February 23, 2023

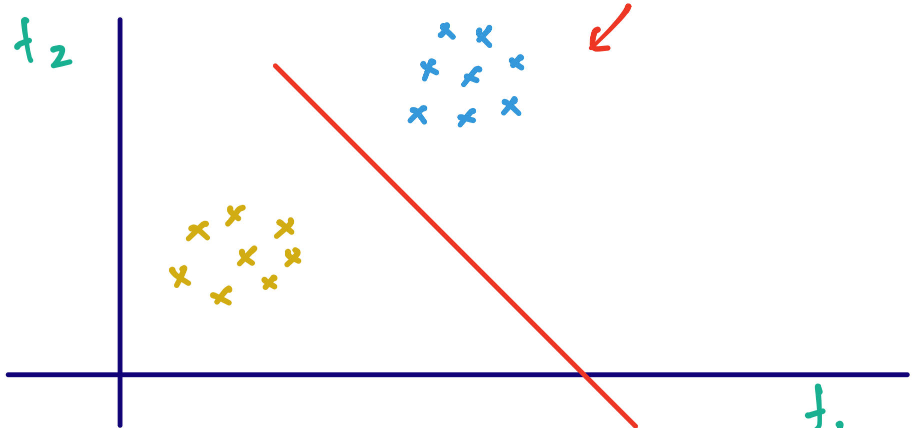
DSML : Math for ML.

Linear Algebra : Problem Solving

Agenda for today :

(a) Recap.
(b) Problems
(c) Perceptron learning rule : Code.

Cheat Sheet (Recap)



classifier.

Features.

f_1	f_2	f_3	f_4	
				1
				−1

Feature vector

label

$$w_1 x_1 + w_2 x_2 + w_0 = 0.$$

parameters.

Co-ordinate geometry.
Visualization.

Linear algebra: To automate the whole thing.

Linear Algebra Cheat Sheet:

(a) Vectors: $\bar{x} = [\underbrace{x_1 \quad x_2}\quad x_3 \quad x_4 \cdots x_d]^T$.

(b) Length/Norm/Magnitude: $\bar{x} \in \mathbb{R}^d$, $\|\bar{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}$

(c) Inner product: $\bar{x}^T \bar{y} = \bar{x} \cdot \bar{y} = \sum_{i=1}^{d} x_i \cdot y_i$.

(d) Angle: $\cos(\theta) = \dfrac{\bar{x}^T \bar{y}}{\|\bar{x}\| \cdot \|\bar{y}\|}$

(e) d-dimensional hyperplane: $\bar{w}^T \bar{x} + w_0 = 0$.

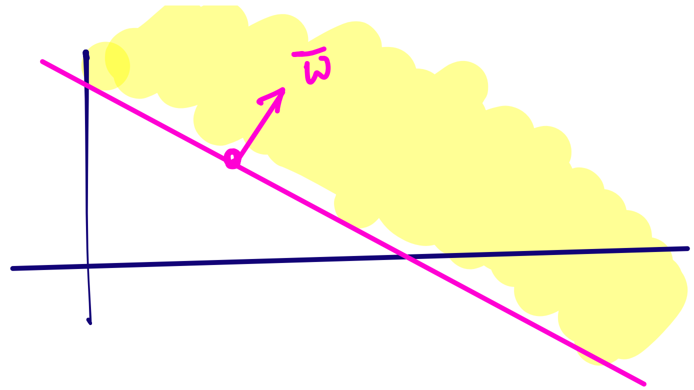
where $\bar{x}, \bar{w} \in \mathbb{R}^d$ $\quad w_0 \in \mathbb{R}$

(f) Halfspace: One half of the plane which gets divided by a d-dimensional hyperplane. sign(d) $\to$ +ve, +ve H.S. $\to$ -ve, -ve H.S.

(g) Distance between a point $\bar{x}$ and a d-dim. hyperplane $\bar{w}^T x + \bar{w}_0$

$\rightarrow d = \dfrac{\bar{w}^T \bar{x} + \bar{w}_0}{\|\bar{w}\|}$

(h) Projection of $\bar{x}$ on $\bar{y}$: $\bar{x}^T \hat{y} = \dfrac{\bar{x}^T \bar{y}}{\|\bar{y}\|}$



visual way. $\bar{w}^T\bar{x} + w_0 = 0.$

Algebra way: $\bar{x}$

$$\bar{d} = \frac{\bar{w}^T\bar{x} + w_0}{\|\bar{w}\|} \begin{cases} \rightarrow +ve, \ +ve \text{ H.S.} \\ \rightarrow -ve, \ -ve \text{ H.S.} \end{cases}$$

$\mapsto 3x + 3y + 3 = 0 \longrightarrow \bar{w} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad w_0 = 3$

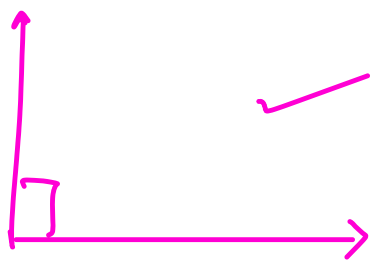
$\mapsto -3x - 3y - 3 = 0. \longrightarrow \bar{w} = \begin{bmatrix} -3 \\ -3 \end{bmatrix} \quad w_0 = -3.$

Q] <u>Two vectors</u> : $\bar{x} = [2, 1, -3]^T$

$\bar{y} = [5, 8, 6]^T$

What is the length of the projection of $\bar{x}$ on $\bar{y}$?

$$\bar{x}^T \hat{y} = \frac{\bar{x}^T \bar{y}}{\|\bar{y}\|} = \frac{(10 + 8 - 18)}{\sqrt{5, \_\_\_}} = \underline{0}$$



Q2] $\hat{a}$ and $\hat{b}$ are unit vectors.

$$\left.\begin{aligned} \bar{c} &= \hat{a} + 2\hat{b} \\ \bar{d} &= 5\hat{a} - 4\hat{b} \end{aligned}\right\} \quad \bar{c}^T \bar{d} = 0.$$

What is the angle between $\bar{a}$ & $\bar{b}$?

$$\cos\theta = \hat{a}^T \hat{b}$$

$$\rightarrow (\hat{a} + 2\hat{b})^T (5.\hat{a} - 4\hat{b}) = 0.$$

$$\underbrace{5 - 8}_{-3} - 4\,\hat{a}^T\hat{b} + 10\,\hat{a}^T\hat{b} = 0$$

$$-3 = -6\cos\theta.$$

$$\cos\theta = \frac{1}{2} \qquad \theta = 60^\circ$$

Q] $\bar{w}_1 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $\qquad \bar{w}_2 = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$ $\qquad w_{01} = 3$

$w_{02} = 7.$

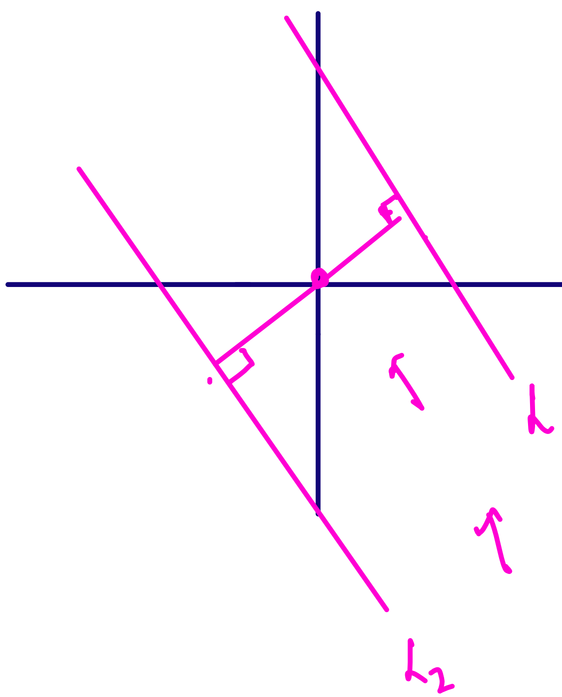
→ $l_1 : \bar{w}_1^T \bar{x} + w_{01}$

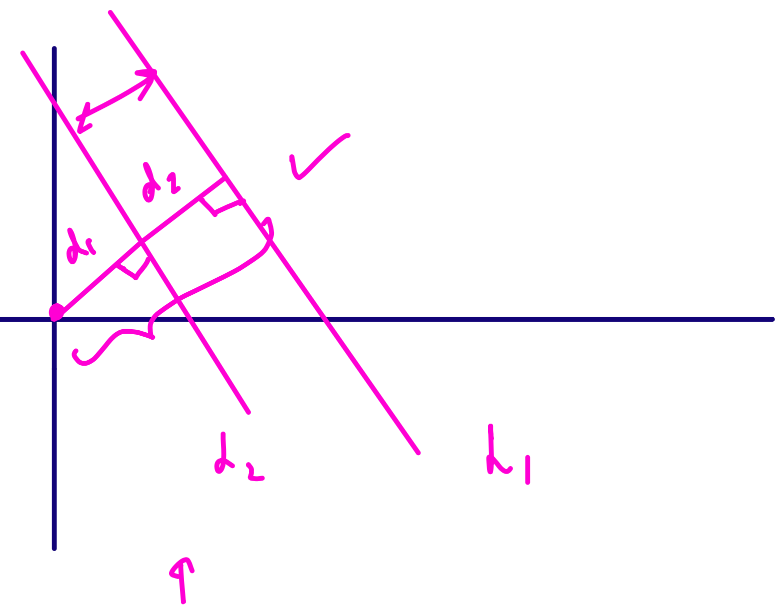
→ $l_2 : \bar{w}_2^T \bar{x} + w_{02}.$

Distance between them.

$$d_1 = \frac{w_{01}}{\|w_1\|} = \frac{3}{5}$$

$$d_2 = \frac{w_{02}}{\|w_2\|} = \frac{7}{20}$$

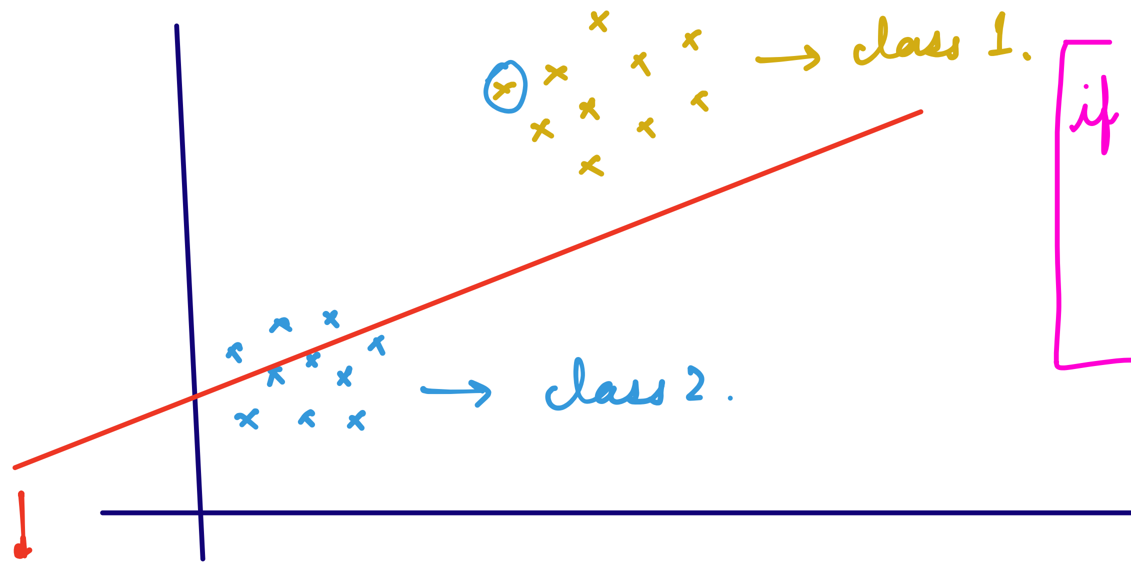




$$\left| \frac{12}{20} - \frac{7}{20} \right| = \frac{5}{20} = \frac{1}{4}.$$

Q] Let us say we have a labelled dataset:

$$X = \{(\bar{x}_i, y_i)\}_{i=1}^{n} \qquad y_i = \begin{cases} +1, & \text{if it is of class 1} \\ -1, & \text{if it is of class 2.} \end{cases}$$

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if y_i != sign(w̄^T x̄_i + w_0):
    w̄ = w̄ + y_i * x̄_i
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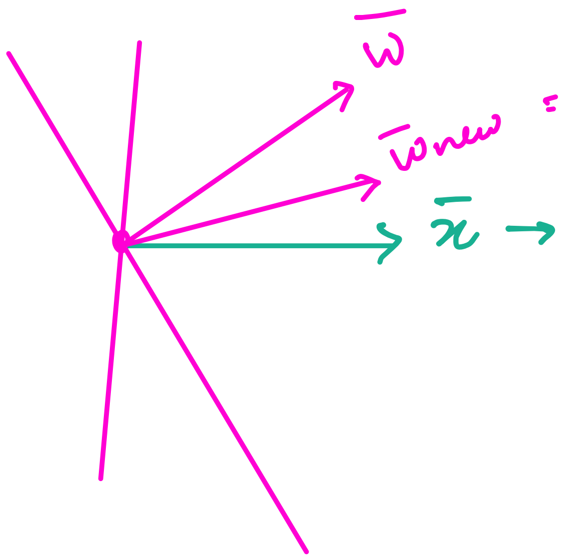
$$\bar{w}^T \bar{x} + w_0 = 0.$$

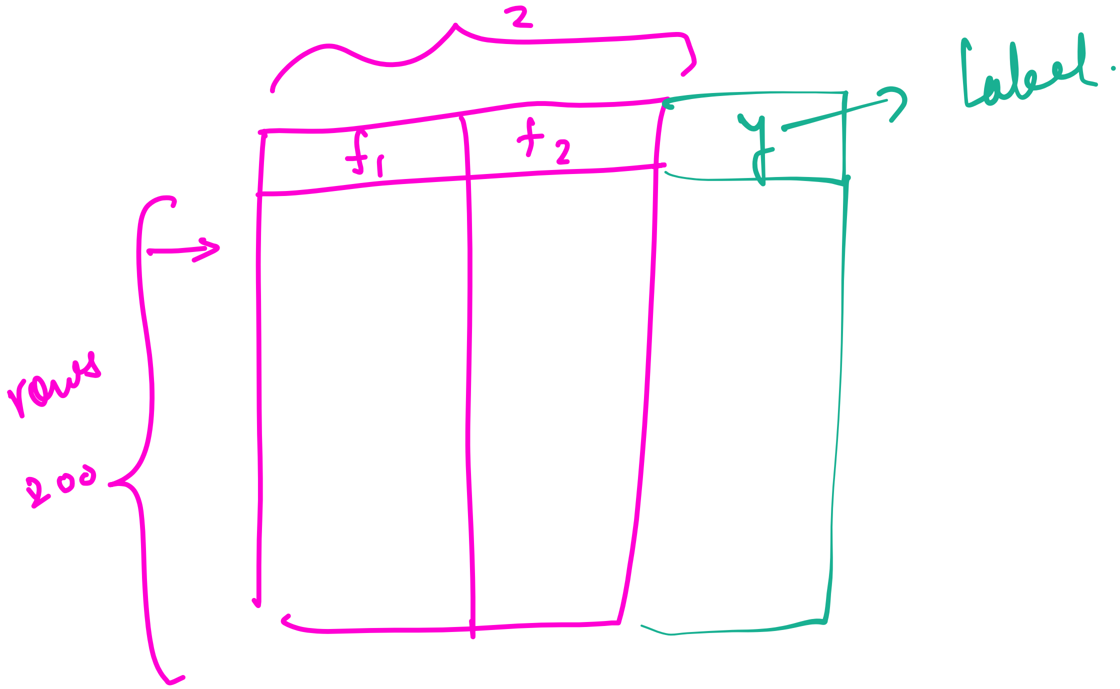
$$f(X; \bar{w}, w_0) = \sum_{i=1}^{n} \left(\frac{\bar{w}^T \bar{x}_i + \bar{w}_0}{\|\bar{w}\|} \right) y_i$$

$$\max_{\bar{w}, w_0} f(X; \bar{w}, w_0)$$



$\bar{w}_{new} = \bar{w} + \bar{x}$

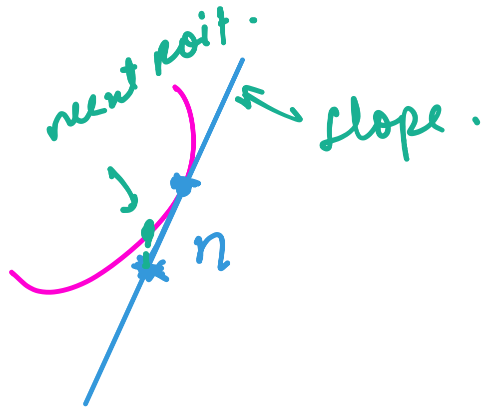
$\bar{x} \rightarrow$ was supposed to be +1
but is given -1



$$w_1 x + w_2 y + w_0 = 0.$$

$$y = -\frac{w_1}{w_2} x - \frac{w_0}{w_2}$$

$$\bar{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = -\frac{w[0]}{w[1]} x - \frac{w_0}{w[1]}$$

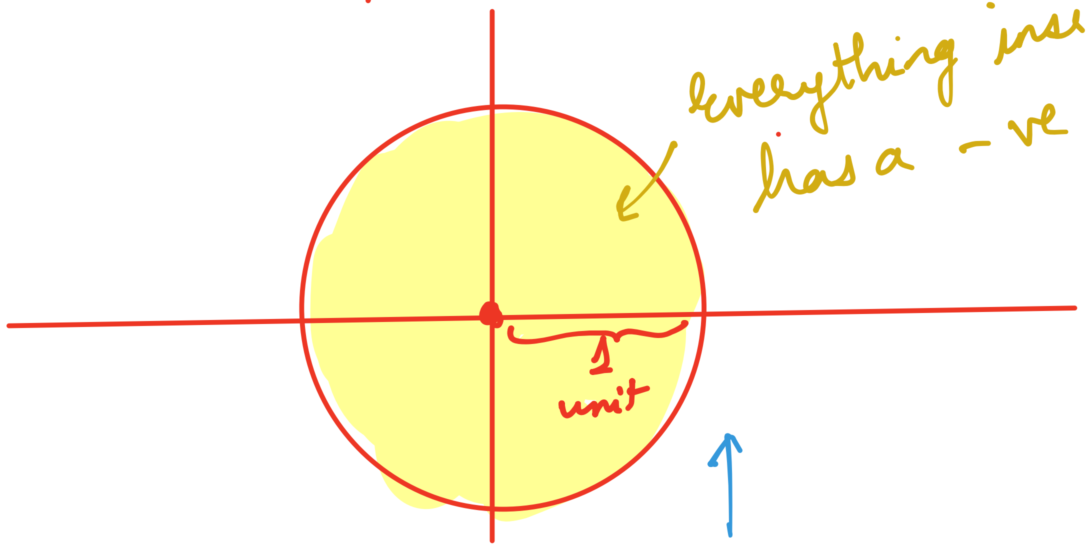


$$\bar{w}^{(t+1)} = \bar{w}^{(t)} - \underbrace{\eta \, \nabla_{\bar{w}} (\text{loss})}$$

Equation of a circle: $\boxed{x^2+y^2-r^2=0.}$

$$x^2+y^2-1=0.$$

(0,0)



everything inside has a -ve sign.

everything outside has a +ve sign.

$$(x-3)^2 + (y-3)^2 - 9 = 0.$$

$$(2-3)^2 + (2-3)^2 - 9 = 0.$$

$$1 + 1 - 9$$

$$= \underline{\underline{-7}}$$