MATH 0520 Study Guide

Brown University

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Introduction

This study guide for Linear Algebra (MATH 0520, taught by J. Usatine in the Fall 2019 semester at Brown University) seeks to cover the content mentioned in the textbook (Linear Algebra and Its Applications, Lay, Lay Macdonald) and in lecture. The textbook material covered corresponds to the sections mentioned in the *"Important Final Exam Information"* email: "The course material corresponds roughly to LLM sections **1.1-1.5, 1.7-1.9, 2.1-2.3, 3.1-3.2, 4.1-4.7, 5.1-5.4.**" Lecture content lines up with textbook content quite well, but information not covered in the textbook has been added. Proofs have been omitted for the most part.

In this guide, A implies the matrix for the system at hand, v, w are always vectors in $\mathbb{R}^n$ unless otherwise implied, and c, d are always scalars in $\mathbb{R}$. Certain parts of the textbook are omitted since they're either better explained elsewhere or hard to type up.

Before studying this guide (or any other Linear Algebra related material), I would highly recommend reviewing the "Essence of linear algebra" videos by 3Blue1Brown (easily found online). **Lastly, please do not distribute this study guide without letting me know first.** Cheers, and good luck on the exam!

Nick Young

Access

This study guide will be hosted in its most up-to-date form on Google Drive. [LINK]

Todo

This study guide is by no means complete, and the following pieces need to be added or amended:

1. Example problems
2. Proofs
3. Diagrams

Contents

Chapter 1 - Linear Equations in Linear Algebra

Chapter 1.1 - Systems of Linear Equations

A **linear equation** in the variables $x_1, ..., x_n$ is an equation in the form:

$$\boxed{a_1x_1 + a_2x_2 + ... + a_nx_n = b} \tag{1}$$

where b and the **coefficients** $a_1...a_n$ are numbers, real or complex, and n is some positive integer.

A **system of linear equations** or **linear system** is a collection of multiple linear equations in the same variable. Example:

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_2 - 4x_3 &= 7 \end{aligned}$$

A **solution** to a linear equation are numbers $s_1...s_n$ that, when substituted for $x_1...x_n$, make the linear system true. The set of all possible solutions is called the **solution set**. Two linear systems are **equivalent** if they have the same solution set. Note that a system of linear equations either has **no solutions**, **one solution**, or **infinitely many solutions**. Solutions can be thought of as intersections between lines, planes, or n-dimensional spaces.

Matrix Notation

A **matrix** contains information about a linear system - specifically, the coefficients. The following is a linear system encoded as a matrix:

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_2 - 4x_3 &= 7 \end{aligned}$$

Is equivalent to the matrix:

$$\begin{bmatrix} 2 & -1 & 1.5 \\ 0 & 1 & -4 \end{bmatrix}$$

Has the augmented matrix (contains the constant term):

$$\begin{bmatrix} 2 & -1 & 1.5 & 8 \\ 0 & 1 & -4 & 7 \end{bmatrix}$$

The **size** of a matrix is denoted as m x n, where m is the number of rows and n is the number of columns.

Solving a Linear System

By using **Elementary Row Operations** on a matrix, we can reduce it to a much simpler linear system. The operations are:

1. Replacement - Add the multiple of a row to another row,
2. Interchange - Swap two rows,

3. Scaling - Multiply all entries in a row by some non-zero constant.

Row operations work on any matrix. Two matrices are **row equivalent** if there is a sequence of row operations that could transform one matrix into the other. Note that row operations are reversible and preserve the solutions of a system. If two matrices are row equivalent, they share the same solution set.

Existence and Uniqueness

Fundamental Questions:

1. Is the system **consistent** (is there at least one solution)?
2. Is the solution **unique**?

As for (1), if row reduction can produce a contradiction, namely, setting 0 equal to some non-zero value, there are no solutions to the system. As for (2), We worry about that later.

Chapter 1.2 - Row Reduction and Echelon Forms

Definitions

A **nonzero** row or column is one that contains at least one nonzero term. A **leading entry** of a row is the leftmost nonzero entry.

A matrix is in **(row) echelon form** if: (1) All nonzero rows are above zero rows, (2) Each leading entry is in a columns to the right of the leading entry in the row above it, (3) All entries below a leading entry in the same column are zeros.

A matrix is in **reduced (row) echelon form** if: (1) The leading entry in each nonzero row is 1, (2) Each leading 1 is the only nonzero entry in its column.

An **(reduced) echelon matrix** is one which is in (reduced) echelon form. Echelon matrices (of a given matrix) are not unique, but:

(Theorem 1) Reduced echelon matrices are unique.

Pivot Positions

A **pivot position** is a location in a matrix that corresponds to a leading 1 in the reduced echelon form of that matrix. A **pivot column** is a column that contains a pivot position.

Omitted here is the row reduction algorithm.

Solutions of Linear Systems

Once an augmented matrix has been taken to reduced echelon form, the equations that come from it can be used to easily deduce a solution. Note that the variables corresponding to a pivot column are called **basic variables**, and ones not corresponding to a pivot column are called **free variables**. Solutions can be described **parametrically**, where basic variables are expressed in terms of the free variables, or in many other ways. Note that, if a system is inconsistent, the solution set is empty.

(Theorem 2) A system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. If there is at least one free variable, and the system is consistent, the system contains infinitely many solutions.

Chapter 1.3 - Vector Equations

Vectors

A **vector**, for now, is an ordered list of numbers. A matrix with only one column is a **column vector**, or just a vector. Examples of this are:

$$u = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} w \\ y \end{bmatrix}$$

The set of all vectors with two entries is $\mathbb{R}^2$. The set of all vectors with n entries is $\mathbb{R}^n$. Vectors are equal if all of their corresponding entries are equal (order matters). The vector whose entries are all zero is called the **zero vector**, denoted 0.

Summing two vectors is obtained by adding entries:

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Scaling a vector is done by scaling each entry:

$$2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

Omitted here is geometric descriptions of vectors, vectors in $\mathbb{R}^3$, and vectors in $\mathbb{R}^n$.

Linear Combinations

Given vectors $v_1, v_2, ...v_n \epsilon \mathbb{R}^n$ and scalars $c_1, c_2, ...c_n \epsilon \mathbb{R}$, the vector y:

$$\boxed{y = c_1 v_1 + c_2 v_2 + ... + c_n v_n} \tag{2}$$

Is a **linear combination** of $v_1, v_2, ...v_n$ with **weights** $c_1, c_2, ...c_n$. The set of all linear combinations of $v_1, v_2, ...v_n$ is denoted by $Span(v_1, v_2, ...v_n)$, and is called the **subset of** $\mathbb{R}^n$ **spanned by** $v_1, v_2, ...v_n$. If b is in $Span(v_1, v_2, ...v_n)$, there is a solution to the equation:

$$\boxed{x_1 v_1 + ... + x_n v_b = b} \tag{3}$$

Or, equivalently, the augmented matrix $[v_1 ... v_n \; b]$ is consistent.

Chapter 1.4 - The Matrix Equation

If A is an m x n matrix, with columns $v_1 ... v_n$, and $x \epsilon \mathbb{R}^n$, then the product of A and x, denoted Ax, is the linear combination of the columns of A using the entries of x as weights. Or:

$$\boxed{Ax = \begin{bmatrix} v_1 ... v_n \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = x_1 v_1 + ... + x_n v_n} \tag{4}$$

A **matrix equation** is in the form $Ax = b$.

Existence of Solutions

$Ax = b$ has a solution if and only if b is a linear combination of the columns of A. Therefore, "Is b in $Span(a_1 ... a_n)$?" is the same as "Is $Ax = b$ consistent?" Important to determine, too, is if the matrix equation is consistent for all b.

(Theorem 3) The matrix equation $Ax = b$ has the same solution set as the vector equation $x_1a_1 + ... + x_na_n = b$ and the same solution set as the augmented matrix $[a_1 \; ... \; a_n \; b]$.

(Theorem 4) The following are logically equivalent:

1. $Ax = b$ is consistent for all b,
2. Every b is a linear combination of the columns of A,
3. The columns of A span $\mathbb{R}^m$,
4. A has a pivot position in every row.

The **identity matrix**, denoted by I, is the n x n matrix where the diagonal is composed of 1s, and every other entry is 0. Note that $Ix = x$ for all x.

(Theorem 5) If A is an m x n matrix, $u, v \epsilon \mathbb{R}^n$, and $c \epsilon \mathbb{R}$, then:

1. $A(u + v) = Au + Av$.
2. $A(cu) = cA(u)$.

Omitted here is the computation of matrix-vector multiplication.

Chapter 1.5 - Solution Sets of Linear Systems

Homogeneous Linear Systems

A system is **homogeneous** if it can be written in the form $Ax = 0$. This system always has the trivial solution $x = 0$. If the system has any free variables, it has a **nontrivial solution**, or some nonzero x that satisfies $Ax = 0$.

The solution set of a homogeneous equation can always be expressed as the span of some vectors.

Solutions of Nonhomogenous Systems

The general solution to a nonhomogenous system with many solutions can be written in **parametric vector form** as one vector plus the span of more vectors. For example, the solutions to the following system, $Ax = b$, where:

$$A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix} \quad b = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$$

Can be written as:

$$x = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

The equation $x = p + tv$ describes the solution set of $Ax = b$ in parametric vector form. See the textbook for a fuller, more geometric explanation of this form.

(Theorem 6) To generalize, if the solution set of $Ax = b$ is the set of all vectors in the form $w = p + v_h$, then v_h is the set of all solutions to the homogenous equation $Ax = 0$. Essentially, p can be thought of as the translation of the solution set of $Ax = 0$ to the solution set of $Ax = b$.

To write a solution set in parametric form: (1) Obtain reduced echelon form, (2) Express basic variables in terms of free variables, (3) Write x as a vector with free variables and their combinations as entries, (4) Decompose x into a linear combination of vectors with the free variables as parameters.

Chapter 1.7 - Linear Independence

A set of vectors are **linearly independent** if the vector equation $c_1v_1 + ... + c_nv_n = 0$ has only the trivial solution. The set is **linearly dependent** if the equation has some solution where not all $c_1...c_n = 0$. The equation above is called a **linear dependence relation** among the vectors $v_1...v_n$ if the system is linearly dependent.

Note that we can treat the columns of a matrix, A, in the same way that we can treat some set of vectors as we did above. Therefore:

The columns of matrix A are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.

A set containing just one vector is linearly independent as long as that one vector isn't the zero vector, since $0x = 0$ has more than just the trivial solution.

A set containing two vectors is linearly independent as long as one vector isn't a scalar multiple of the other. Proof omitted.

(Theorem 7) A set containing n vectors is linearly independent as long as no vector is a linear combination of the other vectors in the set, and no vector is the zero vector. It is linearly dependent if there is some vector that is a linear combination of the others, or if the set contains the zero vector.

(Theorem 8) If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

(Theorem 9) If a set contains the zero vector, it is linearly dependent.

Chapter 1.8 - Introduction to Linear Transformations

While the difference between a matrix equation and the associated vector equation is merely notation, there are matrix equations that are not connected to linear combinations of vectors. From here on, think of A as an object that acts on a vector, x, to produce a new vector, Ax.

To abstract this concept, we define a **transformation** or **function** or **mapping** as a rule that assigns each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of T, and $\mathbb{R}^m$ the **codomain** of T. This is implied by the notation $T : \mathbb{R}^n \to \mathbb{R}^m$. The set of all images $T(x)$ is called the **range** of T.

Mappings associated with matrix multiplication - that is, that $T(x)$ is computed as Ax - are called matrix transformations.

Linear Transformations

A transformation is **linear** if (1) $T(u + v) = T(u) + T(v)$, and (2) $T(cu) = cT(u)$ for all valid inputs. The following naturally follow: (3) T(0) = 0, (4) T(cu + dv) = cT(u) + dT(v)

Every matrix transformation is linear. Linear transformations preserve vector addition and scalar multiplication.

Chapter 1.9 - The Matrix of a Linear Transformation

Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is actually a matrix transformation, and we can find the **standard matrix**, A, of the transformation, T. The key to finding A is observing the effects of T on the columns of the identity matrix, denoted $e_1...e_n$

(Theorem 10)

$$\boxed{T(x) = [T(e_1)...T(e_n)]\begin{bmatrix}x_1\\ ...\\ x_n\end{bmatrix} = Ax} \tag{5}$$

Existence and Uniqueness Questions

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto**, or **surjective**, if each $b \in \mathbb{R}^m$ is the image of at least one $x \in \mathbb{R}^n$. Alternatively, $Ax = b$ is consistent for all b.

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-to-one**, or **injective**, if each $b \in \mathbb{R}^m$ is the image of at most one $x \in \mathbb{R}^n$. Alternatively, the columns of A are linearly independent.

(Theorem 11) A linear map T is one-to-one if and only if $T(x) = 0$ has only the trivial solution.

(Theorem 12) A linear map T is onto if and only if the columns of A span $\mathbb{R}^m$, and is one-to-one if and only if the columns of A are linearly independent.

Chapter 2 - Matrix Algebra

Chapter 2.1 - Matrix Operations

Definitions

The entry in the ith row and jth column of a matrix A is denoted a_{ij} and is called the (i, j)-entry of A. The **main diagonal** of A consists of the **diagonal entries** of A, $a_{11}, a_{22}....$ A **diagonal matrix** is an n x n matrix whose diagonal entries are all nonzero. A matrix whose entries are all zero is a **zero matrix**, denoted 0.

Sums and Scalar Multiples

Two matrices are equal if they are the same size and each corresponding entry is equal. The sum of two matrices $A + B$ is the matrix whose columns are the sums of the corresponding columns in A and B. The scalar multiple cA is the matrix whose entries are all c times the corresponding entry in A.

(Theorem 1) Let A, B, C be matrices of the same size, and let c, d be scalars. Then:

1. $A + B = B + A$.
2. $(A + B) + C = A + (B + C)$.
3. $A + 0 = A$.
4. $c(A + B) = cA + cB$.
5. $(c + d)A = cA + dA$.
6. $c(dA) = (cd)A$.

The above properties stem from the properties of their columns: vectors.

Matrix Multiplication

Applying B onto x produces the vector Bx. Applying A onto Bx produces the vector ABx. Instead, we could apply the matrix AB, produce by matrix multiplication, to x to obtain ABx.

If A is m x n, and B is n x p, and x is in $\mathbb{R}^p$, then:

$$
\begin{aligned}
Bx &= x_1b_1 + ... + x_pb_p \\
A(Bx) &= A(x_1b_1) + ... + A(x_pb_p) \\
A(Bx) &= x_1Ab_1 + ... + x_pAb_p \\
A(Bx) &= [Ab_1...Ab_p]x
\end{aligned}
$$

$$
\boxed{AB = A[b_1...b_p] = [Ab_1...Ab_p]} \tag{6}
$$

Note that AB has the same number of rows as A and the same number of columns as B, or, equivalently, AB is a m x p matrix.

Another way of computing AB is entry by entry using the following definition: $(AB)_{ij} = a_{i1}b_{1j} + ... + a_{in}b_{nj}$.

Properties of Matrix Multiplication

(Theorem 2) Let A be an m x n matrix, and let B and C be appropriate sizes. Then:

1. $A(BC) = (AB)C$.
2. $A(B + C) = AB + AC$.
3. $(B + C)A = BA + CA$.
4. $r(AB) = (rA)B = A(rB)$.
5. $IA = A = AI$.

(Warnings) (1) $AB \neq BA$, (2) $AB = AC$ does not imply $B = C$, (3) $AB = 0$ does not imply $A = 0$ or $B = 0$.

A **power** of a matrix is defined as: $A^k = A...A$, where there are $k - 1$ multiplications of A on A.

The **transpose** of a matrix, A, denoted A^T, is the matrix whose columns are formed by the corresponding rows of A.

(Theorem 3) Let A and B be appropriately sized matrices. Then:

1. $(A^T)^T = A$,
2. $(A + B)^T = A^T + B^T$,
3. $(rA)^T = rA^T$,
4. $(AB)^T = B^TA^T$

Chapter 2.2 - The Inverse of a Matrix

A matrix A is **invertible** if there exists some matrix C such that $AC = CA = I$. The inverse of a matrix is unique, since if there were another inverse of A, denoted B, then $B = BI = B(AC) = (BA)C = IC = C$. The inverse of some matrix A is denoted A^{-1}. A non-invertible matrix is sometimes called **singular**, and an invertible matrix is sometimes called **nonsingular**. Note that only square matrices can be invertible.

For a 2x2 matrix, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the inverse A^{-1} can be computed using the following equation:

$$
\boxed{A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \tag{7}
$$

(Theorem 4) The value $ad - bc$ is known as the determinant, and as long as it is not 0, the matrix is invertible.

(Theorem 5) If A is invertible, then for each b in $\mathbb{R}^n$, $Ax = b$ has the unique solution $x = A^{-1}b$.

(Theorem 6)

1. If A is invertible, then A^{-1} is invertible, and $(A^{-1})^{-1} =$ A;
2. If A and B are invertible, then so is AB. $(AB)^{-1} = B^{-1}A^{-1}$;
3. If A is invertible, so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Elementary Matrices

An **elementary matrix** is one obtained by performing row operations on the identity matrix. If elementary row operations are performed on a matrix A, the resulting matrix can be written as EA, where E is the matrix created by applying the same row operations on I.

(Theorem 7) A is invertible if and only if it is row equivalent to I, and in this case, any series of row operations that transforms A into I also transforms I into A^{-1}.

To find A^{-1} using this fact, row reduce the augmented matrix $[A\ I]$. The resulting matrix will be $[I\ A^{-1}]$.

Chapter 2.3 - Characterizations of Invertible Matrices

(Theorem 8 - The Invertible Matrix Theorem) Let A be a square matrix. The following are equivalent:

1. A is invertible.
2. A is row equivalent to I;
3. A has n pivot positions;
4. $Ax = 0$ has only the trivial solution;
5. The columns of A are linearly independent;
6. The linear transformation $x \to Ax$ is one-to-one;
7. The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$;
8. The columns of A span $\mathbb{R}^n$;
9. The linear transformation $x \to Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$;
10. There is an n x n matrix C such that $CA = I$;
11. There is an n x n matrix D such that $AD = I$;
12. A^T is invertible.

If any of the above are true, the rest must be true; if any are false, then the rest must also be false. Note that the Invertible Matrix Theorem only applies to square matrices.

Invertible Linear Transformations

We can generalize invertibility to linear transformations. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible is there exists a function S such that $S(T(x)) = T(S(x)) = x$. If such a function exists, we call is T^{-1}.

(Theorem 9) If A is the standard matrix for a linear transformation T, then T is invertible if and only if A is invertible. In that case, then $S(x) = A^{-1}x$.

Chapter 3 - Determinants

Chapter 3.1 - Introduction to Determinants

The determinant of a 2x2 matrix was discussed before ($ad - bc$). The determinant of a 3x3 matrix uses a recursive technique:

$$\Delta = a_{11}detA_{11} - a_{12}detA_{12} + a_{13}detA_{13}$$

Where a_{ij} is the (i, j)-entry in A, while A_{ij} is the matrix obtained by deleting row i and column j.

For $n \geq 2$, the **determinant** of an n x n matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{ij}detA_{ij}$. In symbols:

$$\boxed{\sum_{j=1}^{n}(-1)^{i+j}\ a_{ij}\ detA_{ij}} \tag{8}$$

For the following theorem, definiting the **(i, j)-cofactor** of A, C_{ij} is helpful.

$$\boxed{C_{ij} = (-1)^{i+j}\ detA_{ij}} \tag{9}$$

Note that, for any level of determinant computation, i or j should be held constant. That is, you should be traversing a row or column as you compute. If i were chosen to be some arbitrary p, the above formula would be called the **cofactor expansion across the p-th row** of A. If j were chosen to be some arbitrary q, the above formula would be called the **cofactor expansion down the q-th column** of A. Generally, the for or column with the most zeros is chosen, since it simplifies computation.

(Theorem 1) Expansion along the i-th row is:

$$\boxed{detA = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}} \tag{10}$$

And down the j-th column is:

$$\boxed{detA = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}} \tag{11}$$

(Theorem 2) If A is a triangular matrix, then $detA$ is the product of the entries on the main diagonal of A.

Chapter 3.2 - Properties of Determinants

This section doesn't have much in-depth explanation, and the theorems speak for themselves.

(Theorem 3) Let A be a square matrix.

1. If row replacement is performed on A to produce B, $detB = detA$.
2. If two rows of A are interchanged to produce B, $detB = -detA$.
3. If a row of A is multiplied by k to produce B, $detB = kdetA$.

Often times, we can use row operations to reduce a matrix to a more workable one. If we can record the effects of our row operations on our determinant, we can make it much easier to determine the determinant of our matrix.

(Theorem 4) A square matrix A is invertible if and only if $detA \neq 0$

This adds another clause to the Invertible Matrix Theorem.

Column Operations

We can perform operations on the columns of a matrix (**column operations** in the same way that we can perform row operations. This is because:

(Theorem 5) If A is an n x n matrix, then $detA = detA^T$

Determinants and Matrix Products

(Theorem 6) If A and B are n x n matrices, then $detAB = (detA)(detB)$

Proofs omitted.

Chapter 4 - Vector Spaces

Chapter 4.1 - Vector Spaces and Subspaces

The previous chapters have relied heavily on the properties of $\mathbb{R}^n$; however, many of the same theories carry over to general mathematical systems.

A **vector space** is a nonempty set V of vectors on *addition* and *scalar multiplication* are defined. The ten axioms below must hold for a vector space to be valid for some vectors u and v:

1. $u + v$ is in V.
2. $u + v = v + u$.
3. $(u + v) + w = u + (v + w)$.
4. There is a zero vector 0 in V such that $u + 0 = u$.
5. For each u in V, there is a vector $-u$ such that $u + (-u) = 0$.
6. cu is in V.
7. $c(u + v) = cu + cv$.
8. $(c + d)u = cu + du$.
9. $c(du) = (cd)u$.
10. $1u = u$.

Subspaces

Sometimes, a vector space consists of a subset of vectors from a larger vector space. This subset of vectors is defined as a **subspace**, often denoted H. In this case, only three axioms need to be fulfilled:

1. There is a zero vector 0 in H such that $u + 0 = u$.
2. $u + v$ is in H.
3. cu is in H.

A Subspace Spanned by a Set

The most common way of describing a subspace is as the span of a set of vectors. It is guaranteed that the span of a set of vectors is a subspace.

(Theorem 1) If $v_1 ... v_p$ are in V, then $Span(v_1 ... v_p)$ is a subspace of V.

$Span(v_1...v_p)$ is defined as the subspace spanned or generated by $\{v_1...v_p\}$. A spanning set for H is a set $\{v_1...v_p\}$ such that $H = Span(v_1...v_p)$.

Chapter 4.2 - Null Spaces, Column Spaces, and Linear Transformations

Generally, subspaces arise either as a solution set or a span. Two specific kinds of subspaces are null spaces and columns spaces.

The Null Space of a Matrix

The **null space** of a matrix, denoted $NulA$, is defined as the solution set to the homogenous equation $Ax = 0$. Essentially, the null space describes all vectors that, when multiplied by A, return 0. This is an example of a subspace that is a solution set.

(**Theorem 2**) The null space of an m x n matrix A is a subspace of $\mathbb{R}^n$.

Solving for the null space of a matrix is as simple as computing the solution set for $Ax = 0$, then expressing that set as a span.

The Column Space of a Matrix

The **column space** of a matrix, denoted $ColA$, is the set of linear combinations of A. Equivalently: $ColA = Span\{a_1...a_n\}$.

(**Theorem 3**) The column space of an m x n matrix A is a subspace of $\mathbb{R}^n$.

Note that the column space of an m x n matrix A spans $\mathbb{R}^m$ if and only if the equation $Ax = b$ has a solution for all b in R^m.

The Kernel of a Linear Transformation

For some linear transformation T, the **kernel** of T is the set of all u in V such that $T(u) = 0$.

Chapter 4.3 - Linearly Independent Sets; Bases

Linear independence, linear dependence, and the idea of a linear dependence relation were defined in a previous section. We will introduce a new way of thinking about linear dependence:

(**Theorem 4**) A set of two or more vectors, $\{v_1...v_p\}$, with $v_1 \neq 0$, is linearly dependent if and only if some v_j is a linear combination of the preceding vectors, $\{v_1...v_{j-1}\}$.

In a general vector space, we cannot exploit the homogeneous equation $Ax = 0$. Therefore, we must rely on Theorem 4.

Let H be a subspace of a vector space V. An indexed set of vectors $B = \{b_1...b_n\}$ **basis** for H if (1) B is a linearly dependent set, and (2) the subspace spanned by B is H: $H = Span(b_1...b_n)$.

The Spanning Set Theorem

A basis is essentially an "efficient" spanning set of vectors. To construct a basis from a linearly dependent set of vectors that span our desired subspace, we can discard unneeded vectors without affecting our span until we reach linear independence.

(**Theorem 5**) Let $S = v_1...v_p$ be a set in V, and let $H = Span(v_1...v_p)$. (1) If a vector in S is a linear combination of the remaining vectors in S, then the set formed from S by removing that vector still spans H. (2) If $H \neq \{0\}$, some subset of S spans H.

Bases for $NulA$ and $ColA$

Finding a basis for the null space of a matrix is trivial, since the general method for computing $NulA$ produces a linearly independent set. Finding a basis for the column space of a matrix is more difficult, but boils down to the following theorem:

(**Theorem 6**) The pivot columns of A form a basis for $ColA$.

Chapter 4.4 - Coordinate Systems

One important application of bases is to impose **coordinate systems** on a vector space. If some basis B contains n vectors, then the coordinate system will make V act like $\mathbb{R}^n$, or give another view of V if it is already $\mathbb{R}^n$.

(**Theorem 7**) Let $B = \{b_1...b_n\}$ be a basis for V. For each x in V, there exists a set of scalars $c_1...c_n$ such that $x = c_1b_1 + ... + c_nb_n$.

The coordinates of x relative to the basis B, or the B-coordinates of x, are the weights $c_1...c_n$ such that $x = c_1b_1 + ... + c_nb_n$. The **coordinate mapping** is denoted $[x]_B = \begin{bmatrix} c_1 \\ ... \\ c_n \end{bmatrix}$.

A coordinate system can be interpreted as changing the kind of graph paper we represent our vectors on. While typical graph paper is grid-like in nature, with its "basis vectors" being the standard basis vectors e_1 and e_2. On the other hand, we could have B-graph paper, which has different basis vectors, and through which we would have to express points differently.

Coordinates in $\mathbb{R}^n$

Finding the coordinates of some vector x with respect to some basis B is as simple as finding the weights corresponding to the linear combination of the basis vectors of B that sum to x.

Some matrix that changes the B-coordinates of a vector to the standard coordinates of that vector is called the **change-of-coordinates matrix**. For some basis $B = \{b_1...b_n\}$,

$$\boxed{P_B = [b_1\, b_2...b_n]} \tag{12}$$

Such that the following two are equivalent:

$$x = c_1b_1 + ... + c_nb_n$$
$$\boxed{x = P_B[x]_B} \tag{13}$$

Note that the change-of-coordinates matrix is invertible, so there exists a matrix P_B^{-1} such that $P_B^{-1}x = [x]_b$. This inverse matrix is equivalent to the mapping $x \to [x]_B$.

The Coordinate Mapping

(**Theorem 8**) Let $B = \{b_1...b_n\}$ be a basis for V. The coordinate mapping $x \to [x]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

This theorem essentially states that the coordinate mapping is linear, and that the same properties of linear transformations apply to this map. Moreover, this coordinate mapping is an example of **isomorphism** from V onto W, which essentially means that *every vector space calculation in V is accurately reproduced in W, and vice verse.* These two spaces are indistinguishable as vector spaces, and can be treated identically. As an extension, any vector space with a basis of n vectors is indistinguishable from $\mathbb{R}^n$.

Chapter 4.5 - The Dimension of a Vector Space

The number of basis vectors, n, for some vector space, V, is an intrinsic property of the vector space, called its **dimension**. Moreover, the dimension of a basis is independent of the choice of basis.

(**Theorem 9**) If a vector space V has a basis $\{b_1...b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

This theorem is intuitive - it is impossible for a set of vectors in $\mathbb{R}^n$ to remain linearly independent with more than n vectors, which must also be true for a vector space that can be described by $\mathbb{R}^n$.

(**Theorem 10**) If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

This theorem stems from the idea of dimensionality. A vector space must have its dimensionality preserved - any more vectors, and the basis would become linearly dependent; any fewer, and the basis wouldn't span the vector space.

If V is spanned by a finite set of vectors, then V is **finite-dimensional**, and the dimension of V, denoted $dimV$, is the number of vectors in a basis for V. The dimension of the zero vector space is defined as zero. If V cannot be spanned by a finite set, then V is **infinite-dimensional**.

Subspaces of a Finite-Dimensional Space

(**Theorem 11**) Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded into a basis for H. H must also be finite dimensional, and $dimH \leq dimV$.

(**Theorem 12**) Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements in V that spans V is automatically a basis for V.

This theorem lets us determine whether or not a set is a basis without having to determine every property of a basis, as long as the number of vectors is correct.

The Dimensions of $NulA$ and $ColA$

The dimension of $ColA$ for some matrix A is equal to the number of pivot columns in A. The dimension of $NulA$ for some matrix A is equal to the number of free variables in the equation $Ax = 0$. The relation between these two dimensions will be revisited in the next section.

Chapter 4.6 - Rank

The Row Space

The **row space** of a matrix $A \epsilon M_{m,n}(\mathbb{R})$, denoted $RowA$, is the set of all linear combinations of the row vectors of A. Another way of thinking about the row space is the column space of the transpose of A: $RowA = ColA^T$.

(**Theorem 13**) If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B for a basis for the row space of both A and B.

The Rank Theorem

The **rank** of A is the dimension of the column space of A, $ColA$. The dimension of the null space of A is known as the **nullity** of A, but this term won't come up again.

(**Theorem 14**) The dimensions of the column space and the row space of an m x n matrix A are equal. This common dimension, the rank of A, equals the number of pivot positions in A and satisfies the equation $rankA + dimNulA = n$.

Another way of thinking of this theorem is that the number of pivot columns ($rankA$) and the number of nonpivot columns ($dimNulA$) equals the total number of columns (n).

Rank and the Invertible Matrix Theorem

We can now expand our previous theorem:

(**The Invertible Matrix Theorem (*cont.*)**) Let A be an n x n matrix. The following are equivalent:

1. The columns of A for a basis for $\mathbb{R}^n$.
2. $ColA = \mathbb{R}^n$.
3. $dimColA = n$.
4. $rankA = n$.
5. $NulA = \{0\}$.
6. $dimNulA = 0$.

Omitted from this theorem are remarks about A^T and row space. Their inclusion would lead to an incredibly long theorem.

Chapter 4.7 - Change of Basis

We know how to describe a vector, x, with respect to a basis as $[x]_B$. However, if we want to change our basis from, say, B to C, we have to understand how $[x]_B$ and $[x]_C$ are related.

(**Theorem 15**) Let $B = \{b_1...b_n\}$ and $C = \{c_1...c_n\}$ be bases of a vector space V. Then there is a unique n x n matrix $\underset{C \leftarrow B}{P}$ such that

$$\boxed{[x]_C = \underset{C \leftarrow B}{P}[x]_B} \tag{14}$$

The columns of $\underset{C \leftarrow B}{P}$ are the C-coordinate vectors of the basis vectors of B. Or:

$$\boxed{\underset{C \leftarrow B}{P} = [[b_1]_C\ [b_2]_C\ ...\ [b_n]_C]} \tag{15}$$

The matrix $\underset{C \leftarrow B}{P}$ is called the **change-of-coordinates matrix from** B **to** C. Applying it to a vector defined in B-coordinates defines is in C-coordinates. We know that the columns of $\underset{C \leftarrow B}{P}$ are linearly independent, since they are formed from the coordinates of the basis of B. We also know that $\underset{C \leftarrow B}{P}$ is invertible, since it is square. Therefore, there exists a matrix $\underset{C \leftarrow B}{P}^{-1}$ such that:

$$\underset{C \leftarrow B}{P}^{-1}[x]_C = [x]_B$$

Since $\underset{C \leftarrow B}{P}^{-1}$ is the matrix that converts C-coordinates to B-coordinates:

$$\boxed{\underset{C \leftarrow B}{P}^{-1} = \underset{B \leftarrow C}{P}} \tag{16}$$

Change of Basis in $\mathbb{R}^n$

If we were to convert B-coordinates to standard coordinates (E-coordinates), we find that $\underset{E \leftarrow B}{P} = P_B$.

We can also use our knowledge of change-of-coordinates matrices to find the following:

$$P_B[x]_B = x \quad P_C[x]_C = x \quad [x]_C = P_C^{-1}x$$
$$[x]_C = P_C^{-1}x = P_C^{-1}P_Bx$$
$$\boxed{\underset{C \leftarrow B}{P} = P_C^{-1}P_B = \underset{C \leftarrow E}{P}\underset{E \leftarrow B}{P}} \tag{17}$$

Chapter 5 - Eigenvalues and Eigenvectors

Chapter 5.1 - Eigenvectors and Eigenvalues

An **eigenvector** of an n x n matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ. The scalar λ is called an **eigenvalue** of A is there is a nontrivial solution x for $Ax = \lambda x$. Determining whether a vector is an eigenvector or if a scalar is an eigenvalue is easy. For the former, apply the matrix to the vector and see if the result is a scalar multiple of the original vector. For the latter, check if the matrix $A - \lambda I$ is invertible.

For some eigenvalue λ of A, the **eigenspace** corresponding to λ is the set of solutions to the equation $(A-\lambda I)x = 0$, or the null space of $A - \lambda I$.

(**Theorem 1**) The eigenvalues of a triangular matrix are the entries on its main diagonal.

We can verify this theorem quite easily by using the ideas of determinants. If a matrix is triangular, subtracting a value on its diagonal from itself sets the determinant equal to zero, thereby making that value an eigenvalue.

To have an eigenvalue of zero, the equation $Ax = 0$ must have a nontrivial solution; in other words, A is not invertible.

(**Theorem 2**) Is $v_1...v_n$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1...\lambda_n$ of an n x n matrix A, then the set $\{v_1...v_n\}$ is linearly independent.

Eigenvectors with distinct eigenvalues are linearly independent - if they were linearly dependent, they would lie along the same span, and would therefore have the same eigenvalue. This is a contradiction.

Chapter 5.2 - The Characteristic Equation

Previously, we found that verifying whether or not a vector or scalar was an eigenvector or eigenvalue respectively was quite easy. However, computing all eigenvalues and eigenvectors requires more work. For this, we use the **characteristic equation** of a matrix. First, a refresher on determinants.

(**The Invertible Matrix Theorem (*cont.*)**) Let A be an n x n matrix. A is invertible if and only if:

1. 0 is not an eigenvalue of A.
2. The determinant of A is not 0.

(**Theorem 3**) Let A and B be n x n matrices. Then:

1. A is invertible if and only if $detA \neq 0$.
2. $detAB = (detA)(detB)$.
3. $detA = detA^T$.
4. If A is triangular, then $detA$ is the product of the entries on the main diagonal.
5. A row replacement operation on A doesn't change its determinant. A row interchange changes the sign of the determinant. A row scaling scales the determinant by the same scalar factor.

The Characteristic Equation

A scalar λ is an eigenvalue of A if and only if $det(A - \lambda I) = 0$. The characteristic equation of A is determined by taking the determinant of $A - \lambda I$. By setting this **characteristic polynomial** equal to zero and solving for the roots, we can determine the eigenvalues of A. The **multiplicity** of an eigenvalue is its multiplicity as a root.

In this class, only real eigenvalues are discussed, but in theory, an n x n matrix has exactly n eigenvalues, some of which may be complex.

Similarity

Two matrices, A and B, are **similar** if there exists an invertible matrix P such that $A = PBP^{-1}$.

(**Theorem 4**) If n x n matrices A and B are similar, then they have the same characteristic polynomial and the same eigenvalues. Note that if two matrices have the same eigenvalues, they may not necessarily be similar, and if two matrices are similar, they may not necessarily be row equivalent.

Chapter 5.3 - Diagonalization

A diagonal matrix is a matrix that only has entries on its main diagonal. It can often be very useful to display the eigenvalue-eigenvector information in a matrix A through the factorization $A = PDP^{-1}$ where D is a diagonal matrix. This allows us to compute A^k for high values of k quickly and efficiently.

A square matrix A is **diagonalizable** if A is similar to a diagonal matrix; that is, there exists some invertible matrix P and some diagonal matrix D such that $A = PDP^{-1}$.

(**Theorem 5**) An n x n matrix A is diagonalizable if and only if A is n linearly independent eigenvectors. In this case, the columns of P are those linearly independent eigenvectors.

A is diagonalizable if and only if there are enough eigenvectors to form a basis for $\mathbb{R}^n$. Such a basis is called an **eigenvector basis** of $\mathbb{R}^n$.

Diagonalizing Matrices

To diagonalize a matrix, the following steps must be followed:

1. Find the eigenvalues of A.
2. Find three linearly independent eigenvectors of A.
3. Construct P from the linearly independent eigenvectors.
4. Construct D from the corresponding eigenvalues.

(**Theorem 6**) An n x n matrix with n distinct eigenvalues is diagonalizable.

Matrices Whose Eigenvalues Are Not Distinct

We saw from Theorem 6 that if an n x n matrix A has n distinct eigenvectors, and if P consists of the eigenvectors of A, P is invertible because its columns are linearly independent, and A is diagonalizable.

(**Theorem 7**) Let A be an n x n matrix whose distinct eigenvalues are $\lambda_1 ... \lambda_p$. Then:

1. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of λ_k.
2. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces of A equals n, which only happens when eigenvalues are distinct, or when the dimension for all eigenvalues matches their multiplicity.
3. If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the collection of vectors in the sets $B_1 ... B_n$ form an eigenvector basis for $\mathbb{R}^n$.

Chapter 5.4 - Eigenvectors and Linear Transformations

Consider two n-dimensional vector spaces, V and W, with bases B and C for each respectively. Let T be a linear transformation from V to W. We wish to derive a relationship between $[x]_B$ and $[T(x)]_C$.

$$[x]_B = \begin{bmatrix} r_1 \\ ... \\ r_n \end{bmatrix}$$

$$T(x) = T(r_1 b_1 + ... + r_n b_n) = r_1 T(b_1) + ... + r_n T(b_n)$$

$$[T(x)]_C = r_1 [T(b_1)]_C + ... + r_n [T(b_n)]_C$$

$$\boxed{[T(x)]_C = M[x]_B} \tag{18}$$

Where

$$\boxed{M = [[T(b_1)]_C \; [T(b_2)]_C \; ... \; [T(b_n)]_C]} \tag{19}$$

The matrix M is a matrix representation of T, called the **matrix for T relative to the bases B and C**.

Linear Transformation from V into V

In the case where $W = V$ and $B = C$, the matrix M is called the **matrix for T relative to B**, or simply the **B-matrix for T**, denoted by $[T]_B$. The B-matrix for T satisfies, for all x in V:

$$\boxed{[T(x)]_B = [T]_B [x]_B} \tag{20}$$

Linear Transformations on $\mathbb{R}^n$

(**Theorem 8**) Suppose $A = PDP^{-1}$, where D is a diagonal n x n matrix. If B is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the B-matrix for the transformation $x \to Ax$.

Similarity of Matrix Representations

Essentially, if A is similar to a matrix C, with $A = PCP^{-1}$, then C is the B-matrix for the transformation $x \to Ax$ when the basis B is formed from the columns of P. The inverse is true - If B is any basis for $\mathbb{R}^n$, then the B-matrix for T is similar to A.

List of Theorems

Chapter 1

(Theorem 1) Reduced echelon matrices are unique.

(Theorem 2) A system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column. If there is at least one free variable, and the system is consistent, the system contains infinitely many solutions.

(Theorem 3) The matrix equation $Ax = b$ has the same solution set as the vector equation $x_1a_1 + ... + x_na_n = b$ and the same solution set as the augmented matrix $[a_1 \; ... \; a_n \; b]$.

(Theorem 4) The following are logically equivalent:

1. $Ax = b$ is consistent for all b,
2. Every b is a linear combination of the columns of A,
3. The columns of A span $\mathbb{R}^m$,
4. A has a pivot position in every row.

(Theorem 5) If A is an m x n matrix, $u, v \epsilon \mathbb{R}^n$, and $c \epsilon \mathbb{R}$, then:

1. $A(u + v) = Au + Av$.
2. $A(cu) = cA(u)$.

(Theorem 6) To generalize, if the solution set of $Ax = b$ is the set of all vectors in the form $w = p + v_h$, then v_h is the set of all solutions to the homogenous equation $Ax = 0$. Essentially, p can be thought of as the translation of the solution set of $Ax = 0$ to the solution set of $Ax = b$.

(Theorem 7) A set containing n vectors is linearly independent as long as no vector is a linear combination of the other vectors in the set, and no vector is the zero vector. It is linearly dependent if there is some vector that is a linear combination of the others, or if the set contains the zero vector.

(Theorem 8) If a set contains more vectors than there are entries in each vector, then the set is linearly dependent.

(Theorem 9) If a set contains the zero vector, it is linearly dependent.

(Theorem 10)

$$T(x) = [T(e_1)...T(e_n)] \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = Ax$$

(Theorem 11) A linear map T is one-to-one if and only if $T(x) = 0$ has only the trivial solution.

(Theorem 12) A linear map T is onto if and only if the columns of A span $\mathbb{R}^m$, and is one-to-one if and only if the columns of A are linearly independent.

Chapter 2

(Theorem 1) Let A, B, C be matrices of the same size, and let c, d be scalars. Then:

1. $A + B = B + A$.
2. $(A + B) + C = A + (B + C)$.
3. $A + 0 = A$.
4. $c(A + B) = cA + cB$.
5. $(c + d)A = cA + dA$.

6. $c(dA) = (cd)A$.

(Theorem 2) Let A be an m x n matrix, and let B and C be appropriate sizes. Then:

1. $A(BC) = (AB)C$.
2. $A(B + C) = AB + AC$.
3. $(B + C)A = BA + CA$.
4. $r(AB) = (rA)B = A(rB)$.
5. $IA = A = AI$.

(Theorem 3) Let A and B be appropriately sized matrices. Then:

1. $(A^T)^T = A$,
2. $(A + B)^T = A^T + B^T$,
3. $(rA)^T = rA^T$,
4. $(AB)^T = B^T A^T$

(Theorem 4) The value $ad - bc$ is known as the determinant, and as long as it is not 0, the matrix is invertible.

(Theorem 5) If A is invertible, then for each b in $\mathbb{R}^n$, $Ax = b$ has the unique solution $x = A^{-1}b$.

(Theorem 6)

1. If A is invertible, then A^{-1} is invertible, and $(A^{-1})^{-1} =$ A;
2. If A and B are invertible, then so is AB. $(AB)^{-1} = B^{-1}A^{-1}$;
3. If A is invertible, so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

(Theorem 7) A is invertible if and only if it is row equivalent to I, and in this case, any series of row operations that transforms A into I also transforms I into A^{-1}.

(Theorem 8 - The Invertible Matrix Theorem) Let A be a square matrix. The following are equivalent:

1. A is invertible.
2. A is row equivalent to I;
3. A has n pivot positions;
4. $Ax = 0$ has only the trivial solution;
5. The columns of A are linearly independent;
6. The linear transformation $x \to Ax$ is one-to-one;
7. The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$;
8. The columns of A span $\mathbb{R}^n$;
9. The linear transformation $x \to Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$;
10. There is an n x n matrix C such that $CA = I$;
11. There is an n x n matrix D such that $AD = I$;
12. A^T is invertible.

If any of the above are true, the rest must be true; if any are false, then the rest must also be false. Note that the Invertible Matrix Theorem only applies to square matrices.

Chapter 3

(Theorem 1) Expansion along the i-th row is:

$$detA = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}$$

And down the j-th column is:

$$detA = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}$$

(Theorem 2) If A is a triangular matrix, then $detA$ is the product of the entries on the main diagonal of A.

(Theorem 3) Let A be a square matrix.

1. If row replacement is performed on A to produce B, $detB = detA$.
2. If two rows of A are interchanged to produce B, $detB = -detA$.
3. If a row of A is multiplied by k to produce B, $detB = kdetA$.

(Theorem 4) A square matrix A is invertible if and only if $detA \neq 0$

(Theorem 5) If A is an n x n matrix, then $detA = detA^T$

(Theorem 6) If A and B are n x n matrices, then $detAB = (detA)(detB)$

(**Vector Space Axioms**) The ten axioms below must hold for a vector space to be valid:

1. $u + v$ is in V.
2. $u + v = v + u$.
3. $(u + v) + w = u + (v + w)$.
4. There is a zero vector 0 in V such that $u + 0 = u$.
5. For each u in V, there is a vector $-u$ such that $u + (-u) = 0$.
6. cu is in V.
7. $c(u + v) = cu + cv$.
8. $(c + d)u = cu + du$.
9. $c(du) = (cd)u$.
10. $1u = u$.

(**Subspace Axioms**) The three axioms below must hold for a subspace to be valid:

1. There is a zero vector 0 in V such that $u + 0 = u$.
2. $u + v$ is in V.
3. cu is in V.

Chapter 4

(Theorem 1) If $v_1...v_p$ are in V, then $Span(v_1...v_p)$ is a subspace of V.

(**Theorem 2**) The null space of an m x n matrix A is a subspace of $\mathbb{R}^n$.

(**Theorem 3**) The column space of an m x n matrix A is a subspace of $\mathbb{R}^n$.

(**Theorem 4**) A set of two or more vectors, $\{v_1...v_p\}$, with $v_1 \neq 0$, is linearly dependent if and only if some v_j is a linear combination of the preceding vectors, $\{v_1...v_{j-1}\}$.

(**Theorem 5**) Let $S = v_1...v_p$ be a set in V, and let $H = Span(v_1...v_p)$. (1) If a vector in S is a linear combination of the remaining vectors in S, then the set formed from S by removing that vector still spans H. (2) If $H \neq \{0\}$, some subset of S spans H.

(**Theorem 6**) The pivot columns of A form a basis for $ColA$.

(**Theorem 7**) Let $B = \{b_1...b_n\}$ be a basis for V. For each x in V, there exists a set of scalars $c_1...c_n$ such that $x = c_1b_1 + ... + c_nb_n$.

(**Theorem 8**) Let $B = \{b_1...b_n\}$ be a basis for V. The coordinate mapping $x \to [x]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

(**Theorem 9**) If a vector space V has a basis $\{b_1...b_n\}$, then any set in V containing more than n vectors must be linearly dependent.

(**Theorem 10**) If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

(**Theorem 11**) Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded into a basis for H. H must also be finite dimensional, and $dimH \leq dimV$.

(**Theorem 12**) Let V be a p-dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements in V that spans V is automatically a basis for V.

(**Theorem 13**) If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B for a basis for the row space of both A and B.

(**Theorem 14**) The dimensions of the column space and the row space of an m x n matrix A are equal. This common dimension, the rank of A, equals the number of pivot positions in A and satisfies the equation $rankA + dimNulA = n$.

(**The Invertible Matrix Theorem (*cont.*)**) Let A be an n x n matrix. The following are equivalent:

1. The columns of A for a basis for $\mathbb{R}^n$.
2. $ColA = \mathbb{R}^n$.
3. $dimColA = n$.
4. $rankA = n$.
5. $NulA = \{0\}$.
6. $dimNulA = 0$.

(**Theorem 15**) Let $B = \{b_1...b_n\}$ and $C = \{c_1...c_n\}$ be bases of a vector space V. Then there is a unique n x n matrix $\underset{C \leftarrow B}{P}$ such that

$$\boxed{[x]_C = \underset{C \leftarrow B}{P}[x]_B}$$

The columns of $\underset{C \leftarrow B}{P}$ are the C-coordinate vectors of the basis vectors of B. Or:

$$\boxed{\underset{C \leftarrow B}{P} = [[b_1]_C\ [b_2]_C\ ...\ [b_n]_C]}$$

Chapter 5

(**Theorem 1**) The eigenvalues of a triangular matrix are the entries on its main diagonal.

(**Theorem 2**) Is $v_1...v_n$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1...\lambda_n$ of an n x n matrix A, then the set $\{v_1...v_n\}$ is linearly independent.

(**The Invertible Matrix Theorem (*cont.*)**) Let A be an n x n matrix. A is invertible if and only if:

1. 0 is not an eigenvalue of A.

2. The determinant of A is not 0.

(**Theorem 3**) Let A and B be n x n matrices. Then:

1. A is invertible if and only if $detA \neq 0$.
2. $detAB = (detA)(detB)$.
3. $detA = detA^T$.
4. If A is triangular, then $detA$ is the product of the entries on the main diagonal.
5. A row replacement operation on A doesn't change its determinant. A row interchange changes the sign of the determinant. A row scaling scales the determinant by the same scalar factor.

(**Theorem 4**) If n x n matrices A and B are similar, then they have the same characteristic polynomial and the same eigenvalues. Note that if two matrices have the same eigenvalues, they may not necessarily be similar, and if two matrices are similar, they may not necessarily be row equivalent.

(**Theorem 5**) An n x n matrix A is diagonalizable if and only if A is n linearly independent eigenvectors. In this case, the columns of P are those linearly independent eigenvectors.

(**Theorem 6**) An n x n matrix with n distinct eigenvalues is diagonalizable.

(**Theorem 7**) Let A be an n x n matrix whose distinct eigenvalues are $\lambda_1 ... \lambda_p$. Then:

1. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of λ_k.
2. The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces of A equals n, which only happens when eigenvalues are distinct, or when the dimension for all eigenvalues matches their multiplicity.
3. If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the collection of vectors in the sets $B_1 ... B_n$ form an eigenvector basis for $\mathbb{R}^n$.

(**Theorem 8**) Suppose $A = PDP^{-1}$, where D is a diagonal n x n matrix. If B is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the B-matrix for the transformation $x \to Ax$.

List of Equations

$$\boxed{a_1x_1 + a_2x_2 + ... + a_nx_n = b} \tag{1}$$

$$\boxed{y = c_1v_1 + c_2v_2 + ... + c_nv_n} \tag{2}$$

$$\boxed{x_1v_1 + ... + x_nv_b = b} \tag{3}$$

$$\boxed{Ax = \begin{bmatrix} v_1 ... v_n \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = x_1v_1 + ... + x_nv_n} \tag{4}$$

$$\boxed{T(x) = [T(e_1)...T(e_n)] \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = Ax} \tag{5}$$

$$\boxed{AB = A[b_1...b_p] = [Ab_1...Ab_p]} \tag{6}$$

$$\boxed{A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \tag{7}$$

$$\boxed{\sum_{j=1}^{n} (-1)^{i+j}\ a_{ij}\ detA_{ij}} \tag{8}$$

$$\boxed{C_{ij} = (-1)^{i+j}\ detA_{ij}} \tag{9}$$

$$\boxed{detA = a_{i1}C_{i1} + a_{i2}C_{i2} + ... + a_{in}C_{in}} \tag{10}$$

$$\boxed{detA = a_{1j}C_{1j} + a_{2j}C_{2j} + ... + a_{nj}C_{nj}} \tag{11}$$

$$\boxed{P_B = [b_1\, b_2 ... b_n]} \tag{12}$$

$$\boxed{x = P_B[x]_B} \tag{13}$$

$$\boxed{[x]_C = \underset{C \leftarrow B}{P}[x]_B} \tag{14}$$

$$\boxed{\underset{C \leftarrow B}{P} = [[b_1]_C\, [b_2]_C ... [b_n]_C]} \tag{15}$$

$$\boxed{\underset{C \leftarrow B}{P}^{-1} = \underset{B \leftarrow C}{P}} \tag{16}$$

$$\boxed{\underset{C \leftarrow B}{P} = P_C^{-1}P_B = \underset{C \leftarrow E}{P}\underset{E \leftarrow B}{P}} \tag{17}$$

$$\boxed{[T(x)]_C = M[x]_B} \tag{18}$$

$$\boxed{M = [[T(b_1)]_C\, [T(b_2)]_C ... [T(b_n)]_C]} \tag{19}$$

$$\boxed{[T(x)]_B = [T]_B[x]_B} \tag{20}$$

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