1	2	3	4	5	6	7	8	9
80	90	100	80	90	80	75	100	100

Nikhil Patten
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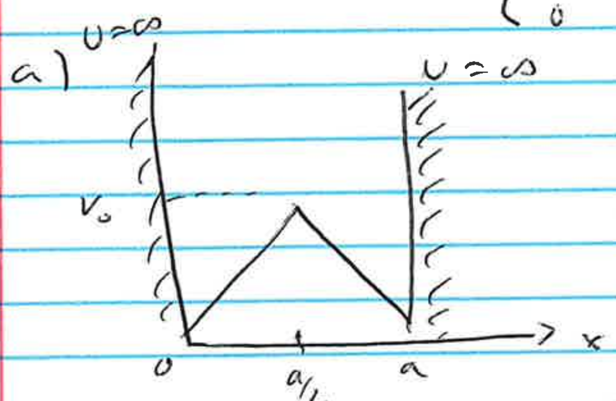
Homework 2
Dr. Dahnovsky

total 84

1. Particle in potential well of width a $(0 < x < a)$, find energy level change in first order of the perturbation theory for

a) $V(x) = \frac{V_0}{2}(a - |2x-a|)$

b) $V(x) = \begin{cases} V_0 & b < x < a-b \\ 0 & 0 < x < b, \ a-b < x \end{cases}$

a) 

$$E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\psi_n^0 = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

a) $E_n^1 = \langle \psi_n^0 | H^1 | \psi_n^0 \rangle$

$$E_n^1 = \int_{-\infty}^{\infty} \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\frac{V_0}{2}(a-|2x-a|)\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)dx$$

$$E_n^1 = \left(\frac{2}{a}\right)\left(\frac{V_0}{2}\right)\int_{-\infty}^{\infty}(a-|2x-a|)\sin^2\left(\frac{n\pi}{a}x\right)dx$$

consider $x > 0$ and $x < a/2$

$$E_n^1 = \frac{V_0}{a}\,2\int_{0}^{a/2}(a-|2x-a|)\sin^2\left(\frac{n\pi x}{a}\right)dx$$

$2x - a < 0$ for interval

$$E_n^1 = \frac{2V_0}{a}\int_{0}^{a/2}(a+2x-a)\sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$E_n^1 = \frac{2V_0}{a}\int_{0}^{a/2} 2x\sin^2\left(\frac{n\pi x}{a}\right)dx$$

$$E_n^1 = \frac{4V_0}{a}\int_0^{a/2} x \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$\sin^2\left(\frac{n\pi x}{a}\right) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{a}\right)$$

$$E_n^1 = \frac{4V_0}{a}\int_0^{a/2} x\left(\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{a}\right)\right) dx$$

$$E_n^1 = \frac{2V_0}{a}\int_0^{a/2}\left[x - x\cos\left(\frac{2n\pi x}{a}\right)\right] dx$$

$$\int x\,dx = \frac{1}{2}x^2$$

$$\int x\cos\left(\frac{2n\pi x}{a}\right) dx$$

$$u = x \qquad dv = \cos\left(\frac{2n\pi x}{a}\right) dx$$

$$du = dx$$

$$v = \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)$$

$$= \frac{a}{2n\pi} x \sin\left(\frac{2n\pi x}{a}\right) - \int \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right) dx$$

$$= \frac{a}{2n\pi} x \sin\left(\frac{2n\pi x}{a}\right) + \frac{a^2}{4n^2\pi^2}\cos\left(\frac{2n\pi x}{a}\right)$$

$$E_n^1 = \frac{2V_0}{a}\left[\frac{1}{2}x^2 + \frac{a}{2n\pi} x\sin\left(\frac{2n\pi x}{a}\right) + \frac{a^2}{4n^2\pi^2}\cos\left(\frac{2n\pi x}{a}\right)\Big|_0^{a/2}\right]$$

$$E_n^1 = \frac{2V_0}{a}\left[\frac{1}{2}\left(\frac{a^2}{4}\right) + \frac{a^2}{4n\pi}\left(\frac{a}{2}\right)\sin(n\pi) + \frac{a^2}{4n^2\pi^2}\cos(n\pi) - 0 - 0 - \frac{a^2}{4n^2\pi^2}\cos(0)\right]$$

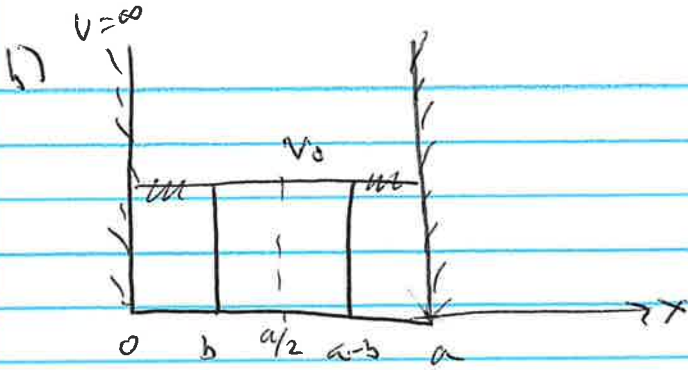
$$E_n^1 = \frac{2V_0}{a}\left[\frac{a^2}{8} + \cancel{\frac{a^3}{8n\pi}\sin(n\pi)}^{\,0} + \frac{a^2}{4n^2\pi^2}\cos(n\pi) - \frac{a^2}{4n^2\pi^2}\right]$$

$$E_n^1 = \frac{2V_0}{a}\left[\frac{a^2}{8} + \frac{a^2}{4n^2\pi^2}(-1)^n - \frac{a^2}{4n^2\pi^2}\right]$$

$$E_n^1 = V_0\left[\frac{a}{4} + \frac{a}{2n^2\pi^2}(-1)^n - \frac{a}{2n^2\pi^2}\right]$$

$$\boxed{E_n^1 = \begin{cases} \frac{V_0 a}{4} & \text{for } n \text{ even} \\ V_0\left[\frac{a}{4} - \frac{a}{n^2\pi^2}\right] & \text{for odd } n\end{cases}}$$

b)



$$V(x) = \begin{cases} V_0 & b < x < a-b \\ 0 & 0<x<b, \; a-b<x \end{cases}$$

$$E_n^1 = \langle \psi_n^0 | V'(x) | \psi_n^0 \rangle$$

$$E_n^1 = \int_{-\infty}^{\infty} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) V_0 \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) dx$$

$$E_n^1 = \frac{V_0 2}{a} \int_b^{a-b} \sin^2\left(\frac{n\pi}{a}x\right) dx$$

$$\sin^2\left(\frac{n\pi}{a}x\right) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi}{a}x\right)$$

$$E_n^1 = \frac{V_0}{a} \int_b^{a-b} 1 - \cos\left(\frac{2n\pi}{a}x\right) dx$$

$$E_n^1 = \frac{V_0}{a}\left[x - \frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}x\right) \Big|_b^{a-b} \right]$$

$$E_n^1 = \frac{V_0}{a}\left[(a-b) - \frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}(a-b)\right) - b + \frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a}b\right)\right]$$

$$E_n^1 = \frac{V_0}{a}\left[(a-2b) - \frac{a}{2n\pi}\sin\left(2n\pi - 2n\pi\frac{b}{a}\right) + \frac{a}{2n\pi}\sin\left(2n\pi\frac{b}{a}\right)\right]$$

$$\sin(x-y) = \sin(x)\cos y - \cos x \sin y$$

$$E_n^1 = \frac{V_0}{a}\left[(a-2b) - \frac{a}{2n\pi}\left[\cancel{\sin(2n\pi)\cos\left(2n\pi\frac{b}{a}\right)} - \underbrace{\cos(2n\pi)}_{1}\sin\left(2n\pi\frac{b}{a}\right)\right] + \frac{a}{2n\pi}\sin\left(2n\pi\frac{b}{a}\right)\right]$$

$$E_n^1 = \frac{V_0}{a}\left[(a-2b) - \cancel{\frac{a}{2n\pi}\sin(2n\pi b/a) + \frac{a}{2n\pi}\sin(2n\pi b/a)}\right]$$

$$E_n^1 = \frac{V_0}{a}(a-2b)$$

$$\boxed{E_n^1 = V_0 - 2V_0 b/a}$$

$$+ \frac{1}{\pi n}\sin\frac{2\pi n b}{a}$$

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2. A charged harmonic oscillator in electric field $\mathcal{E}$. Find first and second order energy corrections.

$$\hat{V} = q\mathcal{E}x$$

$$E_n^1 = \langle \psi_n^0 | \hat{V} | \psi_n^0 \rangle$$

$$E_n^1 = \langle \psi_n^0 | q\mathcal{E}\hat{x} | \psi_n^0 \rangle$$

$$E_n^1 = q\mathcal{E} \langle \psi_n^0 | \hat{x} | \psi_n^0 \rangle$$

for quantum harmonic oscillator:

$\hat{x} = \sqrt{\frac{\hbar}{2\mu\omega}} \left[\hat{a}_+ + \hat{a}_- \right]$ where $\hat{a}_+$ and $\hat{a}_-$ are ladder operators

$$E_n^1 = q\mathcal{E} \langle \psi_n^0 | \hat{a}_+ + a_- | \psi_n^0 \rangle \sqrt{\frac{\hbar}{2\mu\omega}}$$

$$E_n^1 = q\mathcal{E}\sqrt{\frac{\hbar}{2\mu\omega}} \langle \psi_n^0 | \hat{a}_+ + \hat{a}_- | \psi_n^0 \rangle$$

$$E_n^1 = q\mathcal{E}\sqrt{\frac{\hbar}{2\mu\omega}} \left[\langle \psi_n^0 | \hat{a}_+ | \psi_n^0 \rangle + \langle \psi_n | \hat{a}_- | \psi_n^0 \rangle \right]$$

$$E_n^1 = q\mathcal{E}\sqrt{\frac{\hbar}{2\mu\omega}} \left[\sqrt{n+1} \langle \psi_n^0 | \psi_{n+1}^0 \rangle + \sqrt{n} \langle \psi_n^0 | \psi_{n-1}^0 \rangle \right]$$

$$E_n^1 = q\mathcal{E}\sqrt{\frac{\hbar}{2\mu\omega}} \left[0 + 0 \right]$$

$$\boxed{E_n^1 = 0}$$

$$E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | \hat{H}' | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n^2 = \sum_{m \neq n} \frac{|\langle \psi_m^0 | q\mathcal{E}\hat{x} | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n^2 = \sum_{m \neq n} \frac{(q\mathcal{E})^2 |\langle \psi_m^0 | \hat{x} | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n^2 = \sum_{m \neq n} \frac{(q\mathcal{E})^2 |\langle \psi_m^0 | \sqrt{\frac{\hbar}{2\mu\omega}}\, \hat{a}_+ + \hat{a}_- | \psi_n^0 \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n^2 = \sum_{m \neq n} \frac{(q\mathcal{E})^2 \hbar}{2\mu\omega} \frac{|\langle m | \hat{a}_+ + \hat{a}_- | n \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n^2 = \frac{(q\mathcal{E})^2 \hbar}{2\mu\omega} \sum_{n \neq m} \frac{|\langle m | \hat{a}_+ | n \rangle + \langle m | \hat{a}_- | n \rangle|^2}{E_n^0 - E_m^0}$$

$$E_n^2 = \frac{(q\mathcal{E})^2 \hbar}{2\mu\omega} \sum_{m \neq n} \frac{\left|\sqrt{n+1}\langle m|n+1\rangle + \sqrt{n}\langle m|n-1\rangle\right|^2}{E_n^0 - E_m^0}$$

$$E_n^0 = \hbar\omega(n+1/2)$$

$$E_n^2 = \frac{(q\mathcal{E})^2 \hbar}{2\mu\omega\hbar\omega} \sum_{m \neq n} \frac{\left|\sqrt{n+1}\langle m|n+1\rangle + \sqrt{n}\langle m|n-1\rangle\right|^2}{n+1/2 - m - 1/2}$$

$$E_n^2 = \frac{(q\mathcal{E})^2}{2\mu\omega^2} \sum_{m \neq n} \frac{\left|\sqrt{n+1}\langle m|n+1\rangle \sqrt{n}\langle m|n-1\rangle\right|^2}{n-m}$$

Sum zero except for $m=n+1$ and $m=n-1$

$$E_n^2 = \frac{(q\mathcal{E})^2}{2\mu\omega^2}\left[\frac{(n+1)}{n-n-1} + \frac{n}{n-n+1}\right]$$

only?

$$E_n^2 = \left(\frac{q\mathcal{E}}{\omega}\right)^2 \frac{1}{2\mu}\left[\frac{n+1}{-1} + \frac{n}{1}\right]$$

$$E_n^2 = \left(\frac{q\mathcal{E}}{\omega}\right)^2 \frac{1}{2\mu}\left[-n - 1 + n\right]$$

$$\boxed{E_n^2 = -\left(\frac{q\mathcal{E}}{\omega}\right)^2 \frac{1}{2\mu}}$$

9?

3. Suppose quantum harmonic oscillator is perturbed by $\frac{\alpha}{2}x^2$. Find first two corrections to energy levels

$E_n^1 = \langle \psi_n^0 | \hat{V} | \psi_n^0 \rangle$

$E_n^1 = \langle \psi_n^0 | \frac{\alpha}{2} \hat{x}^2 | \psi_n^0 \rangle$

$E_n^1 = \frac{\alpha}{2} \langle n | \hat{x}^2 | n \rangle$ $\qquad \hat{x} = \sqrt{\frac{\hbar}{2\mu\omega}}(\hat{a}_+ + \hat{a}_-)$ for quantum harmonic oscillator

$E_n^1 = \frac{\alpha}{2} \langle n | \left(\sqrt{\frac{\hbar}{2\mu\omega}}(\hat{a}_+ + \hat{a}_-)\right)^2 | n \rangle$

$E_n^1 = \frac{\alpha}{2} \langle n | \frac{\hbar}{2\mu\omega} \left(\hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2\right) | n \rangle$

$E_n^1 = \frac{\alpha\hbar}{4\mu\omega}\left[\cancel{\langle n | \hat{a}_+^2 | n \rangle}^{0} + \langle n | \hat{a}_+\hat{a}_- | n \rangle + \langle n | \hat{a}_-\hat{a}_+ | n \rangle + \cancel{\langle n | \hat{a}_-^2 | n \rangle}^{0}\right]$

$E_n^1 = \frac{\alpha\hbar}{4\mu\omega}\left[\langle n | \hat{a}_+\hat{a}_- | n \rangle + \langle n | \hat{a}_-\hat{a}_+ | n \rangle\right]$

$E_n^1 = \frac{\alpha\hbar}{4\mu\omega}\left[\langle n | \hat{a}_+ | n-1 \rangle \sqrt{n} + \sqrt{n+1}\langle n | \hat{a}_- | n+1 \rangle\right]$

$E_n^1 = \frac{\alpha\hbar}{4\mu\omega}\left[\sqrt{n}\sqrt{(n-1)+1}\,\cancel{\langle n | n \rangle}^{=1} + \sqrt{n+1}\sqrt{n+1}\,\cancel{\langle n | n \rangle}^{=1}\right]$

$E_n^1 = \frac{\alpha\hbar}{4\mu\omega}\left[n + n + 1\right]$

$$\boxed{E_n^1 = \frac{\alpha\hbar}{4\mu\omega}(2n+1)}$$

$E_n^2 = \sum_{m \neq n} \frac{|\langle m | V' | n \rangle|^2}{E_m^0 - E_n^0}$

$\langle \psi_m^0 | V' | \psi_n^0 \rangle = \frac{\alpha}{2} \langle \psi_m^0 | \hat{x}^2 | \psi_n^0 \rangle$

$\langle \psi_m^0 | V' | \psi_n^0 \rangle = \frac{\alpha}{2} \langle \psi_m^0 | \hat{a}_+^2 + \hat{a}_+\hat{a}_- + \hat{a}_-\hat{a}_+ + \hat{a}_-^2 | \psi_n^0 \rangle$

Since $m \neq n$, ignore $\hat{a}_+\hat{a}_-$ and $\hat{a}_-\hat{a}_+$ terms

$\langle \psi_m^0 | V' | \psi_n^0 \rangle = \frac{\alpha}{2} \langle m | \hat{a}_+^2 + \hat{a}_-^2 | n \rangle$

$= \frac{\alpha}{2}\left[\langle m | \hat{a}_+^2 | n \rangle + \langle m | \hat{a}_-^2 | n \rangle\right]$

$= \frac{\alpha}{2}\left[\sqrt{n+1}\sqrt{n+2}\langle m | n+2 \rangle + \sqrt{n}\sqrt{n-1}\langle m | n-2 \rangle\right]$

~~only $m = n+2$ and $m = n-2$~~

$$= \alpha/4 \left[\sqrt{n+1}\sqrt{n+2}\langle m|n+2\rangle + \sqrt{n}\sqrt{n-1}\langle m|n-2\rangle\right]$$

$$-E_m^0 + E_n^0 = \hbar\omega(m+1/2) - \hbar\omega(n+1/2)$$

$$= \hbar\omega(m+n)$$

$$E_n^2 = \sum_{m\neq n} \frac{\left(\alpha/4\left[\sqrt{n+1}\sqrt{n+2}\langle m|n+2\rangle + \sqrt{n}\sqrt{n-1}\langle m|n-2\rangle\right]\right)^2}{\hbar\omega(m+n)}$$

Consider only $m=n+2$ and $m=n-2$

$$E_n^2 = \frac{\alpha^2}{16\hbar\omega}\left[\frac{(n+1)(n+2)}{-n-2+n} + \frac{n(n-1)}{-n+2-n}\right]$$

$$E_n^2 = \frac{\alpha^2}{16\hbar\omega_0^2}\left[\frac{n^2+3n+2}{-2} + \frac{-n^2+n}{-2}\right]$$

$$E_n^2 = \frac{\alpha^2}{32\hbar\omega_0^2}\left(\cancel{n^2}+3n+2-\cancel{n^2}+n\right)$$

$$E_n^2 = \frac{\alpha^2}{32\hbar\omega_0^2}(4n+2)$$

$$\boxed{E_n^2 = \frac{\alpha^2}{16\hbar\omega_0^2}(2n+1)}$$

Compare to exact solution

$$\omega = \sqrt{\frac{(k+\alpha)}{m}}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\omega = \sqrt{\omega_0^2 + \frac{\alpha}{m}}$$

$$\omega = \omega_0\sqrt{1+\frac{\alpha}{m\omega_0^2}}$$

$$\omega = \omega_0\left(1+\frac{\alpha}{m\omega_0^2}\right)^{1/2} \quad \text{(Taylor-expand)}$$

$$\omega \simeq \omega_0\left[1+\frac{1}{2}\left(\frac{\alpha}{m\omega_0^2}\right) - \frac{1}{8}\left(\frac{\alpha}{m\omega_0^2}\right)^2\right]$$

$$E_{exact} = \hbar\omega(n+1/2)$$

$$E_{exact} = \hbar\omega_0\left[1+\frac{1}{2}\left(\frac{\alpha}{m\omega_0^2}\right) - \frac{1}{8}\left(\frac{\alpha}{m\omega_0^2}\right)^2\right](n+1/2)$$

$$E_{exact} = \hbar\omega_0(n+1/2)\left[1+1/2\left(\frac{\alpha}{m\omega_0^2}\right) - \frac{1}{8}\left(\frac{\alpha}{m\omega_0^2}\right)^2\right]$$

$$E_{approx} = E^1_{n} + E^2_n + E^n_0$$

$$E_{approx} = \hbar\omega(n+1/2) + \frac{\alpha\hbar}{4\mu\omega}(2n+1) - \frac{\alpha^2}{16\hbar\omega}(2n+1)$$

$$E_{approx} = \hbar\omega(n+1/2)\left[1 + \frac{2\alpha}{4\mu\omega^2} - \frac{2\alpha}{16\mu\omega^2}\right]$$

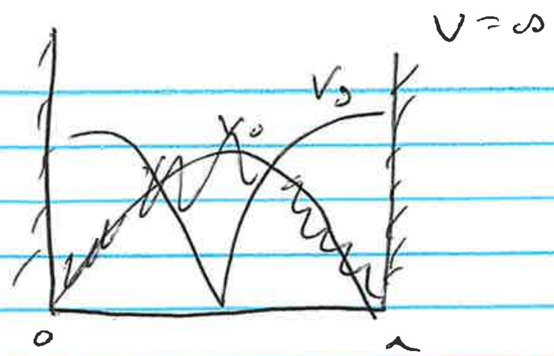
$$\boxed{E_{approx} = \hbar\omega(n+1/2)\left[1 + \frac{\alpha}{2\mu\omega^2} - \frac{\alpha}{8\mu\omega^2}\right]}$$

~~Solutions are~~

Approximation is the same as the exact solution!

100

4.



$$V(x) = V_0 \cos^2\left(\frac{\pi x}{a}\right)$$

$$\psi_n^0 = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

$$E_n^1 = \langle \psi_n | V' | \psi_n \rangle$$

$$E_n^1 = \int_{-\infty}^{\infty} \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right) V_0 \cos^2\left(\frac{\pi}{a} x\right) \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right) dx$$

$$E_n^1 = \frac{2V_0}{a} \int_0^a \sin^2\left(\frac{n\pi}{a} x\right) \cos^2\left(\frac{\pi}{a} x\right)$$

$$E_n^1 = \frac{2V_0}{a} \int_0^a \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2n\pi x}{a}\right)\right]\left[\frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi}{a} x\right)\right] dx$$

$$E_n^1 = \frac{2V_0}{a} \int_0^a \frac{1}{4} + \frac{1}{4}\cos\left(\frac{2\pi}{a} x\right) - \frac{1}{4}\cos\left(\frac{2n\pi}{a} x\right) + \frac{1}{4}\cos\left(\frac{2n\pi}{a} x\right)\cos\left(\frac{2\pi}{a} x\right) dx$$

$$E_n^1 = \frac{V_0}{2a} \int_0^a 1 + \cos\left(\frac{2\pi x}{a}\right) - \cos\left(\frac{2n\pi}{a} x\right) + \cos\left(\frac{2n\pi}{a} x\right)\cos\left(\frac{2\pi}{a} x\right) dx$$

$$\cos\left(\frac{2n\pi}{a} x\right)\cos\left(\frac{2\pi}{a} x\right) = \frac{1}{2}\left[\cos\left((n+1)\left(\frac{2\pi}{a} x\right)\right) + \cos\left((n-1)\left(\frac{2\pi}{a} x\right)\right)\right]$$

$$E_n^1 = \frac{V_0}{2a} \int_0^a 1 + \cos\left(\frac{2\pi}{a} x\right) - \cos\left(\frac{2n\pi}{a} x\right) + \frac{1}{2}\cos\left((n+1)\left(\frac{2\pi}{a} x\right)\right) + \frac{1}{2}\cos\left((n-1)(2\pi/a\, x)\right) dx$$

$$E_n^1 = \frac{V_0}{2a}\left[x + \frac{a}{2\pi}\sin\left(\frac{2\pi}{a} x\right) - \frac{a}{2n\pi}\sin\left(\frac{2n\pi}{a} x\right) + \frac{1}{2}\frac{a}{2\pi(n+1)}\sin\left((n+1)\left(\frac{2\pi}{a} x\right)\right) + \frac{1}{2}\frac{a}{2\pi(n-1)}\sin\left((n-1)2\pi/a\, x\right)\right]_0^a$$

$$E_n^1 = \frac{V_0}{2a}\left[a + \frac{a}{2\pi}\cancel{\sin(2\pi)}^{0} - \frac{a}{2n\pi}\cancel{\sin(2n\pi)}^{0} + \frac{a}{4\pi(n+1)}\cancel{\sin((n+1)2\pi)}^{0} + \frac{a}{4\pi(n-1)}\cancel{\sin((n-1)2\pi)}^{0}\right\} - 0 - \frac{a}{2\pi}\cancel{\sin(0)}^{0} + \frac{a}{2n\pi}\cancel{\sin(0)}^{0} + \frac{a}{4\pi(n-1)}\cancel{\sin(0)}^{0} + \frac{a}{4\pi(n+1)}\cancel{\sin(0)}^{0}\Big]$$

$$E_n^1 = \frac{V_0}{2a}(a) \qquad \boxed{E_n^1 = \frac{V_0}{2}}$$

$m = n \neq 0$

$V/4 \quad m = n = 0$

$$E_n^2 = \sum_{m \neq n} \frac{|\langle m|\hat{V}|n\rangle|^2}{E_n^0 - E_m^0}$$

$$\langle m|\hat{V}|n\rangle = \frac{2V_0}{a}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)\cos^2\left(\frac{\pi}{a}x\right)dx$$

$$\langle m|\hat{V}|n\rangle = \frac{2V_0}{a}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)\left(1-\sin^2\left(\frac{\pi}{a}x\right)\right)dx$$

$$\langle m|\hat{V}|n\rangle = \frac{2V_0}{a}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right) - \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}\right)x\sin^2\left(\frac{\pi}{a}\right)x$$

$$\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right) = \frac{1}{2}\left[\cos\left((n-m)\frac{\pi}{a}x\right) - \cos\left((n+m)\frac{\pi}{a}\right)\right]dx$$

$$\frac{2V_0}{a}\int_0^a \underbrace{\frac{1}{2}\left[\cos\left((n-m)\frac{\pi}{a}x\right) - \cos\left((n+m)\frac{\pi}{a}x\right)\right]}_{} + \underbrace{\frac{1}{2}\left[\cos((n-m)\pi/a\,x) - \cos((n+m)\pi/a\,x)\right]\sin^2((\pi/a)x)\,dx}_{\text{II}}$$

$$\text{I}: \frac{1}{2}\int_0^a \cos((n-m)\pi/a\,x) - \cos((n+m)\pi/a\,x)\,dx$$

$$\cancel{\frac{1}{2}\int_0^a} \frac{1}{2}\left[\frac{a}{\pi(n-m)}\sin((n-m)\pi/a\,x) - \frac{a}{\pi(n+m)}\sin\left((n+m)\frac{\pi}{a}x\right)\right|_0^a\Bigg]$$

$$\frac{1}{2}\left[\frac{a}{\pi(n-m)}\sin\left((n-m)\pi\right) - \frac{a}{\pi(n+m)}\sin\left((n+m)\pi\right) - \sin 0^0 - \sin 0^0\right]$$

$$\frac{1}{2}[0] \qquad \text{I} \rightarrow 0$$

$$\text{II} \quad \int_0^a \frac{1}{2}\left[\cos\left((n-m)\pi/a\,x\right) - \cos\left((n+m)\pi/a\,x\right)\right]\sin^2(\pi/a\,x)\,dx$$

$$\frac{1}{2}\int_0^a \left[\cos((n-m)\pi/a\,x) - \cos((n+m)\pi/a\,x)\right]\left(\frac{1}{2}\left(1-\cos\left(\frac{2\pi}{a}x\right)\right)\right)dx$$

$$\frac{1}{4}\int_0^a \underbrace{\cos((n-m)\pi/a\,x) - \cos((n+m)\pi/a\,x)}_{\text{I}=0} - \cos((n-m)\pi/a\,x)\cos\left(\frac{2\pi}{a}x\right) + \cos((n+m)\pi/a\,x)\cos\left(\frac{2\pi}{a}x\right)dx$$

$$+\frac{1}{4}\int_0^a -\cos((n-m)\pi/a\,x)\cos\left(\frac{2\pi}{a}x\right) + \cos((n+m)\pi/a\,x)\cos\left(\frac{2\pi}{a}x\right)dx$$

$$\frac{1}{4}\int_0^a -\frac{1}{2}\left[\cos\left(\left[\frac{2\pi}{a} + n-m\left(\frac{\pi}{a}\right)\right]x\right) + \cos\left(\left[\frac{2\pi}{a} - (n-m)\frac{\pi}{a}\right]x\right)\right]$$

$$\frac{1}{8}\left[-\frac{1}{\frac{2\pi}{a} + n-m\left(\frac{\pi}{a}\right)}\sin\left(\left[\frac{2\pi}{a} + (n-m)\pi/a\right]x\right) + \frac{1}{\frac{2\pi}{a} - (n-m)\pi/a}\sin\left(\frac{2\pi}{a} - (n-m)\pi/a\,x\right)\right|_0^a$$

$$\frac{1}{8}\left[-\frac{1}{\dots}\sin(2\pi+(n-m)\pi)+\frac{1}{\dots}\sin(2\pi-(n+m)\pi)\right]\Big|_0^a$$

$$-\sin 0^{\to 0}\quad -\sin 0^{\to 0}$$

$$\frac{1}{8}\left[0+0-0-0\right]$$

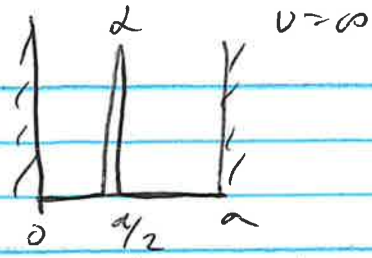
incorrect

$\boxed{E_n^2 = 0}$ because $\langle m|\hat{V}|n\rangle = 0$

no second-order energy correction!

$$E^2 = \frac{2m\alpha^2}{\pi^2\hbar^2 h^2}\qquad n = 2\omega + 1$$

8.0

5. 

$$V(x) = \alpha\,\delta(x - a/2)$$

$$E_n^1 = \langle \psi_n^0 | \hat{V} | \psi_n^0 \rangle$$

$$E_n^1 = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \alpha\, \delta(x - a/2) \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) dx$$

$$E_n^1 = \frac{2\alpha}{a} \int_0^a \sin^2\left(\frac{n\pi}{a}x\right) \delta(x - a/2)\, dx$$

$$\int f(x)\, \delta(x - \alpha)\, dx = f(a)$$

$$E_n^1 = \frac{2\alpha}{a} \sin^2\left(\frac{n\pi}{a}\frac{a}{2}\right)$$

$$\boxed{E_n^1 = \frac{2\alpha}{a}(-1)^n}$$

$$\cancel{E_n^0 = \hbar\omega(n + 1/2)}$$

$$E_n^0 = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$E_n^2 = \sum_{m \neq n} \frac{|\langle m | \hat{V} | n \rangle|^2}{E_n^0 - E_m^0}$$

$$\langle m | \hat{V} | n \rangle = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right) \alpha\, \delta(x - a/2) \sqrt{\frac{2}{a}} \sin\left(\frac{m\pi}{a}x\right) dx$$

$$= \frac{2\alpha}{a} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) \delta(x - a/2)\, dx$$

$$= \frac{2\alpha}{a} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{m\pi}{2}\right)$$

$$= \frac{2\alpha}{a} (\cancel{-1})^m \text{ for } m \text{ is even}$$

$$= \left|\frac{2\alpha}{a}\right| (-1)^{n+m} \text{ for even } m \qquad E_n^0 - E_m^0 = \frac{\pi^2\hbar^2}{2ma^2}(n^2 - m^2)$$

$$\cancel{E_n^2 - E_m^0 = \hbar\omega(n + 1/2) - \hbar\omega(m + 1/2) = \hbar\omega(n - m)}$$

$$\cancel{\boxed{E_n^2 = \sum_{m \neq n} \frac{2\alpha}{a} \frac{(-1)^{n+m}}{\hbar\omega(n - m)}}}$$

$$E_n^2 = \sum_{m \neq n} \frac{2\alpha}{a} \frac{(2ma^2)}{\pi^2\hbar^2} \frac{(-1)^{n+m}}{(n^2 - m^2)}$$

$$\boxed{E_n^2 = \sum_{m \neq n} \frac{4 m a \alpha}{\pi^2 \hbar^2} \frac{\cancel{(-1)^{n+m}}}{(n^2 - m^2)} \quad \text{for } m \text{ odd}}$$

$$\boxed{E_n^2 = 0 \quad \text{for even } m}$$

$$= \frac{-2m\alpha^2}{\pi^2 \hbar^2 4^2} \qquad n = 2k+1$$

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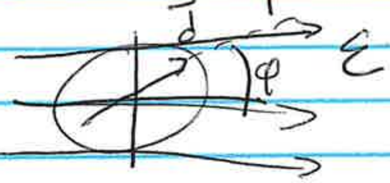
6. Plane rotator with moment of inertia I with dipole moment I is placed in ~~a~~ an electric $\mathcal{E}$ in rotation plane. Find rotator polarizability.



$$\Psi(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

$$m = 0, \pm 1, \pm 2, \ldots$$

$$E_m = \frac{m^2 \hbar^2}{2\mu r^2}$$ (apparently, I am confused on this...)

$$V = -d\mathcal{E}\cos\varphi$$

Polarizer is doubly degenerate for $m \neq 0$! consider $m = 0$ for this problem.

Use nondegenerate perturbation theory.

$$E_n^1 = \langle \Psi_n^0 | \hat{V} | \Psi_n^0 \rangle$$

$$\hat{V} = -|\vec{\mathcal{E}}||\vec{d}| \cos\varphi$$

$$E_n^1 = \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^0 (-|\vec{\mathcal{E}}|)(|\vec{d}|)\cos\varphi \frac{1}{\sqrt{2\pi}} e^0 \, d\varphi$$

$$E_n^1 = -\frac{\mathcal{E}d}{2\pi}\int_0^{2\pi} \cos\varphi \, d\varphi$$

$$E_n^1 = -\frac{\mathcal{E}d}{2\pi}\left[\sin\varphi \Big|_0^{2\pi} \right]$$

$$\boxed{E_n^1 = 0}$$

$$E_n^2 = \frac{|\langle m | \hat{V} | n \rangle|^2}{E_n^0 - E_m^0}$$

$$\langle m | \hat{V} | n \rangle$$

$$= \int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{-im\varphi} \left(-\mathcal{E}d\cos\varphi\right) \frac{1}{\sqrt{2\pi}} e^{in\varphi} \, d\varphi$$

$$= -\frac{\mathcal{E}d}{2\pi} \int_0^{2\pi} e^{i(n-m)\varphi} \cos\varphi \, d\varphi$$

$$= -\frac{\mathcal{E}d}{2\pi} \int_0^{2\pi} \left(\cos((n-m)\varphi) + i\sin((n-m)\varphi)\right) \cos\varphi \, d\varphi$$

$$= -\frac{\mathcal{E}d}{2\pi} \int_0^{2\pi} \cos\varphi \cos((n-m)\varphi) + i\sin((n-m)\varphi)\cos\varphi \, d\varphi$$

$$= -\frac{\mathcal{E}\ell}{2\pi}\int_0^{2\pi} \cos\varphi\cos((n-m)\varphi) + i\sin((n-m)\varphi)\cos\varphi \, d\varphi$$

$$\cos\varphi\cos((n-m)\varphi) = \frac{1}{2}\left[\cos(\varphi(1+(n-m))) - \cos(\varphi(1-(n-m)))\right]$$

$$\int_0^{2\pi} \frac{1}{2}\left[\cos(\varphi(1+(n-m))) - \cos(\varphi(1-(n-m)))\right] d\varphi$$

$$\frac{1}{2}\int_0^{2\pi} \cos(\varphi(1+(n-m))) - \cos(\varphi(1-(n-m))) \, d\varphi$$

$$= \frac{1}{2}\left[\frac{1}{1+(n-m)}\sin(\varphi(1+(n-m))) - \frac{1}{1-(n-m)}\sin(\varphi(1-(n-m)))\Big|_0^{2\pi}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+(n-m)}\sin(2\pi(1+(n-m)))^{\to 0} - \frac{1}{1-(n-m)}\sin(2\pi(1-(n-m)))^{\to 0}\right.$$

$$\left. - \frac{1}{1+(n-m)}\sin 0^{\to 0} + \frac{1}{1-(n-m)}\sin(0)_{\to 0}\right]$$

$$\int_0^{2\pi} \cos\varphi\cos(n-m)\varphi \, d\varphi = 0$$

$$q\langle m|\hat{V}|n\rangle = \frac{\mathcal{E}\ell}{2\pi}\int_0^{2\pi} i\sin((n-m)\varphi)\cos\varphi \, d\varphi$$

$$= \frac{\mathcal{E}\ell i}{2\pi}\int_0^{2\pi} \sin((n-m)\varphi)\cos\varphi \, d\varphi$$

$$\sin((n-m)\varphi)\cos\varphi = \frac{1}{2}\left[\sin(\varphi(1+(n-m))) - \sin(\varphi(1-(n-m)))\right]$$

$$= \frac{1}{2}\int_0^{2\pi} \sin(\varphi(1+(n-m))) - \sin(\varphi(1-(n-m))) \, d\varphi$$

$$= \frac{1}{2}\left[-\frac{1}{1+(n-m)}\cos(\varphi(1+(n-m))) + \frac{1}{1-(n-m)}\cos(\varphi(1-(n-m)))\Big|_0^{2\pi}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{1+(n-m)}\cos(2\pi(1+(n-m))) + \frac{1}{1-(n-m)}\cos(2\pi(1-(n-m)))\right.$$

$$\left. + \frac{1}{1+(n-m)}\cos(0)^{1} - \frac{1}{1-(n-m)}\cos(0)^{1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+(n-m)}\left[1 + (-1)^{(n-m)+1}\right] + \frac{1}{1-(n-m)}\left[-1 + (-1)^{1-(n-m)}\right]\right]$$

$= 0$ for ~~even~~ $|n-m|$

$= \frac{1}{1+(n-m)} + \frac{1}{1-(n-m)}$ for ~~odd~~ even $|n-m|$

$$\langle m|\hat{V}|n\rangle = \frac{\mathcal{E} d i}{n\pi}\left[\frac{2}{1\pm(n-m)}\right] \text{ for } |n-m| \text{ even}$$

$$|\langle m|\hat{V}|n\rangle|^2 = \frac{\mathcal{E}^2 d^2 i^2}{\pi^2}\left[\frac{1}{1\pm(n-m)}\right]^2 \text{ for } |n-m| \text{ even}$$

$n=0$

$$|\langle m|\hat{V}|n\rangle|^2 = \frac{\mathcal{E}^2 d^2}{\pi^2}\left[\frac{1}{1\pm(-m)}\right]^2 \text{ for even } m$$

$$E_n^0 - E_m^0 = \frac{\hbar^2}{2\mu r^2}(n^2-m^2)$$

$$E_n^2 = \sum_{\substack{m\neq n\\ \text{even}}} \frac{\mathcal{E}^2 d^2}{\pi^2}\left[\frac{1}{1-m}\;\frac{1}{1+m}\right]\frac{2\mu r^2}{\hbar^2}\frac{1}{(n^2-m^2)}$$

$n=0$

$$E_n^2 = \sum_{m\neq n,\text{even}} \frac{\mathcal{E}^2 d^2}{\pi^2}\frac{1}{1-m^2}\frac{2\mu r^2}{\hbar^2}\frac{1}{-m^2}$$

Energy ~~due~~ due to $\mathcal{E}$: $-\frac{1}{2}\alpha|\vec{E}|^2$

polarizability

$$-\frac{1}{2}\alpha \cancel{\mathcal{E}^2} = \sum_{m\neq n,\text{even}} \frac{\cancel{\mathcal{E}^2} d^2}{\pi^2}\frac{1}{1-m^2}\frac{2\mu r^2}{\hbar^2}\frac{-1}{m^2}$$

$$\boxed{\alpha = \sum_{m\neq n,\text{even}} \frac{d^2}{\pi^2}\frac{1}{1-m^2}\frac{4\mu r^2}{\hbar^2}\frac{1}{m^2}}$$

The theory is degenerat for all $m\neq 0$

for $m=0$ $\quad E^2 = -\frac{d^2\mathcal{E}_0^2}{2} = -\frac{d^2 \mathcal{E}_0 I}{\hbar^2}$

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7. In previous problem, find energy shift and splitting of rotator excited states. Find correct wavefunction to zeroth order.
Find elements of matrix W:

$W_{ab} = \langle \psi_a^0 | H' | \psi_b^0 \rangle \qquad a = m\ell$

$W_{aa} = \langle e^{-im\ell\varphi} | \mathcal{E} d \cos\varphi | e^{-im\ell\varphi} \rangle = 0 \qquad b = -m\ell$

$W_{bb} = \langle e^{im\ell\varphi} | \mathcal{E} d \cos\varphi | e^{-im\ell\varphi} \rangle = 0$

$W_{ab} = \langle \frac{1}{\sqrt{2\pi}} e^{-im\ell\varphi} | \mathcal{E} d \cos\varphi | e^{-im\ell\varphi} \frac{1}{\sqrt{2\pi}} \rangle$

$W_{ab} = \frac{1}{2\pi} \mathcal{E} d \int_0^{2\pi} e^{i(ab-a)\varphi} \cos\varphi \, d\varphi$

$W_{ab} = \frac{\mathcal{E} d}{2\pi} \int_0^{2\pi} e^{i(b-a)\varphi} \cos\varphi \, d\varphi$

$W_{ab} = \frac{\mathcal{E} d}{2\pi} \int_0^{2\pi} \left(\cos((b-a)\varphi) + i\sin((b-a)\varphi)\right) \cos\varphi \, d\varphi$

$W_{ab} = \frac{\mathcal{E} d}{2\pi} \int_0^{2\pi} \cos\varphi \underbrace{\cancel{\cos((b-a)\varphi)}}_{?\text{ from problem 6}} + i\cos\varphi \sin((b-a)\varphi) \, d\varphi$

$W_{ab} = \frac{\mathcal{E} d}{2\pi} \int_0^{2\pi} i\cos\varphi \sin((b-a)\varphi) \, d\varphi$

$W_{ab} = \frac{\mathcal{E} d i}{2\pi} \int_0^{2\pi} \cos\varphi \sin((b-a)\varphi) \, d\varphi$

Wab should be in the second order because the first order gives 0.

from problem 6:

$\int_0^{2\pi} \cos\varphi \sin(b-a)\varphi \, d\varphi =$

$-\frac{1}{2}\left[\frac{\cos((b-a+1)x)}{b-a+1} + \frac{\cos((b-a-1)x)}{b-a-1}\right]\Big|_0^{2\pi}$

$= -\frac{1}{2}\left[\frac{\cos((b-a+1)2\pi)}{b-a+1} + \frac{\cos((b-a-1)2\pi)}{b-a-1} - \frac{1}{b-a+1} - \frac{1}{b-a-1}\right]$

$= -\frac{1}{2}\left[\frac{(-1)^{(b-a)+1}}{b-a+1} - \frac{1}{b-a+1} + \frac{(-1)^{(b-a)-1}}{b-a-1} - \frac{1}{b-a-1}\right]$

$= 0$ for ~~even~~ odd $(b-a)$

$= -1\left[\frac{1}{b-a+1} + \frac{1}{b-a-1}\right]$ for ~~odd~~ even $(b-a)$

$$W_{ab} = -\frac{\varepsilon d i}{2\pi}\left[\frac{1}{\cancel{a-b}\ b-a+1} + \frac{1}{b-a-1}\right]$$

$$W_{ab} = -\frac{\varepsilon d i}{2\pi}\left[\frac{b-a-1+b-a+1}{(b-a+1)(b-a-1)}\right]$$

$$W_{ab} = -\frac{\varepsilon d i}{2\pi}\left[\frac{2b-2a}{b^2-ab\cancel{-b}-ab+a^2\cancel{+a}\cancel{+b}\cancel{-a}-1}\right]$$

$$W_{ab} = -\frac{\varepsilon d i}{2\pi}\left[\frac{2(b-a)}{b^2-2ab+a^2-1}\right]$$

$$W_{ab} = -\frac{\varepsilon d i}{\pi}\left[\frac{b-a}{b^2-2ab+a^2-1}\right]$$

$a = m_\ell$

$b = -m_\ell$

$$W_{ab} = -\frac{\varepsilon d i}{\pi}\left[\frac{-2m_\ell}{m_\ell^2+2m_\ell m_\ell+m_\ell^2-1}\right]$$

$$W_{ba} = -\frac{\varepsilon d i}{\pi}\left[\frac{2m_\ell}{m_\ell^2+2m_\ell^2+m_\ell^2-1}\right]$$

$$E^1_\pm = \frac{1}{2}\left[W_{aa}+W_{bb} \pm \sqrt{(W_{aa}-W_{bb})^2+4|W_{ab}|^2}\right]$$

$W_{aa} = W_{bb} = 0$

$$E^1_\pm = \frac{1}{2}\sqrt{4|W_{ab}|^2}$$

$$E^0_{m_\ell} = \frac{\hbar^2}{2I}$$

the second α should be

$\downarrow$ incorrect

$$E^1_\pm = |W_{ab}|$$

$$E^1_\pm = \frac{\varepsilon d}{\pi}\left[\frac{2(1)}{4(1)-1}\right]$$

$$E^1_\pm = \pm\frac{2\varepsilon d}{3\pi}$$

$$\boxed{E_1 = \begin{cases} \dfrac{\hbar^2}{2I} + \dfrac{2}{3}\dfrac{\varepsilon d}{\pi} & \text{for } m_\ell = 1 \\[2ex] \dfrac{\hbar^2}{2I} - \dfrac{2}{3}\dfrac{\varepsilon d}{\pi} & \text{for } m_\ell = -1 \end{cases}}$$

$\alpha W_{aa} + \beta W_{ab} = \alpha E^1$ (Griffiths 6.22)

$$\beta = \alpha\left(\frac{E^1 - W_{aa}}{W_{ab}}\right)$$

[illegible]

$$\beta = \alpha \left(\frac{\pm W_{ab} - 0}{W_{ab}} \right)$$

$$\beta = \pm \alpha$$

$$\psi^0 = \alpha \psi_a^0 + \beta \psi_b^0$$

$$\psi_1^0 = \alpha \psi_{a_1} + \alpha \psi_{-1}^0$$

$$\psi_1^0 = \alpha \left(\frac{1}{\sqrt{2\pi}} e^{i\varphi} + \frac{1}{\sqrt{2\pi}} e^{-i\varphi} \right)$$

$$\psi_1^0 = \frac{\alpha}{\sqrt{2\pi}} \left(e^{i\varphi} + e^{-i\varphi} \right)$$

$$\psi_1^0 = \frac{\alpha}{\sqrt{2\pi}} \left(\cos\varphi + i\sin\varphi + \cos\varphi - i\sin\varphi \right)$$

$$\psi_1^0 = \frac{2\alpha}{\sqrt{2\pi}} \left(\cos\varphi \right)$$

$$\psi_{-1}^0 = \alpha \psi_1^0 - \alpha \psi_{-1}^0$$

$$\psi_{-1}^0 = \alpha \left(\psi_1^0 - \psi_{-1}^0 \right)$$

$$\psi_{-1}^0 = \alpha \left(e^{i\varphi} - e^{-i\varphi} \right)$$

$$\psi_{-1}^0 = \frac{\alpha}{\sqrt{2\pi}} \left(\cos\varphi + i\sin\varphi - \cos\varphi + i\sin\varphi \right)$$

$$\psi_{-1}^0 = \frac{2\alpha i}{\sqrt{2\pi}} \left(\sin\varphi \right)$$

$$\boxed{\psi_1^0 = \frac{2\alpha}{\sqrt{2\pi}} \begin{cases} \cos\varphi & \text{for } m = 1 \\ \sin\varphi & \text{for } m = -1 \end{cases}}$$

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8. Determine splitting of the first excited state of a 2D harmonic oscillator in first order ~~a~~ in perturbation theory with perturbation

$$\hat{V} = \alpha \hat{x}\hat{y}$$

~~Assume~~ new $\hat{H}$: $\hat{H} = \left(\frac{p_x^2 + p_y^2}{2\mu}\right) + \frac{1}{2}k(x^2+y^2)$

$$E_{n_x,n_y} = \hbar\omega(n_x+n_y+1)$$

First excited state doubly-degenerate:

$$E_{1,0} = E_{0,1}$$

$$W_{ab} = \langle a|\hat{V}|b\rangle$$

$$W_{ab} = \alpha\langle a|\hat{x}\hat{y}|b\rangle \qquad a=(1,0)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_{x+}+\hat{a}_{x-}) \qquad b=(0,1)$$

$$\hat{y} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_{y+}+\hat{a}_{y-})$$

$$W_{ab} = \frac{\alpha\hbar}{2m\omega}\langle a|(\hat{a}_{x+}+\hat{a}_{x-})(\hat{a}_{y+}+\hat{a}_{y-})|b\rangle$$

$$W_{ab} = \frac{\alpha\hbar}{2m\omega}\langle 10|\hat{a}_{x+}\hat{a}_{y+}+\hat{a}_{x+}\hat{a}_{y-}+\hat{a}_{x-}\hat{a}_{y+}+\hat{a}_{x-}\hat{a}_{y-}|01\rangle$$

$$W_{ab} = \frac{\alpha\hbar}{2m\omega}\Big[\langle 10|\hat{a}_{x+}\hat{a}_{y+}|01\rangle + \langle 10|\hat{a}_{x+}\hat{a}_{y-}|01\rangle + \langle 10|\hat{a}_{x-}\hat{a}_{y+}|01\rangle + \langle 10|\hat{a}_{x-}\hat{a}_{y-}|01\rangle\Big]$$

$$W_{ab} = \frac{\alpha\hbar}{2m\omega}\Big[\cancel{\sqrt{1}\sqrt{2}\langle 10|12\rangle} + \sqrt{1}\sqrt{1}\langle 10|10\rangle + \cancel{\sqrt{0}\sqrt{2}\langle 10|-12\rangle} + \cancel{\langle 10|-10\rangle\sqrt{0}\sqrt{1}}\Big]$$

$$W_{ab} = \frac{\alpha\hbar}{2m\omega}$$

$$W_{aa} = \frac{\alpha\hbar}{2m\omega}\langle 10|\hat{a}_{x+}\hat{a}_{y+}+\hat{a}_{x+}\hat{a}_{y-}+\hat{a}_{x-}\hat{a}_{y+}+\hat{a}_{x-}\hat{a}_{y-}|10\rangle$$

$$W_{aa} = \frac{\alpha\hbar}{2m\omega}\Big[\cancel{\langle 10|21\rangle\sqrt{2}\sqrt{1}} + \cancel{\langle 10|2-1\rangle\sqrt{2}\sqrt{0}} + \cancel{\langle 10|01\rangle\sqrt{1}\sqrt{1}} + \cancel{\langle 10|0-1\rangle\sqrt{1}\sqrt{0}}\Big]$$

$$W_{aa} = 0$$

$$W_{bb} = \frac{\alpha\hbar}{2m\omega}\langle 01|\hat{a}_{x+}\hat{a}_{y+}+\hat{a}_{x+}\hat{a}_{y-}+\hat{a}_{x-}\hat{a}_{y+}+\hat{a}_{x-}\hat{a}_{y-}|01\rangle$$

$$W_{ab} = \frac{\alpha\hbar}{2m\omega}\left[\cancel{\langle 01|\,|2\rangle\sqrt{1}\sqrt{2}} + \cancel{\langle 01|\,|1\rangle\sqrt{1}\sqrt{1}} + \cancel{\langle 01|\,|1\rangle\sqrt{1}\sqrt{2}} + \langle 01|\,|0\;1\rangle\sqrt{1}\sqrt{0}\right]$$

$$W_{bb} = 0$$

$$E^1_\pm = \frac{1}{2}\left[W_{aa} + W_{bb} \pm \sqrt{(W_{aa}-W_{bb})^2 + 4|W_{ab}|^2}\right]$$

$$E^1_\pm = \frac{1}{2}\sqrt{4|W_{ab}|^2}$$

$a = |10\rangle$

$b = |01\rangle$

$$\cancel{E^1_\pm = \pm|W_{ab}|}$$

$$\boxed{E^1_\pm = \pm\frac{\alpha\hbar}{2m\omega}}$$

$$E^0_{n} = \hbar\omega(n_x + n_y + 1)$$

$$E^0_1 = \hbar\omega(2)$$

$$\boxed{E_1 = \hbar\omega\left(2 + \frac{\alpha}{2m\omega^2}\right)}$$

$$\psi^0 = \alpha\psi^0_a + \beta\psi^0_b$$

$$\alpha \cancel{W^0_{aa}} + \beta W_{ab} = \alpha E^1$$

$$\beta \left(\cancel{\frac{\alpha\hbar}{2m\omega}}\right) = \alpha\left(\cancel{\frac{\pm\alpha\hbar}{2m\omega}}\right)$$

$$\beta = \pm\alpha$$

$$\psi = \alpha\left[\psi^0_{1,0} \pm \psi^0_{0,1}\right]$$

$$\cancel{\psi_{1,0} = \psi_{x=1}\,\psi_{y=0}}$$

$$\cancel{\psi_{1,0} = \left(\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2}}H_1(\xi)e^{-\xi^2/2}\right)\left(\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}H_0(\xi)e^{-\xi^2/2}\right)}$$

$$\cancel{\psi_{1,0} = \sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{\sqrt{2}}\,2\xi\,e^{-\xi_x^2/2}e^{-\xi_y^2/2}} \quad \text{where } \xi = \sqrt{\frac{m\omega}{\hbar}}x$$

$$\cancel{\psi_{1,0} = \sqrt{\frac{m\omega}{2\pi\hbar}}\left(2\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}}$$

$$\psi_{0,1} = \sqrt{\frac{m\omega}{2\pi\hbar}}\left(2\sqrt{\frac{m\omega}{\hbar}}y\right)e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

100

$$\Psi_{x=0} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2} \quad \Psi_{x=1} = \frac{2}{\sqrt{2}}\left(\frac{m\omega}{\hbar\hbar}\right)^{1/4}\left(\frac{m\omega}{\hbar}\right)^{1/2} x e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\Psi_{y=0} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}y^2} \quad \Psi_{y=1} = \frac{2}{\sqrt{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{m\omega}{\hbar}\right)^{1/2} y e^{-\frac{m\omega}{2\hbar}y^2}$$

$\alpha = \frac{1}{\sqrt{2}}$ because equal contribution

$$\Psi_1(x,y) = \frac{1}{\sqrt{2}}\left[\left(\Psi_{x=0}\Psi_{y=1}\right) \pm \left(\Psi_{x=1}\Psi_{y=0}\right)\right]$$

$$\Psi_1(x,y) = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{m\omega}{\pi\hbar}}\frac{2}{\sqrt{2}}\sqrt{\frac{m\omega}{\hbar}}\, e^{-\frac{m\omega}{2\hbar}x^2+y^2}\left(y \pm x\right)\right]$$

$$\boxed{\Psi_1(x,y) = \frac{m\omega}{\hbar\sqrt{\pi}} e^{-\frac{m\omega}{2\hbar}x^2+y^2}\left(y \pm x\right)}$$

~~$E_{1,0}$ exact~~:

$$V_{new} = \frac{1}{2}k(x^2+y^2) + \frac{2\alpha xy}{2}$$

$$k' = k + 2\alpha$$

$$\omega' = \omega_0\sqrt{1+\frac{\alpha}{m\omega^2}} \approx \omega_0\left(1+\frac{\alpha}{2m\omega^2}\right)$$

$$E_{1,0}\text{ exact} = \hbar\omega\left(2 \pm \frac{\alpha}{2m\omega^2}\right)$$

corrections match exactly to exact energy!

$c = |0\,2\rangle$
$b = |1\,1\rangle$

9. $a: |2,0\rangle$ triply degenerate.
$b: |0,2\rangle$
$c: |1,1\rangle$

$$W = \frac{\alpha\hbar}{2m\omega}\begin{bmatrix} W_{aa} & W_{ba} & W_{ca} \\ W_{ab} & W_{bb} & W_{cb} \\ W_{ac} & W_{bc} & W_{cc} \end{bmatrix}$$

$$W = \frac{\alpha\hbar}{2m\omega}\begin{bmatrix} \langle a|\hat{x}\hat{y}|a\rangle & \langle b|\hat{x}\hat{y}|a\rangle & \langle c|\hat{x}\hat{y}|a\rangle \\ \langle a|\hat{x}\hat{y}|b\rangle & \langle b|\hat{x}\hat{y}|b\rangle & \langle c|\hat{x}\hat{y}|b\rangle \\ \langle a|\hat{x}\hat{y}|c\rangle & \langle b|\hat{x}\hat{y}|c\rangle & \langle c|\hat{x}\hat{y}|c\rangle \end{bmatrix}$$

$W_{aa} = \langle a|\hat{a}_{x+}\hat{a}_{y+} + \hat{a}_{x+}\hat{a}_{y-} + \hat{a}_{x-}\hat{a}_{y+} + \hat{a}_{x-}\hat{a}_{y-}|a\rangle$

$W_{aa} = \langle 2\,0|\hat{a}_{x+}\hat{a}_{y+} + \hat{a}_{x+}\hat{a}_{y-} + \hat{a}_{x-}\hat{a}_{y+} + \hat{a}_{x-}\hat{a}_{y-}|2\,0\rangle$ ← Switching b and c

$W_{aa} = 0$

$W_{bb} = 0$

$W_{cc} = 0$

$$W = \frac{\alpha\hbar}{2m\omega}\begin{bmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & \sqrt{2} \\ \sqrt{2} & \sqrt{2} & 0 \end{bmatrix}$$

$W_{ba} = \langle 0\,2|\hat{a}_{x+}\hat{a}_{y+} + \hat{a}_{x+}\hat{a}_{y-} + \hat{a}_{x-}\hat{a}_{y+} + \hat{a}_{x-}\hat{a}_{y-}|2\,0\rangle$

$W_{ba} = 0 + 0 + 0 + 0$

$W_{ca} = \langle 1\,1|\hat{a}_{x+}\hat{a}_{y+} + \hat{a}_{x+}\hat{a}_{y-} + \hat{a}_{x-}\hat{a}_{y+} + \hat{a}_{x-}\hat{a}_{y-}|2\,0\rangle$
$= \cancel{\langle 1\,1|3\,1\rangle} + \cancel{\langle 1\,1|3\,-1\rangle} + \langle 1\,1|1\,1\rangle\sqrt{2}\sqrt{1} + \cancel{\langle 1\,1|1\,-1\rangle}$
$= \sqrt{2}$

$W_{ab} = \langle 2\,0|\hat{x}\hat{y}|0\,2\rangle = 0$ Swapped b and c

$W_{ac} = \langle 2\,0|\hat{x}\hat{y}|1\,1\rangle = \sqrt{2}\sqrt{1}$

$W_{bc}\ \langle 0\,2|\hat{x}\hat{y}|1\,1\rangle = \langle 0\,2|0\,2\rangle\sqrt{1}\sqrt{2}$

$W_{cb} = \langle 1\,1|\hat{x}\hat{y}|0\,2\rangle = \sqrt{1}\sqrt{2}$

$$W = \frac{\alpha\hbar}{2m\omega}\begin{bmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

find eigenvalues

$\det(W - \lambda I)$

$$\det(W - \lambda I) = \det\begin{pmatrix} -\lambda & \sqrt{2} & 0 \\ \sqrt{2} & -\lambda & \sqrt{2} \\ 0 & \sqrt{2} & -\lambda \end{pmatrix}$$

$$= -\lambda\det\begin{pmatrix} -\lambda & \sqrt{2} \\ \sqrt{2} & -\lambda \end{pmatrix} + \sqrt{2}\det\begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & -\lambda \end{pmatrix} + 0\,\cancel{\det\begin{pmatrix} \sqrt{2} & -\lambda \\ 0 & \sqrt{2} \end{pmatrix}}$$

$$-\lambda\left(\lambda^2 - 2\left(\frac{\hbar}{2m\omega}\right)^2\right) - \sqrt{2}\frac{\hbar}{2m\omega}\left(\frac{\sqrt{2}\hbar}{-2m\omega}\lambda\right) = 0$$

$$-\lambda^3 + 2\left(\frac{\hbar}{2m\omega}\right)^2\lambda + 2\frac{\hbar^2}{(2m\omega)^2}\lambda = 0$$

$\lambda = 0$ is solution

$$\cancel{-\lambda^3 + 2\left(\frac{\hbar}{2m\omega}\right)^2\lambda - 2\left(\frac{\hbar}{2m\omega}\right)^2} = 0$$

$$(-\lambda \quad)(\lambda \quad) = 0$$

$$-\lambda^2 + 4\left(\frac{\hbar}{2m\omega}\right)^2 = 0$$

$$\lambda = \pm \frac{2\hbar}{2m\omega}$$

Sorry, forgot a term, reinserting α

$$\lambda = \pm\frac{\alpha\hbar}{m\omega}$$

2,3

$\lambda = 0$

$$\boxed{\begin{matrix} E_{20} \\ E_{11} \\ E_{02} \end{matrix} = \begin{cases} \hbar\omega\frac{7}{2} + \alpha\hbar/m\omega \\ \hbar\omega\frac{7}{2} \\ \hbar\omega\frac{7}{2} - \hbar\alpha/m\omega \end{cases}}$$

$$E_{exact} = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)$$

$$\omega = \omega_0\left(\frac{7}{2} + \frac{\alpha\hbar}{m\omega^2}\right)$$

$$\boxed{E_{exact} = \hbar\omega\left(7/2 \quad \frac{\alpha\hbar}{m\omega^2}\right)}$$

exact energies and energy corrections match exactly!

$$\psi_a = \psi_{x=2}\psi_{y=0}$$

$$\psi_b = \psi_{x=1}\psi_{y=1}$$

$$\psi_c = \psi_{x=0}\psi_{y=2}$$

$$\psi_{x=2,y=2} = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{8}}\left[4\left(\sqrt{\frac{m\omega}{\hbar}}x\right)^2 - 2\right]e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_{x=1,y=1} = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}\frac{1}{\sqrt{2}}\left(2\sqrt{\frac{m\omega}{\hbar}}x\right)e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_{x=0,y=0} = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\Psi_a = \sqrt{\frac{m\omega}{\hbar\pi}}\frac{1}{\sqrt{8}}\left[4\frac{m\omega}{\hbar}x^2 - 2\right]e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

$$\Psi_b = \sqrt{\frac{m\omega}{\hbar\pi}}\frac{1}{2}\,2\frac{m\omega}{\hbar}xy\,e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

$$\Psi_c = \sqrt{\frac{m\omega}{\hbar\pi}}\frac{1}{\sqrt{8}}\left[4\frac{m\omega}{\hbar}\cdot y^2 - 2\right]e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

$$\Psi_2^0 = \alpha\Psi_a + \beta\Psi_b + \gamma\Psi_c$$

$$\frac{\alpha\hbar}{2m\omega}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = \lambda\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}$$

$$\sqrt{2}\frac{\alpha\hbar}{2m\omega}\beta = \lambda\beta$$

$$\cancel{\frac{\alpha\hbar}{2m\omega}\sqrt{2}(\alpha+\gamma) = \lambda(\alpha+\gamma)}$$

$\cancel{\text{for}}\ \lambda = 0$ $\quad \lambda\alpha = \sqrt{2}\beta\frac{\alpha\hbar}{2m\omega}$

$\beta = 0$

$(\alpha+\gamma) = 0$ $\quad \lambda\beta = \frac{\sqrt{2}}{2}\frac{\alpha\hbar}{m\omega}\left[\alpha+\gamma\right]$

$\alpha = -\gamma \quad \alpha = \frac{1}{\sqrt{2}}$ $\quad \lambda\gamma = \sqrt{2}\beta\frac{\alpha\hbar}{2m\omega}$

~~for $\lambda = \frac{\alpha\hbar}{m\omega}$~~

$$\cancel{\frac{\sqrt{2}}{2}\frac{\alpha\hbar}{m\omega}\beta = \frac{\alpha\hbar}{m\omega}\lambda(\alpha+\beta+\gamma)\frac{\alpha\hbar}{m\omega}}$$

$$\cancel{\beta = \alpha+\beta+\gamma}$$

$$\alpha+\gamma = 0$$

for $\lambda = 0$

$\alpha^2+\gamma^2 = 1 \qquad \alpha+\gamma = 0 \qquad \beta = 0$

$2\alpha^2 = 1 \qquad \alpha = -\gamma$

$\alpha = \sqrt{2}/2 \qquad \alpha = -\frac{1}{\sqrt{2}} \qquad \gamma = \frac{1}{\sqrt{2}}$

$$\boxed{\Psi_2^0 = \frac{1}{\sqrt{2}}\left(-\Psi_a + \Psi_c\right)}$$

for $\lambda = \frac{\alpha\hbar}{m\omega}$ $\qquad \cancel{\frac{\alpha\hbar}{m\omega}}\alpha = \frac{\sqrt{2}}{2}\cancel{\frac{\alpha\hbar}{m\omega}}\beta$

$$\alpha = \frac{\sqrt{2}}{2}\beta$$

$$\frac{\alpha\hbar}{m\omega}\beta = \frac{\alpha\hbar}{2m\omega}\left(\sqrt{2}\alpha + \sqrt{2}\gamma\right)$$

$$\beta = \frac{\sqrt{2}}{2}\alpha + \frac{\sqrt{2}}{2}\gamma$$

$$\beta = \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\beta + \frac{\sqrt{2}}{2}\gamma$$

$$\beta = \frac{1}{2}\beta + \frac{\sqrt{2}}{2}\gamma$$

$$\frac{\sqrt{2}}{2}\gamma = \frac{1}{2}\beta$$

$$\gamma = \frac{\sqrt{2}}{2}\beta$$

$\cancel{\psi_2 =}$

$$\cancel{\frac{\alpha\hbar}{2m\omega}\sqrt{2}\beta = \frac{\alpha\hbar}{m\omega}\gamma}$$

$$\cancel{\frac{\sqrt{2}}{2}\beta = \gamma}$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

$$\frac{1}{2}\beta^2 + \beta^2 + \frac{1}{2}\beta^2 = 1$$

$$\boxed{\psi_2^0 = \frac{1}{4}\psi_a + \frac{\sqrt{2}}{2}\psi_b + \frac{1}{4}\psi_c}$$

$$2\beta^2 = 1$$

$$\beta = \frac{\sqrt{2}}{2}$$

for $\lambda = -\frac{\alpha\hbar}{m\omega}$

100

$$-\frac{\alpha\hbar}{m\omega}\alpha = \frac{\sqrt{2}}{2}\frac{\alpha\hbar}{m\omega}\beta$$

$$\alpha = -\frac{\sqrt{2}}{2}\beta$$

$$-\frac{\alpha\hbar}{m\omega}\beta = \frac{\sqrt{2}}{2}\frac{\alpha\hbar}{m\omega}\left(\alpha + \gamma\right)$$

$$-\beta = \frac{\sqrt{2}}{2}\alpha + \frac{\sqrt{2}}{2}\gamma$$

$$-\beta = \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}\beta\right) + \frac{\sqrt{2}}{2}\gamma$$

$$-\frac{1}{2}\beta = \frac{\sqrt{2}}{2}\gamma$$

$$\gamma = -\frac{\sqrt{3}}{2}\beta$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

$$\frac{1}{2}\beta^2 + \beta^2 + \frac{1}{2}\beta^2 = 1$$

$$2\beta^2 = 1$$

$$\beta = \frac{\sqrt{2}}{2}$$

$$\boxed{\Psi_2^0 = -\frac{1}{4}\Psi_a + \frac{\sqrt{2}}{2}\Psi_b + \frac{1}{4}\Psi_c}$$

$$\frac{1}{4}$$

$$\Psi_2^{0} = \frac{1}{4\sqrt{8}}\sqrt{\frac{m\omega}{\hbar\pi}}\left[\frac{4m\omega}{\hbar}x^2 - 2\right]e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$
$$+ \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\hbar\pi}}\frac{2m\omega}{\hbar}xy\,e^{-\frac{m\omega}{2\hbar}(x^2+y^2)} \quad \text{for } \lambda = \frac{2\hbar}{m\omega}$$
$$+ \frac{1}{4\sqrt{8}}\sqrt{\frac{m\omega}{\hbar\pi}}\left[\frac{4m\omega}{\hbar}y^2 - 2\right]e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$

$$\Psi_2^{0} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\frac{1}{\sqrt{8}}\sqrt{\frac{m\omega}{\hbar\pi}}e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}\left[\frac{4m\omega}{\hbar}x^2 - 2\left(\frac{4m\omega}{\hbar}y^2 - 2\right)\right]\right]$$

for $\lambda = 0$

$$= -\frac{1}{4\sqrt{8}}\sqrt{\frac{m\omega}{\hbar\pi}}\left[\frac{4m\omega}{\hbar}x^2 - 2\right]e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$
$$+ \frac{1}{\sqrt{2}}\sqrt{\frac{m\omega}{\hbar\pi}}\frac{2m\omega}{\hbar}xy\,e^{-\frac{m\omega}{2\hbar}(x^2+y^2)} \quad \text{for } \lambda = \frac{-2\hbar}{m\omega}$$
$$-\frac{1}{4\sqrt{8}}\sqrt{\frac{m\omega}{\hbar\pi}}\left[\frac{4m\omega}{\hbar}y^2 - 2\right]e^{-\frac{m\omega}{2\hbar}(x^2+y^2)}$$