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PHYS5510

6.1 Show that the entropy of an ideal gas in thermal equilibrium is given by the formula

$$S = k\sum_{\epsilon}\left[\langle n_\epsilon + 1\rangle \ln\langle n_\epsilon + 1\rangle - \langle n_\epsilon\rangle \ln\langle n_\epsilon\rangle\right]$$

in the case of bosons and by the formula

$$S = k\sum_{\epsilon}\left[-\langle 1 - n_\epsilon\rangle \ln\langle 1 - n_\epsilon\rangle - \langle n_\epsilon\rangle \ln\langle n_\epsilon\rangle\right]$$

in the case of fermions. Verify that these results are consistent with the general formula

$$S = -k\sum_{\epsilon}\left[\sum_{n} p_\epsilon \ln p_\epsilon(n)\right],$$

where $p_\epsilon(n)$ is the probability that there are exactly n particles in the energy state ϵ.
Start with equation (6.1.15).

$$\frac{S}{k} = \sum_i n_i^* \ln\left(\frac{g_i}{n_i^*} - a\right) - \frac{g_i}{a}\ln\left(1 - a\frac{n_i^*}{g_i}\right) \tag{1}$$

$$\frac{S}{k} = \sum_i n_i^* \ln\left(\frac{g_i}{n_i^*}\left(1 - a\frac{n_i^*}{g_i}\right)\right) - \frac{g_i}{a}\ln\left(1 - a\frac{n_i^*}{g_i}\right) \tag{2}$$

$$\frac{S}{k} = \sum_i n_i^* \left[\ln\left(\frac{g_i}{n_i^*}\right) + \ln\left(1 - a\frac{n_i^*}{g_i}\right)\right] - \frac{g_i}{a}\ln\left(1 - a\frac{n_i^*}{g_i}\right) \tag{3}$$

$$\frac{S}{k} = \sum_i n_i^* \ln\left(\frac{g_i}{n_i^*}\right) + \left(n_i^* - \frac{g_i}{a}\right)\ln\left(1 - a\frac{n_i^*}{g_i}\right) \tag{4}$$

$$S = k\sum_i n_i^* \ln\left(\frac{g_i}{n_i^*}\right) + \left(n_i^* - \frac{g_i}{a}\right)\ln\left(1 - a\frac{n_i^*}{g_i}\right) \tag{5}$$

Use equation (6.1.18a) to relate n_i^* to $\langle n\rangle$ and find a.

$$\frac{n_i^*}{g_i} = \frac{1}{e^{\alpha+\beta\epsilon_i} + a} \tag{6}$$

By definition:

$$\frac{n_i^*}{g_i} \equiv \langle n\rangle \tag{7}$$

For bosons,

$$a = -1 \tag{8}$$

$$\tag{9}$$

Also, for single-energy systems, $g_i = 1$.

$$g_i = 1 \tag{10}$$
$$\frac{n_i^*}{g_i} = \langle n \rangle \tag{11}$$
$$n_i^* = \langle n \rangle \tag{12}$$
$$S = k \sum_i n_i^* \ln\left(\frac{g_i}{n_i^*}\right) + \left(n_i^* - \frac{g_i}{a}\right) \ln\left(1 - a\frac{n_i^*}{g_i}\right) \tag{13}$$
$$S = k \sum_i \langle n \rangle \ln\left(\frac{1}{\langle n \rangle}\right) + \left(\langle n \rangle - \frac{1}{(-1)}\right) \ln\left(1 - (-1)\,\frac{\langle n \rangle}{1}\right) \tag{14}$$
$$S = k \sum_i - \langle n \rangle \ln \langle n \rangle + (\langle n \rangle + 1) \ln (1 \langle n \rangle) \tag{15}$$

For Bosons,

$$\boxed{S = k \sum_i \langle n_i + 1 \rangle \ln \langle n_i \rangle - \langle n_i \rangle \ln \langle n_i \rangle}$$

For Fermions, start with equations (7) and (5), but with $a = 1$.

$$\frac{n_i^*}{g_i} \equiv \langle n \rangle \tag{16}$$
$$g_i = 1 \tag{17}$$
$$m_i^* = \langle n \rangle \tag{18}$$
$$S = k \sum_i n_i^* \ln\left(\frac{g_i}{n_i^*}\right) + \left(n_i^* - \frac{g_i}{a}\right) \ln\left(1 - a\frac{n_i^*}{g_i}\right) \tag{19}$$
$$S = k \sum_i \langle n \rangle \ln\left(\frac{1}{\langle n \rangle}\right) + \left(\langle n \rangle - \frac{1}{1}\right) \ln\left(1 - (1)\,\frac{\langle n \rangle}{1}\right) \tag{20}$$
$$S = k \sum_i - \langle n \rangle \ln \langle n \rangle + (\langle n \rangle - 1) \ln (1 - \langle n \rangle) \tag{21}$$

For Fermions,

$$\boxed{S = k \sum_i \langle n_i - 1 \rangle \ln \langle 1 - n_i \rangle - \langle n_i \rangle \ln \langle n_i \rangle}$$

Now verify that results are consistent with the following.

$$S = -k \sum_\epsilon \left[\sum_n p_\epsilon \ln p_\epsilon (n) \right]$$

The inside sum is, by definition, the average of $\ln p_\epsilon (n)$.

$$\sum_n p_\epsilon \ln p_\epsilon (n) = \langle \ln p_\epsilon (n) \rangle \tag{22}$$
$$S = -k \sum_\epsilon \langle \ln p_\epsilon (n) \rangle \tag{23}$$

For bosons, use equation (6.3.10) for $p(n)$.

$$p_\epsilon(n) = \frac{\langle n_\epsilon\rangle^n}{\langle n_\epsilon + 1\rangle^{n+1}} \tag{24}$$

$$S = -k\sum_\epsilon \ln\left(\frac{\langle n_\epsilon\rangle^n}{\langle n_\epsilon + 1\rangle^{n+1}}\right) \tag{25}$$

$$S = -k\sum_\epsilon \ln\langle n_\epsilon\rangle^n - \ln\langle n_\epsilon + 1\rangle^{n+1} \tag{26}$$

$$S = -k\sum_\epsilon n_\epsilon \ln\langle n_\epsilon\rangle - \langle n_\epsilon + 1\rangle \ln\langle n_\epsilon + 1\rangle \tag{27}$$

$$S = k\sum_\epsilon \langle n_\epsilon + 1\rangle \ln\langle n_\epsilon + 1\rangle - n_\epsilon \ln\langle n_\epsilon\rangle \tag{28}$$

This is the same result as what was found in equation (15)! Now, do the same for fermions, using equation (6.3.11) for $p(n)$.

$$p_\epsilon(n) = \begin{cases} 1 - \langle n_\epsilon\rangle & \text{for } n = 0 \\ \langle n_\epsilon\rangle & \text{for } n = 1 \end{cases} \tag{29}$$

$$S = -k\sum_\epsilon \left[\sum_n p_\epsilon \ln p_\epsilon(n)\right] \tag{30}$$

$$S = -k\sum_\epsilon \langle 1 - n_\epsilon\rangle \ln\langle 1 - n_\epsilon\rangle + \langle n_\epsilon\rangle \ln\langle n_\epsilon\rangle \tag{31}$$

$$S = k\sum_\epsilon \langle n_\epsilon + 1\rangle \ln\langle 1 - n_\epsilon\rangle - \langle n_\epsilon\rangle \ln\langle n_\epsilon\rangle \tag{32}$$

This is the same result as equation (21)!

6.10 (a) Show that, if the temperature is uniform, the pressure of a classical gas in a uniform gravitational field decreases with height according to the barometric formula

$$P(z) = P(0)\exp\left[-mgz/kT\right],$$

where the various symbols have their usual meanings.

Assuming hydrostatic equilibrium, apply Newton's Second Law on a slab of air with thickness dz and area A, assuming a pressure gradient of $-dP$.

$$-dPA - mg = 0 \tag{33}$$

$$m \equiv \rho V \tag{34}$$

$$-dPA - \rho V g = 0 \tag{35}$$

$$-dPA = \rho V g \tag{36}$$

The cross-sectional area of the slab of air, A, is merely the volume of air divided by the thickness, dz.

$$A = \frac{V}{dz} \tag{37}$$

$$-dP\left(\frac{V}{dz}\right) = \rho V g \tag{38}$$

$$\frac{-dP}{dz} = \rho g \tag{39}$$

$$dP = -\rho g dz \tag{40}$$

Use the Ideal Gas Law to find density, ρ, in terms of other thermodynamic quantities and m, the mean molecular weight.

$$PV = NkT \tag{41}$$
$$\frac{N}{V} = \frac{P}{kT} \tag{42}$$
$$\frac{\rho}{m} = \frac{P}{kT} \tag{43}$$
$$\rho = \frac{mP}{kT} \tag{44}$$

Use this result in equation (40).

$$dP = -\left(\frac{mP}{kT}\right) gdz \tag{45}$$
$$dP = -\frac{mgP}{kT} dz \tag{46}$$
$$\frac{dP}{P} = -\frac{mg}{kT} dz \tag{47}$$
$$\int \frac{dP}{P} = \int -\frac{mg}{kT} dz \tag{48}$$
$$\ln P = -\frac{mg}{kT} z + C \tag{49}$$
$$e^{\ln P} = e^{-\frac{mg}{kT} z + C} \tag{50}$$
$$P = e^C e^{-\frac{mg}{kT} z} \tag{51}$$
$$P = (P(0))\, e^{-\frac{mg}{kT} z} \tag{52}$$

$$\boxed{P = P(0)\, e^{-\frac{mg}{kT} z}}$$

(b) Derive the corresponding formula for an adiabatic atmosphere, that is, the one in which (PV^γ), rather than (PV), stays constant. Also study the variation, with height, of the temperature T and the density n in such an atmosphere.

$$PV^\gamma = C \tag{53}$$

Use the Ideal Gas Law to substitute V in terms of T.

$$PV = NkT \tag{54}$$
$$V = \frac{NkT}{V} \tag{55}$$

Make this substitution in equation (53).

$$P\left(\frac{NkT}{P}\right)^\gamma = C \tag{56}$$
$$P\frac{N^\gamma k^\gamma T^\gamma}{P^\gamma} = C \tag{57}$$
$$\frac{P}{P^\gamma} T^\gamma = C \tag{58}$$
$$P^{1-\gamma} T^\gamma = C \tag{59}$$

Differentiate both sides of this equation.

$$\begin{aligned}
(1-\gamma)\,P^{-\gamma}T^{\gamma}dP+\gamma P^{1-\gamma}T^{\gamma-1}dT &= 0 && (60)\\
(\gamma-1)\,P^{-\gamma}T^{\gamma}dP &= \gamma P^{1-\gamma}T^{\gamma-1}dT && (61)\\
(\gamma-1)\,T^{\gamma}dP &= \gamma PT^{\gamma-1}dT && (62)\\
(\gamma-1)\,dP &= \gamma PT^{-1}dT && (63)\\
(\gamma-1)\,dP &= \gamma\frac{P}{T}dT && (64)\\
dP &= \frac{\gamma}{\gamma-1}\frac{P}{T}dT && (65)
\end{aligned}$$

We are still assuming hydrostatic equilibrium. We can therefore substitute dP with equation (40).

$$-\rho g dz = \frac{\gamma}{\gamma-1}\frac{P}{T}dT \qquad (66)$$

Use the Ideal Gas Law to substitute P in terms of T.

$$\begin{aligned}
PV &= NkT && (67)\\
P &= \frac{NkT}{V} && (68)
\end{aligned}$$

Make this substitution for P in equation (66).

$$\begin{aligned}
-\rho g dz &= \frac{\gamma}{\gamma-1}\left(\frac{NkT}{V}\right)\frac{1}{T}dT && (69)\\
-\rho V g dz &= \frac{\gamma}{\gamma-1}\frac{NkT}{T}dT && (70)\\
-\rho V g dz &= \frac{\gamma}{\gamma-1}NkdT && (71)\\
-\left(\frac{\gamma-1}{\gamma}\right)\frac{\rho V g dz}{Nk} &= dT && (72)\\
dT &= -\left(\frac{\gamma-1}{\gamma}\right)\frac{mgdz}{Nk} && (73)\\
T &= -\left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{Nk}+C && (74)
\end{aligned}$$

Evaluate this expression at $z = 0$ m, knowing the T at $z = 0$ is T_0.

$$\begin{aligned}
T_0 &= -\left(\frac{\gamma-1}{\gamma}\right)\frac{mg\,(0)}{Nk}+C && (75)\\
T_0 &= C && (76)
\end{aligned}$$

The full expresstion for T is:

$$\boxed{T\,(z) = T_0 - \left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{Nk}}$$

Now find an expression of $P(z)$, now knowing $T(z)$. From equation (65), we have:

$$dP = \frac{\gamma}{\gamma-1}\frac{P}{T}dT \tag{77}$$
$$\frac{dP}{P} = \frac{\gamma}{\gamma-1}\frac{dT}{T} \tag{78}$$
$$\int \frac{dP}{P} = \int \frac{\gamma}{\gamma-1}\frac{dT}{T} \tag{79}$$
$$\ln P = \frac{\gamma}{\gamma-1}\ln T + C \tag{80}$$
$$\exp[\ln P] = \exp\left[\frac{\gamma}{\gamma-1}\ln T + C\right] \tag{81}$$
$$P = \exp\left[\frac{\gamma}{\gamma-1}\ln T\right]\exp[C] \tag{82}$$
$$P = C\exp\left[\ln T^{\frac{\gamma}{\gamma-1}}\right] \tag{83}$$
$$P = CT^{\frac{\gamma}{\gamma-1}} \tag{84}$$

We know $T(z)$ already. Make this substitution.

$$P = C\left(T_0 - \left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{Nk}\right)^{\gamma/(\gamma-1)} \tag{85}$$

Evaluate this expression at $z = 0$, knowing that P at $z = 0$ is P_0.

$$P_0 = C\left(T_0 - \left(\frac{\gamma-1}{\gamma}\right)\frac{mg(0)}{Nk}\right)^{\gamma/(\gamma-1)} \tag{86}$$
$$P_0 = C(T_0)^{\gamma/(\gamma-1)} \tag{87}$$
$$C = \frac{P_0}{(T_0)^{\gamma/(\gamma-1)}} \tag{88}$$

Substitute C in equation (84).

$$P = \left(\frac{P_0}{(T_0)^{\gamma/(\gamma-1)}}\right)\left(T_0 - \left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{Nk}\right)^{\gamma/(\gamma-1)} \tag{89}$$
$$P = P_0\left(\frac{T_0}{T_0} - \left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{\gamma/(\gamma-1)} \tag{90}$$
$$P = P_0\left(1 - \left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{\gamma/(\gamma-1)} \tag{91}$$

$$\boxed{P(z) = P_0\left(1 - \left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{\gamma/(\gamma-1)}}$$

Find the density, n. Start with the definition of n.

$$n \equiv \frac{N}{V} \tag{92}$$

Find how V varies with z. Start by how V relates to $P(z)$.

$$PV^{\gamma} = C \tag{93}$$
$$dPV^{\gamma} + \gamma PV^{\gamma-1}dV = 0 \tag{94}$$
$$dPV^{\gamma} = -\gamma PV^{\gamma-1}dV \tag{95}$$
$$\frac{dP}{P}V^{\gamma} = -\gamma V^{\gamma-1}dV \tag{96}$$
$$\frac{dP}{P} = -\gamma V^{-1}dV \tag{97}$$
$$\frac{dP}{P} = -\gamma\frac{dV}{V} \tag{98}$$
$$\int\frac{dP}{P} = \int -\gamma\frac{dV}{V} \tag{99}$$
$$\ln P = -\gamma\ln V + C \tag{100}$$
$$\exp\left[\ln P\right] = \exp\left[-\gamma\ln V + C\right] \tag{101}$$
$$P = \exp\left[-\gamma\ln V\right]\exp\left[C\right] \tag{102}$$
$$P = CV^{-\gamma} \tag{103}$$
$$V = CP^{-1/\gamma} \tag{104}$$

We already know $P(z)$. Substitute this in the above expression.

$$V = C\left(P_0\left(1-\left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{\gamma/(\gamma-1)}\right)^{-1/\gamma} \tag{105}$$
$$V = CP_0^{-1/\gamma}\left(1-\left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{1/(\gamma-1)} \tag{106}$$

Evaluate V at $z = 0$, with $V(0) = V_0$.

$$V_0 = CP_0^{-1/\gamma}\left(1-\left(\frac{\gamma-1}{\gamma}\right)\frac{mg(0)}{NkT_0}\right)^{1/(\gamma-1)} \tag{107}$$
$$V_0 = CP_0^{-1/\gamma}(1)^{1/(\gamma-1)} \tag{108}$$
$$V_0 = CP_0^{-1/\gamma} \tag{109}$$
$$C = V_0P_0^{1/\gamma} \tag{110}$$

Substitute this expression for C in equation (106).

$$V = \left(V_0P_0^{1/\gamma}\right)P_0^{-1/\gamma}\left(1-\left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{1/(\gamma-1)} \tag{111}$$
$$V(z) = V_0\left(1-\left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{1/(\gamma-1)} \tag{112}$$

Use this equation for $V(z)$ to find $n(z)$.

$$n \equiv \frac{N}{V(z)} \tag{113}$$
$$n = \frac{N}{\left(V_0\left(1-\left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{1/(\gamma-1)}\right)} \tag{114}$$

$$\boxed{n = \frac{N}{V_0\left(1-\left(\frac{\gamma-1}{\gamma}\right)\frac{mgz}{NkT_0}\right)^{1/(\gamma-1)}}}$$

7.13 Consider an ideal Bose gas confined to a region of area A in two dimensions. Express the number of particles in the excited states, N_e , and the number of particles in the ground state, N_0 , in terms of z, T , and A, and show that the system does not exhibit Bose–Einstein condensation unless $T \to 0$ K. Refine your argument to show that, if the area A and the total number of particles N are held fixed and we require both N_e and N_0 to be of order N, then we do achieve condensation when

$$T \sim \frac{h^2}{mkl^2}\frac{1}{\ln N},$$

where $l\,[\sim (A/N)]$ is the mean interparticle distance in the system. Of course, if both A and $N \to \infty$, keeping l fixed, then the desired T does go to zero.

Star with the defintion of N_{exc}, for a Bose-Einstein distribution.

$$N_{exc} \equiv \int_{\epsilon_F}^{\infty} n(\epsilon)\, g(\epsilon)\, d\epsilon \tag{115}$$

where,

$$n(\epsilon) \equiv \frac{1}{\exp\left(\frac{\epsilon-\mu}{kT}\right)-1} \tag{116}$$

We can also substitute in for z.

$$z \equiv e^{\mu/kT} \tag{117}$$

$$n(\epsilon)\; z\; \frac{1}{z^{-1}\exp\left(\frac{\epsilon}{kT}\right)-1} \tag{118}$$

In 2-D, the density of states, $g(p)$, can be written as:

$$g(p)\, dp = \frac{A2\pi p dp}{h^2} \tag{119}$$

Knowing all of this, find N_{exc}.

$$N_{exc} = \int_0^{\infty} \frac{1}{z^{-1}\exp(\epsilon/kT)-1}\frac{A2\pi p}{h^2}dp \tag{120}$$

$$N_{exc} = \frac{2\pi A}{h^2}\int_0^{\infty} \frac{1}{z^{-1}\exp\left(\frac{p^2}{2mkT}\right)-1}pdp \tag{121}$$

Make the following change of variables substitution:

$$x = \frac{p^2}{2mkT} \tag{122}$$

$$dx = \frac{p}{mkT}dp \tag{123}$$

$$dp = \frac{mkT}{p}dx \tag{124}$$

$$N_{exc} = \frac{2\pi A}{h^2}\int_0^{\infty} \frac{1}{z^{-1}\exp(x)-1}p\left(\frac{mkT}{p}dx\right) \tag{125}$$

$$N_{exc} = \frac{2\pi A}{h^2}\int_0^{\infty} \frac{1}{z^{-1}\exp(x)-1}(mkT)\,dx \tag{126}$$

$$N_{exc} = \frac{2\pi mkTA}{h^2}\int_0^{\infty} \frac{1}{z^{-1}e^x-1}dx \tag{127}$$

From Appendix D, equation (6) we can solve this integral. Use this result.

$$\int_0^{\infty} \frac{dx}{z^{-1}e^x-1} = -\ln(1-z) \tag{128}$$

$$N_{exc} = \frac{2\pi mkTA}{h^2}\left(-\ln(1-z)\right) \tag{129}$$

$$\boxed{N_{exc} = \frac{2\pi mkTA}{h^2}\left(-\ln\left(1-z\right)\right)}$$

By equation (7.1.22), find the number of particles in the ground state, N_0.

$$z = \frac{N_0}{N_0+1} \tag{130}$$

$$N_0 z + z = N_0 \tag{131}$$

$$N_0\left(1-z\right) = z \tag{132}$$

$$\boxed{N_0 = \frac{z}{1-z}}$$

Now, assuming $N_0 \sim N_{exc} \sim N$, find how temperature reduces. From before we have;

$$N_{exc} = \frac{2\pi mkTA}{h^2}\left(-\ln\left(1-z\right)\right) \tag{133}$$

$$N = \frac{2\pi mkTA}{h^2}\ln\left(\frac{1}{1-z}\right) \tag{134}$$

$$1 = \frac{2\pi mkTA}{Nh^2}\ln\left(\frac{1}{1-z}\right) \tag{135}$$

$$1 = \left(\frac{A}{N}\right)\frac{2\pi mkT}{h^2}\ln\left(\frac{1}{1-z}\right) \tag{136}$$

$$l \equiv \sqrt{\frac{A}{N}} \tag{137}$$

$$1 = \left(l^2\right)\frac{2\pi mkT}{h^2}\ln\left(\frac{1}{1-z}\right) \tag{138}$$

$$1 = \frac{2\pi ml^2kT}{h^2}\ln\left(\frac{1}{1-z}\right) \tag{139}$$

$$z = \frac{N_0}{N_0+1} = \frac{N}{N+1} \tag{140}$$

$$1 = \frac{2\pi ml^2kT}{h^2}\ln\left(\frac{1}{1-\left(\frac{N}{N+1}\right)}\right) \tag{141}$$

$$1 = \frac{2\pi ml^2kT}{h^2}\ln\left(\frac{1}{\left(\frac{N+1-N}{N+1}\right)}\right) \tag{142}$$

$$1 = \frac{2\pi ml^2kT}{h^2}\ln\left(\frac{1}{\left(\frac{1}{N+1}\right)}\right) \tag{143}$$

$$1 = \frac{2\pi ml^2kT}{h^2}\ln\left(N+1\right) \tag{144}$$

$$N >> 1 \tag{145}$$

$$N+1 \sim N \tag{146}$$

$$1 = \frac{2\pi ml^2kT}{h^2}\ln\left(N\right) \tag{147}$$

$$T = \frac{h^2}{2\pi ml^2k\ln\left(N\right)} \tag{148}$$

$$\boxed{T \sim \frac{h^2}{mkl^2\ln N}}$$

7.21 Show that the mean energy per photon in a blackbody radiation cavity is very nearly $2.7kT$. Start with the mean energy of particles.

$$U = \int_0^\infty \epsilon n(\epsilon) g(\epsilon) \, d\epsilon \tag{149}$$
$$U = \int_0^\infty \hbar\omega n(\omega) g(\omega) \, d\omega \tag{150}$$
$$U = \int_0^\infty \hbar\omega n(\omega) g(\omega) \, d\omega \tag{151}$$

Equation (7.3.5) gives $n(\omega)$ and equation (7.3.6) gives $g(\omega)\,d\omega$. Make these substitutions.

$$n(\omega) = \frac{1}{e^{\hbar\omega/kT} - 1} \tag{152}$$
$$g(\omega)\,d\omega = \frac{V\omega^2}{\pi^2 c^3} d\omega \tag{153}$$
$$U = \int_0^\infty \hbar\omega \left(\frac{1}{e^{\hbar\omega/kT} - 1}\right)\left(\frac{V\omega^2}{\pi^2 c^3} d\omega\right) \tag{154}$$
$$U = \int_0^\infty \frac{V\hbar\omega^3}{\pi^2 c^3 \left(e^{\hbar\omega/kT} - 1\right)} d\omega \tag{155}$$
$$U = \frac{V\hbar}{\pi^2 c^3} \int_0^\infty \frac{\omega^3}{e^{\hbar\omega/kT} - 1} d\omega \tag{156}$$
$$x = \frac{\hbar\omega}{kT} \tag{157}$$
$$dx = \frac{\hbar}{kT} d\omega \tag{158}$$
$$d\omega = \frac{kT}{\hbar} dx \tag{159}$$
$$\omega = \frac{kT}{\hbar} x \tag{160}$$
$$U = \frac{V\hbar}{\pi^2 c^3} \int_0^\infty \left(\frac{kT}{\hbar}\right)^3 \frac{x^3}{e^x - 1}\left(\frac{kT}{\hbar} dx\right) \tag{161}$$
$$U = \frac{V\hbar}{\pi^2 c^3} \left(\frac{kT}{\hbar}\right)^4 \int_0^\infty \frac{x^3}{e^x - 1} dx \tag{162}$$

From the book, we can rewrite the integral above.

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15} \tag{163}$$

Make this substitution in the expression for average energy.

$$U = \frac{V\hbar}{\pi^2 c^3} \left(\frac{kT}{\hbar}\right)^4 \left(\frac{\pi^4}{15}\right) \tag{164}$$
$$U = \frac{V\hbar k^4 T^4 \pi^4}{15\hbar^4 \pi^2 c^3} \tag{165}$$
$$U = \frac{V k^4 T^4 \pi^2}{15\hbar^3 c^3} \tag{166}$$

Now that we've found the average energy for a system of photons emitted by a blackbody, find the number of photons emitted from a black body.

$$\langle N\rangle = \int_0^\infty n(\omega)\, g(\omega)\, d\omega \tag{167}$$
$$\langle N\rangle = \int_0^\infty \left(\frac{1}{e^{\hbar\omega/kT}-1}\right)\left(\frac{V\omega^2}{\pi^2 c^3}\right) d\omega \tag{168}$$
$$\langle N\rangle = \int_0^\infty \frac{V\omega^2}{\pi^2 c^3\left(e^{\hbar\omega/kT}-1\right)} d\omega \tag{169}$$
$$\langle N\rangle = \frac{V}{\pi^2 c^3}\int_0^\infty \frac{\omega^2}{e^{\hbar\omega/kT}-1} d\omega \tag{170}$$

Make a substitution inside the integral to make it easier to solve.

$$x = \frac{\hbar\omega}{kT} \tag{171}$$
$$\omega = \frac{kT}{\hbar}x \tag{172}$$
$$d\omega = \frac{kT}{\hbar}dx \tag{173}$$
$$\langle N\rangle = \frac{V}{\pi^2 c^3}\int_0^\infty \left(\frac{kT}{\hbar}x\right)^2 \frac{1}{e^x-1}\left(\frac{kT}{\hbar}dx\right) \tag{174}$$
$$\langle N\rangle = \frac{V}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^3\int_0^\infty \frac{x^2}{e^x-1}dx \tag{175}$$

The integral is well-known, and is related to Apéry's constant, $\zeta(3)$.

$$\int_0^\infty \frac{x^2}{e^x} = 2\zeta(3) = 2\,(1.202056903159594285399...) \tag{176}$$
$$\langle N\rangle = \frac{V}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^3 2\,(1.202056903159594285399...) \tag{177}$$
$$\langle N\rangle = \frac{Vk^3T^3}{\hbar^3\pi^2 c^3}2\,(1.202056903159594285399...) \tag{178}$$

To find the energy per photon from this blackbody, divide U and $\langle N\rangle$ as found before.

$$\frac{U}{\langle N\rangle} = \frac{\frac{Vk^4T^4\pi^2}{15\hbar^3c^3}}{\frac{Vk^3T^3}{\hbar^3\pi^2c^3}2\,(1.202056903159594285399...)} \tag{179}$$
$$\frac{U}{\langle N\rangle} = \frac{Vk^4T^4\pi^2}{15\hbar^3c^3}\,\frac{\hbar^3\pi^2c^3}{Vk^3T^3}\,\frac{1}{2\,(1.202056903159594285399...)} \tag{180}$$
$$\frac{U}{\langle N\rangle} = \frac{kT\pi^4}{15}\,\frac{1}{2\,(1.202056903159594285399...)} \tag{181}$$
$$\tag{182}$$
$$\frac{U}{\langle N\rangle} = \frac{\pi^4}{30\,(1.202056903159594285399...)}kT \tag{183}$$
$$\tag{184}$$

$$\boxed{\frac{U}{\langle N\rangle} \approx 2.70kT}$$

7.23 The sun may be regarded as a black body at a temperature of 5800 **K. Its diameter is about** 1.4×10^9 **m while its distance from the earth is about** 1.5×10^{11} **m.**

(a) Calculate the total radiant intensity (in W/m^2) of sunlight at the surface of the earth.

Equation (7.3.13) gives the luminosity of a blackbody per unit areas at a given temperature T. The total luminosity across the entire surface is then equation (7.3.13) times the surface area of the blackbody.

$$L = 4\pi R_\odot^2 \sigma T^4 \tag{185}$$

To find the intensity of radiation at a distance r away, use the fact that the intensity of radiation times the surface area of the shell surrounding the object radiating as a blackbody is equal to the luminosity of the blackbody.

$$4\pi r^2 I = L \tag{186}$$

$$I = \frac{L}{4\pi r^2} \tag{187}$$

$$I = \frac{4\pi R_\odot^2 \sigma T^4}{4\pi r^2} \tag{188}$$

$$I = \frac{R_\odot^2 \sigma T^4}{r^2} \tag{189}$$

$$I = \frac{\left(7 \times 10^8 \text{ m}\right)^2 \left(5.78 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}\right) (5780 \text{ K})}{\left(1.5 \times 10^{11} \text{ m}\right)^2} \tag{190}$$

$$\boxed{I = 1378 \text{ W m}^{-2}}$$

(b) What pressure would it exert on a perfectly absorbing surface placed normal to the rays of the sun?

Radiation pressure, for perfect absorbers, is defined as:

$$P_{rad} \equiv \frac{I}{c} \tag{191}$$

where I is the flux of electromagnetic radiation and c is the speed of light. Find radiation pressure, P_{rad}, assuming we are at the Earth's distance away from the sun.

$$P_{rad} = \frac{I}{c} \tag{192}$$

$$P_{rad} = \frac{1378 \text{ W m}^{-2}}{3 \times 10^8 \text{ m s}^{-1}} \tag{193}$$

$$\boxed{P_{rad} = 4.593 \times 10^{-6} \text{ Pa}}$$

(c) If a flat surface on a satellite, which faces the sun, were an ideal absorber and emitter, what equilibrium temperature would it ultimately attain?

The power emitted by the satellite is due to blackbody radiation. The Stefan-Boltzmann Law states that the power radiated by a ideal-emitting blackbody across all wavelengths per unit area is:

$$\frac{P}{A} = \sigma T^4 \tag{194}$$

As found in part (b), the incident intensity of sunlight per unit area is a constant 1378 W m^{-2}. Set the incident power per unit area equal to the radiated power per unit area and solve for T to find the equilibrium temperature, T_{eq}.

$$\sigma T_{eq}^4 = 1378 \tag{195}$$

$$T_{eq}^4 = \frac{1378}{\sigma} \tag{196}$$

$$T_{eq} = \left(\frac{1378}{\sigma}\right)^{1/4} \tag{197}$$

$$\boxed{T_{eq} = 394.8 \text{ K}}$$

7.33 Assuming the dispersion relation $\omega = Ak^s$, where ω is the angular frequency and k the wave number of a vibrational mode existing in a solid, show that the respective contribution toward the specific heat of the solid at low temperatures is proportional to $T^{3/s}$.
[Note that while $s = 1$ corresponds to the case of elastic waves in a lattice, $s = 2$ applies to spin waves propagating in a ferromagnetic system.
Start with the definition of C_V and U.

$$C_V \equiv \left(\frac{\partial U}{\partial T}\right)_V \tag{198}$$

$$U \equiv \int \epsilon(\omega)\, n(\omega)\, g(\omega)\, d\omega \tag{199}$$

Since we are talking about phonons, use $n(\omega)$ for Bosons, and $g(p)\,dp$ to convert to $g(\omega)\,d\omega$.

$$n(\omega) = \frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right) - 1} \tag{200}$$

$$g(p)\,dp = \frac{3V}{h^3} 4\pi p^2 dp \tag{201}$$

$$p = \hbar k \tag{202}$$

$$\omega = Ak^s \tag{203}$$

$$k^s = \frac{\omega}{A} \tag{204}$$

$$k = \left(\frac{\omega}{A}\right)^{1/s} \tag{205}$$

$$p = \hbar\left(\frac{\omega}{A}\right)^{1/s} \tag{206}$$

$$dp = \frac{\hbar}{s}\left(\frac{\omega}{A}\right)^{1/s-1} d\omega \tag{207}$$

$$g(\omega)\,d\omega = \frac{12\pi V}{h^3}\left(\hbar\left(\frac{\omega}{A}\right)^{1/s}\right)^2 \left(\frac{\hbar}{s}\left(\frac{\omega}{A}\right)^{1/s-1} d\omega\right) \tag{208}$$

$$g(\omega)\,d\omega = \frac{12\pi V}{h^3}\hbar^2\left(\frac{\omega}{A}\right)^{2/s} \frac{\hbar}{s}\left(\frac{\omega}{A}\right)^{1/s-1} d\omega \tag{209}$$

$$g(\omega)\,d\omega = \frac{12\pi V}{h^3}\frac{\hbar^3}{s}\left(\frac{\omega}{A}\right)^{\frac{2}{s}+\frac{1}{s}-1} d\omega \tag{210}$$

$$g(\omega)\,d\omega = \frac{12\pi V\hbar^3}{sh^3}\left(\frac{\omega}{A}\right)^{\frac{2+1-s}{s}} d\omega \tag{211}$$

$$g(\omega)\,d\omega = \frac{12\pi V}{sh^3}\left(\frac{h}{2\pi}\right)^3\left(\frac{\omega}{A}\right)^{\frac{3-s}{s}} d\omega \tag{212}$$

$$g(\omega)\,d\omega = \frac{12\pi V h^3}{sh^3 8\pi^3}\left(\frac{\omega}{A}\right)^{\frac{3-s}{s}} d\omega \tag{213}$$

$$g(\omega)\,d\omega = \frac{3\pi V}{2s\pi^3}\left(\frac{\omega}{A}\right)^{\frac{3-s}{s}} d\omega \tag{214}$$

$$g(\omega)\,d\omega = \frac{3\pi V}{2s\pi^3 A^{(3-s)/s}}\omega^{(3-s)/s} d\omega \tag{215}$$

Use these results to find C_V.

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V \tag{216}$$

$$C_V = \frac{\partial}{\partial T}\int_0^\omega \hbar\omega n \frac{3\pi V}{2s\pi^3 A^{(3-s)/s}}\omega^{(3-s)/s}d\omega \tag{217}$$

$$C_V = \int_0^\omega \hbar\omega \frac{\partial n}{\partial T}\frac{3\pi V}{2s\pi^3 A^{(3-s)/s}}\omega^{(3-s)/s}d\omega \tag{218}$$

$$C_V = \frac{3\pi V\hbar}{2s\pi^3 A^{(3-s)/s}}\int_0^\omega \omega\frac{\partial n}{\partial T}\omega^{(3-s)/s}d\omega \tag{219}$$

$$C_V = \frac{3V\hbar}{2s\pi^2 A^{(3-s)/s}}\int_0^\omega \omega\frac{\partial n}{\partial T}\omega^{(3-s)/s}d\omega \tag{220}$$

Find $\partial n/\partial T$.

$$\frac{\partial n}{\partial T} = \frac{\partial}{\partial T}\left(\frac{1}{\exp\left(\frac{\hbar\omega}{kT}\right)-1}\right) \tag{221}$$

$$\frac{\partial n}{\partial T} = -\frac{1}{\left(\exp\left(\frac{\hbar\omega}{kT}\right)-1\right)^2}\left(\exp\left(\frac{\hbar\omega}{kT}\right)\right)\left(-\frac{\hbar\omega}{kT^2}\right) \tag{222}$$

$$\frac{\partial n}{\partial T} = \frac{\exp\left(\frac{\hbar\omega}{kT}\right)}{\left(\exp\left(\frac{\hbar\omega}{kT}\right)-1\right)^2}\left(\frac{\hbar\omega}{kT^2}\right) \tag{223}$$

Use this result in equation (220).

$$C_V = \frac{3V\hbar}{2s\pi^2 A^{(3-s)/s}} \int_0^\omega \omega \left(\frac{\exp\left(\frac{\hbar\omega}{kT}\right)}{\left(\exp\left(\frac{\hbar\omega}{kT}\right)-1\right)^2} \left(\frac{\hbar\omega}{kT^2}\right)\right) \omega^{(3-s)/s} d\omega \tag{224}$$

$$C_V = \frac{3V\hbar^2}{2s\pi^2 kT^2 A^{(3-s)/s}} \int_0^\omega \omega^2 \frac{\exp\left(\frac{\hbar\omega}{kT}\right)}{\left(\exp\left(\frac{\hbar\omega}{kT}\right)-1\right)^2} \omega^{(3-s)/s} d\omega \tag{225}$$

$$C_V = \frac{3V\hbar^2}{2s\pi^2 kT^2 A^{(3-s)/s}} \int_0^\omega \frac{\exp\left(\frac{\hbar\omega}{kT}\right)}{\left(\exp\left(\frac{\hbar\omega}{kT}\right)-1\right)^2} \omega^{(3+s)/s} d\omega \tag{226}$$

$$x \equiv \frac{\hbar\omega}{kT} \tag{227}$$

$$\omega = \frac{kT}{\hbar} x \tag{228}$$

$$dx = -\frac{\hbar}{kT} d\omega \tag{229}$$

$$d\omega = -dx \frac{kT}{\hbar} \tag{230}$$

$$C_V = \frac{3V\hbar^2}{2s\pi^2 kT^2 A^{(3-s)/s}} \int_0^\omega \frac{e^x}{(e^x-1)^2} \left(\frac{kT}{\hbar} x\right)^{(3+s)/s} \left(-\frac{kT}{\hbar} dx\right) \tag{231}$$

$$C_V = -\frac{3V\hbar^2}{2s\pi^2 kT^2 A^{(3-s)/s}} \int_0^\omega \frac{e^x}{(e^x-1)^2} \left(\frac{kT}{\hbar}\right)^{(3+s)/s} (x)^{(3+s)/s} \frac{kT}{\hbar} dx \tag{232}$$

$$C_V = -\frac{3V\hbar^2}{2s\pi^2 kT^2 A^{(3-s)/s}} \int_0^\omega \frac{e^x}{(e^x-1)^2} \left(\frac{kT}{\hbar}\right)^{(3+2s)/s} (x)^{(3+s)/s} dx \tag{233}$$

$$C_V = -\frac{3V\hbar^2}{2s\pi^2 kT^2 A^{(3-s)/s}} \left(\frac{kT}{\hbar}\right)^{(3+2s)/s} \int_0^\omega \frac{e^x}{(e^x-1)^2} (x)^{(3+s)/s} dx \tag{234}$$

$$C_V = -\frac{3V\hbar^2}{2s\pi^2 kT^2 A^{(3-s)/s}} \left(\frac{k}{\hbar}\right)^{(3+2s)/s} T^{(3+2s)/s} \int_0^\omega \frac{e^x}{(e^x-1)^2} (x)^{(3+s)/s} dx \tag{235}$$

$$C_V \propto \frac{T^{(3+2s)/s}}{T^2} \tag{236}$$

$$C_V \propto T^{3/s} \tag{237}$$

$$\boxed{C_V \propto T^{3/s}}$$

8.1 Let the Fermi distribution at low temperatures be represented by a broken line, as shown in Figure 8.13, the line being tangential to the actual curve at $\epsilon = \mu$. Show that this approximate representation yields a "correct" result for the low-temperature specific heat of the Fermi gas, except that the numerical factor turns out to be smaller by a factor of $4/\pi^2$. Discuss, in a qualitative manner, the origin of this numerical discrepancy.

Find the equation of the line for $n(x)$ using figure 8.13.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{238}$$

$$m = \frac{1 - 0}{\xi - 2 - (\xi + 2)} \tag{239}$$

$$m = -\frac{1}{4} \tag{240}$$

$$y - y_1 = m(x - x_1) \tag{241}$$

$$y - \frac{1}{2} = -\frac{1}{4}(x - \xi) \tag{242}$$

$$y = \frac{-x + \xi + 2}{4} \tag{243}$$

Using this linear approximation for a segment of the graph, define $n(x)$.

$$n(x) = \begin{cases} 1 & x < \xi \\ \frac{-x+\xi+2}{4} & \xi - 2 < x < \xi + 2 \\ 0 & x > \xi + 2 \end{cases} \tag{244}$$

Find the density of states as a function of x, $g(x)\,dx$.

$$g(p)\,dp = \frac{V}{h^3}4\pi p^2 dp \times 2 \tag{245}$$
$$\epsilon = \frac{p^2}{2m} \tag{246}$$
$$p^2 = 2m\epsilon \tag{247}$$
$$2pdp = 2md\epsilon \tag{248}$$
$$dp = \frac{m}{p}d\epsilon \tag{249}$$
$$dp = \frac{m}{\sqrt{2m\epsilon}}d\epsilon \tag{250}$$
$$g(\epsilon)\,d\epsilon = \frac{V}{h^3}8\pi\,(2m\epsilon)\left(\frac{m}{\sqrt{2m\epsilon}}d\epsilon\right) \tag{251}$$
$$g(\epsilon)\,d\epsilon = \frac{V}{h^3}8\pi\sqrt{2m\epsilon}md\epsilon \tag{252}$$
$$g(\epsilon)\,d\epsilon = \frac{V}{h^3}4\pi\sqrt{2m\epsilon}2md\epsilon \tag{253}$$
$$g(\epsilon)\,d\epsilon = \frac{V}{h^3}4\pi\sqrt{\epsilon}\,(2m)^{3/2}\,d\epsilon \tag{254}$$
$$g(\epsilon)\,d\epsilon = 4\pi V\frac{(2m)^{3/2}}{h^3}\sqrt{\epsilon}d\epsilon \tag{255}$$
$$g(\epsilon)\,d\epsilon = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\sqrt{\epsilon}d\epsilon \tag{256}$$
$$x \equiv \frac{\epsilon}{kT} \tag{257}$$
$$\epsilon = kTx \tag{258}$$
$$d\epsilon = kTdx \tag{259}$$
$$g(x)\,dx = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}\sqrt{(kTx)}\,(kTdx) \tag{260}$$
$$g(x)\,dx = 4\pi V\left(\frac{2m}{h^2}\right)^{3/2}(kT)^{3/2}\sqrt{x}dx \tag{261}$$
$$g(x)\,dx = 4\pi V\left(\frac{2mkT}{h^2}\right)^{3/2}\sqrt{x}dx \tag{262}$$

Find U.

$$U \equiv \int_0^\infty \epsilon n(\epsilon)\,g(\epsilon)\,d\epsilon \tag{263}$$
$$U = \int_0^\infty \epsilon(x)\,n(x)\,g(x)\,dx \tag{264}$$

Use the result of equation (244) in equation (264) to limit the bounds of integration.

$$U = \int_0^{\xi-2} \epsilon(x)\,(1)\,g(x)\,dx + \int_{\xi-2}^{\xi+2} \epsilon(x)\left(\frac{-x+\xi+2}{4}\right)g(x)\,dx \tag{265}$$
$$U = \int_0^{\xi-2} \epsilon(x)\,g(x)\,dx + \int_{\xi-2}^{\xi+2} \epsilon(x)\left(\frac{-x+\xi+2}{4}\right)g(x)\,dx \tag{266}$$

Solve each integral separately.

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = \int_0^{\xi-2} (xkT)\left(4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \sqrt{x}dx\right) \tag{267}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = \int_0^{\xi-2} 4\pi V xkT \left(\frac{2mkT}{h^2}\right)^{3/2} \sqrt{x}dx \tag{268}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = 4\pi V kT \left(\frac{2mkT}{h^2}\right)^{3/2} \int_0^{\xi-2} x\sqrt{x}dx \tag{269}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} \int_0^{\xi-2} x^{3/2}dx \tag{270}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} \left[\frac{2}{5}x^{5/2}\Big|_0^{\xi-2}\right] \tag{271}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} \left[\frac{2}{5}(\xi-2)^{5/2} - 0\right] \tag{272}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = \frac{8\pi V}{5} \left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} (\xi-2)^{5/2} \tag{273}$$

Find the second integral.

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \int_{\xi-2}^{\xi+2} (xkT) \left(\frac{-x+\xi+2}{4}\right) \left(4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \sqrt{x} dx\right) \tag{274}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \int_{\xi-2}^{\xi+2} 4\pi V xkT \left(\frac{-x+\xi+2}{4}\right) \left(\frac{2mkT}{h^2}\right)^{3/2} \sqrt{x} dx \tag{275}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = 4\pi V kT \left(\frac{2mkT}{h^2}\right)^{3/2} \int_{\xi-2}^{\xi+2} x \left(\frac{-x+\xi+2}{4}\right) \sqrt{x} dx \tag{276}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \pi V (kT)^{5/2} \left(\frac{2m}{h^2}\right)^{3/2} \int_{\xi-2}^{\xi+2} x^{3/2} (-x+\xi+2) dx \tag{277}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \pi V (kT)^{5/2} \left(\frac{2m}{h^2}\right)^{3/2} \int_{\xi-2}^{\xi+2} -x^{5/2} + (\xi+2) x^{3/2} dx \tag{278}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \pi V (kT)^{5/2} \left(\frac{2m}{h^2}\right)^{3/2} \left[-\frac{2}{7} x^{7/2} + \frac{2}{5} (\xi+2) x^{5/2} \Big|_{\xi-2}^{\xi+2}\right] \tag{279}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \pi V (kT)^{5/2} \left(\frac{2m}{h^2}\right)^{3/2} \tag{280}$$

$$\times \left[-\frac{2}{7} (\xi+2)^{7/2} + \frac{2}{7} (\xi-2)^{7/2} + \frac{2}{5} (\xi+2) (\xi+2)^{5/2} - \frac{2}{5} (\xi+2) (\xi-2)^{5/2}\right] \tag{281}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \pi V (kT)^{5/2} \left(\frac{2m}{h^2}\right)^{3/2} \tag{282}$$

$$\times \left[-\frac{2}{7} (\xi+2)^{7/2} + \frac{2}{7} (\xi-2)^{7/2} + \frac{2}{5} (\xi+2)^{7/2} - \frac{2}{5} (\xi+2) (\xi-2)^{5/2}\right] \tag{283}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \pi V (kT)^{5/2} \left(\frac{2m}{h^2}\right)^{3/2} \tag{284}$$

$$\times \left[\left(\frac{2}{5} - \frac{2}{7}\right) (\xi+2)^{7/2} + \frac{2}{7} (\xi-2)^{7/2} - \frac{2}{5} (\xi+2) (\xi-2)^{5/2}\right] \tag{285}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x) n(x) g(x) dx = \pi V (kT)^{5/2} \left(\frac{2m}{h^2}\right)^{3/2} \left[\left(\frac{4}{35}\right) (\xi+2)^{7/2} + \frac{2}{7} (\xi-2)^{7/2} - \frac{2}{5} (\xi+2) (\xi-2)^{5/2}\right] \tag{286}$$

Use these results to then solve for U.

$$U = \left(\frac{8\pi V}{5} \left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} (\xi-2)^{5/2}\right) \tag{287}$$

$$+ \left(\pi V (kT)^{5/2} \left(\frac{2m}{h^2}\right)^{3/2} \left[\left(\frac{4}{35}\right) (\xi+2)^{7/2} + \frac{2}{7} (\xi-2)^{7/2} - \frac{2}{5} (\xi+2) (\xi-2)^{5/2}\right]\right) \tag{288}$$

$$U = \pi V \left(\frac{2m}{h^2}\right)^{3/2} (kT)^{5/2} \left[\frac{8}{5} (\xi-2)^{5/2} + \frac{4}{35} (\xi+2)^{7/2} + \frac{2}{7} (\xi-2)^{7/2} - \frac{2}{5} (\xi+2) (\xi-2)^{5/2}\right] \tag{289}$$

Find N.

$$N \equiv \int_0^\infty n(\epsilon) g(\epsilon) d\epsilon \tag{290}$$

$$N = \int_0^\infty n(x) g(x) dx \tag{291}$$

$$N = \int_0^{\xi-2} n(x) g(x) dx + \int_{\xi-2}^{\xi+2} n(x) g(x) dx \tag{292}$$

Solve these integrals separately.

$$\int_0^{\xi-2} n(x)\, g(x)\, dx = \int_0^{\xi-2} (1) \left(4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \sqrt{x}dx\right) \tag{293}$$

$$\int_0^{\xi-2} n(x)\, g(x)\, dx = \int_0^{\xi-2} 4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \sqrt{x}dx \tag{294}$$

$$\int_0^{\xi-2} n(x)\, g(x)\, dx = 4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \int_0^{\xi-2} x^{1/2}dx \tag{295}$$

$$\int_0^{\xi-2} n(x)\, g(x)\, dx = 4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \left[\frac{2}{3}x^{3/2}\Big|_0^{\xi-2}\right] \tag{296}$$

$$\int_0^{\xi-2} n(x)\, g(x)\, dx = 4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \frac{2}{3}(\xi-2)^{3/2} \tag{297}$$

$$\int_0^{\xi-2} n(x)\, g(x)\, dx = \frac{8\pi V}{3} \left(\frac{2mkT}{h^2}\right)^{3/2} (\xi-2)^{3/2} \tag{298}$$

Now find the second integral.

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \int_{\xi-2}^{\xi+2} \left(\frac{-x+\xi+2}{4}\right) \left(4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \sqrt{x}dx\right) \tag{299}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \int_{\xi-2}^{\xi+2} (-x+\xi+2)\sqrt{x}dx \tag{300}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \int_{\xi-2}^{\xi+2} -x\sqrt{x} + (\xi+2)\sqrt{x}dx \tag{301}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \int_{\xi-2}^{\xi+2} -x^{3/2} + (\xi+2)\, x^{1/2}dx \tag{302}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \left[-\frac{2}{5}x^{5/2} + \frac{2}{3}(\xi+2)\, x^{3/2}\Big|_{\xi-2}^{\xi+2}\right] \tag{303}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \tag{304}$$

$$\times \left[-\frac{2}{5}(\xi+2)^{5/2} + \frac{2}{5}(\xi-2)^{5/2} + \frac{2}{3}(\xi+2)(\xi+2)^{3/2} - \frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right] \tag{305}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \tag{306}$$

$$\times \left[-\frac{2}{5}(\xi+2)^{5/2} + \frac{2}{5}(\xi-2)^{5/2} + \frac{2}{3}(\xi+2)^{5/2} - \frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right] \tag{307}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \left[\left(\frac{2}{3}-\frac{2}{5}\right)(\xi+2)^{5/2} + \frac{2}{5}(\xi-2)^{5/2} - \frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right] \tag{308}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \left[\left(\frac{4}{15}\right)(\xi+2)^{5/2} + \frac{2}{5}(\xi-2)^{5/2} - \frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right] \tag{309}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \left[\frac{4}{15}(\xi+2)^{5/2} + \frac{2}{5}(\xi-2)^{5/2} - \frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right] \tag{310}$$

Use these results to find N.

$$N = \left(\frac{8\pi V}{3}\left(\frac{2mkT}{h^2}\right)^{3/2}(\xi-2)^{3/2}\right) \tag{311}$$

$$+\left(\pi V\left(\frac{2mkT}{h^2}\right)^{3/2}\left[\frac{4}{15}(\xi+2)^{5/2}+\frac{2}{5}(\xi-2)^{5/2}-\frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right]\right) \tag{312}$$

$$N = \pi V\left(\frac{2mkT}{h^2}\right)^{3/2}\left(\frac{8}{3}(\xi-2)^{3/2}+\frac{4}{15}(\xi+2)^{5/2}+\frac{2}{5}(\xi-2)^{5/2}-\frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right) \tag{313}$$

Divide equations (289) and (313) to find U/N.

$$\frac{U}{N} = \frac{\pi V\left(\frac{2m}{h^2}\right)^{3/2}(kT)^{5/2}\left[\frac{8}{5}(\xi-2)^{5/2}+\frac{4}{35}(\xi+2)^{7/2}+\frac{2}{7}(\xi-2)^{7/2}-\frac{2}{5}(\xi+2)(\xi-2)^{5/2}\right]}{\pi V\left(\frac{2mkT}{h^2}\right)^{3/2}\left(\frac{8}{3}(\xi-2)^{3/2}+\frac{4}{15}(\xi+2)^{5/2}+\frac{2}{5}(\xi-2)^{5/2}-\frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right)} \tag{314}$$

$$\frac{U}{N} = \frac{(kT)\left[\frac{8}{5}(\xi-2)^{5/2}+\frac{4}{35}(\xi+2)^{7/2}+\frac{2}{7}(\xi-2)^{7/2}-\frac{2}{5}(\xi+2)(\xi-2)^{5/2}\right]}{\left(\frac{8}{3}(\xi-2)^{3/2}+\frac{4}{15}(\xi+2)^{5/2}+\frac{2}{5}(\xi-2)^{5/2}-\frac{2}{3}(\xi+2)(\xi-2)^{3/2}\right)} \tag{315}$$

$$\frac{U}{N} = \frac{(kT)\left(\frac{8}{5}(\xi-2)^{5/2}+\frac{4}{35}(\xi+2)^{7/2}+\frac{2}{7}(\xi-2)^{7/2}-\frac{2}{5}(\xi)(\xi-2)^{5/2}-\frac{4}{5}(\xi-2)^{5/2}\right)}{\left(\frac{8}{3}(\xi-2)^{3/2}+\frac{4}{15}(\xi+2)^{5/2}+\frac{2}{5}(\xi-2)^{5/2}-\frac{2}{3}(\xi)(\xi-2)^{3/2}-\frac{4}{3}(\xi-2)^{3/2}\right)} \tag{316}$$

$$\frac{U}{N} = kT\frac{\left(\frac{4}{5}(\xi-2)^{5/2}+\frac{4}{35}(\xi+2)^{7/2}+\frac{2}{7}(\xi-2)^{7/2}-\frac{2}{5}(\xi)(\xi-2)^{5/2}\right)}{\left(\frac{4}{3}(\xi-2)^{3/2}+\frac{4}{15}(\xi+2)^{5/2}+\frac{2}{5}(\xi-2)^{5/2}-\frac{2}{3}(\xi)(\xi-2)^{3/2}\right)} \tag{317}$$

$$\frac{U}{N} = kT\frac{\frac{1}{5}}{\frac{1}{3}}\frac{\left(4(\xi-2)^{5/2}+\frac{4}{7}(\xi+2)^{7/2}+\frac{10}{7}(\xi-2)^{7/2}-2(\xi)(\xi-2)^{5/2}\right)}{\left(4(\xi-2)^{3/2}+\frac{4}{5}(\xi+2)^{5/2}+\frac{6}{5}(\xi-2)^{5/2}-2(\xi)(\xi-2)^{3/2}\right)} \tag{318}$$

$$\frac{U}{N} = \frac{3kT}{5}\frac{\left(4(\xi-2)^{5/2}+\frac{4}{7}(\xi+2)^{7/2}+\frac{10}{7}(\xi-2)^{7/2}-2(\xi)(\xi-2)^{5/2}\right)}{\left(4(\xi-2)^{3/2}+\frac{4}{5}(\xi+2)^{5/2}+\frac{6}{5}(\xi-2)^{5/2}-2(\xi)(\xi-2)^{3/2}\right)} \tag{319}$$

Apply the limiting case $\xi \to \infty$.

$$\xi \to \infty \tag{320}$$

$$(\xi-2) \approx (\xi+2) \approx \xi \tag{321}$$

$$\frac{U}{N} \approx \frac{3kT}{5}\frac{\left(4(\xi)^{5/2}+\frac{4}{7}(\xi)^{7/2}+\frac{10}{7}(\xi)^{7/2}-2(\xi)(\xi)^{5/2}\right)}{\left(4(\xi)^{3/2}+\frac{4}{5}(\xi)^{5/2}+\frac{6}{5}(\xi)^{5/2}-2(\xi)(\xi)^{3/2}\right)} \tag{322}$$

$$(\xi-2) \approx (\xi+2) \approx \xi \tag{323}$$

$$\frac{U}{N} = \frac{3kT}{5}\frac{\left(4(\xi)^{5/2}+\frac{4}{7}(\xi)^{7/2}+\frac{10}{7}(\xi)^{7/2}-2(\xi)^{7/2}\right)}{\left(4(\xi)^{3/2}+\frac{4}{5}(\xi)^{5/2}+\frac{6}{5}(\xi)^{5/2}-2(\xi)^{5/2}\right)} \tag{324}$$

$$\frac{U}{N} = \frac{3kT}{5}\frac{\xi^{5/2}}{\xi^{3/2}}\frac{\left(4+\frac{4}{7}(\xi)+\frac{10}{7}(\xi)-2(\xi)\right)}{\left(4+\frac{4}{5}(\xi)+\frac{6}{5}(\xi)-2(\xi)\right)} \tag{325}$$

$$\frac{U}{N} = \frac{3kT}{5}\xi\left(\frac{4}{4}\right) \tag{326}$$

$$\frac{U}{N} = \frac{3kT}{5}\xi \tag{327}$$

$$U = \frac{3NkT}{5}\xi \tag{328}$$

Find C_V.

$$C_V \equiv \left(\frac{\partial U}{\partial T}\right)_N \tag{329}$$

$$C_V = \frac{\partial}{\partial T}\left(\frac{3NkT}{5}\xi\right)_N \tag{330}$$

$$C_V = \frac{3Nk}{5}\xi \tag{331}$$

$$\xi \equiv \frac{\mu}{kT} \tag{332}$$

$$C_V = \frac{3Nk}{5}\left(\frac{\epsilon_F}{kT}\right) \tag{333}$$

$$C_V = \frac{3N\epsilon_F}{5T} \tag{334}$$

This expression for C_V is quite different that equation (8.1.39) in the book. I'm not sure why, but my higher order ξ terms canceled out when I don't think they should have. Unfortunately, I am running out of time so I am unable to pursue this problem any further. According to equation (8.1.39), I would've expected C_V to come out to:

$$C_{V,true} = \frac{\pi^2 Nk^2 T}{2\epsilon_F} \tag{335}$$

$$\frac{C_{V,true}}{C_V} = \frac{\frac{\pi^2 Nk^2 T}{2\epsilon_F}}{\frac{3N\epsilon_F}{5T}} \tag{336}$$

$$\frac{C_{V,true}}{C_V} = \frac{\pi^2 Nk^2 T}{2\epsilon_F}\frac{5T}{3N\epsilon_F} \tag{337}$$

$$\frac{C_{V,true}}{C_V} = \frac{5\pi^2 k^2 T^2}{6\epsilon_F^2} \tag{338}$$

As shown, the ratio of the approximated C_V to the true C_V is far from $4/\pi^2$. However, if my result did yield a fraction of $4\pi^2$ difference, I would assume the source of this discrepancy would be a result omitting half of the graph of n, since $4/\pi^2$ is very nearly 1/2.

8.5 Evaluate $\left((\partial^2 P/\partial T^2\right)_V$, $\left(\partial^2\mu/\partial T^2\right)_V$, and $\left(\partial^2\mu/\partial T^2\right)_P$ of an ideal Fermi gas and check that your results satisfy the thermodynamic relations

$$C_V = VT\left(\frac{\partial^2 P}{\partial T^2}\right)_V - NT\left(\frac{\partial^2\mu}{\partial T^2}\right)_V$$

and

$$C_P = -NT\left(\frac{\partial^2\mu}{\partial T^2}\right)_P.$$

Examine the low-temperature behavior of these quantities.

First, use equation (8.1.38), which gives pressure, P, in terms of T. Then take the second derivative with respect to T.

$$P = \frac{2}{5}n\epsilon_F\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + ...\right] \tag{339}$$

$$\frac{\partial P}{\partial T} = \frac{2}{5}n\epsilon_F\left[\frac{5\pi^2}{6}\left(\frac{kT}{\epsilon_F}\right)\left(\frac{k}{\epsilon_F}\right)\right] \tag{340}$$

$$\frac{\partial P}{\partial T} = \frac{2\pi^2 nk^2}{3\epsilon_F}T \tag{341}$$

$$\frac{\partial^2 P}{\partial T^2} = \frac{2\pi^2 nk^2}{3\epsilon_F} \tag{342}$$

Now, use equation (8.1.35) to find μ in terms of T. Then, take the second derivative with respect to T.

$$\mu = \epsilon_F \left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + ...\right] \tag{343}$$

$$\frac{\partial \mu}{\partial T} = \epsilon_F \left[-\frac{\pi^2}{6}\left(\frac{kT}{\epsilon_F}\right)\left(\frac{k}{\epsilon_F}\right)\right] \tag{344}$$

$$\frac{\partial \mu}{\partial T} = \frac{\pi^2 k^2}{6\epsilon_F} T \tag{345}$$

$$\frac{\partial^2 \mu}{\partial T^2} = \frac{\pi^2 k^2}{6\epsilon_F} \tag{346}$$

Since μ has no dependence on V nor P, we can therefore say:

$$\left(\frac{\partial^2 \mu}{\partial T^2}\right)_V = \left(\frac{\partial^2 \mu}{\partial T^2}\right)_P = \frac{\pi^2 k^2}{6\epsilon_F} \tag{347}$$

Now check the first thermodynamic relation, with C_V given by equation (8.3.39).

$$C_V = \frac{\pi^2 N k^2 T}{2\epsilon_F} \tag{348}$$

$$VT\left(\frac{\partial^2 P}{\partial T^2}\right)_V - NT\left(\frac{\partial^\mu}{\partial T^2}\right)_V = VT\left(\frac{2\pi^2 n k^2}{3\epsilon_F}\right) - NT\left(\frac{\pi^2 k^2}{6\epsilon_F}\right) \tag{349}$$

$$VT\left(\frac{\partial^2 P}{\partial T^2}\right)_V - NT\left(\frac{\partial^\mu}{\partial T^2}\right)_V = \frac{2\pi^2 n k^2 VT}{3\epsilon_F} - \frac{NT\pi^2 k^2}{6\epsilon_F} \tag{350}$$

$$VT\left(\frac{\partial^2 P}{\partial T^2}\right)_V - NT\left(\frac{\partial^\mu}{\partial T^2}\right)_V = \frac{2\pi^2 N k^2 T}{3\epsilon_F} - \frac{NT\pi^2 k^2}{6\epsilon_F} \tag{351}$$

$$VT\left(\frac{\partial^2 P}{\partial T^2}\right)_V - NT\left(\frac{\partial^\mu}{\partial T^2}\right)_V = \frac{4\pi^2 N k^2 T - \pi^2 N k^2 T}{6\epsilon_F} \tag{352}$$

$$VT\left(\frac{\partial^2 P}{\partial T^2}\right)_V - NT\left(\frac{\partial^\mu}{\partial T^2}\right)_V = \frac{3\pi^2 N k^2 T}{6\epsilon_F} \tag{353}$$

$$VT\left(\frac{\partial^2 P}{\partial T^2}\right)_V - NT\left(\frac{\partial^\mu}{\partial T^2}\right)_V = \frac{\pi^2 N k^2 T}{2\epsilon_F} \tag{354}$$

Equations (348) and (354) are the same!

$$\boxed{C_V = VT\left(\frac{\partial^2 P}{\partial T^2}\right)_V - NT\left(\frac{\partial^\mu}{\partial T^2}\right)_V}$$

Now, find C_P.

$$C_P \equiv \frac{\partial}{\partial T}(U - PV) \tag{355}$$

From equation (8.1.37), we know:

$$U = \frac{3N\epsilon_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + ...\right] \tag{356}$$

And from equation (8.1.38);

$$PV = \frac{2}{3}U \tag{357}$$

$$PV = \frac{2N\epsilon_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon}\right)^2 + ...\right] \tag{358}$$

Knowing this, find C_P.

$$C_P \equiv \frac{\partial}{\partial T}(U - PV) \tag{359}$$

$$C_P = \frac{\partial}{\partial T}\left(\left(\frac{3N\epsilon_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + ...\right]\right) - \left(\frac{2N\epsilon_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon}\right)^2 + ...\right]\right)\right) \tag{360}$$

$$C_P = \frac{\partial}{\partial T}\left(\frac{3N\epsilon_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + ...\right] - \frac{2N\epsilon_F}{5}\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon}\right)^2 + ...\right]\right) \tag{361}$$

$$C_P = \left(\frac{3N\epsilon_F}{5} - \frac{2N\epsilon_F}{5}\right)\frac{\partial}{\partial T}\left(\left[1 + \frac{5\pi^2}{12}\left(\frac{kT}{\epsilon_F}\right)^2 + ...\right]\right) \tag{362}$$

$$C_P = \left(\frac{N\epsilon_F}{5}\right)\left(\frac{5\pi^2}{6}\left(\frac{kT}{\epsilon_F}\right)\left(\frac{k}{\epsilon_F}\right)\right) \tag{363}$$

$$C_P = \frac{N\pi^2 k^2 T}{6\epsilon_F} \tag{364}$$

$$\boxed{C_P = \frac{N\pi^2 k^2 T}{6\epsilon_F}}$$

8.16 The observed value of γ, see equation (8.3.6), for sodium is 4.3×10^{-4} cal mole^{-1} K^{-2}. Evaluate the Fermi energy ϵ_F and the number density n of the conduction electrons in the sodium metal. Compare the latter result with the number density of atoms (given that, for sodium, $\rho = 0.954$ g cm^{-3} and $M = 23$).

Equation (8.3.5) gives C_V in terms of the Fermi energy ϵ_F.

$$C_V = \frac{\pi^2}{2}Nk\left(\frac{kT}{\epsilon_F}\right) \tag{365}$$

Equation (8.3.6) relates C_V to γ.

$$C_V = \gamma T + \delta T^3 \tag{366}$$

At low temperatures, equation (366) reduces to:

$$C_V \approx \gamma T \tag{367}$$

Set equation (365) equal to (367), and solve for ϵ_F.

$$\frac{\pi^2}{2}Nk\left(\frac{kT}{\epsilon_F}\right) = \gamma T \tag{368}$$

$$\frac{\pi^2}{2}Nk\left(\frac{k}{\epsilon_F}\right) = \gamma \tag{369}$$

$$\epsilon_F = \frac{\pi^2}{2}Nk\left(\frac{k}{\gamma}\right) \tag{370}$$

Find the Fermi energy.

$$\epsilon_F = \frac{\pi^2}{2}\left(6.022 \times 10^{23}\ \text{mol}^{-1}\right)\left(1.38 \times 10^{-23}\ \text{J K}^{-1}\right)\left(\frac{\left(1.38 \times 10^{-23}\ \text{J K}^{-1}\right)}{\left(4.3 \times 10^{-4}\ \text{cal mol}^{-1}\ \text{K}^{-2}\right)}\right) \tag{371}$$

$$\epsilon_F = 1.316 \times 10^{-18}\,\frac{\text{J}^2}{\text{cal}} \times \frac{1\ \text{cal}}{4.186\ \text{J}} \tag{372}$$

$$\boxed{\epsilon_F = 3.14 \times 10^{-19}\ \text{J} = 1.97\ \text{eV}}$$

To find the number density of electrons, use equation (8.1.24) and solve for N/V.

$$\epsilon_F = \left(\frac{6\pi^2 n}{g}\right)^{2/3} \frac{\hbar^2}{2m'} \tag{373}$$

$$\epsilon_F^{3/2} = \frac{6\pi^2 n}{g} \left(\frac{\hbar^2}{2m'}\right)^{3/2} \tag{374}$$

$$\left(\frac{2m'\epsilon_F}{\hbar^2}\right)^{3/2} = \frac{6\pi^2 n}{g} \tag{375}$$

$$n = \frac{g}{6\pi^2} \left(\frac{2m'\epsilon_F}{\hbar^2}\right)^{3/2} \tag{376}$$

The footnote on page 248 gives m' in terms of the mass of a free electron, m_e. Also, $g = 2$.

$$m' = 0.98 m_e \tag{377}$$

$$n = \frac{g}{6\pi^2} \left(\frac{2\left(0.98 m_e\right)\epsilon_F}{\hbar^2}\right)^{3/2} \tag{378}$$

$$n = \frac{(2)}{6\pi^2} \left(\frac{2\left(0.98\left(9.11 \times 10^{-31}\right)\right)\left(3.14 \times 10^{-19}\right)}{\left(1.055 \times 10^{-34}\right)^2}\right)^{3/2} \tag{379}$$

$$\boxed{n = 1.207 \times 10^{28}\ \mathrm{m}^{-3}}$$

Find the number density for sodium atoms and compare.

$$n_{atom} = \frac{\rho_{atom}}{m_{atom}} \tag{380}$$

$$n_{atom} = \frac{954\ \mathrm{kg\ m}^{-3}}{23 m_p} \tag{381}$$

$$n_{atom} = \frac{954\ \mathrm{kg\ m}^{-3}}{23\left(1.67 \times 10^{-27}\ \mathrm{kg}\right)} \tag{382}$$

$$\boxed{n_{atom} = 2.484 \times 10^{28}\ \mathrm{m}^{-3}}$$

$$\frac{n_e}{n_{atom}} = \frac{1.207 \times 10^{28}}{2.484 \times 10^{28}} \tag{383}$$

$$\boxed{\frac{n_e}{n_{atom}} = 0.4862}$$

This result states that the number density of electrons is nearly 1/2 the number density of sodium atoms signifying that for every 2 sodium atoms, one electron is free for conduction in the metal.

8.18 Show that the ground-state energy E_0 of a relativistic gas of electrons is given by

$$E_0 = \frac{\pi V m^4 c^5}{3h^3} B(x),$$

where

$$B(x) = 8x^3 \left[\left(x^2 + 1\right)^{1/2} - 1\right] - A(x),$$

$A(x)$ **and** x **being given by equations** (8.5.13) **and** (8.5.14)**. Check that the foregoing result for** E_0 **and equation** (8.5.12) **for** P_0 **satisfy the thermodynamic relations**

$$E_0 + P_0 V = N\mu_0 \text{ and } P_0 = -\left(\partial E_0/\partial V\right)_N.$$

Start with the definition of energy.

$$E_0 = \int_0^\infty \epsilon n(\epsilon) g(\epsilon) d\epsilon \tag{384}$$

For Fermi-Dirac distribution, $n(\epsilon)$ can be approximated to be a step function, from 1 for $\epsilon < \epsilon_F$ to 0 for $\epsilon > \epsilon_F$.

$$E_0 = \int_0^{\epsilon_F} \epsilon g(\epsilon) d\epsilon \tag{385}$$

Instead of integrating over energy ϵ, integrate over momentum p.

$$E_0 = \int_0^{p_F} \epsilon(p) g(p) dp \tag{386}$$

Where $g(p)$ is defined as:

$$g(p) dp = \frac{2V}{h^3} 4\pi p^2 dp \tag{387}$$

Plugging into equation (386), we get:

$$E_0 = \int_0^{p_F} \epsilon(p) \left(\frac{8\pi V}{h^3} p^2 dp\right) \tag{388}$$

$$E_0 = \frac{8\pi V}{h^3} \int_0^{p_F} \epsilon(p) p^2 dp \tag{389}$$

Relativistic energy in terms of momentum, $\epsilon(p)$, is:

$$\epsilon(p) \equiv \sqrt{p^2c^2 + (mc^2)^2} - mc^2 \tag{390}$$

$$\epsilon(p) = \sqrt{p^2c^2 + m^2c^4} - mc^2 \tag{391}$$

Plug equation (391) into equation (389).

$$E_0 = \frac{8\pi V}{h^3} \int_0^{p_F} \left(\sqrt{p^2c^2 + m^2c^4} - mc^2\right) p^2 dp \tag{392}$$

Use the following substitutions.

$$p = mc\sinh\theta \tag{393}$$
$$dp = mc\cosh\theta d\theta \tag{394}$$
$$p_f = mc\sinh\theta_f \tag{395}$$
$$\theta_f = \sinh^{-1}\left(\frac{p_F}{mc}\right) \tag{396}$$
$$E_0 = \frac{8\pi V}{h^3}\int_0^{\theta_F}\left(\sqrt{(mc\sinh\theta)^2 c^2 + m^2c^4} - mc^2\right)(mc\sinh\theta)^2(mc\cosh\theta d\theta) \tag{397}$$
$$E_0 = \frac{8\pi V}{h^3}\int_0^{\theta_F}\left(\sqrt{(mc\sinh\theta)^2 c^2 + m^2c^4} - mc^2\right)m^2c^2\sinh^2\theta(mc\cosh\theta d\theta) \tag{398}$$
$$E_0 = \frac{8\pi V}{h^3}\int_0^{\theta_F}\left(\sqrt{(mc\sinh\theta)^2 c^2 + m^2c^4} - mc^2\right)m^3c^3\sinh^2\theta\cosh\theta d\theta \tag{399}$$
$$E_0 = \frac{8\pi Vm^3c^3}{h^3}\int_0^{\theta_F}\left(\sqrt{(mc\sinh\theta)^2 c^2 + m^2c^4} - mc^2\right)\sinh^2\theta\cosh\theta d\theta \tag{400}$$
$$E_0 = \frac{8\pi Vm^3c^3}{h^3}\int_0^{\theta_F}\left(\sqrt{\left(m^2c^2\sinh^2\theta\right)c^2 + m^2c^4} - mc^2\right)\sinh^2\theta\cosh\theta d\theta \tag{401}$$
$$E_0 = \frac{8\pi Vm^3c^3}{h^3}\int_0^{\theta_F}\left(\sqrt{m^2c^4\sinh^2\theta + m^2c^4} - mc^2\right)\sinh^2\theta\cosh\theta d\theta \tag{402}$$
$$E_0 = \frac{8\pi Vm^3c^3}{h^3}\int_0^{\theta_F}\left(\sqrt{m^2c^4}\sqrt{\sinh^2\theta + 1} - mc^2\right)\sinh^2\theta\cosh\theta d\theta \tag{403}$$
$$E_0 = \frac{8\pi Vm^3c^3}{h^3}\int_0^{\theta_F}\left(mc^2\sqrt{\sinh^2\theta + 1} - mc^2\right)\sinh^2\theta\cosh\theta d\theta \tag{404}$$
$$E_0 = \frac{8\pi Vm^3c^3}{h^3}\int_0^{\theta_F}mc^2\left(\sqrt{\sinh^2\theta + 1} - 1\right)\sinh^2\theta\cosh\theta d\theta \tag{405}$$
$$E_0 = \frac{8\pi Vm^4c^5}{h^3}\int_0^{\theta_F}\left(\sqrt{\sinh^2\theta + 1} - 1\right)\sinh^2\theta\cosh\theta d\theta \tag{406}$$
$$\sinh^2\theta + 1 = \cosh^2\theta \tag{407}$$
$$E_0 = \frac{8\pi Vm^4c^5}{h^3}\int_0^{\theta_F}\left(\sqrt{\left(\cosh^2\theta\right)} - 1\right)\sinh^2\theta\cosh\theta d\theta \tag{408}$$
$$E_0 = \frac{8\pi Vm^4c^5}{h^3}\int_0^{\theta_F}(\cosh\theta - 1)\sinh^2\theta\cosh\theta d\theta \tag{409}$$
$$E_0 = \frac{8\pi Vm^4c^5}{h^3}\int_0^{\theta_F}\sinh^2\theta\cosh^2\theta - \sinh^2\theta\cosh\theta d\theta \tag{410}$$

Solve the first integral.

$$\int \sinh^2\theta\cosh^2\theta d\theta \tag{411}$$

$$\cosh^2\theta = \frac{1}{2}\cosh 2\theta + \frac{1}{2} \tag{412}$$

$$\int \sinh^2\theta\cosh^2\theta d\theta = \int \sinh^2\theta\left(\frac{1}{2}\cosh 2\theta + \frac{1}{2}\right)d\theta \tag{413}$$

$$\int \sinh^2\theta\cosh^2\theta d\theta = \int \frac{1}{2}\sinh^2\theta\cosh 2\theta + \frac{1}{2}\sinh^2\theta d\theta \tag{414}$$

$$\cosh 2\theta \equiv 2\sinh^2\theta + 1 \tag{415}$$

$$\int \sinh^2\theta\cosh^2\theta d\theta = \int \frac{1}{2}\sinh^2\theta\left(2\sinh^2\theta + 1\right) + \frac{1}{2}\sinh^2\theta d\theta \tag{416}$$

$$\int \sinh^2\theta\cosh^2\theta d\theta = \int \sinh^4\theta + \frac{1}{2}\sinh^2\theta + \frac{1}{2}\sinh^2\theta d\theta \tag{417}$$

$$\int \sinh^2\theta\cosh^2\theta d\theta = \int \sinh^4\theta + \sinh^2\theta d\theta \tag{418}$$

$$\sinh^2\theta \equiv \frac{1}{2}\cosh 2\theta - \frac{1}{2} \tag{419}$$

$$\int \sinh^2\theta\cosh^2\theta d\theta = \int \sinh^4\theta + \left(\frac{1}{2}\cosh 2\theta - \frac{1}{2}\right)d\theta \tag{420}$$

$$\int \sinh^2\theta\cosh^2\theta d\theta = \int \sinh^4\theta + \frac{1}{2}\cosh 2\theta - \frac{1}{2}d\theta \tag{421}$$

Use equation (8.5.12) to help solve this integral.

$$\int_0^{\theta_F} \sinh^4\theta d\theta \equiv \frac{1}{8}A(x) \tag{422}$$

$$\int_0^{\theta_F} \sinh^2\theta\cosh^2\theta d\theta = \frac{1}{8}A(x) + \int_0^{\theta_F} \frac{1}{2}\cosh 2\theta - \frac{1}{2}d\theta \tag{423}$$

$$\int_0^{\theta_F} \sinh^2\theta\cosh^2\theta d\theta = \frac{1}{8}A(x) + \left[\frac{1}{4}\sinh 2\theta - \frac{1}{2}\theta\bigg|_0^{\theta_F}\right] \tag{424}$$

$$\int_0^{\theta_F} \sinh^2\theta\cosh^2\theta d\theta = \frac{1}{8}A(x) + \frac{1}{4}\sinh 2\theta_F - \frac{1}{2}\theta_F \tag{425}$$

Solve the second integral.

$$\int \sinh^2\theta\cosh\theta d\theta \tag{426}$$

$$u = \sinh\theta \tag{427}$$

$$du = \cosh\theta d\theta \tag{428}$$

$$d\theta = \frac{du}{\cosh\theta} \tag{429}$$

$$\int \sinh^2\theta\cosh\theta d\theta = \int \left(u^2\right)\cosh\theta\left(\frac{du}{\cosh\theta}\right) \tag{430}$$

$$\int \sinh^2\theta\cosh\theta d\theta = \int u^2 du \tag{431}$$

$$\int \sinh^2\theta\cosh\theta d\theta = \frac{1}{3}u^3 + C \tag{432}$$

$$\int \sinh^2\theta\cosh\theta d\theta = \frac{1}{3}\sinh^3\theta + C \tag{433}$$

Now use these results in equation (410).

$$E_0 = \frac{8\pi V m^4 c^5}{h^3}\left(\frac{1}{8}A\left(x\right) + \frac{1}{4}\sinh 2\theta_F - \frac{1}{2}\theta_F + \left[-\frac{1}{3}\sinh^3\theta + C\Big|_0^{\theta_F}\right]\right) \tag{434}$$

$$E_0 = \frac{8\pi V m^4 c^5}{h^3}\left(\frac{1}{8}A\left(x\right) + \frac{1}{4}\sinh 2\theta_F - \frac{1}{2}\theta_F - \frac{1}{3}\sinh^3\theta_F\right) \tag{435}$$

$$E_0 = \frac{\pi V m^4 c^5}{h^3}\left(A\left(x\right) + 2\sinh 2\theta_F - 4\theta_F - \frac{8}{3}\sinh^3\theta_F\right) \tag{436}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3A\left(x\right) + 6\sinh 2\theta_F - 12\theta_F - 8\sinh^3\theta_F\right) \tag{437}$$

Write equation (437) in terms of x.

$$x \equiv \sinh\theta_F \tag{438}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3A\left(x\right) + 6\sinh 2\theta_F - 12\sinh^{-1}x - 8x^3\right) \tag{439}$$

$$\sinh 2\theta_F \equiv \sinh\theta_F\cosh\theta_F \tag{440}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3A\left(x\right) + 6\left(2\sinh\theta_F\cosh\theta_F\right) - 12\sinh^{-1}x - 8x^3\right) \tag{441}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3A\left(x\right) + 12\sinh\theta_F\cosh\theta_F - 12\sinh^{-1}x - 8x^3\right) \tag{442}$$

$$\cosh^2\theta - \sinh^2\theta \equiv 1 \tag{443}$$

$$\cosh^2\theta = 1 + \sinh^2\theta \tag{444}$$

$$\cosh\theta = \sqrt{1 + \sinh^2\theta} \tag{445}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3A\left(x\right) + 12\sinh\theta_F\left(\sqrt{1 + \sinh^2\theta_F}\right) - 12\sinh^{-1}x - 8x^3\right) \tag{446}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3A\left(x\right) + 12\sinh\theta_F\sqrt{1 + \sinh^2\theta_F} - 12\sinh^{-1}x - 8x^3\right) \tag{447}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3A\left(x\right) + 12x\sqrt{1 + x^2} - 12\sinh^{-1}x - 8x^3\right) \tag{448}$$

$$A\left(x\right) \equiv x\left(x^2 + 1\right)^{1/2}\left(2x^2 - 3\right) + 3\sinh^{-1}x \tag{449}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3\left(x\left(x^2 + 1\right)^{1/2}\left(2x^2 - 3\right) + 3\sinh^{-1}x\right) + 12x\sqrt{1 + x^2} - 12\sinh^{-1}x - 8x^3\right) \tag{450}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3x\left(x^2 + 1\right)^{1/2}\left(2x^2 - 3\right) + 9\sinh^{-1}x + 12x\sqrt{1 + x^2} - 12\sinh^{-1}x - 8x^3\right) \tag{451}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(3x\left(x^2 + 1\right)^{1/2}\left(2x^2 - 3\right) + 12x\sqrt{1 + x^2} - 3\sinh^{-1}x - 8x^3\right) \tag{452}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(\left(x^2 + 1\right)^{1/2}\left(6x^3 - 9x\right) + 12x\sqrt{1 + x^2} - 3\sinh^{-1}x - 8x^3\right) \tag{453}$$

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(\left(x^2 + 1\right)^{1/2}\left(6x^3 + 3x\right) - 3\sinh^{-1}x - 8x^3\right) \tag{454}$$

Define $B(x)$.

$$B(x) \equiv 8x^3\left[\left(x^2+1\right)^{1/2}-1\right]-A(x) \tag{455}$$

$$B(x) = 8x^3\left[\left(x^2+1\right)^{1/2}-1\right]-\left(x\left(x^2+1\right)^{1/2}\left(2x^2-3\right)+3\sinh^{-1}x\right) \tag{456}$$

$$B(x) = 8x^3\left[\left(x^2+1\right)^{1/2}-1\right]-x\left(x^2+1\right)^{1/2}\left(2x^2-3\right)-3\sinh^{-1}x \tag{457}$$

$$B(x) = 8x^3\left(x^2+1\right)^{1/2}-8x^3-x\left(x^2+1\right)^{1/2}\left(2x^2-3\right)-3\sinh^{-1}x \tag{458}$$

$$B(x) = \left(8x^3\left(x^2+1\right)^{1/2}-x\left(x^2+1\right)^{1/2}\left(2x^2-3\right)\right)-8x^3-3\sinh^{-1}x \tag{459}$$

$$B(x) = \left(8x^3\left(x^2+1\right)^{1/2}-\left(x^2+1\right)^{1/2}\left(2x^3-3x\right)\right)-8x^3-3\sinh^{-1}x \tag{460}$$

$$B(x) = \left(\left(x^2+1\right)^{1/2}\left(6x^3+3x\right)\right)-8x^3-3\sinh^{-1}x \tag{461}$$

$$B(x) = \left(x^2+1\right)^{1/2}\left(6x^3+3x\right)-8x^3-3\sinh^{-1}x \tag{462}$$

Rewrite equation (454) in terms of $B(x)$.

$$E_0 = \frac{\pi V m^4 c^5}{3h^3}\left(B(x)\right) \tag{463}$$

$$\boxed{E = \frac{\pi V m^4 c^5}{3h^3} B(x)}$$

Now, check that:

$$E_0 + P_0 V = N\mu_0 \text{ and } P_0 = -\left(\frac{\partial E_0}{\partial V}\right)_N$$

with

$$P_0 = \frac{\pi m^4 c^5}{3h^3} A(x)$$

From equation (8.5.11), find N.

$$N = \frac{8\pi V m^3 c^3}{3h^3} x^3 \tag{464}$$

Now, find $N\mu_0$.

$$N\mu_0 = N\epsilon_F \tag{465}$$

$$\epsilon_F = \sqrt{p_F^2 c^2 + m^2 c^4} - mc^2 \tag{466}$$

$$\epsilon_F = \sqrt{(mcx)^2 c^2 + m^2 c^4} - mc^2 \tag{467}$$

$$\epsilon_F = \sqrt{(m^2c^2x^2)\, c^2 + m^2 c^4} - mc^2 \tag{468}$$

$$\epsilon_F = \sqrt{m^2c^4x^2 + m^2 c^4} - mc^2 \tag{469}$$

$$\epsilon_F = \left(mc^2\sqrt{x^2+1}\right) - mc^2 \tag{470}$$

$$\epsilon_F = mc^2\left(\left(x^2+1\right)^{1/2}-1\right) \tag{471}$$

$$N\mu_0 = \left(\frac{8\pi V m^3 c^3}{3h^3}x^3\right)\left(mc^2\left(\left(x^2+1\right)^{1/2}-1\right)\right) \tag{472}$$

$$N\mu_0 = \left(\frac{8\pi V m^4 c^5}{3h^3}x^3\right)\left(\left(x^2+1\right)^{1/2}-1\right) \tag{473}$$

$$N\mu_0 = \left(\frac{\pi V m^4 c^5}{3h^3}\right)\left(8x^3\left(x^2+1\right)^{1/2}-8x^3\right) \tag{474}$$

Now, find $E_0 + P_0V$.

$$E_0 + P_0V = \frac{\pi V m^4 c^5}{3h^3} B(x) + \left(\frac{\pi m^4 c^5}{3h^3} A(x)\right) V \tag{475}$$

$$E_0 + P_0V = \frac{\pi V m^4 c^5}{3h^3} B(x) + \frac{\pi V m^4 c^5}{3h^3} A(x) \tag{476}$$

$$E_0 + P_0V = \frac{\pi V m^4 c^5}{3h^3} (B(x) + A(x)) \tag{477}$$

$$E_0 + P_0V = \frac{\pi V m^4 c^5}{3h^3} \left(\left(8x^3 \left[\left(x^2+1\right)^{1/2} - 1\right] - A(x)\right) + A(x)\right) \tag{478}$$

$$E_0 + P_0V = \frac{\pi V m^4 c^5}{3h^3} \left(8x^3 \left[\left(x^2+1\right)^{1/2} - 1\right] - A(x) + A(x)\right) \tag{479}$$

$$E_0 + P_0V = \frac{\pi V m^4 c^5}{3h^3} \left(8x^3 \left[\left(x^2+1\right)^{1/2} - 1\right]\right) \tag{480}$$

$$E_0 + P_0V = \frac{\pi V m^4 c^5}{3h^3} \left(8x^3 \left(x^2+1\right)^{1/2} - 8x^3\right) \tag{481}$$

Equations (474) and (481) are exactly the same!

$$\boxed{E_0 + P_0V = N\mu_0}$$

Now, verify $P_0 = -\left(\partial E_0/\partial V\right)_N$.

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\partial}{\partial V}\left(\frac{\pi V m^4 c^5}{3h^3}\left(\left(x^2+1\right)^{1/2}\left(6x^3+3x\right) - \sinh^{-1} x - 8x^3\right)\right)_N \tag{482}$$

x has an implicit dependence on V. Therefore, use the product rule with implicit differentiation to find this derivative. I am running out of time, so I will use a derivative calculator to differentiate this. Use equation

(6.5.5) to relate p_F to n.

$$u \equiv \left(x^2+1\right)^{1/2}\left(6x^3+3x\right) - \sinh^{-1} - 8x^3 \tag{483}$$

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + V\frac{\partial u}{\partial x}\frac{\partial x}{\partial p_F}\frac{\partial p_F}{\partial n}\frac{\partial n}{\partial V}\right) \tag{484}$$

$$\frac{\partial u}{\partial x} = -\frac{24x^2\sqrt{x^2+1} - 24x^4 - 24x^2}{\sqrt{x^2+1}} \tag{485}$$

$$x = \frac{p_F}{mc} \tag{486}$$

$$\frac{\partial x}{\partial p_F} = \frac{1}{mc} \tag{487}$$

$$p_F = \left(\frac{3n}{8\pi}\right)^{1/3} h \tag{488}$$

$$\left(\frac{8\pi}{3n}\right)^{1/3} = \frac{h}{p_F} \tag{489}$$

$$\frac{\partial p_F}{\partial n} = \frac{h}{3}\left(\frac{8\pi}{3n}\right)^{2/3}\frac{3}{8\pi} \tag{490}$$

$$\frac{\partial p_F}{\partial n} = \frac{h}{8\pi}\left(\frac{h}{p_F}\right)^2 \tag{491}$$

$$\frac{\partial p_F}{\partial n} = \frac{h^3}{8\pi p_F^2} \tag{492}$$

$$n \equiv \frac{N}{V} \tag{493}$$

$$\frac{\partial n}{\partial V} = -\frac{N}{V^2} \tag{494}$$

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + V\left(-\frac{24x^2\sqrt{x^2+1} - 24x^4 - 24x^2}{\sqrt{x^2+1}}\right)\left(\frac{1}{mc}\right)\left(\frac{h^3}{8\pi p_F^2}\right)\left(-\frac{N}{V^2}\right)\right) \tag{495}$$

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + \left(\frac{24x^2\sqrt{x^2+1} - 24x^4 - 24x^2}{\sqrt{x^2+1}}\right)\left(\frac{1}{mc}\right)\left(\frac{h^3}{8\pi p_F^2}\right)\left(\frac{N}{V}\right)\right) \tag{496}$$

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + \left(\frac{24x^2\sqrt{x^2+1} - 24x^4 - 24x^2}{\sqrt{x^2+1}}\right)\left(\frac{1}{mc}\right)\left(\frac{h^3}{8\pi\left(mcx\right)^2}\right)\left(\frac{N}{V}\right)\right) \tag{497}$$

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + \left(\frac{24x^2\sqrt{x^2+1} - 24x^4 - 24x^2}{\sqrt{x^2+1}}\right)\left(\frac{1}{mc}\right)\left(\frac{h^3}{8\pi m^2 c^2 x^2}\right)\left(\frac{N}{V}\right)\right) \tag{498}$$

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + \left(\frac{24x^2\sqrt{x^2+1} - 24x^4 - 24x^2}{\sqrt{x^2+1}}\right)\left(\frac{Nh^3}{8\pi m^3 c^3 x^2 V}\right)\right) \tag{499}$$

However, we already know N from equation (8.5.11).

$$N = \frac{8\pi V m^3 c^3}{3h^3} x^3 \tag{500}$$
$$N\left(\frac{h^3}{8\pi m^3 c^3 x^2 V}\right) = \left(\frac{8\pi V m^3 c^3}{3h^3} x^3\right)\left(\frac{h^3}{8\pi m^3 c^3 x^2 V}\right) \tag{501}$$
$$N\left(\frac{h^3}{8\pi m^3 c^3 x^2 V}\right) = \frac{8\pi V m^3 c^3 x^3 h^3}{3h^3\left(8\pi m^3 c^3 x^2 V\right)} \tag{502}$$
$$N\left(\frac{h^3}{8\pi m^3 c^3 x^2 V}\right) = \frac{V m^3 c^3 x^3}{3m^3 c^3 x^2 V} \tag{503}$$
$$N\left(\frac{h^3}{8\pi m^3 c^3 x^2 V}\right) = \frac{x}{3} \tag{504}$$

Use this result in equation (499).

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + \left(\frac{24x^2\sqrt{x^2+1} - 24x^4 - 24x^2}{\sqrt{x^2+1}}\right)\left(\frac{x}{3}\right)\right) \tag{505}$$
$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + \left(\frac{8x^3\sqrt{x^2+1} - 8x^5 - 8x^3}{\sqrt{x^2+1}}\right)\right) \tag{506}$$
$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(u + \frac{8x^3\sqrt{x^2+1} - 8x^5 - 8x^3}{\sqrt{x^2+1}}\right) \tag{507}$$

Find the term inside the parenthesis.

$$u + \ldots = 8x^3\left(x^2+1\right)^{1/2} - 8x^3 - A(x) + \frac{8x^3\sqrt{x^2+1} - 8x^5 - 8x^3}{\sqrt{x^2+1}} \tag{508}$$
$$u + \ldots = 8x^3\left(x^2+1\right)^{1/2} - 8x^3 - A(x) + 8x^3 + \frac{-8x^5 - 8x^3}{\sqrt{x^2+1}} \tag{509}$$
$$u + \ldots = 8x^3\left(x^2+1\right)^{1/2} - A(x) + \frac{-8x^5 - 8x^3}{\sqrt{x^2+1}} \tag{510}$$
$$u + \ldots = 8x^3\left(x^2+1\right)^{1/2} - A(x) - 8x^3\left(\frac{x^2+1}{\sqrt{x^2+1}}\right) \tag{511}$$
$$u + \ldots = 8x^3\left(x^2+1\right)^{1/2} - A(x) - 8x^3\left(x^2+1\right)^{1/2} \tag{512}$$
$$u + \ldots = -A(x) \tag{513}$$

Use this result in equation (507).

$$-\left(\frac{\partial E_0}{\partial V}\right)_N = -\frac{\pi m^4 c^5}{3h^3}\left(-A(x)\right) \tag{514}$$
$$-\left(\frac{\partial E_0}{\partial V}\right)_N = \frac{\pi m^4 c^5}{3h^3} A(x) \tag{515}$$

Compare this result with equation (8.5.12) in the book.

$$P_0 = \frac{\pi m^4 c^5}{3h^3} \tag{516}$$

This is exaclty the same as equation (515)!

$$P_0 \equiv \left(\frac{\partial E_0}{\partial V}\right)_N \tag{517}$$