Applications of semidefinite programming to combinatorial problems

Norihide Tokushige (University of the Ryukyus)

Spectral graph theory and related topics

January 26–28th, 2025 @Waseda University

Outline

- References and Software for SDP
- Applications of SDP for intersecting families
  - A toy problem (independence number of the Petersen graph)
  - Wilson's proof of Erdős–Ko–Rado Theorem
  - Recent results on cross intersecting families

References

Applications of SDP/LP for some concrete problems

- Wagner, Adam Zsolt. Refuting conjectures in extremal combinatorics via linear programming. JCTA (2020)
- Schrijver, Alexander. New code upper bounds from the Terwilliger algebra and semidefinite programming. IEEE (2005)
- 田中太初. Terwilliger 代数に基づく符号の半正定値計画限界. 代数学シンポ (2006)
- Bansal, Nikhil. Constructive algorithms for discrepancy minimization. FOCS (2010)

Semidefinite programming

- M. J. Todd.
  Semidefinite optimization.
  Acta Numer. 10 (2001) 515–560.
- B. Gärtner, J. Matoušek.
  Approximation algorithms and semidefinite programming.
  Springer, 2012. xii+251 pp.

Intersecting families / Association Schemes

- Godsil–Meagher. Erdős–Ko–Rado theorems: algebraic approaches. Cambridge Stud. Adv. Math., 2016.

Software

SDP solver

- SDPA on NEOS server

- SDPA の使い方 (How to start using SDPA)

An example

- Consider the following primary program:

$$
\begin{aligned}
\text{minimize} \quad & \alpha \\
\text{subject to} \quad & S := -\begin{bmatrix} \frac{9}{16} & \frac{3}{16} \\ \frac{3}{16} & \frac{1}{16} \end{bmatrix} + \alpha \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \\ x_2 & 0 \end{bmatrix} - \begin{bmatrix} z_1 & z_2 \\ z_2 & z_3 \end{bmatrix}, \\
& S \succeq 0, \quad z_1, z_2, z_3 \geq 0.
\end{aligned}
$$

- true value for $\min \alpha$ is $\frac{1}{4}$.

```
6
2
(2,-3)
{1,0,0,0,0,0}
0 1 1 1 0.5625
0 1 1 2 0.1875
0 1 2 2 0.0625
1 1 1 1 0.75
1 1 2 2 0.25
2 1 1 1 1
3 1 1 2 1
4 1 1 1 -1
4 2 1 1 1
5 1 1 2 -1
5 2 2 2 1
6 1 2 2 -1
6 2 3 3 1
```

```
phase.value = dFEAS
   Iteration = 35
          mu = 6.4453585130804993e-18
relative gap = 1.3840612359273038e-17
         gap = 3.2226792565402498e-17
      digits = 1.6256784703351158e+01
objValPrimal = 2.5000000000000000e-01
objValDual   = 2.5000000000000000e-01
p.feas.error = 4.3368086899420177e-19
d.feas.error = 3.3787568866817071e-31
relative eps = 4.9303806576313200e-32
total time   = 0.000
```

Applications of SDP to intersecting families

- $[n] = \{1, 2, \ldots, n\}$.
- $\binom{[n]}{k} = \{F \subset [n] : |F| = k\}$. (the set of k-element subsets)
- $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$.
- $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$.

What is the maximum size of a t-intersecting family $\mathcal{F} \subset \binom{[n]}{k}$?

- The answer is known. The complete intersection theorem by Ahlswede and Khachatrian (1995).
- Wilson (1983) proved one of the main cases by solving an SDP.

Cross t-intersecting families

- Let $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$.
- $\mathcal{F}$ and $\mathcal{G}$ are cross t-intersecting if $|F \cap G| \geq t$ for $\forall F \in \mathcal{F}, G \in \mathcal{G}$.
- What is $\max |\mathcal{F}||\mathcal{G}|$?

Theorem (Zhang–Wu arXiv:2410.22792)
Let $n \geq (t+1)(k-t+1)$, $t \geq 3$.
Under above conditions we have $|\mathcal{F}||\mathcal{G}| \leq \binom{n-t}{k-t}^2$.

We settled the case $t = 2$ by solving an SDP problem.

Theorem. The above result is also true for $t = 2$.

The case $t = 1$ was settled by Pyber (1986), T (2013).

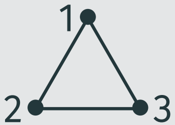
A toy problem

A family $\mathcal{F}$ is intersecting if $F \cap F' \neq \emptyset$ for all $F, F' \in \mathcal{F}$.

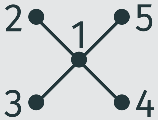
What is the maximum size of an intersecting family $\mathcal{F} \subset \binom{[5]}{2}$?

- $\binom{[5]}{2} = \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\}$. (I write 12 for $\{1, 2\}$.)
- examples of intersecting families in $\binom{[5]}{2}$:

$\{12, 13, 23\}$

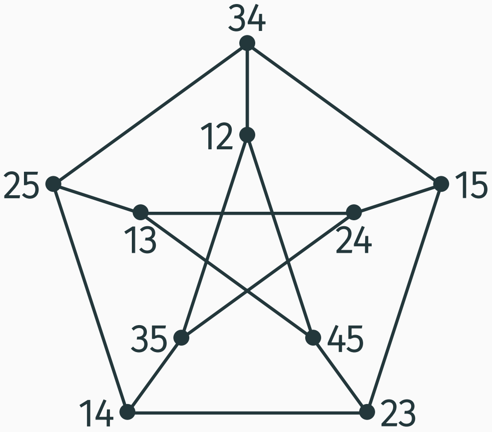


$\{12, 13, 14, 15\}$



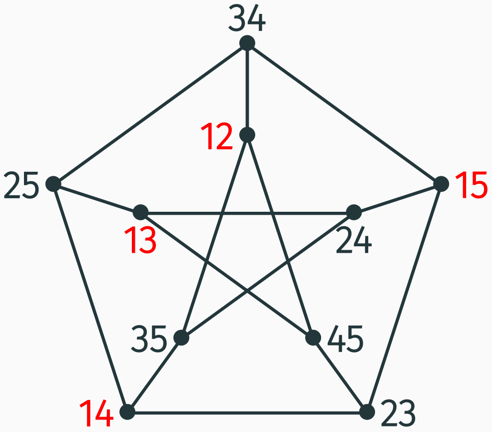
What is the maximum size of an intersecting family $\mathcal{F} \subset \binom{[5]}{2}$?

- Kneser graph $G = G(5,2)$
- $V(G) = \binom{[5]}{2}$
- $x \sim y \iff x \cap y = \emptyset$.



What is the maximum size of an intersecting family $\mathcal{F} \subset \binom{[5]}{2}$?

- $U \subset V(G)$ is independent if no edges inside U.
- $\text{indep}(G) :=$ $\max\{|U| : U \text{ is independent}\}$.
- $\max |\mathcal{F}| = \text{indep}(G) \geq 4$.



What is indep(G) ?

- adjacency matrix A of G: $(A)_{x,y} = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{if } x \not\sim y. \end{cases}$
- For $G = G(5,2)$,

$$A = \begin{bmatrix} 0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&1&1&0&0&1 \\ 0&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&1&1&0&1&0&0 \\ 0&0&1&1&0&0&0&0&0&1 \\ 0&1&0&1&0&0&0&0&1&0 \\ 0&1&1&0&0&0&1&0&0&0 \\ 1&0&0&1&0&0&1&0&0&0 \\ 1&0&1&0&1&0&0&0&0&0 \\ 1&1&0&0&1&0&0&0&0&0 \end{bmatrix}$$

- eigenvalues are $3, 1, -2$.

positive semidefinite matrix

- Let M be an $n \times n$ real symmetric matrix.
- M is positive semidefinite if $x^{\mathsf{T}} M x \geq 0$ for all $x \in \mathbb{R}^n$.
- We write $M \succeq 0$ if M is positive semidefinite.
- $M \succeq 0$ iff all eigenvalues are non-negative.
- For two matrices A, B, let $A \bullet B = \sum_{x,y} (A)_{x,y} (B)_{x,y}$.
  ex. $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \bullet \begin{bmatrix} x & y \\ y & z \end{bmatrix} = ax + 2by + cz$.
- $x^{\mathsf{T}} M x = M \bullet (x x^{\mathsf{T}})$.
- If $M \succeq 0$ then $M \bullet (x x^{\mathsf{T}}) \geq 0$.

Bounding indep(G)

- Let $U \subset V(G)$ be an independent set in G.
- Let $\boldsymbol{u} \in \{0,1\}^{10}$ be the indicator (column) vector of U, that is,

$$(\boldsymbol{u})_x = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if not.} \end{cases}$$

- Let $X = \boldsymbol{u}\boldsymbol{u}^\mathsf{T}$. Then $(X)_{x,y} = 0$ if $x \sim y$.

Bounding indep(G)

- Let $U \subset V(G)$ be an independent set in G.
- Let $\boldsymbol{u}$ be the indicator of U, and let $X = \boldsymbol{u}\boldsymbol{u}^\mathsf{T}$.
- Let A be the adjacency matrix of the Kneser graph $G = G(5, 2)$.
- $I \bullet X = |U|$.
- $J \bullet X = |U|^2$.
- $A \bullet X = 0$, that is, $(A)_{x,y}\,(X)_{x,y} = 0$ for all x, y.
- Let $S := \alpha\, I - J + \beta\, A$. Then $S \bullet X = \alpha|U| - |U|^2$.
- If $S \succeq 0$ then $S \bullet X \geq 0$, so $|U| \leq \alpha$, i.e., $\mathrm{indep}(G) \leq \alpha$.

What is indep(G) ?

- Let $S := \alpha I - J + \beta A$.
- If $S \succeq 0$, then indep(G) $\leq \alpha$.
- For $G = G(5, 2)$, let $S = 4I - J + 2A$. Then $S \succeq 0$.
- That is, indep(G) ≤ 4. (So, indep(G) $= 4$)

An SDP problem for indep($G(5, 2)$)

minimize α

subject to $S := \alpha I - J + \beta A \succeq 0$. (variables are α, β.)

- A feasible solution α satisfies indep(G) $\leq \alpha$.

Extending adjacency matrix

- To bound $\mathrm{indep}(G)$ we used $A \bullet X = 0$.
- For this, we didn't use $(A)_{x,y} = 1$ if $x \sim y$.
- This means that if $x \sim y$, then $(A)_{x,y}$ is not necessarily 1.
- Redefine an "adjacency matrix" by

$$(A)_{x,y} = \begin{cases} 0 & \text{if } x \not\sim y, \\ * & \text{if not,} \end{cases}$$

where $*$ is any number (provided A is symmetric).

Bounding the size of t-intersecting families in $\binom{[n]}{k}$

- $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting if $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$.
- $\mathcal{F} = \{F \in \binom{[n]}{k} : [t] \subset F\}$ is t-intersecting, and $|\mathcal{F}| = \binom{n-t}{k-t}$.
- What is the maximum size of t-intersecting families $\mathcal{F} \subset \binom{[n]}{k}$?
- Kneser graph $G = G(n, k, t)$: $V(G) = \binom{[n]}{k}$, $x \sim y \iff |x \cap y| < t$.
- What is indep(G) ? By construction, indep$(G) \geq \binom{n-t}{k-t}$.
- (An SDP problem) minimize α
  subject to $S := \alpha I - J + A \succeq 0$, where $(A)_{x,y} = 0$ if $x \not\sim y$.
- A feasible solution α satisfies indep$(G) \leq \alpha$.
- Wilson found an $S \succeq 0$ with $\alpha = \binom{n-t}{k-t}$ if n is not too small.

- Let $n \geq (t+1)(k-t+1)$.
- Wilson found an A satisfying $(A)_{x,y} = 0$ for $|x \cap y| \geq t$ and

$$S = \binom{n-t}{k-t} I - J + A \succeq 0.$$

- ex. Wilson's matrix for $G(8,3,2)$: $(A)_{x,y} = \begin{cases} \frac{1}{2} & \text{if } |x \cap y| = 0, \\ \frac{3}{2} & \text{if } |x \cap y| = 1, \\ 0 & \text{if } |x \cap y| \geq 2. \end{cases}$
- This implies that $\mathrm{indep}(G) \leq \binom{n-t}{k-t}$, so $\mathrm{indep}(G) = \binom{n-t}{k-t}$.

Theorem. Let $n \geq (t+1)(k-t+1)$.
If $\mathcal{F} \subset \binom{[n]}{k}$ is t-intersecting, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$.

Cross 2-intersecting families

- $\mathcal{F}$ and $\mathcal{G}$ are cross 2-intersecting if $|F \cap G| \geq 2$ for $\forall F \in \mathcal{F}, G \in \mathcal{G}$.
- If $\mathcal{F} = \mathcal{G} = \{F \in \binom{[n]}{k} : \{1, 2\} \subset F\}$, then they are cross 2-intersecting, and $|\mathcal{F}| = |\mathcal{G}| = \binom{n-2}{k-2}$.

We got the following result by solving an SDP problem.

Theorem. Let $\mathcal{F}, \mathcal{G} \subset \binom{[n]}{k}$ and $n \geq 3(k-1)$.
If $\mathcal{F}$ and $\mathcal{G}$ are cross 2-intersecting, then $|\mathcal{F}||\mathcal{G}| \leq \binom{n-2}{k-2}^2$.

SDP for Cross 2-intersecting families

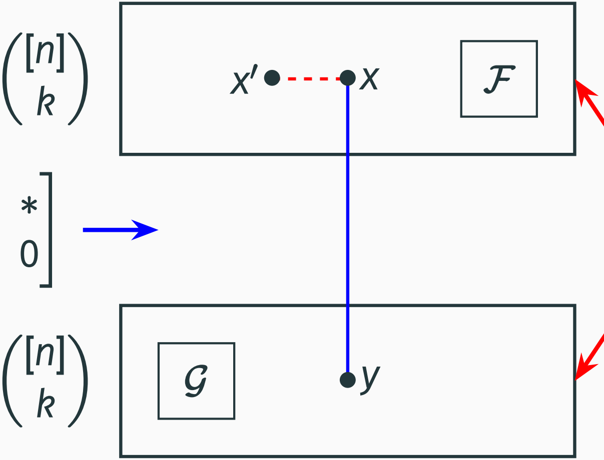
- (Suda–Tanaka 2014) minimize α, subject to

$$S := \frac{1}{2}\begin{bmatrix} \alpha I & -J \\ -J & \alpha I \end{bmatrix} + A - Z \succeq 0, \quad Z \geq 0, \quad (A)_{x,y} = 0 \text{ for } |x \cap y| \geq 2.$$

- $(\star)$ A feasible solution α satisfies $|\mathcal{F}||\mathcal{G}| \leq \alpha^2$.
- If $n \geq 3(k-1)$, then we can find A and $Z \geq 0$ so that $S \succeq 0$ with $\alpha = \binom{n-2}{k-2}$.
- What is this Z anyway?

(talk at RIMS, Kyoto University, 6th March)

Bipartite Kneser graph



$$x \sim y \Leftrightarrow |x \cap y| \leq 1$$

$$A = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$$

$$Z = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}$$

Conjecture

Let $k \geq l$ and $n \geq 3(k-1)$. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ and $\mathcal{G} \subset \binom{[n]}{l}$ are cross 2-intersecting. Then $|\mathcal{F}||\mathcal{G}| \leq \binom{n-2}{k-2}\binom{n-2}{l-2}$.

Problems

- families of subspaces in a vector space
- families of permutations
- other structures, e.g., partitions, perfect matching, etc.
- See also Godsil–Meagher (Chapter 16, Open problems).