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2.7 Derive (i) an asymptotic expression for the number of ways in which a given energy E can be distributed among a set of N one-dimensional harmonic oscillators, the energy eigenvalues of the oscillators being $\left(n+\frac{1}{2}\right)\hbar\omega$; $n = 0, 1, 2, ...$, and (ii) the corresponding expression for the "volume" of the relevant region of the phase space of this system. Establish the correspondence between the two results, showing that the conversion factor ω_0 is precisely h^N.

(i)

The energy for a given n for a harmonic oscillator is given by:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \tag{1}$$

Total energy is therefore

$$E = \sum_{i=1}^{N}\left(n_i + \frac{1}{2}\right)\hbar\omega \tag{2}$$

which can be expanded into

$$E = \hbar\omega\sum_{i=1}^{N} n_i + \frac{1}{2}\hbar\omega\sum_{i=1}^{N} \tag{3}$$

$$E = q\hbar\omega + \frac{N}{2}\hbar\omega \tag{4}$$

where $q = \sum_{i=1}^{N} n_i$. As shown in class, for harmonic oscillator, $\Omega = \frac{(q+N-1)!}{q!(N-1)!}$. Find Ω.

$$\Omega = \frac{(q+N-1)!}{q!\,(N-1)!} \tag{5}$$

$$\log\Omega = \log{(q+N-1)!} - \log q! - \log{(N-1)!} \tag{6}$$

Invoke Stirling's Approximation $\log n! \approx n\log n - n$, for $n >> 1$.

$$\log\Omega = (q+N-1)\log{(q+N-1)} - (q+N-1) - q\log q + q - (N-1)\log{(N-1)} + (N-1) \tag{7}$$

For many oscillators, $q > N >> 1$. Therefore, $(q+N-1) \approx (q+N)$ and $(N-1) \approx N$.

$$\log\Omega = (q+N)\log{(q+N)} - (q+N) - q\log q + q - N\log N + N \tag{8}$$

$$\log\Omega = (q+N)\log{(q+N)} - q\log q - N\log N \tag{9}$$

For $q > N$, $\log q + N \approx \log q + \frac{N}{q}$.

$$\log\Omega = (q+N)\left[\log q + \frac{N}{q}\right] - q\log q - N\log N \tag{10}$$

$$\log\Omega = q\log q + N + N\log q + \frac{N^2}{q} - q\log q - N\log N \tag{11}$$

$$\log\Omega = N + N\log q + \frac{N^2}{q} - N\log N \tag{12}$$

$$\log\Omega = N + N\log\frac{q}{N} + \frac{N^2}{q} \tag{13}$$

$$\log\Omega = N\left(1 + \log\frac{q}{N} + \frac{N}{q}\right) \tag{14}$$

Assuming $q > N$, we can therefore conclude $\frac{N}{q} \approx 0$.

$$\log \Omega \approx N \left(1 + \log \frac{q}{N}\right) \tag{15}$$
$$\log \Omega = \left(\log \frac{q}{N} + 1\right)^N \tag{16}$$

Solve for q in terms of energy. Use the approximation $q + \frac{N}{2} \approx q$, for $q > N$.

$$E = \left(q + \frac{N}{2}\right) \hbar\omega \tag{17}$$
$$E \approx q\hbar\omega \tag{18}$$
$$q = \frac{E}{\hbar\omega} \tag{19}$$

Plug in this result into equation (16).

$$\log \Omega = \left(\log \frac{q}{N} + 1\right)^N \tag{20}$$
$$\Omega = e^N \left(\frac{q}{N}\right)^N \tag{21}$$
$$\Omega = e^N \left(\left(\frac{E}{\hbar\omega}\right) \frac{1}{N}\right)^N \tag{22}$$

$$\boxed{\Omega = \left(\frac{Ee}{\hbar\omega N}\right)^N}$$

(ii)
The total energy of N harmonic oscillators can be written as

$$E = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \frac{m\omega^2 q_i^2}{2}\right) \tag{23}$$

The equation of an ellipse is

$$1 = \frac{x^2}{a^2} + \frac{y^2}{a^2} \tag{24}$$

Equation (23) resembles the equation of an ellipse. Manipulate equation (23) to match the equation of an ellipse.

$$1 = \sum_{i=1}^{N} \left(\frac{p_i^2}{2mE} + \frac{m\omega^2 q_i^2}{2E}\right) \tag{25}$$
$$a = \sqrt{2mE} \tag{26}$$
$$b = \sqrt{\frac{2E}{m\omega^2}} \tag{27}$$

From equation (25), we can see that the ellipse in the phase space is $2N$ dimensional. Starting from the equation for the volume of an n-dimensional ellipse, find the volume of the ellipse in the phase space.

$$V_n(\vec{a}) = \left(\Pi_{k=1}^{n} a_k\right) \frac{2\pi^{n/2}}{n\Gamma\left(\frac{n}{2}\right)} \tag{28}$$

$$n = 2N \tag{29}$$

$$V_n = \left(\sqrt{2mE}\sqrt{\frac{2E}{m\omega^2}}\right)^N \frac{2\pi^N}{2N\Gamma(N)} \tag{30}$$

$$V_n = \left(\sqrt{\frac{4mE^2}{m\omega^2}}\right)^N \frac{\pi^N}{N(N-1)!} \tag{31}$$

$$V_n = \left(\sqrt{\frac{4E^2}{\omega^2}}\right)^N \frac{\pi^N}{N!} \tag{32}$$

$$\boxed{V_n = \left(\frac{2E\pi}{\omega}\right)^N \frac{1}{N!}}$$

Compare this result to Ω found before.

$$\log\left(\frac{V_n}{\Omega}\right) = \log V_n - \log\Omega \tag{33}$$

$$\log\left(\frac{V_n}{\Omega}\right) = \log\left(\left(\frac{2E\pi}{\omega}\right)^N \frac{1}{N!}\right) - \log\left(\left(\frac{Ee}{\hbar\omega N}\right)^N\right) \tag{34}$$

$$\log\left(\frac{V_n}{\Omega}\right) = \log\left(\left(\frac{2E\pi}{\omega}\right)^N\right) - \log(N!) - N\log\left(\frac{Ee}{\hbar\omega N}\right) \tag{35}$$

$$\log\left(\frac{V_n}{\Omega}\right) = N\log\left(\frac{2E\pi}{\omega}\right) - N\log N + N - N\log\left(\frac{Ee}{\hbar\omega N}\right) \tag{36}$$

$$\log\left(\frac{V_n}{\Omega}\right) = N\log\left(\frac{2E\pi}{\omega}\frac{1}{N}\frac{\hbar\omega N}{Ee}\right) + N \tag{37}$$

$$\log\left(\frac{V_n}{\Omega}\right) = N\log\left(\frac{2\pi\hbar}{e}\right) + N \tag{38}$$

$$\log\left(\frac{V_n}{\Omega}\right) = N\log(2\pi\hbar) - N\log e + N \tag{39}$$

$$\log\left(\frac{V_n}{\Omega}\right) = N\log 2\pi\hbar - N + N \tag{40}$$

$$\log\left(\frac{V_n}{\Omega}\right) = N\log h \tag{41}$$

$$\log\left(\frac{V_n}{\Omega}\right) = \log h^N \tag{42}$$

$$\frac{V_n}{\Omega} = h^N \tag{43}$$

Assuming that $\frac{V_n}{\Omega}$ is equal to some quantity ω_0, we have just shown that ω_0 must be h^N

$$\boxed{\omega_0 = h^N}$$

2.8 Following the method of Appendix C, replacing equation (C.4) by the integral

$$\int_0^\infty e^{-r} r^2 dr = 2$$

show that

$$V_{3N} = \underset{0 \le \sum_{i=1}^{3N} |x_i| \le R}{\int \quad \ldots \quad \int} \Pi_{i=1}^N \left(4\pi r_i^2 dr_i\right) = \frac{\left(8\pi R^3\right)^N}{(3N)!}$$

Using this result, compute the "volume" of the relevant region of the phase space of an extreme relativistic gas ($\epsilon = pc$) of N particles moving in three dimensions. Hence, derive expressions for the various thermodynamic properties of this system and compare your results with those of Problem 1.7.

We know that V_{3N} should be of the form.

$$V_{3N} = c_{3N} R^{3N} \tag{44}$$

Find dV_{3N}/dR.

$$dV_{3N} = 3N c_{3N} R^{3N-1} \tag{45}$$

Given

$$\int_0^\infty e^{-r} r^2 dr = 2 \tag{46}$$

Multiplying equation (46) by N integrals we get

$$\int_0^\infty \ldots \int_0^\infty \exp\left(\sum_{i=1}^N -r_i\right) \Pi_{i=1}^N r_i^2 = 2^N \tag{47}$$

$$\frac{1}{(4\pi)^N} \int_0^\infty \ldots \int_0^\infty \exp\left(\sum_{i=1}^N -r_i\right) \Pi_{i=1}^N 4\pi r_i^2 = 2^N \tag{48}$$

For 3 dimensions

$$\frac{dV_{3N}}{dR} = \Pi_{i=1}^N 4\pi r_i^2 \tag{49}$$

Therefore

$$\frac{1}{(4\pi)^N} \int_0^\infty \ldots \int_0^\infty \exp\left(\sum_{i=1}^N -r_i\right) \Pi_{i=1}^N 4\pi r_i^2 = \frac{1}{(4\pi)^N} \int_0^\infty e^{-R} 3N c_{3N} R^{3N-1} dR \tag{50}$$

$$\frac{1}{(4\pi)^N} \int_0^\infty e^{-R} 3N c_{3N} R^{3N-1} dR = 2^N \tag{51}$$

$$\int_0^\infty e^{-R} 3N c_{3N} R^{3N-1} dR = (8\pi)^N \tag{52}$$

$$3N c_{3N} \int_0^\infty e^{-R} R^{3N-1} dR = (8\pi)^N \tag{53}$$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \tag{54}$$

$$3N c_{3N} \Gamma(3N) = (8\pi)^N \tag{55}$$

$$3N c_{3N} (3N-1)! = (8\pi)^N \tag{56}$$

$$c_{3N} (3N)! = (8\pi)^N \tag{57}$$

$$c_{3N} = \frac{(8\pi)^N}{(3N)!} \tag{58}$$

Use this result in equation (44).

$$V_{3N} = \left(\frac{(8\pi)^n}{(3N)!}\right) R^{3N} \tag{59}$$

$$\boxed{V_{3N} = \frac{\left(8\pi R^3\right)^N}{(3N)!}}$$

Show various thermodynamic properties. From class, we know

$$\Omega = \frac{V_{3N} V^N}{h^{3N} N!} \tag{60}$$

$$\Omega = \frac{\left(8\pi R^3\right)^N}{(3N)!} \frac{V^N}{h^{3N} N!} \tag{61}$$

Find the entropy.

$$S = k \log \Omega \tag{62}$$

$$S = k \left(\log\left(\left(8\pi R^3\right)^N\right) + \log\left(V^N\right) - \log\left((3N)!\right) - \log\left(h^{3N}\right) - \log\left(N!\right)\right) \tag{63}$$

However, since R is related to momentum, and this is a relativistic gas, we can therefore say

$$\epsilon = \sum_{i=1}^{N} p_i c \tag{64}$$

$$\epsilon = Rc \tag{65}$$

$$R = \frac{\epsilon}{c} \tag{66}$$

Plugging this result into the entropy found in equation (63) yields

$$S = k \left(\log\left(\left(8\pi \left(\frac{\epsilon}{c}\right)^3\right)^N\right) + \log\left(V^N\right) - \log\left((3N)!\right) - \log\left(h^{3N}\right) - \log\left(N!\right)\right) \tag{67}$$

Find energy as a function of temperature.

$$\left(\frac{\partial S}{\partial \epsilon}\right)_{V,N} = \frac{1}{T} \tag{68}$$

$$\frac{\partial}{\partial \epsilon}\left(k \log\left(\left(8\pi\epsilon^3 \frac{1}{c^3}\right)^N\right)\right) = \frac{1}{T} \tag{69}$$

$$Nk \frac{\partial}{\partial \epsilon}\left(\log\left(\frac{8\pi\epsilon^3}{c^3}\right)\right) = \frac{1}{T} \tag{70}$$

$$Nk \frac{c^3}{8\pi\epsilon^3} \frac{24\pi\epsilon^2}{c^3} = \frac{1}{T} \tag{71}$$

$$\frac{3Nk}{\epsilon} = \frac{1}{T} \tag{72}$$

$$\boxed{\epsilon = 3NkT}$$

Find γ. For constant Ω...

$$\Omega = \frac{\left(8\pi R^3\right)^N}{(3N)!} \frac{V^N}{h^{3N} N!} \tag{73}$$

$$R^{3N} V^N \propto c \tag{74}$$

$$R^3 V \propto c \tag{75}$$

$$R = \frac{\epsilon}{c} \tag{76}$$

$$\frac{\epsilon^3}{c^3} V \propto c \tag{77}$$

$$\epsilon^3 V \propto c \tag{78}$$

$$\epsilon \propto T \tag{79}$$

$$T^3 V \propto c \tag{80}$$

$$\left(T^3 V\right)^{1/3} \propto c \tag{81}$$

$$T V^{1/3} \propto c \tag{82}$$

For constant c. From thermodynamics, we know that $TV^{\gamma-1} = c$. Therefore

$$\gamma - 1 = \frac{1}{3} \tag{83}$$

$$\boxed{\gamma = \frac{4}{3}}$$

Where $\gamma = \frac{c_P}{c_V}$.