)}{\not{p} - M(p^2)} \tag{11.60}$$

Derivation 11.10 (Photon propagator) The photon propagator has rank 2, and the only possible structures are the metric $g^{\mu\nu}$ and the momenta product $q^{\mu}q^{\nu}$.
Let us again investigate the terms one by one. In the Feynman gauge, the free photon propagator is

$$D_0^{\mu\nu}(q) = \frac{-ig^{\mu\nu}}{q^2 + i\epsilon} \tag{11.61}$$

Its inverse with a factor of i is then

$$i\left(D_0^{-1}\right)^{\mu\nu}(q) = q^2 g^{\mu\nu} \tag{11.62}$$

Previously in the Ward identity, we have seen that $\Pi^{\mu\nu}(q)$ must be transverse. As such, it takes the form

$$\Pi^{\mu\nu}(q) = \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2) \tag{11.63}$$

where $\Pi(q^2)$ is a scalar function known as the *vacuum polarisation scalar*.
Substituting into the inverse propagator, we find

$$i(D^{-1})^{\mu\nu}(q) = q^2 g^{\mu\nu} + \left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Pi(q^2) = q^2\left[g^{\mu\nu} + \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\Pi(q^2)\right] \tag{11.64}$$

Inverting this again in the Feynman gauge gives

$$D^{\mu\nu}(q) = \frac{-i}{q^2\left[1+\Pi(q^2)\right]}\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) + \lambda^{-1}\frac{-iq^\mu q^\nu}{(q^2)^2} \tag{11.65}$$

We can write a shorthand of this by defining the transverse part of the inverse photon propagator:

$$T^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \tag{11.66}$$

which gives the decomposition as

$$D^{\mu\nu}(q) = \frac{-i}{q^2\left[1+\Pi(q^2)\right]}T^{\mu\nu} + \lambda^{-1}\frac{-iq^\mu q^\nu}{(q^2)^2} \tag{11.67}$$

Derivation 11.11 (Vertex) As per the Feynman rules, the tree-level vertex factor is $-ie\Gamma^\mu = -ie\gamma^\mu$, making the tree-level contribution to the vertex simply the gamma matrices γ^μ. Denoting loop corrections as $\Lambda^\mu(p', p)$, we then have the full, corrected vertex

$$\Gamma^\mu(p', p) = \gamma^\mu + \Lambda^\mu_{\text{1-loop}}(p', p) + \Lambda^\mu_{\text{2-loop}}(p', p) + \cdots \tag{11.68}$$

where the 1-loop correction reads, for free indices α and β

$$\Lambda^\mu(p', p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\alpha \frac{i}{\not{k} - m}\gamma^\mu \frac{i}{\not{k} + \not{q} - m}\gamma^\beta\left(\frac{-ig_{\alpha\beta}}{(k-p)^2}\right) \tag{11.69}$$

and so on.
This decomposition is often given a more systematic treatment. Ultimately, we construct Γ^μ from available vectors and matrices: vectors p^μ, $p'^\mu = p^\mu + q^\mu$ and q^μ; Dirac matrices: γ^μ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. Finally, we have scalar functions $F_i(q^2)$ known as *form factors*, which depend only on the Lorentz scalar q^2.

$$\Gamma^\mu(p, q) = F_1(q^2)\gamma^\mu + F_2(q^2)i\sigma^{\mu\nu}q_\nu + F_3(q^2)q^\mu \tag{11.70}$$

where:

- $F_1(q^2)$ is the *Dirac form factor* concerned with charge and normalisation.
- $F_2(q^2)$ is the *Pauli form factor* concerned with anomalous magnetic moment.
- $F_3(q^2)$ which vanishes due to the Ward identity in Equation 11.18.

Due to the vanishing of $F_3(q^2)$, the physically relevant vertex decomposition is

$$\Gamma^\mu(p, q) = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i}{2m}\sigma^{\mu\nu}q_\nu \tag{11.71}$$

This is identical to Equation 11.68. To see this, we consider the tree-level case, where $F_1(q^2) = 1$ and $F_2(q^2) = 0$. This recovers the tree-level vertex:

$$\Gamma^\mu(p, q)\big|_{\text{tree}} = \gamma^\mu = \Gamma^\mu_0(p, q) \tag{11.72}$$

The same idea follows at higher orders. The equivalence lies in the fact that the correction $\Lambda^\mu(p', p)$ for each order can be recovered by setting $F_1(q^2)$ and $F_2(q^2)$ to their specific incarnation in that order.

11.5 Counterterms

At this point, it should be obivous that in QED, we are more interested in the corrections to the propagators and the vertex than the counterterms of a specific value (save for charge). This is for practical reasons. We therefore have the following renormalisation conditions:

Definition 11.2 (QED OS renormalisation conditions)

- **Mass counterterm:** The electron propagator has a pole at the physical (measured) mass m

$$\Sigma(\not{p})\big|_{\not{p}=m} = 0 \tag{11.73}$$

  where $\Sigma(\not{p})$ is the electron self-energy.

- **Fermion field counterterm:** The residue of the electron propagator at the physical pole is unity

$$\frac{d}{d\not{p}}\left[\not{p} - m - \Sigma(\not{p})\right]\Bigg|_{\not{p}=m} = 1 \to \frac{d\Sigma(\not{p})}{d\not{p}}\Bigg|_{\not{p}=m} = 0 \tag{11.74}$$

- **Photon field counterterm:** The photon remains massless, i.e., its propagator has a pole at $p^2 = 0$

$$\Pi^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi(q^2) \quad \text{with} \quad \Pi(0) = 0 \tag{11.75}$$

  where $\Pi^{\mu\nu}(q)$ is the photon self-energy (vacuum polarisation).

- **Coupling (i.e. charge) counterterm:**

$$Z_1\Gamma^\mu(p',p)\big|_{p=p',q=0} = \gamma^\mu \tag{11.76}$$

  This is known as the *Thomson limit* and ensures that the vertex function reduces to the bare interaction in the limit of zero momentum exchange.

From here on, we are in a position to calculate specific corrections. As of the time of writing, QED has been renormalised up to the fifth loop order in key quantities, particularly the anomalous magnetic moment of the electron. Calculations for fourth and fifth loop order renormalisations, pioneered by the much-celebrated Toichiro Kinoshita, were only developed from the 1990s on.

11.6 Beyond perturbation theory

> **Quote 11.3** Chromodynamics is weird, with couplings that grow with distances and...
>
> *Alessio Serafini, 27 March 2025*

While powerful, perturbation theory has limitations. It fails for large coupling constants ($\lambda \ll 1$). In cases where the coupling constant is not strictly *small* or new physics appears at different energy scales, perturbation theory can still be applied in a restricted sense. *Effective field theories* (EFTs) use perturbative methods to focus on low-energy phenomena by integrating out high-energy degrees of freedom. We conclude with some philosophical remarks:

- As we proceeded through Part III, we saw how everything became increasingly arbitrary and handwavy. There is first a breakdown of tree-level QFT at the loop level. However, instead of overthrowing this theory, we introduced renormalisation, which is effectively an 'extension' of tree-level QFT that becomes increasingly unwieldy as the diagrams increase in their complexity.

- This is because QFT is actually an EFT[2]. Unlike fundamental theories, EFTs are valid up to a certain energy scale but are expected to break down beyond that scale, giving rise to new physics. Higher-energy effects then manifest as *suppressed corrections* in the form of higher-dimensional

[2]This idea, considered revolutionary at the time of its inception, was formulated by Ken Wilson, whose general ideas are often known as *Wilsonian renormalisation*.

operators. In this sense, all classical physics is nothing but effective theories w.r.t. modern physics. In the same vein, GR and modified theories of gravity are likewise EFTs of a future grand unified theory.

- As the standard model is effectively a collection of QFTs, it, too, is an EFT. Most of its extensions, like supersymmetry (SUSY), are likewise EFTs. In the 1970s, physicists initially thought that supergravity (SUGRA), a gravitational extension of SUSY, might provide a complete quantum theory of gravity by itself. However, it was later realised that supergravity alone is not sufficient to quantise gravity at all energy scales. Instead, string theory emerged as a more complete framework, with supergravity appearing as a low-energy effective theory of string theory.

Ultimately, the goal of physics is then to construct a single fundamental theory, the candidates of which include string theoryas well as non-perturbative quantum gravity theories[3] asymptotically safe quantum gravity and loop quantum gravity.

> **Quote 11.4** Our two statements actually describe the same, but are written down on two different sides of the same medal.
> As they said, it's not the difficulty of the territory. You can avoid the most dangerous cliffs if you take the right route. It's more the sheer size which bends your knee, as you have to cross long distances in the realm of QFT.
>
> *Felix Halbwedl, on Quote 1.1 and Quote 1.2, 17 November 2024*

[3]This is motivated by the fact that perturbation theory makes GR non-renormalisable.

Figure 11.1: 'There is nothing we can do...'