UMR 8579

Efficient solvers for constrained optimization in structural mechanics

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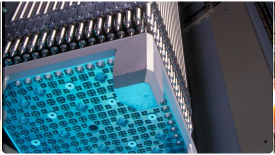
Industrial framework

- EDF : an operator of electric power production.



- Ensure proper functioning of production structures and optimize their availability.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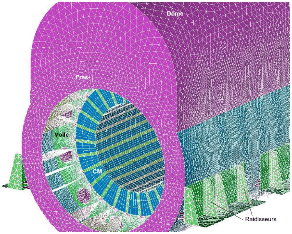
Observed variability on a nominally identical machine park



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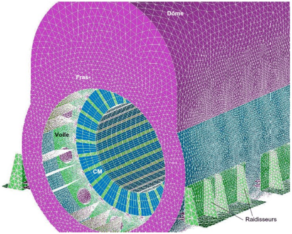

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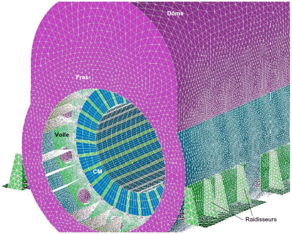

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It is difficult to build a Model

Toward a new model

Prediction of dynamic response levels...

- Complex system : assembly, Multiphysics

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... In a context of severe lack-of-knowledge

- Manufacturing process inducing high variability
- Imprecise tests for model validation in operation
- Absence of plan, materials data, etc.
- Evolutionary behaviour

➜ **Need for a predictive model**

Construction of the hybrid model

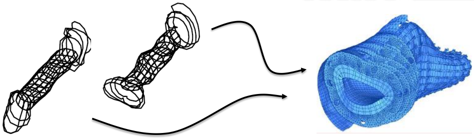
Ingredients

- Hybrid model = numerical model + experimental model
- Finite element numerical model of the structure with the mass matrix $M = M(\theta) \in R^n$ and the stiffness matrix $K = K(\theta) \in R^n$
- Each numerical couple of eigenvalue and eigenvector $(\omega_\theta, \varphi_\theta)$ satisfies :

$$(K(\theta) - \omega_\theta^2 M(\theta))\varphi_\theta = 0, \varphi_\theta \neq 0$$

- Experimental modal basis is available $(\omega_{exp}, \phi_{exp})$

➜ Expansion of the experimental modes on the numerical model in order to compute the response



Constrained optimization problem

- φ is the best estimation of φ_θ, minimizing the distance with the ϕ_{exp} at the pulsation ω_{exp}.
- ψ is an error in stiffness in the model. It satisfies :

$$K(\theta)\psi = (K(\theta) - \omega_{exp}^2 M(\theta))\varphi,$$

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- Quadratic problem ➜ Model error + numerical/experimental distance

$$e_\omega(\varphi, \psi, \theta) = \frac{1}{2}\psi^T K(\theta)\psi + \frac{r}{2(1-r)}(\Pi\varphi - \phi_{exp})^T K_r(\Pi\varphi - \phi_{exp})$$

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- there are kinematic linear constraints which are described as follows :

$$C\varphi = 0, \qquad C\psi = 0$$

where $C \in R^{m \times n}$ represents m linear relations coming from the kinematic boundary conditions.

Constrained optimization problem

- Minimizing the cost function :

$$\begin{cases} f_\omega(\varphi,\psi,\lambda,\lambda_1,\lambda_2,\theta) = e_\omega(\varphi,\psi,\theta) + c_\omega(\varphi,\psi,\lambda,\lambda_1,\lambda_2,\theta) \\ c_\omega(\varphi,\psi,\lambda,\lambda_1,\lambda_2,\theta) = \lambda^T((K(\theta)-\omega_{exp}^2 M(\theta))\varphi - K(\theta)\psi) - \lambda_1^T C\psi + \lambda_2^T(C\psi - C\varphi) \end{cases}$$

Stationarity conditions :

$$\begin{cases} \frac{\partial f_\omega}{\partial \varphi} = 0 \iff \frac{r}{1-r}\Pi^T K_r(\Pi\varphi - \phi_{exp}) + (K(\theta)-\omega_{exp}^2 M(\theta))\lambda - C^T\lambda_2 = 0 \\ \frac{\partial f_\omega}{\partial \psi} = 0 \iff K(\theta)\psi - K(\theta)\lambda + C^T\lambda_2 - C^T\lambda_1 = 0 \\ \frac{\partial f_\omega}{\partial \lambda} = 0 \iff -K(\theta)\psi + (K(\theta)-\omega_{exp}^2 M(\theta))\varphi = 0 \\ \frac{\partial f_\omega}{\partial \lambda_1} = 0 \iff C\psi = 0 \\ \frac{\partial f_\omega}{\partial \lambda_2} = 0 \iff C\psi - C\varphi = 0 \end{cases}$$

Searching a solution in the whole space

- Minimizing the cost function yields the following saddle-point linear system :

$$\begin{bmatrix} -K(\theta) & -C^T & K(\theta)-\omega_{exp}^2 M(\theta) & C^T \\ -C & 0 & C & 0 \\ K(\theta)-\omega_{exp}^2 M(\theta) & C^T & \frac{r}{1-r}\Pi^T K_r \Pi & 0 \\ C & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \lambda_1 \\ \varphi \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{r}{1-r}\Pi^T K_r \phi_{exp} \\ 0 \end{bmatrix}$$

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- If we consider the constrained stiffness and mass matrices :

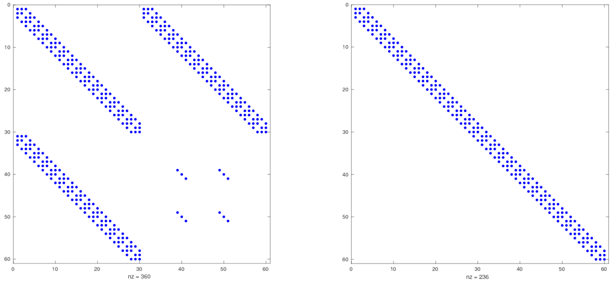
$$\widetilde{K} = \begin{bmatrix} K(\theta) & C^T \\ C & 0 \end{bmatrix}, \qquad \widetilde{M} = \begin{bmatrix} M(\theta) & 0 \\ 0 & 0 \end{bmatrix}$$

Then :

$$\begin{bmatrix} -\widetilde{K}(\theta) & \widetilde{K}(\theta)-\omega_{exp}^2 \widetilde{M}(\theta) \\ \widetilde{K}(\theta)-\omega_{exp}^2 \widetilde{M}(\theta) & \frac{r}{1-r}\widetilde{\Pi}^T \widetilde{K}_r \widetilde{\Pi} \end{bmatrix} \begin{bmatrix} \widetilde{\psi} \\ \widetilde{\varphi} \end{bmatrix} = \begin{bmatrix} \widetilde{0} \\ \frac{r}{1-r}\widetilde{\Pi}^T \widetilde{K}_r \widetilde{\phi}_{exp} \end{bmatrix}$$

Searching a solution in the whole space

- Nonsingular matrix
- But : Large band, bad fill-in ratio, highly indefinite ...

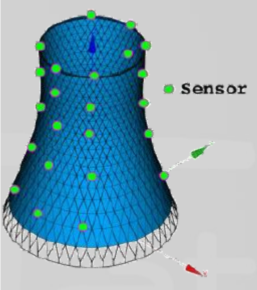


The pattern of the studied saddle point matrix (left) and a finite element matrix in (right)

Searching a solution in the whole space

Mechanical solvers

- for an industrial structure model with more than 10^6 dofs and few hundreds of measurement points (i.e. $N \approx 10^6$ and $n \approx 100$), *MD Nastran*® provides a huge computation cost for a single calculation.



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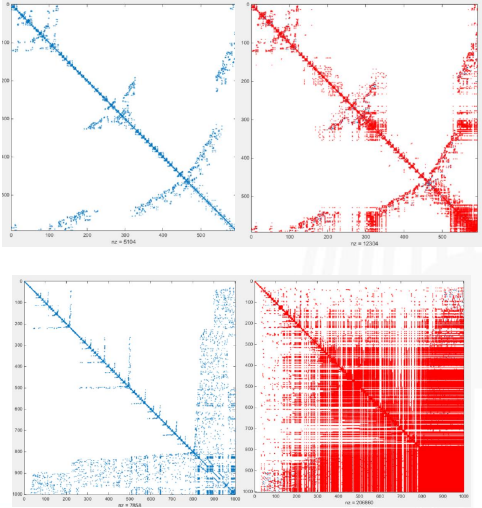

Searching the solution in the kinematic conditions nullspace

The linear system could be described in equivalent form as follows :

$$\mathcal{A} = \begin{bmatrix} -A & -C^T & B^T & C^T \\ -C & 0 & C & 0 \\ B & C^T & T & 0 \\ C & 0 & 0 & 0 \end{bmatrix} \equiv \left[\begin{array}{cc|cc} -A & B^T & -C^T & C^T \\ B & T & C^T & 0 \\ \hline -C & C & 0 & 0 \\ C & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} \widetilde{A} & \widetilde{C}^T \\ \widetilde{C} & 0 \end{bmatrix}$$

where $A = K(\theta) \in \mathbb{R}^{n\times n}$, $B = K(\theta) - \omega_{exp}^2 M(\theta) \in \mathbb{R}^{n\times n}$ and $T = \frac{r}{1-r}\Pi^T K_r \Pi$.

The fundamental nullspace basis of $\widetilde{C}$, is described using :

- $\widetilde{Z} \in \mathbb{R}^{2n\times 2(n-m)}$ such that $range(\widetilde{Z}) = Ker(\widetilde{C})$
- $\widetilde{Y} \in \mathbb{R}^{2n\times 2m}$ such that $Im(\widetilde{Y}) = Im(\widetilde{C}^T)$. We can take $\widetilde{Y} = \widetilde{C}^T$ which is the description of $Im(\widetilde{C}^T)$ in canonical basis.

$$NB = \begin{bmatrix} \widetilde{C}^T & \widetilde{Z} & 0 \\ 0 & 0 & \mathbb{I}_{2m} \end{bmatrix}$$

Searching the solution in the kinematic conditions nullspace

- The equivalent linear system is as follows :

$$\mathcal{A} \equiv \begin{bmatrix} \widetilde{C} & 0 \\ \widetilde{Z}^T & 0 \\ 0 & \mathbb{I}_{2m} \end{bmatrix} \begin{bmatrix} \widetilde{A} & \widetilde{C}^T \\ \widetilde{C} & 0 \end{bmatrix} \begin{bmatrix} \widetilde{C}^T & \widetilde{Z} & 0 \\ 0 & 0 & \mathbb{I}_{2m} \end{bmatrix} = \begin{bmatrix} \widetilde{C}\widetilde{A}\widetilde{C}^T & \widetilde{C}\widetilde{A}\widetilde{Z} & \widetilde{C}\widetilde{C}^T \\ \widetilde{Z}^T\widetilde{A}\widetilde{C}^T & \widetilde{Z}^T\widetilde{A}\widetilde{Z} & 0 \\ \widetilde{C}\widetilde{C}^T & 0 & 0 \end{bmatrix}$$

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- Let $Z \in R^{n\times(n-m)}$ be a matrix such that $range(Z) = Ker(C)$. It is trivial to see that $\widetilde{Z} = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}$ where $Im(Z) = Ker(C)$. We obtain that :

$$\widetilde{Z}^T\widetilde{A}\widetilde{Z} = \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix}^T \begin{bmatrix} -A & B \\ B & T \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} = \begin{bmatrix} -Z^TAZ & Z^TBZ \\ Z^TBZ & Z^TTZ \end{bmatrix}$$

Searching the solution in the kinematic conditions nullspace

Computing a sparse nullspace basis of C

- Using skinny LU technique : Perform LU on the "skinny" matrix C^T

$$PC^TQ = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} U_1$$

where P and Q are permutations, and define the nulspace to be :

$$Z = P^T \begin{bmatrix} -L_1^{-T} L_2^T \\ \mathbb{I} \end{bmatrix}$$

- Implementation of a fast sparse nullspace basis generation algorithm on SuperLU

Searching the solution in the kinematic conditions nullspace

- The reduced linear system to solve is :

$$\begin{bmatrix} -Z^T A Z & Z^T B Z \\ Z^T B Z & Z^T T Z \end{bmatrix} \begin{bmatrix} x_{Z1} \\ x_{Z2} \end{bmatrix} = \begin{bmatrix} -A_Z & B_Z \\ B_Z & T_Z \end{bmatrix} \begin{bmatrix} x_{Z1} \\ x_{Z2} \end{bmatrix} = \begin{bmatrix} 0 \\ f_Z \end{bmatrix}$$

Searching the solution in the kinematic conditions nullspace

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- The coefficient matrix is also a saddle point one :

$$\mathcal{A} = \begin{bmatrix} \mathbb{A} & \mathbb{B}^T \\ \mathbb{B} & \mathbb{D} \end{bmatrix}$$

  - $\mathbb{A}$ is SPD.
  - $\mathbb{B}$ is a symmetric indefinite matrix that shares the same pattern as $\mathbb{A}$.
  - $\mathbb{D}$ is symmetric positive semidefinite, composed of a dense $c \times c$ sub-block scattered into a $n \times n$ matrix, where $c << n$ is the number of sensors.

Iterative solution of the reduced system

The constraint preconditioner

- We use the following constraint preconditioner :

$$\begin{bmatrix} Diag(-A_Z) & B_Z \\ B_Z & T_Z \end{bmatrix}$$

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- The more convenient factorized form is :

$$\begin{bmatrix} Diag(-A_Z) & B_Z \\ B_Z & T_Z \end{bmatrix} = \begin{bmatrix} \mathbb{I} & 0 \\ -B_Z D^{-1} & S_Z \end{bmatrix} \begin{bmatrix} -D & B_Z \\ 0 & \mathbb{I} \end{bmatrix}$$

Where $S_Z = T_Z + B_Z D^{-1} B_Z$ is the shur complement of $Diag(-A_Z)$.

$$P_{full}^{-1} = \begin{bmatrix} -D & B_Z \\ B_Z & T_Z \end{bmatrix}^{-1} = \begin{bmatrix} -D^{-1} & D^{-1}B_Z \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbb{I} & 0 \\ L^{-T}L^{-1}B_Z D^{-1} & L^{-T}L^{-1} \end{bmatrix}$$

Iterative solution of the reduced system

The constraint preconditioner

- We approximate the matrix $B_Z Diag(-A_Z)^{-1} B_Z$ by only calculating its diagonal. Suppose that $B_Z = (bz_{ij}) \in \mathbb{R}^{n-m}$ and $D = (d_{ij}) \in \mathbb{R}^{n-m}$ then :

$$D_{Bz} = \text{Diag}(B_Z D^{-1} B_Z) = \underset{1 \leq j \leq n-m}{\text{Diag}} \left(\sum_{i=1}^{n-m} \frac{bz_{ij}^2}{d_{ii}} \right)$$

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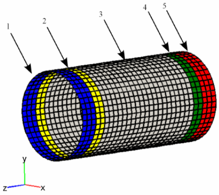
- We consider the approximation $\tilde{S}_Z = T_Z + D_{Bz}$ of the Shur complement. Where $S_Z = T_Z + B_Z D^{-1} B_Z$ is the shur complement of $Diag(-A_Z)$.

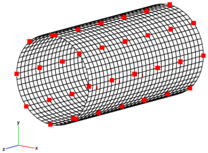
$$P_{diag}^{-1} = \begin{bmatrix} -D^{-1} & D^{-1} B_Z \\ 0 & \mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbb{I} & 0 \\ \tilde{L}^{-T} \tilde{L}^{-1} B_Z D^{-1} & \tilde{L}^{-T} \tilde{L}^{-1} \end{bmatrix}$$

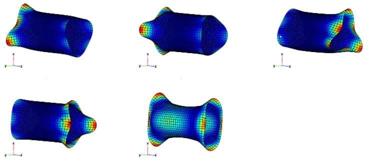
Iterative solution of the reduced system

Numerical results

- We use Petsc (block user implementation) and Matlab to solve the above linear system. We apply GMRES method with this setting : restart = 30, maximum iterations = 200, relative tolerance = 1e-08.
- Academic application : A small prototype of an alternator







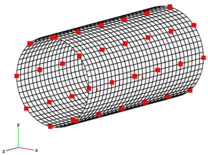
Iterative solution of the reduced system

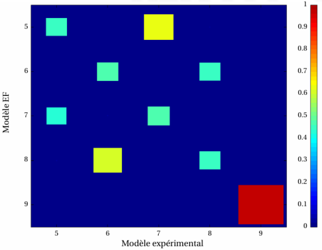
Numerical results

Matrix A	Physical dofs (n)	Lagrange dofs (m)	Global system size	non-zero elements (nnz)	The preconditioner	Petsc (Gmres) # iterations	CPU Time (sec)	Flops
A_1	9,564	873	17,382	1,824,767	P_{full}	10	1.122e+01	2.962e+08
					P_{diag}	21	6.046e-01	3.711e+07
A_2	12,927	1,089	23,676	2,599,697	P_{full}	13	2.277e+01	4.521e+08
					P_{diag}	21	1.349e+00	4.358e+07
A_3	22,101	1,593	41,016	4,872,240	P_{full}	12	6.974e+01	1.065e+09
					P_{diag}	25	1.263e+00	8.184e+07
A_4	146,781	5,553	282,456	38,919,746	P_{full}	10	3.753e+03	2.317e+10
					P_{diag}	22	1.179e+01	9.432e+08
A_5	452,721	11,913	881,616	517,245,560	P_{full}	15	+10h	1.127e+11
					P_{diag}	26	7.778e+01	3.289e+09

Iterative solution of the reduced system

Model updating for the numerical model associated with matrix A_5 ($Physical\ dofs = 452,721$; $Boudary\ conditions = 11,913$)





Conclusions and perspectives

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Some perspectives

- Need to interface different implementations to be used within the mechanical code CodeAster ®
- Application to industrial structures
- Many ideas of preconditioners to be tested