Simulation and Modeling

Analysis of Simulation Output

Er. Narayan Sapkota, M.Sc.
narayan.sapkota@eemc.edu.np

Everest Engineering College (Senior Lecturer)
Kathmandu University (Visiting Faculty)

January 5, 2025

Syllabus

1. Nature of the Problem
2. Estimation Methods
3. Simulation Run Statistics
4. Replication of Run
5. Elimination of Initial Bias

Table of Contents

Nature of the Problem (1)

The nature of the problem in simulation modeling involves understanding the system being modeled and identifying how its characteristics influence the simulation and analysis methods. This ensures the simulation produces useful and meaningful results.

When stochastic variables are introduced in a simulation model, most system variables also become stochastic due to interdependencies. As the simulation progresses, these variables fluctuate, so a single observation cannot represent their true value. Multiple observations are required to estimate the true value and construct a confidence interval around it.

However, traditional statistical methods assume independence between observations, which does not hold true in simulations. In simulations, variables are often dependent on each other, such as in the case of waiting line simulations where one entity's wait time depends on the number of entities

Nature of the Problem (2)

already in the queue. This interdependence requires adjustments to standard statistical methods.

Nature of problem focuses on:

- Consistency: Ensuring estimates become more accurate as the sample size grows.
- Bias Control: Reducing bias in estimates of means and variances.
- Sequential Testing: Determining how long a simulation should run to achieve a desired level of confidence in the results.
- Problem Identification
  - Clarify the objectives of the simulation (e.g., studying queueing systems, optimizing inventory, evaluating manufacturing systems).
  - Define system inputs (e.g., random variables, system parameters) and outputs (e.g., performance metrics), and consider real-world complexities like randomness, interdependencies, and feedback loops.

Nature of the Problem (3)

- Model Type
  - Discrete Event Simulation (DES): Models systems where changes happen at specific times (e.g., customer arrivals in a queue).
  - Continuous Simulation: Models systems with continuous changes (e.g., temperature changes in a tank).
  - Monte Carlo Simulation: Uses random sampling to model systems affected by randomness (e.g., stock price behavior).
- Inputs and Outputs
  - Inputs: Random variables, system parameters, or deterministic parameters.
  - Outputs: Performance metrics like waiting time, throughput, or cost.
- Assumptions
  - Assumptions about system behavior are crucial for accurate simulations. For example, assuming Poisson-distributed inter-arrival times in a queueing system may not always match real-world data, impacting simulation accuracy.

Nature of the Problem (4)

Example Consider a Queuing System (M/M/1), where the arrival rate (λ) and service rate (μ) influence the system's behavior. The goal might be to estimate the average waiting time for customers, where the average waiting time is given by the formula:

$$W = \frac{1}{\mu - \lambda}$$

where $\mu > \lambda$ ensures system stability.

This summary emphasizes identifying the problem's characteristics, choosing an appropriate model, and considering assumptions to ensure the accuracy and relevance of simulation results.

Table of Contents

Estimation Methods (1)

Estimation methods are statistical techniques used to make inferences from simulation data. The goal is to estimate parameters like the *mean*, *variance*, or other performance metrics that describe the system.

Point Estimation Point estimation involves using sample data to estimate a single value for a parameter of interest, such as the mean or variance. The most common point estimator is the *sample mean*.
The sample mean $\hat{\mu}$ is the most common point estimator for the population mean and is calculated as:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

where x_i are the observations, and n is the number of samples.

Estimation Methods (2)

Example: After running a simulation to model *waiting times* for customers in a queue, you might collect the following data:

$$x = [5.2, 4.8, 5.5, 6.1, 5.0]$$

The sample mean waiting time is:

$$\hat{\mu} = \frac{5.2 + 4.8 + 5.5 + 6.1 + 5.0}{5} = 5.12 \text{ minutes}$$

Confidence Intervals Confidence intervals are used to estimate a range within which the true population parameter is likely to lie. For the mean of a normally distributed variable, the confidence interval is calculated as:

$$\hat{\mu} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$

where:

Estimation Methods (3)

- $\hat{\mu}$ = sample mean
- s = sample standard deviation
- n = sample size
- $z_{\alpha/2}$ = critical value for the desired confidence level (for 95% confidence, $z_{\alpha/2} = 1.96$).

Example: Suppose we have the following sample data:

$$x = [5.2, 4.8, 5.5, 6.1, 5.0]$$

Sample mean: $\hat{\mu} = 5.12$, sample standard deviation: $s = 0.69$, and $n = 5$. For a 95% confidence level:

$$CI = 5.12 \pm 1.96 \cdot \frac{0.69}{\sqrt{5}} = [4.35, 5.89]$$

So, the true mean waiting time is likely between 4.35 and 5.89 minutes.

Estimation Methods (4)

Method of Moments The method of moments involves equating sample moments (e.g., sample mean, sample variance) with the corresponding moments of a theoretical distribution to estimate parameters. For example, the sample mean can be equated to the population mean, and the sample variance can be equated to the population variance to solve for unknown parameters.

Maximum Likelihood Estimation (MLE) Maximum likelihood estimation (MLE) is a method for estimating the parameters of a statistical model by maximizing the likelihood function. The likelihood function gives the probability of observing the sample data as a function of the parameters. For example, in a queuing system with exponential service times, the MLE for the rate parameter μ is:

$$\hat{\mu} = \frac{1}{\bar{X}}$$

Estimation Methods (5)

where $\bar{X}$ is the sample mean of the service times.

Example: Suppose we have the following sample service times from a queuing system:

$$x = [1.1, 1.5, 1.4, 1.2, 1.3]$$

The sample mean $\bar{X}$ is:

$$\bar{X} = \frac{1.1 + 1.5 + 1.4 + 1.2 + 1.3}{5} = 1.3$$

Thus, the MLE for the service rate μ is:

$$\hat{\mu} = \frac{1}{1.3} = 0.769 \, \text{per time unit}$$

This is the estimated service rate for the system based on the observed data.

Estimation Methods (6)

Typically, a random variable is drawn from an infinite population with a stationary probability distribution that has a finite mean, μ, and finite variance, σ^2. This means that the population distribution is unaffected by the number of samples already taken, and it does not change over time. Additionally, if the value of one sample does not influence the value of any other sample, the random variables are said to be mutually independent. Random variables that meet these conditions are termed *independently and identically distributed* (i.i.d.).

Under broad conditions expected to hold for simulation data, the Central Limit Theorem (CLT) can be applied to i.i.d. data. The theorem states that the sum of n i.i.d. variables drawn from a population with a mean μ and variance σ^2 is approximately distributed as a normal variable with mean $n\mu$ and variance $n\sigma^2$. As any normal distribution can be transformed into a

Estimation Methods (7)

standard normal distribution with mean 0 and variance 1.

Let x_i $(i = 1, 2, \ldots, n)$ represent the n i.i.d. random variables. Using the Central Limit Theorem and applying the transformation, we obtain the following (approximate) normal variate:

$$z = \frac{\overline{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

where $\overline{x}$ is the sample mean. The sample mean is a consistent estimator for the population mean. Since the sample mean itself is a random variable, a confidence interval around its computed value needs to be established.
The probability density function of the standard normal variate is shown in Figure 1.

Estimation Methods (8)

The integral from $-\infty$ to a value u gives the probability that $z \leq u$, often denoted as $\Phi(u)$. Tables for $\Phi(u)$ are widely available. Suppose the value of u is chosen such that:

$$\Phi(u_\alpha) = 1 - \frac{\alpha}{2}$$

where α is a constant less than 1, and denote this value as u_α. The probability that $z \geq u_\alpha$ is then $\frac{\alpha}{2}$. The normal distribution is symmetric about its mean, so the probability that z is between $-u_\alpha$ and u_α is:

$$P(-u_\alpha \leq z \leq u_\alpha) = 1 - \alpha$$

Estimation Methods (9)

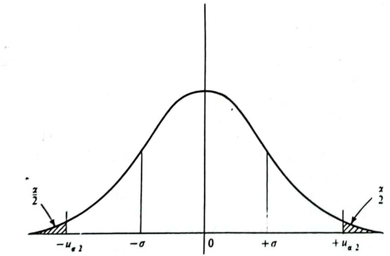

Figure 1: PDF of standard normal variate

Estimation Methods (10)

In terms of the sample mean, this probability statement becomes:

$$P\left(\mu - u_\alpha \cdot \frac{\sigma}{\sqrt{n}} \leq \overline{x} \leq \mu + u_\alpha \cdot \frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha$$

This defines the confidence interval, $\left[\overline{x} - u_\alpha \cdot \frac{\sigma}{\sqrt{n}}, \overline{x} + u_\alpha \cdot \frac{\sigma}{\sqrt{n}}\right]$, where α is the level of significance (usually expressed as a percentage).

For example, with a confidence level of 90%, $u_\alpha = 1.65$, which means that, if the experiment is repeated many times, the confidence interval will cover the true population mean 90% of the time.
In practice, the population variance σ^2 is typically unknown, so it is replaced by an estimate s^2, calculated from the sample:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2$$

Estimation Methods (11)

The normalized random variable based on s^2 follows a Student's t-distribution with $n-1$ degrees of freedom. The quantity u_α used in defining the confidence interval is replaced by a similar quantity from the Student's t-distribution, for which tables are also available. The Student's t-distribution is most accurate when the population is normally distributed, but it is commonly used when invoking the Central Limit Theorem.

Expressed in terms of the estimated variance s^2, the confidence interval for μ is given by:

$$\overline{x} \pm t_{\alpha/2,n-1} \cdot \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2,n-1}$ is the critical value from the Student's t-distribution with $n-1$ degrees of freedom.

Table of Contents

Simulation Run Statistics (1)

Simulation run statistics refer to the collection and analysis of output data generated from a single simulation run. These statistics help describe the system's performance, summarizing key metrics such as the system's efficiency, utilization, throughput, and others. The goal is to understand how the system behaves under the given conditions, and what insights can be gained from the simulation.

Descriptive Statistics Descriptive statistics provide summary information about the simulation output. The basic statistics include measures of central tendency, variability, and the shape of the data distribution.

Sample Mean: The sample mean, $\hat{\mu}$, is a measure of central tendency and is calculated as:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Simulation Run Statistics (2)

Sample Variance: The sample variance, s^2, measures the spread or dispersion of the data. It is given by:

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \hat{\mu})^2$$

Sample Standard Deviation: The standard deviation is simply the square root of the variance and provides an intuitive measure of dispersion in the same units as the original data:

$$s.d = \sqrt{\sigma^2} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2}$$

Simulation Run Statistics (3)

Skewness: Skewness measures the asymmetry of the distribution. If skewness is positive, the right tail is longer than the left; if negative, the left tail is longer. It is defined as:

$$\text{Skewness} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \hat{\mu}}{s.d}\right)^3$$

Kurtosis: Kurtosis measures the "tailedness" of the distribution. A distribution with high kurtosis tends to have more extreme values (outliers) than a normal distribution. It is given by:

$$\text{Kurtosis} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{x_i - \hat{\mu}}{s}\right)^4 - 3$$

where subtracting 3 makes the kurtosis of a normal distribution equal to 0.

Simulation Run Statistics (4)

Histograms A histogram is a graphical representation of the distribution of a dataset. It shows how often different values appear in the simulation output and helps to identify the shape of the data distribution (e.g., normal, skewed, bimodal).

Example: If you are analyzing the `waiting times` in a queue, a histogram can reveal whether the waiting times follow a normal distribution or if there are long waiting times (outliers).

Simulation Run Statistics (5)

Time-Series Analysis Time-series analysis is used when simulation output is collected over time, such as inventory levels over a set period, or the number of customers in a system at different time points. It can help identify trends, seasonality, and patterns in the data.

Methods of time-series analysis include:

- Moving averages: A simple method to smooth time-series data and identify trends.
- Autocorrelation: Measures how a time-series is correlated with its past values. This helps to identify periodic or repetitive patterns.
- Fourier Transforms: A method to convert the time-domain data into frequency-domain data, often used to identify cycles or oscillations in the data.

Simulation Run Statistics (6)

Example: Consider a simulation of an inventory system, where you track inventory levels at the end of each day. A time-series analysis can be used to identify trends such as increasing or decreasing stock levels, or periodic spikes during certain times of the year.

Additional Descriptive Metrics

Beyond central tendency and spread, other key metrics can be used to understand the simulation's performance:

Throughput: Throughput refers to the rate at which the system produces its output. For example, in a manufacturing system, throughput would be the number of items produced per unit of time.

Simulation Run Statistics (7)

Utilization: Utilization refers to the extent to which system resources (e.g., servers, machines) are being used. It is calculated as:

$$\text{Utilization} = \frac{\text{Total time resource is in use}}{\text{Total available time}}$$

Queue Length: Queue length represents the number of entities (e.g., customers, jobs) waiting in a queue at any given time. It is an important metric for systems like call centers, manufacturing lines, or service desks.

Visualizing Simulation Output

Boxplots A boxplot (or box-and-whisker plot) is a useful graphical tool for visualizing the distribution of data, particularly in terms of the quartiles. It highlights the median, quartiles, and potential outliers.
The boxplot consists of:

Simulation Run Statistics (8)

- The central box representing the interquartile range (IQR) between the first quartile (Q1) and third quartile (Q3).
- A line inside the box representing the median (Q2).
- Whiskers extending from the box to the minimum and maximum values that are within a defined range (usually 1.5 times the IQR).
- Outliers displayed as points outside the whiskers.

Example: If you are measuring the `service times` in a queue, a boxplot can give you a visual summary of the data distribution, and show if there are any significant outliers.

Density Plots A density plot is a smoothed version of a histogram. It is used to estimate the probability density function of a continuous random variable. This can give a clearer picture of the distribution's shape, especially when

there are a lot of data points. The density function is an estimate of the probability density of the variable at different points.
Example: If the simulation output represents the `waiting times` in a queue, a density plot can help reveal whether the waiting times follow a normal distribution, or if there are multiple peaks (e.g., bimodal distribution).

Empirical CDF (Cumulative Distribution Function) The empirical CDF is used to estimate the cumulative probability of observing a value less than or equal to a given data point. It is particularly useful for identifying the percentiles of the distribution.
For a set of data points $x_1, x_2, \ldots, x_n$, the empirical CDF $F(x)$ is defined as:

$$F(x) = \frac{\text{Number of values} \leq x}{n}$$

Simulation Run Statistics (10)

This function provides the cumulative probability for each value in the dataset.

Example: If you're analyzing the `response times` of a system, the CDF can show you the probability that the system's response time is below a certain threshold, which helps in understanding performance limits.

Table of Contents

Replication of Run (1)

Replication of runs involves performing multiple independent simulations to reduce the variability caused by random processes and provide more reliable estimates of system performance. By running multiple independent simulations, we can reduce variance, estimate confidence intervals, and apply statistical tests to assess the performance of the system. While the number of replications must be balanced with computational resources, the insights gained from replication lead to more precise and robust conclusions from the simulation study.

- Independent Replications: Replicating the simulation multiple times with different random seeds ensures the results are not biased by a particular sequence of random numbers. Average the results from multiple replications to improve the precision of the estimates.
- Variance Reduction Techniques:

Replication of Run (2)

- Antithetic Variates: This involves pairing random variables so that they counterbalance each other, thus reducing variability in the estimation. For example, if one simulation run results in a high value, the paired run is designed to give a low value, leading to reduced variance.
- Control Variates: This method uses a variable that is correlated with the output and whose expected value is known. By adjusting the simulation outputs using the control variate, we reduce the variance of the estimate.

- Analysis of Replicates: After running multiple replications, the results are averaged. The variance of the average estimate decreases as the number of replications increases, improving the accuracy of the simulation output.

Replication of Run (3)

Sample Mean of Multiple Replications If $Y_1, Y_2, \ldots, Y_m$ are the results of m independent replications, the sample mean $\hat{\mu}$ of these runs is given by:

$$\hat{\mu} = \frac{1}{m} \sum_{i=1}^{m} Y_i$$

This provides an estimate of the mean of the performance metric over all replications.

Sample Variance of Multiple Replications The sample variance of the outcomes $Y_1, Y_2, \ldots, Y_m$ is given by:

$$s_Y^2 = \frac{1}{m-1} \sum_{i=1}^{m} (Y_i - \hat{\mu})^2$$

where $\hat{\mu}$ is the sample mean. The variance gives an estimate of the spread or dispersion of the outcomes from the replications.

Replication of Run (4)

Standard Error of the Mean (SE) The standard error of the mean (SE) is a measure of how much the sample mean $\hat{\mu}$ varies from the true population mean, and it is given by:

$$\text{SE} = \frac{s_Y}{\sqrt{m}}$$

where s_Y is the sample standard deviation of the replication outcomes and m is the number of replications.

Example: Consider a simulation of a queueing system where we wish to estimate the average waiting time. Suppose we perform $m = 5$ independent replications and obtain the following results for the average waiting time in each run (in minutes):

$$Y_1 = 4.2, \quad Y_2 = 5.1, \quad Y_3 = 4.8, \quad Y_4 = 4.9, \quad Y_5 = 5.3$$

Replication of Run (5)

The sample mean is:

$$\hat{\mu} = \frac{1}{5}(4.2 + 5.1 + 4.8 + 4.9 + 5.3) = 4.86 \,\text{minutes}$$

Next, we calculate the sample variance:

$$s_Y^2 = \frac{1}{5-1}\left((4.2 - 4.86)^2 + (5.1 - 4.86)^2 + \ldots + (5.3 - 4.86)^2\right)$$

$$s_Y^2 = \frac{0.692}{4} = 0.173$$

Thus, the sample variance is $s_Y^2 = 0.173$.
The standard error of the mean is:

$$\text{SE} = \frac{\sqrt{0.173}}{\sqrt{5}} = \frac{0.415}{2.236} = 0.186$$

Replication of Run (6)

This means that the estimate of the mean waiting time has a standard error of 0.186 minutes.

Confidence Interval for the Mean Using the results from the replication runs, we can construct a confidence interval for the true population mean. Assuming a normal distribution for the output data, the 95% confidence interval for the mean is given by:

$$\hat{\mu} \pm z_{\alpha/2} \cdot \mathrm{SE}$$

where $\hat{\mu}$ is the sample mean, SE is the standard error of the mean, and $z_{\alpha/2}$ is the critical value for the standard normal distribution (for a 95% confidence level, $z_{\alpha/2} = 1.96$).

In our example, the confidence interval is:

$$4.86 \pm 1.96 \cdot 0.186 = [4.49, 5.23]$$

Replication of Run (7)

This means that the true mean waiting time is likely between 4.49 and 5.23 minutes with 95% confidence.

Replications and the Law of Large Numbers
The Law of Large Numbers (LLN) states that as the number of replications increases, the sample mean of the replications will converge to the true population mean. This property ensures that more replications result in more reliable estimates.

Effect of Increasing Replications As the number of replications m increases, the standard error SE decreases, which leads to a more accurate estimate of the mean. This can be mathematically observed from the formula for SE:

$$\text{SE} = \frac{s_Y}{\sqrt{m}}$$

Replication of Run (8)

Thus, increasing the number of replications reduces the variability in the mean estimate, improving the precision of the results.

Practical Considerations In practice, the number of replications is often chosen based on a trade-off between computational cost and the desired level of precision. A larger number of replications will provide more reliable results, but at the cost of increased computation time. Statistical tools or heuristics can help determine an appropriate number of replications based on the desired confidence level and acceptable margin of error.

Replication of Run (9)

Common Issues with Replications

- Run-to-Run Correlation: If the results of different simulation runs are correlated (i.e., they are not independent), the estimates from the replications may be biased. It is important to ensure that each replication is independent, which can be achieved by randomizing initial conditions and other parameters.
- Convergence: Sometimes, the simulation may not have reached a steady state by the time the first replication is completed, especially in long-running simulations. In such cases, it is important to allow the system to "warm up" or reach equilibrium before starting to collect data for analysis.
- Practical Implementation: In most cases, replications are run in parallel to save time and computational resources. Modern simulation software allows for easy implementation of multiple replications, often with built-in functions for collecting and analyzing the results.

Table of Contents

Elimination of Initial Bias (1)

When running a simulation, especially in systems that evolve over time, the initial bias can skew results if the system is not in a steady state at the start. The warm-up period is the time required for the system to stabilize.

- Warm-Up Period: The initial phase of the simulation is discarded to avoid the influence of initial conditions (such as empty queues or newly initialized systems). After the warm-up period, data is collected for analysis.
- Statistical Adjustment: If the warm-up period is not explicitly known or discarded, regression models or moving averages can be used to adjust for any trends observed in the initial part of the simulation.
- Eliminating Bias in Simulation: Regression can model the relationship between time and the output, adjusting for initial conditions. A common technique is to remove the initial data that shows trends that are unrepresentative of steady-state behavior.

Elimination of Initial Bias (2)

Why Initial Bias Exists ?

Initial bias arises because the system may not start in a steady state. This is particularly true for systems that involve queues, inventories, or other time-dependent processes. Some common causes of initial bias include:

- Transients in the system: The system may need time to reach a stable state where the distribution of variables (such as queue lengths or resource utilization) matches the steady-state behavior.
- Non-equilibrium conditions: If the system starts from an arbitrary initial state, such as an empty queue or a full inventory, the behavior during the first few time steps may not be representative of the long-term behavior.
- Warm-up period: Some simulations require an initial period where the system "warms up" before it reaches equilibrium.

Including data from this transient phase can distort the results and lead to erroneous conclusions about the system's performance.

Methods for Eliminating Initial Bias

1. Discarding the Initial Part of the Run The most straightforward method for eliminating initial bias is to discard the data collected during the initial "warm-up" period. In this approach, first, identify the warm-up period during which the system is not in equilibrium. Then, discard the results from this period and only use data from the steady-state phase of the simulation for analysis.

Example: In a simulation of a queueing system, suppose the first 500 time steps are considered the warm-up period. In this case, data from time steps 1 to 500 should be discarded, and the analysis should focus on the data from time step 501 onward.

2. Heuristic Approach to Identifying the Warm-Up Period Sometimes, it is difficult to determine a precise end point for the warm-up period. In such cases, heuristic methods can be used to estimate when the system has reached a steady state. A commonly used approach is to examine the behavior of key performance metrics (e.g., average queue length, server utilization) and look for when they stabilize.
Example: If the average queue length fluctuates greatly in the initial part of the simulation and then levels off after some time, the point at which the fluctuation decreases significantly could be considered the end of the warm-up period.

3. Graphical Methods Graphical methods can also help in determining when the system reaches a steady state. One common technique is to plot the

Elimination of Initial Bias (5)

performance measure of interest (e.g., the average queue length or waiting time) over time and visually inspect the plot for stabilization.
Example: A plot of the average queue length over time might show an initial period of high variability followed by a plateau. The point at which the plot stabilizes marks the end of the warm-up period.

4. Statistical Tests for Steady-State Behavior Statistical tests can be used to determine whether the system has reached a steady state. One such test is the *Chi-square goodness-of-fit test*, which compares the observed data with the expected steady-state distribution. If the observed data significantly differs from the expected steady-state distribution, the system has likely not reached equilibrium.
Example: Consider a simulation of a manufacturing system where the output rate is being monitored. If a chi-square test indicates that the output rate has significantly deviated from its expected steady-state distribution

Elimination of Initial Bias (6)

during the first few replications, this suggests that the system is still in the warm-up phase.

5. Determining the Length of the Warm-Up Period Determining the length of the warm-up period can be a challenge. There are several techniques available to estimate this length, including:

- Empirical methods: Based on visual inspection of performance measure plots or using known system dynamics to estimate when equilibrium is reached.
- Confidence interval methods: Using confidence intervals for key performance measures to determine when they stabilize to a point where the system no longer shows significant changes.
- Batch means method: Dividing the simulation run into multiple batches and using statistical tests to determine when the results across batches converge.

6. Batch Means Method The batch means method involves dividing the simulation run into smaller batches and checking whether the mean of each batch becomes consistent over time. Once the batch means stabilize, it indicates that the system has reached equilibrium.
Example: Suppose you divide your simulation into 10 batches, and calculate the average queue length for each batch. If the means of the first few batches are significantly different from those of the later batches, this suggests that the warm-up period has not yet ended. If the means stabilize after a certain number of batches, the warm-up period is over.

7. Autocorrelation Analysis Autocorrelation analysis involves examining the correlation between a performance measure at different time points. If the correlation is high in the initial phase of the simulation and diminishes over time, this suggests that the system is still in the warm-up phase.

Elimination of Initial Bias (8)

Example: For a simulation of a job scheduling system, the autocorrelation function of the number of jobs in the system can be computed. A high autocorrelation at early time steps and low autocorrelation at later steps indicates that the system is transitioning from a non-equilibrium state to equilibrium.

8. Sensitivity Analysis After eliminating the initial bias, it may be useful to perform a sensitivity analysis to assess how the results change with different lengths of the warm-up period. This helps ensure that the conclusions drawn from the simulation are robust and not overly sensitive to the choice of warm-up length.

Elimination of Initial Bias (9)

Practical Considerations

- Computational Cost: Eliminating initial bias can increase the computational cost of the simulation because it often requires running the simulation for a longer period or conducting multiple replications. However, the benefit of more accurate and reliable results typically outweighs the cost of additional computation.
- Choosing a Warm-Up Period: In practice, the length of the warm-up period is often chosen based on experimentation or domain knowledge. It may require a trial-and-error approach to determine the optimal number of time steps or replications to discard.