94%

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ASTR5420

Please show all work. If you collaborate with other students, write their names at the top of your homework. Please also write the total number of hours you spent on this. If you use any code or plotting routines, print those out and attach them to your hand-written solutions.
I collaborated with Alex on this homework.
This took me approximately 8 hours to complete.

1. **(20%) Luminosity Relations**

20/20

(a) **The first-order homologous relation for the mass conservation equation $dM(r)/dr = 4\pi r^2\rho(r)$ becomes $M/R \propto R^2\rho$ or $\rho \propto M/R^3$. Write first-order homologous relations for hydrostatic equilibrium (P in terms of M and R) and radiative energy transport (T in terms of L, M, and R; you can assume opacity k_R is a constant due to Thompson electron scattering).**

$$\frac{dP}{dR} = -\rho g$$
$$\frac{dP}{dR} \propto \frac{M}{R^3}\frac{M}{R^2}$$
$$dP \propto \frac{M^2}{R^5}dR$$

$$\boxed{P \propto \frac{M^2}{R^4}}$$

$$\frac{dT}{dR} = -\frac{3k_R\rho}{64\pi R^2\sigma T^3}L$$
$$\frac{dT}{dR} \propto \frac{M}{R^3}\frac{1}{R^2T^3}L$$
$$T^3 dT \propto ML\frac{dR}{R^5}$$
$$T^4 \propto \frac{ML}{R^4}$$

$$\boxed{T \propto (ML)^{1/4}\frac{1}{R}}$$

(b) **Solar-type main-sequence stars are pressure supported by nearly an ideal gas, i.e., $P \propto \rho T$. Now solve for the mass-luminosity $L \propto M^x$ relation. Recall that $x = 3.5$ is the actual exponent for solar-type MS stars. Is your approximation close to the true value?**

$$\frac{M^2}{R^4} \propto (ML)^{1/4}\frac{1}{R}\frac{M}{R^3}$$
$$M^2 \propto L^{1/4}M^{1.25}$$
$$M^{0.75} \propto L^{1/4}$$

$$\boxed{L \propto M^3}$$

This approximation is reasonable close to the actual luminosity-mass relations for MS stars. We found a value of 3 for the exponent using the first order homologous approximations when the actual value is 3.5.

(c) Massive, luminous MS stars are supported by radiation pressure such that $P = P_{rad} \propto T^4$. Now solve for the mass-luminosity $L \propto M^x$ relation. Recall that the luminosities of massive stars are regulated by their Eddington limit. Does your solution for x make sense?

$$\frac{M^2}{R^4} \propto \frac{ML}{R^4}$$
$$M^2 \propto ML$$
$$\boxed{L \propto M}$$

The Eddington Luminosity is defined as:

$$L_{edd} = \frac{4\pi G M m_p c}{\sigma}$$

This answer is completely consistent with the Luminosity-Mass relation in the Eddington Luminosity.

2. **(15%) Determine how the temperature gradient ∇_{rad} in the upper layers of MS stars varies with stellar mass (i.e., determine a in $\nabla_{rad} \propto M^a$). Assume Kramer's opacity law ($k_R \propto \rho T^{-7/2}$), ideal gas ($P \propto \rho T$), standard MS relations ($L \propto M^{3.5}; R \propto M^{0.8}$), and density and pressure relations from #1a. How much does ∇_{rad} decrease between the Sun and the Kraft break? This question should demonstrate why the envelopes of MS stars slightly more massive than the Sun become fully radiative.**

+15/15

$$\nabla_{rad} = \frac{3k_R P}{64\pi R^2 g\sigma T^4} L$$
$$\nabla_{rad} \propto \frac{k_R P}{R^2 g T^4} L$$
$$\nabla_{rad} \propto \frac{\rho}{T^{3.5}} \frac{\rho T}{R^2 g T^4} L$$
$$\nabla_{rad} \propto \left(\frac{M}{R^3}\right)^2 \frac{1}{R^2 g T^{6.5}} L$$
$$\nabla_{rad} \propto \frac{M^2}{R^8 T^{6.5}} \frac{R^2}{M} L$$
$$\nabla_{rad} \propto \frac{M}{R^6} \left(\frac{R}{(ML)^{0.25}}\right)^{6.5} L$$
$$\nabla_{rad} \propto \frac{M R^{0.5}}{M^{1.625} L^{1.625}} L$$
$$\nabla_{rad} \propto \frac{R^{0.5}}{M^{0.625} L^{0.625}}$$
$$\nabla_{rad} \propto \frac{(M^{0.8})^{0.5}}{M^{0.625} (M^{3.5})^{0.625}}$$
$$\nabla_{rad} \propto M^{0.4} M^{-0.625} M^{-2.1875}$$
$$\boxed{\nabla_{rad} \propto M^{-2.4125}}$$

From the above relation, the temperature gradient would decrease by about 2.25 times going from the Sun's temperature gradient to a star at the Kraft break (1.4 solar masses). In other words, the upper layers of a star at the Kraft break would have ∇_{rad} of approximately 0.444 times the ∇_{rad} of the Sun.

3. **(15%) Jovian planets ($M = 0.3$–$13 M_J$), brown dwarfs (13–$80 M_J$), and fully convective late-M dwarfs ($80 - 300 M_J$) have an overall equation of state of $P = K\rho^\gamma$ with adiabatic index $\gamma = 2$ (polytropic**

+12/15

index $n = 1$). **More massive stars develop radiative cores while the rocky cores of less massive planets strongly affect their equation of state. For $n = 1$ polytropes, solve for the radius R in terms of K, γ, and other constants, independent of mass M. This question should demonstrate why Jovian planets, brown dwarfs, and late-M dwarfs all have roughly the same radius of $R = 0.1R_\odot = 1R_J$ despite spanning 3 orders of magnitude in mass $M = 0.3 - 300M_J$.**

$$P \propto K\rho^\gamma \tag{1}$$
$$\frac{M^2}{R^4} \propto K\left(\frac{M}{R^3}\right)^2 \tag{2}$$
$$\frac{M^2}{R^4} \propto K\frac{M^2}{R^6} \tag{3}$$
$$R^2 \propto K \tag{4}$$
$$\boxed{R \propto K^{0.5}}$$

Wanted you to keep constants through.
$R = \sqrt{\frac{K\pi}{2G}}$ ⟹ but right idea

As shown above, radius is independent of mass for fully-convective cores (Jovian planets, brown dwarfs, late-M dwarfs).

4. **(20%) White Dwarfs (WDs)**

+20/20 (a) **Low-mass WDs are supported by degeneracy pressure of non-relativistic electrons, which have an equation of state of $P \propto \rho^\gamma$ with $\gamma = 5/3$(polytropic index $n = 1.5$). Show that such WDs shrink with increasing mass according to a $R \propto M^{-1/3}$ mass-radius relation.**

$$P \propto \rho^{1.5}$$
$$\frac{M^2}{R^4} \propto \left(\frac{M}{R^3}\right)^{3/2}$$
$$\frac{M^2}{R^4} \propto \frac{M^{5/3}}{R^5}$$
$$R \propto \frac{M^{5/3}}{M^2}$$
$$\boxed{R \propto M^{-1/3}}$$

(b) **The centers of massive WDs are supported by degeneracy pressure of relativistic electrons, which have an EOS of $P = K\rho^\gamma$ with $\gamma = 4/3$ ($n = 3$) and $K = (3/\pi)^{1/3}hc/\left[8(\mu_e m_H)^{4/3}\right]$. Show that the radius dependence of a $n = 3$ polytrope disappears, and solve for the mass (in $M_\odot$) of a fully relativistic WD composed of fully ionized C, O, and Ne. What is the significance of this result?**

$$M_* = -4\pi\alpha^3\left[\frac{(n+1)K}{4\pi G\alpha^2}\right]^{n/n-1}\xi_0^2\left(\frac{d\theta}{d\xi}\right)_{\xi_0}$$
$$\alpha = R_*/\xi_0$$
$$\xi_0^2\left(\frac{d\theta}{d\xi}\right)_{\xi_0} = -2.02$$
$$M_* = -4\pi\frac{R_*^3}{\xi_0^3}\frac{(3+1)^{3/2}K^{3/2}}{(4\pi G)^{3/2}\left(\frac{R_*}{\xi_0}\right)^3}(-2.02)$$
$$M_* = (2.02)\frac{4\pi(8)}{(4\pi)^{3/2}G^{3/2}}\left[\left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{8(\mu_e m_H)^{4/3}}\right]^{3/2}$$
$$M_* = \frac{(2.02)\,8}{\sqrt{4\pi}G^{3/2}}\left(\frac{3}{\pi}\right)^{1/2}\frac{(hc)^{3/2}}{8^{3/2}(\mu_e m_H)^2}$$

$$\boxed{M_* = 1.44 M_\odot}$$

Code:

```
In [10]: R_o = 6.957e8
         # NP Radius of SUn
         G = 6.67e-11
         # NP Gravatational constant
         L_o = 3.898e26
         # NP Luminosity of Sun
         M_o = 2e30
         # NP Mass of Sun
         m_p = 1.67e-27
         # NP Mass of proton
         h = 6.63e-34
         # NP Planck's constant
         c = 3e8
         # NP Speed of light
         # NP Useful constants in SI (mks) units

In [16]: M = ((2.02 *8) /(np.sqrt(4 *np.pi) *G **(3/2)))\
             *np.sqrt(3 /np.pi) *((h *c) **(3/2) /(8\
             **(3/2) *(2 *m_p) **2))
         # NP Calculating mass of relativistic white dwarf
         print('Fully relativistic mass: ' +format(M /M_o,\
             '.2E') +' Solar Masses')
         # NP Printing result

         Fully relativistic mass: 1.44E+00 Solar Masses
```

This is the Chandrasekhar limit! Beyond this mass, the neutron degeneracy pressure is unable to counteract the gravitational force and the white dwarf collapses into a blackhole.

+27/30

5. (30%) Download the interior model of the Sun (identical to Table 5.1 in the textbook) from WyoCourses. You will also need the table of Rosseland mean opacities k_R for solar abundances in terms of $\log T$ and $\log R = \log \rho$–$3logT_6$ given in HW #4.

(a) **Compute ∇_{rad} for each of the 16 radial data points in the solar model and plot $\nabla_{rad}(r)$.**
Reading in opacity table.
Code:

```
In [148]: opa = pd.read_csv('/d/users/nikhil/Downloads/OPAL_SolarComposition.txt')
          # NP Reading in raw text file
          goodopa = opa.to_numpy()
          # NP Converting to numpy array
          betteropa = [np.fromstring(goodopa[i][0], dtype = float, sep = ' ').tolist()\
              for i in range(len(goodopa))]
          # NP Creating a table for opacities and temps., skipping first row
          bestopa = [betteropa[i+1][1:] for i in range(len(betteropa) -1)]
          # NP Creating a list of only opacities
          row_lengths = []
          for row in bestopa:
              row_lengths.append(len(row))
          max_length = max(row_lengths)
          for row in bestopa:
              while len(row) < max_length:
                  row.append(0)
          pristineopa = np.array(bestopa)
          # NP Creating numpy array of opacities to better parse through array
          logTs = np.array([betteropa[i+1][0] for i in range(len(betteropa)-1)])
          # NP Creating array of temperatures
          logRs = np.array(np.fromstring(goodopa[0][0][6:], dtype = float, sep = ' '))
          # NP Creating array of R's
          Tgrid = np.array([[t for r in range(len(logRs))] for t in logTs])
          # NP Creating 2D numpy array of temperatures
          Rgrid = np.array([[r for r in logRs] for t in range(len(logTs))])
          # NP Creating 2D numpy array of temperatures
          rhos = np.array([np.array([np.round(r +3 *(t -6) ,3) for r in logRs])\
              for t in logTs])
          # NP Creating density grip for each opacity

/tmp/ipykernel_3361101/2143050322.py:5: DeprecationWarning: string or file coulc
ched data; this will raise a ValueError in the future.
  betteropa = [np.fromstring(goodopa[i][0], dtype = float, sep = ' ').tolist()\
```

Interpolating over opacity table.
Code:

```
In [149]: fillinopa = interp2d(logRs, logTs, pristineopa, kind = 'cubic')
          # NP Interpolating over Rs and Ts to fill in all values for opacity
```

Reading in Solar Model.
Code:

```
In [150]: r, m_r, l_r, T_r, rho_r, logP_r = np.loadtxt('/d/users/nikhil/'
              'Downloads/SolarModel.csv', unpack = True, skiprows = 1,\
              usecols = (0, 1, 2, 3, 4, 5), delimiter = ',')
          # NP Reading in table quantities
```

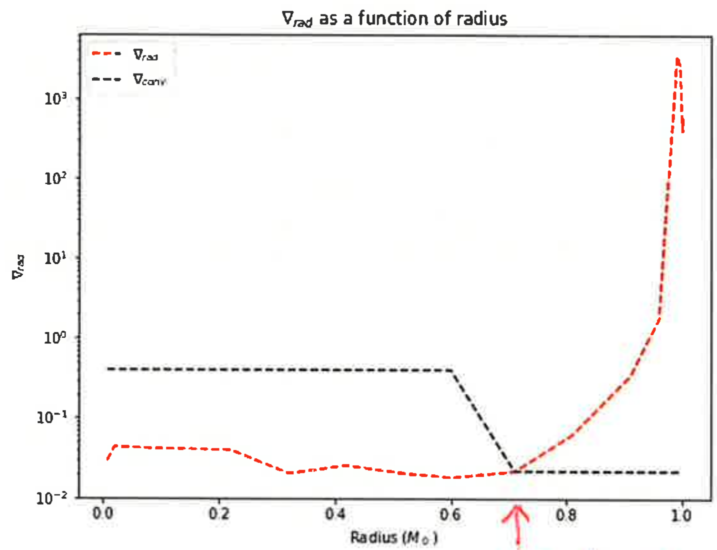
Defining temperature gradient. Note, the convective temperature gradient changes in different layers of the Sun corresponding to the different states of matter in different layers.
Code:

```
In [197]: n_rad = np.array([(3e-2 *fillinopa(usefulRs[i], usefulTs[i])\
              *10 **(logP_r[i]) *l_r[i] *L_o) /(64 *np.pi\
              *G *m_r[i] *M_o *o *(10 **(usefulTs[i])) **4)\
              for i in range(len(r))])
          # NP Calculating temperature gradient for the Sun for all layers
          n_conv = [0.4 for ra in r if ra < .7]+ [0.0216 for ra in r\
              if ra >= .7]
          # NP Calculating convective temperature gradient for the whole
          # NP Sun
```

Plotting ∇_{rad} as a function of radius.
Code:

```
In [155]: plt.figure(figsize = [8, 6])
          plt.plot(r, n_rad, '--r', label = r'$\nabla_{rad}$')
          plt.plot(r, n_conv, '--k', label = r'$\nabla_{conv}$')
          plt.yscale('log')
          plt.xlabel(r'Radius $(M_\odot)$')
          plt.ylabel(r'$\nabla_{rad}$')
          plt.title(r'$\nabla_{rad}$ as a function of radius')
          plt.legend()
          plt.savefig('/d/www/nikhil/public_html/ASTR5420/images/nablaradradius.png')
```

Plot:



yes - lower than $\nabla_{adi} = 0.4$ from $\gamma = 5/3$, but a little too low

(b) **What are the values of ∇_{rad} in the layers $r = 0.7$–$1.0R_\odot$ where the Sun is convective? Do they satisfy the Schwarzschild criterion for convection assuming a neutral, ideal monoatomic gas ($\gamma = 5/3$)? Explain why or why not.**

interpolation issue

Code:

```
In [207]: print('Minimum radiative temperature gradient: '\
              +format(min(n_rad[r > .7])[0], '.2E'))
          # NP Printing minimum temperature gradient after 0.7 R_o
          print('gamma: ' +format(1 /(1 -min(n_rad[r > 0.7])[0]), '.2E'))

          Minimum radiative temperature gradient: 2.16E-02
          gamma: 1.02E+00
```

→ should be closer to $\nabla_{adi} = 0.2$ at $r = 0.7R_\odot$

As shown, the radiative temperature gradient past $0.7R_\odot$ is as low as 0.0216. This corresponds to $\gamma = 1.02$, roughly consistent to the $\gamma \approx 1.1 - 1.5$ partially ionized polytrope.

but right profile

(c) **Consider the $r = 0.09R_\odot$ layer in the core. Assuming all other parameters remain constant, how much more luminous would this layer have to be to become convective, i.e. reach the minimum value of ∇_{rad} across $r = 0.7$–$1.0R_\odot$ that is already convective? Assuming all stellar layers scale with the overall mass-luminosity relation $L \propto M^{3.5}$, what minimum mass MS star begins to develop a convective core?**

To make the core fully convective ($r = 0.09R_\odot$), the luminosity would have to increase by a factor of 9.68. Assuming luminosity at a given layer scales with total luminosity, we would expect a star with a fully convective core to be 9.68 times more luminous. Using the mass-luminosity relation, we can calculate how many times more massive a star would have to be to achieve a fully convective core.

$$L \propto M^{3.5}$$

$$\frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^{3.5}$$

$$\frac{L}{L_\odot} = 9.68$$

$$\left(\frac{M}{M_\odot}\right)^{3.5} = 9.68$$

$$M = (9.68)^{1/3.5} M_\odot$$

$$\boxed{M = 1.91 M_\odot}$$

Code:

```
In [213]: print('Luminosity difference: ' +format(.4/n_rad\
              [r == 0.09][0][0], '.2E') +' times.')
          # NP Calculating luminosity difference to make core convective
          print('Mass difference: ' +format((.4 /n_rad[r ==\
              0.09][0][0]) **(1 /3.5), '.2E') +' times.')
          # NP Calculating mass difference to make core convective

          Luminosity difference: 9.68E+00 times.
          Mass difference: 1.91E+00 times.
```