

# Erdős–Ko–Rado type problems: old and new

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Shadow functions and their limit

Erdős–Ko–Rado theorem

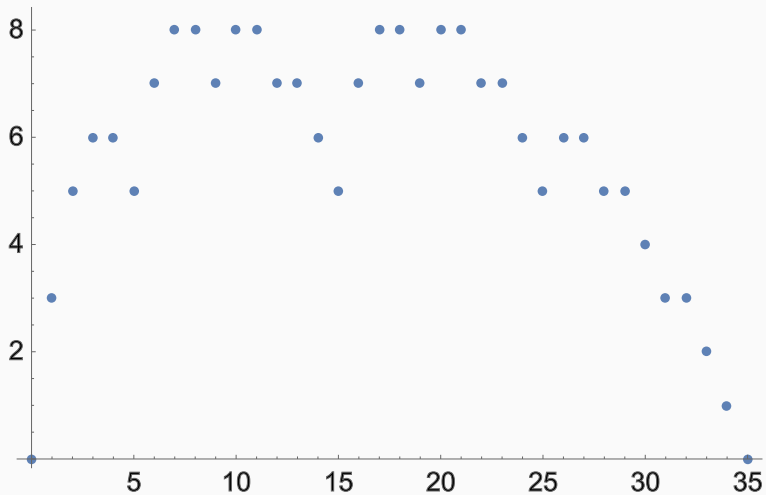
Cross intersecting EKR

$t$ -intersecting families

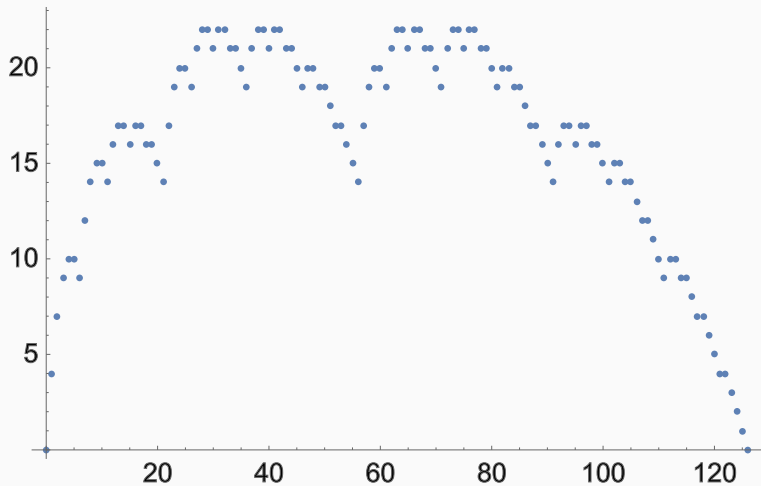
Cross  $t$ -intersecting EKR, and more

# Shadow functions and their limit

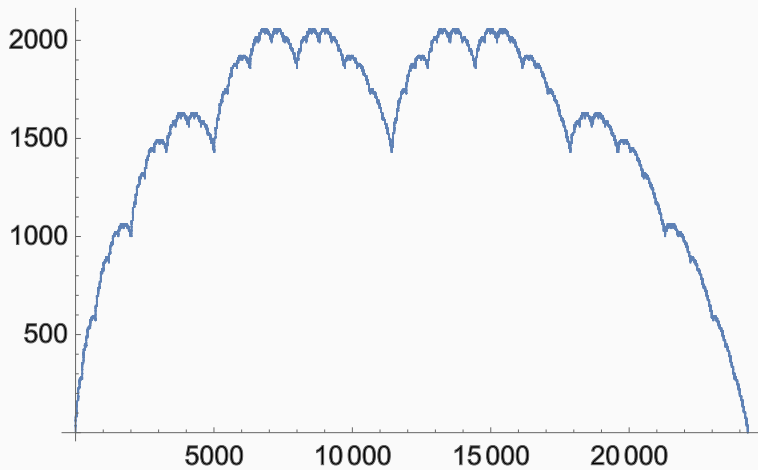
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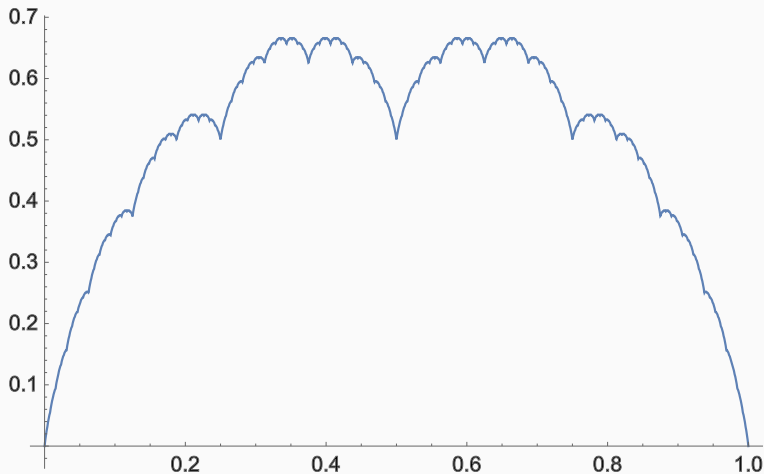
The shadow function  $f_4$



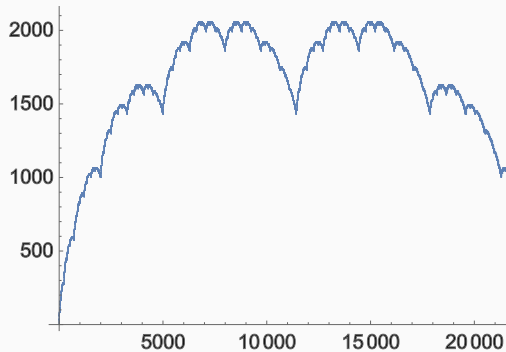
The shadow function  $f_5$



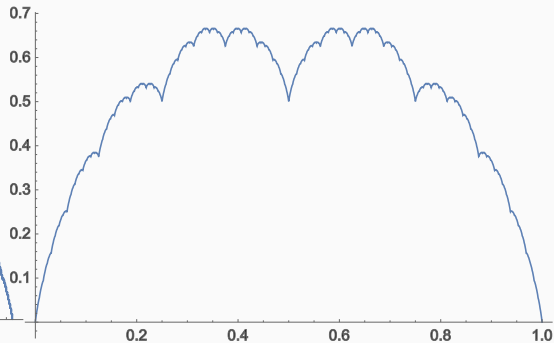
The shadow function  $f_9$



The Takagi function  
(a nowhere differentiable continuous function)



The shadow function  $f_9$

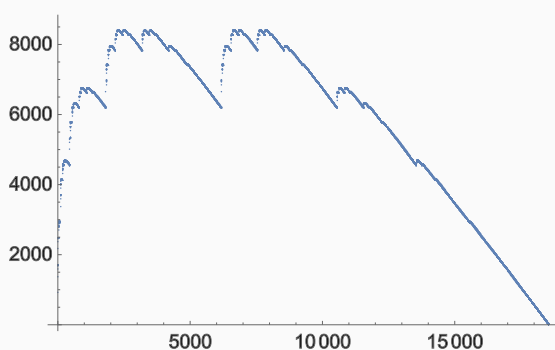


The Takagi function

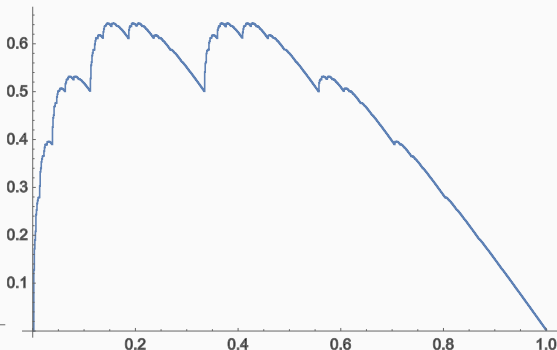
### Theorem (Matsumoto, 1990)

By normalizing appropriately, the shadow functions  $f_k$  converges to the Takagi function uniformly (as  $k \rightarrow \infty$ ).





A biased shadow function



A biased Takagi function

### Theorem (Ruzsa, 1991)

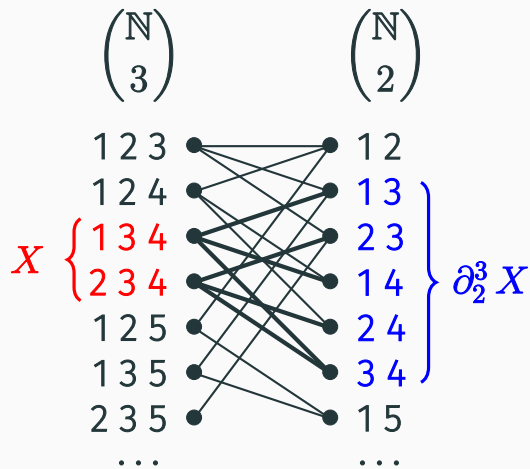
The left functions converges to the right function uniformly.

P. Frankl, M. Matsumoto, I. Ruzsa, N. Tokushige.  
Minimum shadows in uniform hypergraphs and a generalization  
of the Takagi function.  
Journal of Combinatorial Theory (A), Vol 69, (1995) 125–148.

D. E. Knuth, The art of computer programming, Vol. 4, Fasc. 3,  
(2005) p.21.

# Discrete isoperimetric problem

- For given  $x > 0$ , what is  $\min \partial X$  such that  $\text{vol}(X) = x$  ?
- $\binom{\mathbb{N}}{k} := \{u \subset \mathbb{N} : |u| = k\}$ .
- For  $X \subset \binom{\mathbb{N}}{k}$  and  $k > l$ , let  $\partial_l^k X := \{v \in \binom{\mathbb{N}}{l} : v \subset \exists u \in X\}$ .
- For given positive integers  $k > l$  and  $x$ , Kruskal–Katona theorem tells us  $\min |\partial_l^k X|$  such that  $X \subset \binom{\mathbb{N}}{k}$  with  $|X| = x$ .
- Let  $\partial_l^k(x)$  denote this minimal size of boundary (or  $l$ -shadow).
- Then, the shadow function is defined by  $f_k(x) := \partial_{k-1}^k(x) - x$ .



$$\partial_2^3(1) = 3, \quad \partial_2^3(2) = 5, \quad \partial_2^3(3) = 6, \quad \partial_2^3(4) = 6.$$

# Kruskal–Katona theorem

$k$ -cascade form of  $m$ :

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s} \text{ with } a_k > a_{k-1} > \cdots > a_s \geq s \geq 1.$$

Example. 3-cascade form of 100.  $100 = \binom{9}{3} + \binom{6}{2} + \binom{1}{1}$ .

- Choose maximal  $a_3$  such that  $\binom{a_3}{3} \leq 100$ .  
 $\binom{9}{3} = 84$ ,  $\binom{10}{3} = 120$ , so  $a_3 = 9$ .
- $100 - 84 = 16$ . Choose maximal  $a_2$  such that  $\binom{a_2}{2} \leq 16$ .  
 $\binom{6}{2} = 15$ ,  $\binom{7}{2} = 21$ , so  $a_2 = 6$ .

# Kruskal–Katona theorem

$k$ -cascade form of  $m$ :

$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$  with  $a_k > a_{k-1} > \cdots > a_s \geq s \geq 1$ .

**Theorem (Kruskal 1963, Katona 1968)**

If  $X \subset \binom{\mathbb{N}}{k}$  with  $|X| = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_s}{s}$ , then

$$|\partial_l^k X| \geq \binom{a_k}{l} + \binom{a_{k-1}}{l-1} + \cdots + \binom{a_s}{s - (k-l)}.$$

# Erdős–Ko–Rado theorem

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# Erdős–Ko–Rado theorem

- Let  $[n] := \{1, 2, \dots, n\}$ .
- A family of subsets  $F \subset 2^{[n]}$  is called **intersecting** if
$$u \cap v \neq \emptyset \text{ for all } u, v \in F.$$
- ex.  $F = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$ ,  $F = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$ .

## Theorem (Erdős–Ko–Rado, 1930')

If  $n \geq 2k$  and  $F \subset \binom{[n]}{k}$  is intersecting, then  $|F| \leq \binom{n-1}{k-1}$ .

ex.  $F = \{u \in \binom{[n]}{k} : 1 \in u\}$ .



# Hoffman's bound

- Let  $G = (V, E)$  be a regular graph.
- Let  $A \in \mathbb{R}^{|V| \times |V|}$  be the adjacency matrix, i.e.,  
 $(A)_{u,v}$  is 1 if  $u \sim v$ , and 0 if  $u \not\sim v$ .
- $F \subset V$  is called **independent** if  $u \not\sim v$  for all  $u, v \in F$ .

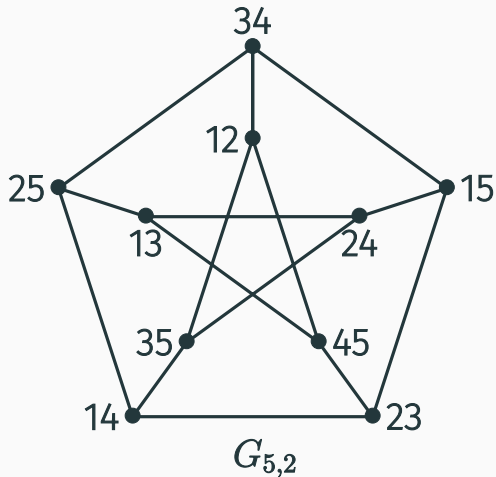
Theorem (Hoffman 1974? cf. Haemers 2021)

If  $F \subset V$  is independent, then  $|F| \leq \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} |V|$ .

# Kneser graph

Kneser graph  $G = G_{n,k}$  :

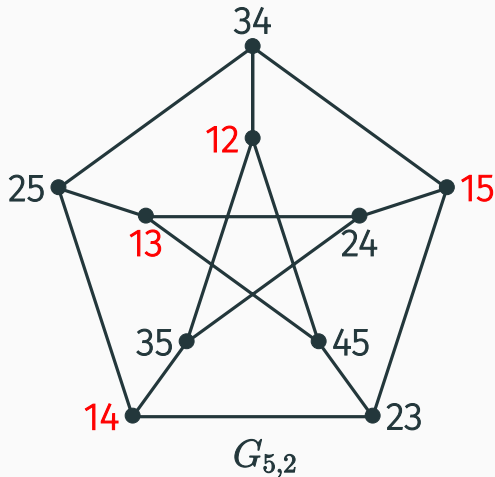
- $V(G) = \binom{[n]}{k}$ ,
- $u \sim v \iff u \cap v = \emptyset$ .



# Kneser graph

Kneser graph  $G_{5,2}$  :

- $F = \{12, 13, 14, 15\}$  is independent.
- $\lambda_{\max} = 3, \lambda_{\min} = -2$ .
- $|F| \leq \frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} |V| = 4$ .



# EKR from Hoffman's bound of Kneser graph

- Kneser graph  $G = G_{n,k} : V(G) = \binom{[n]}{k}$ , and  $u \sim v \iff u \cap v = \emptyset$ .
- In this case,  $F \subset V(G)$  is independent iff it is intersecting.
- $\lambda_{\max} = \binom{n-k}{k}$ ,  $\lambda_{\min} = -\binom{n-k-1}{k-1}$ .
- Hoffman's bound is  $\frac{-\lambda_{\min}}{\lambda_{\max} - \lambda_{\min}} \binom{n}{k} = \binom{n-1}{k-1}$ .
- Thus, if  $F \subset \binom{[n]}{k}$  is intersecting, then  $|F| \leq \binom{n-1}{k-1}$ .

# Cross intersecting EKR

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# cross intersecting EKR

- Two families  $F, G \subset 2^{[n]}$  are called **cross intersecting** if  $u \cap v \neq \emptyset$  for all  $u \in F, v \in G$ .

## Theorem (Pyber, 1986)

If  $n \geq 2k$ , and  $F, G \subset \binom{[n]}{k}$  are cross int., then  $|F||G| \leq \binom{n-1}{k-1}^2$ .

## Hoffman-like bound

$$\sqrt{|F||G|} \leq \frac{\lambda_2}{\lambda_{\max} + \lambda_2} \binom{n}{k}, \text{ where } \lambda_2 = \max\{|\lambda| : \lambda \neq \lambda_{\max}\}.$$

# cross intersecting EKR

## Theorem (Pyber, 1986)

If  $n \geq 2k$ , and  $F, G \subset \binom{[n]}{k}$  are cross int., then  $|F||G| \leq \binom{n-1}{k-1}^2$ .

- What about if  $F \subset \binom{[n]}{k}$  and  $G \subset \binom{[n]}{l}$  are cross intersecting?
- Pyber showed that if  $n \geq n_0(k, l)$  then  $|F||G| \leq \binom{n-1}{k-1} \binom{n-1}{l-1}$ .

## Theorem (Matsumoto-T, 1987)

If  $n \geq 2k \geq 2l$ , and  $F \subset \binom{[n]}{k}$  and  $G \subset \binom{[n]}{l}$  are cross intersecting, then  $|F||G| \leq \binom{n-1}{k-1} \binom{n-1}{l-1}$ .

**Homological isoperimetry in the torus.** Let  $X$  be the  $N$ -torus. Then  $A = H^*(\mathbb{T}^N; \mathbb{F})$  is isomorphic to the exterior algebra  $\wedge^* \mathbb{F}^N = \mathbb{F}[\Delta_{\circ}^{N-1}]$  for the graded semigroup  $G_{\circ} = \Delta_{\circ}^{N-1}$  associated to the simplex  $\Delta^{N-1}$  on  $N$ -vertices (see 2.1).

Namely,  $G_{\circ}$  equals  $2^{\{1, \dots, N\}}$  that is the set of subsets  $g$  in  $\{1, \dots, N\}$ , where  $G(n) \subset G_{\circ}$  consists of all subsets of cardinality  $n$  and where the product  $g_i \smile g_j$  for  $g_i \subset \{1, \dots, N\}$  is defined as follows:

*If  $g_1$  intersects  $g_2$ , then  $g_1 \smile g_2 = 0$ ; otherwise,  $g_1 \smile g_2 = \pm g_1 \cup g_2$ .*

(If  $\text{char } \mathbb{F} = 2$  one does not have to bother with the specification of the  $\pm$  sign.)

Thus,

*bounds on cardinalities of subsets  $G_i \subset 2^{\{1, \dots, N\}}$  established in extremal set theory in terms of the numbers of non-intersecting members  $g_i \in 2^{\{1, \dots, N\}}$  regarded as subsets  $g_i \subset \{1, \dots, N\}$  imply corresponding inequalities between the cohomology masses of subsets  $X_i \subset \mathbb{T}^N$  and of their intersections.*

**EXAMPLE: MATSUMOTO–TOKUSHIGE INEQUALITY [MatT1].**

*Let  $G_i \subset G(n_i) \subset G_{\circ} = 2^{\{1, \dots, N\}}$ ,  $i = 0, 1$ , be subsets such that the intersections  $g_0 \cap g_1$  in  $\{1, \dots, N\}$  are non-empty for all  $g_0 \in G_0$  and  $g_1 \in G_1$ . If  $n_0, n_1 \leq N/2$ , then the cardinalities of these sets satisfy*

$$|G_0| \cdot |G_1| \leq \binom{N-1}{n_0-1} \binom{N-1}{n_1-1}.$$



# Original proof ideas

- Suppose that  $F \subset \binom{[n]}{k}$  and  $G \subset \binom{[n]}{l}$  are cross intersecting.
- Let  $F^c := \{[n] \setminus u : u \in F\} \subset \binom{[n]}{n-k}$ .
- $$\begin{aligned}\partial_l^{n-k} F^c &= \{v \in \binom{[n]}{l} : \exists w \in F^c, v \subset w\} \\ &= \{v \in \binom{[n]}{l} : \exists u \in F, v \cap u = \emptyset\}.\end{aligned}$$
- We have  $G \cap \partial_l^{n-k} F^c = \emptyset$ .
- Let  $x := |F| = |F^c|$ . Then  $|F||G| \leq x \cdot (\binom{n}{l} - \partial_l^{n-k}(x))$ .
- We can compute  $\partial_l^{n-k}(x)$  using Kruskal–Katona (in principle).
- The actual computation is not easy.

- We have  $|F||G| \leq x \cdot \left( \binom{n}{l} - \partial_l^{n-k}(x) \right)$ .
- We want to show that the RHS is  $\leq \binom{n-1}{k-1} \binom{n-1}{l-1}$ .
- The hardest part is the case  $\binom{n-1}{k-1} < x \leq \binom{n-1}{k-1} + \binom{n-2}{n-k-1}$ .
- For  $y \in \mathbb{R}$  and  $k \in \mathbb{N}$ , let  $\binom{y}{k} := y(y-1) \cdots (y-k+1)/k!$ .

### Lemma (Matsumoto, 1987)

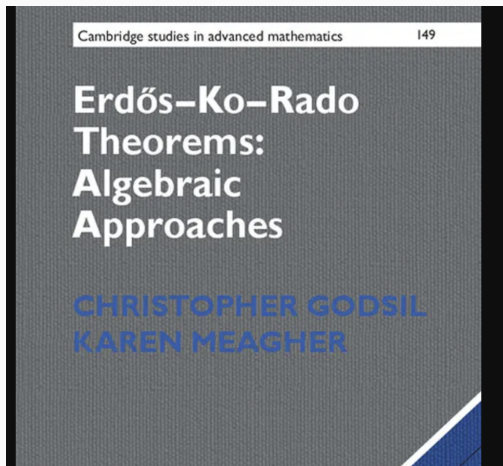
Let  $n \geq 2k \geq 2l$  and  $n - k - 1 \leq y \leq n - 2$ . Then

$$\left( \binom{n-1}{k-1} + \binom{y}{n-k-1} \right) \left( \binom{n-1}{l-1} - \binom{y}{l-1} \right) < \binom{n-1}{k-1} \binom{n-1}{l-1}.$$

- Legendre multiplier is utilized in the proof.

## Theorem (Matsumoto–T, 1987)

If  $n \geq 2k \geq 2l$ , and  $F \subset \binom{[n]}{k}$  and  $G \subset \binom{[n]}{l}$  are cross intersecting, then  $|F||G| \leq \binom{n-1}{k-1} \binom{n-1}{l-1}$ .



“We give only a brief outline of their proof, since it involves careful and detailed manipulations with binomial coefficients.”

# Alternative proof

- Suda and Tanaka translated the problem into a semidefinite programming problem.
- Then they constructed an optimal solution to the dual problem. They also solved the vector space version.

## Theorem (Suda–Tanaka, 2014)

Let  $\Omega = \mathbb{F}_q^n$ . If  $n \geq 2k \geq 2l$ , and  $F \subset \begin{bmatrix} \Omega \\ k \end{bmatrix}$  and  $G \subset \begin{bmatrix} \Omega \\ l \end{bmatrix}$  are cross intersecting, then  $|F||G| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}$ .

# t-intersecting families

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# EKR for $t$ -intersecting families

- A family of subsets  $F \subset 2^{[n]}$  is called  **$t$ -intersecting** if  $|u \cap v| \geq t$  for all  $u, v \in F$ .

Problem: Let  $F \subset \binom{[n]}{k}$  be  $t$ -intersecting. Is  $\max |F| = \binom{n-t}{k-t}$  ?

- EKR (1930') Yes, if  $n \geq n_0(k, t)$ .
- Frankl (1978) Yes, if  $t \geq 15$  and  $n \geq (t+1)(k-t+1)$ .
- **Wilson** (1983) Yes, if  $t \geq 1$  and  $n \geq (t+1)(k-t+1)$ .
- Ahlswede–Khachatrian (1996) determined  $\max |F|$  for ALL  $n$ . It is called “the complete intersection theorem.”

# Wilson's proof

- Kneser graph  $G = G_{n,k,t} : V = \binom{[n]}{k}$ , and  $u \sim v \iff |u \cap v| < t$ .
- In this case,  $F \subset V$  is independent iff it is  $t$ -intersecting.
- Let  $A \in \mathbb{R}^{V \times V}$  be a symmetric matrix with  $(A)_{u,v} = 0$  if  $u \not\sim v$  with a constant row sum. (pseudo-adjacency matrix)

**An SDP problem for independent sets in  $G_{n,k,t}$**

minimize  $\alpha$ , subject to  $S := \alpha I - J + A \succeq 0$ . (variables are  $\alpha, A$ )

- A feasible solution satisfies  $|F| \leq \alpha$  if  $F$  is independent.
- Wilson constructed an  $A$  with  $\alpha = \binom{n-t}{k-t}$ .

## positive semidefinite matrix

- Let  $M = \mathbb{R}^{V \times V}$  be a real symmetric matrix.
- $M$  is positive semidefinite if  $\mathbf{x}^\top M \mathbf{x} \geq 0$  for all  $\mathbf{x} \in \mathbb{R}^V$ .
- We write  $M \succeq 0$  if  $M$  is positive semidefinite.
- $M \succeq 0$  iff all eigenvalues are non-negative.
- For symmetric matrices  $A, B$ , let  $A \bullet B = \sum_{u,v} (A)_{u,v} (B)_{u,v}$ .  
ex.  $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \bullet \begin{bmatrix} x & y \\ y & z \end{bmatrix} = ax + 2by + cz$ .
- $\mathbf{x}^\top M \mathbf{x} = M \bullet (\mathbf{x} \mathbf{x}^\top)$ .
- If  $M \succeq 0$  then  $M \bullet (\mathbf{x} \mathbf{x}^\top) \geq 0$ .



## Bounding independent sets in $G_{n,k,t}$

- Kneser graph  $G = G_{n,k,t} : V = \binom{[n]}{k}$ , and  $u \sim v \iff |u \cap v| < t$ .
- Let  $A$  be a pseudo-adjacency matrix of  $G$ .
- Let  $F \subset V$  be an independent set in  $G$ .
- Let  $\mathbf{x}$  be the indicator of  $F$ , and let  $X = \mathbf{x}\mathbf{x}^\top$ .
- If  $u \sim v$  then  $(X)_{u,v} = 0$ , and if  $u \not\sim v$  then  $(A)_{u,v} = 0$ .
- $I \bullet X = |F|$ .
- $J \bullet X = |F|^2$ .
- $A \bullet X = 0$  because  $(A)_{u,v}(X)_{u,v} = 0$  for all  $u, v$ .
- Let  $S := \alpha I - J + A$ . Then  $S \bullet X = \alpha|F| - |F|^2$ .
- If  $S \succeq 0$ , then  $S \bullet X \geq 0$ , and so  $|F| \leq \alpha$ .

# Cross t-intersecting EKR, and more

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- Two families  $F, G \subset 2^{[n]}$  are called **cross  $t$ -intersecting** if  $|u \cap v| \geq t$  for all  $u \in F, v \in G$ .

Let  $F, G \subset \binom{[n]}{k}$  be cross  $t$ -intersecting. Is  $\max |F||G| \leq \binom{n-t}{k-t}^2$  ?

- Yes, if  $n > k/(1 - 2^{-1/t})$  by the Hoffman-like bound. (T 2013)
- Yes, if  $n \geq (t+1)k$  and  $t \geq 14$  by probabilistic method. (Frankl–Lee–Siggers–T, 2014)
- Yes, **if  $n \geq (t+1)(k-t+1)$  and  $t \geq 3$**  using ideas by Ahlswede and Khachatrian. (Zhang–Wu 2024, arXiv:2410.22792)
- Yes, **if  $t = 2$  and  $n \geq 3(k-1)$**  by solving a SDP problem. (Tanaka–T 2025, arXiv:2503.14844)
- The cases  $t \geq 2$  and  $n < (t+1)(k-t+1)$  are wide open. (Some partial results by Lee–Siggers–T, 2015, 2019)

Let  $F \subset \binom{[n]}{k}$  and  $G \subset \binom{[n]}{l}$  be cross  $t$ -intersecting.  
 Suppose that  $k \geq l$  and  $n \geq (t+1)(k-t+1)$ .  
 Is it true that  $|F||G| \leq \binom{n-t}{k-t} \binom{n-t}{l-t}$ ?

- Yes?, if  $t \geq 3$ . The proof is similar to Zhang–Wu.  
 (Bao–Ji 2025, arXiv:2510.11724)
- Is it true that if  $t = 2$ ,  $k \geq l$ ,  $n \geq 3(k-1)$ , and  $F \subset \binom{[n]}{k}$  and  $G \subset \binom{[n]}{l}$  are cross 2-intersecting, then  $|F||G| \leq \binom{n-2}{k-2} \binom{n-2}{l-2}$ ?
- Yes? if  $t = 2$  and  $n \geq 3.38k$ . (Chen–Li–Wu–Zhang 2025, arXiv:2503.15971)

# More families

- Three families  $F_1, F_2, F_3 \subset 2^{[n]}$  are called **3-cross intersecting** if  $u_1 \cap u_2 \cap u_3 \neq \emptyset$  for all  $u_1 \in F_1, u_2 \in F_2, u_3 \in F_3$ .
- If  $n \geq \frac{3}{2}k$ , and  $F_1, F_2, F_3 \subset \binom{[n]}{k}$  are 3-cross intersecting, then  $|F_1||F_2||F_3| \leq \binom{n-1}{k-1}^3$ . (Frankl-T 2011)
- Is it true that if  $k_1 \geq k_2 \geq k_3$ ,  $n \geq \frac{3}{2}k_1$ , and  $F_1 \subset \binom{[n]}{k_1}, F_2 \subset \binom{[n]}{k_2}, F_3 \subset \binom{[n]}{k_3}$  are 3-cross intersecting, then  $|F_1||F_2||F_3| \leq \binom{n-1}{k_1-1} \binom{n-1}{k_2-1} \binom{n-1}{k_3-1}$  ?

- Let  $q$  be a fixed prime power.
- For  $x \in \mathbb{R}$  and  $k \in \mathbb{N}$ , let  $\begin{bmatrix} x \\ k \end{bmatrix} := \prod_{j=0}^{k-1} \frac{q^{x-j} - 1}{q^{k-j} - 1}$ .

### Theorem (Suda–Tanaka, 2014)

Let  $\Omega = \mathbb{F}_q^n$ . If  $n \geq 2k \geq 2l$ , and  $F \subset \begin{bmatrix} \Omega \\ k \end{bmatrix}$  and  $G \subset \begin{bmatrix} \Omega \\ l \end{bmatrix}$  are cross intersecting, then  $|F||G| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}$ .

- Three families  $F_1, F_2, F_3 \subset \Omega$  are called **3-cross intersecting** if  $\dim(u_1 \cap u_2 \cap u_3) \geq 1$  for all  $u_1 \in F_1, u_2 \in F_2, u_3 \in F_3$ .
- Let  $n \geq \frac{3}{2}k$ , and  $F_1, F_2, F_3 \subset \begin{bmatrix} \Omega \\ k \end{bmatrix}$  be 3-cross intersecting. Is it true that  $|F_1||F_2||F_3| \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}^3$ ?

- Let  $n = 3l$ ,  $V = \mathbb{F}_q^n$ ,  $G_1, G_2, G_3 \subset \begin{bmatrix} V \\ l \end{bmatrix}$ .
- Define  $y_i \in \mathbb{R}$  by  $|G_i| = \begin{bmatrix} y_i \\ l \end{bmatrix}$ .
- Let  $\mathcal{S}$  be the set of geometric  $l$ -spreads on  $V$ .

### Conjecture

Suppose that  $g_1 + g_2 + g_3 \neq V$  for all  $g_i \in G_i$  and  $S \in \mathcal{S}$ .

Then we have  $y_1 + y_2 + y_3 \leq 3(n - 1)$ .

(Equality holds if  $G_1 = G_2 = G_3 = \begin{bmatrix} U \\ l \end{bmatrix}$  for some  $U \cong \mathbb{F}_q^{n-1}$ .)

If this is true, then the answer to the problem in the previous slide is affirmative.