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1.1

any two vectors $U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$ that conforms to the properties of vector space and scalar multiplcation with one another.

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ These two vectors are linearly dependant because they live in the same vector space and conform to the scalar multiplication principle, e.g. you can get to one from the other.

$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$ The two vectors are linearly independant because it is impossible to multiply to one of the vectors to get the other.

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ These two vectors form a basis because they can be used to describe any other vector of the same dimension

1.2

First one forms a basis
Second one is not linearly independant and does not form a basis.
Third one has 3 vectors and cannot form a basis
Fourth one forms a basis because it can only be solved when c_1 & c_2 are 0.

1.3

$e_1 = a_1\vec{u} + a_2 v$
$e_2 = b_1\vec{u} + b_2 v$

$a_1 + a_2 = 1$
$a_1 - a_2 = 0$

$(a_1 + a_2) + (a_1 - a_2) = 1 + 0$
$2a_1 = 1$
$a_1 = \frac{1}{2}$

$$\frac{1}{2} + a_2 = 1$$
$$a_2 = \frac{1}{2}$$

$$e_1 = \frac{1}{2}\vec{u} + \frac{1}{2}v$$

same math can be applied to b_1 *and* b_2

$$e_2 = \frac{1}{2}\vec{u} + \frac{1}{2}v$$

$e_1 \times u = 1$ *simplified matrix multipcation, easier to do in head than write it all out :)*
$$u \times u = 2$$
$$Proj_u e_1 = \frac{1}{2}u = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$e_1 \times v = 1$$
$$v \times v = 2$$
$$Proj_v e_1 = \frac{1}{2}v = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$e_2 \times u = 1$$
$$u \times u = 2$$
$$Proj_u e_2 = \frac{1}{2}u = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$e_2 \times v = 1$$
$$v \times v = 2$$

$$Proj_v e_2 = \frac{1}{2}v = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

Nikka Note, that took too long, going to go shower

1.4

when $x = i$, the vector does not conform to $\Re^3$

1.6

a union of linear subspaces is not neccarcily a linear subspace itself because for example, if we take the sum of two example linear subspaces, e.g. $\begin{bmatrix} 1 & 1 \end{bmatrix}$ does not fit in either linear subspace therefore, they are not a linear subspace itself

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$$Ha = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1(1) + 1(0) \\ 1(1) + \ -1(0) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$Yb = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 0(1) + i(i) \\ -i(1) + 0(i) \end{pmatrix} = \begin{pmatrix} -1 \\ -i \end{pmatrix}$$

$$HY = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1(0) + 1(-i) & 1(i) + 1(0) \\ 1(0) + \ -1(-i) & 1(i) + \ -1(0) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ i & i \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} -\frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \end{pmatrix}$$

$$H^T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Y^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$H^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Y^{\dagger} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

$$Hermitian\ H^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$Hermitian\ Y^{\dagger} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$HH^{\dagger} = H^{\dagger}H = I$$

$$HH^{\dagger} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1(1)+1(1) & 1(1)+1(-1) \\ 1(1)-1(1) & 1(1)-1(-1) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H^{\dagger}H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$YY^{\dagger} = Y^{\dagger}Y = I$$

$$YY^{\dagger} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0(0)+i(-i) & 0(i)+i(0) \\ -i(0)+0(-i) & -i(i)+0(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Y^{\dagger}Y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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$$a_1 \times b_1 + a_2 \times b_2 = 1(1) + 0(i) = 1$$

Extra credit:

$$\vec{v} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix},\ y = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$\{1\ \ 2\}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} 1(0)+2(-i) & 1(i)+2(0) \end{bmatrix} = \begin{bmatrix} -1 & i \end{bmatrix}$$

Now, when we apply the same unitary gate to the resulting matrix

$$\begin{bmatrix} -1 & i \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = \begin{bmatrix} -1(0)+i(-i) & -1(i)+i(0) \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Boom!