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PHYS5510

8.1 Let the Fermi distribution at low temperatures be represented by a broken line, as shown in Figure 8.13, the line being tangential to the actual curve at $\epsilon = \mu$. Show that this approximate representation yields a "correct" result for the low-temperature specific heat of the Fermi gas, except that the numerical factor turns out to be smaller by a factor of $4/\pi^2$. Discuss, in a qualitative manner, the origin of this numerical discrepancy.

Find the equation of the line for $n\left(x\right)$ using figure 8.13.

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{1}$$

$$m = \frac{1-0}{\xi - 2 - (\xi + 2)} \tag{2}$$

$$m = -\frac{1}{4} \tag{3}$$

$$y - y_1 = m\left(x - x_1\right) \tag{4}$$

$$y - \frac{1}{2} = -\frac{1}{4}\left(x - \xi\right) \tag{5}$$

$$y = \frac{-x + \xi + 2}{4} \tag{6}$$

Using this linear approximation for a segment of the graph, define $n\left(x\right)$.

$$n\left(x\right) = \begin{cases} 1 & x < \xi \\ \frac{-x+\xi+2}{4} & \xi - 2 < x < \xi + 2 \\ 0 & x > \xi + 2 \end{cases} \tag{7}$$

Find the density of states as a function of x, $g(x)\,dx$.

$$g(p)\,dp = \frac{V}{h^3} 4\pi p^2 dp \times 2 \tag{8}$$
$$\epsilon = \frac{p^2}{2m} \tag{9}$$
$$p^2 = 2m\epsilon \tag{10}$$
$$2pdp = 2md\epsilon \tag{11}$$
$$dp = \frac{m}{p} d\epsilon \tag{12}$$
$$dp = \frac{m}{\sqrt{2m\epsilon}} d\epsilon \tag{13}$$
$$g(\epsilon)\,d\epsilon = \frac{V}{h^3} 8\pi\,(2m\epsilon)\left(\frac{m}{\sqrt{2m\epsilon}} d\epsilon\right) \tag{14}$$
$$g(\epsilon)\,d\epsilon = \frac{V}{h^3} 8\pi\sqrt{2m\epsilon}\, m d\epsilon \tag{15}$$
$$g(\epsilon)\,d\epsilon = \frac{V}{h^3} 4\pi\sqrt{2m\epsilon}\, 2m d\epsilon \tag{16}$$
$$g(\epsilon)\,d\epsilon = \frac{V}{h^3} 4\pi\sqrt{\epsilon}\,(2m)^{3/2}\, d\epsilon \tag{17}$$
$$g(\epsilon)\,d\epsilon = 4\pi V \frac{(2m)^{3/2}}{h^3}\sqrt{\epsilon} d\epsilon \tag{18}$$
$$g(\epsilon)\,d\epsilon = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2}\sqrt{\epsilon} d\epsilon \tag{19}$$
$$x \equiv \frac{\epsilon}{kT} \tag{20}$$
$$\epsilon = kTx \tag{21}$$
$$d\epsilon = kTdx \tag{22}$$
$$g(x)\,dx = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2}\sqrt{(kTx)}\,(kTdx) \tag{23}$$
$$g(x)\,dx = 4\pi V \left(\frac{2m}{h^2}\right)^{3/2}(kT)^{3/2}\sqrt{x} dx \tag{24}$$
$$g(x)\,dx = 4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2}\sqrt{x} dx \tag{25}$$

Simplify this by grouping all the constants into one, α.

$$\alpha = 4\pi V \left(\frac{2mkT}{h^2}\right)^{3/2} \tag{26}$$
$$g(x)\,dx = \alpha\sqrt{x} dx \tag{27}$$

Find U.

$$U \equiv \int_0^\infty \epsilon n(\epsilon)\, g(\epsilon)\, d\epsilon \tag{28}$$
$$U = \int_0^\infty \epsilon(x)\, n(x)\, g(x)\, dx \tag{29}$$

Use the result of equation (7) in equation (29) to limit the bounds of integration.

$$U = \int_0^{\xi-2} \epsilon(x)(1)\, g(x)\, dx + \int_{\xi-2}^{\xi+2} \epsilon(x) \left(\frac{-x+\xi+2}{4}\right) g(x)\, dx \tag{30}$$

$$U = \int_0^{\xi-2} \epsilon(x)\, g(x)\, dx + \int_{\xi-2}^{\xi+2} \epsilon(x) \left(\frac{-x+\xi+2}{4}\right) g(x)\, dx \tag{31}$$

$$\epsilon \equiv kTx \tag{32}$$

$$d\epsilon = kTdx \tag{33}$$

Solve each integral separately.

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = \int_0^{\xi-2} (kTx)\left(\alpha\sqrt{x}dx\right) \tag{34}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = \int_0^{\xi-2} kT\alpha x^{3/2} dx \tag{35}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = kT\alpha \int_0^{\xi-2} x^{3/2} dx \tag{36}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = kT\alpha \left[\frac{2}{5} x^{5/2}\Big|_0^{\xi-2}\right] \tag{37}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = kT\alpha \left[\frac{2}{5}(\xi-2)^{5/2}\right] \tag{38}$$

$$\int_0^{\xi-2} \epsilon(x)\, g(x)\, dx = \frac{2}{5} kT\alpha (\xi-2)^{5/2} \tag{39}$$

Find the second integral.

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = \int_{\xi-2}^{\xi+2} (kTx)\left(\frac{(\xi+2-x)}{4}\right)\left(\alpha\sqrt{x}dx\right) \tag{40}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = \int_{\xi-2}^{\xi+2} kTx\frac{(\xi+2-x)}{4}\alpha\sqrt{x}dx \tag{41}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = kT\frac{\alpha}{4}\int_{\xi-2}^{\xi+2} (\xi+2-x)\, x^{3/2} dx \tag{42}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = kT\frac{\alpha}{4}\int_{\xi-2}^{\xi+2} (\xi+2)\, x^{3/2} - x^{5/2} dx \tag{43}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = kT\frac{\alpha}{4}\left[\frac{2}{5}(\xi+2)\, x^{5/2} - \frac{2}{7}x^{7/2}\Big|_{\xi-2}^{\xi+2}\right] \tag{44}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = kT\frac{\alpha}{4}\left[\frac{2}{5}(\xi+2)(\xi+2)^{5/2} - \frac{2}{7}(\xi+2)^{7/2} - \frac{2}{5}(\xi+2)(\xi-2)^{5/2} + \frac{2}{7}(\xi-2)^{7/2}\right] \tag{45}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = kT\frac{\alpha}{4}\left[\frac{2}{5}(\xi+2)^{7/2} - \frac{2}{7}(\xi+2)^{7/2} - \frac{2}{5}(\xi+2)(\xi-2)^{5/2} + \frac{2}{7}(\xi-2)^{7/2}\right] \tag{46}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = kT\frac{\alpha}{2}\left[\frac{1}{5}(\xi+2)^{7/2} - \frac{1}{7}(\xi+2)^{7/2} - \frac{1}{5}(\xi+2)(\xi-2)^{5/2} + \frac{1}{7}(\xi-2)^{7/2}\right] \tag{47}$$

$$\int_{\xi-2}^{\xi+2} \epsilon(x)\, n(x)\, g(x)\, dx = kT\frac{\alpha}{2}\left[\frac{2}{35}(\xi+2)^{7/2} - \frac{1}{5}(\xi+2)(\xi-2)^{5/2} + \frac{1}{7}(\xi-2)^{7/2}\right] \tag{48}$$

Use the results of equations (39) and (48) to then solve for U.

$$U = \left(\frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2}\right) + \left(kT\frac{\alpha}{2}\left[\frac{2}{35}\left(\xi+2\right)^{7/2} - \frac{1}{5}\left(\xi+2\right)\left(\xi-2\right)^{5/2} + \frac{1}{7}\left(\xi-2\right)^{7/2}\right]\right) \tag{49}$$

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{2}\left[\frac{2}{35}\left(\xi+2\right)^{7/2} - \frac{1}{5}\left(\xi+2\right)\left(\xi-2\right)^{5/2} + \frac{1}{7}\left(\xi-2\right)^{7/2}\right] \tag{50}$$

Now, invoke the approximation that we are at low temperatures.

$$\xi \equiv \frac{\mu}{kT} \tag{51}$$

$$T \to 0 \tag{52}$$

$$\xi >> 1 \tag{53}$$

This is quite useful. Firstly:

$$\left(\xi+2\right) \approx \left(\xi-2\right) \tag{54}$$

$$\left(\xi+2\right)\left(\xi-2\right)^{5/2} \approx \left(\xi-2\right)^{7/2} \tag{55}$$

Use this approximation in equation (50).

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{2}\left[\frac{2}{35}\left(\xi+2\right)^{7/2} - \frac{1}{5}\left[\left(\xi+2\right)\left(\xi-2\right)^{5/2}\right] + \frac{1}{7}\left(\xi-2\right)^{7/2}\right] \tag{56}$$

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{2}\left[\frac{2}{35}\left(\xi+2\right)^{7/2} - \frac{1}{5}\left[\left(\xi-2\right)^{7/2}\right] + \frac{1}{7}\left(\xi-2\right)^{7/2}\right] \tag{57}$$

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{2}\left[\frac{2}{35}\left(\xi+2\right)^{7/2} - \frac{1}{5}\left(\xi-2\right)^{7/2} + \frac{1}{7}\left(\xi-2\right)^{7/2}\right] \tag{58}$$

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{2}\left[\frac{2}{35}\left(\xi+2\right)^{7/2} - \frac{1}{5}\left(\xi-2\right)^{7/2} + \frac{1}{7}\left(\xi-2\right)^{7/2}\right] \tag{59}$$

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{2}\left[\frac{2}{35}\left(\xi+2\right)^{7/2} - \frac{2}{35}\left(\xi-2\right)^{7/2}\right] \tag{60}$$

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{2}\frac{2}{35}\left[\left(\xi+2\right)^{7/2} - \left(\xi-2\right)^{7/2}\right] \tag{61}$$

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{2}\frac{2}{35}\left[\left(\xi+2\right)^{7/2} - \left(\xi-2\right)^{7/2}\right] \tag{62}$$

$$U = \frac{2}{5}kT\alpha\left(\xi-2\right)^{5/2} + kT\frac{\alpha}{35}\left[\left(\xi+2\right)^{7/2} - \left(\xi-2\right)^{7/2}\right] \tag{63}$$

Use the approximation in equation (53) to simplify our expression for U to only consider higher orders of ξ.

$$\xi >> 1 \tag{64}$$

$$U \approx kT\frac{\alpha}{35}\left[\left(\xi+2\right)^{7/2} - \left(\xi-2\right)^{7/2}\right] \tag{65}$$

$$U \approx kT\frac{\alpha}{35}\left[\xi^{7/2}\left(1+\frac{2}{\xi}\right)^{7/2} - \xi^{7/2}\left(1-\frac{2}{\xi}\right)^{7/2}\right] \tag{66}$$

$$U \approx kT\frac{\alpha}{35}\xi^{7/2}\left[\left(1+\frac{2}{\xi}\right)^{7/2} - \left(1-\frac{2}{\xi}\right)^{7/2}\right] \tag{67}$$

Since $\xi >> 1$, we can Taylor expand $\left(1+\frac{2}{\xi}\right)^{7/2}$ and $\left(1-\frac{2}{\xi}\right)^{7/2}$ because $\frac{2}{\xi} << 1$. Use the binomial expansion.

$$(1+x)^n \approx 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 \tag{68}$$

$$(1-x)^n \approx 1 - nx + \frac{(n)(n-1)}{2!}x^2 - \frac{(n)(n-1)(n-2)}{3!}x^3 \tag{69}$$

$$(1+x)^n - (1-x)^n \approx 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 \tag{70}$$

$$-\left(1 - nx + \frac{(n)(n-1)}{2!}x^2 - \frac{(n)(n-1)(n-2)}{3!}x^3\right) \tag{71}$$

$$(1+x)^n - (1-x)^n \approx 1 + nx + \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 \tag{72}$$

$$-1 + nx - \frac{(n)(n-1)}{2!}x^2 + \frac{(n)(n-1)(n-2)}{3!}x^3 \tag{73}$$

$$(1+x)^n - (1-x)^n \approx 2nx + \frac{2(n)(n-1)(n-2)}{3!}x^3 \tag{74}$$

$$(1+x)^n - (1-x)^n \approx 2nx + \frac{(n)(n-1)(n-2)}{3}x^3 \tag{75}$$

Using this binomial expansion:

$$\left(1+\frac{2}{\xi}\right)^{7/2} - \left(1-\frac{2}{\xi}\right)^{7/2} \approx 2\left(\frac{7}{2}\right)\left(\frac{2}{\xi}\right) + \frac{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)}{3}\left(\frac{2}{\xi}\right)^3 \tag{76}$$

$$\left(1+\frac{2}{\xi}\right)^{7/2} - \left(1-\frac{2}{\xi}\right)^{7/2} \approx 2\left(\frac{7}{\xi}\right) + \frac{(7)(5)(3)}{3}\left(\frac{1}{\xi}\right)^3\left(\frac{2}{2}\right)^3 \tag{77}$$

$$\left(1+\frac{2}{\xi}\right)^{7/2} - \left(1-\frac{2}{\xi}\right)^{7/2} \approx 2\left(\frac{7}{\xi}\right) + (7)(5)\left(\frac{1}{\xi}\right)^3 \tag{78}$$

$$\left(1+\frac{2}{\xi}\right)^{7/2} - \left(1-\frac{2}{\xi}\right)^{7/2} \approx \frac{14}{\xi} + \frac{35}{\xi^3} \tag{79}$$

Use the result of equation (79) in equation (67).

$$U \approx kT\frac{\alpha}{35}\xi^{7/2}\left[\left(1+\frac{2}{\xi}\right)^{7/2} - \left(1-\frac{2}{\xi}\right)^{7/2}\right] \tag{80}$$

$$U \approx kT\frac{\alpha}{35}\xi^{7/2}\left[\frac{14}{\xi} + \frac{35}{\xi^3}\right] \tag{81}$$

$$U \approx kT\frac{\alpha}{35}\xi^{5/2}\left[14 + \frac{35}{\xi^2}\right] \tag{82}$$

$$U \approx kT\frac{\alpha}{35}\xi^{5/2}\left[14 + 35\xi^{-2}\right] \tag{83}$$

$$U \approx kT\alpha\xi^{5/2}\left[\frac{14}{35} + \xi^{-2}\right] \tag{84}$$

$$U \approx kT\alpha\xi^{5/2}\left[\frac{2}{5} + \xi^{-2}\right] \tag{85}$$

$$U \approx \frac{2}{5}kT\alpha\xi^{5/2}\left[1 + \frac{5}{2}\xi^{-2}\right] \tag{86}$$

Now find N.

$$N \equiv \int_0^\infty n(\epsilon)\, g(\epsilon)\, d\epsilon \tag{87}$$

$$N = \int_0^\infty n(x)\, g(x)\, dx \tag{88}$$

Use the result of equation (7) in equation (88) to limit the bounds of integration.

$$N = \int_{0}^{\xi-2} (1)\, g(x)\, dx + \int_{\xi-2}^{\xi+2} \left(\frac{-x+\xi+2}{4}\right) g(x)\, dx \tag{89}$$

$$N = \int_{0}^{\xi-2} g(x)\, dx + \int_{\xi-2}^{\xi+2} \left(\frac{-x+\xi+2}{4}\right) g(x)\, dx \tag{90}$$

Solve each integral separately.

$$\int_{0}^{\xi-2} g(x)\, dx = \int_{0}^{\xi-2} \left(\alpha\sqrt{x}dx\right) \tag{91}$$

$$\int_{0}^{\xi-2} g(x)\, dx = \int_{0}^{\xi-2} \alpha x^{1/2} dx \tag{92}$$

$$\int_{0}^{\xi-2} g(x)\, dx = \alpha \int_{0}^{\xi-2} x^{1/2} dx \tag{93}$$

$$\int_{0}^{\xi-2} g(x)\, dx = \alpha \left[\frac{2}{3} x^{3/2} \Big|_{0}^{\xi-2} \right] \tag{94}$$

$$\int_{0}^{\xi-2} g(x)\, dx = \alpha \left[\frac{2}{3} (\xi-2)^{3/2} \right] \tag{95}$$

$$\int_{0}^{\xi-2} g(x)\, dx = \frac{2}{3} \alpha (\xi-2)^{3/2} \tag{96}$$

Find the second integral.

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \int_{\xi-2}^{\xi+2} \left(\frac{(\xi+2-x)}{4}\right) \left(\alpha\sqrt{x}dx\right) \tag{97}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \int_{\xi-2}^{\xi+2} \frac{(\xi+2-x)}{4} \alpha\sqrt{x}dx \tag{98}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \frac{\alpha}{4} \int_{\xi-2}^{\xi+2} (\xi+2-x)\, x^{1/2} dx \tag{99}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \frac{\alpha}{4} \int_{\xi-2}^{\xi+2} (\xi+2)\, x^{1/2} - x^{3/2} dx \tag{100}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \frac{\alpha}{4} \left[\frac{2}{3} (\xi+2)\, x^{3/2} - \frac{2}{5} x^{5/2} \Big|_{\xi-2}^{\xi+2} \right] \tag{101}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \frac{\alpha}{4} \left[\frac{2}{3} (\xi+2)(\xi+2)^{3/2} - \frac{2}{5} (\xi+2)^{5/2} - \frac{2}{3} (\xi+2)(\xi-2)^{3/2} + \frac{2}{5} (\xi-2)^{5/2} \right] \tag{102}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \frac{\alpha}{4} \left[\frac{2}{3} (\xi+2)^{5/2} - \frac{2}{5} (\xi+2)^{5/2} - \frac{2}{3} (\xi+2)(\xi-2)^{3/2} + \frac{2}{5} (\xi-2)^{5/2} \right] \tag{103}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \frac{\alpha}{2} \left[\frac{1}{3} (\xi+2)^{5/2} - \frac{1}{5} (\xi+2)^{5/2} - \frac{1}{3} (\xi+2)(\xi-2)^{3/2} + \frac{1}{5} (\xi-2)^{5/2} \right] \tag{104}$$

$$\int_{\xi-2}^{\xi+2} n(x)\, g(x)\, dx = \frac{\alpha}{2} \left[\frac{2}{15} (\xi+2)^{5/2} - \frac{1}{3} (\xi+2)(\xi-2)^{3/2} + \frac{1}{5} (\xi-2)^{5/2} \right] \tag{105}$$

Use the results of equations (96) and (105) to then solve for N.

$$N = \left(\frac{2}{3}\alpha (\xi-2)^{3/2}\right) + \left(\frac{\alpha}{2} \left[\frac{2}{15} (\xi+2)^{5/2} - \frac{1}{3} (\xi+2)(\xi-2)^{3/2} + \frac{1}{5} (\xi-2)^{5/2} \right]\right) \tag{106}$$

$$N = \frac{2}{3}\alpha (\xi-2)^{3/2} + \frac{\alpha}{2} \left[\frac{2}{15} (\xi+2)^{5/2} - \frac{1}{3} (\xi+2)(\xi-2)^{3/2} + \frac{1}{5} (\xi-2)^{5/2} \right] \tag{107}$$

Now, invoke the approximation that we are at low temperatures.

$$\xi \equiv \frac{\mu}{kT} \tag{108}$$
$$T \to 0 \tag{109}$$
$$\xi >> 1 \tag{110}$$
$$(\xi + 2) \approx (\xi - 2) \tag{111}$$
$$(\xi + 2)(\xi - 2)^{3/2} \approx (\xi - 2)^{5/2} \tag{112}$$

Use this approximation in equation (107).

$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2}\left[\frac{2}{15}(\xi + 2)^{5/2} - \frac{1}{3}\left[(\xi + 2)(\xi - 2)^{3/2}\right] + \frac{1}{5}(\xi - 2)^{5/2}\right] \tag{113}$$
$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2}\left[\frac{2}{15}(\xi + 2)^{5/2} - \frac{1}{3}\left[(\xi - 2)^{5/2}\right] + \frac{1}{5}(\xi - 2)^{5/2}\right] \tag{114}$$
$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2}\left[\frac{2}{15}(\xi + 2)^{5/2} - \frac{1}{3}(\xi - 2)^{5/2} + \frac{1}{5}(\xi - 2)^{5/2}\right] \tag{115}$$
$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2}\left[\frac{2}{15}(\xi + 2)^{5/2} - \frac{2}{15}(\xi - 2)^{5/2}\right] \tag{116}$$
$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{2}\frac{2}{15}\left[(\xi + 2)^{5/2} - (\xi - 2)^{5/2}\right] \tag{117}$$
$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{15}\left[(\xi + 2)^{5/2} - (\xi - 2)^{5/2}\right] \tag{118}$$

Use the approximation in equation (53) to simplify our expression for U to only consider higher orders of ξ.

$$\xi >> 1 \tag{119}$$
$$N = \frac{2}{3}\alpha(\xi - 2)^{3/2} + \frac{\alpha}{15}\left[(\xi + 2)^{5/2} - (\xi - 2)^{5/2}\right] \tag{120}$$
$$N \approx \frac{\alpha}{15}\left[(\xi + 2)^{5/2} - (\xi - 2)^{5/2}\right] \tag{121}$$
$$N \approx \frac{\alpha}{15}\left[\xi^{5/2}\left(1 + \frac{2}{\xi}\right)^{5/2} - \xi^{5/2}\left(1 - \frac{2}{\xi}\right)^{5/2}\right] \tag{122}$$
$$N \approx \frac{\alpha}{15}\xi^{5/2}\left[\left(1 + \frac{2}{\xi}\right)^{5/2} - \left(1 - \frac{2}{\xi}\right)^{5/2}\right] \tag{123}$$

Use the same binomial expansion as before.

$$(1 + x)^n - (1 - x)^n \approx 2nx + \frac{(n)(n-1)(n-2)}{3}x^3 \tag{124}$$
$$\left(1 + \frac{2}{\xi}\right)^{5/2} - \left(1 - \frac{2}{\xi}\right)^{5/2} \approx 2\left(\frac{5}{2}\right)\left(\frac{2}{\xi}\right) + \frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{3}\left(\frac{2}{\xi}\right)^3 \tag{125}$$
$$\left(1 + \frac{2}{\xi}\right)^{5/2} - \left(1 - \frac{2}{\xi}\right)^{5/2} \approx 2\left(\frac{5}{\xi}\right) + \frac{(5)(3)(1)}{3}\left(\frac{1}{\xi}\right)^3\left(\frac{2^3}{2^3}\right) \tag{126}$$
$$\left(1 + \frac{2}{\xi}\right)^{5/2} - \left(1 - \frac{2}{\xi}\right)^{5/2} \approx 2\left(\frac{5}{\xi}\right) + \frac{15}{3}\left(\frac{1}{\xi}\right)^3 \tag{127}$$
$$\left(1 + \frac{2}{\xi}\right)^{5/2} - \left(1 - \frac{2}{\xi}\right)^{5/2} \approx \frac{10}{\xi} + \frac{5}{\xi^3} \tag{128}$$

Use the result of equation (128) in equation (123).

$$N \approx \frac{\alpha}{15}\xi^{5/2}\left[\left(1+\frac{2}{\xi}\right)^{5/2}-\left(1-\frac{2}{\xi}\right)^{5/2}\right] \tag{129}$$

$$N \approx \frac{\alpha}{15}\xi^{5/2}\left[\frac{10}{\xi}+\frac{5}{\xi^3}\right] \tag{130}$$

$$N \approx \frac{\alpha}{15}\xi^{3/2}\left[10+\frac{5}{\xi^2}\right] \tag{131}$$

$$N \approx \frac{\alpha}{15}\xi^{3/2}\left[10+5\xi^{-2}\right] \tag{132}$$

$$N \approx \frac{10}{15}\alpha\xi^{3/2}\left[1+\frac{1}{2}\xi^{-2}\right] \tag{133}$$

$$N \approx \frac{2}{3}\alpha\xi^{3/2}\left[1+\frac{1}{2}\xi^{-2}\right] \tag{134}$$

Use equations (86) and (134) to rewrite U in terms of N.

$$U \approx \frac{2}{5}kT\alpha\xi^{5/2}\left[1+\frac{5}{2}\xi^{-2}\right] \tag{135}$$

$$N \approx \frac{2}{3}\alpha\xi^{3/2}\left[1+\frac{1}{2}\xi^{-2}\right] \tag{136}$$

$$\frac{U}{N} = \frac{\frac{2}{5}kT\alpha\xi^{5/2}}{\frac{2}{3}\alpha\xi^{3/2}}\frac{\left[1+\frac{5}{2}\xi^{-2}\right]}{\left[1+\frac{1}{2}\xi^{-2}\right]} \tag{137}$$

$$\frac{U}{N} = \frac{\frac{2}{5}}{\frac{2}{3}}\frac{\alpha}{\alpha}\frac{\xi^{5/2}}{\xi^{3/2}}kT\frac{\left[1+\frac{5}{2}\xi^{-2}\right]}{\left[1+\frac{1}{2}\xi^{-2}\right]} \tag{138}$$

$$\frac{U}{N} = \frac{2}{5}\frac{3}{2}\xi kT\frac{\left[1+\frac{5}{2}\xi^{-2}\right]}{\left[1+\frac{1}{2}\xi^{-2}\right]} \tag{139}$$

$$\frac{U}{N} = \frac{3}{5}\xi kT\frac{\left[1+\frac{5}{2}\xi^{-2}\right]}{\left[1+\frac{1}{2}\xi^{-2}\right]} \tag{140}$$

Use long division of polynomials to evalue the last fraction in equation (140) (I can't write long division in LATEX, look it up).

$$\frac{\left[1+\frac{5}{2}\xi^{-2}\right]}{\left[1+\frac{1}{2}\xi^{-2}\right]} = 1+\frac{2\xi^{-2}}{1+\frac{1}{2}\xi^{-2}} \tag{141}$$

$$\frac{\left[1+\frac{5}{2}\xi^{-2}\right]}{\left[1+\frac{1}{2}\xi^{-2}\right]} = 1+2\xi^{-2}+\ldots \tag{142}$$

$$\frac{\left[1+\frac{5}{2}\xi^{-2}\right]}{\left[1+\frac{1}{2}\xi^{-2}\right]} \approx 1+2\xi^{-2} \tag{143}$$

Use the result of (143) in (140).

$$\frac{U}{N} \approx \frac{3}{5}\xi kT\left(1+2\xi^{-2}\right) \tag{144}$$

$$U \approx \frac{3}{5}\xi NkT\left(1+2\xi^{-2}\right) \tag{145}$$

However, we can further reduce this from the definition of ξ. Start with this first approximation, which is kind of nonsensical.

$$2\xi^{-2} \approx \frac{5}{3}\xi^{-2} \tag{146}$$
$$U \approx \frac{3}{5}\xi NkT\left(1 + 2\xi^{-2}\right) \tag{147}$$
$$U \approx \frac{3}{5}\xi NkT\left(1 + \frac{5}{3}\xi^{-2}\right) \tag{148}$$
$$\xi \equiv \frac{\epsilon_F}{kT} \tag{149}$$
$$U \approx \frac{3}{5}\left(\frac{\epsilon_F}{kT}\right) NkT\left(1 + \frac{5}{3}\left(\frac{\epsilon_F}{kT}\right)^{-2}\right) \tag{150}$$
$$U \approx \frac{3}{5}\epsilon_F N\left(1 + \frac{5}{3}\left(\frac{\epsilon_F}{kT}\right)^{-2}\right) \tag{151}$$
$$U \approx \frac{3}{5}\epsilon_F N\left(1 + \frac{5}{3}\left(\frac{kT}{\epsilon_F}\right)^{2}\right) \tag{152}$$

Now find C_V. By definition:

$$C_V \equiv \left(\frac{\partial U}{\partial T}\right)_V \tag{153}$$
$$C_V = \frac{\partial}{\partial T}\left(\frac{3}{5}\epsilon_F N\left(1 + \frac{5}{3}\left(\frac{kT}{\epsilon_F}\right)^{2}\right)\right) \tag{154}$$
$$C_V = \frac{\partial}{\partial T}\left(\frac{3}{5}\epsilon_F N\right) + \frac{\partial}{\partial T}\left(\epsilon_F N\left(\frac{kT}{\epsilon_F}\right)^{2}\right) \tag{155}$$
$$C_V = \frac{\partial}{\partial T}\left(\epsilon_F N\left(\frac{kT}{\epsilon_F}\right)^{2}\right) \tag{156}$$
$$C_V = 2\epsilon_F N\left(\frac{kT}{\epsilon_F}\right)\left(\frac{k}{\epsilon_F}\right) \tag{157}$$
$$C_V = 2N(kT)\left(\frac{k}{\epsilon_F}\right) \tag{158}$$
$$C_V = \frac{2Nk^2T}{\epsilon_F} \tag{159}$$
$$\frac{C_V}{Nk} = \frac{2kT}{\epsilon_F} \tag{160}$$

Equation (8.1.39) in the book gives:

$$\frac{C_V}{Nk} = \frac{\pi^2}{2}\frac{kT}{\epsilon_F} + \ldots \tag{161}$$
$$\frac{C_{V,\text{book}}}{Nk} \approx \frac{\pi^2 kT}{2\epsilon_F} \tag{162}$$

Find the difference between our approximation and what's given in the book.

$$\frac{\frac{C_{V,\text{approx}}}{Nk}}{\frac{C_{V,\text{book}}}{Nk}} = \frac{\frac{2kT}{\epsilon_F}}{\frac{\pi^2 kT}{2\epsilon_F}} \tag{163}$$

$$\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{\frac{2kT}{\epsilon_F}}{\frac{\pi^2 kT}{2\epsilon_F}} \tag{164}$$

$$\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{2kT}{\epsilon_F}\frac{2\epsilon_F}{\pi^2 kT} \tag{165}$$

$$\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{4}{\pi^2}\frac{kT}{kT}\frac{\epsilon_F}{\epsilon_F} \tag{166}$$

$$\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{4}{\pi^2} \tag{167}$$

$$\boxed{\frac{C_{V,\text{approx}}}{C_{V,\text{book}}} = \frac{4}{\pi^2}}$$

Our approximation is smaller than what's given in the book by a factor of $4/\pi^2$! This means that in the process of approximating $n\left(x\right)$, we are under-counting the number of excited particles by a factor of $\sim \frac{1}{2}$.