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DOES CURVATURE ENHANCE FORECASTING?

CAIO ALMEIDA*

Graduate School of Economics, Getulio Vargas Foundation
calmeida@fgv.br

ROMEU GOMES† and ANDRÉ LEITE‡

Central Bank of Brazil
†*romeu.gomes@bcb.gov.br*
‡*andreluis.leite@bcb.gov.br*

AXEL SIMONSEN

Graduate School of Economics
Getulio Vargas Foundation
axelsimonsen@fgvmail.br

JOSÉ VICENTE

Central Bank of Brazil and Faculdades, Ibmec-RJ
jose.valentim@bcb.gov.br

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In this paper, we analyze the importance of curvature term structure movements on forecasts of interest rates. An extension of the exponential three-factor Diebold and Li (2006) model is proposed, where a fourth factor captures a second type of curvature. The new factor increases model ability to generate volatility and to capture nonlinearities in the yield curve, leading to a significant improvement of forecasting ability. The model is tested against the original Diebold and Li model and some other benchmarks. Based on a forecasting experiment with Brazilian fixed income data, it obtains significantly lower bias and root mean square errors for most examined maturities, and under three different forecasting horizons. Robustness tests based on two sub-sample analyses partially confirm the favorable results.

Keywords: Parametric term structure models; principal components; vector autoregressive models; interest rate mean forecasting.

1. Introduction

Understanding the evolution of the term structure of interest rates is important for a variety of reasons. Portfolio managers adopt yield curve models for allocation

*Corresponding author.

purposes; risk managers and macroeconomists extract term structure movements to mimic their behavior or use them as a tool for monetary policy. For each specific application, there are innumerous statistical procedures available to model the term structure.[1] Identifying the perfect matching between a model and the corresponding empirical application is a hard task, which in general demands creativity and pragmatism.

In particular, forecasting interest rates is an issue that has recently attracted the attention of a large number of researchers. Predictability questions raised by Fama and Bliss [16] have been revisited through the lens of dynamic term structure models (see Duffee [14]; Dai and Singleton [10]). Ang and Piazzesi [3] have proposed and analyzed a macro-finance affine Gaussian model showing that macroeconomic variables contribute to improve interest rate forecasts. More recently, Diebold and Li (DL) [11] have suggested a variation of the Nelson and Siegel [28] model to forecast the yield curve. In their model the term structure is parameterized as a sum of three basic movements (level, slope and curvature) and time-series for these movements are extracted to forecast the future evolution of interest rates.

Although most authors adopt three-factor term structure models on empirical forecasting applications,[2] Cochrane and Piazzesi (CP) [7] suggest that the fourth principal component of the U.S. zero-coupon curve should not be neglected since it explains a large portion of bond return predictability. CP obtain a tent-shaped factor, common to all bonds, which predicts excess returns with R^2's over 40%. Then they identify that the fourth principal component explains more than 20% of the return predictability captured by this factor. CP further stress that this fourth factor is usually neglected by fixed income researchers because it only explains a tiny portion of in-sample interest rates variability.

Motivated by the results provided by CP, our goal is to make use of higher order movements of the term structure to improve model forecasting ability. We extend the DL model to incorporate a fourth factor driving a second type of curvature obtaining a dynamic version of the Svensson [29] model. The role of this new factor is to possibly enhance the fit of interest rate variability and nonlinearities on the yield curve. The effective result obtained is a modification of the original dynamics of the slope and curvature factors, culminating in a change on the bond risk premia structure, a fundamental component in forecasting related exercises (see Duffee, [14]; Almeida and Vicente, [2]). A forecasting empirical application adopting high frequency (daily) Brazilian fixed income data indicates that our model outperforms

[1]To cite a few, McCulloch [26] presented a cubic splines model to estimate a cross-sectional term structure; Vasicek [30] proposed one of the first affine dynamic term structure models, a one-factor Gaussian model; Litterman and Scheinkman [25] adopted principal component analysis to extract term structure movements; Heath *et al.* [20] proposed a general theory for arbitrage-free dynamic models; Duffie and Kan [15] proposed affine multi-factor models; Ahn *et al.* [1] proposed quadratic term structure models, one of the most recently developed multi-factor dynamic models.
[2]Some exceptions include Svensson [29], Fan *et al.* [17], Bester [5], Collin Dufresne *et al.* [9], and Han [19].

the DL model on both bias and Root Mean Square Errors (RMSE) criteria, for most maturities analyzed and under three different forecasting horizons (1-day, 1- and 3-month). The superior forecasting ability is confirmed to be statistically significant with a Diebold and Mariano [12] test under a quadratic loss function.

Of course, a number of criticisms to the empirical exercise could be raised and some robustness tests are in order here. First, one could argue that the superior predictive ability of the dynamic Svensson model might not be consistent across different time frames. In order to address this question, we perform a robustness check and show that the results present considerable stability under one of the subsamples analyzed but are weakened under another one. Another possible criticism says that the DL model might not be the most appropriate benchmark. Although it quickly became a benchmark on forecasting exercises since it outperforms a variety of reliable candidates,[3] it does not outperform the Random Walk model for more recent term structure data (see Moench [27]). In order to shed some lights on this question, we also compare the predictions of the dynamic Svensson model with two additional alternative benchmarks, namely the Random Walk and an autoregressive process on yields. The proposed model outperforms these two competitors indicating that it provides sensible forecasts.

There are also some other recent studies that have extended the DL model. Fontaine and Garcia [18] include an extra liquidity factor to the model. Huse [22] maps the three DL extracted term structure movements into observable macroeconomic variables. And, Koopman *et al.* [24] introduce time-varying parameters in the DL model. Notwithstanding the fact that all these papers suggest generalizations of the DL model aiming to improve forecasts, they differ from ours in that we analyze the specific impact on forecasts of introducing a second curvature factor. We believe that this extra factor is important to capture more complex features of the yield curve, and that it can become especially important to emerging fixed income markets. Our intuition is reassured by the empirical results obtained.

In general, due to in-sample over-fitting problems, more complex models fail to correctly capture the dynamics of observed phenomena, and end up achieving worse out-of-sample forecasting records. In clear contrast with the idea of in-sample over-fitting, our results indicate that curvature and more generally other higher order movements of the term structure should be seriously considered as tools for a better identification of bond risk premia. In fact, our results reinforce the findings of Cochrane and Piazzesi [7] but for a different category of models.[4]

The paper is organized as follows. Section 2 presents the DL and the dynamic Svensson models. Section 3 explains how to estimate and forecast with these models. Section 4 describes the dataset adopted. Section 5 analyzes the empirical results

[3]See Almeida and Vicente [2], Bowsher and Meeks [6], Huse [22], Kargin and Onatski [23], and Moench [27] for comparisons of the DL model to other forecasting methods.

[4]For a deeper analysis of bond risk premia in dynamic models, see Cochrane and Piazzesi [8], who construct an affine model consistent with the stylized facts observed by Cochrane and Piazzesi [7].

where results on estimation are presented and a forecasting exercise is performed. Section 6 offers some concluding comments and possible topics for future research.

2. The Models

We implement a dynamic version of the Svensson [29] model and compare its ability to forecast out-of-sample interest rates to that of the Diebold and Li [11] model (DL model). In this section, the two models are described in details.

2.1. *The DL model*

DL modified the exponential model proposed by Nelson and Siegel [28], considering the following parametric form for the dynamic evolution of the term structure:

$$R_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right). \tag{2.1}$$

A nice interpretation of Eq. (2.1) can be obtained if we fix the parameter λ_t. The panel on the left-hand side of Fig. 1 presents the loadings of the three movements captured when $\lambda_t = \lambda = 0.123\ \forall t$.[5] The dashed line represents the loadings of the level factor. A shock on the variable β_1 changes yields of all maturities in the same direction. The solid line represents the loadings of the slope factor. A positive shock to β_2 increases short-term yields approximately preserving long-term yields the same. The dotted line captures the loadings of the curvature factor. A positive

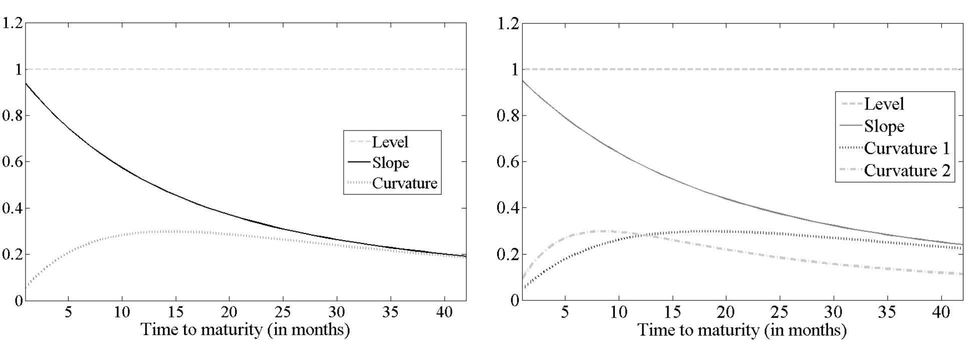

Fig. 1. Loadings of the DL and Svensson models.

This picture presents the term structure movements under the DL and Svensson models. The panel on the left-hand side shows loadings of the level, slope and curvature movements under the DL model when the decay parameter is fixed at $\lambda = 0.123$. The panel on the right-hand side shows loadings of the level, slope and the two curvatures movements under the Svensson model when the decay parameters are fixed at $\lambda = 0.0976$ and $\tilde{\lambda} = 0.2105$.

[5]All λ values in this paper are specified considering one month as the time unit.

shock to β_3 primarily makes medium-term yields go up, preserving the two extremes of the curve approximately the same. Note that the parameter λ controls the decay rate of the slope loading and the maximum value of the curvature loading.

The choice of λ_t is a very important and difficult issue to solve. How should one choose the λ_t process? Should it be a stochastic process like the betas, a deterministic process, or simply a constant value for all dates? In their work, DL decided for the last and simpler solution, to fix it to a constant value ($\lambda_t = \lambda \; \forall \, t$), advocating in favor of simplicity and parsimoniousness.[6] DL argue that historically the curvature has been linked to changes of medium term yields, and that usually 2- and 3-year yields were used to represent medium term yields. For this reason, they decided to choose λ to maximize the curvature loadings at the average of these two maturities, that is, at 30 months. The λ_t value that maximizes the loading of the curvature factor at 30 months is $\lambda_t = \lambda = 0.061 \; \forall \, t$.

The choice of the decay parameter suggested by DL is arbitrary and does not necessarily work well for yields with features different from the US data. In this work, the value of λ is also kept fixed, but we use a less arbitrary way to choose its fixed value. The idea is to search for a λ under which the model generates its best in-sample fit by considering the whole panel of yield observations in both time and maturity dimensions.

Note that if λ is fixed, there will be linear regressions relating yields to β parameters:

$$R_t(\tau)_{\texttt{observed}} = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) + \epsilon_t(\tau). \quad (2.2)$$

Therefore, for each fixed date the parameters (betas) can be obtained by minimizing the sum of squared residuals of these cross-section linear regressions

$$\hat{\beta}_t = \arg\min_{\beta} \sum_{j=1}^{N_t} \epsilon_t(\tau_j, \lambda)^2, \quad (2.3)$$

where $\hat{\beta}_t$ represents a vector with stacked betas, N_t represents the number of observed yields for date t, τ_j is the time to maturity of the jth yield on that same date, and λ is fixed at a constant value.

Thus for each λ, we have a two-dimensional surface of residuals $\epsilon_t(\tau, \lambda)$. Then we choose λ such that the weighted sum of squared residuals is minimized:

$$\hat{\lambda} = \underset{\lambda \in [0.03\ 0.42]}{\arg\min} \sum_{t=1}^{T} \left(\sum_{j=1}^{N_t} \omega_{t,j} \epsilon_t(\tau_j, \lambda)^2 \right), \quad (2.4)$$

[6] Note that if λ_t varies through time this will imply a change on the loadings of the slope and curvature factors and the procedure will not be exactly consistent with a raw application of principal component analysis as done by Litterman and Scheinkman [25]. However, there might be cases where indeed dynamic loadings will better capture the dynamics of certain term structures (see Koopman *et al.* [24]).

where T is the number of daily observations in the in-sample period and $\omega_{t,j}$ is the weight of the residual on day t corresponding to the yield with time to maturity τ_j. We limit the λ values between 0.03 and 0.42. These bounds correspond to a maximum of the curvature loading at 5 years and 0.05 years (0.6 months), respectively.[7] There is no reason to search for optimal λ values outside this interval, since the greatest time to maturity of yields in our database is 3.5 years, and the shortest maturity of fixed income instruments available is 1 month.

The different maturities are weighted by a liquidity factor. As will be explained in Sec. 4 we extract the observed yields from a future contract of interest rates negotiated in the Brazilian market. Then we set $\omega_{t,j}$ equal to the trading volume in day t of the contract with time to maturity τ_j. Figure 2 presents the average number of traded contracts as function of the maturity for the full sample (between January 3, 2003 and October 18, 2007) and two different subsamples (the first between January 3, 2003 and May 2, 2005 and the second between May 3, 2005 and October 18, 2007).[8] As we can see the traded volume is a decreasing function of maturity. The liquidity for maturities greater than 3.5 years is very poor. Therefore we decided to consider in this study only interest rates with time to maturity less than or equal to 3.5 years. Moreover, note that the liquidity in the first sub-sample is smaller than

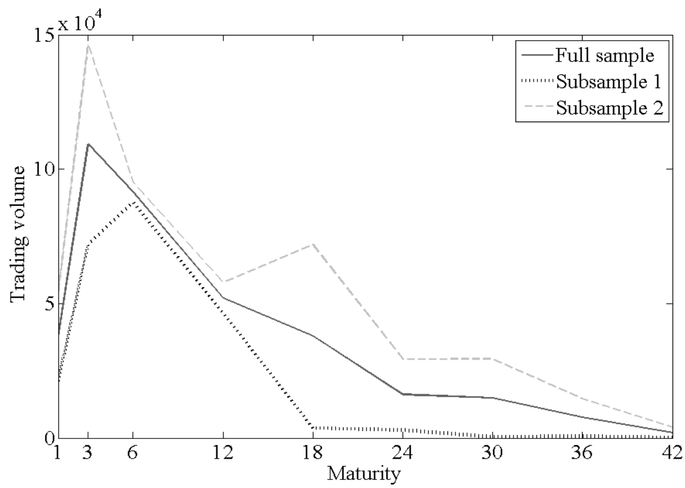

Fig. 2. Liquidity of ID-Future.

This picture presents the average number of traded ID-Future contracts for different maturities and samples.

[7]Given a value for λ, to obtain the time to maturity where the curvature achieves its maximum one has to solve the equation $1 + \lambda\tau + (\lambda\tau)^2 = e^{\lambda\tau}$ on the maturity variable τ.

[8]In Fig. 2 the number of traded contracts corresponding to the maturity τ_j is actually the average number of traded contracts with maturities between τ_{j-1} and τ_j. We adopted this procedure to clarify the exposition because the time to maturity of each contract varies everyday.

the liquidity in the second sub-sample. This fact is a possible justification for the weak performance of the models in the first sub-sample as will be shown in Sec. 5.

2.2. *The dynamic Svensson model*

The dynamic Svensson model is just an extension of the DL model defined by the incorporation of a fourth factor, which represents a second type of curvature. In this case, the term structure will dynamically evolve according to the following equation:

$$\begin{aligned} R_t(\tau) = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda_t\tau}}{\lambda_t\tau} - e^{-\lambda_t\tau}\right) \\ + \beta_{4t}\left(\frac{1-e^{-\tilde{\lambda}_t\tau}}{\tilde{\lambda}_t\tau} - e^{-\tilde{\lambda}_t\tau}\right). \end{aligned} \tag{2.5}$$

Note that the first three movements are exactly the ones that appear in the DL model. The fourth term is a copy of the third one, with a different λ, though. We argue that this subtle change will be very important to model term structures of interest rates that are more volatile than the U.S. curve. This will generally be the case for emerging markets curves, corporate bond curves, and credit derivative markets, indicating that this small extension might potentially produce a huge gain in forecasting abilities. The panel on the right-hand side of Fig. 1 presents the loadings of those four movements when the fixed value for $\lambda = 0.0976$ maximizes the first curvature loadings at 18.4 months, and the fixed value for $\tilde{\lambda} = 0.2105$ maximizes the second curvature loadings at 8.5 months.

Similarly to DL, in the dynamic Svensson model we fix the parameters λ's to minimize the in-sample adjustment errors. For each λ and $\tilde{\lambda}$, the cross-section regressions are implemented writing the observed yields as a linear combination of the four proposed movements plus an error term:

$$\begin{aligned} R_t(\tau)_{\texttt{observed}} = \beta_{1t} + \beta_{2t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \beta_{3t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \\ + \beta_{4t}\left(\frac{1-e^{-\tilde{\lambda}\tau}}{\tilde{\lambda}\tau} - e^{-\tilde{\lambda}\tau}\right) + \tilde{\epsilon}_t(\tau). \end{aligned} \tag{2.6}$$

The λ's are estimated by minimizing the weighted sum of squared residuals $\tilde{\epsilon}_t(\cdot)$:

$$\left(\hat{\lambda}, \hat{\tilde{\lambda}}\right) = \underset{\lambda,\tilde{\lambda}\in[0.03\ 0.42]}{\arg\min} \sum_{t=1}^{T}\left(\sum_{j=1}^{N_t}\omega_{t,j}\tilde{\epsilon}_t(\tau_j,\lambda,\tilde{\lambda})^2\right), \tag{2.7}$$

where ω is specified in the same way as in the DL model.

3. Forecasting

DL propose univariate AR(1) processes as time series models to forecast the future evolution of the Nelson and Siegel factors. Under forecasting horizons larger then one time unit, there are two alternatives to be adopted. The first is to estimate the model with the original data frequency and produce multi-step forecasts. The second is to estimate the model by regressing movements at time $t+h$ ($\beta_{i,t+h}$'s) on movements at time t (β_{it}'s), were h is the number of time slots within each particular forecasting horizon. For instance, if one is interested in one-month horizon forecasts, and is using daily data to estimate the model, a regression of factors on their 21-day lagged values should be performed. DL suggest this last method as the optimal one when the purpose is to minimize the RMSE. Following their suggestion, the lagged-values method is adopted in this work. For $i = 1, 2, 3$, DL estimate

$$\hat{\beta}_{i,t+h/t} = c_i^{DL} + \phi_i^{DL} \hat{\beta}_{it} + \eta_{i,t+h}^{DL}, \tag{3.1}$$

where c_i^{DL} and ϕ_i^{DL} are constants, $\eta_{i,t+h}^{DL}$ is a univariate zero mean Gaussian error, and h is the number of time slots within each particular forecasting horizon.

In their work, DL also experiment with vector autoregressive (VAR) models but identify that, for the particular period of the U.S. term structure analyzed in their paper, the independent univariate autoregressive processes are better forecasters than the vector autoregressive model. In contrast with their results, considering the Brazilian term structure analyzed in this work, the VAR version of the model presents superior forecasting results when compared to the univariate version. The VAR is fitted by

$$\hat{\beta}_{t+h/t} = C^{DL} + \Phi^{DL} \hat{\beta}_t + \bar{\eta}_{t+h}^{DL}, \tag{3.2}$$

where C^{DL} is a 3×1 vector of constants, Φ^{DL} is a 3×3 matrix, and $\bar{\eta}$ is a multivariate zero mean Gaussian error, with a free correlation structure, not necessarily the identity matrix.[9]

Under the univariate model, for each specific movement ($i = 1, 2, 3$), forecasts for its conditional mean is produced by

$$\hat{\hat{\beta}}_{i,t+h/t} = \hat{c}_i^{DL} + \hat{\phi}_i^{DL} \hat{\beta}_{it}. \tag{3.3}$$

Similarly, for the VAR multivariate model, forecasts for the conditional means of all movements are jointly produced by

$$\hat{\hat{\beta}}_{t+h/t} = \hat{C}^{DL} + \hat{\Phi}^{DL} \hat{\beta}_t. \tag{3.4}$$

Once knowing the conditional forecasts for each movement, for any fixed maturity, model implied yield forecasts can be easily produced with the use of

$$R_{t+h/t}(\tau) = \hat{\hat{\beta}}_{1,t+h/t} + \hat{\hat{\beta}}_{2,t+h/t} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \hat{\hat{\beta}}_{3,t+h/t} \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right). \tag{3.5}$$

[9]In the empirical exercise presented in Sec. 5 the constant term of the VAR process is set to zero for both DL and Svensson models. The results obtained with the complete VAR model were slightly worse under both models.

For the dynamic Svensson model, the procedure is exactly the same. A univariate autoregressive time series model is fitted to the four term structure movements by

$$\hat{\beta}_{i,t+h/t} = c_i^S + \phi_i^S \hat{\beta}_{it} + \eta_{i,t+h}^S, \quad i = 1, \dots 4, \tag{3.6}$$

and a VAR process is fitted by

$$\hat{\beta}_{t+h/t} = C^S + \Phi^S \hat{\beta}_t + \bar{\eta}_{t+h}^S. \tag{3.7}$$

Finally, yields forecasts are given by

$$\begin{aligned} R_{t+h/t}(\tau) = {} & \hat{\hat{\beta}}_{1,t+h/t} + \hat{\hat{\beta}}_{2,t+h/t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau}\right) + \hat{\hat{\beta}}_{3,t+h/t}\left(\frac{1-e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau}\right) \\ & + \hat{\hat{\beta}}_{4,t+h/t}\left(\frac{1-e^{-\tilde{\lambda}\tau}}{\tilde{\lambda}\tau} - e^{-\tilde{\lambda}\tau}\right). \end{aligned} \tag{3.8}$$

As discussed before, while DL obtained better results with the use of univariate models instead of a VAR, their results were not confirmed for the Brazilian term structure data. Similarly, for the dynamic Svensson model the VAR forecasting ability is higher than that of univariate autoregressive models.

4. Data

In this section, the Brazilian market of ID Futures and the dataset adopted are briefly described. For more detailed information on the products and available datasets see www.bmf.com.br/portal/portal_english.asp.

The One-Day Inter Bank Deposit Future Contract (ID-Future) with maturity T is a future contract whose underlying asset is the accumulated daily ID rate[10] capitalized between the trading time t ($t \leq T$) and T. The contract size corresponds to R\$ 100,000.00 (one hundred thousand Brazilian Reals) discounted by the accumulated rate negotiated between the buyer and the seller of the contract.

This contract is very similar to a zero coupon bond, except that it pays margin adjustments every day. Each daily cash flow is the difference between the settlement price[11] on the current day and the settlement price on the day before corrected by the ID rate of the day before.

The Brazilian Mercantile and Futures Exchange (BM&F) is the entity that offers the ID-Future. The number of authorized contract-maturity months is fixed by BM&F (on average, there are about twenty authorized contract-maturity months for each day but only around ten are liquid). Contract-maturity months are the first four months subsequent to the month in which a trade has been made and, after

[10] The ID rate is the average one-day inter bank borrowing/lending rate, calculated by CETIP (Central of Custody and Financial Settlement of Securities) every workday. The ID rate is expressed in effective rate per annum, based on 252 business-days.

[11] The settlement price at time t of an ID-Future with maturity T is equal to R\$ 100,000.00 discounted by its closing price quotation.

that, the months that initiate each following quarter. Expiration date is the first business-day of the contract-maturity month.

Data consisted of 1191 daily observations of ID Future yields between January 3, 2003 and October 18, 2007, and represent the most liquid ID contracts traded during these five years. We split this database in two parts. The first, composed of 596 daily observations from January 3, 2003 to May 3, 2005, is the in-sample period in which the estimations of λ's are made. The second, from May 4, 2005 to October 18, 2007 is the out-of-sample period in which the forecast power of the models is tested. We also consider two additional sub-samples in order to verify the robustness of our results. The sub-sample 1 starts in January 3, 2003 and extends until May 2, 2005 with in-sample period between January 3, 2003 and February 20, 2004. The sub-sample 2 covers the rest of the full sample, with in-sample period extending from May 3, 2005 to July 10, 2006.

Figure 3 plots the time series of yields maturating at 1, 6, 12, and 24 months over the five years of the sample period. It also illustrates the in-sample and out-of-sample periods for the full sample and the two sub-samples. Because not every day has the same maturities available, we obtain these times series using a spline interpolation procedure. Note that the yields plummeted in the first year (in-sample of sub-sample 1). Next (out-of-sample of sub-sample 1), they stayed almost steady with a slight increase. Finally, they have fallen slowly until reaching a minimum at the end of the sample.

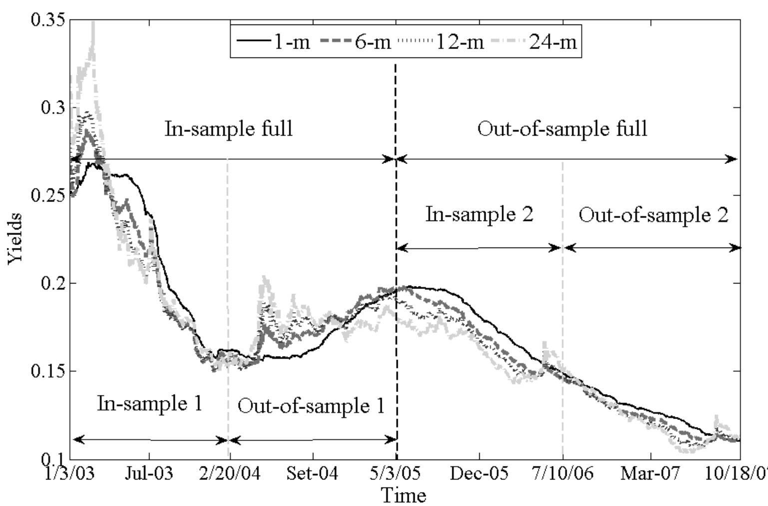

Fig. 3. Time series of some yields.

This picture presents the time series of the yields with time to maturity of 1, 6, 12, and 24 months from January 3, 2003 until October 18, 2007. The yields were obtained by means of a spline interpolation procedure. Vertical dashed lines indicate the in-sample and out-of-sample periods for the full sample and two different sub-samples.

5. Empirical Results

5.1. *Model fitting*

Figure 4 illustrates four examples of in-sample fitting of the DL and Svensson models, for arbitrarily chosen moments. Within these graphs there are increasing, inverted and twisted yield curves, demanding a significantly flexible curvature factor to capture all these shapes. Note that the Svensson model is better able to capture the twist in the short-term yields of the term structure on 08/17/2007 than the DL model. Also, observe that the fitting errors under the Svensson model are lower than the ones under the DL model. The average RMSE across observed yields in these four days are 5.46, 5.96, 3.70 and 7.36 bps for the DL model, and 3.87, 3.01, 2.63 and 1.94 bps for the Svensson model. Figure 5 presents the time series of the fitting RMSE over the full sample. Two observations can be draw from the analysis of this figure. First the Svensson model fits better the in-sample data. Second, the errors in the first half are higher than the errors in the second half. The worse fitting performance on the first part of the sample is probably due to the low liquidity of ID-Future contracts in the biennium 2003–2004.

Figure 6 illustrates the time series of the term structure movements extracted adopting respectively the DL and the Svensson models. Under both models, the

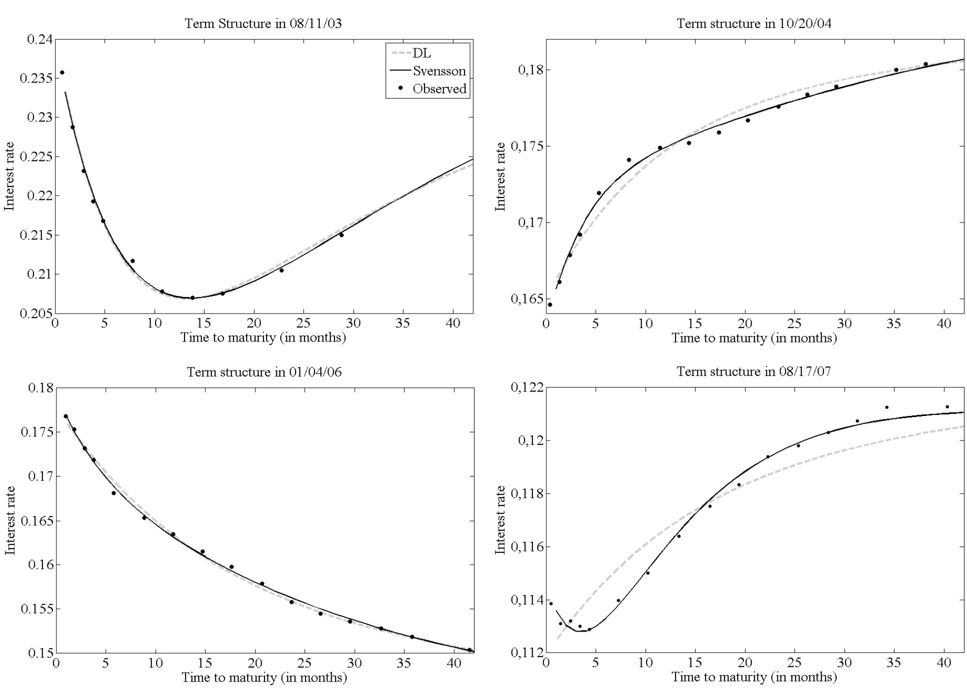

Fig. 4. Pictures of term structure cross-sections.

This picture presents observed, DL and dynamic Svensson models implied term structures for four different moments in time.

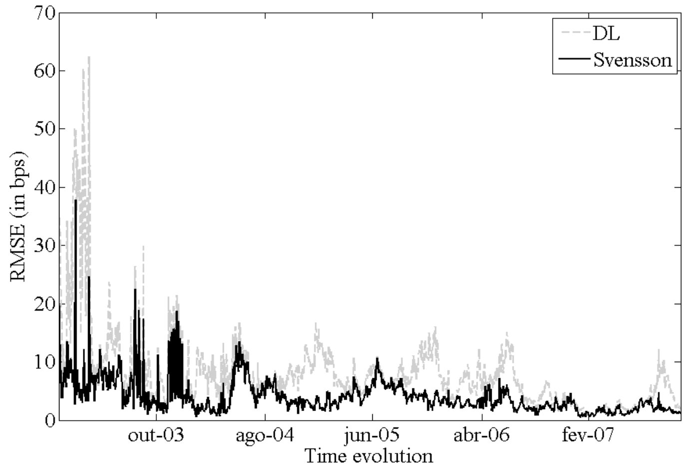

Fig. 5. RMSE for in-sample fit.

This picture presents the average RMSE in bps across nine different maturities (1, 3, 6, 12, 18, 24, 30, 36, and 42 months) for in-sample fit under the DL and Svensson models from January 3, 2003 to October 18, 2007.

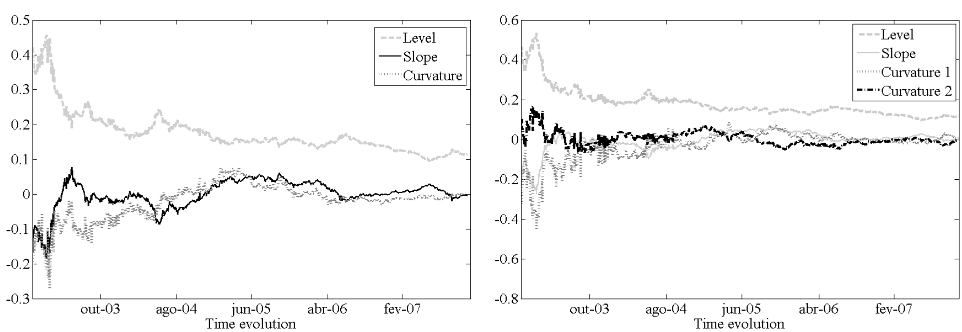

Fig. 6. Term structure movements.

This picture presents the term structure movements under the DL and Svensson models. The panel on the left-hand side shows time series of the level, slope and curvature captured by DL model with the decay parameter fixed at $\lambda = 0.123$. The panel on the right-hand side shows the time series of the level, slope, first and second curvatures captured by the Svensson model with the decay parameters fixed at $\lambda = 0.0976$ and $\tilde{\lambda} = 0.2105$.

level is the most stable movement oscillating around a long-term mean of 0.20 with a peak in the beginning of the sample, while slope and curvature switch signs along time. For the Svensson model, the second curvature appears to be a mirror of the first curvature and indeed Table 2 confirms this fact. This table shows the correlation coefficients between the time-series of any two movements extracted under the Svensson model, and indicates a negative correlation of -0.69 between the two curvature factors. In addition, note that all the movements are highly

Table 1. Correlation matrix of movements — DL model.

Movement	Level	Slope	Curvature
Level	1.00	−0.75	−0.76
Slope	−0.75	1.00	0.67
Curvature	−0.76	0.67	1.00

This table presents the correlation coefficients between any two term structure movements extracted using the DL three factor model. The decay parameter is fixed at $\lambda = 0.123$. Movements come from daily Brazilian IDs term structure data ranging from January 3, 2003 to October 18, 2007.

Table 2. Correlation matrix of movements — Svensson model.

Movement	Level	Slope	Curvature 1	Curvature 2
Level	1.00	−0.84	−0.87	0.59
Slope	−0.84	1.00	0.79	0.58
Curvature 1	−0.87	0.79	1.00	−0.69
Curvature 2	0.59	−0.58	−0.69	1.00

This table presents the correlation coefficients between any two term structure movements extracted using the Svensson four factor model. The decay parameters are fixed at $\lambda = 0.0976$ and $\tilde{\lambda} = 0.2105$. Movements come from daily Brazilian IDs term structure data ranging from January 3, 2003 to October 18, 2007.

correlated indicating that a VAR structure is more suited to capture the time series behavior of the four movements. Similarly Table 1 indicates that level, slope and curvature are strongly correlated under the DL model, also suggesting that a VAR would be a process more adequate than univariate autoregressions to fit to the time series of those three movements together.

Figure 7 presents for the level, slope and first curvature, the distance between the time series obtained under the two models. It shows that once a second curvature is included in the model it changes the behavior of the previously extracted level, slope, and curvature movements. This effect is stronger for the curvature factor, but it also appears significant on the slope factor. When the second curvature factor is included it produces potentially two effects on the dynamics of the term structure: the first, an inclusion of a new movement, and the second, a change on all existing movements. Note that this effect is more complex than simply including a fourth principal component to capture a second type of curvature because by the orthogonality of the principal components there wouldn't be any change to the time series of the previous three principal components already used. In fact, the introduction of a second curvature factor has a disciplinary effect on the previously extracted term structure movements, providing higher ability to capture bond risk premia and consequently interest rate conditional means. Almeida and Vicente [2] find that imposing no-arbitrage restrictions to a polynomial term structure model, induces

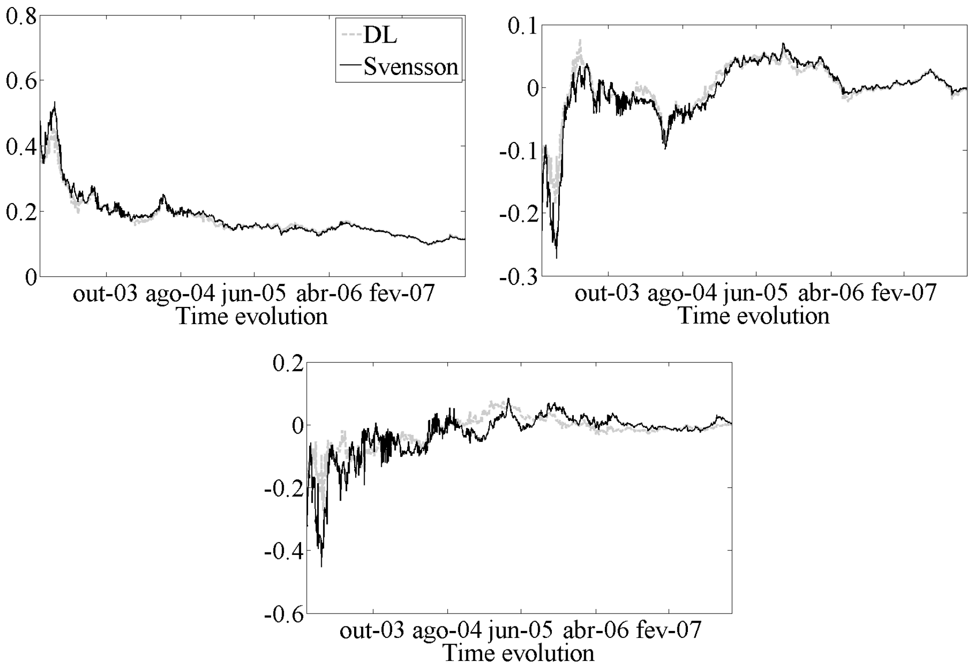

Fig. 7. Distance between movements across models.

This picture presents the time series of the three main term structure movements extracted under the DL and Svensson models.

the existence of conditionally deterministic movements for the term structure that increase model forecasting ability, when compared to a corresponding version that allows for arbitrages. They associate this improvement in model forecasting to a better ability in capturing bond risk premia. We will observe with the proposed forecasting exercise that extending the DL model to include a second curvature term has a similar effect here.

5.2. *Forecasting exercise*

In this section, we formulate a forecasting exercise using the in-sample and the out-of-sample periods defined in Sec. 4. Forecasts are performed for three different horizons: one day, one month (21 business-days), and three months (63 business-days). For each horizon and each out-of-sample observation, the four models were re-estimated keeping fixed the λ values obtained in the in-sample period. The re-estimation consists in running the AR e VAR process defined in Sec. 3 using a moving window procedure whenever a new observation is available. The length of the estimation window is fixed at 596 observations. Table 3 presents the λ values estimated by the criteria defined in Sec. 2.

The results of the comparison between the DL and the dynamic Svensson models tell us what is the importance of adding a second curvature factor for forecasting

Table 3. Decay parameters for the DL and Svensson models.

Sample	λ_{DL}	λ_{SV}^1	λ_{SV}^2
Full sample	0.1233/14.5	0.0975/18.4	0.2108/8.5
Subsample 1	0.1067/16.8	0.0900/19.9	0.2558/7.0
Subsample 2	0.2017/8.9	0.0992/18.1	0.1633/11.0

This table presents the decay parameter (λ's) and the corresponding maturity where the curvature achieves its maximum, for the full sample and for the two analyzed sub-samples. The first element in each cell is the decay parameter and the second is the maturity (in months) where the curvature achieves its maximum value. The estimation period ranges from January 3, 2003 to May 3, 2005 for the full sample, from January 3, 2003 to February 20, 2004 for the sub-sample 1, and from May 3, 2005 to July 10, 2006, for the sub-sample 2. The λ's are estimated by minimization of the in-sample RMSE.

purposes. We would also like to find out whether the suggested model is a good forecasting technique. To this end, we compare its predictions with the Random Walk and an autoregressive process of lag one on yields. The Random Walk prediction at time t for a forecasting horizon h and a yield with maturity τ, consists of simply repeating this yield ($R^{RW}_{t+h/t}(\tau) = R_t(\tau)_{\text{observed}}$). The AR(1) process uses the same moving window procedure specified above.

In short, we have six different models: DL model with AR-factor (DL-AR1) and VAR-factor (DL-VAR1) dynamics, Svensson model with AR-factor (S-AR1) and VAR-factor (S-VAR1) dynamics, Random Walk (RW), and an autoregressive process with lag one on yield levels (AR1). In order to analyzed the quality of out-of-sample forecasts of all models, we compute the bias and Root Mean Squared Error (RMSE) for yields with time to maturity of 1, 3, 6, 12, 18, 24, 30, 36, and 42 months. Since we do not observe all these yields in a certain day, we use an interpolation scheme based in the spline method to evaluate the difference between the actual and the predicted value.

Tables 4–6 present the bias and RMSE for all models, for the nine different chosen maturities and the three forecasting horizons. Bold values indicate that the model outperforms the random walk and a box around an element indicates the best model for a given maturity. Three important conclusions can be drawn from the analysis of these tables. Firstly, in contrast with DL, we find that models with VAR-factor dynamics are more efficient than models with AR-factor dynamics. Particularly, for the 1-month and 3-month forecasting horizons the VAR models completely dominate AR models both on bias and RMSE. Moreover, for these two forecasting horizons, VAR models consistently outperform the RW and AR1 competitors. Secondly, the Random Walk outperforms all models on 1-day ahead forecasts. The absolute value of bias is smaller under the Random Walk for short-term yields and under the S-VAR1 for long-term yields. In terms of RMSE, the Random Walk is unbeatable for all maturities. However, for longer forecasting horizons (1- and 3-month) both VAR models outperform the RW. Furthermore, they outperform the

Table 4. Bias and RMSE for 1-day ahead out-of-sample forecasts — Full sample.

Maturity	RW	DL-AR1	DL-VAR1	S-AR1	S-VAR1	AR1
1-m	−1.45/$\boxed{2.79}$	$\boxed{\mathbf{-1.08}}$/7.92	2.43/8.10	−6.53/8.92	−2.30/5.74	−1.74/4.04
3-m	$\boxed{-1.46}$/$\boxed{3.57}$	−5.73/10.70	−2.21/8.20	−2.13/8.97	1.87/6.04	−1.80/5.11
6-m	$\boxed{-1.42}$/$\boxed{5.37}$	−12.66/18.72	−9.17/16.20	−5.41/11.65	−1.64/8.35	−1.93/7.36
12-m	$\boxed{-1.26}$/$\boxed{8.72}$	−10.63/16.28	−7.21/14.53	−7.51/14.28	−4.08/12.75	−2.08/12.04
18-m	−1.12/$\boxed{11.48}$	−2.87/15.99	$\boxed{\mathbf{0.47}}$/15.67	−4.13/16.56	**−1.05**/15.50	−2.33/15.70
24-m	−1.04/$\boxed{12.51}$	**0.22**/18.11	3.48/17.91	−2.55/18.94	$\boxed{\mathbf{0.17}}$/17.42	−2.52/17.38
30-m	−0.99/$\boxed{12.69}$	**−0.27**/18.74	2.92/18.53	−2.58/19.26	$\boxed{\mathbf{-0.20}}$/18.19	−2.88/18.07
36-m	−0.93/$\boxed{13.45}$	−1.67/19.98	1.47/19.67	−2.56/20.12	$\boxed{\mathbf{-0.50}}$/19.43	−4.74/19.80
42-m	−0.88/$\boxed{14.04}$	−3.29/21.08	$\boxed{\mathbf{-0.20}}$/20.65	−2.45/20.87	**−0.66**/20.42	−15.88/30.05

This table presents the bias and RMSE in bps for 1-day ahead out-of-sample forecasts of yields with time to maturity of 1, 3, 6, 9, 12, 18, 24, 30, 36, and 42 months. Both error metrics were computed for the random walk (RW), DL model with AR(1) dynamics (DL-AR1), DL model with VAR(1) dynamics (DL-VAR1), Svensson model with AR(1) dynamics (S-AR1), Svensson model with VAR(1) dynamics (S-VAR1), and AR(1) process on yields level (AR1). First element on each cell indicates bias; second element indicates RMSE. Bold values mean that the model outperform the random walk. A box around an element indicates the best competitor for a specific maturity. Out-of-sample data ranges from May 4, 2005 to October 18, 2007, with a total of 595 observations.

Table 5. Bias and RMSE for 1-month ahead out-of-sample forecasts — Full sample.

Maturity	RW	DL-AR1	DL-VAR1	S-AR1	S-VAR1	AR1
1-m	−31.74/36.05	−57.83/71.66	$\boxed{\mathbf{-5.03}}$/**18.46**	−61.15/78.97	**−1.99**/$\boxed{\mathbf{13.00}}$	**−8.81/25.98**
3-m	−31.90/36.86	−64.81/78.90	**−11.90/29.31**	−59.54/79.35	$\boxed{\mathbf{-1.04}}$/$\boxed{\mathbf{18.68}}$	**−9.81/28.37**
6-m	−30.72/38.59	−73.11/85.84	**−20.77**/39.77	−63.79/82.18	$\boxed{\mathbf{-6.69}}$/$\boxed{\mathbf{26.08}}$	**−12.56/35.15**
12-m	−26.58/44.04	−69.49/84.95	**−19.54**/44.69	−62.75/83.25	$\boxed{\mathbf{-9.03}}$/$\boxed{\mathbf{37.52}}$	**−18.12**/50.13
18-m	−24.00/52.37	−59.97/82.79	**−12.82/49.45**	−56.67/83.72	$\boxed{\mathbf{-6.31}}$/$\boxed{\mathbf{46.99}}$	−25.40/63.99
24-m	−22.67/58.06	−55.55/84.94	**−10.96/55.25**	−53.53/85.81	$\boxed{\mathbf{-6.16}}$/$\boxed{\mathbf{53.10}}$	−30.93/71.73
30-m	−21.68/60.72	−54.85/86.57	**−12.40/59.10**	−52.51/85.89	$\boxed{\mathbf{-7.68}}$/$\boxed{\mathbf{56.92}}$	−38.13/76.84
36-m	−20.54/63.39	−54.96/89.02	**−14.26/63.12**	−51.54/86.67	$\boxed{\mathbf{-8.83}}$/$\boxed{\mathbf{60.62}}$	−62.87/103.61
42-m	−19.80/64.31	−55.58/90.68	**−16.28**/65.53	−50.81/87.02	$\boxed{\mathbf{-9.84}}$/$\boxed{\mathbf{62.66}}$	−135.20/179.89

This table presents the bias and RMSE in bps for 1-month ahead out-of-sample forecasts of yields with time to maturity of 1, 3, 6, 9, 12, 18, 24, 30, 36, and 42 months. Both error metrics were computed for the random walk (RW), DL model with AR(1) dynamics (DL-AR1), DL model with VAR(1) dynamics (DL-VAR1), Svensson model with AR(1) dynamics (S-AR1), Svensson model with VAR(1) dynamics (S-VAR1), and AR(1) process on yields level (AR1). First element on each cell indicates bias; second element indicates RMSE. Bold values mean that the model outperform the random walk. A box around an element indicates the best competitor for a specific maturity. Out-of-sample data ranges from May 30, 2005 to October 18, 2007, with a total of 575 observations.

Table 6. Bias and RMSE for 3-month ahead out-of-sample forecasts — Full sample.

Maturity	RW	DL-AR1	DL-VAR1	S-AR1	S-VAR1	AR1
1-m	−100.35/107.92	−175.54/233.60	**−19.24/57.98**	−181.31/233.06	**−1.21**/**29.76**	**−31.62/71.42**
3-m	−100.14/107.94	−188.61/235.73	**−33.35/70.24**	−185.77/232.32	**−3.11**/**35.78**	**−33.39/76.66**
6-m	−94.86/105.63	−199.96/238.23	**−47.73/85.11**	−190.88/233.11	**−9.45**/**49.99**	**−39.70/92.16**
12-m	−83.08/106.98	−195.66/233.62	**−51.75/96.34**	−183.45/228.34	**−11.50**/**75.25**	**−55.83**/123.79
18-m	−75.30/115.89	−182.91/226.68	**−47.52/104.15**	−170.08/221.61	**−8.82**/**93.63**	−73.88/149.19
24-m	−70.29/123.59	−175.07/223.18	**−47.11/111.92**	−161.21/216.40	**−9.12**/**103.22**	−86.04/162.76
30-m	−65.69/125.83	−169.99/220.30	**−48.12/116.89**	−154.59/211.30	**−9.89**/**110.23**	−100.95/172.32
36-m	−61.87/128.72	−166.38/218.99	**−49.39/121.60**	−149.23/207.59	**−10.42**/**114.87**	−138.22/205.87
42-m	−59.51/129.76	−164.38/217.90	**−51.28/124.90**	−145.50/204.352	**−11.41**/**117.35**	−208.53/269.86

This table presents the bias and RMSE in bps for 1-day ahead out-of-sample forecasts of yields with time to maturity of 1, 3, 6, 9, 12, 18, 24, 30, 36, and 42 months. Both error metrics were computed for the random walk (RW), DL model with AR(1) dynamics (DL-AR1), DL model with VAR(1) dynamics (DL-VAR1), Svensson model with AR(1) dynamics (S-AR1), Svensson model with VAR(1) dynamics (S-VAR1), and AR(1) process on yields level (AR1). First element on each cell indicates bias; second element indicates RMSE. Bold values mean that the model outperform the random walk. A box around an element indicates the best competitor for a specific maturity. Out-of-sample data ranges from June 27, 2005 to October 18, 2007, with a total of 533 observations.

Table 7. Relative RMSE of the S-VAR1 model with respect to each candidate model.

Horizon	RW	DL-AR1	DL-VAR1	S-AR1	AR1
1 day	46%	−16%	−11%	−11%	−4%
1 month	−17%	−50%	−12%	−50%	−41%
3 months	−31%	−64%	−18%	−63%	−45%

This table presents the average RMSE across maturities of the S-VAR1 model compared to the average RMSE across maturities of each candidate model ($\frac{RMSE_{\text{SVAR-1}}}{RMSE_{\text{Candidate Model}}} - 1$) where the candidate model assumes RW, DL-AR1, DL-VAR1, S-AR1, or AR1. Therefore negative values indicate a reduction of the RMSE by the S-VAR1 when compared with the other competitors. The forecasting period is from May 4, 2005 to October 18, 2007 for 1-day horizon, from May 30, 2005 to October 18, 2007 for 1-month horizon, and from June 27, 2005 to October 18, 2007, for 3-months horizon.

other competitors with the S-VAR1 clearly being the best of all models. Table 7 presents the average RMSE (across maturities) for all competitors relatively to the S-VAR1 ($\frac{RMSE(\text{S-VAR1})-RMSE(\text{competitor})}{RMSE(\text{competitor})}$). Observe that the suggested model reduces RMSE forecast errors for longer horizons, and, apart from the RW, it also improves forecasts for the 1-day horizon. Finally, note that the bias of all models is almost always negative, indicating that they tend to overestimate future yields. Throughout the full sample, yields are most of the time declining. The negative bias points out that the models are not being able to capture this persistent decline on yields.

An issue of major interest is the effect on forecasts coming from the introduction of the new curvature factor. By the results described above, it seems that the more elaborate model improves the predictive ability. In order to statistically confirm the superior forecasting ability of the dynamic Svensson model, we present on Table 8 the Diebold and Mariano [12] S1 and S2 (size corrected) statistics using a quadratic loss function. Since the models with VAR dynamics have superior predictive ability when compared to models with AR dynamics, we perform the statistical analysis only considering the VAR-type models. Positive values are in favor of the S-VAR1 model, with values higher than 1.64 indicating significance at a 90% confidence level, and values higher than 1.96 indicating significance at a 95% confidence level. Note that 11 out of 27 values of the S1 statistics are higher than 1.64. Similarly, 14 out of 27 values are significant under the S2 statistics, in favor of the S-VAR1 model. This shows that about half of the RMSE are statistically smaller under the Svensson model at a 90% confidence interval. On the other hand, for long maturities under the 3-month forecasting horizon, although the S-VAR1 produces smaller RMSE than the DL-VAR1 model, the S2 statistic indicates that the DL-VAR1 model forecasts better in a larger number of out-of-sample points.

The three mentioned tables clearly evidence the superior performance of the dynamic Svensson model, consistent with the fact that higher order principal components of the yield curve might have an important role on capturing bond risk premium. Indeed, in ideas related to ours other studies indicate the importance of

Table 8. Out-of-sample statistical significance of difference in forecasting ability.

Maturity	1-m	3-m	6-m	12-m	18-m	24-m	30-m	36-m	42-m
Horizon					*1-day ahead forecast*				
S1	7.08	7.35	14.13	5.43	3.22	2.15	0.52	0.17	1.05
S2	6.53	8.34	15.56	3.65	0.45	0.94	0.21	0.86	2.01
Horizon					*1-month ahead forecast*				
S1	0.92	1.64	1.93	1.17	0.49	0.67	0.85	1.04	1.20
S2	2.97	7.23	4.80	2.13	1.71	1.88	1.55	1.04	0.21
Horizon					*3-month ahead forecast*				
S1	4.62	2.59	1.83	0.99	0.47	0.47	0.41	0.50	0.63
S2	11.01	9.28	4.94	0.52	−2.60	−2.51	−2.77	−1.99	−1.21

This table presents the Diebold and Mariano [12] S1 and S2 size statistics for 1-day, 1-month and 3-month ahead out-of-sample forecasts considering the DL-VAR1 and S-AR1 models. Comparisons are done as functions of Mean Squared Errors (MSE). Out-of-sample data ranges from May 4, 2005 to October 18, 2007. Positive values are in favor of the S-VAR1 model. Large values for $S1$ and $S2$ indicate high probability of rejecting the null hypothesis that the difference in Mean Square Errors is negligible. Absolute values larger than 1.64 indicate significance at a 90% confidence level, and larger than 1.96, indicate significance at a 95% level.

residuals in term structure models in explaining interest rate portfolio returns (Bali *et al.* [4]), and even derivative prices (Heidari and Wu [21]).

A possible criticism to the results obtained above relates to the sample period considered. One might ask whether the superior forecast accuracy of the S-VAR1 model is stable when different periods of time are used. In order to address this question, we carried out a pair of sub-sample analyses. We consider the two sub-samples specified in Sec. 4. Sub-sample 1 covers the first half of the full sample while sub-sample 2 covers the second half. Tables 9–11 present the RMSE relative to the Random Walk for these two sub-samples under the same horizon forecasting and maturities employed in the full sample analysis. Note that for the sub-sample 2 the three main results reported previously are qualitatively confirmed. For the sub-sample 1 some of our findings are not reproduced. Firstly, the S-VAR1 does not outperforms the RW not even for longer forecasting horizon. Moreover, all models are outperformed by Random Walk. Secondly, for medium-term and long-term yields the VAR models do not provide superior predictive ability when compared to the AR models. We believe that all these differences in behavior are due to at least two reasons: Liquidity, and variability of yields. In what regards liquidity, Figure 2 shows that the liquidity of ID Futures for maturities greater than 18 months is very small in the first sub-sample, what might be distorting prices and consequently influencing the results. This is in accordancy to the results by Bali *et al.* [4] who suggest that liquidity factors are a source of increase in interest rate predictability. The second reason for a worse performance of the dynamic Svensson model in sub-sample 1 is that the yields in the out-of-sample period of sub-sample 1 vary much less than in the corresponding in-sample period without a specific trend, obviously favoring the Random Walk process (see Fig. 3).

Table 9. RMSE relative to RW for 1-day ahead forecasts — Sub-samples.

Maturity	DL-AR1	DL-VAR1	S-AR1	S-VAR1	AR1
1-m	174.1%/170.6%	122.9%/136.5%	166.7%/120.5%	48.5%/100.9%	$\boxed{47.1\%}$/$\boxed{30.4\%}$
3-m	55.0%/113.8%	44.3%/75.4%	82.3%/70.7%	$\boxed{25.7\%}$/43.9%	27.3%/$\boxed{34.9\%}$
6-m	86.1%/89.3%	92.6%/63.2%	93.5%/45.6%	48.0%/33.3%	$\boxed{31.8\%}$/$\boxed{33.0\%}$
12-m	57.2%/48.5%	60.5%/35.8%	74.5%/45.1%	64.9%/37.6%	$\boxed{35.2\%}$/$\boxed{33.3\%}$
18-m	39.2%/41.8%	40.8%/37.6%	43.3%/37.7%	45.0%/32.3%	$\boxed{31.3\%}$/$\boxed{31.9\%}$
24-m	38.1%/46.5%	39.7%/43.4%	39.3%/40.8%	43.7%/34.1%	$\boxed{36.8\%}$/$\boxed{33.8\%}$
30-m	$\boxed{13.7\%}$/48.4%	14.4%/45.2%	18.4%/45.9%	18.5%/$\boxed{38.3\%}$	15.7%/38.5%
36-m	$\boxed{-41.1\%}$/44.6%	−41.4%/42.6%	−38.0%/43.9%	−39.2%/$\boxed{37.4\%}$	−25.5%/37.8%
42-m	−72.2%/39.6%	$\boxed{-72.5\%}$/37.8%	−71.2%/41.0%	−72.0%/$\boxed{35.8\%}$	−40.9%/36.1%

This table presents the RMSE relative to random walk for 1-day ahead out-of-sample forecasts of yields with time to maturity of 1, 3, 6, 9, 12, 18, 24, 30, 36, and 42 months for two different sub-samples ($\frac{RMSE(\text{Model})-RMSE(\text{RW})}{RMSE(\text{RW})}$). First element on each cell refers to the sub-sample 1; second element refers to the sub-sample 2. A box around an element indicates the best competitor, apart the random walk, for a specific maturity. Sub-sample 1 ranges from January 3, 2003 to May 2, 2005, while sub-sample 2 extends from May 3, 2005 until October 18, 2007.

Table 10. RMSE relative to RW for 1-month ahead forecasts — Sub-samples.

Maturity	DL-AR1	DL-VAR1	S-AR1	S-VAR1	AR1
1-m	52.3%/112.5%	−10.9%/$\boxed{-59.1\%}$	77.1%/66.5%	$\boxed{-41.1\%}$/−53.7%	51.9%/−58.8%
3-m	17.2%/116.2%	1.1%/−49.7%	42.7%/59.0%	$\boxed{-12.7\%}$/−40.7%	48.2%/$\boxed{-57.9\%}$
6-m	$\boxed{-19.7\%}$/102.0%	31.8%/$\boxed{-38.2\%}$	−3.1%/48.0%	19.7%/−28.9%	35.2%/−36.3%
12-m	−7.0%/76.4%	37.1%/−19.2%	$\boxed{-14.7\%}$/47.3%	40.5%/$\boxed{-20.6\%}$	19.1%/−10.8%
18-m	−1.2%/58.6%	33.6%/−9.0%	$\boxed{-15.0\%}$/45.4%	42.4%/$\boxed{-17.3\%}$	9.7%/2.0%
24-m	2.8%/54.2%	34.5%/−3.5%	$\boxed{-10.8\%}$/48.5%	43.3%/$\boxed{-15.3\%}$	−1.3%/8.2%
30-m	3.2%/50.4%	35.7%/−2.2%	$\boxed{-7.0\%}$/48.3%	42.4%/$\boxed{-16.2\%}$	−3.2%/12.0%
36-m	1.8%/47.4%	34.0%/−1.0%	$\boxed{-5.2\%}$/47.4%	39.0%/$\boxed{-16.1\%}$	10.7%/15.8%
42-m	−11.5%/46.2%	16.8%/−0.8%	$\boxed{-15.5\%}$/47.1%	20.2%/$\boxed{-16.6\%}$	4.7%/19.5%

This table presents the RMSE relative to random walk for 1-month ahead out-of-sample forecasts of yields with time to maturity of 1, 3, 6, 9, 12, 18, 24, 30, 36, and 42 months for two different sub-samples ($\frac{RMSE(\text{Model})-RMSE(\text{RW})}{RMSE(\text{RW})}$). First element on each cell refers to the sub-sample 1; second element refers to the sub-sample 2. A box around an element indicates the best competitor, apart the random walk, for a specific maturity. Sub-sample 1 ranges from January 3, 2003 to May 2, 2005, while sub-sample 2 extends from May 3, 2005 until October 18, 2007.

Table 11. RMSE relative to RW for 3-month ahead forecasts — Sub-samples.

Maturity	DL-AR1	DL-VAR1	S-AR1	S-VAR1	AR1
1-m	95.6%/185.5%	23.7%/$\boxed{-68.2\%}$	98.9%/130.4%	$\boxed{4.4\%}$/−58.3%	68.4%/−56.5%
3-m	65.4%/193.3%	28.9%/$\boxed{-70.9\%}$	103.2%/134.6%	$\boxed{18.8\%}$/−61.3%	63.0%/−64.1%
6-m	$\boxed{23.4\%}$/199.0%	58.9%/$\boxed{-68.7\%}$	106.1%/140.2%	47.0%/−65.4%	64.6%/−48.5%
12-m	$\boxed{-31.6\%}$/183.2%	160.5%/−46.7%	64.7%/134.5%	146.3%/$\boxed{-53.7\%}$	61.3%/−11.3%
18-m	$\boxed{7.1\%}$/161.9%	233.5%/−32.6%	8.7%/123.9%	224.4%/$\boxed{-40.3\%}$	32.7%/10.8%
24-m	48.5%/150.1%	274.2%/−26.6%	$\boxed{-16.4\%}$/118.2%	275.1%/$\boxed{-32.1\%}$	−14.3%/22.9%
30-m	60.1%/142.1%	265.9%/−23.1%	−14.5%/114.0%	275.5%/$\boxed{-27.4\%}$	$\boxed{-52.3\%}$/30.8%
36-m	56.6%/136.5%	235.2%/−20.8%	−7.5%/110.9%	250.4%/$\boxed{-23.6\%}$	$\boxed{-32.0\%}$/36.4%
42-m	28.3%/133.4%	163.7%/−19.3%	−16.7%/109.1%	179.6%/$\boxed{-21.2\%}$	$\boxed{-25.8\%}$/40.8%

This table presents the RMSE relative to random walk for 3-months ahead out-of-sample forecasts of yields with time to maturity of 1, 3, 6, 9, 12, 18, 24, 30, 36, and 42 months for two different sub-samples ($\frac{RMSE(\mathrm{Model})-RMSE(\mathrm{RW})}{RMSE(\mathrm{RW})}$). First element on each cell refers to the sub-sample 1; second element refers to the sub-sample 2. A box around an element indicates the best competitor, apart the Random Walk, for a specific maturity. Sub-sample 1 ranges from January 3, 2003 to May 2, 2005, while sub-sample 2 extends from May 3, 2005 until October 18, 2007.

6. Conclusion

In a recent paper, Cochrane and Piazzesi [7] show the importance of the fourth principal component of zero-coupon yields to predict bond excess returns. They find a tent-shaped return forecasting factor common to all zero-coupon bonds and show that the fourth principal component that explains only 0.02% of the variability of yields, explains more than 20% of bond risk premia captured by this return forecasting factor.

Motiveted by these results we extended the parametric three-factor exponential model of Diebold and Li [11] to contain a fourth factor, related to a second type of curvature. We show that the introduction of a second curvature significantly improves model forecasting ability by lowering both bias and RMSE on out-of-sample forecasts. An empirical exercise adopting high frequency (daily) fixed income data from the most liquid Brazilian fixed income market documents the superior performance of the suggested model. It outperforms the Diebold and Li [11] model for most maturities under three different forecasting horizons. We also found that the dynamic Svensson model has superior predictive ability when compared to some traditional benchmarks.

The results presented in this paper confirm in a dynamic econometric setting the findings of Cochrane and Piazzesi [7], and suggest that this extended model should be considered on forecasting exercises, specially on markets with more volatile yield curves, like emerging markets, corporate bond markets, and credit derivative markets.

Some interesting topics for future research include testing other econometric estimation techniques, like substituting the two-step estimation procedure by a one-step estimation adopting a Kalman filter like in Diebold *et al.* [13], or considering a more detailed analysis of the bond risk premia structure implied across compared models, similarly to Almeida and Vicente [2].

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