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Dr. Moe
ASTR5420

99%

Please show all work. If you collaborate with other students, write their names at the top of your homework. Please also write the total number of hours you spent on this. If you use any code or plotting routines, print those out and attach them to your hand-written solutions.
Collaborated with Alex
This took me approximately 16 hours to complete. thanks for putting in the hard work

+15/15

1. **(15%) Assuming a grey atmosphere, calculate the average effective temperature of the Sun assuming the effective temperature near the center of the Sun's disk is $T_{eff}(u = 1) = 6380$ K while the effective temperature near the solar limb is $T_{eff}(u = 0) = 5080$ K. Your answer should be within 30 K of the true value of 5778 K (the Sun does not have perfect grey atmosphere).**

$$\frac{I(0,u)}{I(0,1)} = \frac{3}{5}\left[u + \frac{2}{3}\right]$$

$$I(0,u) \propto T(u)^4$$

$$\frac{T(u)^4}{T^4_{u=1}} = \frac{3}{5}\left[u + \frac{2}{3}\right]$$

$$\frac{T(u)}{T_{u=1}} = \left(\frac{3}{5}\left[u + \frac{2}{3}\right]\right)^{0.25}$$

$$T(u) = T_{u=1}\left(\frac{3}{5}\left[u + \frac{2}{3}\right]\right)^{0.25}$$

$$< T_{eff} > = \frac{\int_{u_1}^{u_2} T(u)}{u_2 - u_1}$$

$$< T_{eff} > = T_{u=1}\int_0^1 \left(\frac{3}{5}\left[u + \frac{2}{3}\right]\right)^{0.25} du \left(\frac{1}{1-0}\right)$$

$$< T_{eff} > = T_{u=1}\int_0^1 \left(\frac{3}{5}\left[u + \frac{2}{3}\right]\right)^{0.25} du$$

$$v = \frac{3}{5}\left[u + \frac{2}{3}\right], dv = \frac{3}{5}du$$

$$< T_{eff} > = T_{u=1}\int_0^1 v^{1/4}\frac{5}{3}dv$$

$$< T_{eff} > = \frac{5}{3}T_{u=1}\int_0^1 v^{1/4}dv$$

$$< T_{eff} > = \frac{5}{3}T_{u=1}\left[\frac{4}{5}v^{5/4}\Big|_0^1\right]$$

$$< T_{eff} > = \frac{4}{3}T_{u=1}\left[\left(\frac{3}{5}\left[u + \frac{2}{3}\right]\right)^{5/4}\Big|_0^1\right]$$

$$< T_{eff} > = \frac{4}{3}T_{u=1}\left[1 - \left(\frac{2}{5}\right)^{5/4}\right]$$

$$< T_{eff} > = \frac{4}{3}\left[1 - \left(\frac{2}{5}\right)^{5/4}\right](6380)\text{ K}$$

Code:

(b) **Calculate the most probable speed V_0 as defined in Equation 4.53.**

$$
\begin{aligned}
\left.\frac{\partial f}{\partial V}\right|_{V=V_0} &= 0 \\
\frac{\partial f}{\partial V} &= 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left[-V^2\left(\frac{mV}{kT}\right)\exp\left(\frac{-mV^2}{2kT}\right)+2V\exp\left(-\frac{mV^2}{2kT}\right)\right] \\
0 &= 4\pi\left(\frac{m}{2\pi kT}\right)^{3/2}\left[-V_0^2\left(\frac{mV_0}{kT}\right)\exp\left(\frac{-mV_0^2}{2kT}\right)+2V_0\exp\left(-\frac{mV_0^2}{2kT}\right)\right] \\
0 &= -V_0^2\left(\frac{mV_0}{kT}\right)\exp\left(\frac{-mV_0^2}{2kT}\right)+2V_0\exp\left(-\frac{mV_0^2}{2kT}\right) \\
0 &= V_0\exp\left(-\frac{mV_0^2}{2kT}\right)\left[2-V_0\left(\frac{mV_0}{kT}\right)\right] \\
0 &= 2-V_0^2\frac{m}{kT} \\
V_0^2\frac{m}{kT} &= 2 \\
V_0^2 &= \frac{2kT}{m}
\end{aligned}
$$

$$\boxed{V_0 = \sqrt{\frac{2kT}{m}}}$$

Code used for rest of problem:

In [119]:
```
fescHEarth = np.trapz(f_VHEarth[iiE],\
    V[iiE])/np.trapz(f_VHEarth, V)
# NP Calculating fraction of escaping H_2
print('Fraction of escaping H_2: '\
    +format(fescHEarth, '.2E'))
# NP Printing result
fescOEarth = np.trapz(f_VOEarth[iiE],\
    V[iiE])/np.trapz(f_VOEarth, V)
# NP Calculating fraction of escaping O_2
print('Fraction of escaping O_2: '\
    +format(fescOEarth, '.2E'))
# NP Printing result
```

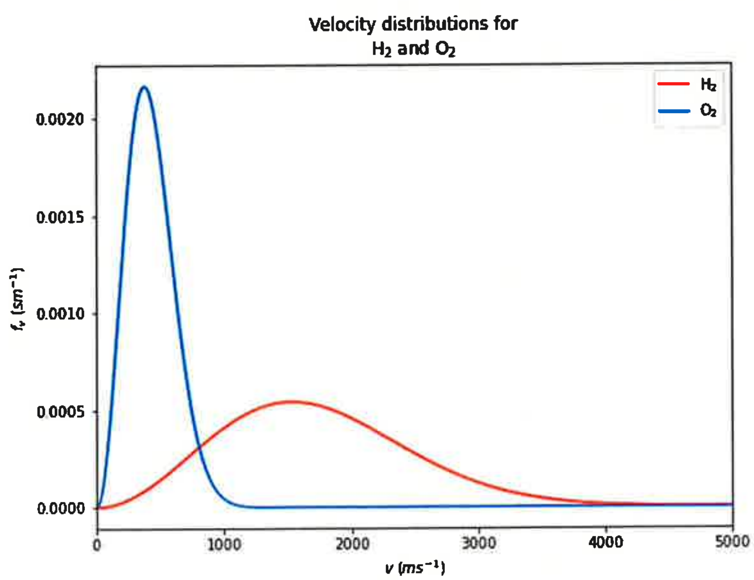
```
Fraction of escaping H_2: 6.74E-23
Fraction of escaping O_2: 0.00E+00
```

(d) **What fraction of molecular oxygen O_2 exceeds Earth's escape velocity?**

$$f_{esc} = \frac{\int_{v_{esc}}^{\infty} f dV}{\int_0^{\infty} f dV}$$

$$\boxed{f_{esc} = 0}$$

Plot:



(e) **Given Jupiter's atmosphere has mean T = 120 K, what fraction of molecular hydrogen H_2 exceeds Jupiter's escape velocity $v_{esc} = \sqrt{2GM_J/R_J}$.**

$$f_{esc} = \frac{\int_{v_{esc}}^{\infty} f dV}{\int_0^{\infty} f dV}$$

$$\boxed{f_{esc} = 0}$$

Code:

10/10

3. **(10%) The lifetime of an electron in the $n = 2$ excited state of hydrogen is 10^{-8} s before it spontaneously decays to the $n = 1$ ground state. Compute the natural line width of Lyman α in units of $\Delta\lambda$ (Å), $\Delta\lambda/\lambda = v/c$, and v (cm s^{-1}). What resolution $R = \lambda/\Delta\lambda$ spectrograph would you need to resolve natural line broadening?**

Calculate Full-width at half-maximum.

$$\phi_\nu = \frac{\frac{\Gamma}{4\pi}}{(\nu-\nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

$$\phi_{\nu,max} = \frac{4\pi}{\Gamma}$$

$$\frac{1}{2}\left(\frac{4\pi}{\Gamma}\right) = \frac{\frac{\Gamma}{4\pi}}{(\nu-\nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

$$\frac{2\pi}{\Gamma} = \frac{\frac{\Gamma}{4\pi}}{(\nu-\nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

$$1 = \frac{\frac{\Gamma^2}{8\pi^2}}{(\nu-\nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2}$$

$$(\nu-\nu_0)^2 + \left(\frac{\Gamma}{4\pi}\right)^2 = 2\left(\frac{\Gamma}{4\pi}\right)^2$$

$$(\nu-\nu_0)^2 = \left(\frac{\Gamma}{4\pi}\right)^2$$

$$\nu_2 - \nu_0 = \frac{\Gamma}{4\pi}$$

$$\nu_2 = \frac{\Gamma}{4\pi} + \nu_0$$

$$\nu_1 - \nu_0 = -\frac{\Gamma}{4\pi}$$

$$\nu_1 = -\frac{\Gamma}{4\pi} + \nu_0$$

$$\Delta\nu = \nu_2 - \nu_1$$

$$\Delta\nu = \left(\frac{\Gamma}{4\pi} + \nu_0\right) - \left(-\frac{\Gamma}{4\pi} + \nu_0\right)$$

$$\Delta\nu = \frac{\Gamma}{2\pi}$$

Convert this width into wavelengths.

$$\nu = \frac{c}{\lambda}$$

$$\partial\nu = -\frac{c}{\lambda^2}\partial\lambda$$

$$|\Delta\nu| = |-\frac{c}{\lambda^2}\Delta\lambda|$$

$$\Delta\nu = \frac{c}{\lambda^2}\Delta\lambda$$

$$\Delta\lambda = \frac{\lambda^2}{c}\Delta\nu$$

$$\Delta\lambda = \frac{\lambda^2}{c}\frac{\Gamma}{2\pi}$$

$$\Delta\lambda = \frac{(1216 \times 10^{-10})^2}{3 \times 10^8}\frac{\frac{1}{10^{-8}}}{2\pi} \times 10^{10} \text{ Å}$$

$$\boxed{\Delta\lambda = 7.84 \times 10^{-6} \text{ Å}}$$

$$\boxed{R = 1.55 \times 10^8}$$

Code:

```
In [32]: R = lambda1 /dlambda
         # NP Calculating required resoltion to see broadening
         print('R: ' +format(R, '.2E'))
         # NP Printing result

R: 1.55E+08
```

+20/20

4. **(20%) Curve of Growth.**

   (a) **For small $\tau(\nu_0) < 1$ at line center, show that the equivalent width $W \propto N$. Hint: Doppler core dominates W; Taylor expand $e^{-\tau(\nu)}$ where $\tau(\nu)$ is a Gaussian profile.**

$$e^{-\tau(\nu)} \approx 1 - \tau(\nu)$$

$$1 - \tau(\nu) = 1 - N\sigma \exp\left(\frac{-(\nu-\nu_0)^2}{2\sigma^2}\right)$$

$$W = \int_0^\infty 1 - \left(1 - \exp\left(-\frac{(\nu-\nu_0)^2}{2\sigma^2}\right)\right) d\nu$$

$$W = N\sigma \int_0^\infty \exp\left(-\frac{(\nu-\nu_0)^2}{2\sigma}\right) d\nu$$

$$\boxed{W \propto N}$$

   (b) **For large $\tau(\nu_0) > 10$ at line center, show that the equivalent width $W \propto N^{1/2}$. Hint: Lorentzian wings dominate W; integrate $\left[1 - e^{-\tau(\nu)}\right] d\nu$ where $\tau(\nu)$ is a Lorentzian profile in the limit $|\nu - \nu_0| >> \Gamma$.**

$$W = \int_0^\infty 1 - \exp(-\tau(\nu))\, d\nu$$

$$W = \int_0^\infty 1 - \exp\left(-N\sigma\frac{\gamma^2}{(\nu-\nu_0)^2 + \gamma^2}\right) d\nu$$

$$W \approx \int_0^\infty 1 - \exp\left(-N\sigma\frac{\gamma^2}{(\nu-\nu_0)^2}\right) d\nu$$

?

$$W = \int_0^\infty 1 - \exp\left(N\sigma\,\nu^2\right) d\nu$$

$$W \propto \sqrt{N\sigma}$$

+34/35

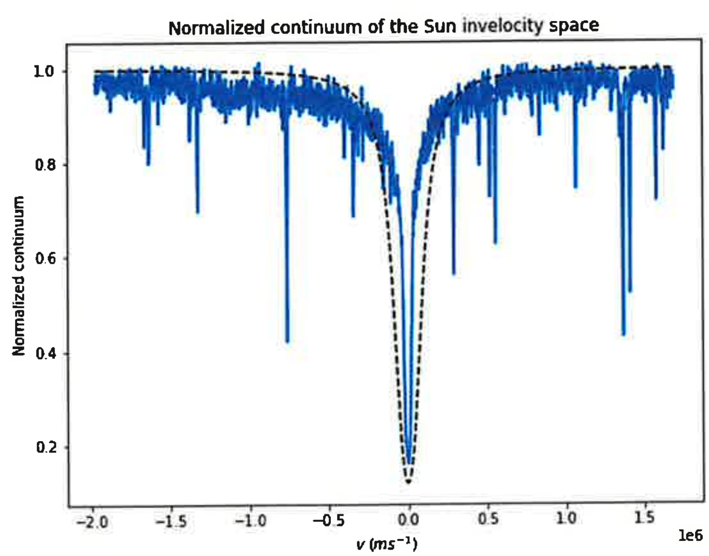
$$\boxed{W \propto \sqrt{N}}$$

5. **(35%) Download the normalized spectra of the Sun ($G2V$), epsilon Virginis ($G8III$), and Vega ($A0V$) across wavelengths $\lambda = 652 - 660$ nm from WyoCourses.**

   (a) **Plot the normalized spectrum of the Sun. Label at least 4 absorption lines in addition to $H\alpha$. To do this, go to the NIST Atomic Spectral Line Database (https://physics.nist.gov/PhysRefData/ASD/lines_form.html) and search for atomic transitions across the covered wavelength range. Consider only strong (allowed) transitions from neutral and singly ionized atoms with better than a B accuracy.**
   Read in spectra.

```
In [74]: f = plt.figure(figsize = [8, 6])
         # NP Making figure large
         plt.plot(ws[1], fs[1], 'r')
         # NP Plotting Sun values
         plt.axvline(x = 656.3, ymin = 0.00, ymax = 0.9,\
             color = 'k', lw = 1)
         plt.axvline(x = 659.2609, ymin = 0.00, ymax = 0.9,
             color = 'k', lw = 1)
         plt.axvline(x = 659.38701, ymin = 0.00, ymax = 0.9,
             color = 'k', lw = 1)
         plt.axvline(x = 657.4228, ymin = 0.00, ymax = 0.9,\
             color = 'k', lw = 1)
         plt.axvline(x = 654.5973, ymin = 0.00, ymax = 0.9,\
             color = 'k', lw = 1)
         # NP Plotting lines for different absorption features
         plt.text(656.3 - 0.2, 1.15, r'H$\alpha$')
         plt.text(659.2609 - 0.3, 1.15, r'Fe I')
         plt.text(659.38701 - 0.0, 1.15, r'Fe I')
         plt.text(657.4228 - 0.2, 1.15, r'Fe I')
         plt.text(654.5973 - 0.2, 1.15, r'Mg II')
         # NP Labeling features
         plt.xlabel(r'$\lambda$ (nm)')
         # NP Labeling x-axis
         plt.ylabel('Normalized flux')
         # NP Labelin y-axis
         plt.title('Normalized flux continuum of the Sun')
         # NP Labeling figure
         plt.ylim(0.1, 1.2)
         # NP Changing y-bounds to see labels
         plt.savefig('/d/www/nikhil/public_html/ASTR5420/images'
             '/Sunspec.png')
         # NP Saving figure
```

(b) **Plot the spectrum of the sun in velocity space $v = c\Delta\lambda/\lambda$ relative to $H\alpha$ (656.28 nm). By eye, fit a velocity profile $e^{-\tau(v)}$ to the normalized flux where $\tau(v; v_0, \sigma, \gamma)$ is a Voigt function with line center v_0, Gaussian standard deviation σ, and Lorentzian damping factor γ. Overplot your best-fit Voigt function and report the three best-fit parameters.** Plot:



Code:

(e) **Suppose your spectrograph has a poor resolution with a Gaussian line-split function of $\sigma_{res} =$ 100 km/s. What is the resolution R of the spectrograph? Convolve the spectrum of the Sun with this line-split function. Plot the convolved spectrum on top of the original spectrum (both in velocity space but in different colors). Describe the differences in the $H\alpha$ profile, i.e., does σ of the Voigt profile change?**

Calculate convolution in pixel space:

$$\sigma_{pix} = \sigma_v p \tag{1}$$

Where p is pixel scale.

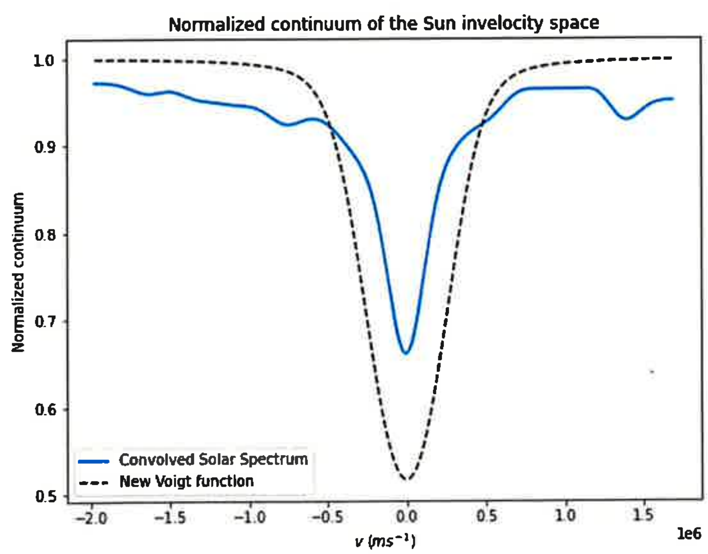
$$\boxed{\sigma_{pix} = 219 \text{ pix}}$$ ✓

Code:

```
In [493]: print('Pixel scale: ' +format(len(v_space[1])\
              /(max(v_space[1]) -min(v_space[1])), '.2E')\
              +' pixels / m/s')
          # NP Calculating pixel scale
          print('Convolution for models: ' +format(100000/\
              ((max(v_space[1])-min(v_space[1]))/\
              (len(v_space[1]))), '.2E') +' pix')
          # NP Calculating convolution for models in pixel space

          Pixel scale: 2.19E-03 pixels / m/s
          Convolution for models: 2.19E+02 pix
```

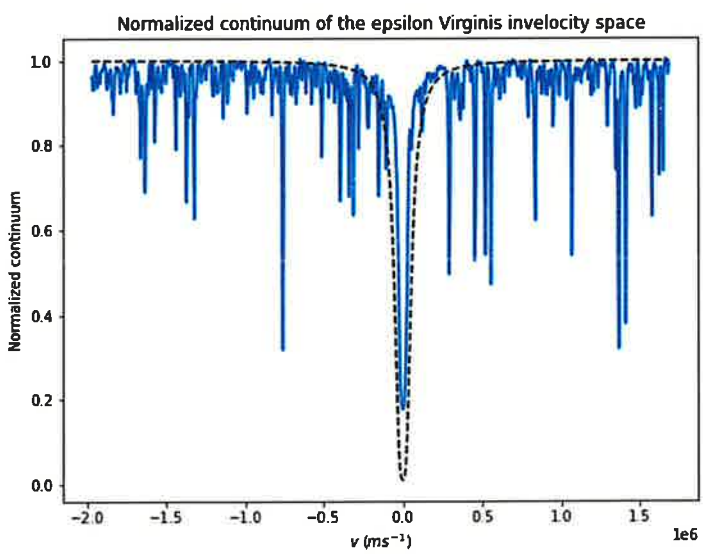
Convolved spectrum:



Plotting code:

```
In [665]: v_space = [[c *((l - 656.28) /(l)) for l in w]\
              for w in ws]
          # NP Converting wavelengths to velocity space
          f = plt.figure(figsize = [8, 6])
          # NP Making figure larger
          plt.plot(v_space[2], fs[2], 'r')
          # NP Plotting epsilon virginis continuum in v space
          plt.plot(v_space[2], np.exp(-350000 \
              *(vp(v_space[2], 10000, 20000))), '--k')
          # NP Plotting best-fit Voigt
          plt.xlabel(r'$v$ ($ms^{-1}$)')
          # NP Labeling x-axis
          plt.ylabel('Normalized continuum')
          # NP Labeling y-axis
          plt.title('Normalized continuum of the epsilon Virginis in'
              'velocity space')
          # NP Labeling Plot
          plt.savefig('/d/www/nikhil/public_html/ASTR5420'
              '/images/epsvir.png')
          # NP Saving figure
```

Plot:



$$v_0 \approx v_0$$
$$\sigma_{vir} \approx \sigma_{\odot}$$
$$\gamma_{vir} \approx 0.4\gamma_{\odot}$$

Not surprisingly, v_0 is the same is it is for the sun. This is because v_0 corresponds to the wavelength of $H\alpha$, which is the same for every star. We can also see that σ is roughly the same as it is for the Sun. This would make sense if epsilon Virginis is roughly the same temperature as the sun. The only difference between the Sun's spectrum and epsilon Virginis' spectrum is the γ parameter. I found that this value was roughly half that of the Sun. This corresponds to pressure broadening. Since this value was lower than that of the Sun, I would expect epsilon Virginis to have a surface gravity less than the Sun's surface gravity.

(h) **Repeat part (b) for Vega. Report and discuss the differences in the parameters v_0, σ, and γ compared to the solar values. Can you derive the effective temperature of Vega from σ? If**

$$\boxed{T \approx 27200 \text{ K}}$$

Code:

```
In [671]: Tvega = (15000) **2 *(mH2 /2) /(1.38e-23)
          # NP Calculating temperature
          print('Temperature: ' +format(Tvega, '.2E') +' K')
          # NP Printing result

          Temperature: 2.72E+04 K
```