BACHELOR OF SCIENCE THESIS IN PHYSICS

Optimal Cloning Quantum Channels

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Abstract

This thesis reviews the optimal cloning device for copying the pure quantum states. We begin with reviewing the No Cloning theorem [24]. The proof of optimality is given for the studied cloning machine following the arguments presented in [22]. The proposed optimal cloning device is proven to be a covariant quantum channel.

Furthermore, the entropy of images of the states generated by the covariant quantum channel is studied, inspired by the Lieb-Solovej theorem [8]. It is shown numerically that the images of coherent states majorize all other images under the action of the covariant quantum channel.

Contents

Chapter 1

Introduction

We have seen a significant surplus in the quantity and complexity of the scientific problems in the last few decades in all fields from space exploration to medical research, and these are all waiting for the new set of technologies to happen. Digital computers have already reached to point, where they can perform 442×10^{15} floating-point operations per second [20]. Meanwhile, quantum computers have also been realized, which are now capable of solving classically hard problems with exponential complexity in a polynomial time [18].

The advancement in quantum computing is primarily influenced by the progress in quantum information science or quantum information theory. It primarily studies the limitations and possibilities of quantum computation based on the underlying laws of physics. Alike, the limitation on copying a quantum state is known. The paper in 1982 by Wooters and Zurek firmly established the theorem of No Cloning, limited due to the linearity principle in Quantum Mechanics [24].

The essence of the No Cloning Theorem is that not all arbitrary quantum states can be copied to their atomic level i.e. making the perfect clone. The perfect copy or clone of only basis states is shown to be possible. Then, it becomes logical to try to make the optimal copies of the quantum states. Bužek and Hillary first hinted at the concept of imperfect copying of quantum states in the year 1996 [4]. The unitary transformation described in Bužek-Hillery paper was the first Quantum Cloning Machine. Later in 1999, Reinhard Werner generalized the optimal quantum cloning to arbitrary dimensional systems [22].

In the present thesis, the optimal quantum cloning is studied according to Werner's [22], Gisin and Masar's [6] reasoning. The studied quantum cloning device has been proven to be covariant quantum channel.

The other part of the thesis analyzes the relation of the covariant quantum channel to the entropy, particularly Wehrl entropy, defined by POVM of a coherent states. In 1979, Alfred Wehrl conjectured that Gaussian coherent states minimize the Wehrl entropy [21]. In 1978[1], Elliot Lieb proved the Wehrl conjecture [9] and further conjectured that this holds true for $SU(2)$ Bloch coherent states too. It took 35 years for the experts in the field, after many attempts. In 2012, Elliot Lieb and Jan Philip Solovej proved the Lieb-conjecture and generalized in 2016 for symmetric $SU(N)$ [8]. Lieb-Solovej theorem is an interesting plot for optimal quantum cloning, studied independently then. Lieb-Solovej theorem proves that the images of coherent states majorize the images of random pure states under the action of a covariant

[1]inconsistency in dates here is quite surprising

quantum channel. Following the lines of arguments of Quantum HLP Lemma and Schur's Horn theorem, we can conclude that the images of coherent states would have minimal entropy. The uniqueness of the minimum still remains an open question in the field [2]. It is numerically shown in the present thesis that images of coherent states have the minimum angular entropy under the action of a covariant quantum channel.

In **Chapter 2**, the necessary concepts in physics and mathematics are reviewed. The section can be skipped by any advanced reader of quantum mechanics. The relevant theory is always referenced when required in the subsequent chapters of the thesis.

In **Chapter 3**, we will restate the No Cloning Theorem. The results of optimal cloning are stated, and the proof of optimality with respect to fidelity, as a figure of merit, is provided. Furthermore, it is shown that the optimal quantum cloning is a covariant quantum channel.

In **Chapter 4**, the Wehrl entropy relation is stated following the arguments of Alfred Wehrl. The essence of the Lieb-Solovej theorem is further emphasized. The majorization of density matrices is studied to establish the relation between entropy and the quantum cloning machine. The eigenvalues plot of the images of random pure states and coherent states is studied and analyzed.

In **Chapter 5**, the concluding remarks of this thesis are presented. Few applications of optimal quantum cloning are also discussed.

Chapter 2

Mathematical Preliminaries

In this chapter, we shall deal with all the necessary mathematical and physical concepts required in this thesis. First, we start with mathematical tools to craft our optimal cloning device. We also need a bit of Group Theory. Moreover, we will also discuss Lie algebra of Angular Momentum representation.

2.1 Review of Quantum Mechanics

We will start building the framework for the quantum systems. Let's begin with the definition of Hilbert space [12].

Definition 2.1.1 (Hilbert Space)**.** Let $\mathcal{H}$ be a vector space over $\mathbb{C}$ equipped with a map $\langle \cdot, \cdot \rangle : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}$, called a scalar product or inner product over $\mathcal{H}$, is complete, then the pair $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is called a Hilbert space.

In order to manipulate our quantum systems, we need tools.

Definition 2.1.2 (Operator)**.** Suppose X and Y be two normed spaces. A linear map $L : X \mapsto Y$ between X and Y is called bounded operator if there exists $C < \infty$ such that:

$$\|Lx\|_Y \leq C\|x\|_X, \qquad \text{for all} \quad x \in X$$

For our finite dimensional $\mathcal{H}$, we denote the set of bounded operators with $\mathcal{L}(\mathcal{H})$.

Definition 2.1.3 (Positive operators)**.** An operator A is called positive if $\langle \psi, A\psi \rangle \geq 0 \quad \forall \psi \in \mathcal{H}$.

We will primarily deal with the density matrices as our operators.

Definition 2.1.4 (Density matrix)**.** [11] The density matrix is defined as a positive operator ρ with unit trace. For our study, we consider only pure states $|\psi\rangle$ on $\mathcal{H}$, and the density matrix is defined as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

where p_i is the probability of the system to be in a state $|\psi_i\rangle$, hence $\sum_i p_i = 1$ and all $p_i > 0$.

To work on the composite system $\mathcal{H}_1 \otimes \mathcal{H}_2$, we need to know how our tools work on such space, and we will also review few properties of our tools.

Definition 2.1.5 (Tensor product of operators)**.** [11] Consider A and B as the two vector spaces with the tensor product $A \otimes B$. Given C on A and D on B, then the operator $C \otimes D : A \otimes B \mapsto A \otimes B$ is defined by

$$C \otimes D \sum_i c_i a_i \otimes b_i = \sum_i c_i C(a_i) \otimes D(b_i) \qquad \forall a_i \in A, \quad \forall b_i \in B, \quad \forall c_i \in \mathbb{C}$$

Definition 2.1.6 (Positivity)**.** [13] Given a map $T : \mathcal{O} \mapsto \mathcal{P}$ that is linear, where $\mathcal{O} \in \mathcal{L}(\mathcal{H}_1)$ and $\mathcal{P} \in \mathcal{L}(\mathcal{H}_2)$ are operators on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, and T is said to be positive if for all non-negative $O \in \mathcal{O}, T(O)$ is also non-negative.

This tells us that the map takes non-negative operators to non-negative operators. In the thesis, we study primarily of density matrices as operators, and thus the positivity is automatically established.

Definition 2.1.7 (Complete Positivity)**.** [13] A positive map T is called completely positive on $\mathcal{H}$ if the map $T_{\mathcal{H}_1} \otimes \mathbb{1}_{\mathcal{H}_2}$ is positive acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$

This essentially tells us that $T_{\mathcal{H}_1} \otimes \mathbb{1}_{\mathcal{H}_2}$ is positive even if we extended the system.

Definition 2.1.8 (Quantum Channel)**.** [13] A linear map T is called a quantum channel if it preserves:

- Hermiticity
- Trace
- Positivity i.e. T is completely positive.

Remark. Quantum channels are often represented in Kraus form [17], i.e

$$\rho_{out} = T(\rho_{in}) = \sum_i A_i \rho_{in} A_i^\dagger \qquad , \qquad \sum_i A_i^\dagger A_i = \mathbb{1}$$

where A is called a Kraus operator.

Definition 2.1.9 (Covariant Quantum Channel)**.** [17] A quantum channel is said to be covariant or unitary covariant if

$$T(U_1 X U_1^*) = U_2 T(X) U_2^* \qquad \forall X \in \mathcal{L}(\mathcal{H})$$

That says it is equivalent to applying the unitary transformation before or after the cloning.

2.2 Group Theory

We need to employ few techniques from Group theory [5] to discuss the permutation invariance of the proposed cloning map.

Definition 2.2.1 (Group)**.** A set G with $\circ : G \times G \mapsto G$, has an identity element e and inverse for every element in G is said to form a group.

We will primarily focus on $SU(2)$ group, where all elements are 2 by 2 unitary matrices with unit determinant. For our study of $SU(2)$ Bloch coherent states, the operators are Pauli spin matrices, discussed in Section 4.2.

The other example is a symmetric group $\mathcal{S}$, which is the set of permutations.

Definition 2.2.2 (Representation of G). Let G be a finite group. A representation of G is a finite-dimensional complex vector space V with a group homomorphism $\pi : G \mapsto GL(E)$ with $\pi(gg') = \pi(g)\pi(g') \, \forall g, g' \in G$.

Lemma 2.2.3 (Schur's lemma). *Let V and W be vector spaces with irreducible representation π_V and π_W respectively of group G with linear map $T : V \mapsto W$, then*

- *If π_V and π_W are not equivalent, then $T = 0$*
- *If $V = W$ and $\pi_V = \pi_W$, then $T = \lambda \mathbb{1}$ with $\lambda \in \mathbb{R}$*

Definition 2.2.4 (Symmetric subspace). [11] The symmetric subspace $\mathcal{H}_+^{\otimes n}$ is

$$\mathcal{H}_+^{\otimes n} := \{\psi \in \mathcal{H}^{\otimes n} : P_\pi \psi = \psi \forall \pi \in \mathcal{S}\}$$

where P_π is known as permutation or exchange operator such as

$$P_{(12)}^2 |x_1, x_2\rangle = P_{(12)} |x_2, x_1\rangle = |x_1, x_2\rangle$$

Proposition 2.2.5 (Projection). *[11] An orthogonal projection $S_n : \mathcal{H}^{\otimes N} \mapsto \mathcal{H}_+^{\otimes N}$ is defined as:*

$$S_N = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}} P_\sigma$$

Proof. Let's show $S_N \mathcal{H}^{\otimes N} = \mathcal{H}_+^{\otimes N}$ first.
Linearity is evident from the structure of S_N. Let's show $S_N \mathcal{H}^{\otimes N} \subset \mathcal{H}_+^{\otimes N}$. We first need to show $P_\pi S_N \psi = S_N \psi$ for any $\psi \in \mathcal{H}^{\otimes N}, \pi \in \mathcal{S}$.

$$P_\pi S_N \psi = \frac{1}{N!} \sum P_\pi P_\sigma \psi = \frac{1}{N!} \sum P_{\pi\sigma} \psi = \frac{1}{N!} \sum P_\tau \psi = S_N \psi$$

with $P_\pi P_\sigma = P_{\pi\sigma} = P_\tau$, where $\pi\sigma$ is the composition of the group operation.
Next, we prove $\mathcal{H}_+^{\otimes N} \subset S_N \mathcal{H}^{\otimes N}$. For any $\psi \in \mathcal{H}_+^{\otimes N} \subset \mathcal{H}^{\otimes N}$, and $\pi \in \mathcal{S}_N$, we have $P_\pi \psi = \psi$, also $S_N \psi = \psi$. Hence, $S_N \mathcal{H}^{\otimes N} = \mathcal{H}_+^{\otimes N}$.
Furthermore we will show $S_N^2 = S_N$ and $S_N^* = S_N$,

$$S_N S_N \psi = \frac{1}{N!} \sum P_\sigma \cdot \frac{1}{N!} \sum P_\pi \psi = \left(\frac{1}{N!}\right)^2 \sum \sum P_{\sigma\pi} \psi$$

Rewriting as composite operation τ,

$$S_N^2 \psi = \left(\frac{1}{N!}\right)^2 \cdot N! \cdot \sum P_\tau \psi = S_N \psi$$

Thus, $S_N^2 = S_N$. Also,

$$S_N^* = \frac{1}{N!} \sum P_\sigma^* = \frac{1}{N!} \sum P_{\sigma^{-1}} = \frac{1}{N!} \sum P_\tau = S_N$$

with $\sigma \mapsto \sigma^{-1} = \tau$ as a bijection.

□

Theorem 2.2.6 (Stinespring dilation theorem). *[19] Let A be a unital C^*- algebra, $\mathcal{H}$ be a Hilbert space, and $\mathcal{L}(\mathcal{H})$ be the bounded operators on $\mathcal{H}$. For every completely positive map*

$$T : A \mapsto \mathcal{L}(\mathcal{H})$$

there exists an auxiliary Hilbert space $\mathcal{K}$ and a unital $*-$homomorphism $\pi : A \mapsto \mathcal{L}(\mathcal{K})$ such that $T(a) = V^*\pi(a)V$, where $V : \mathcal{H} \mapsto \mathcal{K}$ is a bounded operator. Furthermore, we have $\|T(1)\| = \|V\|^2$

Theorem 2.2.7 (Stinespring Theorem for Covariant CP-Maps). *[7] Let G be a group with finite dimensional unitary representations $\pi_i : G \mapsto \mathcal{L}(\mathcal{H}_i)(i = 1, 2)$ and $T : \mathcal{L}(\mathcal{H}_2) \mapsto \mathcal{L}(\mathcal{H}_1)$ a completely positive map with the covariance property $\pi_1(g)T(X)\pi_1(g)^* = T(\pi_2(g)X\pi_2(g)^*)$. Then, there is another finite dimensional unitary representation $\tilde{\pi} : G \mapsto \mathcal{L}(\tilde{\mathcal{H}})$ and an intertwiner $V : \mathcal{H}_1 \mapsto \mathcal{H}_2 \otimes \tilde{\mathcal{H}}$ with $V\pi_1(g) = \pi_2 \otimes \tilde{\pi}V$ such that $T(X) = V^*(X \otimes \mathbb{1})V$ holds.*

Proof. We first employ the theorem in its first form. There exits a representation $\eta : \mathcal{L}(\mathcal{H}_2) \mapsto \mathcal{L}(\mathcal{K})$ with bounded operator $V : \mathcal{H}_1 \mapsto \mathcal{K}\, s.t.\, T(X) = V^*\eta(X)V$ is true. By unitary equivalence, there exits only one such representation, say, $\tilde{\pi}$. Let $V_g = V\pi_1(g)$ and $\eta_g(X) = \eta(\pi_2(g)X\pi_2(g)^*)$ form Stinespring dilation of CP map $T_g(X) = \pi_1(g)^*T(\pi_2(g)X\pi_2(g)^*)\pi_1(g) = T$. We used the covariance property. By uniqueness of the unitary, let $U_g \in \mathcal{L}(\mathcal{K})\, s.t.\, V_g = V\pi_1(g) = U_gV$ and $\eta_g(X) = \eta(\pi_2(g)X\pi_2(g)^*) = U_g\eta(X)U_g^*$. Let $\tilde{U}_g = \eta(\pi_2(g))^*U_g$, which commutes with all $\eta(X)$. We will show now that $\tilde{U}$ is also representation like U_g. Since $\tilde{U}_g\tilde{U}_h = \eta(\pi_2(g))^*U_g\eta(\pi_2(h)^*)U_h = \eta(\pi_2(g))^*\eta(\pi_2(g)\pi_2(h)^*\pi_2(g)^*)U_gU_h = \eta(\pi_2(g)^*\pi_2(g)\pi_2(h)^*\pi_2(g)^*)U_{gh} = \eta(\pi_2(gh)^*)U_{gh} = \tilde{U}_{gh}$.

Now, we can say that all representations of $\mathcal{L}(\mathcal{H}_2)$ can be expressed as $\eta \simeq \mathrm{id} \otimes \mathbb{1}$ with $K = \mathcal{H}_2 \otimes \tilde{\mathcal{H}}$, where $\simeq$ is the unitary equivalence, id is the identity element of the group. Since $\tilde{U}_g$ commutes with all $\eta(X) = X \otimes \mathbb{1}$ of the form $\tilde{U}_g = \mathbb{1} \otimes \tilde{\pi}(g)$. Finally, $T_g(X) = V_g^*\eta_g(X)V_g = V^*U_g^*U_g\eta(X)U_g^*U_gV = V^*(X \otimes \mathbb{1})V = T$, by using covariance property again. □

2.3 Angular Momentum Representation

We will review the basics of angular momentum representation [16]. In all our settings, we will use $\hbar = 1$. The Lie Algebra is

$$[L_i, L_j] = iL_k \qquad \text{and} \qquad L^2 = L_1^2 + L_2^2 + L_3^2$$

with the relations

$$L^2|l, m\rangle = l(l + 1)|l, m\rangle$$
$$L_3|l, m\rangle = m|l, m\rangle$$

where $l \in \mathbb{N}$ is a spin representation and $m = \{-l, -l + 1, \ldots, l - 1, l\}$.
Furthermore, the Lie Algebra can be complexified as

$$L_3 \quad , \quad L_+ := L_1 + iL_2 \quad , \quad L_- := L_1 - iL_2 \quad , \quad L_+^\dagger = L_-$$

with

$$L_+|l, m\rangle = \sqrt{l(l + 1) - m(m + 1)}|l, m + 1\rangle$$
$$L_-|l, m\rangle = \sqrt{l(l + 1) - m(m - 1)}|l, m - 1\rangle$$

Chapter 3

Optimal Cloning Device

Before we introduce the optimal cloning device, it is very logical to restate the proof of the No cloning theorem. Although there were few instances [15] that shows the No Cloning theorem was understood before 1982 paper from Wooters and Zurek [24], the theorem got widely known only after "No cloning" theorem was firmly stated in the 1982 paper.

3.1 The No Cloning Theorem

The No Cloning Theorem in quantum mechanics has been well established. This theorem says that we cannot copy perfectly any arbitrary states. We'll restate the No Cloning Theorem here.

Theorem 3.1.1 (No Cloning Theorem)**.** *[24] Consider a finite $d > 1$ dimensional Hilbert space with orthonormal basis $|i\rangle_{i=0}^{d-1}$ and $|\psi\rangle \in \mathcal{H}$, then there does not exists any unitary map such that $\psi\rangle \otimes |0\rangle \mapsto |\psi\rangle \otimes |\psi\rangle$.*

Proof. Suppose there is unitary transformation U. Consider the basis $|0\rangle, |1\rangle \in \mathcal{H}$:, and we act with U following its linearity. Then,

$$U|0\rangle \otimes |0\rangle = |0\rangle \otimes |0\rangle \qquad\qquad U|1\rangle \otimes |0\rangle = |1\rangle \otimes |1\rangle$$

where the second register is considered to be the ancilla space.
Also, take $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\psi\rangle \in \mathcal{H}\, s.t.$ the state is normalized. Following the linearity gives

$$U|\psi\rangle \otimes |0\rangle = U(\alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |0\rangle) = \alpha|0\rangle \otimes |0\rangle + \beta|1\rangle \otimes |1\rangle$$

However, the perfect clone would be the composite system of the identical states,

$$|\psi\rangle\otimes|\psi\rangle = (\alpha|0\rangle+\beta|1\rangle)\otimes(\alpha|0\rangle+\beta|1\rangle) = \alpha^2|0\rangle\otimes|0\rangle+\beta^2|1\rangle\otimes|1\rangle+\alpha\beta(|0\rangle\otimes|1\rangle+|1\rangle\otimes|0\rangle)$$

If true, $\alpha\beta = 0$, which means only the perfect copy of the basis is possible, but not any state that is in superposition, which is a contradiction. □

Now, we will closely follow the arguments presented in Reinhard Werner's 1999 paper [22], which generalizes the optimal cloning to arbitrary dimensions.

3.2 Quantum Cloning Device

For any cloning procedure, we need three components. The first is the original quantum state that we want to copy. In our thesis, we focus only on pure states. The second is the ancilla where we want our copied state to live, and the third obviously is the cloning machine. It is essential to note that the cloning machine has no prior information about the state that it is about to copy. Let's restate the input and output space of our optimal cloning map.

Input/Output Spaces

In our quantum cloning device, the quantum states are assumed to be in finite dimensional Hilbert space. Only the pure states are subject to our interests here.

In general, we prepare N separate copies of the original quantum states as our input, and the cloning device makes its M copies with $M > N$. Thus, the input quantum systems are denoted as $\sigma^{\otimes N}$ where $\sigma = |\psi\rangle\langle\psi|$, for any pure state $|\psi\rangle \in \mathcal{H}$. They sit in symmetric Hilbert space $\mathcal{H}_+^{\otimes N}$ with $\dim \mathcal{H}_+^{\otimes N} = \binom{d+N-1}{N}$, and the output M copies are in $\mathcal{H}^{\otimes M}$, not necessarily symmetric.

We'll state our results here, and prove its optimality in the next section.

Theorem 3.2.1 (Optimal cloning map)**.** *[22] The optimal cloning map $\hat{T}_*$ is denoted as*

$$\hat{T}_*(\rho) = \frac{d[N]}{d[M]} S_M \left(\rho \otimes \mathbb{1}_{\mathcal{H}}^{\otimes(M-N)}\right) S_M$$

where ρ is a pure d.m., $S_M : \mathcal{H}^{\otimes M} \mapsto \mathcal{H}_+^{\otimes M}$ is the symmetric projection , and we let $d[n] = \binom{d+n-1}{n} = \dim(\mathcal{H}_+^{\otimes N})$, where $n \in \mathbb{N}$.

Definition 3.2.2 (Fidelity)**.** [22] The fidelity of T_* is given by:

$$\mathcal{F}(T_*) = \mathrm{Tr}[\sigma^{\otimes M} T_*\left(\sigma^{\otimes N}\right)]$$

where the infimum is considered among all pure states.

Fidelity is the figure of merit to measure the quality of our clones. It compares the overlap between the ideal clones (which are forbidden by the No Cloning Theorem), and the copies we receive from our cloning machine.

Theorem 3.2.3 (Unitary average)**.** *[11] For CP map $T_* : \mathcal{L}(\mathcal{H}) \mapsto \mathcal{L}(\mathcal{H})$. The unitary averages $\overline{T}$ with dual $\overline{T}_*$ are*

$$\overline{T}(A) := \int_{U(d)} u^{*\otimes N} T\left(u^{\otimes M} A u^{*\otimes M}\right) u^{\otimes N} du$$

$$\overline{T}_*(\rho) := \int_{U(d)} u^{*\otimes M} T\left(u^{\otimes N} \rho u^{*\otimes N}\right) u^{\otimes M} du$$

where $A \in \mathcal{L}(\mathcal{H}^{\otimes M}), \rho$ is a pure d.m., du is the normalized Haar measure such that $\mathcal{F}(\overline{T}) \geq \mathcal{F}(T)$.

Proof. Let $\sigma_u := u\sigma u^* \implies u^{\otimes N}\sigma^{\otimes N}u^{*\otimes N} = \sigma_u^{\otimes N}$. Then following the definition 3.2.2, the fidelity of an average cloning map would be

$$\mathcal{F}(\overline{T}_*) = \mathrm{Tr}[\sigma^{\otimes M}\overline{T}_*\left(\sigma^{\otimes N}\right)] = \mathrm{Tr}[\sigma^{\otimes M}\int_{U(d)} u^{*\otimes M}T_*\left(\sigma_u^{\otimes N}\right)u^{\otimes M}du]$$

Using linearity of trace and the integral, then the cyclic property of the trace gives

$$\mathcal{F}(\overline{T}_*) = \int_{U(d)} \mathrm{Tr}[\sigma^{\otimes M}u^{*\otimes M}T_*\left(\sigma_u^{\otimes N}\right)u^{\otimes M}]du = \int_{U(d)} \mathrm{Tr}[\sigma_u^{\otimes M}T_*\left(\sigma_u^{\otimes N}\right)]du$$

For a pure state σ_u , then from definition 3.2.2,

$$\mathrm{Tr}[\sigma_u^{\otimes M}T_*\left(\sigma_u^{\otimes N}\right)] \geq \mathcal{F}(T_*) \qquad \forall\sigma\,\mathrm{and}\,,\forall u \in U(d)$$

Then, the integral becomes

$$\mathcal{F}(\overline{T}_*) = \mathrm{Tr}[\sigma^{\otimes M}\overline{T}_*\left(\sigma^{\otimes N}\right)] \geq \int_{U(d)} \mathcal{F}(T_*)du = \mathcal{F}(T_*) \qquad \forall\sigma$$

.

□

This shows that the proposed optimal cloning map is a unitary covariant. We need to also show that it is also a quantum channel.

3.3 Proof that $\hat{T}_*$ is a Quantum Channel

We need to prove the complete positivity and trace preserving properties of the optimal cloning map as discussed in Section 2.1.8. The complete positivity can be understood from the structure of $\hat{T}_*$, due to linearity of $\hat{T}_*$ and since $S_M^* = S_M$ and $\left(\rho \otimes \mathbb{1}_{\mathcal{H}}^{\otimes(M-N)}\right)$ being positive, $\hat{T}_*$ is a completely positive map. The proof of trace preserving follows a long argument [22]. In summary, we write the trace $\hat{T}_*\left(\rho\right)$ with $\hat{T}_*\left(\rho X\right)$ for some positive operator X. It is shown X commutes with all unitaries, then from Schur's lemma, $X = \lambda\mathbb{1}$. Finally, it is shown that $\lambda = 1$, which means $X = \mathbb{1}$. Hence, the optimal cloning preserves the trace.

3.4 Proof of optimality with respect to fidelity

The proof of the Theorem 3.2.1 is stated here, revising [22].

Let's first define our input and output states. Let $\tau_N := d[N]^{-1}S_N$ be the input and $\tau_M := d[M]^{-1}S_M$ be the output state i.e, $\hat{T}_*\left(\tau_N\right) = \tau_M$.

Proof. Let us consider τ_N on $\mathcal{H}_+^{\otimes N}$.

$$\overline{T}_*(\tau_N) = \overline{T}_*(u^{\otimes N}\tau_N u^{*\otimes N}) = u^{\otimes M}\overline{T}_*(\tau_N)u^{*\otimes M} \iff [\overline{T}_*(\tau_N), u^{\otimes M}] = 0 \qquad \forall \in U(d)$$

The first equality follows from the commutation relation of S_M with all unitaries $u^{\otimes N}$. We use the covariance property of the optimal cloning map in the second equality. Writing the output space in terms of orthogonal components, and, then using Schur's lemma gives

$$\overline{T}_*(\tau_N) = \frac{\lambda}{d[M]} S_M + (1-\lambda)\chi$$

where $S_M = \mathbb{1}_{\mathcal{H}_+^{\otimes M}}$, and $\chi \in \mathcal{H}_+^{\otimes M,\perp}$ is a d.m., and $0 \leq \lambda \leq 1$. Thus, the trace over the second quantity with $\sigma^{\otimes M}$ cancels by orthogonality.

$$0 \leq \mathrm{Tr}[\sigma^{\otimes M}\overline{T}_*(S_N - \sigma^{\otimes N})] = d[N]\,\mathrm{Tr}[\sigma^{\otimes M}\overline{T}_*(\tau_N)] - \mathrm{Tr}[\sigma^{\otimes M}\overline{T}_*(\sigma^{\otimes N})]$$

We used $\tau_N = d[N]^{-1}S_N$ Let's evaluate the second component first.

$$\mathrm{Tr}[\sigma^{\otimes M}\overline{T}_*(\sigma^{\otimes N})] = \int_{U(d)} \mathrm{Tr}[\sigma_u^{\otimes M} T_*(\sigma_u^{\otimes N})]du \geq \int_{U(d)} \mathcal{F}(T)du = \hat{\mathcal{F}}$$

Since the expression is for any pure σ, then the expression also equals to $\mathcal{F}(\overline{T}) \implies \mathcal{F}(\overline{T}) \geq \hat{\mathcal{F}}$. From the definition of $\hat{\mathcal{F}}$, we can say $\hat{\mathcal{F}} \geq \mathcal{F}(\overline{T})$, which means $\mathcal{F}(\overline{T}) = \hat{\mathcal{F}}$. Then, the whole expression can be written as

$$0 \leq \lambda\frac{d[N]}{d[M]} - \hat{\mathcal{F}} \Rightarrow \hat{\mathcal{F}} \leq \frac{d[N]}{d[M]} \leq \frac{d[N]}{d[M]}$$

and equality if $\lambda = 1$. Hence, $\overline{T}_*(\tau_N) = \tau_M$ in the optimal case, and $\hat{\mathcal{F}} \leq \frac{d[N]}{d[M]}$. Also if $\lambda = 1$, then

$$\mathrm{Tr}[\sigma^{\otimes M}\overline{T}_*(S_N - \sigma^{\otimes N})] = \frac{d[N]}{d[M]} - \frac{d[N]}{d[M]} = 0 \Rightarrow \int_{U(d)} \mathrm{Tr}[\sigma_u^{\otimes M} T_*(S_N - \sigma_u^{\otimes N})]du = 0$$

where $u^{\otimes N}S_N u^{*\otimes N} = S_N$ is used, due to their commutation relation. The integrand then becomes

$$\mathrm{Tr}[\sigma^{\otimes M} T_*(S_N - \sigma^{\otimes N})] = 0 \qquad \forall\sigma$$

. We can now employ the covariant version of Stinespring's dilation theorem 2.2.7,

$$T_*(\rho) = \hat{\mathcal{F}} \cdot V^*(\rho \otimes \mathbb{1}_{\mathcal{K}})V \qquad , \qquad \rho \text{ is a pure d.m.}$$

for $V : \mathcal{H}_+^{\otimes M} \mapsto \mathcal{H}_+^{\otimes N} \otimes \mathcal{K}$ with ancilla $\mathcal{K}$. $\hat{\mathcal{F}}$ is relevant in the optimal condition, where $V^*V = \mathbb{1}_{\mathcal{H}_+^{\otimes M}}$. Rewriting:

$$\langle \psi^{\otimes M}, V^*[(S_N - \sigma^{\otimes N}) \otimes \mathbb{1}_{\mathcal{K}}]V, \psi^{\otimes M}\rangle = 0 \qquad \forall\sigma = |\psi\rangle\langle\psi|$$

Also,

$$[(S_N - \sigma^{\otimes N}) \otimes \mathbb{1}_{\mathcal{K}}]^2 = \left(S_N^2 - S_N\sigma^{\otimes N} - \sigma^{\otimes N}S_N + (\sigma^2)^{\otimes N}\right) \otimes \mathbb{1}_{\mathcal{K}} = (S_N - \sigma^{\otimes N}) \otimes \mathbb{1}_{\mathcal{K}}$$

where we have used $S_N^2 = S_N, \sigma^{\otimes N} = S_N\sigma^{\otimes N} = \sigma^{\otimes N}S_N$, and $\sigma^2 = \sigma$ for pure states. Then, we can write

$$\langle \psi^{\otimes M}V^*[\left(S_N - \sigma^{\otimes N}\right)\otimes\mathbb{1}_{\mathcal{K}}]^*, [\left(S_N - \sigma^{\otimes N}\right)\otimes\mathbb{1}_{\mathcal{K}}]V\psi^{\otimes M}\rangle = \|[\left(S_N - \sigma^{\otimes N}\right)\otimes\mathbb{1}_{\mathcal{K}}]V\psi^{\otimes M}\|$$

Calculating the inner product yields

$$\begin{aligned}\langle\phi,\psi\rangle^M &= \langle\phi^{\otimes M}, \psi^{\otimes M}\rangle = \langle\phi^{\otimes M}, V^*V\psi^{\otimes M}\rangle = \langle V\phi^{\otimes M}, V\psi^{\otimes M}\rangle \\ &= \langle\phi^{\otimes N} \otimes \xi(\phi), \psi^{\otimes N} \otimes \xi(\psi)\rangle = \langle\phi,\psi\rangle^N\langle\xi(\phi),\xi(\psi)\rangle_{\mathcal{K}}\end{aligned}$$

where $V^*V = \mathbb{1}_{\mathcal{H}_+^{\otimes M}}$ is used . We can write it as

$$\langle \xi(\phi), \xi(\psi) \rangle_{\mathcal{K}} = \langle \phi, \psi \rangle^{M-N}$$

Thus, the optimal cloning channel is uniquely determined with respect to the fidelity. □

We also want to present one example of optimal quantum cloning.

3.5 Example of qubit copying

Let's take an example of $1 \to 2$ cloning, $N = 1, M = 2$.
Here, $d[N] = \binom{2}{1} = 2, d[M] = \binom{3}{2} = 3$.
The symmetric projector S_2:

$$S_2 = \frac{1}{2} \sum_{\pi \in \mathcal{S}} P_\pi = \frac{1}{2} P_{(1)} + \frac{1}{2} P_{(12)}$$

where (1) is the identity element and subscript (12) denotes the swap operator between 1 and 2.

$$P_{(1)} = \mathbb{1}_{\mathcal{H}} \otimes \mathbb{1}_{\mathcal{H}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ; P_{(12)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then,

$$S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Consider a pure state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ and density matrix $\sigma = |\psi\rangle\langle\psi|$. Then,

$$\sigma \otimes \mathbb{1}_{\mathcal{H}} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & 0 & \alpha\beta^* & 0 \\ 0 & |\alpha|^2 & 0 & \alpha\beta^* \\ \alpha^*\beta & 0 & |\beta|^2 & 0 \\ 0 & \alpha^*\beta & 0 & |\beta|^2 \end{pmatrix}$$

Substituting all the calculated quantities yields

$$\hat{T}_*(\sigma) = \frac{2}{3} S_2 (\sigma \otimes \mathbb{1}_{\mathcal{H}}) S_2 = \frac{2}{3} \cdot \frac{1}{4} \begin{pmatrix} 4|\alpha|^2 & 2\alpha\beta^* & 2\alpha\beta^* & 0 \\ 2\alpha^*\beta & 1 & 1 & 2\alpha\beta^* \\ 2\alpha^*\beta & 1 & 1 & 2\alpha\beta^* \\ 0 & 2\alpha^*\beta & 2\alpha^*\beta & 4|\beta|^2 \end{pmatrix}$$

This is the optimal clone for any pure state with $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ in 2 dimensions for $1 \to 2$ cloning.

Chapter 4

Coherent States, Majorization, and Wehrl Entropy

4.1 Canonical Coherent States

A canonical coherent state $|z\rangle$ or Gaussian coherent state [17] is an eigenstate of the annihilation operator $\hat{a}$ with eigenvalue z.

$$\hat{a}|z\rangle = z|z\rangle$$

The key properties of the coherent states are

- The coherent states form a complete basis with resolution of unity

$$\int dz|z\rangle\langle z| = \mathbb{1}$$

- The diagonal matrix elements (lower symbol) of an operator A uniquely determines the operator.

$$A(z) = \langle z|A|z\rangle$$

- Any operator A can be expanded diagonally in coherent states.

$$A = \int dz h_A(z)|z\rangle\langle z|$$

  where $h_A(z)$ is called the upper symbol of A.

- The coherent states saturate the Heisenberg uncertainty relation.

$$\Delta q \Delta p = \frac{1}{2}$$

The minimal Heisenberg uncertainty also hints that there must be something "classical" about the coherent states. We will see later how Wehrl made use of these properties of the coherent states to define the Wehrl entropy.

4.2 Bloch Coherent States

The Bloch coherent states [17] are similar to the canonical coherent states. They are also known as Spin Coherent States. They are defined by angular momentum operators $\vec{S} = (S_1, S_2, S_3)$ with commutation relations

$$[S_i, S_j] = iS_k \qquad \text{and} \qquad S^2 = S_1^2 + S_2^2 + S_3^2 = l(l+1)\mathbb{1}$$

in a spin-l irreducible representation $[l] := \mathbb{C}^{2l+1}$ of $SU(2)$ with $2l+1 \in \mathbb{N}$. We denote $\mathbb{S}$ the unit sphere in three dimensions

$$\mathbb{S} = \{(x, y, z)|x^2 + y^2 + z^2 = 1\}$$

and let

$$\Omega = (\theta, \phi) \qquad s.t. \qquad 0 \leq \theta \leq \pi; 0 \leq \phi \leq 2\pi$$

. The Bloch coherent state $|\Omega\rangle \in \mathbb{C}^{2l+1}$ is the highest weight vector $|l, l\rangle$ and defined as

$$|\Omega\rangle := e^{\frac{1}{2}\theta e^{i\phi}S_- - \frac{1}{2}\theta e^{-i\phi}S_+}|l, l\rangle = e^{zS_-}e^{-\ln\left(1+|z|^2\right)S_z}e^{-z^*S_+}|l, l\rangle$$

where $z = \tan\frac{\theta}{2}e^{i\phi}$. The above expression can be simplified to

$$|\Omega\rangle = (1+z^2)^{-l}\sum_{m=-l}^{l} z^{l-m}\binom{2l}{l+m}^{1/2}|m\rangle$$

where $|m\rangle$ is the normalized state

$$|m\rangle = \binom{2l}{l+m}^{-\frac{1}{2}}\frac{1}{(l-m)!}S_-^{l-m}|l\rangle \qquad \text{and} \qquad S_3|m\rangle = m|m\rangle$$

Expressing z in terms of θ and ϕ gives

$$|\Omega\rangle = \sum_{m=-l}^{l}\binom{2l}{l+m}^{\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{l+m}\left(\sin\frac{\theta}{2}\right)^{l-m}e^{i(l-m)}|m\rangle$$

Spin coherent states are complete via Schur's lemma

$$(2l+1)\int\frac{d\Omega}{4\pi}|\Omega_l\rangle\langle\Omega_l| = S_l = \mathbb{1}$$

where S_l is the projector onto $[l]$.

Remark (Dicke States). [2] Dicke states have a special form. They are in the form $|l+m, l-m\rangle$, a two level systems. They are defined as the eigenstates of

$$J_x = \frac{1}{2}\sum_{i=1}^{N} X_i$$

Similarly J_y and J_z for Pauli matrices Y and Z.

4.3 Wehrl Entropy

The von Neumann entropy is defined as

$$S(\rho) = -\operatorname{Tr}[\rho \ln \rho]$$

is a functional of the density matrix ρ. From its construction, it is always non-negative and is exactly 0 for pure states [17]. For our covariant quantum channel, we are interested in studying pure and coherent states separately. Since coherent states are also pure states, their von Neumann entropy is also zero. Coherent states are studied because of their "classical" nature. There is Boltzmann entropy that is classical and defined for a continuous phase space distribution $\rho\,(q,p)$

$$S_{cl} = -\int \frac{dqdp}{2\pi} \rho\,(q,p) \ln \rho\,(q,p)$$

The problem with using Boltzmann entropy is it can take arbitrary negative values.

Wehrl addressed this problem by redefining the quantum entropy in terms of density matrix of Gaussian coherent states [21]. He interpreted the density matrix of coherent states as the probability density for a measurement.

$$\rho(z) = \operatorname{Tr}[\rho|z\rangle\langle z|]$$

Moreover, we also have,

$$\int \rho(z)dz = \mathbb{1}$$

Thus, the Wehrl entropy is

$$S_W = -\int dz \rho(z) \ln \rho(z)$$

It is shown to behave like classical Shannon entropy, even if the underlying distribution $\rho(z)$ of S_W is continuous. It is also important to note that $S_W \geq 1$ and clearly larger than von Neumann entropy [2]. The natural question then to ask is what states minimize the Wehrl entropy. Wehrl conjectured that Gaussian coherent states are the minimizers for Wehrl entropy. This has been generalized to all coherent states [8]. The Lieb conjecture stated that

$$S_W(|\psi\rangle\langle\psi|) \geq \frac{2j}{2j+1}$$

with equality if $|\psi\rangle$ is a coherent state. Lieb and Solovej have finally proved the conjecture in 2012, after several attempts from different experts in the field.

4.4 Majorization of Density Matrices

Consider positive vectors $\vec{x}$ and $\vec{y}$ with descending order $x_1 \geq x_2 \geq, \ldots, x_N; y_1 \geq y_2 \geq, \ldots, y_N$, denoted by $x_i^{\downarrow}$ and $y_i^{\downarrow}$ respectively. Then $\vec{x}$ is majorized by $\vec{y}$ is written as

$$x \prec y \qquad \Longleftrightarrow \qquad \begin{cases} (i) \sum_{i=1}^{k} x_i^{\downarrow} \leq \sum_{i=1}^{k} y_i^{\downarrow} & \text{for} \quad k = 1, \ldots, N \\ (ii) \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i \end{cases}$$

The geometrical meaning of a majorization tells us that if $x \prec y$, then $\vec{x}$ is in the convex hull of all vectors obtained by permuting the coordinates of $\vec{y}$ [2]. We will make use of this property later in our connection with the covariant quantum channel to the entropy.

For the study of our optimal cloning map, we need to analyze the majorization relation of the density matrix among random pure states and coherent states. For our density matrices of pure states, conditions (ii) is automatically satisfied.

Theorem 4.4.1. *[2] The state σ is majorized by the state ρ iff the eigenvalue vector of σ is majorized by the eigenvalue vector of ρ.*

$$\sigma \prec \rho \iff \vec{\lambda}(\sigma) \prec \vec{\lambda}(\rho)$$

Lemma 4.4.2 (Quantum HLP Lemma)**.** *[2] There exists a completely positive bistochastic map transforming ρ into σ iff $\sigma \prec \rho$*

$$\rho \xrightarrow{bistochastic} \iff \sigma \prec \rho$$

For our purpose, then we need to show that our optimal cloning map is indeed a bistochastic map.

Theorem 4.4.3. *[2] A TPCP map is called bistochastic map if it is also unital.*

The proof of TPCP for our optimal cloning device is presented in Section 3.3. The proof of unital is also stated in Stinespring's dilation theorem for covariant quantum channel in Theorem 2.2.7.

Then by the implication of Quantum HLP Lemma, it is indeed true that the output density matrix majorizes the input density matrix.

This gives us now insight into the entropy of our optimal cloning map. The entropy of a density matrix can be defined with respect to any pure POVM-$\{E_i\}$. For example, Shannon entropy

$$S(\rho) := S(\vec{p}) \qquad , \qquad p_i = \mathrm{Tr}[E_i \rho]$$

Minimizing over all possible POVMs gives the definition of von-Neumann entropy

$$S(\rho) = \min_{\mathrm{POVM}} S(\vec{p})$$

And the Wehrl entropy is the special case of this where POVM is defined by the coherent states [2].

This also concludes that the entropy of an output state would be larger than the entropy of an input state. The output with the lowest entropy is conjectured to be of coherent states. This is the Lieb-Solovej theorem.

4.5 Angular Entropy and Covariant Quantum Channel

Wehrl entropy is computationally difficult as being dependent on the size of the density matrix. Fortunately, there is angular entropy whose calculation would involve only 3 by 3 matrices, and it is numerically shown that the angular entropy has all the properties of Wehrl entropy [10].

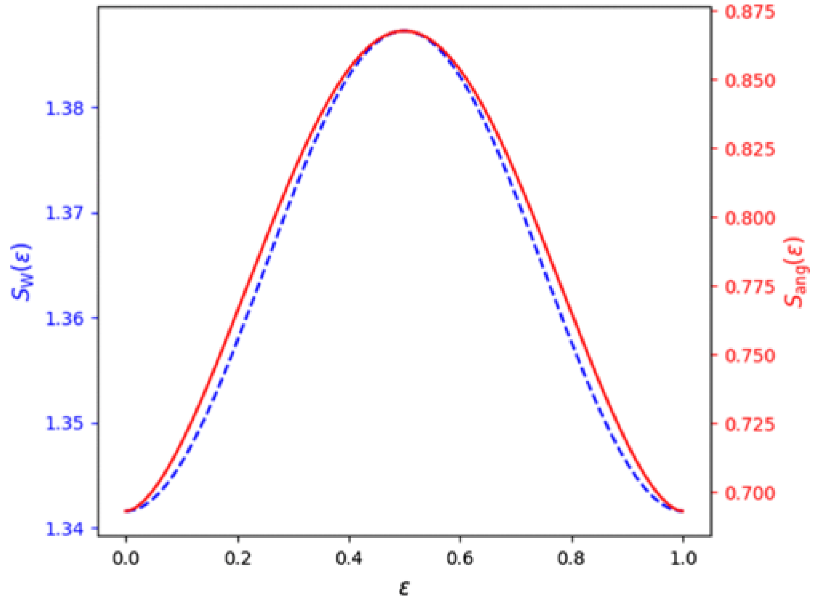

Figure 4.1: Comparison of Wehrl and Angular Entropy [10]

We will now follow the arguments presented in [10] to explicitly calculate the eigenvalues of the images of states generated by the covariant quantum channel. We start by defining a generalized density matrix

$$\rho_{ang} = \frac{1}{l(l+1)} \sum_{i=1}^{3} L_i \rho L_i^\dagger$$

with

$$S_{ang} = -\operatorname{Tr}[\rho_{ang} \ln \rho_{ang}]$$

where L_1, L_2, L_3 are standard angular momentum operators. The transformation being in Kraus form tells us that it is completely positive. The trace preserving property emerges from $C = \sum L_i^\dagger L_i$ has the value $l(l+1)$ in spin-l representation.

For pure state $|\psi\rangle$, we can also express entropy in terms of dual Gramm matrix formulation

$$G_{ij} = \langle \psi | C^{-1} L_i^\dagger L_j | \psi \rangle \qquad , \qquad S_{ang} = -\operatorname{Tr}[G \ln G]$$

Expressing our state in the form where l is fixed,

$$|\psi\rangle = \sum_m a_m |l, m\rangle$$

Then, rewriting the Hermitian 3 by 3 dual Gramm matrix G

$$G = \begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{23} & G_{33} \end{pmatrix}$$

with

$$G_{ij} = \frac{1}{2l(l+1)} g_{ij}(l, \{a_m\})$$

and

$$g_{11} = \sum_{m=-l}^{l} (l+m+1)(l-m) \cdot |a_m|^2$$

$$g_{12} = \sum_{m=-l+1}^{l-1} \sqrt{(l^2-m^2)((l+1)^2-m^2)} \cdot a_{m-1}a_{m+1}^*$$

$$g_{13} = \sqrt{2} \sum_{m=-l}^{l-1} (m+1)\sqrt{(l+m+1)(l-m)} \cdot a_m \cdot a_{m+1}^*$$

$$g_{22} = \sum_{m=-l}^{l} (l-m+1)(l+m) \cdot |a_m|^2$$

$$g_{23} = \sqrt{2} \sum_{m=-l+1}^{l} (m-1)\sqrt{(l-m+1)(l+m)} \cdot a_m a_{m-1}^*$$

$$g_{33} = 2 \sum_{m=-l}^{l} m^2 |a_m|^2$$

The eigenvalue plot is presented below, and the calculations are shown in Appendix A.

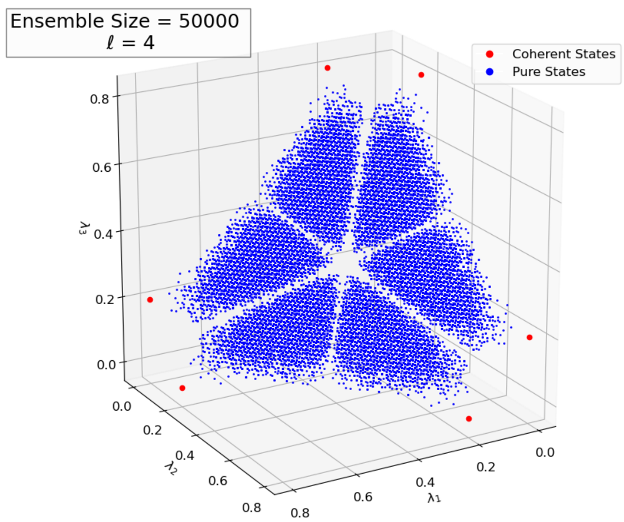

Figure 4.2: Spectrum of the images of coherent and random pure states via covariant quantum channel

The figure is deemed to be of highly academic interest. We know pure states majorize mixed states, and coherent states majorize pure states; it is also important that this property holds true even for their images under covariant quantum channel. It is also interesting to see for larger l that Dicke states form a line segment. The empty convex hull created by the intersection of these lines is also an interesting area to observe. One such natural candidate is a diagonal matrix χ with eigenvalues $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, which is a mixed density matrix, $\chi^2 \neq \chi$. It is of great interest to know which states transform to maximally mixed one under the action of covariant quantum channel as such states are supposed of great subjects to study in other fields of physics [23], [1].

On the other hand, the image of coherent states majorizing the images of states tells us that the entropy is minimum for the images of coherent states. This gives us a new definition of coherent states i.e. they are the states whose images majorize all other states under the covariant quantum channels. It has an advantage now that covariant quantum channel like our studied optimal quantum cloning can be used to understand more about the classical properties of the coherent states. This is not covered in the present thesis.

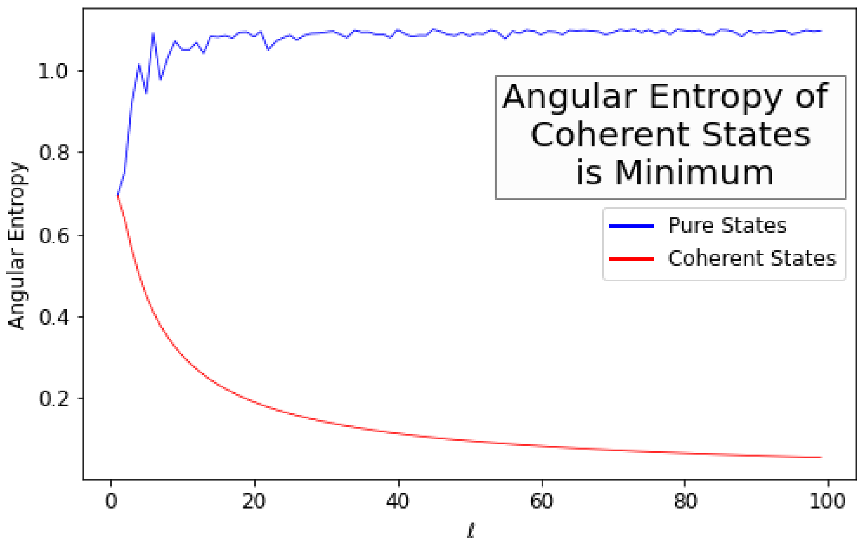

Figure 4.3: Entropy of images of coherent and random pure states under the action of covariant quantum channel

Chapter 5

Conclusion

The thesis primarily studied the copying of quantum states, which is restrained due to the No Cloning Theorem by analyzing the paper from [22] and [6]. The perfect copy of the states that are in superposition is not possible [24]. Then it becomes logical to think of the optimal copies. The optimal cloning map, which is proven to be a covariant quantum channel, can make optimal copies of pure states. It was first studied by Bužek-Hillery [4], and later generalized by Reinhard Werner [22] in 1999. Fidelity is considered a figure of merit to determine the quality of our clones. It compares the overlap between the ideal copies, which are forbidden by the No cloning theorem, and the copies we receive from the optimal cloning device. The other figure of merit such as single clone fidelity is also studied in [6].

It is also shown that the optimal cloning map is “simple” by its functionality. It takes the input state and on the other hand prepares the maximally mixed states but universal copies and distributes them among all copies. Thus, it is permutation invariant. Moreover, the fidelity of all copies remains the same i.e. the fidelity is independent on the state. Thus, our optimal cloning map is also proven to be universal.

The thesis also studies the entropy of our optimal cloning device as a covariant quantum channel. Coherent states, which minimize the Wehrl entropy [8], also become a primary study for the covariant quantum channel. The images of coherent states majorize the images of random pure states. This is the primary essence of the Lieb-Solovej theorem. Following the majorization theorem, we are also able to conclude that the entropy of images of coherent states would be smaller compared to the entropy of images of pure states. The eigenvalue plot of the images under the covariant quantum channel is also great progress in the thesis. We were able to see the relation between the quantum cloning machine, which was studied independently, and the astounding theorem like the Lieb-Solovej one.

The research into optimal quantum cloning has interests in the applications field too. It has been used in metrology to increase the sensitivity of devices like radiometers [14]. In quantum communication, it can be used to eavesdrop on the communication between Alice and Bob [3].

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Appendix A

Calculation of Dual Gramm Matrix entries

The calculations are easy in the basis $\{\frac{1}{\sqrt{2}}L_+, \frac{1}{\sqrt{2}}L_-, L_3\}$. We will thus first show the unitary map between $\{L_1, L_2, L_3\}$ and the above mentioned basis.

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}L_+ \\ \frac{1}{\sqrt{2}}L_- \\ L_3 \end{pmatrix}$$

Then, we can rewrite all relations in terms of L_+, L_-, and L_3.

$$\begin{aligned} L_{11} &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}L_+^2 \\ L_{12} &= \frac{1}{2}L_+L_- \\ L_{13} &= \frac{1}{\sqrt{2}}L_+L_3 \\ L_{23} &= \frac{1}{\sqrt{2}}L_-L_3 \end{aligned}$$

Then we can explicitly calculate the dual Gramm matrix entries for a fixed l.

$$\begin{aligned} \langle\psi|L_1L_1|\psi\rangle &= \frac{1}{2}\sum_{m=-l-1}^{l-1}(l(l+1) - m(m+1))a_{m+1}^*a_{m+1} \\ &= \frac{1}{2}\sum_{m=-l}^{l}((l+m)(l-m) + (l-m))\,|a_m|^2 \\ &= \frac{1}{2}\sum_{m=-l}^{l}(l-m)(l+m+1)|a_m|^2 \end{aligned}$$

$$
\begin{aligned}
\langle\psi|L_1L_2|\psi\rangle &= \frac{1}{2}\sum_{m=-l+1}^{l+1}\sqrt{\left(l^2+l-m^2-m\right)\left(l^2+l-m^2+m\right)}a^*_{m+1}a_{m-1} \\
&= \frac{1}{2}\sum_{m=-l+1}^{l-1}\sqrt{(l-m)(l+m+1)(l+m)(l-m+1)}a^*_{m+1}a_{m-1} \\
&= \frac{1}{2}\sum_{m=-l+1}^{l+1}\sqrt{\left(l^2-m^2\right)\left((l+1)^2-m^2\right)}a^*_{m+1}a_{m-1}
\end{aligned}
$$

The other relations are straightforward to carry out.

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