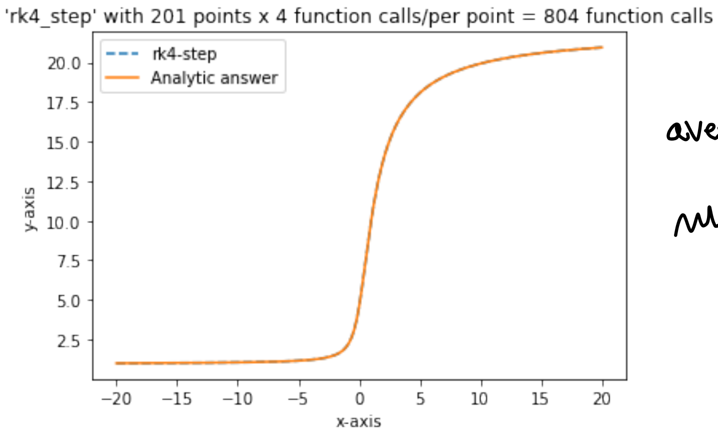
The first stepper is just like what we did in class. So I a just going to put my plot.



average of residuals = 1e-4

max residual = 2e-4

For rk4_stepd observe that if

y_1 is given by a big step of $2h$ than

$$y(x+2h) = y_1 + (2h)^5 \text{ stuff} + O(6)$$

and if y_2 is given by two sucessive steps of length h than

$$y(x+2h) = y_2 + 2h^5 \text{ stuff} + O(6)$$

so ignoring $O(6)$ we get

$$y(x+2h) - 16\,y(x+2h) = y_1 - 16\,y_2$$

$$y(x+2h) = \frac{-1}{15}(y_1 - y_2) - \frac{1}{15}(-15\,y_2)$$

$$\boxed{y(x+2h) = y_2 + \frac{y_2 - y_1}{15}}$$

This what we are going to use for rk4_stepd()

For rk4_step there was 4 function calls per points

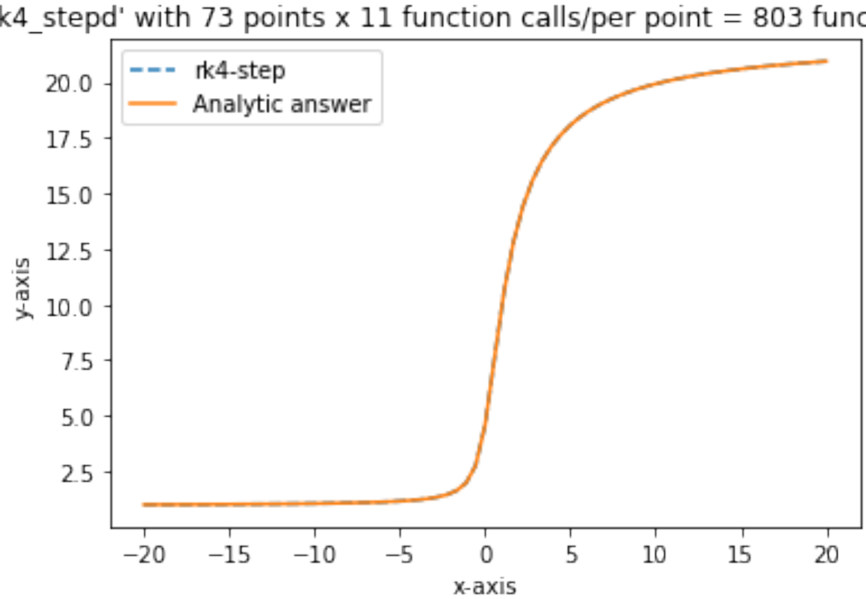
For rk4_step there is $4+4+4-1 = 11$ calls the minus one is because two repeats so we can save it

so $201 \cdot 4 = 804$ calls

$804 / 11 \approx 73$ points

We want to plot 73 points to get 803 function calls

'rk4_stepd' with 73 points x 11 function calls/per point = 803 function calls



avg residuals = 3e-6
max residual = 7e-6

So this is more accurate given the same number of function calls

Question 2

I am going to start with nothing that if we neglect all the intermediary step. Then its a 2-states system

$$\frac{dy_1}{dt} = -\frac{1}{\tau} y_1 \qquad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} -1/\tau & 0 \\ 1/\tau & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\frac{dy_2}{dt} = \tau y_1$$

$$\lambda_1 = -1/\tau, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \lambda_2 = 0, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 e^{-t/\tau} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 e^{0} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

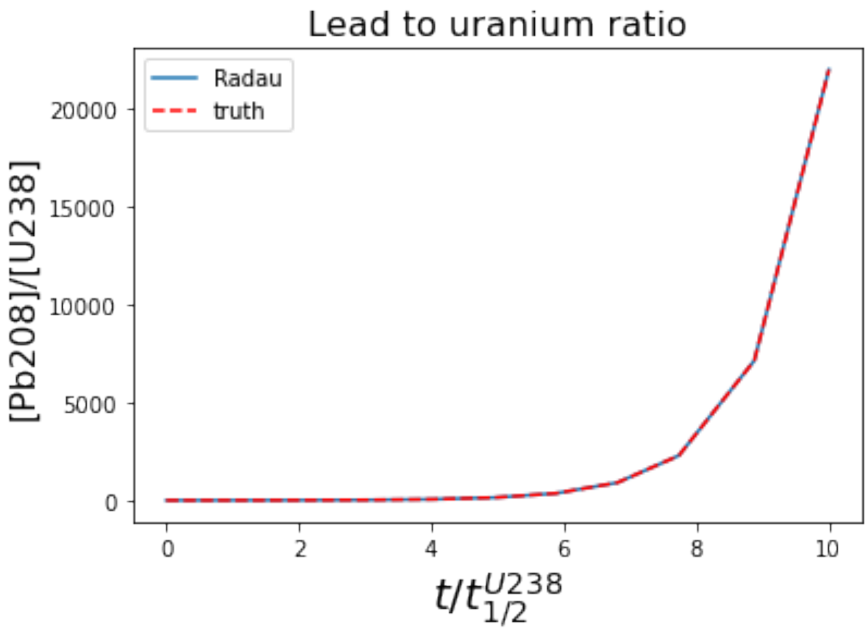
$$\Rightarrow \quad c_1 = -1 \quad \& \quad c_2 = 1$$

$$y_2 / y_1 = \frac{1 - e^{-t/\tau}}{e^{-t/\tau}} = e^{-t/\tau} - 1$$

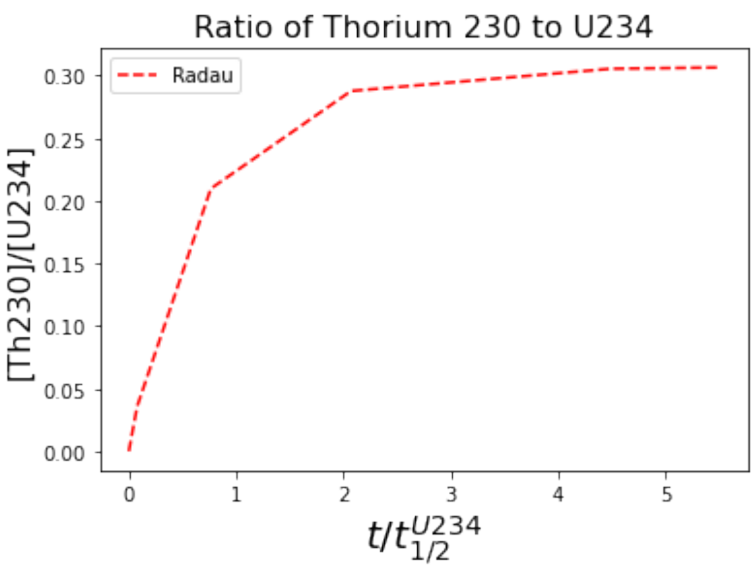
$$\boxed{\frac{[Pb208]}{[U238]} = e^{-t/\tau} - 1}$$

where τ is the half life of U238

This is indeed what we get



The thorium 230 to U234 plots looks like



Question 3

so we get

$$z - z_0 = a\left((x-x_0)^2 + (y-y_0)^2\right)$$

$$\Rightarrow z = (a x_0^2 + a y_0^2 + z_0) + (-2a x_0) x + (-2a y_0) y + a(x^2+y^2)$$

$$z = A + Bx + Cy + + D(x^2+y^2)$$

where $a = D$ $\quad y_0 = C/-2D$

$x_0 = B/-2D$ $\quad Z_0 = A - D\left((B/-2D)^2 + (C/-2D)^2\right)$
$$= A - \left(\frac{B^2}{4D} + \frac{C^2}{4D}\right)$$

Now to solve for $A, B, C, \& D$

we set

$$A = \begin{array}{c} 1,\ x,\ y,\ x^2+y^2 \\ \begin{pmatrix} 1 & x_1 & y_1 & x_1^2+y_1^2 \\ 1 & x_2 & y_2 & x_2^2+y_2^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & y_n & x_n^2+y_n^2 \end{pmatrix} \end{array}$$

Now from class we know that the best fit estimator is given by

$\vec{m} = (A^T N A)^{-1}(A^T N^{-1})x$ here I assumed that

$N =$ identity to compute fit

I get $A = 3.12\,e{-1}$ $\quad D = 6.45\,e{-8}$

$B = 1.25\,e{-4}$

$C = 1.192\,e{-4}$

$$\Rightarrow \boxed{\begin{aligned} a &= 6.45e-8 \\ x_0 &= -1.36 \text{ mm} \\ y_0 &= 58.2 \text{ mm} \\ z_0 &= -1512.8 \text{ mm} \end{aligned}}$$

Now to the focal lenghth

$$y = \frac{x^2}{4f} \Rightarrow a = \frac{1}{4f}$$

$$\Rightarrow f = \frac{1}{4a}$$

$$f = \frac{1}{4(1.67e-4)} = 1499.66 \text{ mm}$$

for the error we will take the diagonal entry of the covariance matrix

Now we need to evaluate the noise matrix N

will just make a matrix with the mean of the residuals

I get $\sigma_a = 6.45e-8$

so $\sigma_f = f \cdot \dfrac{\sigma_a}{a} = 0.6$

$$\boxed{f = 1499.7 \pm 0.6 \text{ mm}}$$