ROB2004
Robotic Manipulation and Locomotion

Laboratory II: Forward Kinematics

The goal of the laboratory is to find the position of the foot as a function of the joint angles

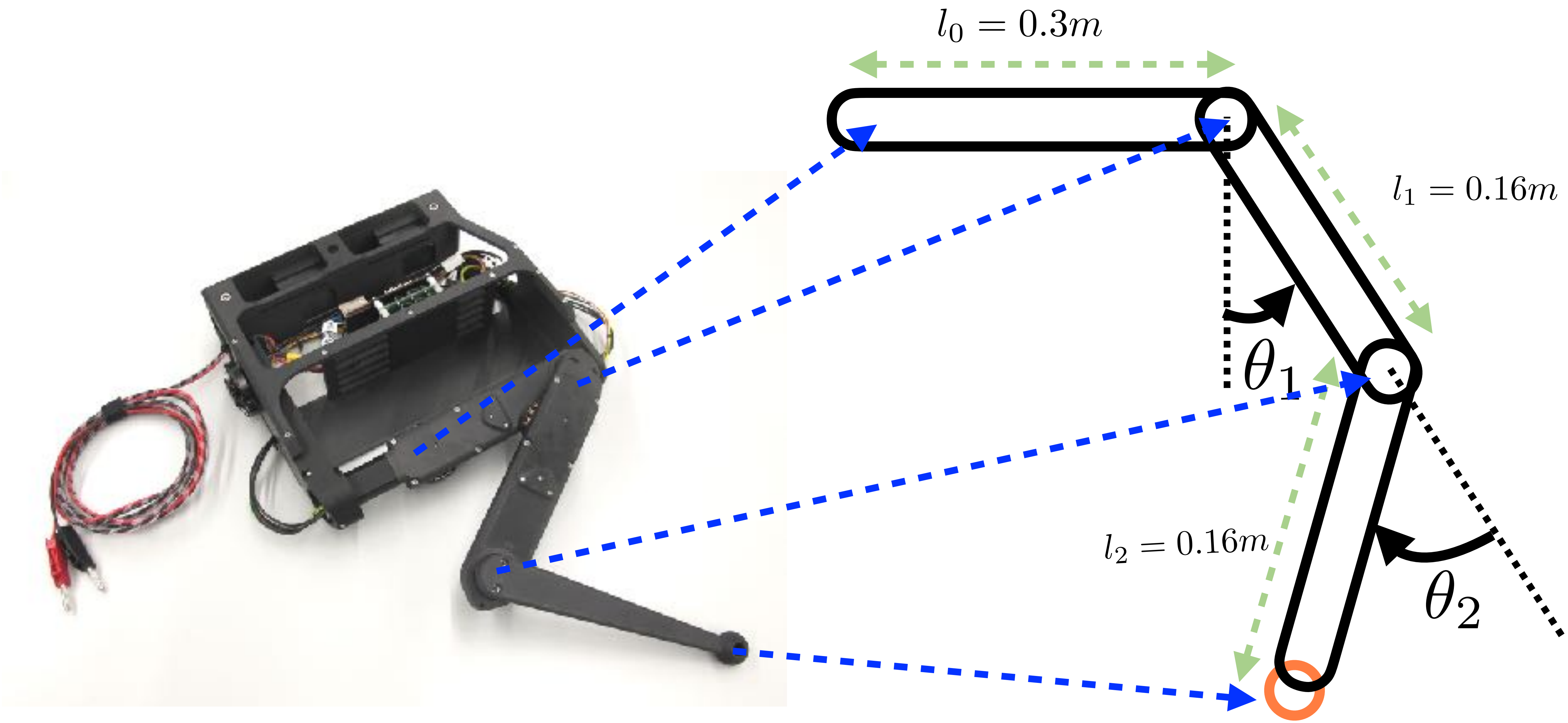
The computation of the foot (or any other point on the robot) given the configuration (i.e. joint angles) is called <u>forward kinematics</u>

The goal of the laboratory is to find the position of the foot as a function of the joint angles

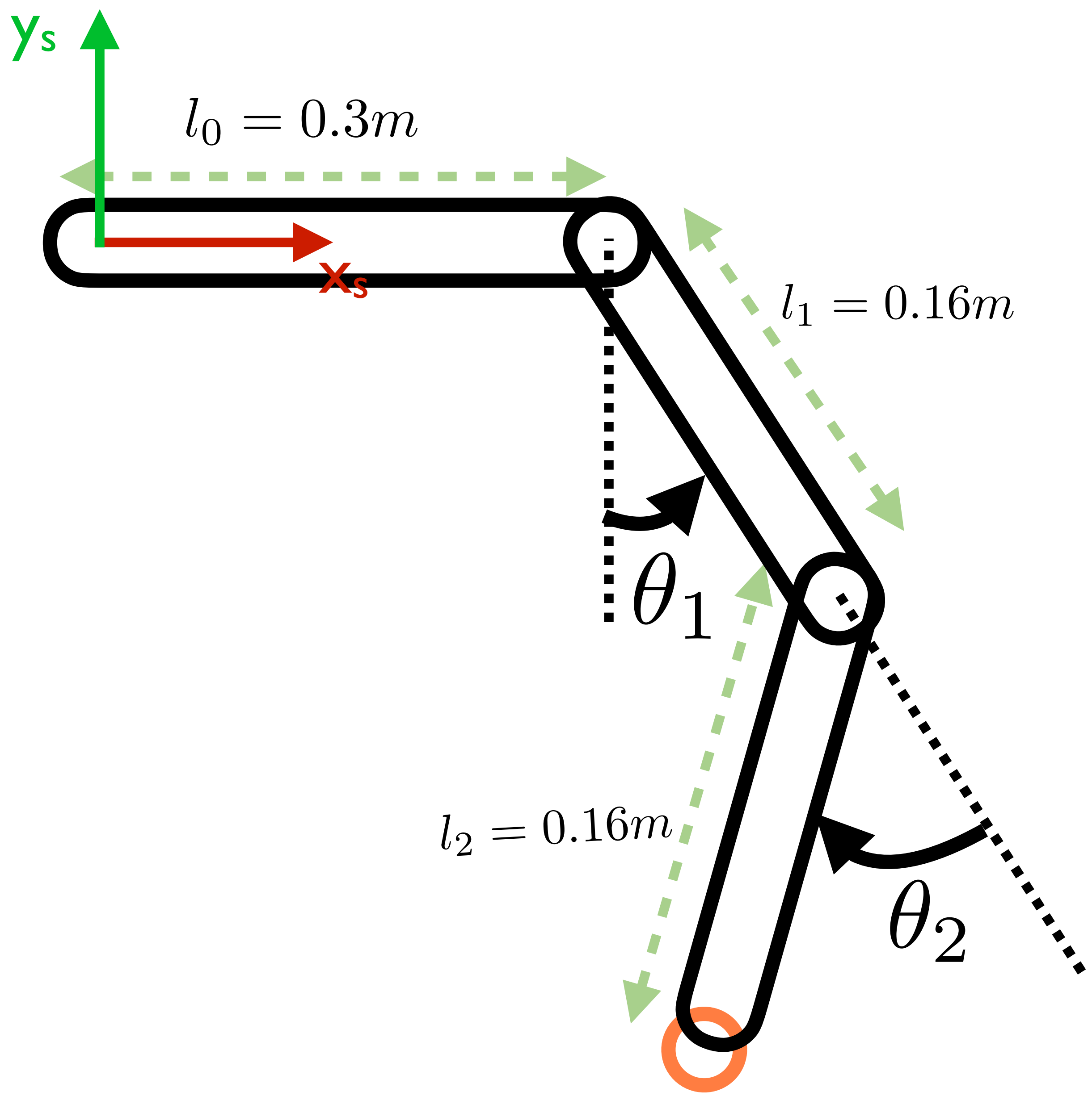
Assume we have the first DOF fixed at position 0

With the remaining joints, the leg/finger moves in a 2D plane

First: let's draw a 2D model of the leg



Second: let's attach a few coordinate frames



Frame {s} is our fixed frame, i.e. the spatial frame

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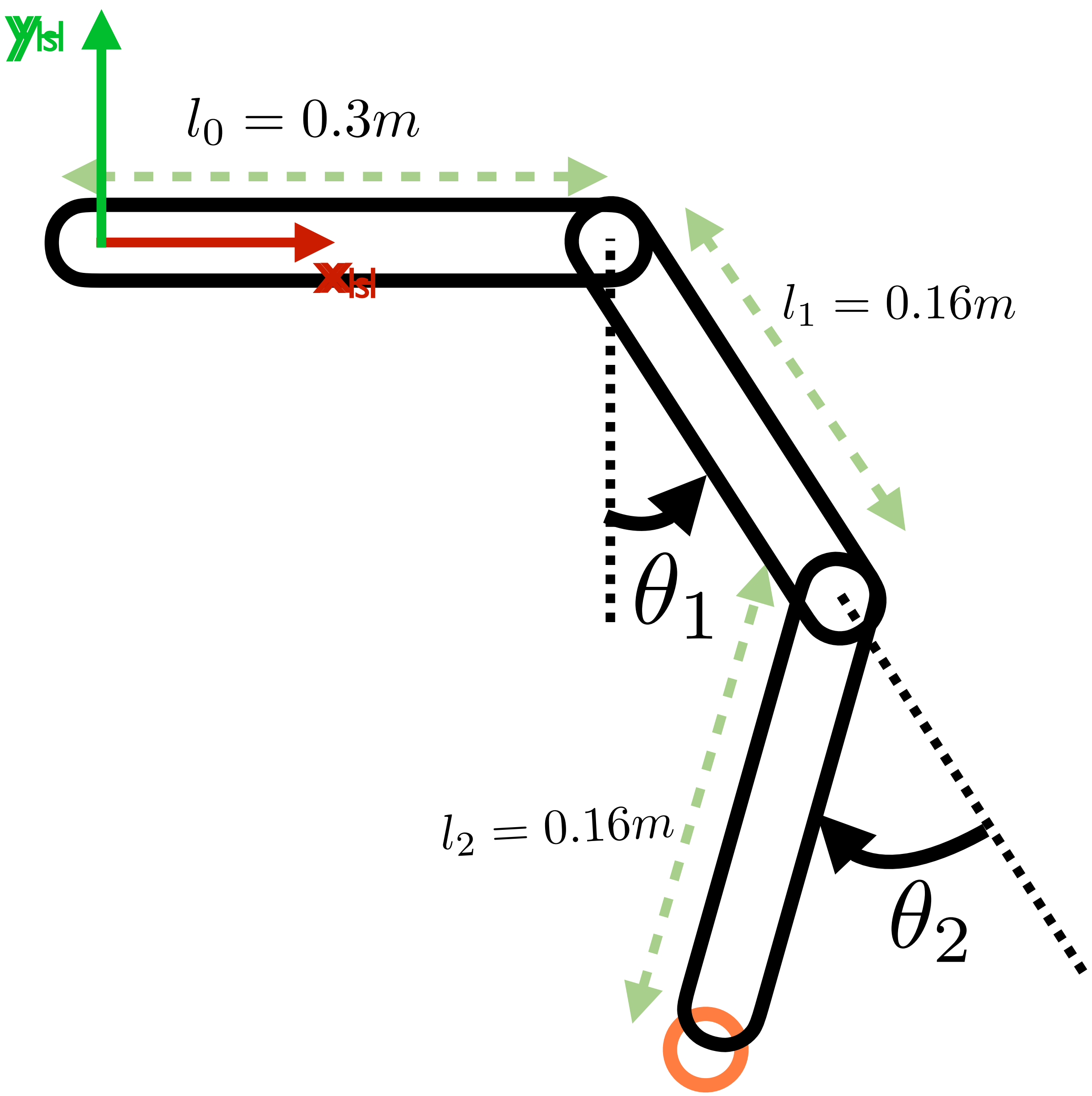

Frame {s} is our fixed frame, i.e. the spatial frame
The hip frame {H} is translated by l_0 and rotated by θ_0 with respect to frame $\{s\}$

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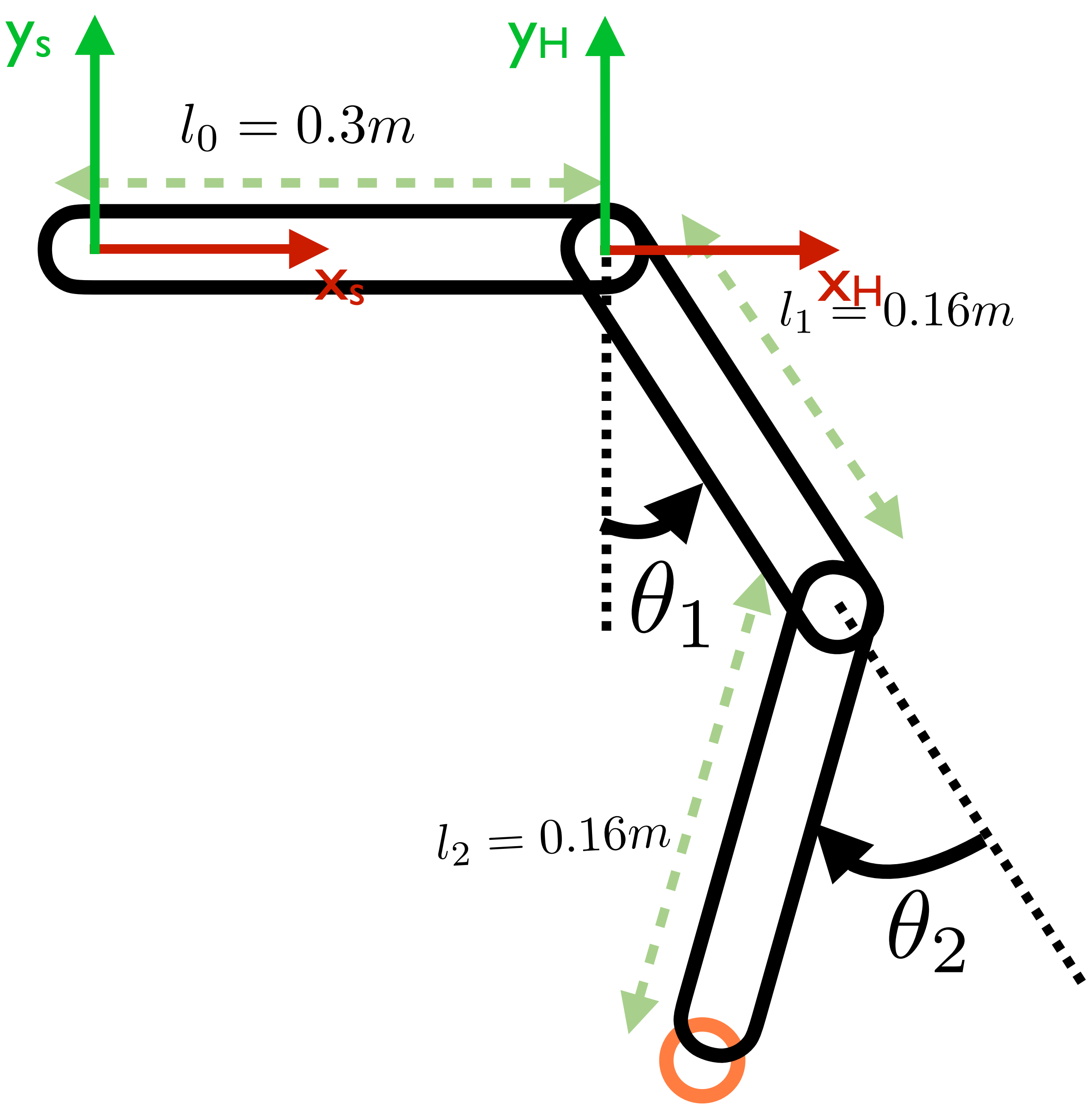

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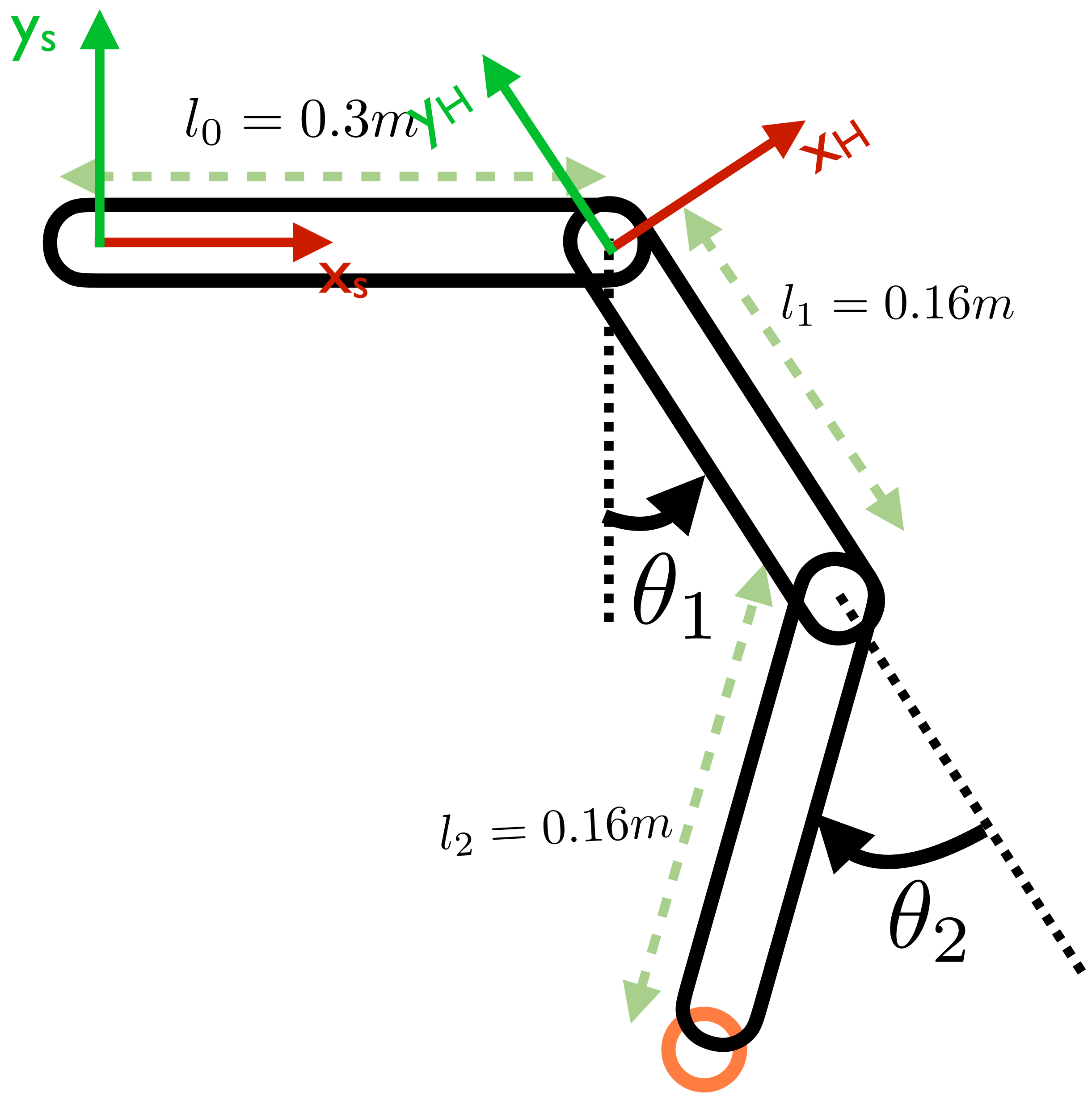

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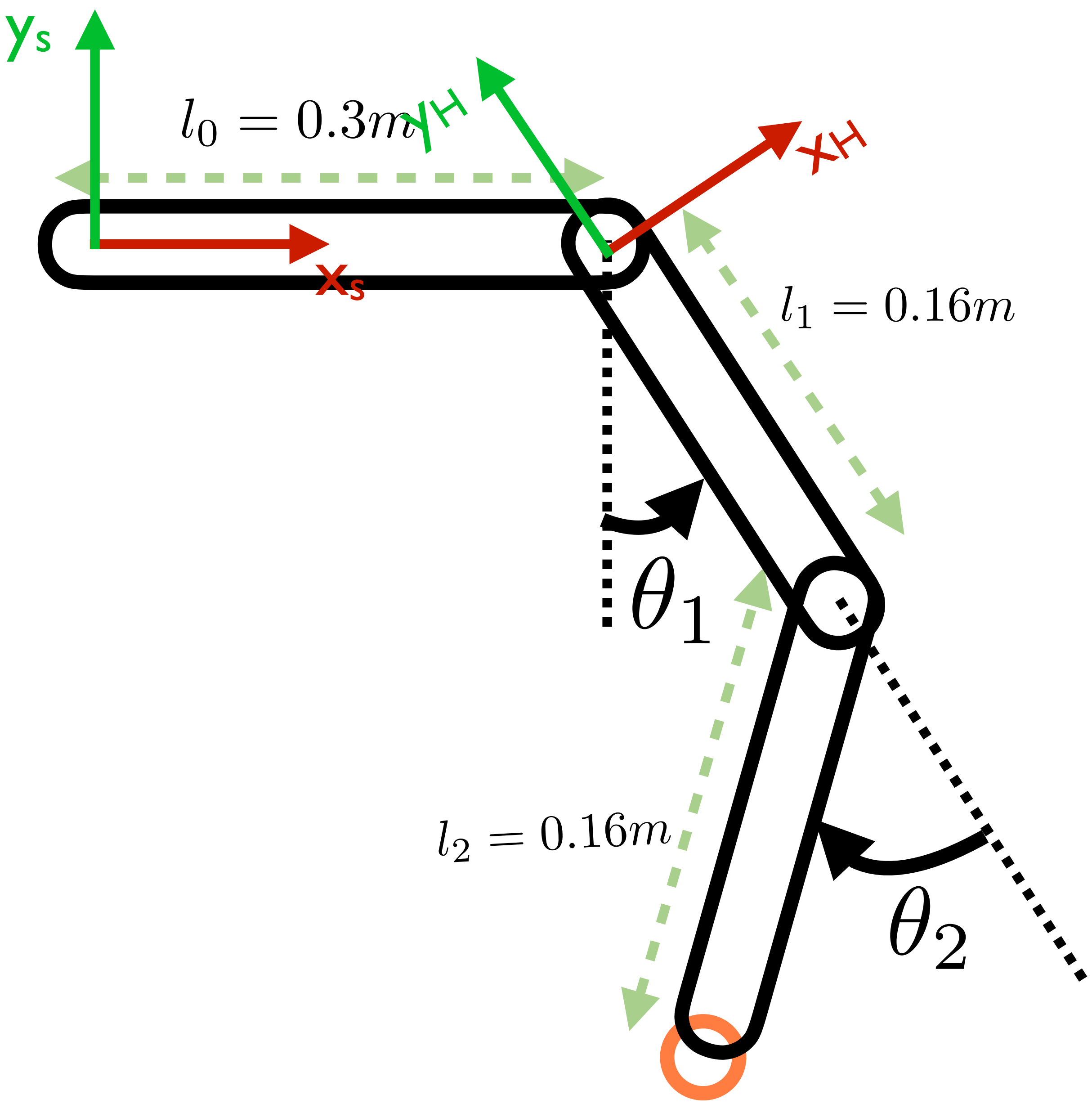

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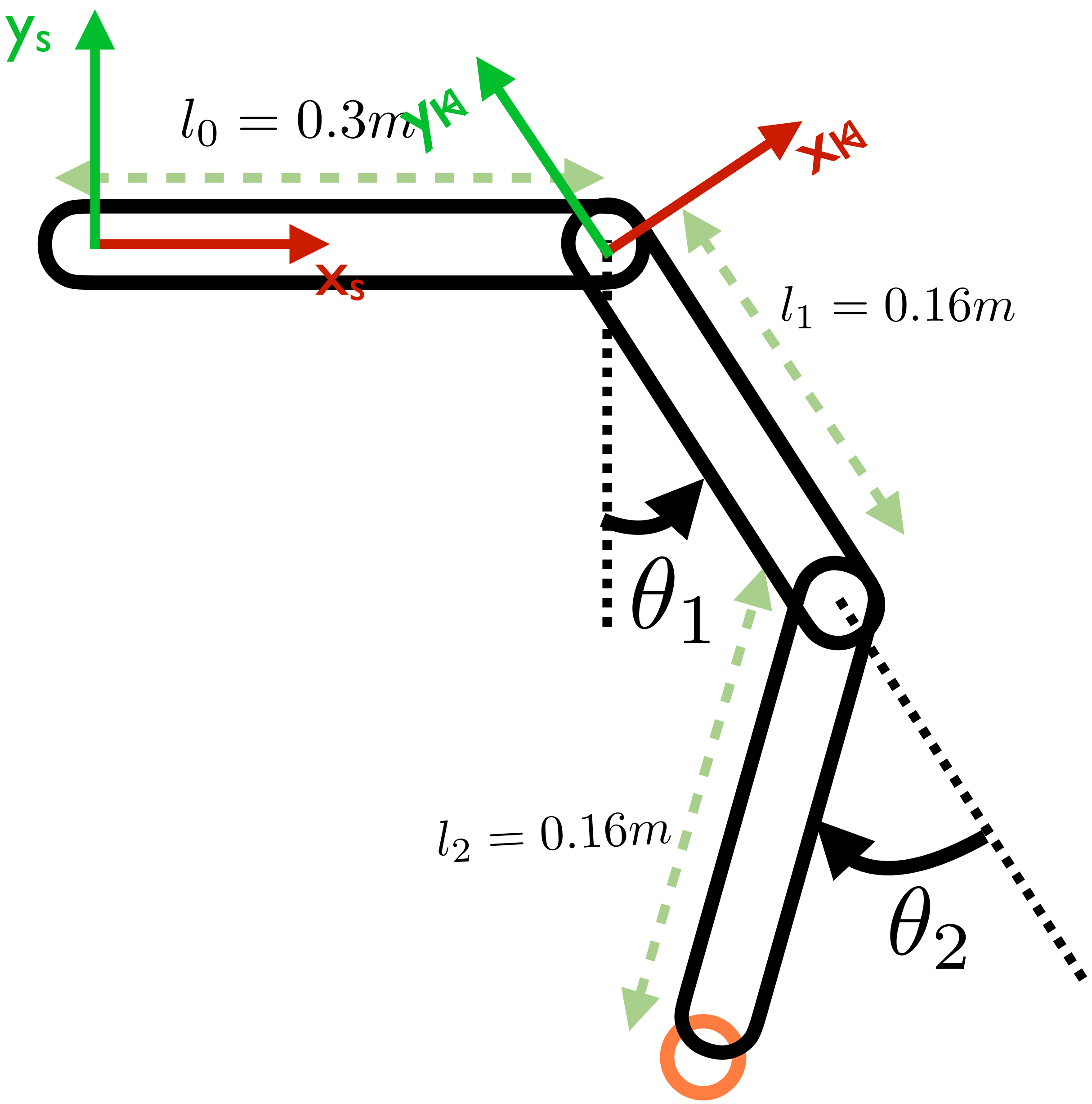

Frame {s} is our fixed frame, i.e. the spatial frame
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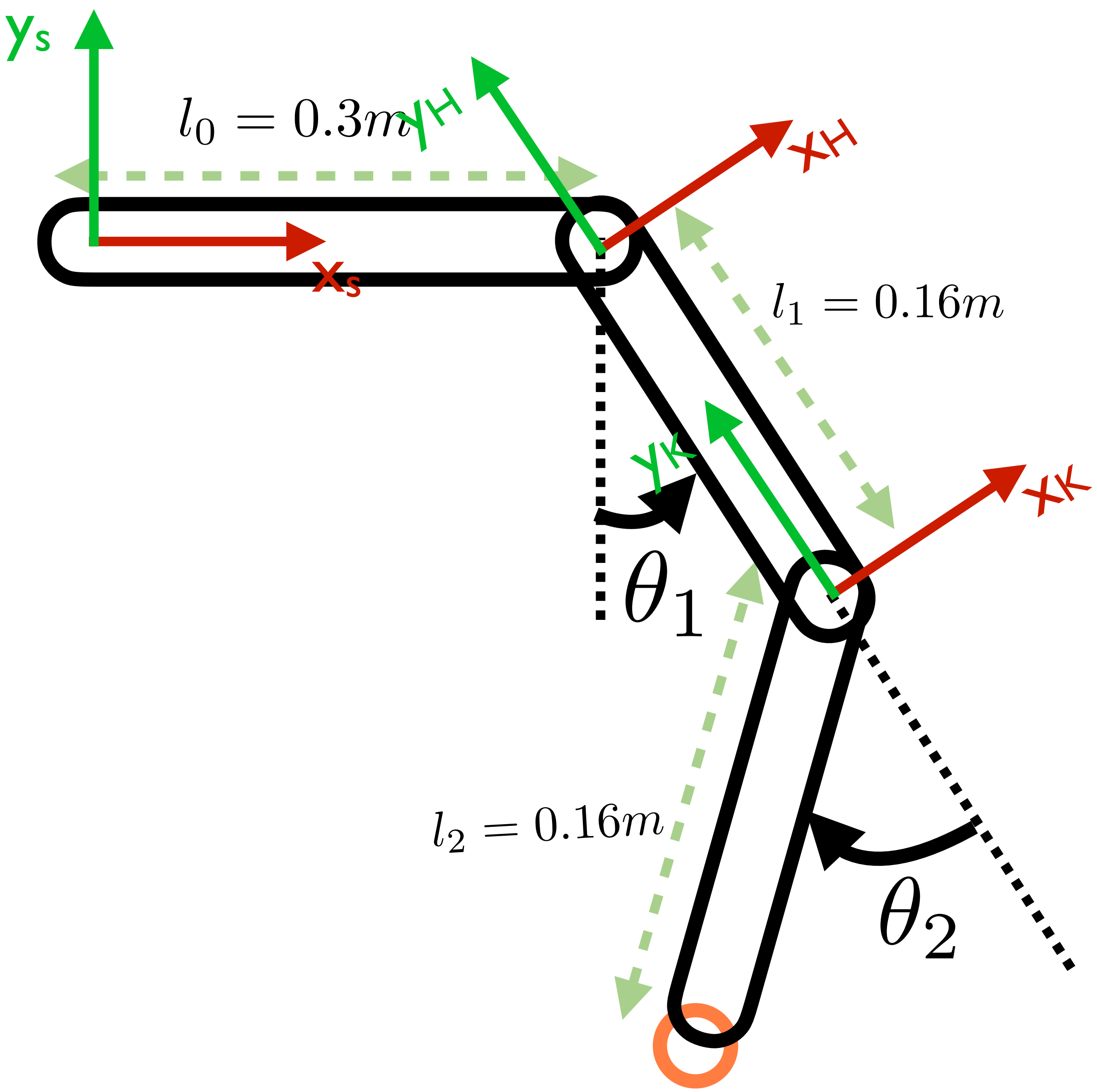

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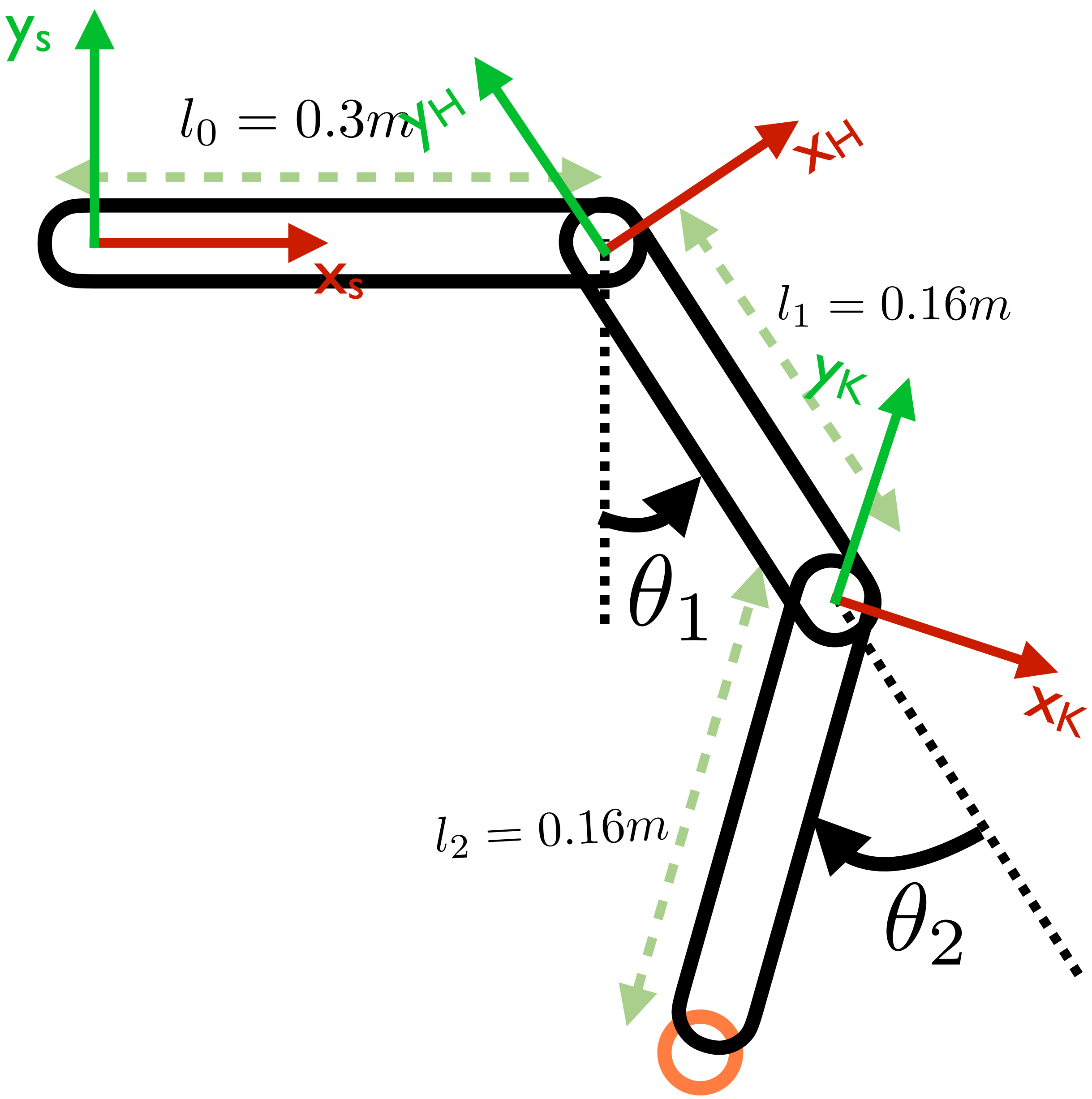

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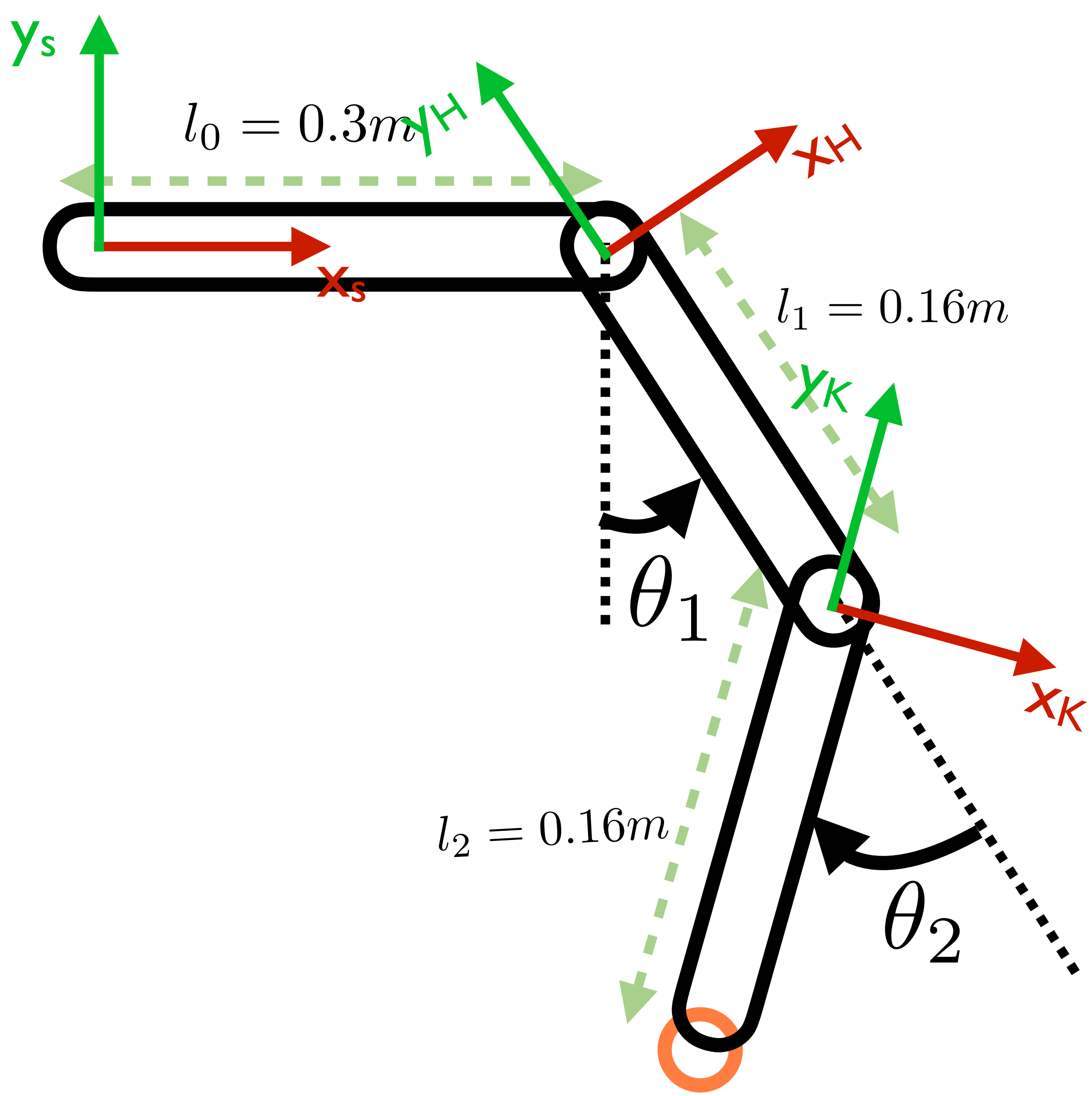

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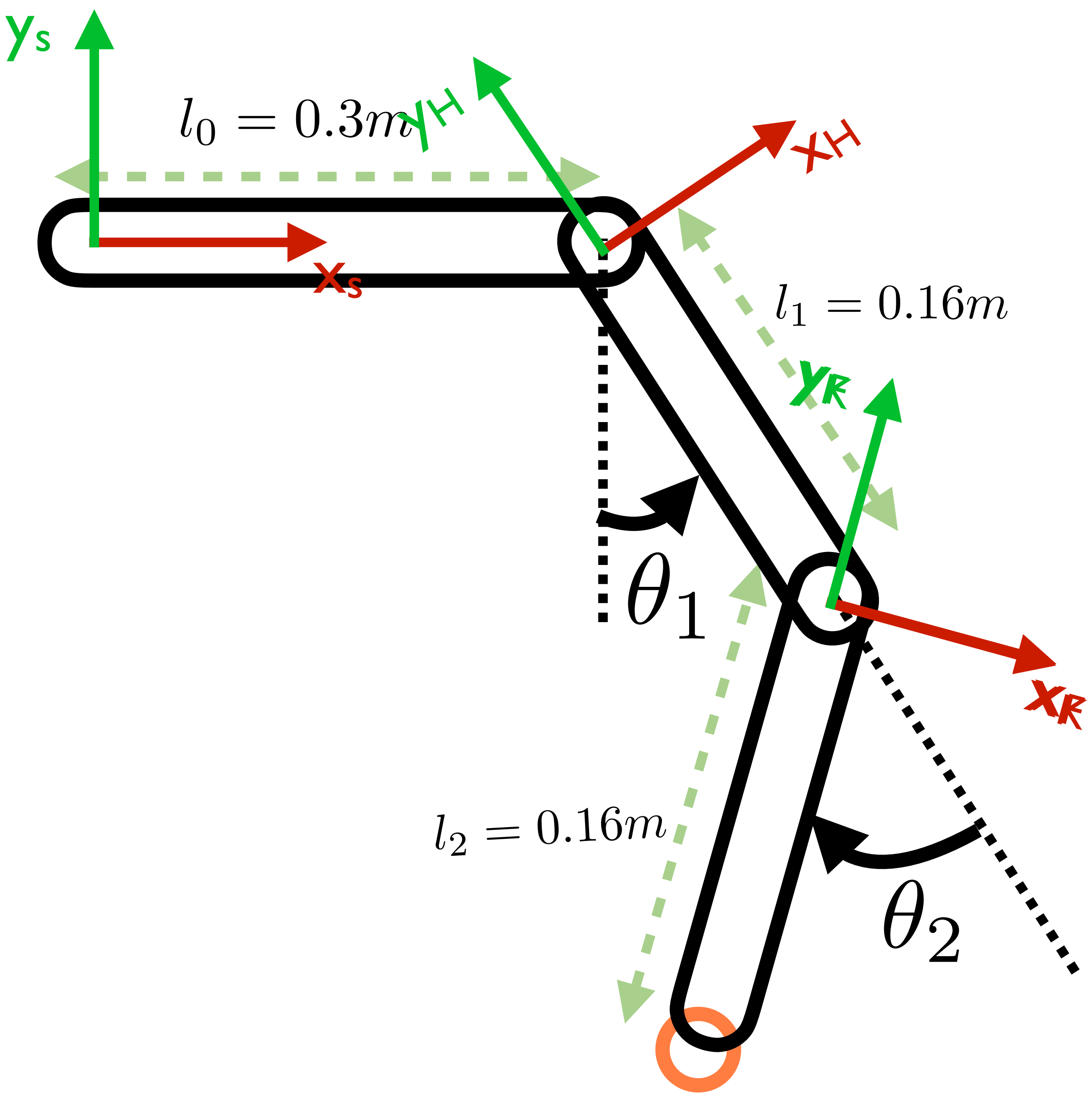

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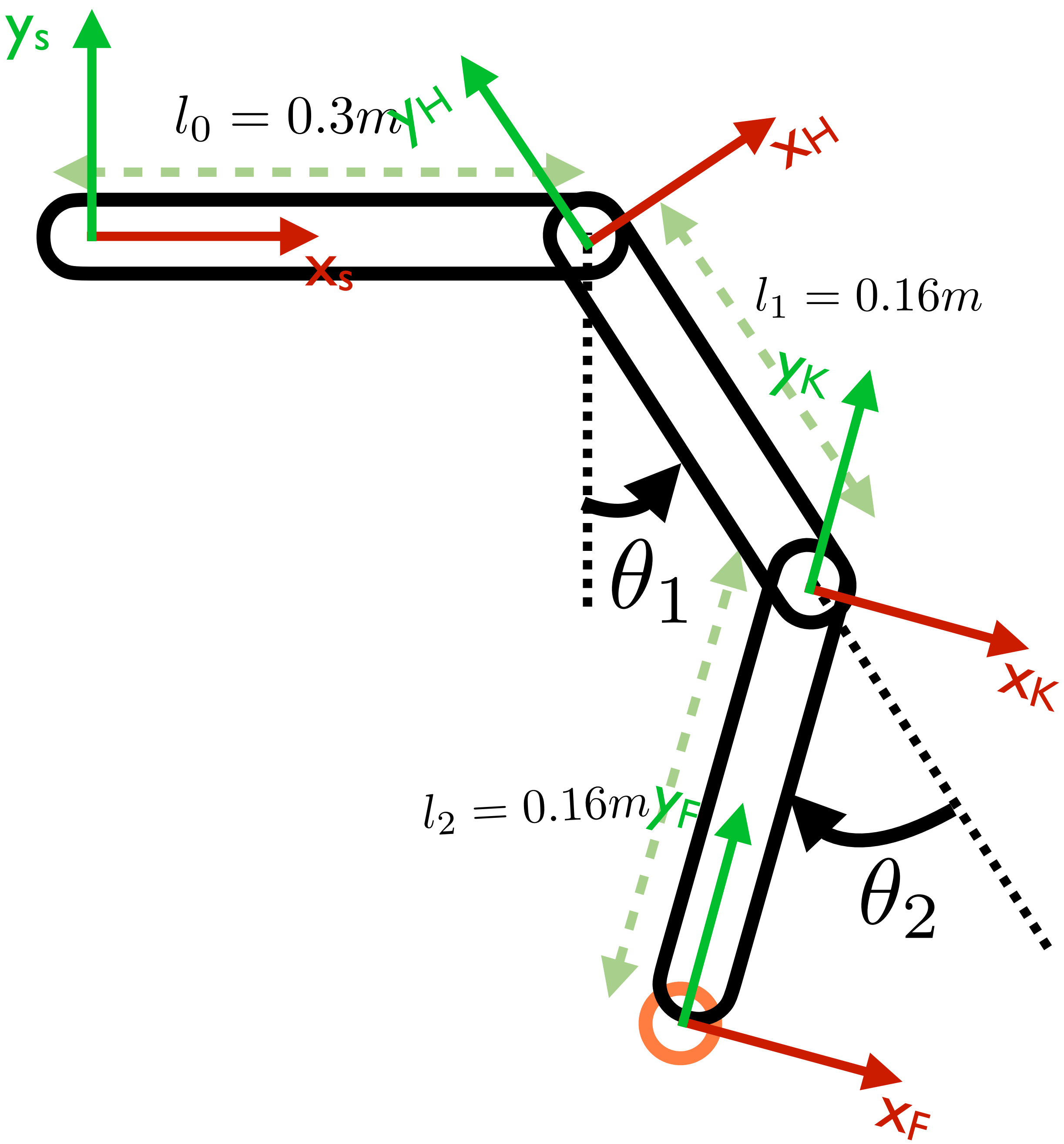

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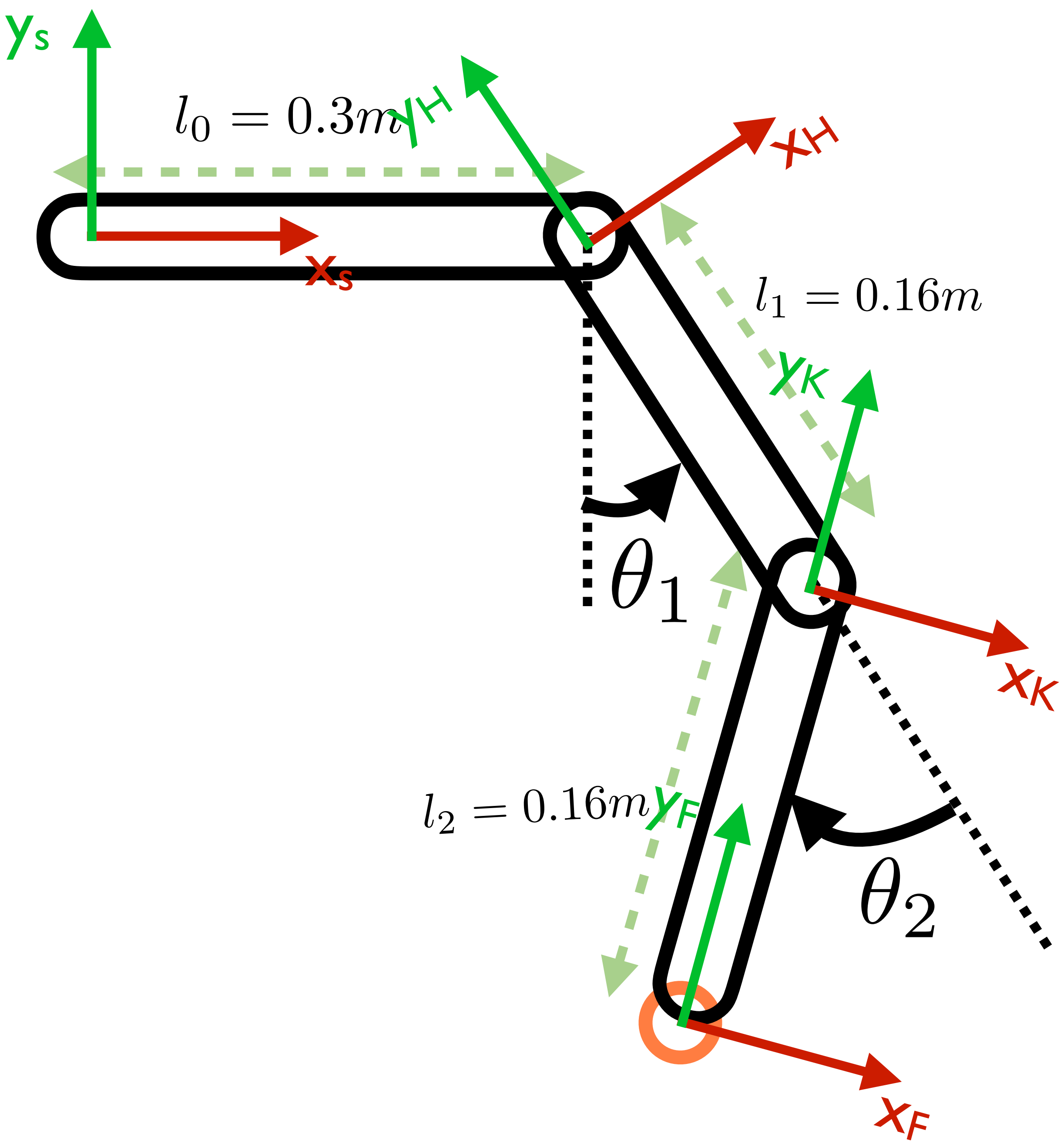

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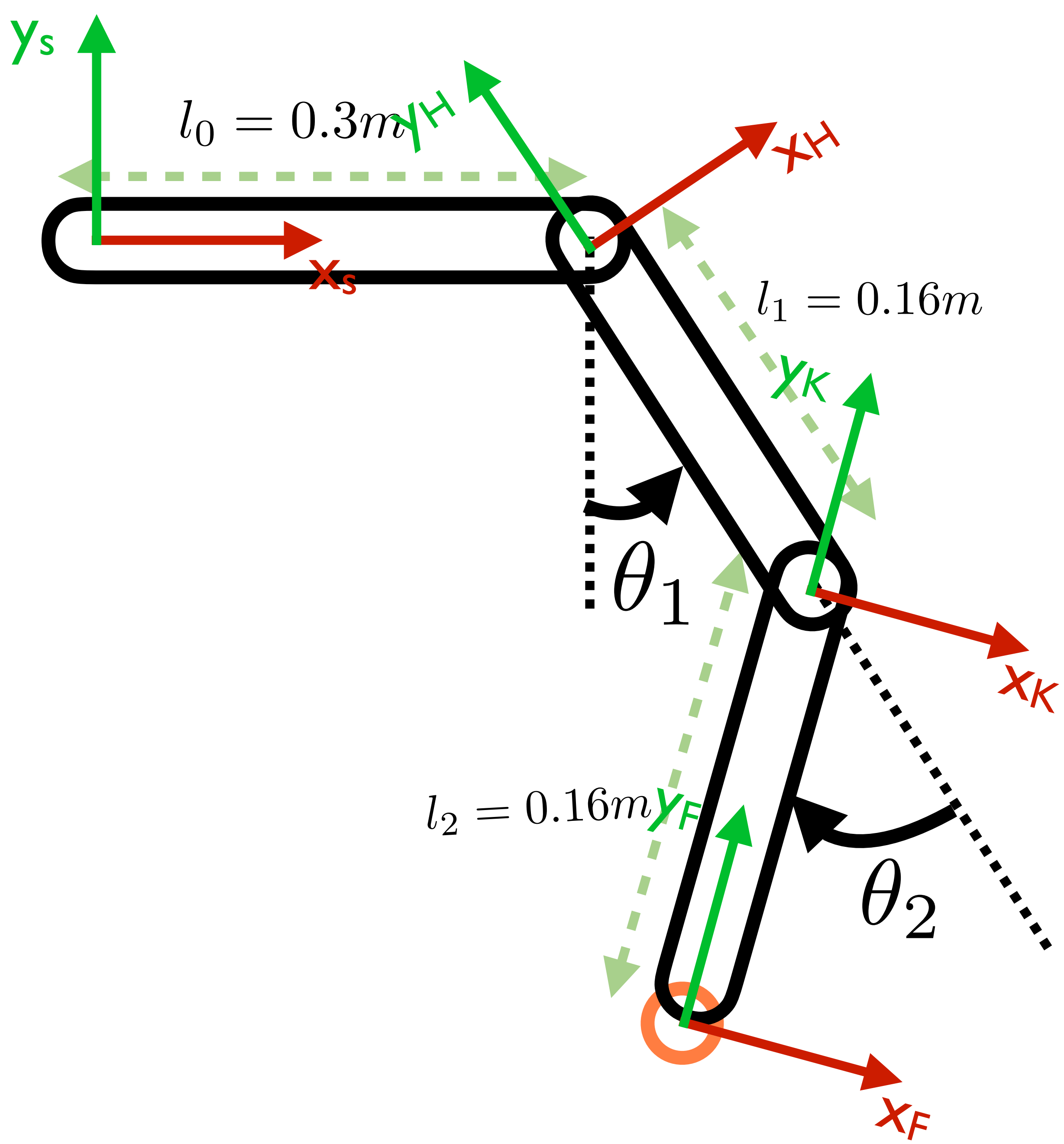
Third: let's compute the homogeneous transforms between frames



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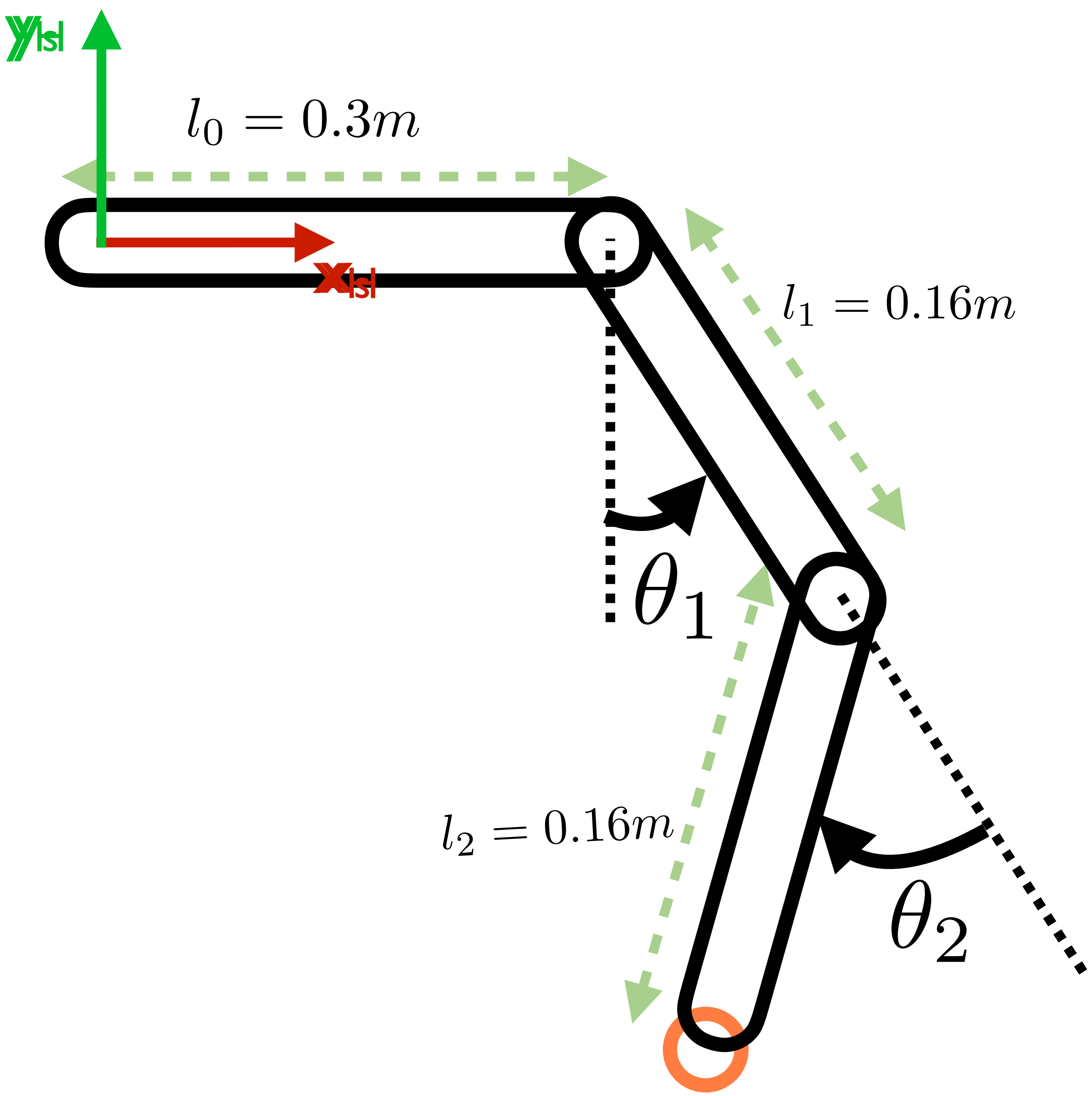

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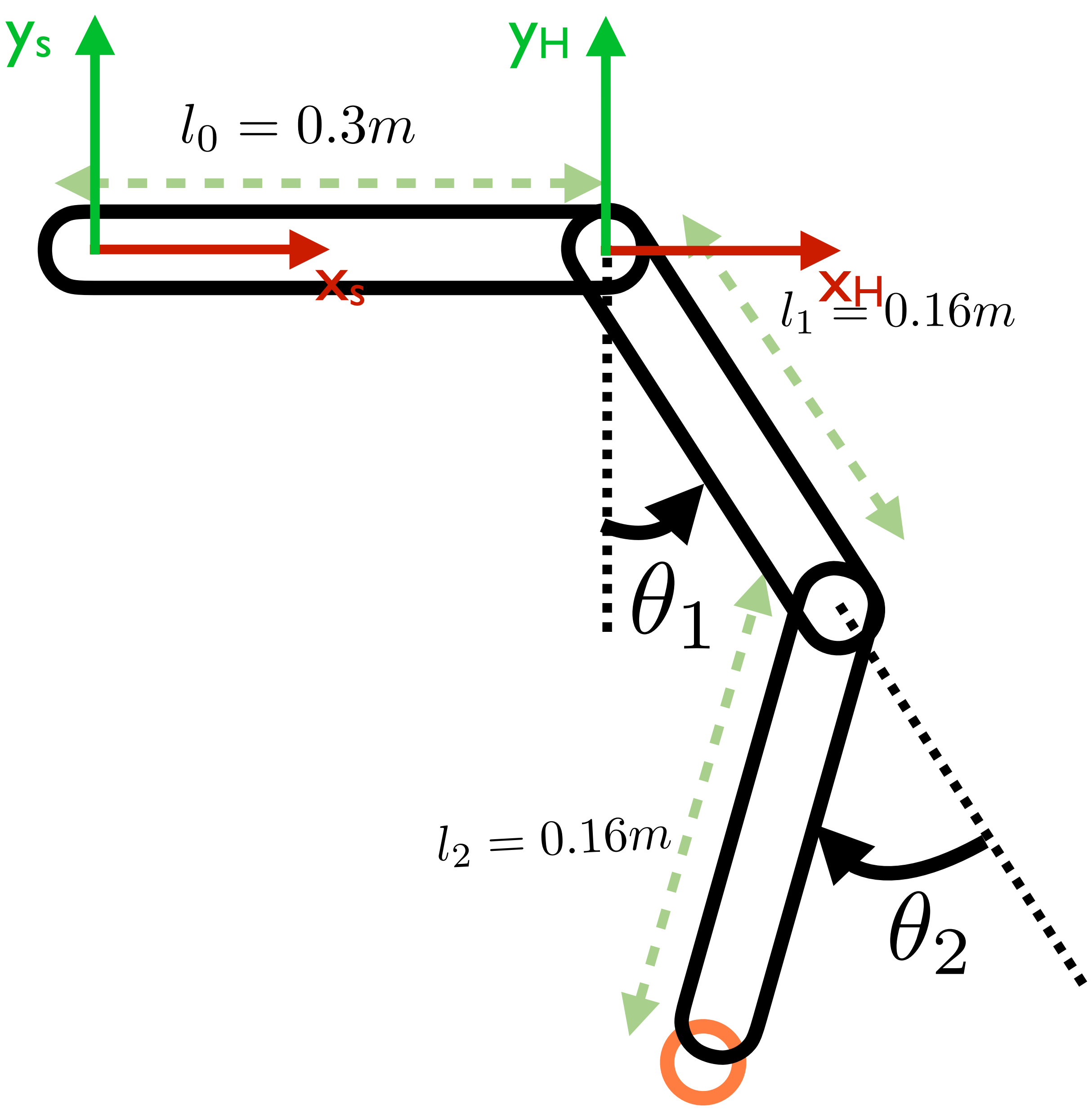

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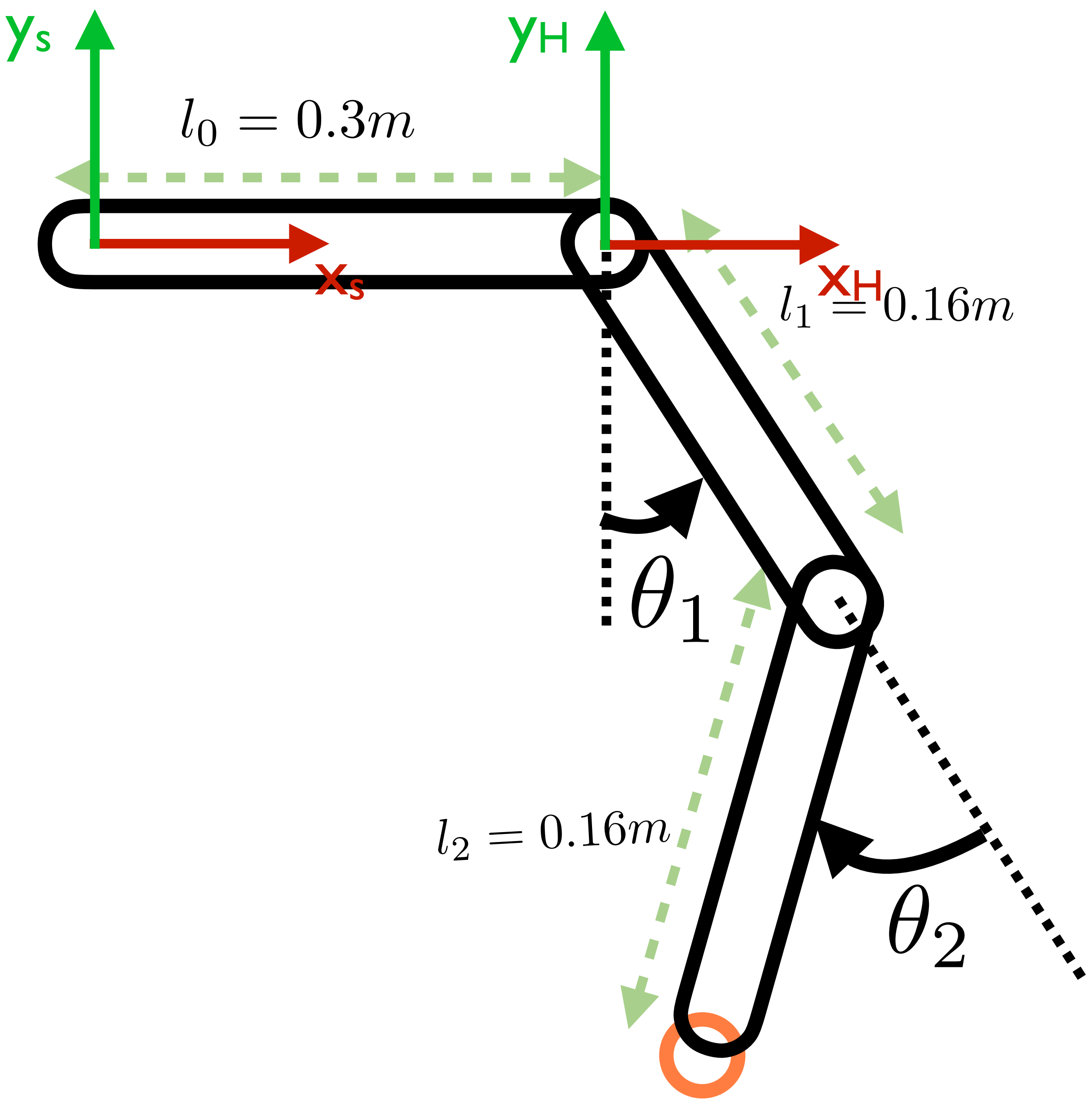

The hip frame $\{H\}$ is translated by l_0 and rotated by θ_0 with respect to frame $\{s\}$

$$T_{SH} = \begin{bmatrix} 1 & 0 & l_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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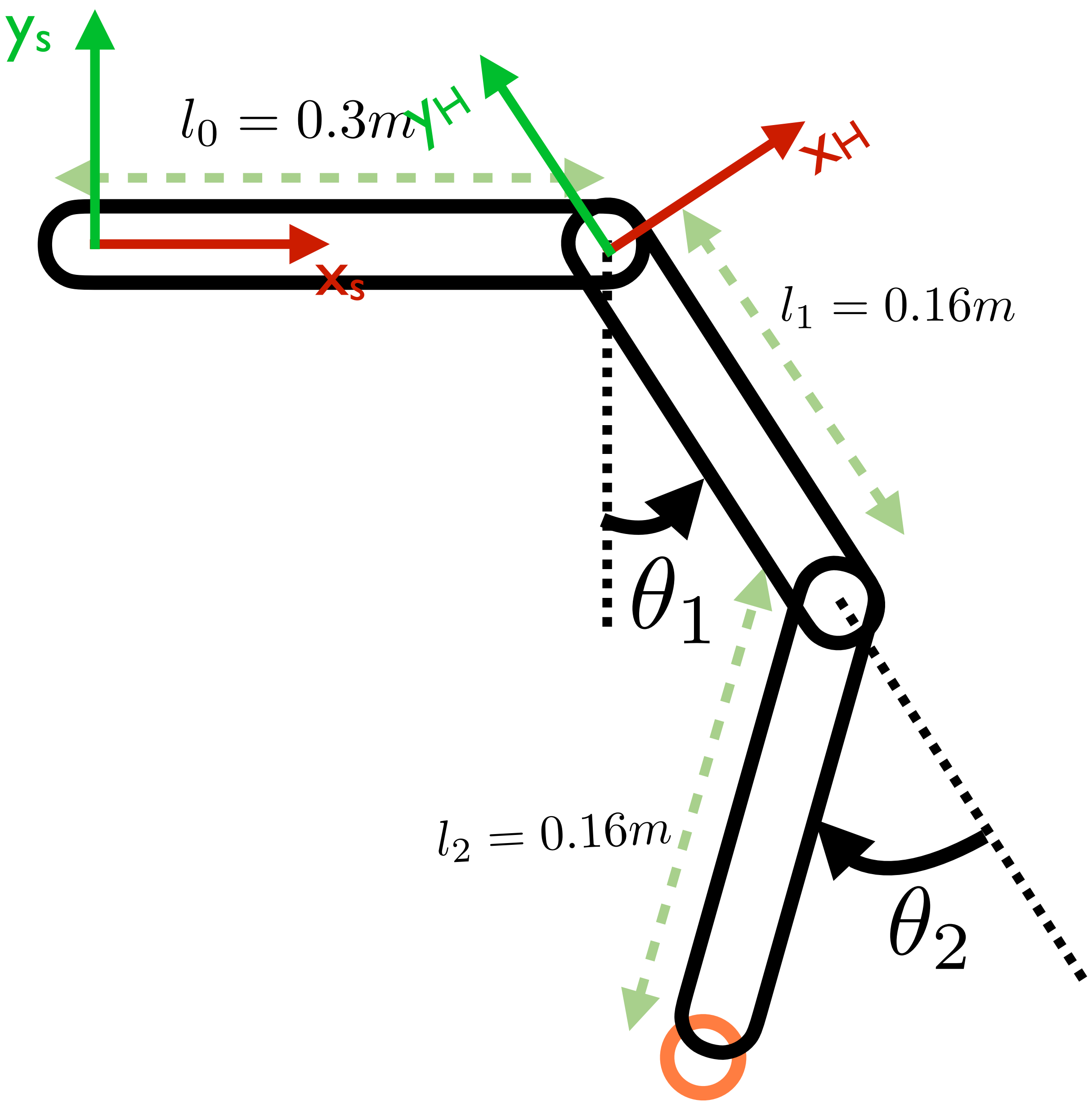

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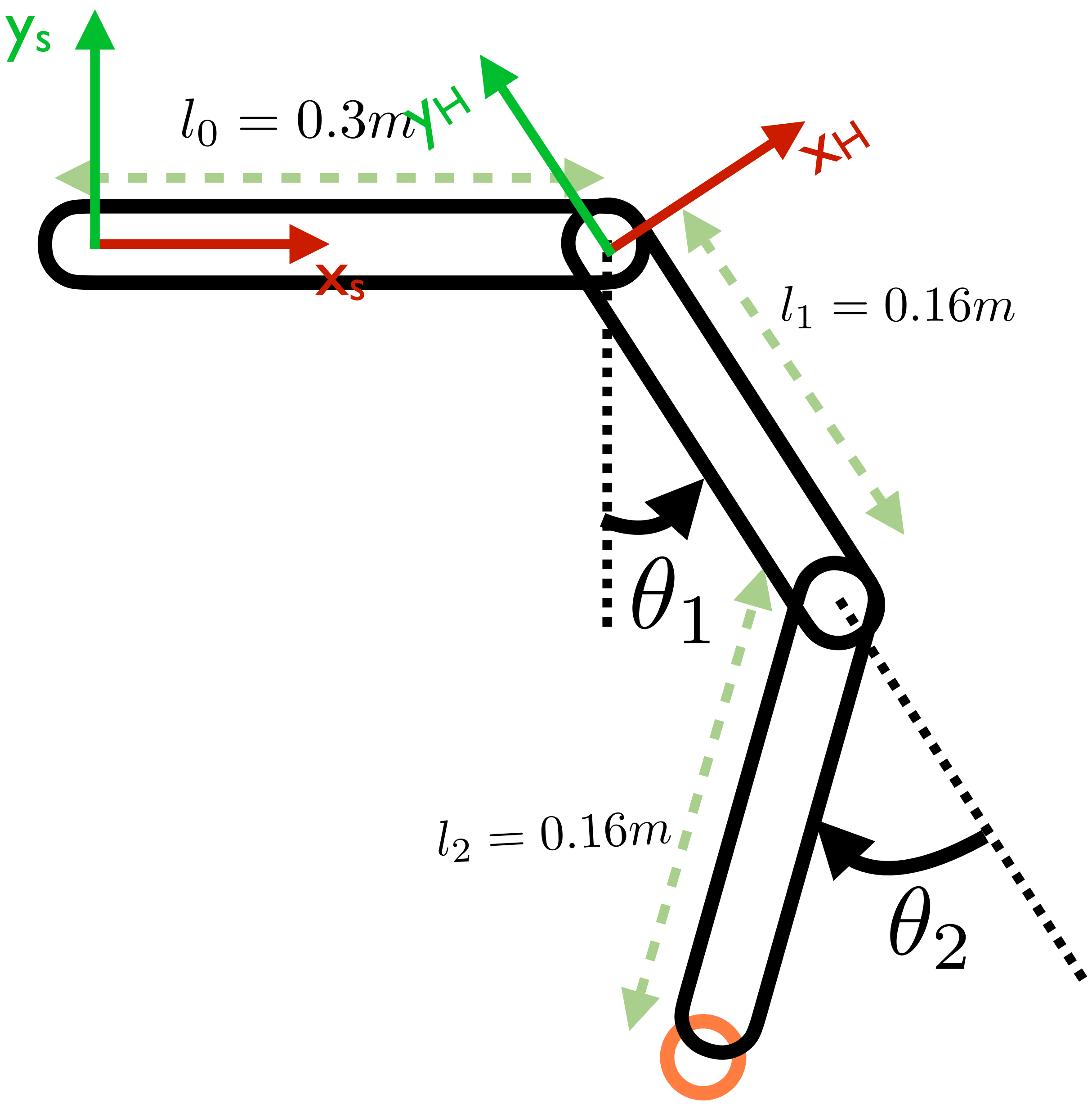

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$$T_{SH} = \begin{bmatrix} 1 & 0 & l_0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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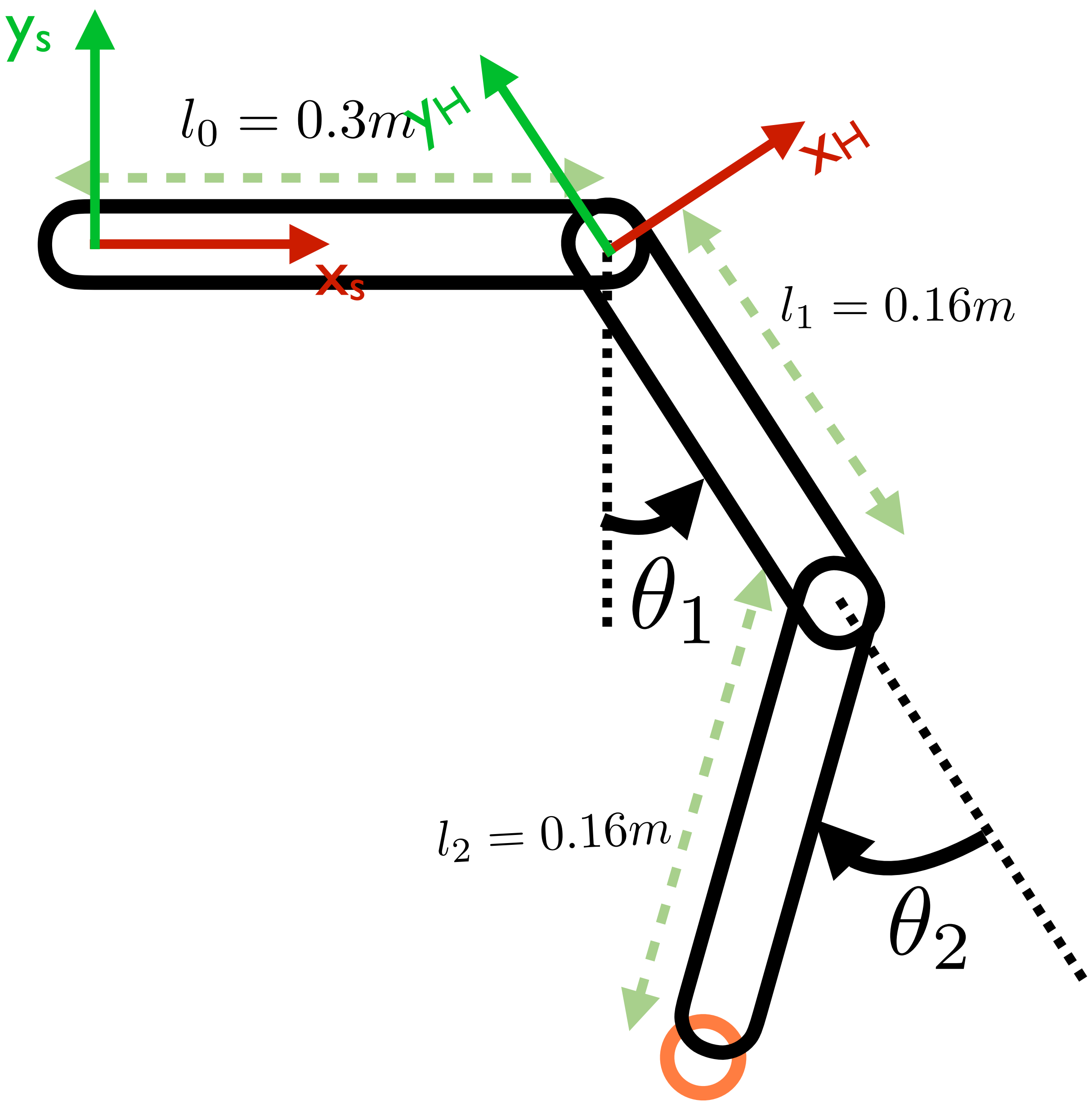

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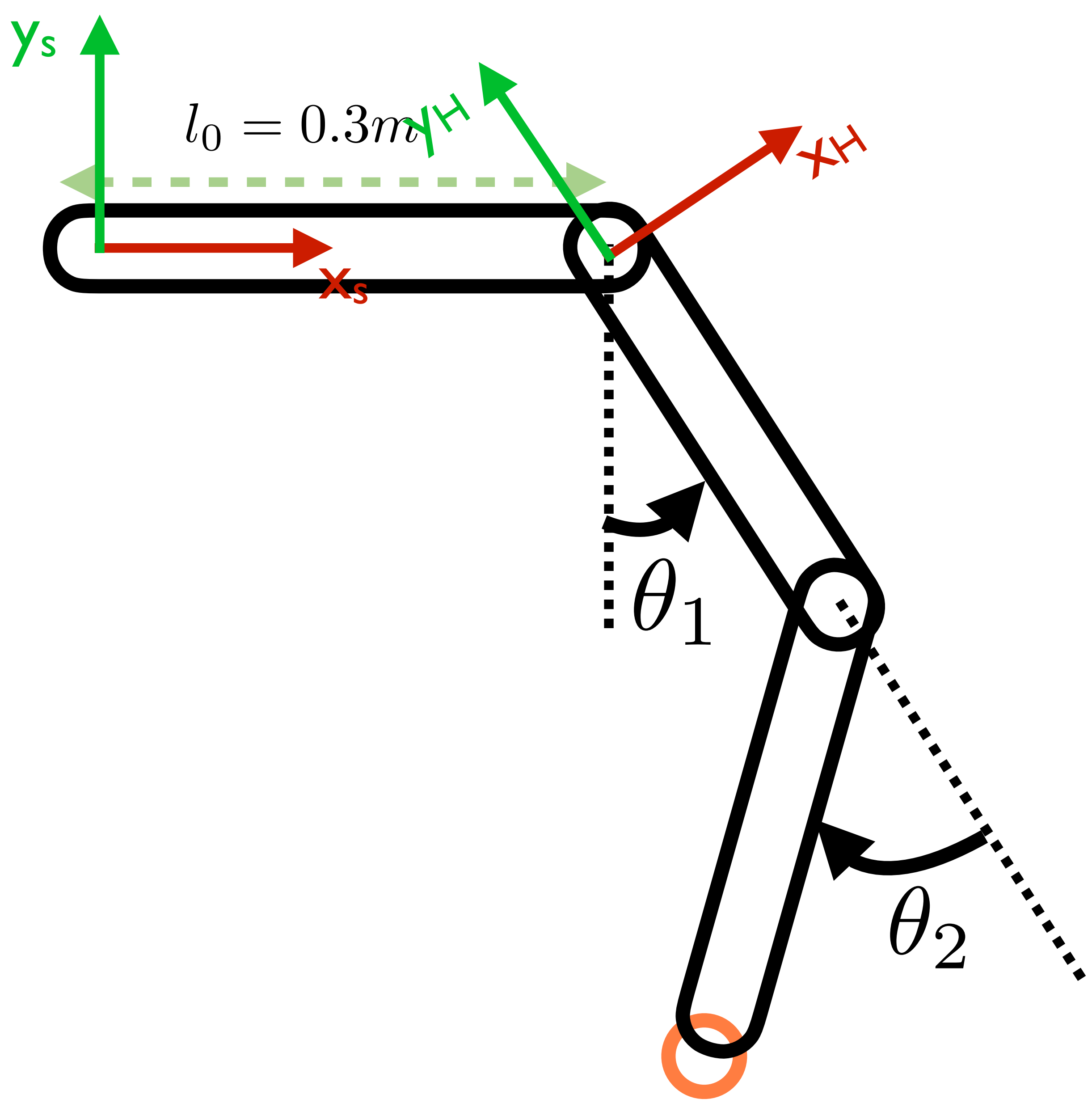

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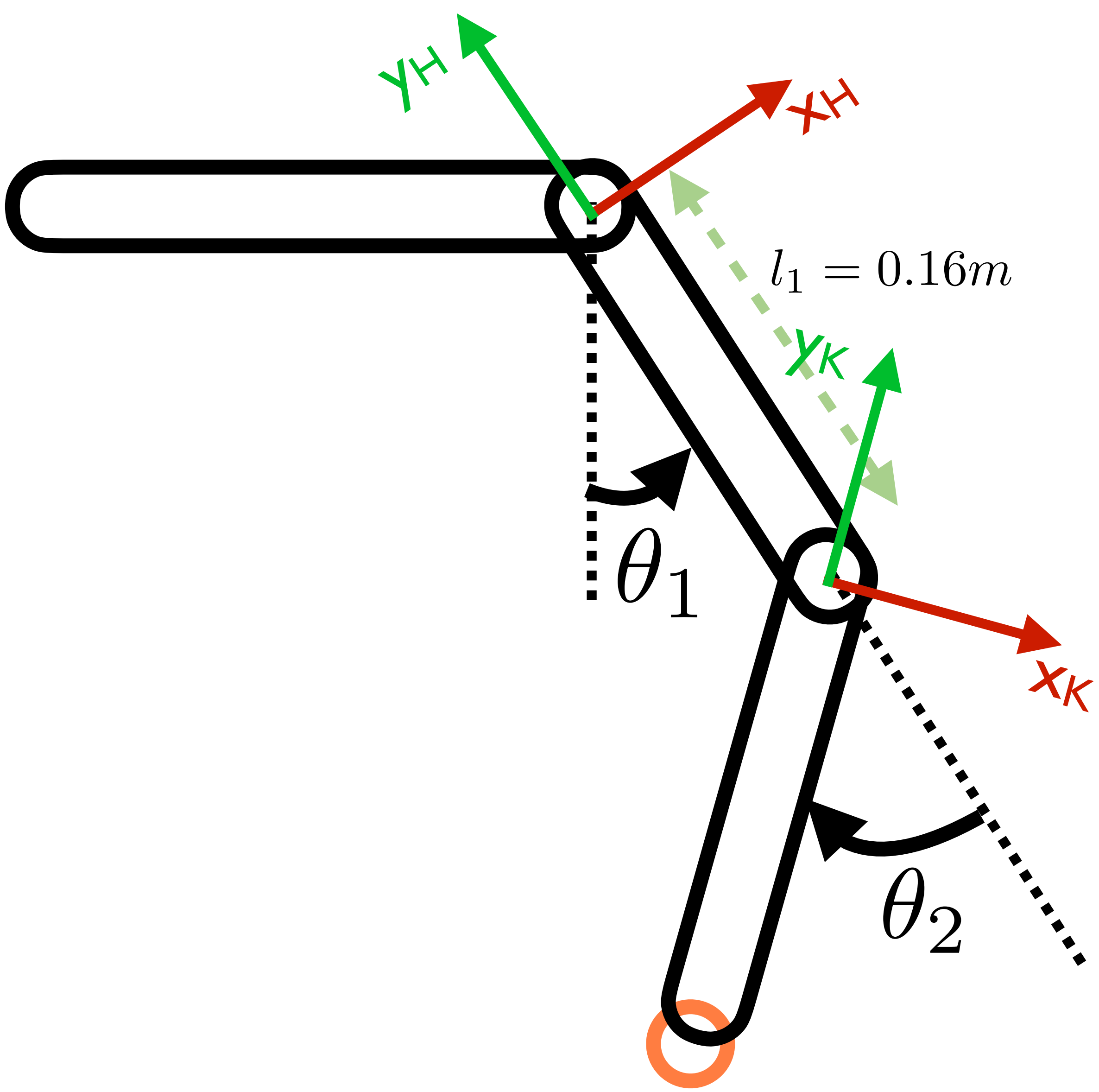

The knee frame {K} is translated by l_1 and rotated by θ_1 with respect to frame $\{H\}$

$$T_{HK} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & -l_1 \\ 0 & 0 & 1 \end{bmatrix}$$

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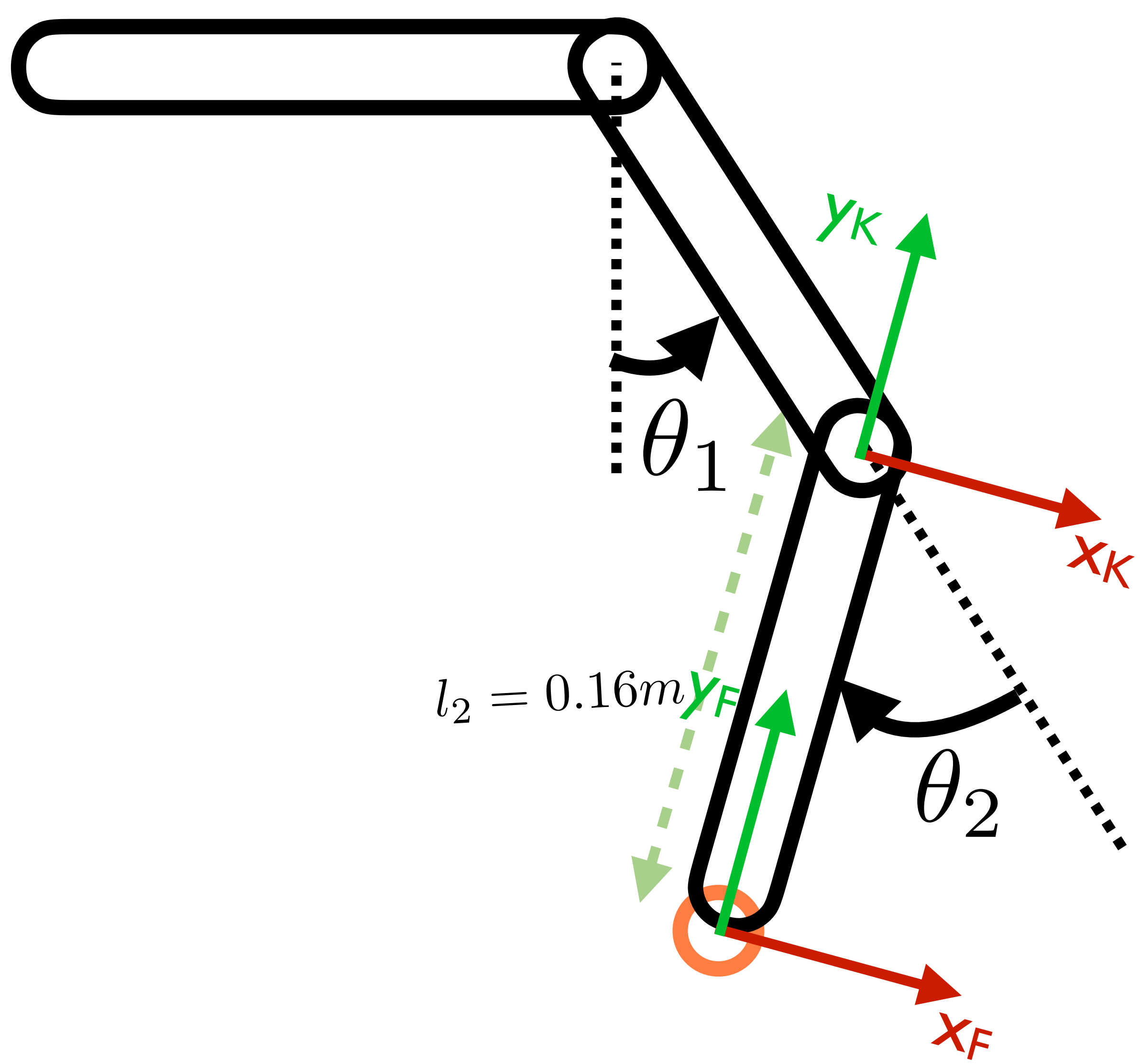

The foot frame {F} is translated by l_2 with respect to frame $\{K\}$

$$T_{KF} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -l_2 \\ 0 & 0 & 1 \end{bmatrix}$$

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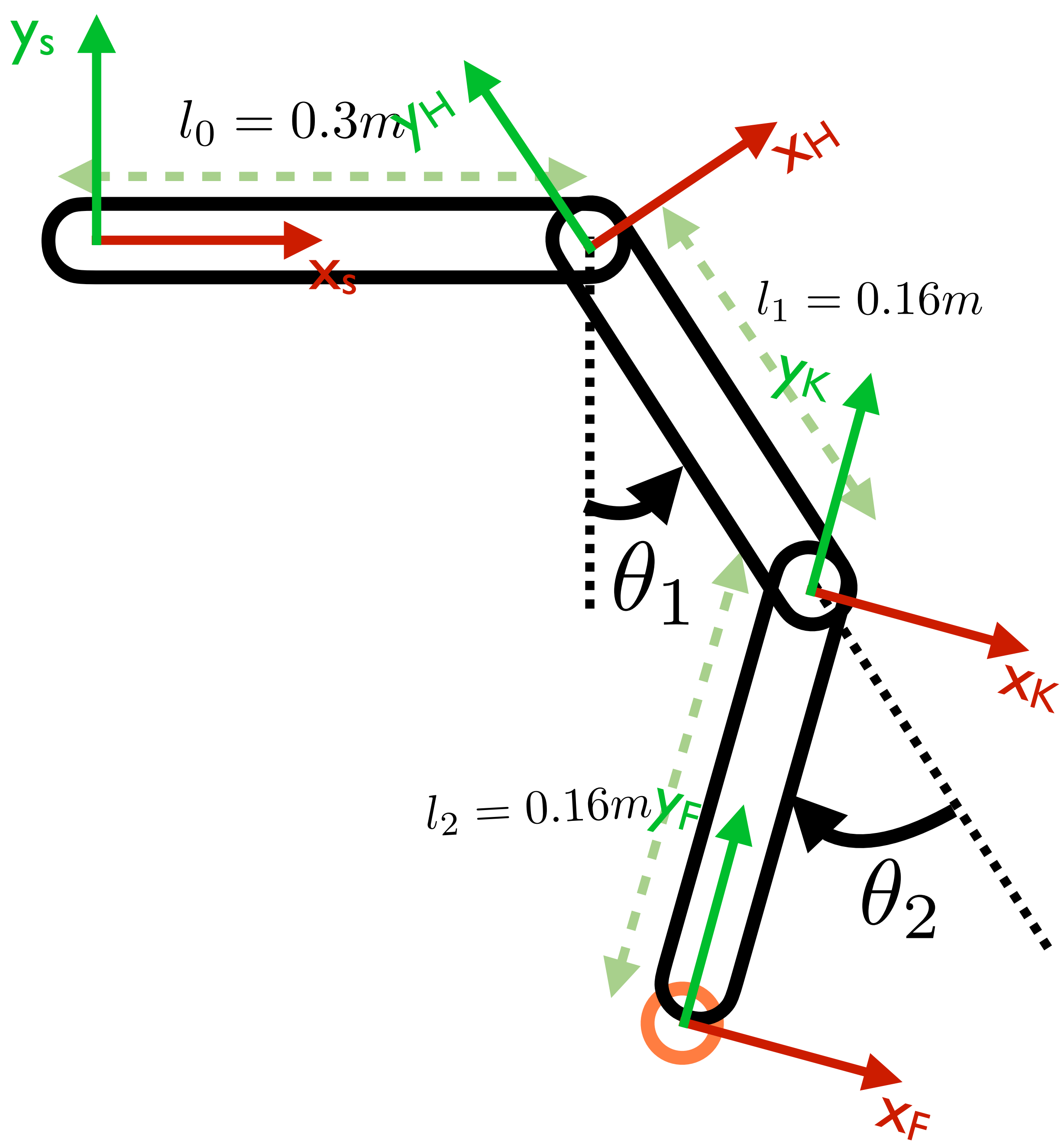

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$$T_{KF} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -l_2 \\ 0 & 0 & 1 \end{bmatrix}$$

The pose of the foot in frame $\{S\}$ is $T_{SF} = T_{SH} \cdot T_{HK} \cdot T_{KF}$

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$$T_{SF} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & l_0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & -l_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -l_2 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$\begin{aligned} T_{SF} &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & l_0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & -l_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -l_2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & l_0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & l_2 \sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 & -l_2 \cos\theta_2 - l_1 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

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The foot orientation $R_{SF} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix}$ is a rotation of $\theta_1 + \theta_2$

The foot position is $p_{SF} = \begin{pmatrix} l_2\sin(\theta_1+\theta_2) + l_1\sin\theta_1 + l_0 \\ -l_2\cos\theta_1+\theta_2) - l_1\cos\theta 1 \end{pmatrix}$

Forward kinematics of a robot

The pose of the foot in frame $\{S\}$ is $T_{SF} = T_{SH} \cdot T_{HK} \cdot T_{KF}$

In practice we never compute explicitly the formula for T_{SF}.
This would become very tedious for robots with many degrees of freedom!
Instead we typically attach frames at each joint and iteratively compute their composition numerically. This is what we will do in the rest of the laboratory.

$$
\begin{aligned}
T_{SF} &= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & l_0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & -l_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -l_2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & l_0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & l_2 \sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 & -l_2\cos\theta_2 - l_1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & l_2\sin(\theta_1+\theta_2) + l_1 \sin\theta_1 + l_0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & -l_2 \cos\theta_1 + \theta_2) - l_1 \cos\theta 1 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$