Erdős–Ko–Rado type problems: old and new

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Shadow functions and their limit

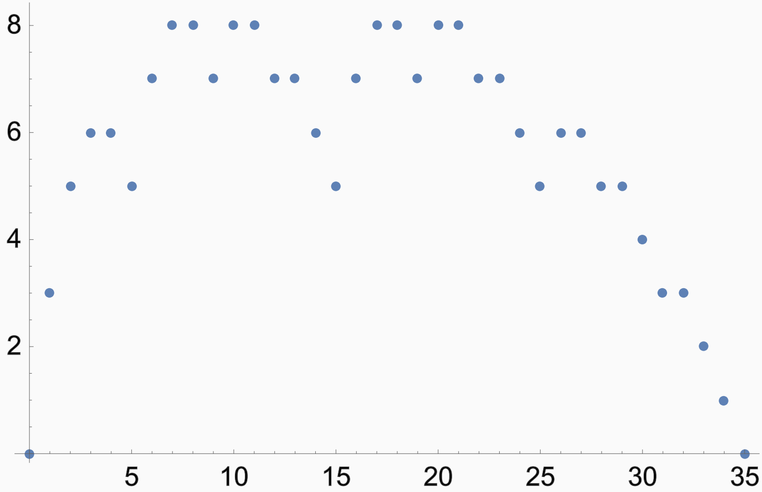
Erdős–Ko–Rado theorem

Cross intersecting EKR

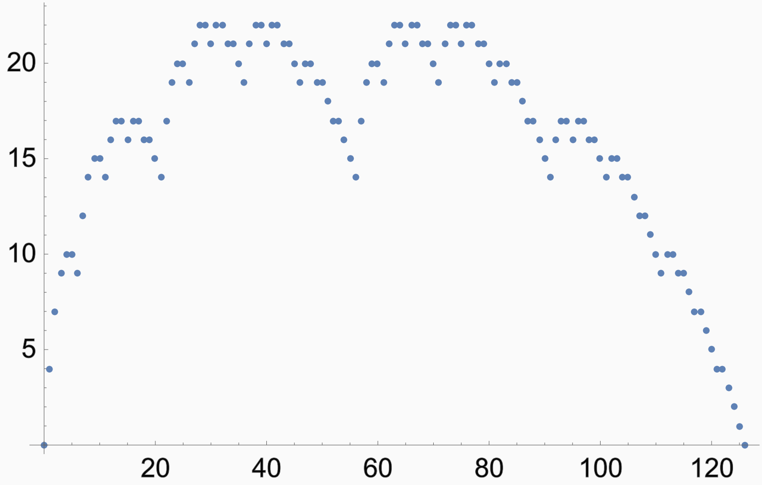
t-intersecting families

Cross t-intersecting EKR, and more

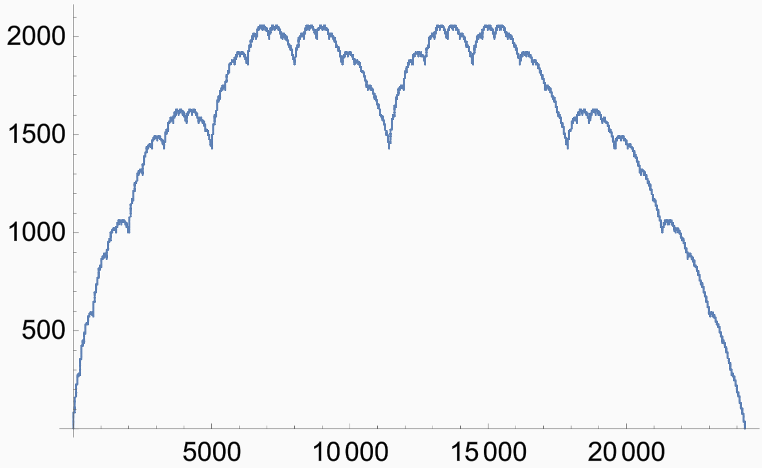
Shadow functions and their limit



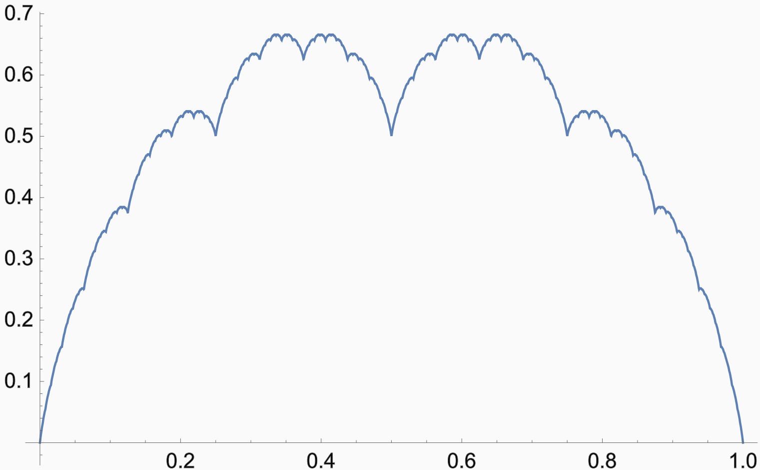
The shadow function f_4



The shadow function f_5

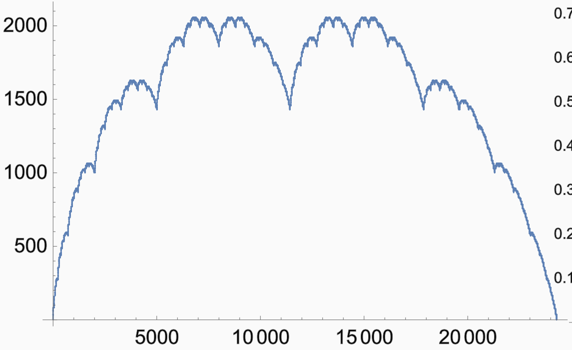


The shadow function f_9

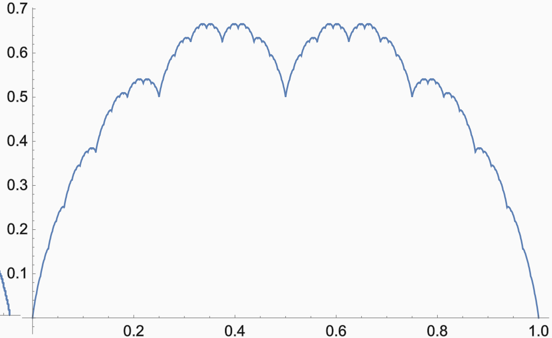


The Takagi function

(a nowhere differentiable continuous function)



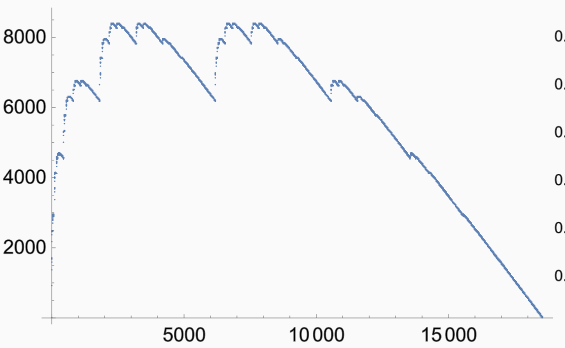
The shadow function f_9



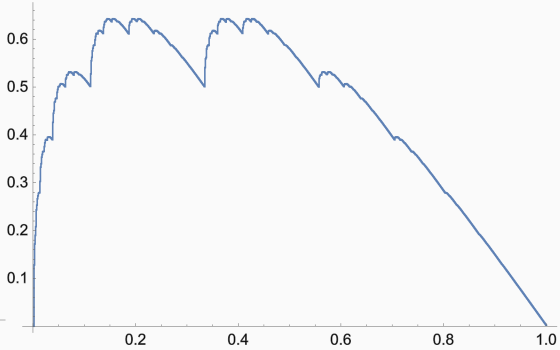
The Takagi function

Theorem (Matsumoto, 1990)

By normalizing appropriately, the shadow functions f_k converges to the Takagi function uniformly (as $k \to \infty$).



A biased shadow function



A biased Takagi function

Theorem (Ruzsa, 1991)

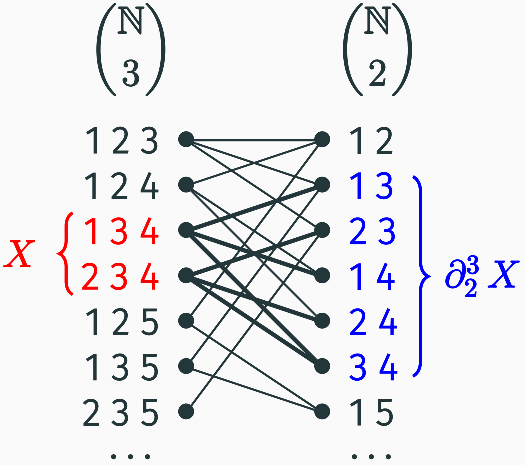
The left functions converges to the right function uniformly.

P. Frankl, M. Matsumoto, I. Ruzsa, N. Tokushige.
Minimum shadows in uniform hypergraphs and a generalization of the Takagi function.
Journal of Combinatorial Theory (A), Vol 69, (1995) 125–148.

D. E. Knuth, The art of computer programming, Vol. 4, Fasc. 3, (2005) p.21.

Discrete isoperimetric problem

- For given $x > 0$, what is $\min \partial X$ such that $\mathrm{vol}(X) = x$?
- $\binom{\mathbb{N}}{k} := \{u \subset \mathbb{N} : |u| = k\}$.
- For $X \subset \binom{\mathbb{N}}{k}$ and $k > l$, let $\partial_l^k X := \{v \in \binom{\mathbb{N}}{l} : v \subset \exists\, u \in X\}$.
- For given positive integers $k > l$ and x, Kruskal–Katona theorem tells us $\min |\partial_l^k X|$ such that $X \subset \binom{\mathbb{N}}{k}$ with $|X| = x$.
- Let $\partial_l^k(x)$ denote this minimal size of boundary (or l-shadow).
- Then, the shadow function is defined by $f_k(x) := \partial_{k-1}^k(x) - x$.



$$\partial_2^3(1) = 3, \quad \partial_2^3(2) = 5, \quad \partial_2^3(3) = 6, \quad \partial_2^3(4) = 6.$$

Kruskal–Katona theorem

k-cascade form of m:
$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$ with $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$.

Example. 3-cascade form of 100. $100 = \binom{9}{3} + \binom{6}{2} + \binom{1}{1}$.

- Choose maximal a_3 such that $\binom{a_3}{3} \leq 100$.
  $\binom{9}{3} = 84$, $\binom{10}{3} = 120$, so $a_3 = 9$.
- $100 - 84 = 16$. Choose maximal a_2 such that $\binom{a_2}{2} \leq 16$.
  $\binom{6}{2} = 15$, $\binom{7}{2} = 21$, so $a_2 = 6$.

Kruskal–Katona theorem

k-cascade form of m:
$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$ with $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$.

Theorem (Kruskal 1963, Katona 1968)

If $X \subset \binom{\mathbb{N}}{k}$ with $|X| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$, then

$$|\partial_l^k X| \geq \binom{a_k}{l} + \binom{a_{k-1}}{l-1} + \cdots + \binom{a_s}{s-(k-l)}.$$

Erdős–Ko–Rado theorem

Erdős–Ko–Rado theorem

- Let $[n] := \{1, 2, \ldots, n\}$.
- A family of subsets $F \subset 2^{[n]}$ is called intersecting if

$$u \cap v \neq \emptyset \text{ for all } u, v \in F.$$

- ex. $F = \{\{1,2\}, \{1,3\}, \{1,4\}\}$, $F = \{\{1,2\}, \{1,3\}, \{2,3\}\}$.

Theorem (Erdős–Ko–Rado, 1930')

If $n \geq 2k$ and $F \subset \binom{[n]}{k}$ is intersecting, then $|F| \leq \binom{n-1}{k-1}$.

ex. $F = \{u \in \binom{[n]}{k} : 1 \in x\}$.

Hoffman's bound

- Let $G = (V, E)$ be a regular graph.
- Let $A \in \mathbb{R}^{|V| \times |V|}$ be the adjacency matrix, i.e.,

  $$(A)_{u,v} \text{ is } 1 \text{ if } u \sim v, \text{ and } 0 \text{ if } u \not\sim v.$$

- $F \subset V$ is called independent if $u \not\sim v$ for all $u, v \in F$.

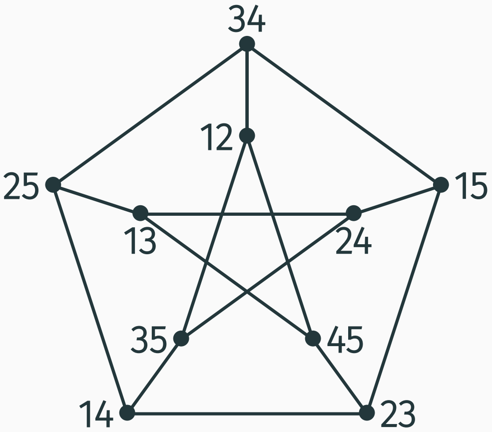
Theorem (Hoffman 1974? cf. Haemers 2021)

If $F \subset V$ is independent, then $|F| \leq \dfrac{-\lambda_{\mathrm{min}}}{\lambda_{\mathrm{max}} - \lambda_{\mathrm{min}}} |V|$.

Kneser graph

Kneser graph $G = G_{n,k}$:

- $V(G) = \binom{[n]}{k}$,
- $u \sim v \iff u \cap v = \emptyset$.

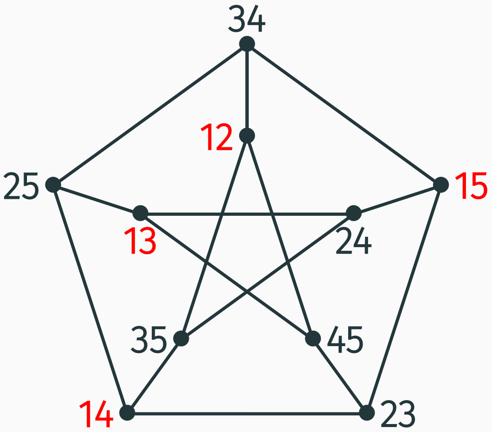


$G_{5,2}$

Kneser graph

Kneser graph $G_{5,2}$:

- $F = \{12, 13, 14, 15\}$ is independent.
- $\lambda_{\max} = 3$, $\lambda_{\min} = -2$.
- $|F| \leq \dfrac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}}|V| = 4$.



$G_{5,2}$

EKR from Hoffman's bound of Kneser graph

- Kneser graph $G = G_{n,k} : V(G) = \binom{[n]}{k}$, and $u \sim v \iff u \cap v = \emptyset$.
- In this case, $F \subset V(G)$ is independent iff it is intersecting.
- $\lambda_{\max} = \binom{n-k}{k}$, $\lambda_{\min} = -\binom{n-k-1}{k-1}$.
- Hoffman's bound is $\dfrac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} \dbinom{n}{k} = \dbinom{n-1}{k-1}$.
- Thus, if $F \subset \binom{[n]}{k}$ is intersecting, then $|F| \leq \binom{n-1}{k-1}$.

Cross intersecting EKR

cross intersecting EKR

- Two families $F, G \subset 2^{[n]}$ are called cross intersecting if
$$u \cap v \neq \emptyset \text{ for all } u \in F,\ v \in G.$$

Theorem (Pyber, 1986)

If $n \geq 2k$, and $F, G \subset \binom{[n]}{k}$ are cross int., then $|F||G| \leq \binom{n-1}{k-1}^2$.

Hoffman-like bound

$$\sqrt{|F||G|} \leq \frac{\lambda_2}{\lambda_{\max} + \lambda_2} \binom{n}{k},$$ where $\lambda_2 = \max\{|\lambda| : \lambda \neq \lambda_{\max}\}$.

cross intersecting EKR

Theorem (Pyber, 1986)

If $n \geq 2k$, and $F, G \subset \binom{[n]}{k}$ are cross int., then $|F||G| \leq \binom{n-1}{k-1}^2$.

- What about if $F \subset \binom{[n]}{k}$ ard $G \subset \binom{[n]}{l}$ are cross intersecting?
- Pyber showed that if $n \geq n_0(k, l)$ then $|F||G| \leq \binom{n-1}{k-1}\binom{n-1}{l-1}$.

Theorem (Matsumoto–T, 1987)

If $n \geq 2k \geq 2l$, and $F \subset \binom{[n]}{k}$ and $G \subset \binom{[n]}{l}$ are cross intersecting, then $|F||G| \leq \binom{n-1}{k-1}\binom{n-1}{l-1}$.

Homological isoperimetry in the torus. Let X be the N-torus. Then $A = H^*(\mathbb{T}^N;\mathbb{F})$ is isomorphic to the exterior algebra $\wedge^*\mathbb{F}^N = \mathbb{F}[\Delta_\circ^{N-1}]$ for the graded semigroup $G_\circ = \Delta_\circ^{N-1}$ associated to the simplex Δ^{N-1} on N-vertices (see 2.1).

Namely, $G_\circ$ equals $2^{\{1,\dots,N\}}$ that is the set of subsets g in $\{1,\dots,N\}$, where $G(n) \subset G_\circ$ consists of all subsets of cardinality n and where the product $g_i \smile g_2$ for $g_i \subset \{1,\dots N\}$ is defined as follows:

If g_1 intersects g_2, then $g_1 \smile g_2 = 0$; otherwise, $g_1 \smile g_2 = \pm g_1 \cup g_2$.

(If $\operatorname{char}\mathbb{F} = 2$ one does not have to bother with the specification of the $\pm$ sign.)

Thus,

bounds on cardinalities of subsets $G_i \subset 2^{\{1,\dots,N\}}$ established in extremal set theory in terms of the numbers of non-intersecting members $g_i \in 2^{\{1,\dots,N\}}$ regarded as subsets $g_i \subset \{1,\dots,N\}$ imply corresponding inequalities between the cohomology masses of subsets $X_i \subset \mathbb{T}^N$ and of their intersections.

EXAMPLE: MATSUMOTO–TOKUSHIGE INEQUALITY [MatT1].

Let $G_i \subset G(n_i) \subset G_\circ = 2^{\{1,\dots,N\}}$, $i = 0,1$, be subsets such that the intersections $g_0 \cap g_1$ in $\{1,\dots,N\}$ are non-empty for all $g_0 \in G_0$ and $g_1 \in G_1$. If $n_0, n_1 \le N/2$, then the cardinalities of these sets satisfy

$$|G_0| \cdot |G_1| \le \binom{N-1}{n_0-1}\binom{N-1}{n_1-1}.$$

Original proof ideas

- Suppose that $F \subset \binom{[n]}{k}$ ard $G \subset \binom{[n]}{l}$ are cross intersecting.
- Let $F^c := \{[n] \setminus u : u \in F\} \subset \binom{[n]}{n-k}$.
- $\partial_l^{n-k} F^c = \{v \in \binom{[n]}{l} : \exists\, w \in F^c,\, v \subset w\}$
  $= \{v \in \binom{[n]}{l} : \exists\, u \in F,\, v \cap u = \emptyset\}$.
- We have $G \cap \partial_l^{n-k} F^c = \emptyset$.
- Let $x := |F| = |F^c|$. Then $|F||G| \leq x \cdot (\binom{n}{l} - \partial_l^{n-k}(x))$.
- We can compute $\partial_l^{n-k}(x)$ using Kruskal–Katona (in principle).
- The actual computation is not easy.

- We have $|F||G| \le x \cdot (\binom{n}{l} - \partial_l^{n-k}(x))$.
- We want to show that the RHS is $\le \binom{n-1}{k-1}\binom{n-1}{l-1}$.
- The hardest part is the case $\binom{n-1}{k-1} < x \le \binom{n-1}{k-1} + \binom{n-2}{n-k-1}$.
- For $y \in \mathbb{R}$ and $k \in \mathbb{N}$, let $\binom{y}{k} := y(y-1)\cdots(y-k+1)/k!$.

Lemma (Matsumoto, 1987)

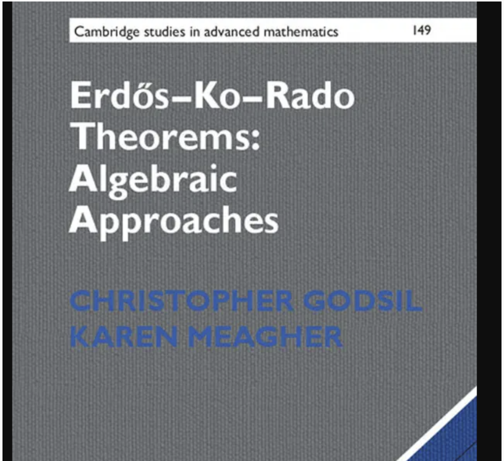
Let $n \ge 2k \ge 2l$ and $n-k-1 \le y \le n-2$. Then

$$\left(\binom{n-1}{k-1}+\binom{y}{n-k-1}\right)\left(\binom{n-1}{l-1}-\binom{y}{l-1}\right) < \binom{n-1}{k-1}\binom{n-1}{l-1}.$$

- Legendre multiplier is utilized in the proof.

Theorem (Matsumoto–T, 1987)

If $n \geq 2k \geq 2l$, and $F \subset \binom{[n]}{k}$ and $G \subset \binom{[n]}{l}$ are cross intersecting, then $|F||G| \leq \binom{n-1}{k-1}\binom{n-1}{l-1}$.



"We give only a brief outline of their proof, since it involves careful and detailed manipulations with binomial coefficients."

Alternative proof

- Suda and Tanaka translated the problem into a semidefinite programming problem.
- Then they constructed an optimal solution to the dual problem. They also solved the vector space version.

Theorem (Suda–Tanaka, 2014)

Let $\Omega = \mathbb{F}_q^n$. If $n \geq 2k \geq 2l$, and $F \subset \begin{bmatrix}\Omega \\ k\end{bmatrix}$ and $G \subset \begin{bmatrix}\Omega \\ l\end{bmatrix}$ are cross intersecting, then $|F||G| \leq \begin{bmatrix}n-1 \\ k-1\end{bmatrix}\begin{bmatrix}n-1 \\ l-1\end{bmatrix}$.

t-intersecting families

EKR for t-intersecting families

- A family of subsets $F \subset 2^{[n]}$ is called t-intersecting if
$$|u \cap v| \geq t \text{ for all } u, v \in F.$$

Problem: Let $F \subset \binom{[n]}{k}$ be t-intersecting. Is $\max |F| = \binom{n-t}{k-t}$?

- EKR (1930') Yes, if $n \geq n_0(k, t)$.
- Frankl (1978) Yes, if $t \geq 15$ and $n \geq (t+1)(k-t+1)$.
- Wilson (1983) Yes, if $t \geq 1$ and $n \geq (t+1)(k-t+1)$.
- Ahlswede–Khachatrian (1996) determined $\max |F|$ for ALL n.
  It is called "the complete intersection theorem."

Wilson's proof

- Kneser graph $G = G_{n,k,t} : V = \binom{[n]}{k}$, and $u \sim v \iff |u \cap v| < t$.
- In this case, $F \subset V$ is independent iff it is t-intersecting.
- Let $A \in \mathbb{R}^{V \times V}$ be a symmetric matrix with $(A)_{u,v} = 0$ if $u \not\sim v$ with a constant row sum. (pseudo-adjacency matrix)

An SDP problem for independent sets in $G_{n,k,t}$

minimize α, subject to $S := \alpha I - J + A \succeq 0$. (variables are α, A.)

- A feasible solution satisfies $|F| \leq \alpha$ if F is independent.
- Wilson constructed an A with $\alpha = \binom{n-t}{k-t}$.

positive semidefinite matrix

- Let $M = \mathbb{R}^{V\times V}$ be a real symmetric matrix.
- M is positive semidefinite if $\boldsymbol{x}^\top M\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{R}^V$.
- We write $M \succeq 0$ if M is positive semidefinite.
- $M \succeq 0$ iff all eigenvalues are non-negative.
- For symmetric matrices A, B, let $A \bullet B = \sum_{u,v}(A)_{u,v}(B)_{u,v}$.
  ex. $\left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right] \bullet \left[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right] = ax + 2by + cz$.
- $\boldsymbol{x}^\top M\boldsymbol{x} = M \bullet (\boldsymbol{x}\boldsymbol{x}^\top)$.
- If $M \succeq 0$ then $M \bullet (\boldsymbol{x}\boldsymbol{x}^\top) \geq 0$.

Bounding independent sets in $G_{n,k,t}$

- Kneser graph $G = G_{n,k,t} : V = \binom{[n]}{k}$, and $u \sim v \iff |u \cap v| < t$.
- Let A be a pseudo-adjacency matrix of G.
- Let $F \subset V$ be an independent set in G.
- Let $\boldsymbol{x}$ be the indicator of F, and let $X = \boldsymbol{x}\boldsymbol{x}^{\mathsf{T}}$.
- If $u \sim v$ then $(X)_{u,v} = 0$, and if $u \not\sim v$ then $(A)_{u,v} = 0$.
- $I \bullet X = |F|$.
- $J \bullet X = |F|^2$.
- $A \bullet X = 0$ because $(A)_{u,v}(X)_{u,v} = 0$ for all u, v.
- Let $S := \alpha I - J + A$. Then $S \bullet X = \alpha|F| - |F|^2$.
- If $S \succeq 0$, then $S \bullet X \geq 0$, and so $|F| \leq \alpha$.

Cross t-intersecting EKR, and more

- Two families $F, G \subset 2^{[n]}$ are called cross t-intersecting if
$$|u \cap v| \geq t \text{ for all } u \in F,\ v \in G.$$

Let $F, G \subset \binom{[n]}{k}$ be cross t-intersecting. Is $\max |F||G| \leq \binom{n-t}{k-t}^2$?

- Yes, if $n > k/(1-2^{-1/t})$ by the Hoffman-like bound. (T 2013)
- Yes, if $n \geq (t+1)k$ and $t \geq 14$ by probabilistic method. (Frankl–Lee–Siggers–T, 2014)
- Yes, if $n \geq (t+1)(k-t+1)$ and $t \geq 3$ using ideas by Ahlswede and Khachatrian. (Zhang–Wu 2024, arXiv:2410.22792)
- Yes, if $t = 2$ and $n \geq 3(k-1)$ by solving a SDP problem. (Tanaka–T 2025, arXiv:2503.14844)
- The cases $t \geq 2$ and $n < (t+1)(k-t+1)$ are wide open. (Some partial results by Lee–Siggers–T, 2015, 2019)

Let $F \subset \binom{[n]}{k}$ and $G \subset \binom{[n]}{l}$ be cross t-intersecting.
Suppose that $k \geq l$ and $n \geq (t+1)(k-t+1)$.
Is it true that $|F||G| \leq \binom{n-t}{k-t}\binom{n-t}{l-t}$?

- Yes$^?$, if $t \geq 3$. The proof is similar to Zhang–Wu.
  (Bao–Ji 2025, arXiv:2510.11724)
- Is it true that if $t = 2$, $k \geq l$, $n \geq 3(k-1)$, and $F \subset \binom{[n]}{k}$ and $G \subset \binom{[n]}{l}$ are cross 2-intersecting, then $|F||G| \leq \binom{n-2}{k-2}\binom{n-2}{l-2}$?
- Yes$^?$ if $t = 2$ and $n \geq 3.38k$. (Chen–Li–Wu–Zhang 2025, arXiv:2503.15971)

More families

- Three families $F_1, F_2, F_3 \subset 2^{[n]}$ are called 3-cross intersecting if $u_1 \cap u_2 \cap u_3 \neq \emptyset$ for all $u_1 \in F_1, u_2 \in F_2, u_3 \in F_3$.
- If $n \geq \frac{3}{2}k$, and $F_1, F_2, F_3 \subset \binom{[n]}{k}$ are 3-cross intersecting, then $|F_1||F_2||F_3| \leq \binom{n-1}{k-1}^3$. (Frankl-T 2011)
- Is it true that if $k_1 \geq k_2 \geq k_3$, $n \geq \frac{3}{2}k_1$, and $F_1 \subset \binom{[n]}{k_1}$, $F_2 \subset \binom{[n]}{k_2}$, $F_3 \subset \binom{[n]}{k_3}$ are 3-cross intersecting, then $|F_1||F_2||F_3| \leq \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}\binom{n-1}{k_3-1}$?

- Let q be a fixed prime power.
- For $x \in \mathbb{R}$ and $k \in \mathbb{N}$, let $\begin{bmatrix} x \\ k \end{bmatrix} := \prod_{j=0}^{k-1} \frac{q^{x-j} - 1}{q^{k-j} - 1}$.

Theorem (Suda–Tanaka, 2014)

Let $\Omega = \mathbb{F}_q^n$. If $n \geq 2k \geq 2l$, and $F \subset \begin{bmatrix} \Omega \\ k \end{bmatrix}$ and $G \subset \begin{bmatrix} \Omega \\ l \end{bmatrix}$ are cross intersecting, then $|F||G| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}$.

- Three families $F_1, F_2, F_3 \subset \Omega$ are called 3-cross intersecting if $\dim(u_1 \cap u_2 \cap u_3) \geq 1$ for all $u_1 \in F_1, u_2 \in F_2, u_3 \in F_3$.
- Let $n \geq \frac{3}{2}k$, and $F_1, F_2, F_3 \subset \begin{bmatrix} \Omega \\ k \end{bmatrix}$ be 3-cross intersecting. Is it true that $|F_1||F_2||F_3| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^3$?

- Let $n = 3l, \quad V = \mathbb{F}_q^n, \quad G_1, G_2, G_3 \subset \begin{bmatrix} V \\ l \end{bmatrix}$.
- Define $y_i \in \mathbb{R}$ by $|G_i| = \begin{bmatrix} y_i \\ l \end{bmatrix}$.
- Let $\mathcal{S}$ be the set of geometric l-spreads on V.

Conjecture

Suppose that $(G_1 \cap S) + (G_2 \cap S) + (G_3 \cap S) \neq V$ for all $S \in \mathcal{S}$. Then we have $y_1 + y_2 + y_3 \leq 3(n-1)$.

(Equality holds if $G_1 = G_2 = G_3 = \begin{bmatrix} U \\ l \end{bmatrix}$ for some $U \cong \mathbb{F}_q^{n-1}$.)

If this is true, then the answer to the problem in the previous slide is affirmative.