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 PHYS5510

8.1 Let the Fermi distribution at low temperatures be represented by a broken line, as shown in Figure 8.13, the line being tangential to the actual curve at $\epsilon = \mu$. Show that this approximate representation yields a “correct” result for the low-temperature specific heat of the Fermi gas, except that the numerical factor turns out to be smaller by a factor of $4/\pi^2$. Discuss, in a qualitative manner, the origin of this numerical discrepancy.

Find the equation of the line for $n(x)$ using figure 8.13.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad (1)$$

$$m = \frac{1 - 0}{\xi - 2 - (\xi + 2)} \quad (2)$$

$$m = -\frac{1}{4} \quad (3)$$

$$y - y_1 = m(x - x_1) \quad (4)$$

$$y - \frac{1}{2} = -\frac{1}{4}(x - \xi) \quad (5)$$

$$y = \frac{-x + \xi + 2}{4} \quad (6)$$

Using this linear approximation for a segment of the graph, define $n(x)$.

$$n(x) = \begin{cases} 1 & x < \xi \\ \frac{-x + \xi + 2}{4} & \xi - 2 < x < \xi + 2 \\ 0 & x > \xi + 2 \end{cases} \quad (7)$$

Find the density of states as a function of x , $g(x) dx$.

$$g(p) dp = \frac{V}{h^3} 4\pi p^2 dp \times 2 \quad (8)$$

$$\epsilon = \frac{p^2}{2m} \quad (9)$$

$$p^2 = 2m\epsilon \quad (10)$$

$$2p dp = 2m d\epsilon \quad (11)$$

$$dp = \frac{m}{p} d\epsilon \quad (12)$$

$$dp = \frac{m}{\sqrt{2m\epsilon}} d\epsilon \quad (13)$$

$$g(\epsilon) d\epsilon = \frac{V}{h^3} 8\pi (2m\epsilon) \left(\frac{m}{\sqrt{2m\epsilon}} d\epsilon \right) \quad (14)$$

$$g(\epsilon) d\epsilon = \frac{V}{h^3} 8\pi \sqrt{2m\epsilon} m d\epsilon \quad (15)$$

$$g(\epsilon) d\epsilon = \frac{V}{h^3} 4\pi \sqrt{2m\epsilon} 2m d\epsilon \quad (16)$$

$$g(\epsilon) d\epsilon = \frac{V}{h^3} 4\pi \sqrt{\epsilon} (2m)^{3/2} d\epsilon \quad (17)$$

$$g(\epsilon) d\epsilon = 4\pi V \frac{(2m)^{3/2}}{h^3} \sqrt{\epsilon} d\epsilon \quad (18)$$

$$g(\epsilon) d\epsilon = 4\pi V \left(\frac{2m}{h^2} \right)^{3/2} \sqrt{\epsilon} d\epsilon \quad (19)$$

$$x \equiv \frac{\epsilon}{kT} \quad (20)$$

$$\epsilon = kT x \quad (21)$$

$$d\epsilon = kT dx \quad (22)$$

$$g(x) dx = 4\pi V \left(\frac{2m}{h^2} \right)^{3/2} \sqrt{(kT x)} (kT dx) \quad (23)$$

$$g(x) dx = 4\pi V \left(\frac{2m}{h^2} \right)^{3/2} (kT)^{3/2} \sqrt{x} dx \quad (24)$$

$$g(x) dx = 4\pi V \left(\frac{2mkT}{h^2} \right)^{3/2} \sqrt{x} dx \quad (25)$$

Simplify this by grouping all the constants into one, α .

$$\alpha = 4\pi V \left(\frac{2mkT}{h^2} \right)^{3/2} \quad (26)$$

$$g(x) dx = \alpha \sqrt{x} dx \quad (27)$$

Find U .

$$U \equiv \int_0^\infty \epsilon n(\epsilon) g(\epsilon) d\epsilon \quad (28)$$

$$U = \int_0^\infty \epsilon(x) n(x) g(x) dx \quad (29)$$

Use the result of equation (7) in equation (29) to limit the bounds of integration.

$$U = \int_0^{\xi-2} \epsilon(x) (1) g(x) dx + \int_{\xi-2}^{\xi+2} \epsilon(x) \left(\frac{-x + \xi + 2}{4} \right) g(x) dx \quad (30)$$

$$U = \int_0^{\xi-2} \epsilon(x) g(x) dx + \int_{\xi-2}^{\xi+2} \epsilon(x) \left(\frac{-x + \xi + 2}{4} \right) g(x) dx \quad (31)$$

$$\epsilon \equiv kTx \quad (32)$$

$$d\epsilon = kTdx \quad (33)$$

Solve each integral separately.

$$\int_0^{\xi-2} \epsilon(x) g(x) dx = \int_0^{\xi-2} (kTx) (\alpha\sqrt{x}dx) \quad (34)$$

$$\int_0^{\xi-2} \epsilon(x) g(x) dx = \int_0^{\xi-2} kT\alpha x^{3/2} dx \quad (35)$$

$$\int_0^{\xi-2} \epsilon(x) g(x) dx = kT\alpha \int_0^{\xi-2} x^{3/2} dx \quad (36)$$

$$\int_0^{\xi-2} \epsilon(x) g(x) dx = kT\alpha \left[\frac{2}{5} x^{5/2} \right]_0^{\xi-2} \quad (37)$$

$$\int_0^{\xi-2} \epsilon(x) g(x) dx = kT\alpha \left[\frac{2}{5} (\xi - 2)^{5/2} \right] \quad (38)$$

$$\int_0^{\xi-2} \epsilon(x) g(x) dx = \frac{2}{5} kT\alpha (\xi - 2)^{5/2} \quad (39)$$

Find the second integral.

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \int_{\xi-2}^{\xi+2} (kTx) \left(\frac{(\xi + 2 - x)}{4} \right) (\alpha\sqrt{x}dx) \quad (40)$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \int_{\xi-2}^{\xi+2} kTx \frac{(\xi + 2 - x)}{4} \alpha\sqrt{x}dx \quad (41)$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = kT\frac{\alpha}{4} \int_{\xi-2}^{\xi+2} (\xi + 2 - x) x^{3/2} dx \quad (42)$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = kT\frac{\alpha}{4} \int_{\xi-2}^{\xi+2} (\xi + 2) x^{3/2} - x^{5/2} dx \quad (43)$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = kT\frac{\alpha}{4} \left[\frac{2}{5} (\xi + 2) x^{5/2} - \frac{2}{7} x^{7/2} \right]_{\xi-2}^{\xi+2} \quad (44)$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = kT\frac{\alpha}{4} \left[\frac{2}{5} (\xi + 2) (\xi + 2)^{5/2} - \frac{2}{7} (\xi + 2)^{7/2} - \frac{2}{5} (\xi + 2) (\xi - 2)^{5/2} + \frac{2}{7} (\xi - 2)^{7/2} \right] \quad (45)$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = kT\frac{\alpha}{4} \left[\frac{2}{5} (\xi + 2)^{7/2} - \frac{2}{7} (\xi + 2)^{7/2} - \frac{2}{5} (\xi + 2) (\xi - 2)^{5/2} + \frac{2}{7} (\xi - 2)^{7/2} \right] \quad (46)$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = kT\frac{\alpha}{2} \left[\frac{1}{5} (\xi + 2)^{7/2} - \frac{1}{7} (\xi + 2)^{7/2} - \frac{1}{5} (\xi + 2) (\xi - 2)^{5/2} + \frac{1}{7} (\xi - 2)^{7/2} \right] \quad (47)$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = kT\frac{\alpha}{2} \left[\frac{2}{35} (\xi + 2)^{7/2} - \frac{1}{5} (\xi + 2) (\xi - 2)^{5/2} + \frac{1}{7} (\xi - 2)^{7/2} \right] \quad (48)$$

Use the results of equations (39) and (48) to then solve for U .

$$U = \left(\frac{2}{5} kT \alpha (\xi - 2)^{5/2} \right) + \left(kT \frac{\alpha}{2} \left[\frac{2}{35} (\xi + 2)^{7/2} - \frac{1}{5} (\xi + 2) (\xi - 2)^{5/2} + \frac{1}{7} (\xi - 2)^{7/2} \right] \right) \quad (49)$$

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{2} \left[\frac{2}{35} (\xi + 2)^{7/2} - \frac{1}{5} (\xi + 2) (\xi - 2)^{5/2} + \frac{1}{7} (\xi - 2)^{7/2} \right] \quad (50)$$

Now, invoke the approximation that we are at low temperatures.

$$\xi \equiv \frac{\mu}{kT} \quad (51)$$

$$T \rightarrow 0 \quad (52)$$

$$\xi \gg 1 \quad (53)$$

This is quite useful. Firstly:

$$(\xi + 2) \approx (\xi - 2) \quad (54)$$

$$(\xi + 2) (\xi - 2)^{5/2} \approx (\xi - 2)^{7/2} \quad (55)$$

Use this approximation in equation (50).

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{2} \left[\frac{2}{35} (\xi + 2)^{7/2} - \frac{1}{5} [(\xi + 2) (\xi - 2)^{5/2}] + \frac{1}{7} (\xi - 2)^{7/2} \right] \quad (56)$$

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{2} \left[\frac{2}{35} (\xi + 2)^{7/2} - \frac{1}{5} [(\xi - 2)^{7/2}] + \frac{1}{7} (\xi - 2)^{7/2} \right] \quad (57)$$

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{2} \left[\frac{2}{35} (\xi + 2)^{7/2} - \frac{1}{5} (\xi - 2)^{7/2} + \frac{1}{7} (\xi - 2)^{7/2} \right] \quad (58)$$

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{2} \left[\frac{2}{35} (\xi + 2)^{7/2} - \frac{1}{5} (\xi - 2)^{7/2} + \frac{1}{7} (\xi - 2)^{7/2} \right] \quad (59)$$

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{2} \left[\frac{2}{35} (\xi + 2)^{7/2} - \frac{2}{35} (\xi - 2)^{7/2} \right] \quad (60)$$

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{2} \frac{2}{35} [(\xi + 2)^{7/2} - (\xi - 2)^{7/2}] \quad (61)$$

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{2} \frac{2}{35} [(\xi + 2)^{7/2} - (\xi - 2)^{7/2}] \quad (62)$$

$$U = \frac{2}{5} kT \alpha (\xi - 2)^{5/2} + kT \frac{\alpha}{35} [(\xi + 2)^{7/2} - (\xi - 2)^{7/2}] \quad (63)$$

Use the approximation in equation (53) to simplify our expression for U to only consider higher orders of ξ .

$$\xi \gg 1 \quad (64)$$

$$U \approx kT \frac{\alpha}{35} [(\xi + 2)^{7/2} - (\xi - 2)^{7/2}] \quad (65)$$

$$U \approx kT \frac{\alpha}{35} \left[\xi^{7/2} \left(1 + \frac{2}{\xi} \right)^{7/2} - \xi^{7/2} \left(1 - \frac{2}{\xi} \right)^{7/2} \right] \quad (66)$$

$$U \approx kT \frac{\alpha}{35} \xi^{7/2} \left[\left(1 + \frac{2}{\xi} \right)^{7/2} - \left(1 - \frac{2}{\xi} \right)^{7/2} \right] \quad (67)$$

Since $\xi \gg 1$, we can Taylor expand $\left(1 + \frac{2}{\xi}\right)^{7/2}$ and $\left(1 - \frac{2}{\xi}\right)^{7/2}$ because $\frac{2}{\xi} \ll 1$. Use the binomial expansion.

$$(1+x)^n \approx 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 \quad (68)$$

$$(1-x)^n \approx 1 - nx + \frac{(n)(n-1)}{2!}x^2 - \frac{(n)(n-1)(n-2)}{3!}x^3 \quad (69)$$

$$(1+x)^n - (1-x)^n \approx 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 \quad (70)$$

$$- \left(1 - nx + \frac{(n)(n-1)}{2!}x^2 - \frac{(n)(n-1)(n-2)}{3!}x^3 \right) \quad (71)$$

$$(1+x)^n - (1-x)^n \approx 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 \quad (72)$$

$$- 1 + nx - \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 \quad (73)$$

$$(1+x)^n - (1-x)^n \approx 2nx + \frac{2(n)(n-1)(n-2)}{3!}x^3 \quad (74)$$

$$(1+x)^n - (1-x)^n \approx 2nx + \frac{(n)(n-1)(n-2)}{3}x^3 \quad (75)$$

Using this binomial expansion:

$$\left(1 + \frac{2}{\xi}\right)^{7/2} - \left(1 - \frac{2}{\xi}\right)^{7/2} \approx 2 \left(\frac{7}{2}\right) \left(\frac{2}{\xi}\right) + \frac{\left(\frac{7}{2}\right) \left(\frac{5}{2}\right) \left(\frac{3}{2}\right)}{3} \left(\frac{2}{\xi}\right)^3 \quad (76)$$

$$\left(1 + \frac{2}{\xi}\right)^{7/2} - \left(1 - \frac{2}{\xi}\right)^{7/2} \approx 2 \left(\frac{7}{\xi}\right) + \frac{(7)(5)(3)}{3} \left(\frac{1}{\xi}\right)^3 \left(\frac{2}{\xi}\right)^3 \quad (77)$$

$$\left(1 + \frac{2}{\xi}\right)^{7/2} - \left(1 - \frac{2}{\xi}\right)^{7/2} \approx 2 \left(\frac{7}{\xi}\right) + (7)(5) \left(\frac{1}{\xi}\right)^3 \quad (78)$$

$$\left(1 + \frac{2}{\xi}\right)^{7/2} - \left(1 - \frac{2}{\xi}\right)^{7/2} \approx \frac{14}{\xi} + \frac{35}{\xi^3} \quad (79)$$

Use the result of equation (79) in equation (67).

$$U \approx kT \frac{\alpha}{35} \xi^{7/2} \left[\left(1 + \frac{2}{\xi}\right)^{7/2} - \left(1 - \frac{2}{\xi}\right)^{7/2} \right] \quad (80)$$

$$U \approx kT \frac{\alpha}{35} \xi^{7/2} \left[\frac{14}{\xi} + \frac{35}{\xi^3} \right] \quad (81)$$

$$U \approx kT \frac{\alpha}{35} \xi^{5/2} \left[14 + \frac{35}{\xi^2} \right] \quad (82)$$

$$U \approx kT \frac{\alpha}{35} \xi^{5/2} [14 + 35\xi^{-2}] \quad (83)$$

$$U \approx kT \alpha \xi^{5/2} \left[\frac{14}{35} + \xi^{-2} \right] \quad (84)$$

$$U \approx kT \alpha \xi^{5/2} \left[\frac{2}{5} + \xi^{-2} \right] \quad (85)$$

$$U \approx \frac{2}{5} kT \alpha \xi^{5/2} \left[1 + \frac{5}{2} \xi^{-2} \right] \quad (86)$$

Now find N .

$$N \equiv \int_0^\infty n(\epsilon) g(\epsilon) d\epsilon \quad (87)$$

$$N = \int_0^\infty n(x) g(x) dx \quad (88)$$

Use the result of equation (7) in equation (88) to limit the bounds of integration.

$$N = \int_0^{\xi-2} (1) g(x) dx + \int_{\xi-2}^{\xi+2} \left(\frac{-x + \xi + 2}{4} \right) g(x) dx \quad (89)$$

$$N = \int_0^{\xi-2} g(x) dx + \int_{\xi-2}^{\xi+2} \left(\frac{-x + \xi + 2}{4} \right) g(x) dx \quad (90)$$

Solve each integral separately.

$$\int_0^{\xi-2} g(x) dx = \int_0^{\xi-2} (\alpha \sqrt{x} dx) \quad (91)$$

$$\int_0^{\xi-2} g(x) dx = \int_0^{\xi-2} \alpha x^{1/2} dx \quad (92)$$

$$\int_0^{\xi-2} g(x) dx = \alpha \int_0^{\xi-2} x^{1/2} dx \quad (93)$$

$$\int_0^{\xi-2} g(x) dx = \alpha \left[\frac{2}{3} x^{3/2} \right]_0^{\xi-2} \quad (94)$$

$$\int_0^{\xi-2} g(x) dx = \alpha \left[\frac{2}{3} (\xi - 2)^{3/2} \right] \quad (95)$$

$$\int_0^{\xi-2} g(x) dx = \frac{2}{3} \alpha (\xi - 2)^{3/2} \quad (96)$$

Find the second integral.

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \int_{\xi-2}^{\xi+2} \left(\frac{(\xi + 2 - x)}{4} \right) (\alpha \sqrt{x} dx) \quad (97)$$

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \int_{\xi-2}^{\xi+2} \frac{(\xi + 2 - x)}{4} \alpha \sqrt{x} dx \quad (98)$$

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \frac{\alpha}{4} \int_{\xi-2}^{\xi+2} (\xi + 2 - x) x^{1/2} dx \quad (99)$$

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \frac{\alpha}{4} \int_{\xi-2}^{\xi+2} (\xi + 2) x^{1/2} - x^{3/2} dx \quad (100)$$

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \frac{\alpha}{4} \left[\frac{2}{3} (\xi + 2) x^{3/2} - \frac{2}{5} x^{5/2} \right]_{\xi-2}^{\xi+2} \quad (101)$$

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \frac{\alpha}{4} \left[\frac{2}{3} (\xi + 2) (\xi + 2)^{3/2} - \frac{2}{5} (\xi + 2)^{5/2} - \frac{2}{3} (\xi + 2) (\xi - 2)^{3/2} + \frac{2}{5} (\xi - 2)^{5/2} \right] \quad (102)$$

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \frac{\alpha}{4} \left[\frac{2}{3} (\xi + 2)^{5/2} - \frac{2}{5} (\xi + 2)^{5/2} - \frac{2}{3} (\xi + 2) (\xi - 2)^{3/2} + \frac{2}{5} (\xi - 2)^{5/2} \right] \quad (103)$$

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \frac{\alpha}{2} \left[\frac{1}{3} (\xi + 2)^{5/2} - \frac{1}{5} (\xi + 2)^{5/2} - \frac{1}{3} (\xi + 2) (\xi - 2)^{3/2} + \frac{1}{5} (\xi - 2)^{5/2} \right] \quad (104)$$

$$\int_{\xi-2}^{\xi+2} n(x) g(x) dx = \frac{\alpha}{2} \left[\frac{2}{15} (\xi + 2)^{5/2} - \frac{1}{3} (\xi + 2) (\xi - 2)^{3/2} + \frac{1}{5} (\xi - 2)^{5/2} \right] \quad (105)$$

Use the results of equations (96) and (105) to then solve for N .

$$N = \left(\frac{2}{3} \alpha (\xi - 2)^{3/2} \right) + \left(\frac{\alpha}{2} \left[\frac{2}{15} (\xi + 2)^{5/2} - \frac{1}{3} (\xi + 2) (\xi - 2)^{3/2} + \frac{1}{5} (\xi - 2)^{5/2} \right] \right) \quad (106)$$

$$N = \frac{2}{3} \alpha (\xi - 2)^{3/2} + \frac{\alpha}{2} \left[\frac{2}{15} (\xi + 2)^{5/2} - \frac{1}{3} (\xi + 2) (\xi - 2)^{3/2} + \frac{1}{5} (\xi - 2)^{5/2} \right] \quad (107)$$

Now, invoke the approximation that we are at low temperatures.

$$\xi \equiv \frac{\mu}{kT} \quad (108)$$

$$T \rightarrow 0 \quad (109)$$

$$\xi \gg 1 \quad (110)$$

$$(\xi + 2) \approx (\xi - 2) \quad (111)$$

$$(\xi + 2)(\xi - 2)^{3/2} \approx (\xi - 2)^{5/2} \quad (112)$$

Use this approximation in equation (107).

$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2} \left[\frac{2}{15}(\xi + 2)^{5/2} - \frac{1}{3}[(\xi + 2)(\xi - 2)^{3/2}] + \frac{1}{5}(\xi - 2)^{5/2} \right] \quad (113)$$

$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2} \left[\frac{2}{15}(\xi + 2)^{5/2} - \frac{1}{3}[(\xi - 2)^{5/2}] + \frac{1}{5}(\xi - 2)^{5/2} \right] \quad (114)$$

$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2} \left[\frac{2}{15}(\xi + 2)^{5/2} - \frac{1}{3}(\xi - 2)^{5/2} + \frac{1}{5}(\xi - 2)^{5/2} \right] \quad (115)$$

$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2} \left[\frac{2}{15}(\xi + 2)^{5/2} - \frac{2}{15}(\xi - 2)^{5/2} \right] \quad (116)$$

$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2} \frac{2}{15} [(\xi + 2)^{5/2} - (\xi - 2)^{5/2}] \quad (117)$$

$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{15} [(\xi + 2)^{5/2} - (\xi - 2)^{5/2}] \quad (118)$$

Use the approximation in equation (53) to simplify our expression for U to only consider higher orders of ξ .

$$\xi \gg 1 \quad (119)$$

$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{15} [(\xi + 2)^{5/2} - (\xi - 2)^{5/2}] \quad (120)$$

$$N \approx \frac{\alpha}{15} [(\xi + 2)^{5/2} - (\xi - 2)^{5/2}] \quad (121)$$

$$N \approx \frac{\alpha}{15} \left[\xi^{5/2} \left(1 + \frac{2}{\xi} \right)^{5/2} - \xi^{5/2} \left(1 - \frac{2}{\xi} \right)^{5/2} \right] \quad (122)$$

$$N \approx \frac{\alpha}{15} \xi^{5/2} \left[\left(1 + \frac{2}{\xi} \right)^{5/2} - \left(1 - \frac{2}{\xi} \right)^{5/2} \right] \quad (123)$$

Use the same binomial expansion as before.

$$(1+x)^n - (1-x)^n \approx 2nx + \frac{(n)(n-1)(n-2)}{3}x^3 \quad (124)$$

$$\left(1 + \frac{2}{\xi} \right)^{5/2} - \left(1 - \frac{2}{\xi} \right)^{5/2} \approx 2 \left(\frac{5}{2} \right) \left(\frac{2}{\xi} \right) + \frac{\left(\frac{5}{2} \right) \left(\frac{3}{2} \right) \left(\frac{1}{2} \right)}{3} \left(\frac{2}{\xi} \right)^3 \quad (125)$$

$$\left(1 + \frac{2}{\xi} \right)^{5/2} - \left(1 - \frac{2}{\xi} \right)^{5/2} \approx 2 \left(\frac{5}{\xi} \right) + \frac{(5)(3)(1)}{3} \left(\frac{1}{\xi} \right)^3 \left(\frac{2^3}{2^3} \right) \quad (126)$$

$$\left(1 + \frac{2}{\xi} \right)^{5/2} - \left(1 - \frac{2}{\xi} \right)^{5/2} \approx 2 \left(\frac{5}{\xi} \right) + \frac{15}{3} \left(\frac{1}{\xi} \right)^3 \quad (127)$$

$$\left(1 + \frac{2}{\xi} \right)^{5/2} - \left(1 - \frac{2}{\xi} \right)^{5/2} \approx \frac{10}{\xi} + \frac{5}{\xi^3} \quad (128)$$

Use the result of equation (128) in equation (123).

$$N \approx \frac{\alpha}{15} \xi^{5/2} \left[\left(1 + \frac{2}{\xi}\right)^{5/2} - \left(1 - \frac{2}{\xi}\right)^{5/2} \right] \quad (129)$$

$$N \approx \frac{\alpha}{15} \xi^{5/2} \left[\frac{10}{\xi} + \frac{5}{\xi^3} \right] \quad (130)$$

$$N \approx \frac{\alpha}{15} \xi^{3/2} \left[10 + \frac{5}{\xi^2} \right] \quad (131)$$

$$N \approx \frac{\alpha}{15} \xi^{3/2} [10 + 5\xi^{-2}] \quad (132)$$

$$N \approx \frac{10}{15} \alpha \xi^{3/2} \left[1 + \frac{1}{2} \xi^{-2} \right] \quad (133)$$

$$N \approx \frac{2}{3} \alpha \xi^{3/2} \left[1 + \frac{1}{2} \xi^{-2} \right] \quad (134)$$

Use equations (86) and (134) to rewrite U in terms of N .

$$U \approx \frac{2}{5} kT \alpha \xi^{5/2} \left[1 + \frac{5}{2} \xi^{-2} \right] \quad (135)$$

$$N \approx \frac{2}{3} \alpha \xi^{3/2} \left[1 + \frac{1}{2} \xi^{-2} \right] \quad (136)$$

$$\frac{U}{N} = \frac{\frac{2}{5} kT \alpha \xi^{5/2} [1 + \frac{5}{2} \xi^{-2}]}{\frac{2}{3} \alpha \xi^{3/2} [1 + \frac{1}{2} \xi^{-2}]} \quad (137)$$

$$\frac{U}{N} = \frac{\frac{2}{5} \alpha \xi^{5/2}}{\frac{2}{3} \alpha \xi^{3/2}} kT \frac{[1 + \frac{5}{2} \xi^{-2}]}{[1 + \frac{1}{2} \xi^{-2}]} \quad (138)$$

$$\frac{U}{N} = \frac{2}{5} \frac{3}{2} \xi kT \frac{[1 + \frac{5}{2} \xi^{-2}]}{[1 + \frac{1}{2} \xi^{-2}]} \quad (139)$$

$$\frac{U}{N} = \frac{3}{5} \xi kT \frac{[1 + \frac{5}{2} \xi^{-2}]}{[1 + \frac{1}{2} \xi^{-2}]} \quad (140)$$

Use long division of polynomials to evaluate the last fraction in equation (140) (I can't write long division in LATEX, look it up).

$$\frac{[1 + \frac{5}{2} \xi^{-2}]}{[1 + \frac{1}{2} \xi^{-2}]} = 1 + \frac{2\xi^{-2}}{1 + \frac{1}{2}\xi^{-2}} \quad (141)$$

$$\frac{[1 + \frac{5}{2} \xi^{-2}]}{[1 + \frac{1}{2} \xi^{-2}]} = 1 + 2\xi^{-2} + \dots \quad (142)$$

$$\frac{[1 + \frac{5}{2} \xi^{-2}]}{[1 + \frac{1}{2} \xi^{-2}]} \approx 1 + 2\xi^{-2} \quad (143)$$

Use the result of (143) in (140).

$$\frac{U}{N} \approx \frac{3}{5} \xi kT (1 + 2\xi^{-2}) \quad (144)$$

$$U \approx \frac{3}{5} \xi N kT (1 + 2\xi^{-2}) \quad (145)$$

However, we can further reduce this from the definition of ξ . Start with this first approximation, which is kind of nonsensical.

$$2\xi^{-2} \approx \frac{5}{3}\xi^{-2} \quad (146)$$

$$U \approx \frac{3}{5}\xi NkT (1 + 2\xi^{-2}) \quad (147)$$

$$U \approx \frac{3}{5}\xi NkT \left(1 + \frac{5}{3}\xi^{-2}\right) \quad (148)$$

$$\xi \equiv \frac{\epsilon_F}{kT} \quad (149)$$

$$U \approx \frac{3}{5} \left(\frac{\epsilon_F}{kT}\right) NkT \left(1 + \frac{5}{3} \left(\frac{\epsilon_F}{kT}\right)^{-2}\right) \quad (150)$$

$$U \approx \frac{3}{5}\epsilon_F N \left(1 + \frac{5}{3} \left(\frac{\epsilon_F}{kT}\right)^{-2}\right) \quad (151)$$

$$U \approx \frac{3}{5}\epsilon_F N \left(1 + \frac{5}{3} \left(\frac{kT}{\epsilon_F}\right)^2\right) \quad (152)$$

Now find C_V . By definition:

$$C_V \equiv \left(\frac{\partial U}{\partial T}\right)_V \quad (153)$$

$$C_V = \frac{\partial}{\partial T} \left(\frac{3}{5}\epsilon_F N \left(1 + \frac{5}{3} \left(\frac{kT}{\epsilon_F}\right)^2\right) \right) \quad (154)$$

$$C_V = \frac{\partial}{\partial T} \left(\frac{3}{5}\epsilon_F N \right) + \frac{\partial}{\partial T} \left(\epsilon_F N \left(\frac{kT}{\epsilon_F}\right)^2 \right) \quad (155)$$

$$C_V = \frac{\partial}{\partial T} \left(\epsilon_F N \left(\frac{kT}{\epsilon_F}\right)^2 \right) \quad (156)$$

$$C_V = 2\epsilon_F N \left(\frac{kT}{\epsilon_F}\right) \left(\frac{k}{\epsilon_F}\right) \quad (157)$$

$$C_V = 2N (kT) \left(\frac{k}{\epsilon_F}\right) \quad (158)$$

$$C_V = \frac{2Nk^2T}{\epsilon_F} \quad (159)$$

$$\frac{C_V}{Nk} = \frac{2kT}{\epsilon_F} \quad (160)$$

Equation (8.1.39) in the gives:

$$\frac{C_V}{Nk} = \frac{\pi^2}{2} \frac{kT}{\epsilon_F} + \dots \quad (161)$$

$$\frac{C_{V,\text{book}}}{Nk} \approx \frac{\pi^2 kT}{2\epsilon_F} \quad (162)$$

Find the difference between our approximation and what's given in the book.

$$\frac{\frac{C_{V,\text{approx}}}{Nk}}{\frac{C_{V,\text{book}}}{Nk}} = \frac{\frac{2kT}{\epsilon_F}}{\frac{\pi^2 kT}{2\epsilon_F}} \quad (163)$$

$$\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{\frac{2kT}{\epsilon_F}}{\frac{\pi^2 kT}{2\epsilon_F}} \quad (164)$$

$$\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{2kT}{\epsilon_F} \frac{2\epsilon_F}{\pi^2 kT} \quad (165)$$

$$\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{4}{\pi^2} \frac{kT}{kT} \frac{\epsilon_F}{\epsilon_F} \quad (166)$$

$$\pi^2 kT \quad (167)$$

$$\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{4}{\pi^2} \quad (168)$$

$$\boxed{\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{4}{\pi^2}}$$

Our approximation is smaller than what's given in the book by a factor of $4/\pi^2$! This means that in the process of approximating $n(x)$, we are under-counting the number of excited particles by a factor of $\sim \frac{1}{2}$.