Taylor Polynomial is Algebra

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1 Agenda

We want to show that any function can be approximated by a polynomial to arbitrary precision using projections from the space of real-valued functions to the subspace of real-valued polynomials.

2 Setup

Let V be the vector space of all continuous real-valued functions, and W_n be the space of all real-valued polynomials of degree at least n. W_n is a subspace of V and W_m for all $n \leq m$.

Define the inner-product on V as

$$f(x) \cdot g(x) = \int_{-1}^{1} f(x)g(x)\, dx.$$

3 Main Theorem

We want to prove that for any real-valued continuous function $f \in V$ we can pick an infinite series of polynomials $(p_i)_{i\geq 0}$, $p_i \in W_n$ such that for all $x \in [-1, 1]$

$$\lim_{n\to\infty} f(x) - p_n = 0.$$

By Gram-Schmit we know that any vector space has an orthogonal basis. Moreover, we can construct this basis so that it contains $v_1, v_2, ..., v_n$ where v_i's form an arbitrary orthogonal set within the vector space. Using this fact, let's construct a basis $\{\alpha_1, \alpha_2, ...\}$ for V such that α_i is a polynomial of degree i. It is not entirely obvious that this is possible, but, it turns out it is. Also, assuming we start with $\alpha_1 = 1$ the basis will be the set of all Legendre polynomials.

We see that the span of $\{\alpha_1, ..., \alpha_n\}$ is W_n as any polynomial of degree n can be represented as a sum of linearly independent polynomials of degrees 1 through n. So, let p_n be the projection of the function f onto the subspace W_n. In other words

$$p_n = E(f) = (f \cdot \alpha_1)\alpha_1 + ... + (f \cdot \alpha_n)\alpha_n$$

Note that

$$(\forall x \in [-1, 1]) \quad \lim_{n\to\infty} f(x) - p_n = 0 \iff \lim_{n\to\infty} ||E_n(f) - f|| = 0.$$

Proof of this is just some basic algebra assuming the continuity of $E_n(f) - f$, which is given since it is in V. Let $g = \lim_{n\to\infty} E_n(f)$:

$$\lim_{n\to\infty} ||E_n(f) - f|| = \int_{-1}^{1} (E_n(f) - f)^2\, dx = \int_{-1}^{1} (g - f)^2\, dx$$
$$\iff g - f = 0$$

What remains is to prove the convergence of the norm of $E_n(f) - f$.