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19 September 2022
Dr. Dahnovsky
Homework 1

1	2	3	4	5
100	100	110	110	80

Total 96

1. $U(x) = U_0 \left|\frac{x}{a}\right|^\nu$ $\quad U_0 > 0, \quad \nu > 0$

a) Determine energy levels in the semi-classical approximation

(25.12) $\oint p\,dx = 2\pi\hbar(n+\tfrac{1}{2})$, $\quad p(x) = \sqrt{2\mu\,(E-U(x))}$

$p(x) = \sqrt{2\mu\left(E - U_0\left|\frac{x}{a}\right|^\nu\right)}$

$p(x) = \sqrt{2\mu}\left(E - U_0\left|\frac{x}{a}\right|^\nu\right)^{1/2}$

$\oint \sqrt{2\mu}\left(E - U_0\left|\frac{x}{a}\right|^\nu\right)^{1/2} dx = 2\pi\hbar(n+\tfrac{1}{2})$

for $x > 0$, $U(x) = U_0\left(\frac{x}{a}\right)^\nu$

at $x_t \rightarrow E = U(x_t)$, $E = U_0\left(\frac{x_t}{a}\right)^\nu$

$x_t = \left(\frac{E}{U_0}\right)^{1/\nu} a$

Substitute $t = \frac{x}{x_t} \rightarrow \frac{x}{a}\left(\frac{U_0}{E}\right)^{1/\nu}$

$t^\nu = \left(\frac{x}{a}\right)^\nu \frac{U_0}{E}$

$\frac{dt}{dx} = \frac{1}{x_t}$

$dt = \frac{dx}{x_t}$

$dx = x_t\,dt$

$$\cancel{\int_{-x_t}^{x_t}} \oint p\,dx = 2\int_{-x_t}^{x_t} p(x)\,dx = 4\int_0^{x_t} p(x\,dx) = 4\int_0^1 p(t)\,dt$$

$$4\int_0^1 \sqrt{2m\left(E - V_0\left(\frac{x}{a}\right)^{\nu}\right)}\,dt\,x_t$$

Substitute in terms of t

$$4\int_0^1 \sqrt{2m}\sqrt{E - E\left(\frac{V_0}{E}\left(\frac{x}{a}\right)^{\nu}\right)}\,dt\,x_t$$

$$4\int_0^1 \sqrt{2m}\sqrt{E}\left(\sqrt{1-t^{\nu}}\right)dt\,x_t$$

$$4x_t\sqrt{2\mu E}\underbrace{\int_0^1 \sqrt{1-t^{\nu}}\,d\nu}_{C_\nu \text{ - Sure constant}} = 2\pi\hbar(n+1/2)$$

$$4x_t\sqrt{2\mu E} = 2\pi\hbar(n+1/2)$$

$$16x_t^2(2\mu E) = 2\pi\hbar(n+1/2)$$

$$x_t = \left(\frac{E}{V_0}\right)^{1/\nu} a$$

$$x_t^2 = \frac{E^{2/\nu}}{V_0^{2/\nu}}a^2$$

$$32\mu E\left(\frac{E^{2/\nu}}{V^{2/\nu}}a^2\right) = 4\pi^2\hbar^2(n+1/2)^2$$

$$\frac{E^{2+\nu/\nu}}{V^{2/\nu}}a^2 = \frac{\pi^2\hbar^2(n+1/2)^2}{8\mu}$$

$$E^{2+\nu/\nu} = \frac{V^{2/\nu}\pi^2\hbar^2(n+1/2)^2}{8\mu a^2}$$

$$\boxed{E_n = \left(\frac{V^{2/\nu}\pi^2\hbar^2(n+1/2)^2}{8\mu a^2}\right)^{\nu/2+\nu}}$$

b) $$\boxed{E_{n+1} - E_n = \left[\frac{V^{2/\nu}\pi^2\hbar^2}{8\mu a^2}\right]^{\nu/2+\nu}\left[(n+3/2)^2 - (n+1/2)^2\right]^{\nu/2+\nu}}$$

c) Density of states

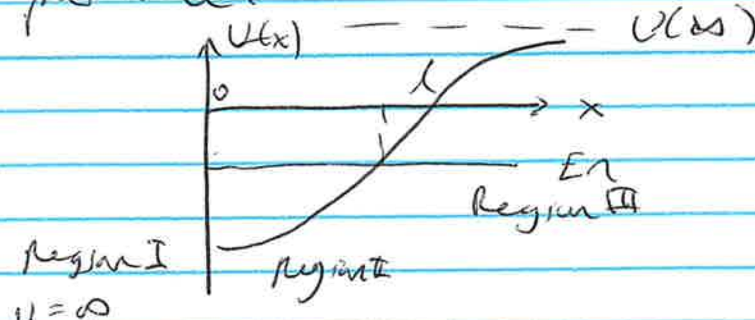
$$\rho = \frac{dn}{dE} \approx \frac{\Delta n}{\Delta E} \sim \frac{1}{E_{n+1} - E_n}$$

$$\rho_n \sim \frac{1}{E_{n+1} - E_n}$$

$$\boxed{\rho_n \cancel{\alpha} \sim \left[\frac{U^{2/\nu} \pi^2 \hbar^2}{8 \mu a^2}\right]^{-\nu/2+\nu} \left[(n+3/2)^2 - (n+1/2)^2\right]^{-\nu/2+\nu}}$$

100

2. Determine quantization rule for energy levels and find corresponding wavefunction for particle.



Region I: $\Psi_I = 0$, wave function not permitted

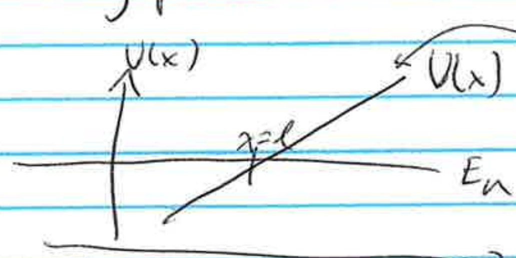
Region II: Classically allowed $\quad \Psi_{II} = \dfrac{A}{\sqrt{p}} \sin\left(\int_{\ell}^{x} k(x')\,dx' + \alpha\right)$ $\quad$ 0 ? x or ∞

$0 < x < \ell$

Region III: Classically forbidden: $\quad \Psi_{III} = \dfrac{C}{\sqrt{|p|}} \exp\left(-\int k(x')\,dx'\right)$

$x > \ell$

turning point: $x = \ell$



linear approximation for potential:

$U(x) \sim E - F(\ell)(x-\ell) + \ldots$

$$\Psi_p(x) = \begin{cases} \dfrac{1}{\sqrt{p(x)}}\left[B\exp\left(\int_x^{\ell} k(x')\,dx'\right) + C\exp\left(-\int_x^{\ell} k(x')\,dx'\right)\right], & x < \ell \\ \dfrac{1}{\sqrt{|p(x)|}} D\exp\left(-K(x')\,dx'\right), & x > \ell \end{cases}$$

$$\cancel{U(x)} + \frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\Psi_p + U\Psi_p = E\Psi_p$$

$$+\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\Psi_p + (\cancel{E} - F(\ell)(x-\ell))\Psi_p = \cancel{E\Psi_p}$$

$$\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2}\Psi_p = F_\ell(x-\ell)\Psi_p$$

$$\frac{d^2}{dx^2}\Psi_p = \frac{2\mu}{\hbar^2}F_\ell(x-\ell)\Psi_p$$

solution is Airy function with $\dfrac{d^2}{dx^2}\Psi_p = 2\alpha(x-\ell)\Psi_p$

where $\alpha = \left(\dfrac{2\mu F_\ell}{\hbar^2}\right)^{1/3}$

$$\Psi_p = \cancel{A}^{aA} i\, z + \cancel{B}^{bB} i\, z$$

$$\Psi_p = {}^{a}\cancel{A}{}^{A} i\,(\alpha(x-\ell)) + {}^{b}\cancel{B}{}^{\beta} i\,(\alpha(x-\ell)) \quad \text{for } \alpha(x-\ell) \gg 0$$

$$p(x) = \sqrt{2\mu(-F_\ell(x-\ell))}$$

$$\alpha = \left(\frac{2\mu F_\ell}{\hbar^2}\right)^{1/3}$$

$$p(x) \sim \sqrt{\hbar^2\alpha^3(-(x-\ell))} \qquad x<\ell$$

$$p(x) \sim \hbar\, \alpha^{3/2}\sqrt{-y} \qquad \text{substitute } y = x-\ell$$

$y<0$ in region

$$p(x) = \hbar\alpha^{3/2}\sqrt{y}$$

$$\int_{y}^{0} \cancel{\hbar} p(x')\,dx' = \hbar\alpha^{3/2}\int_y^0 y^{1/2}\,dy = \frac{2}{3}\hbar\alpha^{3/2}\, y^{3/2}\Big|_y^0$$

$$= \frac{2}{3}\hbar(-\alpha y)^{3/2}$$

for $y<0$ $(x<\ell)$

$$\Psi_p(x) = \frac{1}{\sqrt{p(x)}}\left[B\exp\left(\frac{i}{\hbar}\int_{y}^{0} p(x')\,dx'\right) + C\exp\left(-\frac{i}{\hbar}\int_y^0 p(x')\,dx'\right)\right]$$

(1) $$\Psi_{p\, x<\ell} = \frac{1}{\sqrt{\hbar}\,\alpha^{3/4}(-y)^{1/4}}\left[B\exp\left(\frac{i}{\hbar}\frac{2}{3}\hbar(-\alpha y)^{3/2}\right) + C\exp\left(-\frac{i}{\hbar}\frac{2}{3}\hbar(-\alpha y)^{3/2}\right)\right]$$

Use Ψ_p to find Ψ_{III}

$$\int_0^x p(x')\,dx', \quad p(x) \sim \sqrt{2\mu(F_\ell(x-\ell)}$$

$$p(x) \sim \sqrt{\hbar^2\alpha^3(-(x-\ell))}$$

$$p(x) \sim \hbar\alpha^{3/2}\sqrt{y} \qquad y>0 \quad (x-\ell>0)$$

$$\int_0^x |p(x')|\,dx' = \int_0^x \hbar\alpha^{3/2}\sqrt{x'}\,dx' = \frac{2}{3}\hbar(\alpha y)^{3/2}$$

for $x>\ell$

$$\Psi_p = \frac{1}{\sqrt{|p(x)|}}\, D\exp\left(-\frac{1}{\hbar}\int_0^x |p(x')|\,dx'\right)$$

$$\Psi_p(x>\ell) = \frac{D}{\sqrt{\hbar}\,\alpha^{3/4}(-x)^{3/4}}\exp\left(-\frac{2}{3}(\alpha y)^{3/2}\right)$$

$$\Psi_p(x) = \frac{a}{2\sqrt{\pi}(\alpha y)^{1/4}} e^{-2/3(\alpha x)^{3/2}} + \frac{b}{\sqrt{\pi}(\alpha y)^{1/4}} e^{2/3(\alpha x)^{3/2}} = \frac{D}{\sqrt{\hbar}\,\alpha^{3/4}(-x)^{1/4}} e^{-2/3(\alpha x)^{3/2}}$$

b must be 0

$$\frac{D}{\sqrt{\hbar}\,\alpha^{3/4}(-y)^{1/4}}\cancel{\exp\left(-\frac{2}{3}(\alpha y)^{3/2}\right)} = \frac{a}{2\sqrt{\pi}(\alpha y)^{1/4}}\cancel{\exp\left(-\frac{2(\alpha x)^{3/2}}{3}\right)}$$

$$\frac{D}{\sqrt{\hbar}\,\alpha^{3/4}\cancel{(-y)^{1/4}}} = \frac{a}{2\sqrt{\pi}\,\cancel{(\alpha y)^{1/4}}}$$

$$\frac{2\sqrt{\pi}\,D}{\sqrt{\hbar}\,(\alpha^{2 3/4})} = \frac{a}{\cancel{\alpha^{1/4}}}$$

$$\boxed{a = 2\sqrt{\frac{\pi}{\hbar\alpha}}\,D}$$

for $x < \lambda$

$$\psi_p(x<\lambda) = \frac{a}{\sqrt{\pi}(-\alpha y)^{1/4}}\sin\left(\frac{2}{3}(-\alpha y)^{3/2} + \frac{\pi}{4}\right) \quad (\text{Airy})$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\psi_p = \frac{a}{2i\sqrt{\pi}(-\alpha y)^{1/4}}\left[e^{i\pi/4}e^{\frac{2i}{3}(-\alpha y)^{3/2}} - e^{(-i\pi/4)}e^{(\frac{2i}{3}(-\alpha y)^{3/2})}\right]$$

Relate this result to ① to find B and C

$$\frac{1}{\sqrt{\hbar}\,\alpha^{3/4}(-y)^{1/4}}\left[B\exp\left(i\frac{2}{3}(-\alpha y)^{3/2}\right) + C\exp\left(-i\frac{2}{3}(-\alpha y)^{3/2}\right)\right]$$

$$= \frac{a}{2i\sqrt{\pi}(-\alpha y)^{1/4}}\left[e^{i\pi/4}e^{\frac{2i}{3}(-\alpha y)^{3/2}} - e^{-(i\pi/4)}e^{-\frac{2i}{3}(-\alpha y)^{3/2}}\right]$$

Set equal corresponding exponential terms

$$\frac{a}{2i\sqrt{\pi}\,\cancel{(-\alpha y)^{1/4}}}e^{i\pi/4}\cancel{e^{2/3\,i(-\alpha y)^{3/2}}} = \frac{B}{\sqrt{\hbar}\,\alpha^{2 3/4}\cancel{(-y)^{1/4}}}\cancel{e^{2/3\,i(-\alpha y)^{3/2}}}$$

$$\frac{a}{2i\sqrt{\pi}}e^{i\pi/4} = \frac{B}{\sqrt{\hbar}\sqrt{\alpha}}$$

$$B = \frac{a}{2i}\sqrt{\frac{\hbar\alpha}{\pi}}\,e^{i\pi/4}$$

$$B = \left(\cancel{2}\sqrt{\frac{\pi}{\hbar\alpha}}\,D\right)\frac{1}{\cancel{2}i}\sqrt{\frac{\hbar\alpha}{\pi}}\,e^{i\pi/4}$$

$$\boxed{B = \frac{D}{i}e^{i\pi/4}}$$

$$-\frac{a}{2i\sqrt{\pi}\,\cancel{(-\alpha y)^{1/4}}}e^{-i\pi/4}\cancel{e^{-\frac{2}{3}i(-\alpha y)^{3/2}}} = \frac{C\cancel{e^{-i2/3(-\alpha y)^{3/2}}}}{\sqrt{\hbar}\,\alpha^{2 3/4}\cancel{(-y)^{1/4}}}$$

$$-\frac{a}{2i\sqrt{\pi}}e^{-i\pi/4} = \frac{C}{\sqrt{\hbar\alpha}}$$

$$\cancel{\frac{C}{\sqrt{\hbar\alpha}} = e^{-i\pi/4} - \frac{(2\sqrt{\frac{\pi}{\hbar\alpha}}\,D)}{2i\sqrt{\pi}}}$$

$$\boxed{C = e^{-i\pi/4}\left(-\frac{D}{i}\right)}$$

~~Finally,~~

$$\cancel{\psi_p(x) = \frac{1}{\sqrt{p(x)}}\left[B e^{i/\hbar\int_{\infty}^{0} p(x')dx'} + C e^{-i/\hbar\int_x^0 p(x')dx'}\right]}$$

from before:

$$\underset{(x<l)}{\psi(x)_p} = \frac{1}{\sqrt{p(x)}}\left[B e^{i/\hbar\int_x^0 p(x')dx'} + C e^{-i/\hbar\int_x^0 p(x')dx'}\right]$$

$$B = \frac{D}{i}e^{i\pi/4}$$

$$C = e^{-i\pi/4}\left(-\frac{D}{i}\right)$$

$$\psi_p(x) = \frac{1}{\sqrt{p(x)}}\left[\frac{D}{i}e^{i\pi/4}e^{i/\hbar\int_x^l p(x')dx'} - \frac{D}{i}e^{-i\pi/4}e^{i/\hbar\int_x^l p(x')dx'}\right]$$

$$\underset{(x<l)}{\psi_p(x)} = \frac{2D}{2\sqrt{p(x)}}\sin\left(\frac{1}{\hbar}\int_x^l p(x')dx' + \pi/4\right)$$

$$\underset{(x>l)}{\psi_p(x)} = \frac{1}{\sqrt{p(x)}}\left[D e^{-i/\hbar\int_l^x p(x')dx'}\right]$$

$$\underset{(x>l)}{\psi_p(x)} = \frac{D}{\sqrt{p(x)}}\exp\left[-\frac{1}{\hbar}\int_l^x p(x')dx'\right]$$

$$\boxed{\psi_p(x) = \begin{cases}\frac{2D}{2\sqrt{p(x)}}\sin\left(\frac{1}{\hbar}\int_x^l p(x')dx' + \pi/4\right), & x<l\\ \frac{D}{\sqrt{p(x)}}\exp\left(-\frac{1}{\hbar}\int_l^x p(x')dx'\right), & x>l\end{cases}}$$

To find quantization, $\psi(x)=0$ for $x<0$

$$\sin\left(\frac{1}{\hbar}\int_x^l p(x')dx' + \pi/4\right) = 0$$

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$\sin\theta = 0$ for $\theta = n\pi$ $(n = 0, 1, 2, \ldots)$

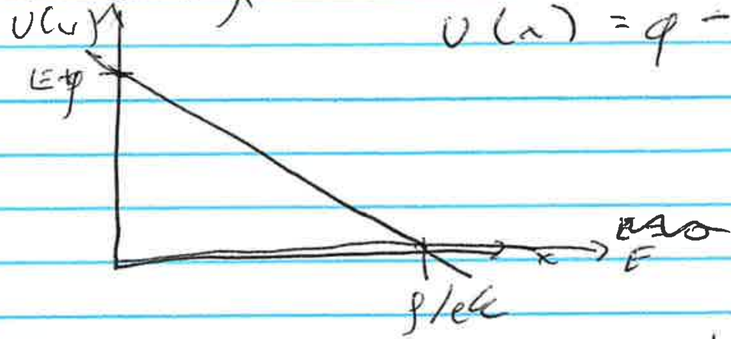
$$\frac{1}{\hbar}\int_0^l p(x')dx' + \pi/4 = n\pi$$

$$\boxed{\int_0^l p(x')dx' = \pi\hbar(n - 1/4)}$$ quantization rule

3. Determine validity of Equation 26.14

$U(x) - E = \varphi - e\mathcal{E}x$

$U(x) = \varphi - e\mathcal{E}x + E$



Semi-classical approximation valid from

$|x - x_0| \gg \frac{1}{2}\left[\frac{\hbar^2}{\mu\left|\frac{dU}{dx}\right|}\right]^{1/3}$ (Eq. 24.8)

$x_0 = \varphi/e\mathcal{E}$

$\frac{dU}{dx} = -e\mathcal{E}$

$|x - x_0| \gg \frac{1}{2}\left[\frac{\hbar^2}{\mu e\mathcal{E}}\right]^{1/3}$

$\boxed{|x - \varphi/e\mathcal{E}| \gg \frac{1}{2}\left[\frac{\hbar^2}{\mu e\mathcal{E}}\right]^{1/3}}$ Condition of validity

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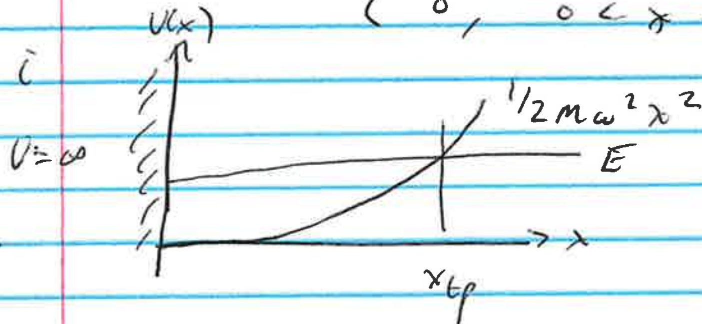
4. Use semi-classical approximation to find energy levels in the following potentials:

i) $U(x) = \begin{cases} \infty, & x < 0 \\ \frac{1}{2} m\omega^2 x^2, & x > 0 \end{cases}$

ii) $U(x) = \begin{cases} \infty, & x < 0,\ x > a \\ 0, & 0 < x < a \end{cases}$

i



$$E = U(x_t)$$
$$E = \tfrac{1}{2} m\omega^2 x_t^2$$
$$x_{tp} = \sqrt{\frac{2E}{m\omega^2}}$$
$$x_t^2 = \frac{2E}{m\omega^2}$$

From problem 2:

$$\int_0^{x_t} p(x')\,dx' = (n - \tfrac{1}{4})\pi\hbar$$

$$p(x) = \sqrt{2\mu(E - \tfrac{1}{2} m\omega^2 x^2)}$$

$$p(x) = \sqrt{\cancel{2}\mu\left(\tfrac{1}{\cancel{2}}\mu\omega^2 x_t^2 - \cancel{\tfrac{1}{2}}\mu\omega^2 x^2\right)}$$

$$p(x) = \sqrt{\mu^2\omega^2(x_t^2 - x^2)}$$

$$p(x) = \mu\omega\sqrt{x_t^2 - x^2}$$

$$\int_0^{x_t} p(x)\,dx = \int_0^{x_t} \mu\omega\sqrt{x_t^2 - x^2}\,dx$$

$$= \mu\omega\int_0^{x_t}\sqrt{x_t^2 - x^2}\,dx$$

$$= \mu\omega\left[\frac{1}{2}\left(x_t^2 \arcsin\left(\frac{x}{x_t}\right) + \frac{x}{|x_t|}\sqrt{-(x-x_t)(x+x_t)}\right)\right]_0^{x_t}$$

$$= \frac{\mu\omega}{2}\left[x_t^2\arcsin(1) + \frac{x}{\cancel{x_t}}\sqrt{0} - x_t\arcsin(0) - \frac{\cancel{(0)}\sqrt{x_t^2}}{\cancel{x_t}}\right]_0$$

$$= \frac{\mu\omega}{2}\left[x_t^2\left(\frac{\pi}{2}\right) - \cancel{x_t(0)}\right]$$

$$\int_0^{x_t} p(x)\,dx = \mu\omega\left(\frac{x_t^2\pi}{4}\right) = \frac{\cancel{\mu}\omega\pi}{4}\left(\frac{2E}{\cancel{\mu}\omega^{\cancel{2}}}\right) = \frac{2\pi E}{4\omega} = \frac{\pi E}{2\omega}$$

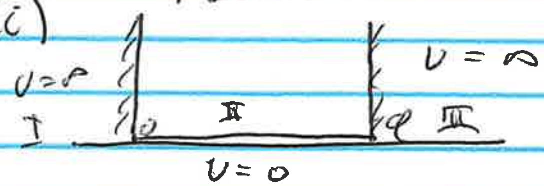
$$\int_0^{x_t} p(x)\,dx = \frac{\pi E}{2\omega}$$

$$\int_0^{x_t} = (n - 1/4)\pi\hbar$$

$$\frac{\cancel{\pi} E_n}{2\omega} = (n - 1/4)\cancel{\pi}\hbar$$

$$\boxed{E_n = 2\omega\hbar(n - 1/4)}$$

ii)



$$\psi_I = \psi_{III} = 0$$

$$\psi_{II}(x) = \frac{1}{\sqrt{p(x)}}\left[c_1 \sin\left(\frac{1}{\hbar}\int_0^x p(x')\,dx'\right) + c_2 \cos\left(\int_0^x \frac{1}{\hbar} p(x')\,dx'\right)\right]$$

$$\psi_{II}(0) = 0$$

$$\frac{1}{\sqrt{p(x)}}\left[c_1 \sin(0) + c_2 \cos(0)\right] = 0$$

$$c_1 \cancel{\sin(0)}^{\,0} + c_2 \cos(0) = 0$$

$$c_2 = 0$$

$$\psi_{II} = \frac{1}{\sqrt{p(x)}}\left[c_1 \sin\left(\frac{1}{\hbar}\int_0^x p(x')\,dx'\right)\right]$$

$$x = \varphi,\ \psi_{II} = 0$$

$$\frac{1}{\sqrt{p(\varphi)}}\left[c_1 \sin\left(\frac{1}{\hbar}\int_0^\varphi p(x')\,dx'\right)\right] = 0$$

$$c_1 \sin\left(\frac{1}{\hbar}\int_0^\varphi p(x')\,dx'\right) = 0$$

$$\frac{1}{\hbar}\int_0^\varphi p(x')\,dx' = n\pi$$

$$\int_0^\varphi p(x')\,dx' = n\pi\hbar$$

$$p = \sqrt{2\mu(E - U)} \qquad U = 0$$

$$\int_0^\varphi \sqrt{2\mu E_n}\,dx = n\pi\hbar$$

$$\sqrt{2\mu E_n}\,\varphi = n\pi\hbar$$

$E_n = \frac{n\pi\hbar}{2\mu\varphi}$ $\quad 2\mu E_n \varphi^2 = n^2\pi^2\hbar^2$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2\mu\varphi^2}}$$

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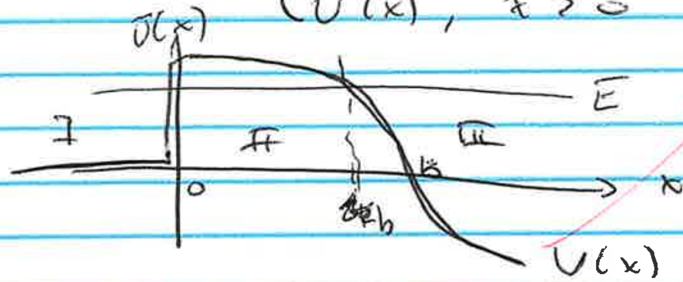
5. Find pre-exponential factor for transparency coefficient for the potential energy:

$$U(x) = \begin{cases} 0, & x < 0 \\ \tilde{U}(x), & x > 0 \end{cases}$$



in semiclassical approximation.

the arbitrary potential

$$\frac{C}{\sqrt{p}} e^{\frac{i}{\hbar}\int p\,dx - i\frac{5}{4}}$$

Region I: $\psi_I = Ae^{ik_0x} + Be^{-ik_0x}$ – transmitted and reflected

Region III: $\psi_{III} = Ce^{ik_2x}$ – only outgoing particles

$$D = \left|\frac{C}{A}\right|^2$$

$$p = \hbar k_0$$

$$\psi_{II}(x) = \frac{1}{\sqrt{K(x)}}\left[\alpha \exp\left(\int_0^x K(y)\,dy\right) + \beta \exp\left(-\int_0^x K(y)\,dy\right)\right]$$

Set ψ's and $\frac{\partial}{\partial x}\psi$'s equal at 0

$$\psi_I = \psi_{II}$$

$$Ae^{ik_0x} + Be^{-ik_0x} = \frac{1}{\sqrt{K(x)}}\left[\alpha \exp\left(\int_0^x K(y)\,dy\right) + \beta \exp\left(-\int_0^x K(y)\,dy\right)\right]$$

$$Ae^{0} + Be^{-0} = \frac{1}{\sqrt{K(0)}}\left[\alpha \exp(0) + \beta \exp(-0)\right]$$

$$A + B = \frac{1}{\sqrt{K(0)}}(\alpha + \beta)$$

$$K(0) = a = \frac{1}{\hbar}\sqrt{2\mu(U(0) - E)}$$

$$\boxed{\sqrt{a}(A + B) = \alpha + \beta}$$

assume $\frac{\partial}{\partial x}\frac{1}{\sqrt{K(x)}} \sim 0$

$$Aik_0e^{ik_0x} - Bik_0e^{-ik_0x} = \frac{1}{\sqrt{K(x)}}\left[K(x)\alpha\exp\left(\int_0^x K(y)\,dy\right) - K(x)\beta\exp\left(-\int_0^x K(y)\,dy\right)\right]$$

$$Aik_0 - Bik_0 = \frac{1}{\sqrt{a}}a(\alpha - \beta)$$

$$\boxed{ik_0(A - B) = \sqrt{a}(\alpha - \beta)}$$

at $x = \cancel{a}\, b$

$$\frac{1}{\sqrt{k(x)}}\left(\alpha \exp \int_0^x k(y)\,dy + \beta \exp - \int_0^x k(y)\,dy\right) = C e^{ik_0 x}$$

$$\frac{1}{\sqrt{k(b)}}\left[\alpha \exp \int_0^b k(y)\,dy + \beta \exp - \int_0^x k(y)\,dy\right) = C e^{ik_0 b}$$

$$b = k(b) = \frac{1}{\hbar}\sqrt{2\mu(U(b)-E)}$$

$$\gamma = \int_0^b k(y)\,dy$$

x-part

$$\boxed{\frac{1}{\sqrt{b}}\left[\alpha \exp(\gamma) + \beta \exp(-\gamma)\right] = C e^{ik_0 b}}$$

Substitution

$$\alpha e^{\gamma} + \beta e^{-\gamma} = \sqrt{b}\, C e^{ik_0 b}$$

$$C i k_0 e^{ik_0 x} = \frac{1}{\sqrt{k(x)}}\left[k(x)\alpha \exp\left(\int_0^b k(x)\,dy\right) - k(x)\beta \exp\left(-\int_0^b k(y)\,dy\right)\right]$$

$$C i k_0 e^{ik_0 b} = \frac{k(b)}{\sqrt{k(b)}}\left[\alpha \exp(\gamma) - \beta \exp(-\gamma)\right]$$

$$\boxed{C i k_0 e^{ik_0 b} = \sqrt{b}\left[\alpha \exp(\gamma) - \beta \exp(-\gamma)\right]}$$

$$\alpha \exp(\gamma) - \beta \exp(-\gamma) = \frac{C}{\sqrt{b}} i k_0 e^{ik_0 b}$$

$$\alpha \exp(\gamma) + \beta \exp(-\gamma) = C\sqrt{b} e^{ik_0 b}$$

$$2\alpha \exp(\gamma) = C e^{ik_0 b}\left(\frac{ik_0}{\sqrt{b}} + \sqrt{b}\right)$$

$$\boxed{\alpha = \frac{1}{2}\left[C e^{ik_0 b - \gamma}\right]\left[\frac{ik_0}{\sqrt{b}} + \sqrt{b}\right]}$$

$$2\beta \exp(-\gamma) = C\sqrt{b} e^{ik_0 b} - \frac{C}{\sqrt{b}} i k_0 e^{ik_0 b}$$

$$2\beta \exp(-\gamma) = \left(\sqrt{b} - \frac{ik_0}{\sqrt{b}}\right) C e^{ik_0 b}$$

$$\boxed{\beta = \frac{1}{2}\left[\sqrt{b} - \frac{ik_0}{\sqrt{b}}\right] C e^{ik_0 b + \gamma}}$$

wide barrier

$$\gamma = \bar{\kappa} l >> 1$$

$$\alpha << \beta$$

From before:

$$A + B = \frac{\alpha + \beta}{\sqrt{a}}$$

$$A - B = \frac{\sqrt{a}}{ik_0}(\alpha - \beta)$$

$$2A = \frac{\alpha + \beta}{\sqrt{a}} + \frac{\sqrt{a}}{ik_0}(\alpha - \beta)$$

$$A = \frac{1}{2}\left[\frac{\alpha + \beta}{\sqrt{a}} + \frac{\sqrt{a}}{ik_0}(\alpha - \beta)\right]$$

$$C = \frac{2\beta}{\left[\sqrt{b} - \frac{ik_0}{\sqrt{b}}\right]} e^{-ik_0 b - \gamma}$$

$$\frac{C}{A} = \frac{\frac{2\beta}{\left[\sqrt{b} - \frac{ik_0}{\sqrt{b}}\right]} e^{-ik_0 b - \gamma}}{\frac{1}{2}\left[\frac{\alpha + \beta}{\sqrt{a}} + \frac{\sqrt{a}}{ik_0}(\alpha - \beta)\right]}$$

$$\frac{C}{A} = \frac{4\beta}{\left[\sqrt{b} - \frac{ik_0}{\sqrt{b}}\right]\left[\frac{\alpha+\beta}{\sqrt{a}} + \frac{\sqrt{a}}{ik_0}(\alpha - \beta)\right]} e^{-ik_0 b - \gamma}$$

$\beta >> \alpha$

$$\frac{C}{A} = \frac{4\beta}{\left[\sqrt{b} - \frac{ik_0}{\sqrt{b}}\right]\left[\frac{\beta}{\sqrt{a}} - \frac{\sqrt{a}}{ik_0}\beta\right]} e^{-ik_0 b - \gamma}$$

$$\frac{C}{A} = \frac{4}{\left[\sqrt{b} - \frac{ik_0}{\sqrt{b}}\right]\left[\frac{1}{\sqrt{a}} - \frac{\sqrt{a}}{ik_0}\right]} e^{-ik_0 b - \gamma}$$

$$T = \left|\frac{C}{A}\right|^2 \frac{k_2}{k_0} = \frac{4e^{-ik_0 b - \gamma}}{\left[\sqrt{b} - \frac{ik_0}{\sqrt{b}}\right]\left[\frac{1}{\sqrt{a}} - \frac{\sqrt{a}}{ik_0}\right]} \frac{4e^{ik_0 b - \gamma}}{\left[\sqrt{b} + \frac{ik_0}{\sqrt{b}}\right]\left[\frac{1}{\sqrt{a}} + \frac{\sqrt{a}}{ik_0}\right]}$$

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$$D = \frac{16 e^{-ik_0 b - \gamma + ik_0 b - \gamma}}{\left[\sqrt{b} - \frac{ik_{02}}{\sqrt{b}}\right]\left[\frac{1}{\sqrt{a}} - \frac{\sqrt{a}}{ik_0}\right]\left[\sqrt{b} + \frac{ik_{02}}{\sqrt{b}}\right]\left[\frac{1}{\sqrt{a}} + \frac{\sqrt{a}}{ik_0}\right]}$$

$$D = \frac{16 e^{-2\gamma}}{\left(b + ik_{02} - ik_{02} - \frac{i^2 k_{02}^2}{b}\right)\left(\frac{1}{a} + \frac{1}{ik_0} - \frac{1}{ik_0} - \frac{a}{i^2 k_0^2}\right)}$$

$$D = \frac{16 e^{-2\gamma}}{\left(b + \frac{k_{02}^2}{b}\right)\left(\frac{1}{a} + \frac{a}{k_0^2}\right)}$$

$$D = \frac{16 e^{-2\gamma} k_2}{\left(\frac{b}{a} + \frac{ab}{k_0^2} + \frac{k_0^2}{ab} + \frac{a}{b}\right)}$$

$$\boxed{D \sim \exp\left(-\frac{2}{\hbar}\int_0^b \sqrt{2m(\tilde{u}(x) - E}\, dx\right)} \quad \text{(equation 26.8)}$$