Monte Carlo Simulation Method for Calculation of Integral

1 Objectives

- To understand and apply the Monte Carlo method for numerical integration.
- To approximate the value of a definite integral using random sampling.
- To evaluate the accuracy of the Monte Carlo simulation in comparison with the analytical solution.
- To analyze the effect of increasing the number of random samples on the accuracy of the result.

2 Prerequisites

- Basic understanding of definite integrals and probability theory.
- Familiarity with Python programming, especially generating random numbers using libraries such as `NumPy`.
- Understanding the concept of convergence and the Law of Large Numbers in the context of simulations.

3 Theory

The Monte Carlo method is a statistical technique that uses random sampling to estimate numerical results. It is particularly useful for evaluating definite integrals where analytical solutions are difficult or impossible to obtain.
Consider the integral:

$$I = \int_a^b f(x)\, dx$$

Using Monte Carlo simulation, this integral can be approximated by:

$$I \approx (b - a) \cdot \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

where x_i are uniformly distributed random samples in the interval $[a, b]$, and N is the number of samples.
Suppose we have a function $f(x)$ that is positive and bounded within the interval $[a, b]$, and the function is bounded above by some value c. The graph of the function lies within a rectangle of width $(b-a)$ and height c. We can estimate the integral of $f(x)$ by randomly sampling points within this rectangle.

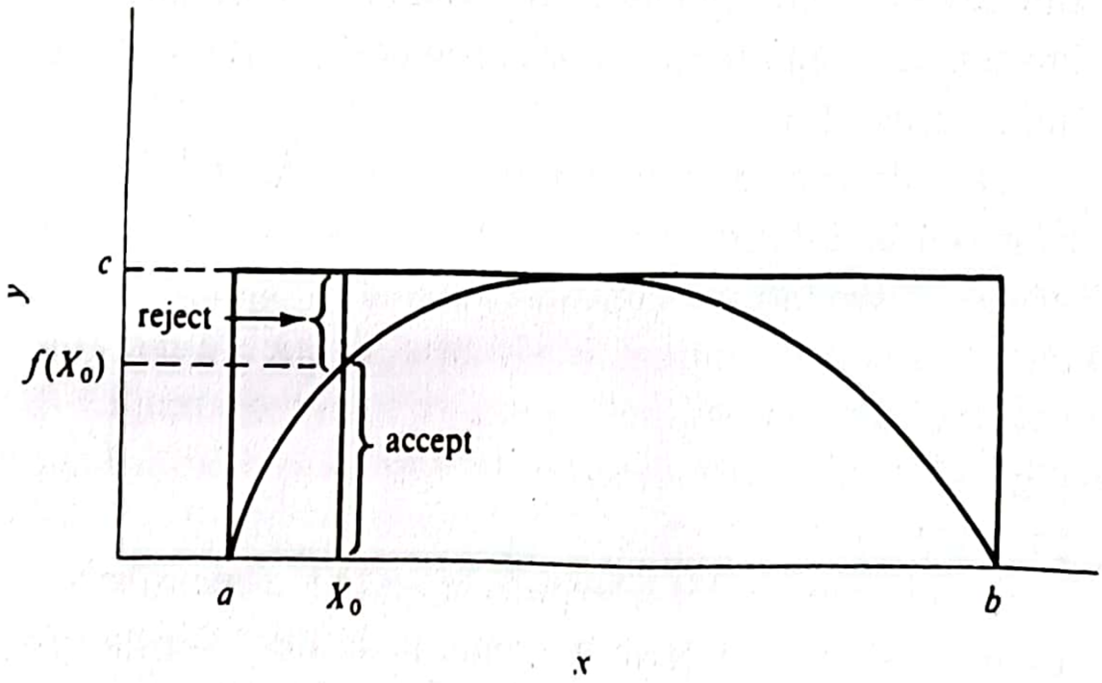

Figure 1:

Steps of Monte Carlo Integration

1. **Define the Area of the Rectangle:**
   The area of the rectangle in which we are sampling is given by:
   $$A_{\text{rectangle}} = c(b - a)$$
   where c is the height of rectangle, and $b - a$ is the length of the integration interval.

2. **Generate Random Points:**
   - For each random point, select x_0 randomly from the interval $[a, b]$.
   - Select a random y-coordinate Y from the interval $[0, c]$.

3. **Check if the Point is Under the Curve:**
   - If the point (x_0, Y) lies below the curve, i.e., if $Y \leq f(x_0)$, then it is counted as a point under the curve.
   - If $Y > f(x_0)$, the point is discarded.

4. **Estimate the Integral:**
   After generating N random points, the fraction of points that lie under the curve is approximately:
   $$\frac{n}{N}$$
   where n is the number of points under the curve, and N is the total number of points generated. The estimated value of the integral is then given by:
   $$I = \frac{n}{N} \times c(b - a)$$

where $c(b-a)$ is the area of the rectangle.

5. **Improvement of Accuracy:**
As the number N of random points increases, the accuracy of the estimate improves due to the Law of Large Numbers. With larger values of N, the fraction $\frac{n}{N}$ converges to the actual area under the curve.

4 Tasks

1. **Simulate Integral Calculation Using Monte Carlo Method**
    - Define a continuous function $f(x)$ to integrate over a specified interval $[a, b]$.
    - Generate a large number of random points x_i in the interval.
    - Approximate the integral using the average value of $f(x_i)$.
2. **Analyze Convergence and Accuracy**
    - Repeat the simulation with varying sample sizes (e.g., $N = 100, 1000, 10000$).
    - Compare the Monte Carlo results with the exact value of the integral, if known.
    - Observe how the accuracy improves with increasing sample size.
3. **Visualize the Monte Carlo Process**
    - Create a graphical representation of random points and the function curve.
    - Plot convergence results (error vs number of samples).
    - Analyze the efficiency of the method and factors affecting its performance.

5 Expected Outcomes

- Estimation of definite integrals using Monte Carlo simulations with increasing accuracy as the number of samples increases.
- Visualization of the random sampling approach used in the simulation.
- Better understanding of statistical approximation methods and how they compare to traditional analytical techniques.

6 Student Self-Assessment Criteria

- **Accuracy of Estimated Integral:** Compare the simulated result with the exact or numerical solution.
- **Understanding of the Monte Carlo Method:** Evaluate the ability to explain the simulation process and how randomness contributes to the estimation.
- **Analysis of Convergence:** Assess how well the student can analyze and report the relationship between sample size and result accuracy.