UCLA CS 289 Final

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1 Problem 1

We know that $\text{disc}(f) \leq \text{disc}_\mu(f)$ for all probability distributions μ over M_f. Thus it is enough to find a μ for which $\text{disc}_\mu(f) \leq O(1/\sqrt{n})$. Let's do that.

Let

$$\mu(X,Y) = \begin{cases} \left[\binom{n}{n/2}n\right]^{-1}, & \text{if } X \in \binom{\{1,2,\ldots,n\}}{1}, Y \in \binom{\{1,2,\ldots,n\}}{n/2} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Note that μ's non-zero values form a combinatorial rectangle, from here on when we say M_f, we will only analyze properties within this rectangle and ignore the rest of M_f.

Claim 1. Each row of the rectangle has an equal number of 0 and 1 entries.

Proof. For each row X, we have $f(X,Y) = 1 \implies f(X,Y^C) = 0$, and $f(X,Y) = 0 \implies f(X,Y^C) = 1$. Thus, since the function $g(Y) = Y^C$ is a bijection over $\binom{\{1,2,\ldots,n\}}{n/2}$, we prove the claim. □

Claim 2. If all combinatorial rectangles R with $\leq \frac{n}{2}$ rows satisfy $|\mu(R \cap f^{-1}(0)) - \mu(R \cap f^{-1}(1))| \leq O(1/\sqrt{n})$, then all rectangles satisfy that propery.

Proof. Let R be a rectangle with more than $\frac{n}{2}$ rows. Let $R_1 \cup R_2 = R$, so that both R_1 and R_2 contain all the columns of R, but both R_1 and R_2 contain at most $\frac{n}{2}$ rows. We have

$$\begin{aligned} |\mu(R \cap f^{-1}(0)) - \mu(R \cap f^{-1}(1))| &\leq |\mu(R_1 \cap f^{-1}(0)) - \mu(R_1 \cap f^{-1}(1))| + |\mu(R_2 \cap f^{-1}(0)) - \mu(R_2 \cap f^{-1}(1))| \\ &\leq O(1/\sqrt{n}) \end{aligned}$$

□

Claim 3. For all combinatorial rectangles with less than $\frac{n}{2}$ rows, $|\mu(R \cap f^{-1}(0)) - \mu(R \cap f^{-1}(1))| \leq O(1/\sqrt{n})$.

Proof. Notice first that whichever row we select, we cannot have more than $\frac{1}{2n}$ gain from it, as it has an equal number of ones and zeros.

Now assume that the rectangle R contradicts the claim. Let's assume we selected k rows for R. Also, assume that we are trying to find an R such that $\mu(R \cap f^{-1}(0)) \leq \mu(R \cap f^{-1}(1))$. This can be done without loss of generality, as each rectangle that does not have this property can be mapped to a rectangle with complement columns that does have this property and has the same value for $|\mu(R \cap f^{-1}(0)) - \mu(R \cap f^{-1}(1))|$ (each zero in the original maps to a one in the complement and vis versa).

Thus, we are trying to maximize $\mu(R \cap f^{-1}(1)) - \mu(R \cap f^{-1}(0))$. With this in mind, which columns does it make sense to include for R? Clearly, we should only include columns that represent sets that contain at least $\frac{k}{2}$ values of the selected rows, as otherwise we would have selected a majority of zeros in the column, which would make $\mu(R \cap f^{-1}(1)) - \mu(R \cap f^{-1}(0))$ smaller (remember, the rows are single-element sets). Thus, R can only select $\sum_{i=0}^{k/2} \binom{k}{k/2+i}\binom{n-k}{\frac{n-k}{2}-i}$ columns where the ith component of the sum is associated with the number of sets that contain $2i$ rows more than they don't contain. Thus, the ith component will have $2i$ more ones than zeros and thus will contribute to the total sum by $2i\left[\binom{n}{n/2}n\right]^{-1}\binom{k}{\frac{k}{2}+i}\binom{n-k}{\frac{n}{2}-(\frac{k}{2}+i)}$. Thus, the final sum will be

$$\sum_{i=0}^{k} 2i\left[\binom{n}{n/2}n\right]^{-1}\binom{k}{\frac{k}{2}+i}\binom{n-k}{\frac{n}{2}-(\frac{k}{2}+i)}$$

and we need to prove that

$$\max_{k\in\{2,4,\ldots,\lfloor\frac{n}{2}\rfloor\}}\left\{\lim_{n\to\infty}\sqrt{n}\sum_{i=0}^{k/2} 2i\left[\binom{n}{n/2}n\right]^{-1}\binom{k}{\frac{k}{2}+i}\binom{n-k}{\frac{n}{2}-(\frac{k}{2}+i)}\right\} < \infty.$$

Now, notice that

$$f(k) = \left\{ \lim_{n\to\infty} \sqrt{n} \sum_{i=0}^{k/2} 2i \left[\binom{n}{n/2} n \right]^{-1} \binom{k}{\frac{k}{2}+i} \binom{n-k}{\frac{n}{2} - (\frac{k}{2}+i)} \right\} < \infty.$$

is monotonically increasing for $0 < k \leq \frac{n}{2}$. This is because for $0 < k_1 < k_2 \leq \frac{n}{2}$ and all i in the appropriate range, we have

$$\binom{k_1}{\frac{k_1}{2}+i} \binom{n-k_1}{\frac{n}{2} - \left(\frac{k_1}{2}+i\right)} < \binom{k_2}{\frac{k_2}{2}+i} \binom{n-k_2}{\frac{n}{2} - \left(\frac{k_2}{2}+i\right)},$$

(proof by Wolfram Alpha, seriously - the function is very well-behaved over even $k < \frac{n}{2}$, $i \in \{0, ..., \frac{k}{2}\}$). Note that also, k_2 will have additional factors in the sum that k_1 will not have.

Thus

$$\left\lfloor \frac{n}{2} \right\rfloor = \operatorname*{arg\,max}_{k \in \{2,4,\ldots,\lfloor \frac{n}{2} \rfloor\}} \left\{ \lim_{n\to\infty} \sqrt{n} \sum_{i=0}^{k/2} 2i \left[\binom{n}{n/2} n \right]^{-1} \binom{k}{\frac{k}{2}+i} \binom{n-k}{\frac{n}{2} - (\frac{k}{2}+i)} \right\}.$$

From here we have

$$\lim_{n\to\infty} \sqrt{n} \sum_{i=0}^{\lfloor \frac{n}{4} \rfloor} 2i \left[\binom{n}{n/2} n \right]^{-1} \binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{4} \rfloor + i} \binom{n - \lfloor \frac{n}{2} \rfloor}{\frac{n}{2} - (\lfloor \frac{n}{4} \rfloor + i)} = \frac{1}{2\sqrt{2\pi}}.$$

This is again from Wolfram Alpha. □

Claim 4. $\text{disc}_\mu(DISJ_n) \leq O(1/\sqrt{n})$.

Proof. We have from claims 3 and 4 that

$$\text{disc}_\mu(DISJ_n) = \max_R |\mu(R \cap f^{-1}(0)) - \mu(R \cap f^{-1}(1))| \leq \frac{1}{2\sqrt{2\pi n}} \leq O(1/\sqrt{n}). \quad \square$$

2 Problem 2

Was not able to solve.

3 Problem 3

Was not able to solve.

4 Problem 4

We will prove the following equivalent claim:

$$\exists \phi \neq 0 : \frac{\langle (-1)^f, \phi \rangle - 2\epsilon ||\phi||_1}{\max_{\text{rectangle } R} |\langle R, \phi \rangle|} > 2^c \implies R_\epsilon(f) > c \tag{2}$$

Claim. Proving 2 will be sufficient.

Proof. Suppose that proving 2 is not sufficient, *i.e.* that for some f

$$\exists \phi \neq 0 : \frac{\langle (-1)^f, \phi \rangle - 2\epsilon ||\phi||_1}{\max_{\text{rectangle } R} |\langle R, \phi \rangle|} > 2^{R_\epsilon(f)}$$

Clearly, this implies by 2 that $R_\epsilon(f) > R_\epsilon(f)$, a contradiction. □

Now, let's prove 2: Define

$$P_c = 2^c \text{conv}\{\pm M_R : \text{rectangle } R\}$$
$$F_\epsilon = \{M : M_{xy} \in [-1-2\epsilon, -1+2\epsilon] \text{ for } (x,y) \in f^{-1}(1); M_{xy} \in [1-2\epsilon, 1+2\epsilon] \text{ for } (x,y) \in f^{-1}(0)\}$$

We have that $F_\epsilon \cap P_c = \emptyset \implies R_\epsilon(f) > c$, as the set of all protocols of cost c that decide $(-1)^{f(x,y)}$ is contained in P_c and the matrix M defined by $M_{xy} = (-1)^{f(x,y)}$, that is

$$M_{xy} = \begin{cases} 1, & \text{if } f(x,y) = 1 \\ -1, & \text{otherwise} \end{cases}$$

is in F_ϵ.

Claim.

$$\exists \phi \neq 0 : \frac{\langle (-1)^f, \phi \rangle - 2\epsilon ||\phi||_1}{\max_{\text{rectangle } R} |\langle R, \phi \rangle|} > 2^c \iff F_\epsilon \cap P_c = \emptyset.$$

Proof. By applying the strict separating hyperplane theorem, we have

$$\begin{aligned} F_\epsilon \cap P_c = \emptyset &\iff \exists \phi \neq 0 : \langle F_\epsilon, \phi \rangle > \langle P_c, \phi \rangle \\ &\iff \exists \phi \neq 0 : \sum_{x,y} \langle (-1)^{f(x,y)}, \phi(x,y) \rangle + \sum_{x,y} (-2\epsilon) |\phi(x,y)| > 2^c \max_{\text{Rectangle } R} |\langle R, \phi \rangle| \\ &\iff \exists \phi \neq 0 : \frac{\langle (-1)^f, \phi \rangle - 2\epsilon ||\phi||_1}{\max_{\text{rectangle } R} |\langle R, \phi \rangle|} > 2^c \end{aligned}$$

This is all because

$$\max_{p \in P_c} \langle p, \phi \rangle = 2^c \max_{\text{Rectangle } R} |\langle R, \phi \rangle|$$

$$\left\{ \arg\min_{g \in F_\epsilon} \langle g, \phi \rangle \right\}_{x,y} = \begin{cases} 1 - 2\epsilon, & \text{if } \phi(x,y) \geq 0, f(x,y) = 0 \\ 1 + 2\epsilon, & \text{if } \phi(x,y) < 0, f(x,y) = 0 \\ -1 - 2\epsilon, & \text{if } \phi(x,y) \geq 0, f(x,y) = 1 \\ -1 + 2\epsilon, & \text{if } \phi(x,y) < 0, f(x,y) = 1 \end{cases}$$

where $\max_{p \in P_c} \langle p, \phi \rangle$ follows from the fact that the maximum inner product of vectors in a convex hull with any vector in the space is achieved for at least one of the edge points that define the convex hull. Here, the edge points are defined by combinatorial rectangles. □

This completes the proof.

5 Problem 5

Here is the algorithm: While there is at least 1 set of 2018 consecutive numbers, change the signs of one such set.

Assuming that the algorithm terminates, it clearly gives a valid configuration in the end. Now, let's prove it will terminate.

Notice that after each step the total sum S of all numbers on the circle will increase (the sum of the 2018 consecutive values who's sign is flipped in this move will strictly increase; all other numbers will be unchanged). The total sum of all the numbers after any step is bounded from above by $n \max_i |a_i|$, where the a_is are the numbers on the circle. Thus, the algorithm must terminate. □

6 Problem 6

First notice that $E(n,d) = 0$ for all $d \geq n$ (just set $p = x_1 x_2 \cdot ... x_n$). Let's look at $d < n$ then.

Claim. $E(n,d) = 1$ for all $d < n$.

Proof. First, notice that $E(n,d) \leq 1$ as for $p \equiv 0$, we have $\sum_{x \in \{0,1\}^n} |p(x_1, x_2, ..., x_n) - x_1 x_2 ... x_n| = 1$. Now let's prove that this is the minimum possible. Specifically, we will prove that for all $m < 1$, $E(n,d) > m$.

Define $\phi \in \mathbb{R}^{\{0,1\}^n}$ as

$$\phi(x) = (-1)^{n - |\{i : x_i = 1\}|}$$

Notice that $||\phi||_\infty = 1$ and that for all $\tilde{f}$ such that $||f - \tilde{f}||_1 < m < 1$ we have that there exists a ξ such that $||\xi||_1 \le 1$ and

$$\begin{aligned}
\langle \tilde{f}, \phi \rangle &= \langle f + \xi, \phi \rangle \\
&= \langle f, \phi \rangle + \langle \xi, \phi \rangle \\
&= 1 + \langle \xi, \phi \rangle \\
&> 1 - \sum_i |\xi_i| \cdot ||\phi||_\infty \\
&\ge 1 - m \cdot 1 = 1 - m > 0
\end{aligned}$$

Also notice that $\{\tilde{f} : ||f - \tilde{f}||_1 < 1\}$ is convex and bounded.

Now we will prove that for all polynomials p such that $d = deg(p) < n$, we will have $\langle \phi, p \rangle = 0$. Intuitively, this is true because of the principle of inclusion and exclusion. Namely, $p(1,1,...,1) = \sum_{k \in \{0,1\}^n} \lambda_k$, where λ_k is the coeficient in p associated with the monomial $\prod_{i:k_i=1} x_i$ and for all k such that $|\{i : k_i = 1\}| > d$, $\lambda_k = 0$ in order for the degree of the polynomial to be d.

By the principle of inclusion and exclusion

$$\begin{aligned}
p(1,1,...,1) &= \sum_{k \in \{0,1\}^n} \lambda_k \\
&= \sum_{i=0}^{n-1} (-1)^{n-i-1} \sum_{k \in \{0,1\}^n : |\{i:k_i=1\}| = i} p(k).
\end{aligned}$$

How exactly is the inclusion-exclusion principle applied? First, notice that for all $k \in \{0,1\}^n$,

$$p(k) = \sum_{l \in \{0,1\}^n : l_i = 1 \implies k_i = 1} \lambda_l$$

This is because the monomial associated with λ_l will be zero if for some i we have $l_i = 1$ and $k_i = 0$, as the factor x_i in the monomial will be equal to zero. On the other hand, if all the factors in the monomial are ones, the monomial will contribute λ_l to the final sum.

From here we can define a set

$$A_k = \bigcup_{l \in \{0,1\}^n : l_i = 1 \implies k_i = 1} B_l$$

where for all l, B_l is a set such that $|B_l| = \lambda_l$ and for all $l_1, l_2 \in \{0,1\}^n$ such that $l_1 \ne l_2$ we have that B_{l_1} and B_{l_2} are disjoint. This is all with the goal of having $|A_k| = \sum_l \lambda_l = p(k)$ and also

$$A_{\{1,1,...,1\}} = \bigcup_{k \in \{0,1\}^n : |\{i:k_i=1\}| = n-1} A_k$$

for all polynomials of degree $< n$, because $\lambda_{\{1,1,...,1\}} = 0$. We also need the property that $A_k \cap A_m = A_s$, where

$$s_i = \begin{cases} 1, & \text{if } k_i = m_i = 1 \\ 0, & \text{otherwise} \end{cases}$$

These properties give the above expression for $p(1,1,...,1)$ when the inclusion-exclusion principle is applied on all A_k, $k \in \{0,1\}^n : |\{i : k_i = 1\}| = n - 1$, *i.e.* for $\{0,1,1,...,1\}$, $\{1,0,1,...,1\}$, ..., $\{1,1,1,...,0\}$. Why? Let $M_i = A_k$, where $k_i = 0$, and $k_j = 1$ for all $j \ne i$. We have that for $J \subset \{1,2,...,n\}$

$$\left| \bigcap_{j \in J} M_j \right| = A_s$$

where

$$s_i = \begin{cases} 1, & \text{if } i \notin J \\ 0, & \text{otherwise} \end{cases}$$

by the property for the intersection of A_k's. Thus, we see that $|i : s_i = 1| = n - |J|$. We are now able to make a logical mapping from the expression for $p(1, 1, ..., 1)$ and the general statement of the principle of inclusion and exclusion:

$$\begin{aligned}
p(1, 1, ..., 1) &= |A_{\{1,1,...,1\}}| \\
&= \left| \bigcup_{k \in \{0,1\}^n : |\{i : k_i = 1\}| = n-1} A_k \right| \\
&= \sum_{\emptyset \neq J \subset \{1,2,...,n\}} (-1)^{|J|-1} \left| \bigcap_{j \in J} M_j \right| \\
&= \sum_{i=0}^{n-1} (-1)^{n-i-1} \sum_{k \in \{0,1\}^n : |\{i : k_i = 1\}| = i} p(k).
\end{aligned}$$

Now we have that for all p with degree less than n

$$\begin{aligned}
\langle \phi, p \rangle &= \phi_{\{1,1,...,1\}} p(1, 1, ..., 1) + (-1) \sum_{i=0}^{n-1} (-1)^{n-i-1} \sum_{k \in \{0,1\}^n : |\{i : k_i = 1\}| = i} p(k) \\
&= 1 \cdot \left[\sum_{i=0}^{n-1} (-1)^{n-i-1} \sum_{k \in \{0,1\}^n : |\{i : k_i = 1\}| = i} p(k). \right] - \left[\sum_{i=0}^{n-1} (-1)^{n-i-1} \sum_{k \in \{0,1\}^n : |\{i : k_i = 1\}| = i} p(k). \right] \\
&= 0.
\end{aligned}$$

Finally, note that the set $\{p : deg(p) < n\}$ is convex and closed. Also, $\inf \langle \{\tilde{f} : ||f - \tilde{f}|| < m\}, \phi \rangle > 1 - m$ and $\sup \langle \{p : deg(p) < n\}, \phi \rangle = 0$ This implies that we can apply the strict separating hyperplane theorem and conclude that $\{\tilde{f} : ||f - \tilde{f}|| < m\} \cap \{p : deg(p) < n\} = \emptyset$ for all $m < 1$. Thus, we are done. $\square$

7 Problem 7

Here is the protocol that each prisoner will follow and that guarantees that exactly one prisoner will always give a correct color:

1. Each prisoner is assigned a unique number in the set $\{1, 2, ..., n\}$
2. Each color is assigned a unique number in the set $\{1, 2, ..., n\}$
3. Prisoner i will sum up all the colors he sees (call this sum S_i) and say that his color is $i - S_i \mod n$.

Why does this protocol work? Clearly, the actual sum of all the colors on the hats will be some $S \equiv \sum_i \text{color}_i \mod n$. Since each prisoner has a different guess for S, and there are n prisoners and n possible values for S, exactly one prisoner will guess S correctly (say that is prisoner i). Since there is exactly one number $c \in \{1, 2, ..., n\}$ such that $c + S_i \equiv S \mod n$, when prisoner i guesses the color $c \equiv S - S_i \mod n$, he is guaranteed to be correct. $\square$