CSC 225

Algorithms and Data Structures I
Fall 2014
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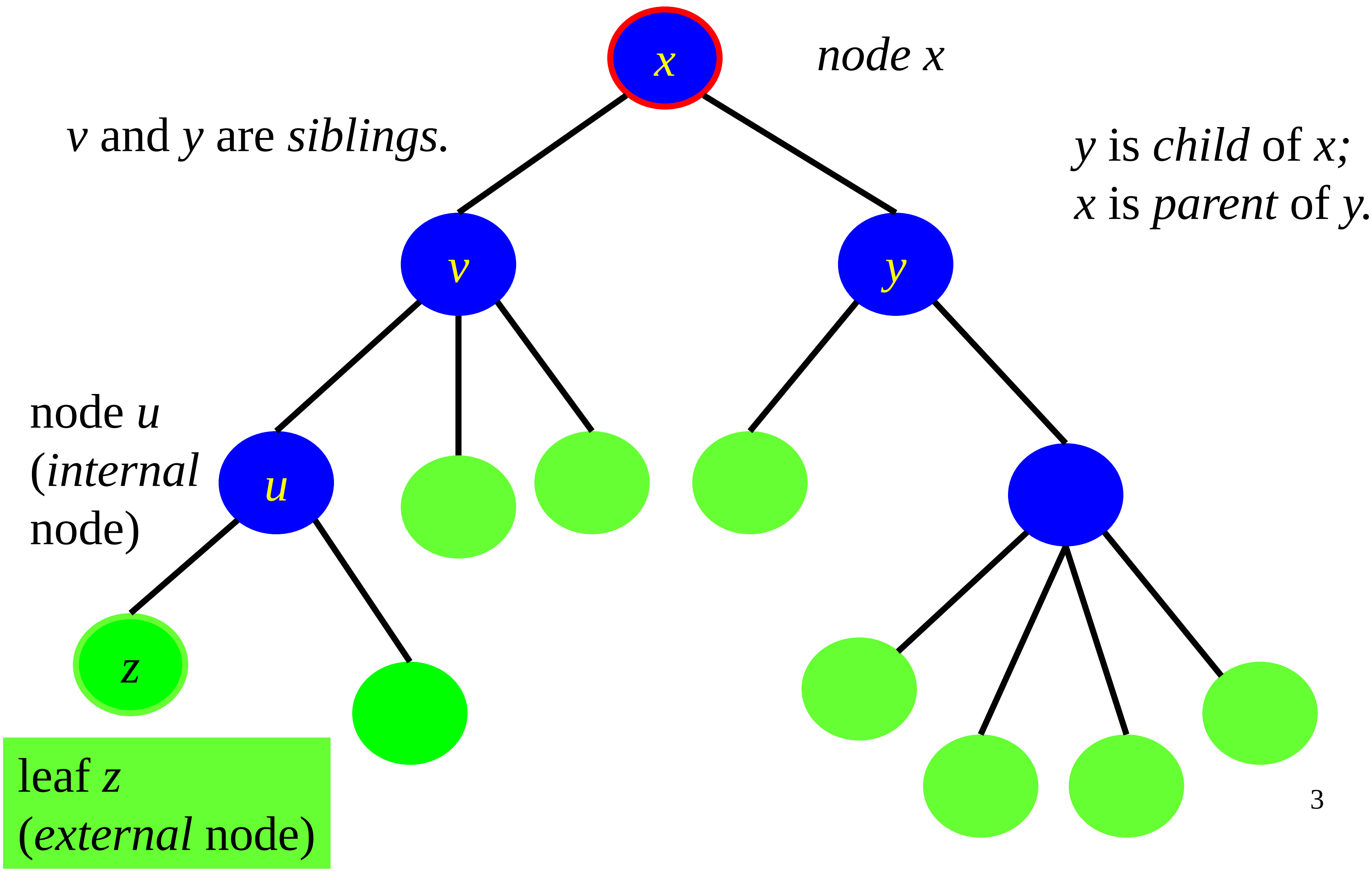
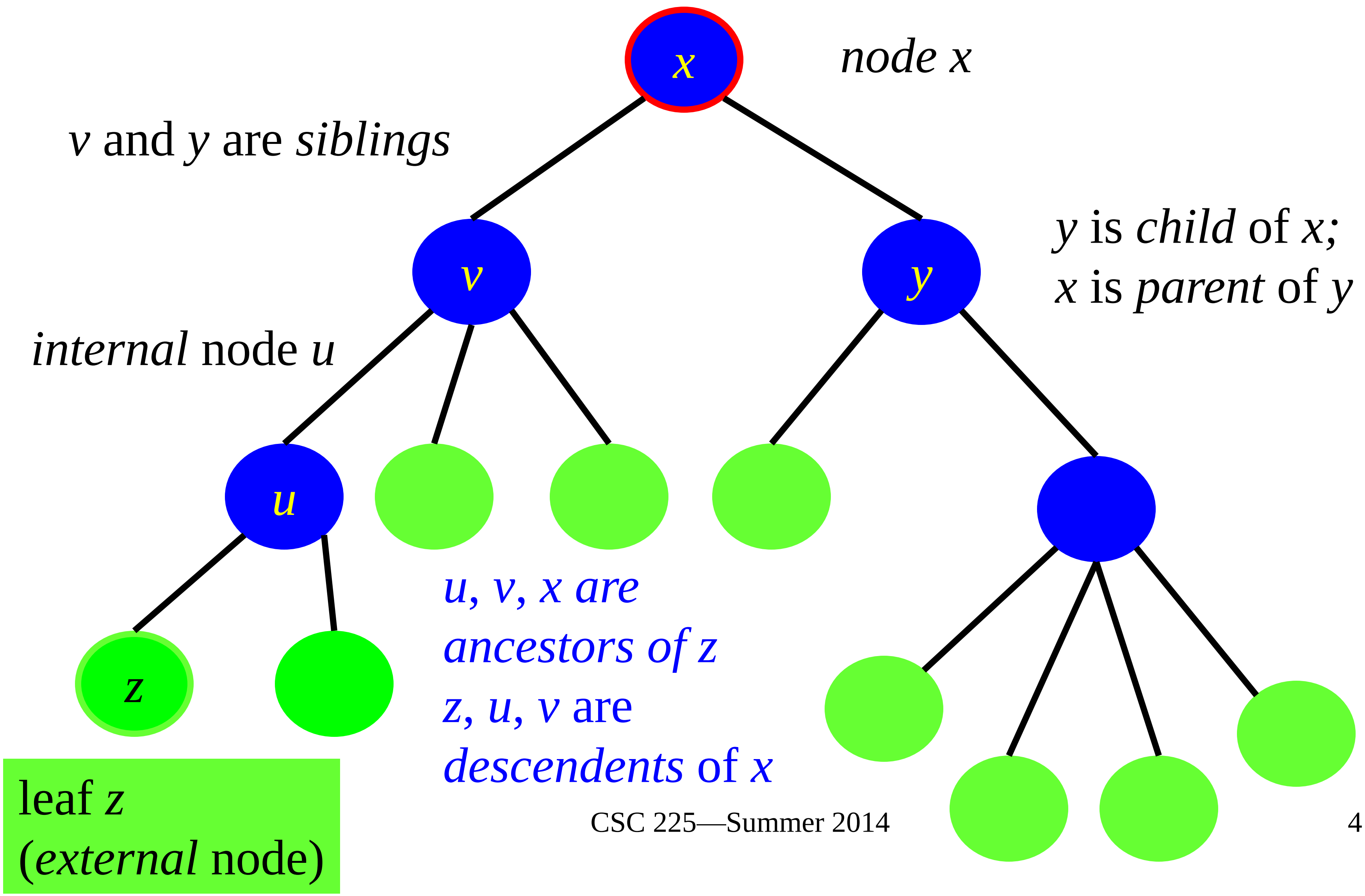
Trees

- A (*rooted*) *tree T* is a set of *nodes* in a *parent-child relationship* with the following properties:
  - ➢ *T* has a special node *r*, called the *root* of *T*
  - ➢ Each node *v* of *T* different from *r* has a *parent* node *u*

Trees


node *x*
v and *y* are *siblings*.
y is *child* of *x*;
x is *parent* of *y*.
x
v
y
node *u*
(*internal*
node)
u
z
leaf *z*
(*external* node)

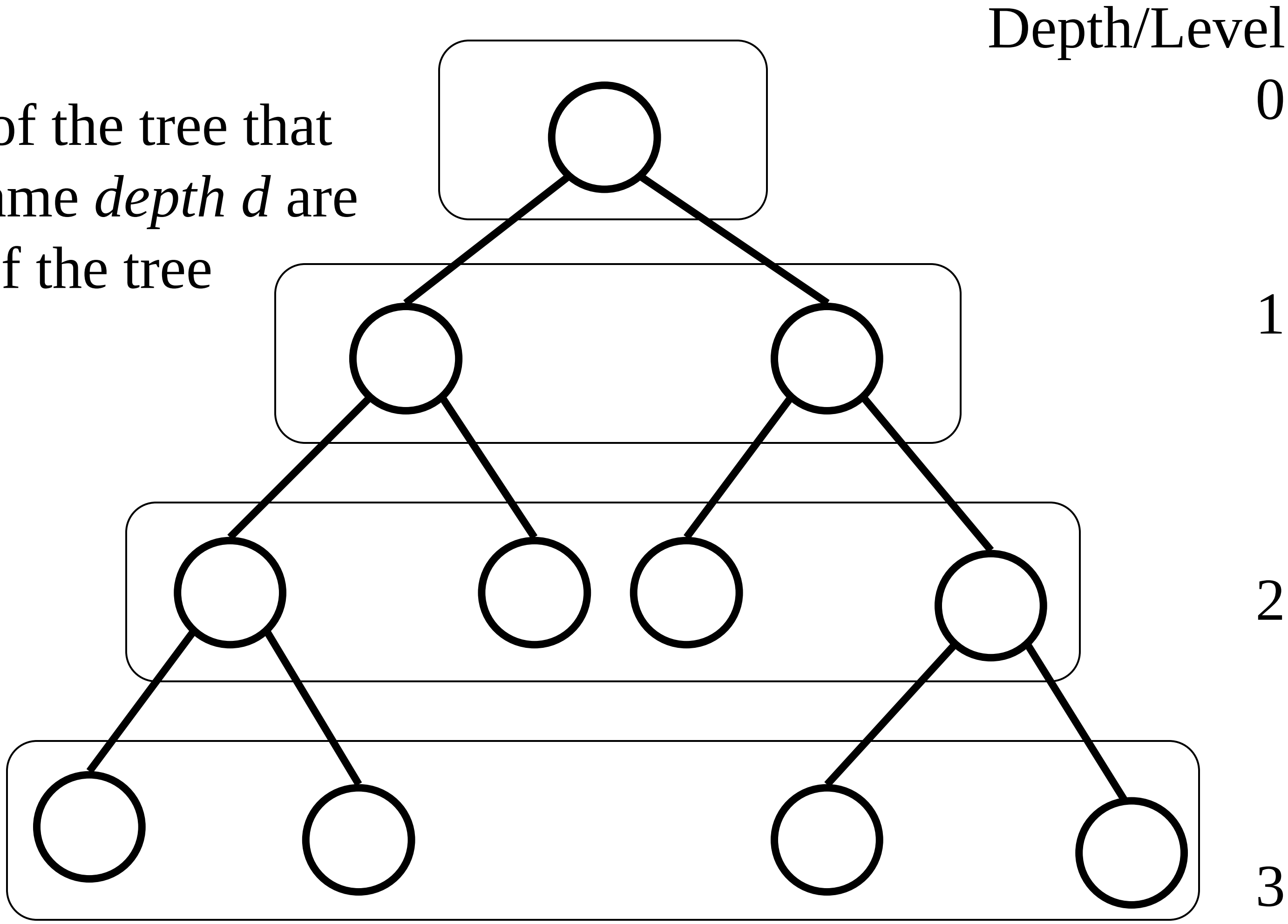
Trees



x
node x
v and y are siblings
v
y
y is child of x;
x is parent of y
internal node u
u
z
u, v, x are
ancestors of z
z, u, v are
descendents of x
leaf z
(external node)

Depth and Levels in Trees

All nodes of the tree that have the same *depth d* are at *level d* of the tree



Height of a Tree

Definition: The *height* of a a tree *T* rooted at node *v* is (recursively) defined to be

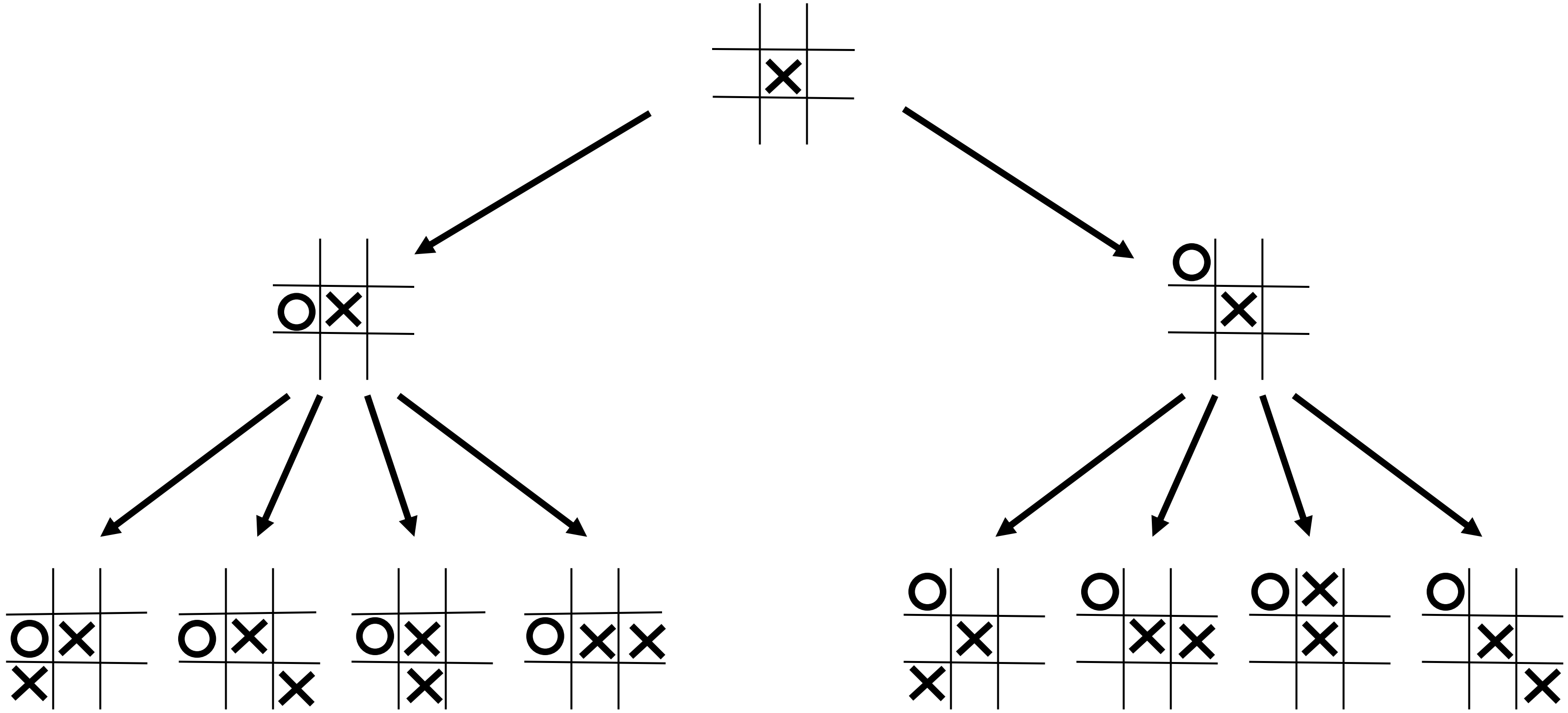
- The height is 0 if *v* is a leaf node
- The height is equal to 1 plus the maximum height of any child of *v*, otherwise.

Applications of Trees

- Phylogenetic trees
- Data structure (search trees)
- Search trees for exponential algorithms (branch-and-bound techniques, fixed-parameter tractable algorithms)
- Visualization of algorithms (and tool for complexity analysis)
- Decision trees
- Parse trees
- Expression trees
- Forests

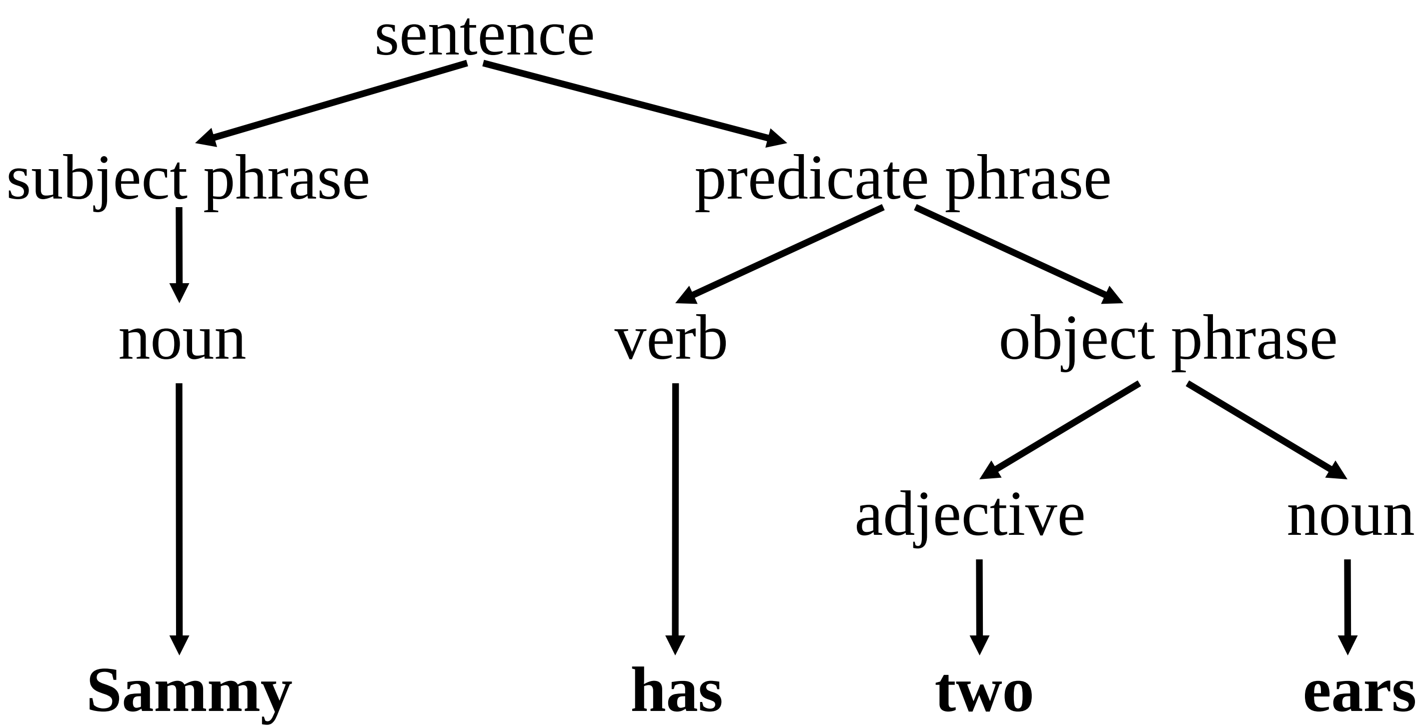
Decision Trees

(after Gross & Yellen, 1999, p. 93)



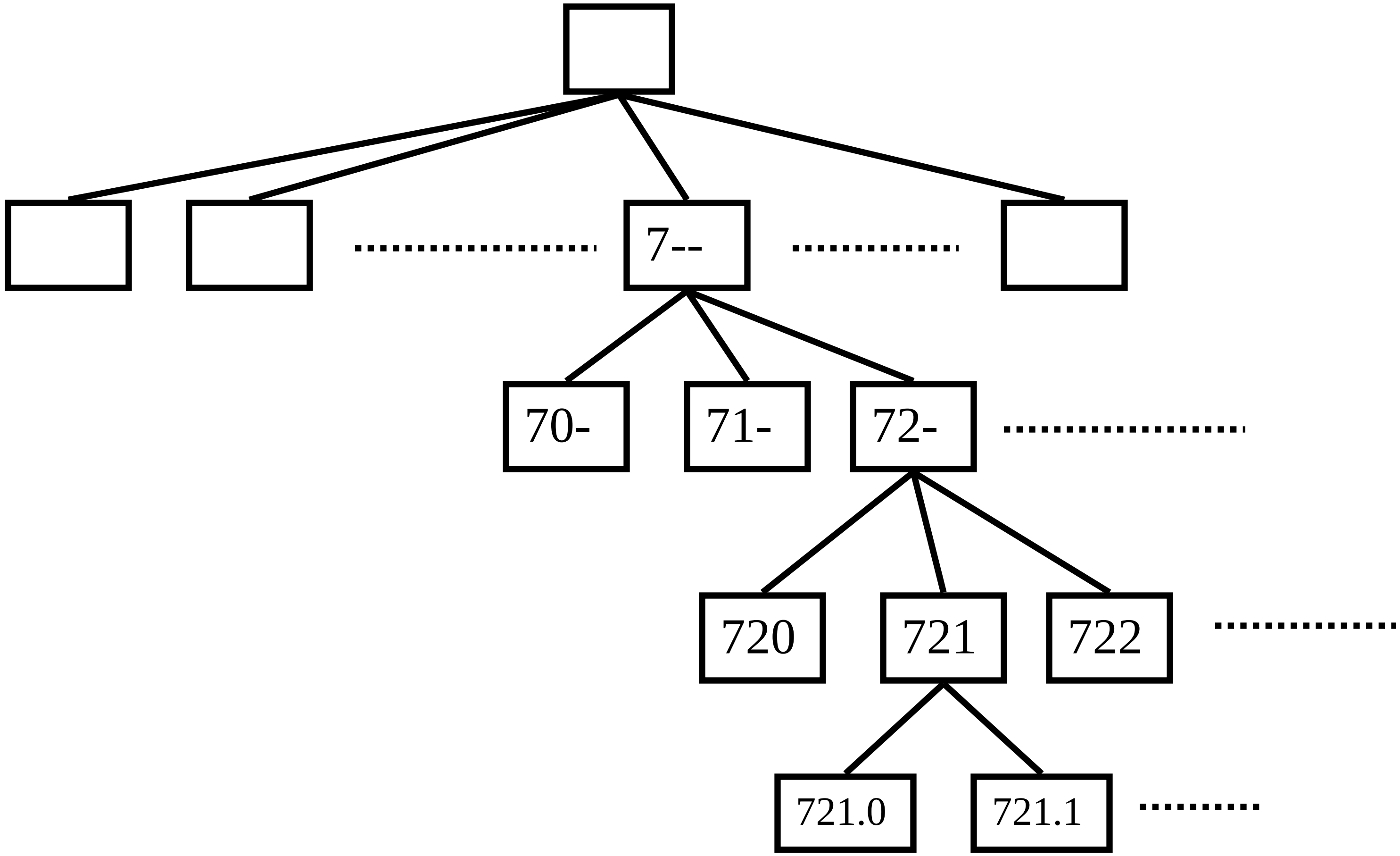
The first three moves of tic-tac-toe

Sentence Parsing
(after Gross & Yellen, 1999, p. 93)



Data Organization: DDCS for libraries

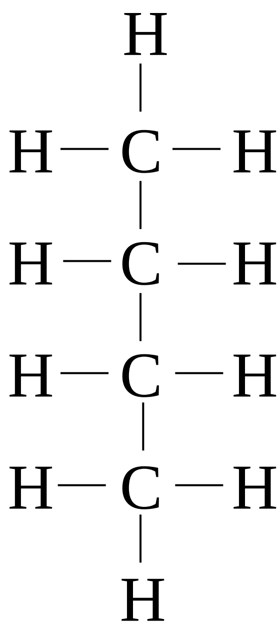
(after Gross & Yellen, 1999, p. 93)



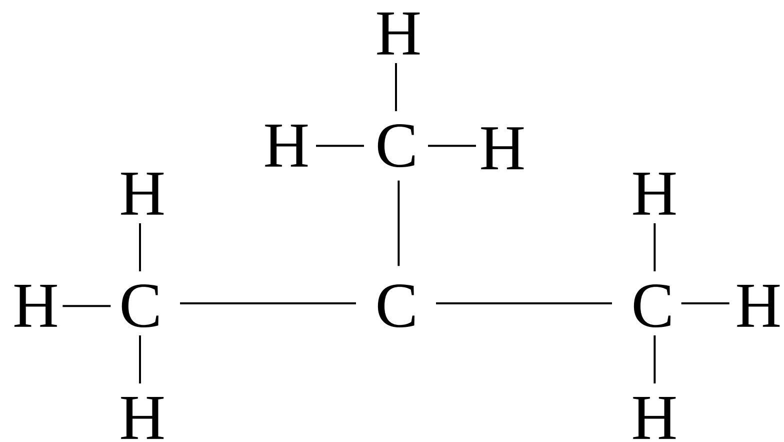
DDDC = *Dewey Decimal Classification System*

Chemical Isomers

(after Grimaldi, 1994, p. 610)



(a)

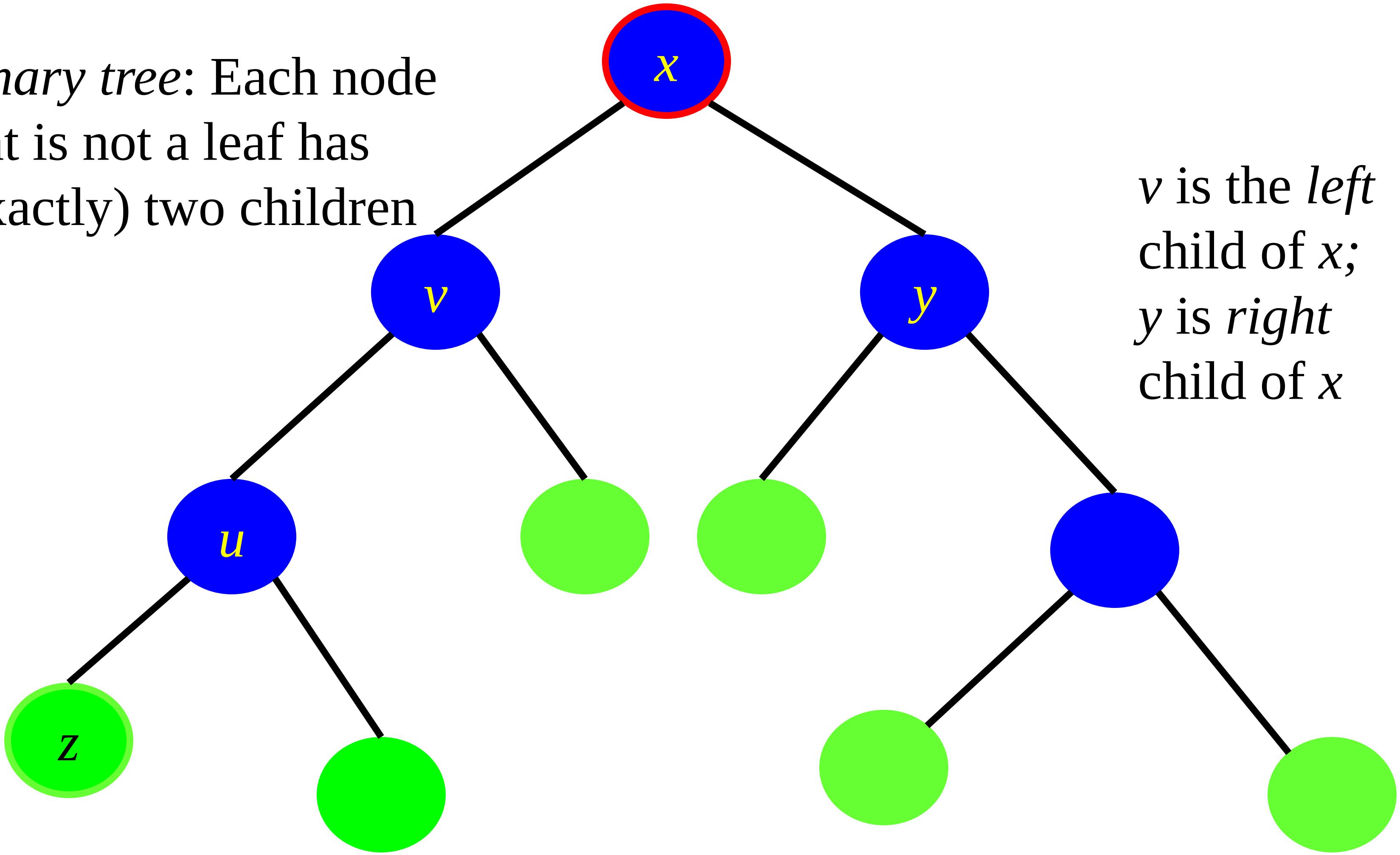


(b)

Two trees, each with 14 vertices (labeled with C's and H's) and 13 edges.
Each vertex has degree 4 (C, carbon atom) or degree 1 (H, hydrogen atom).

Binary Trees

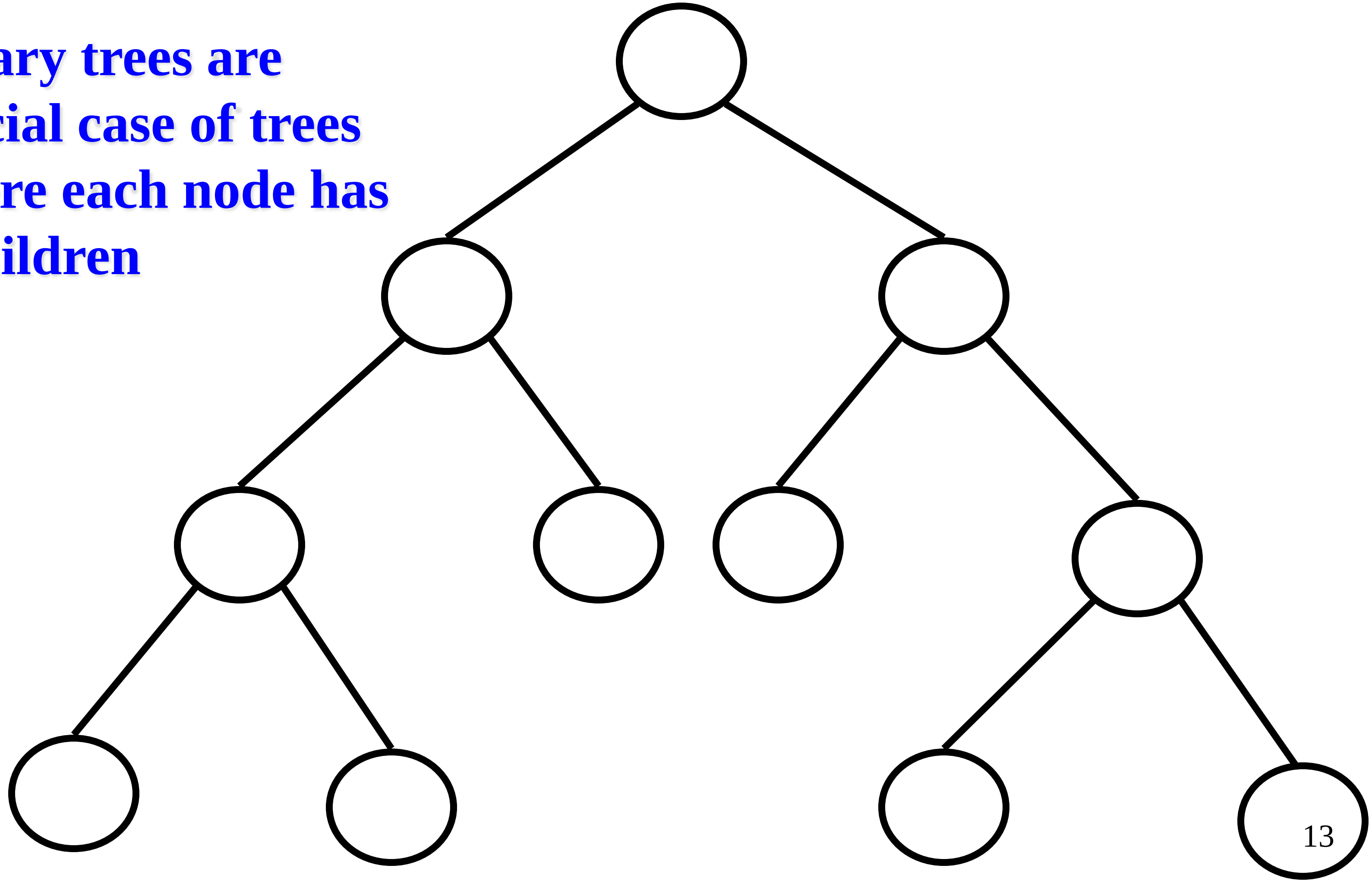
Binary tree: Each node that is not a leaf has (exactly) two children



v is the *left* child of *x*; *y* is *right* child of *x*

Binary Trees

Binary trees are special case of trees where each node has 2 children

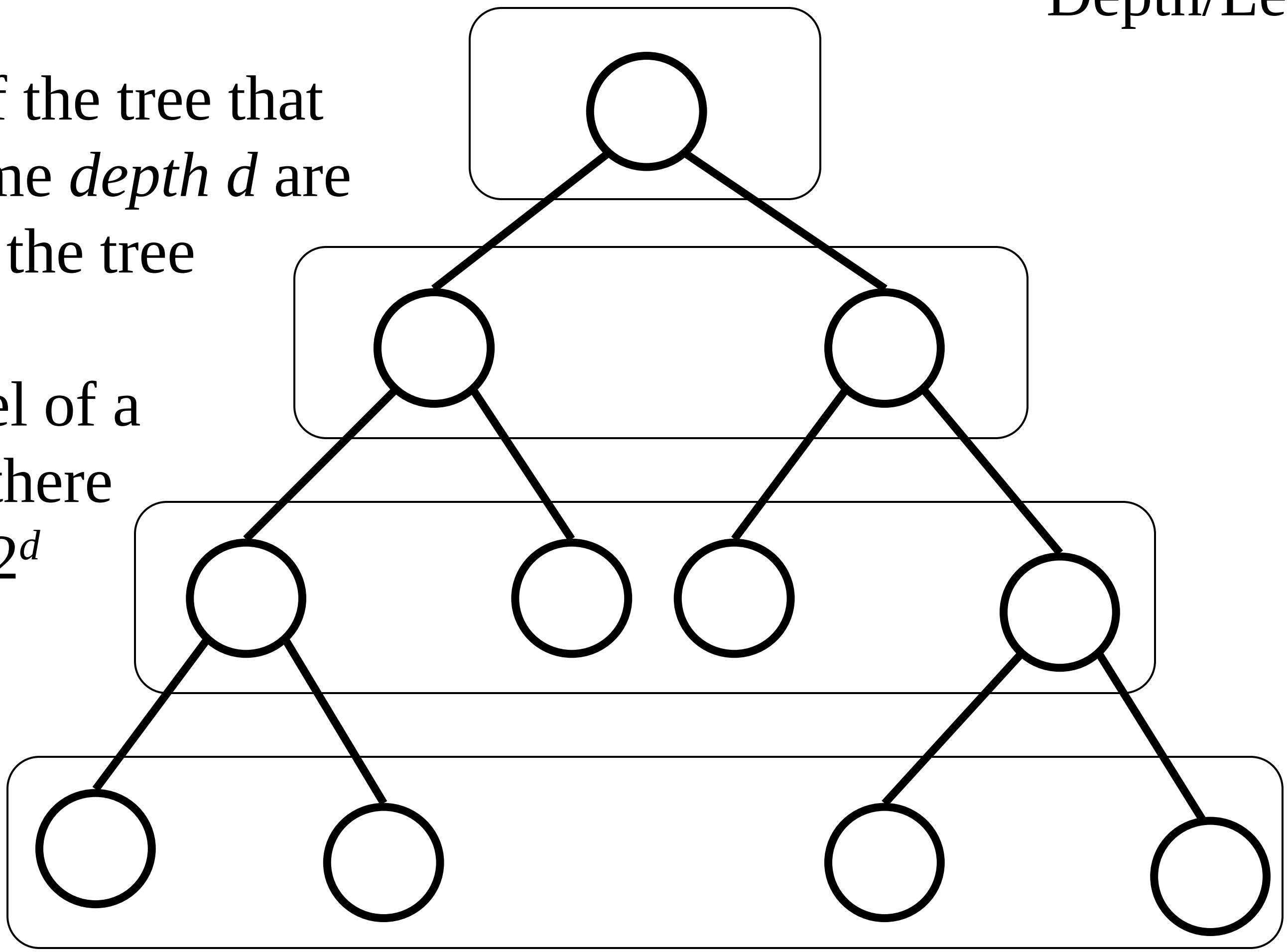


Depth and Levels in Trees

All nodes of the tree that have the same *depth d* are at *level d* of the tree

At each level of a *binary* tree there are at most 2^d nodes

Depth/Level



0

1

2

3

Properties of Binary Trees

- Let T be a binary tree with n nodes and let h denote the height of T
  - The number of *leaves* in T is at least $h + 1$
  - The number of internal nodes in T is at least h and at most $2^h - 1$
  - $n \leq 2^{h+1} - 1$
  - $\log(n+1) - 1 \leq h \leq (n-1)/2$
  - # of leaves = 1 + # of internal nodes

ADT Tree

- Stores *elements* at positions, which are defined relative to their neighbour positions (i.e., parents, children, siblings)
- The *positions* in a tree are its nodes
- Supported methods by a node/position object:
  - **root():** returns the root of the tree
  - **element(t):** returns the object at this position
  - **parent(*v*):** returns the parent of node *v;* an error occurs if *v* is root
  - **children(*v*):** returns an iterator to enumerate the children of node *v*

ADT Tree …

- **isInternal():** Test whether node *v* is internal
- **isExternal():** Test whether node *v* is external (i.e., leaf)
- **isRoot(*v*):** Test whether node *v* is root.
- **size():** returns the number of nodes in the tree
- **elements():** returns an iterator of all the elements stored at nodes of the tree (i.e., pre-, in-, post-, level-order)
- **positions():** returns an iterator of all the positions of the tree (i.e., pre-, in-, post-, level-order)
- **swapElements(*v*,*w*):** Swap the elements stored at nodes *v* and *w*
- **replaceElements(*v*,*e*):** Replace the element stored at node *v* with *e* and return the original element stored at node *v*

ADT Binary Tree

- Specialization of a tree ADT that supports the accessor methods
  - **leftChild(*v*):** returns the left child of *v;* an error occurs if *v* is a leaf.
  - **rightChild(*v*):** returns the right child of *v;* an error occurs if *v* is a leaf.
  - **sibling(*v*):** returns the sibling of *v;* an error occurs if *v* is the root.

Tree Algorithms: depth

Definition: The *depth* of a node v in a tree T is (recursively) defined to be

- 0, if v is the root of T
- The depth of the parent of v + 1, otherwise

Algorithm depth(T,v):

Input: Tree T, node v in T

Output: the depth of v in T (i.e., the number of ancestors of v in T, excluding v itself)

Tree Algorithms: height

Definition: The *height* of a a tree *T* rooted at node *v* is (recursively) defined to be

- 0 if *v* is a leaf.
- 1+ the maximum height of a child of *v*, otherwise.

Algorithm height(*T*,*v*):

Input: Tree *T*, node *v* in *T*

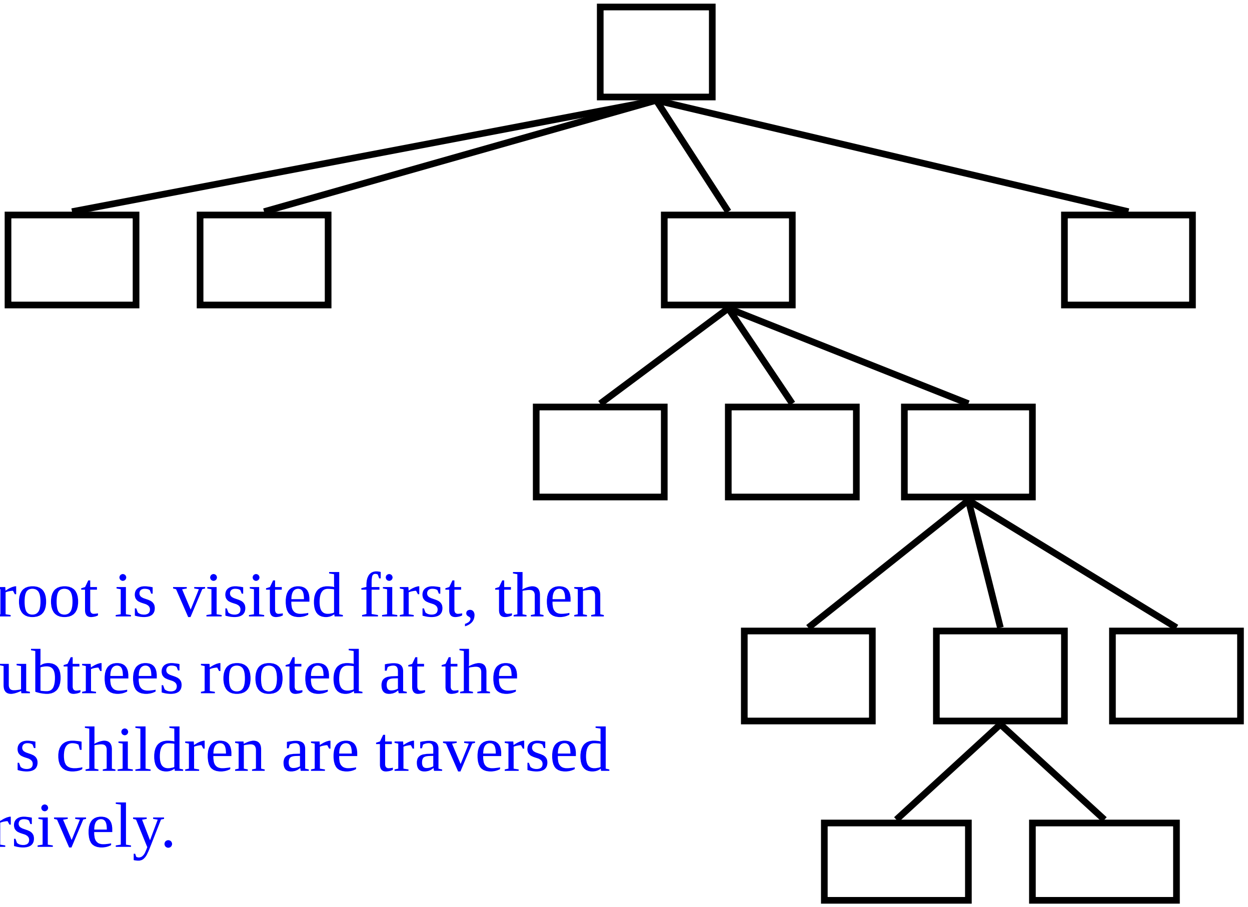
Output: the height of tree *T* rooted at node *v* (i.e., the maximum depth of a leaf of the tree *T* rooted by *v* in *T*)

Tree Traversals

- n-ary tree traversals
  - Preorder
  - Postorder
  - Level order

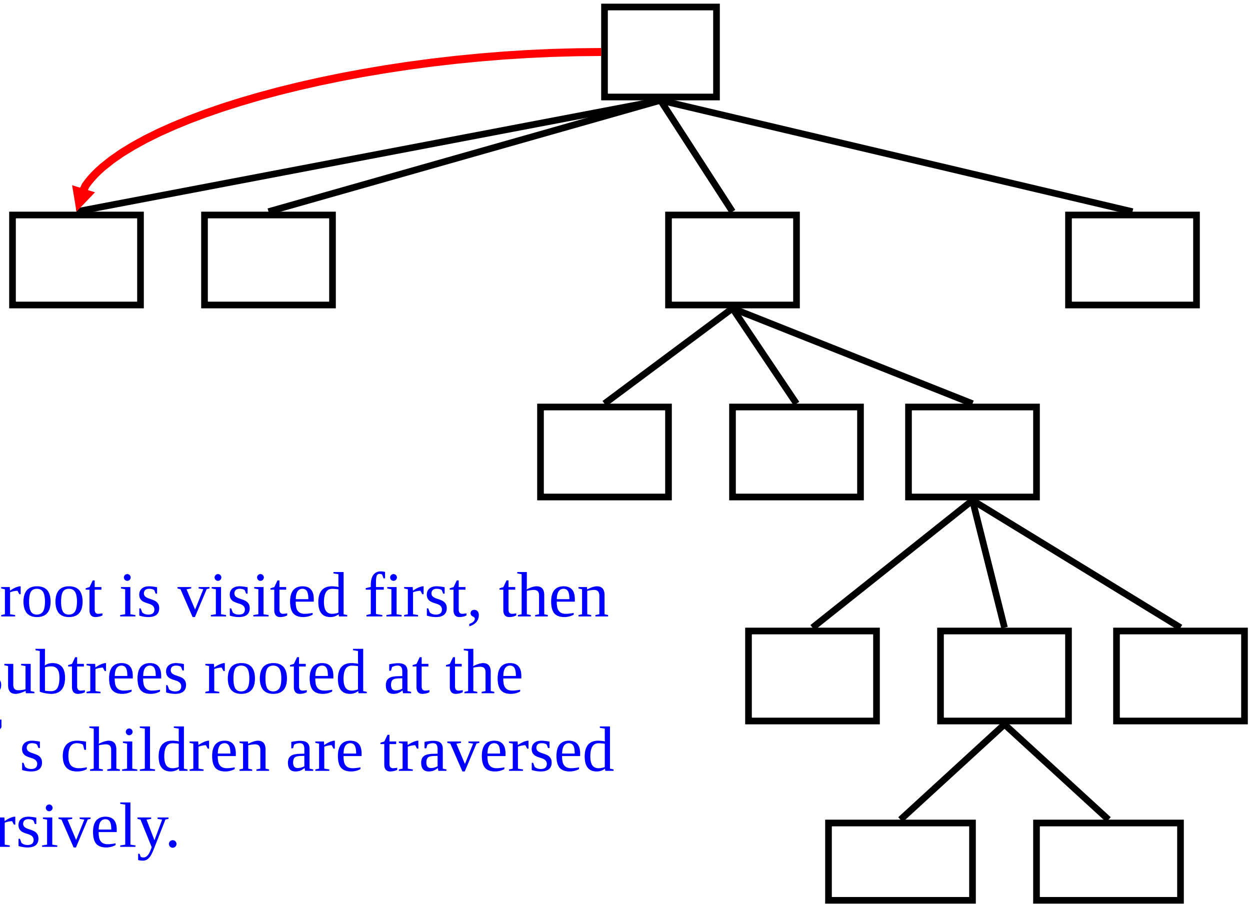
- Binary tree traversals
  - Preorder
  - Postorder
  - Inorder
  - Level order

Preorder Traversal
Depth First Search



The root is visited first, then the subtrees rooted at the root's children are traversed recursively.

Preorder Traversal
Depth First Search



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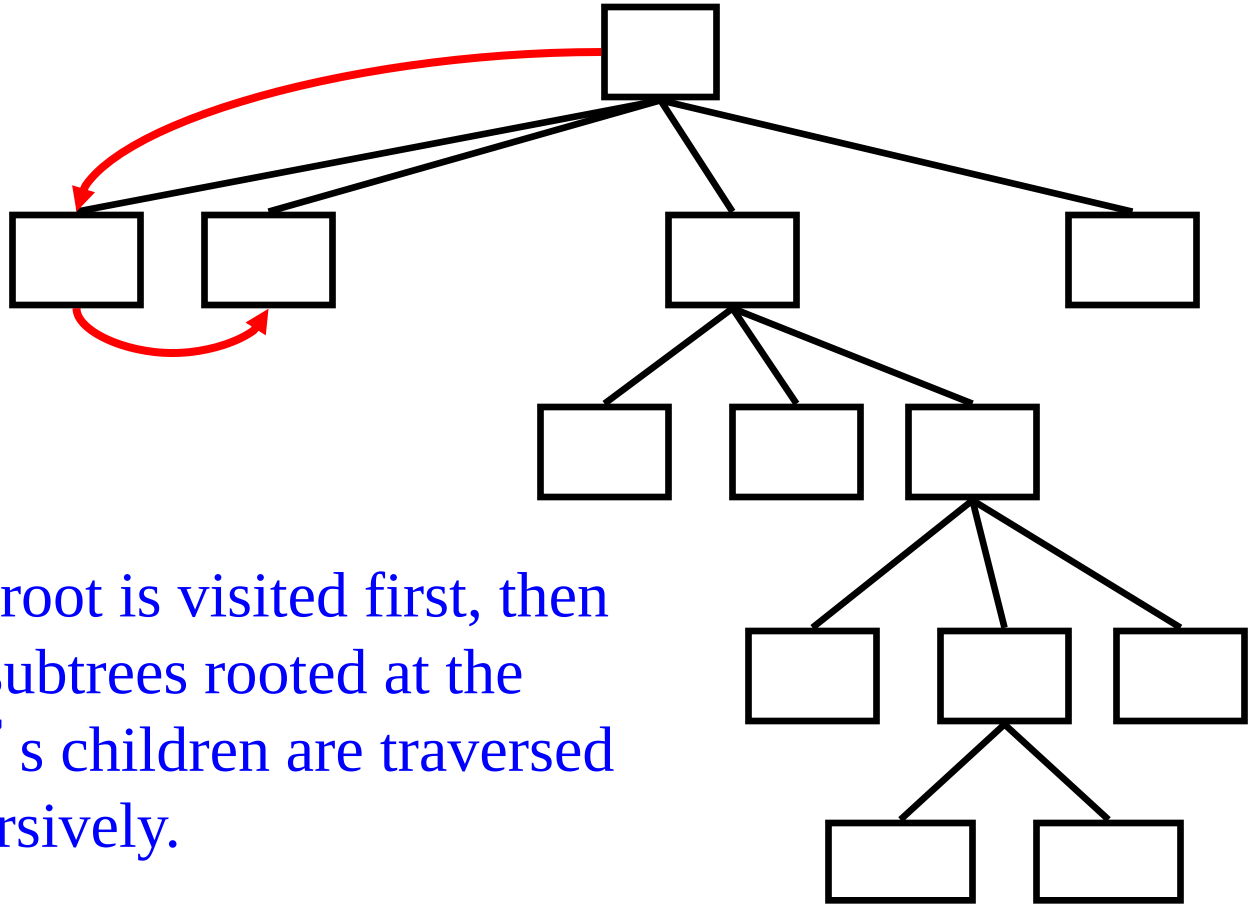

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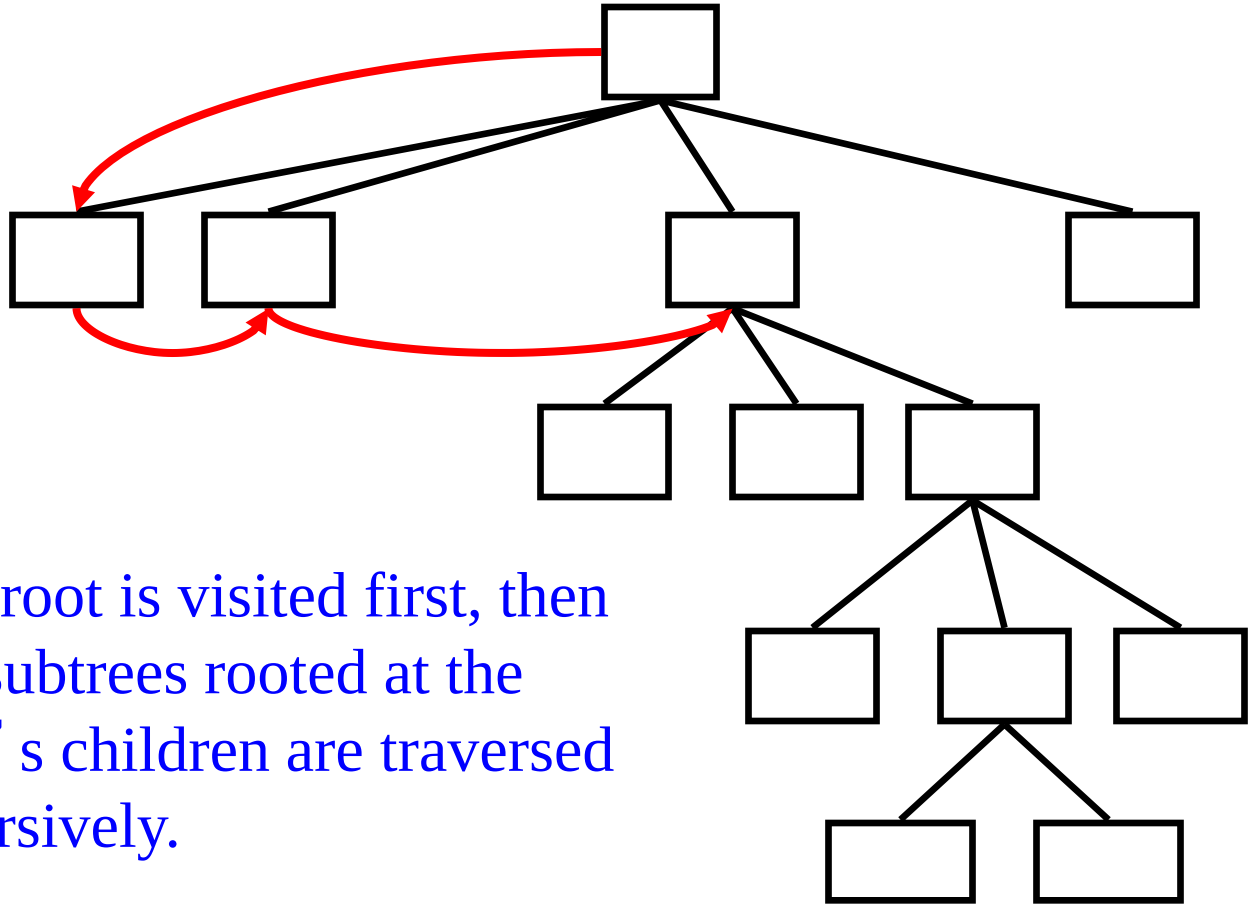

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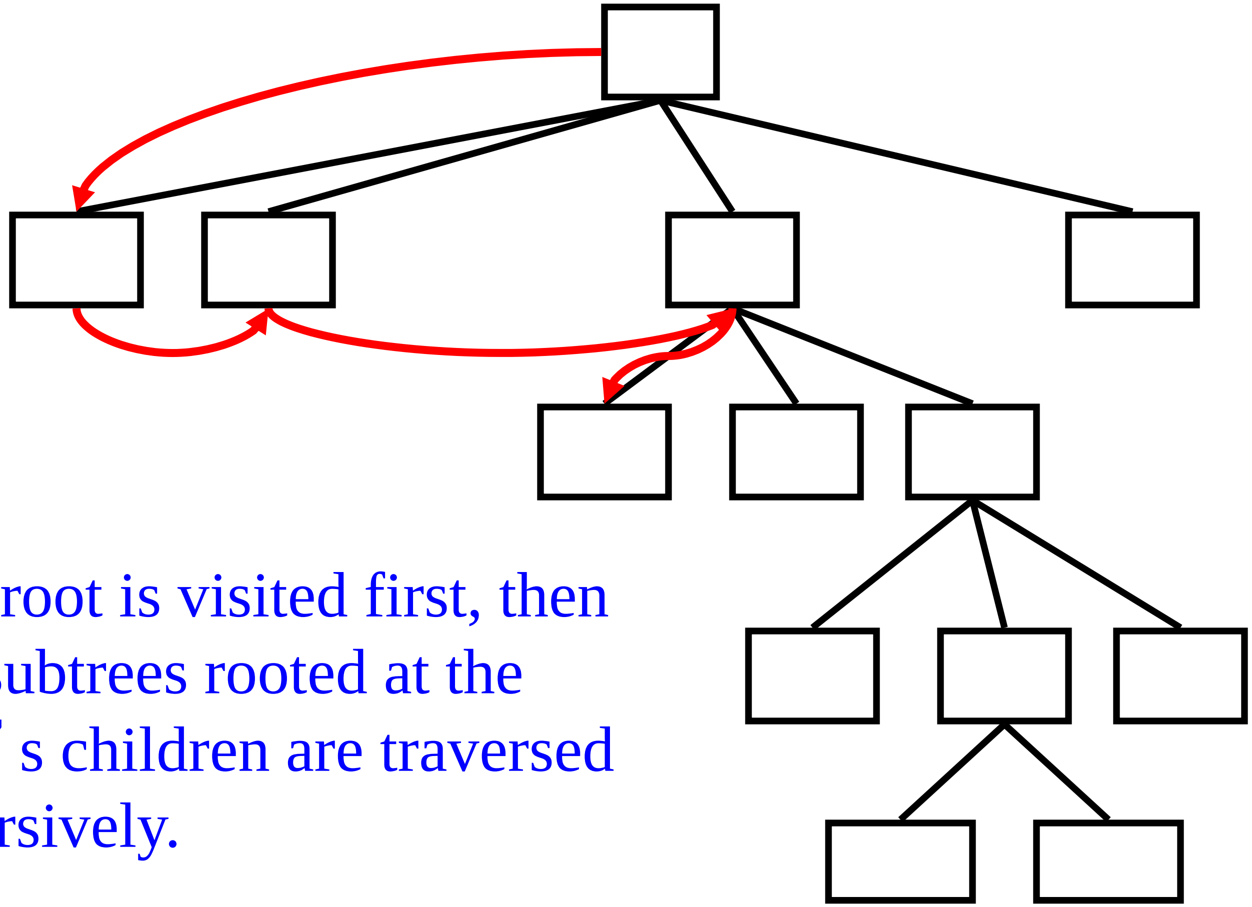

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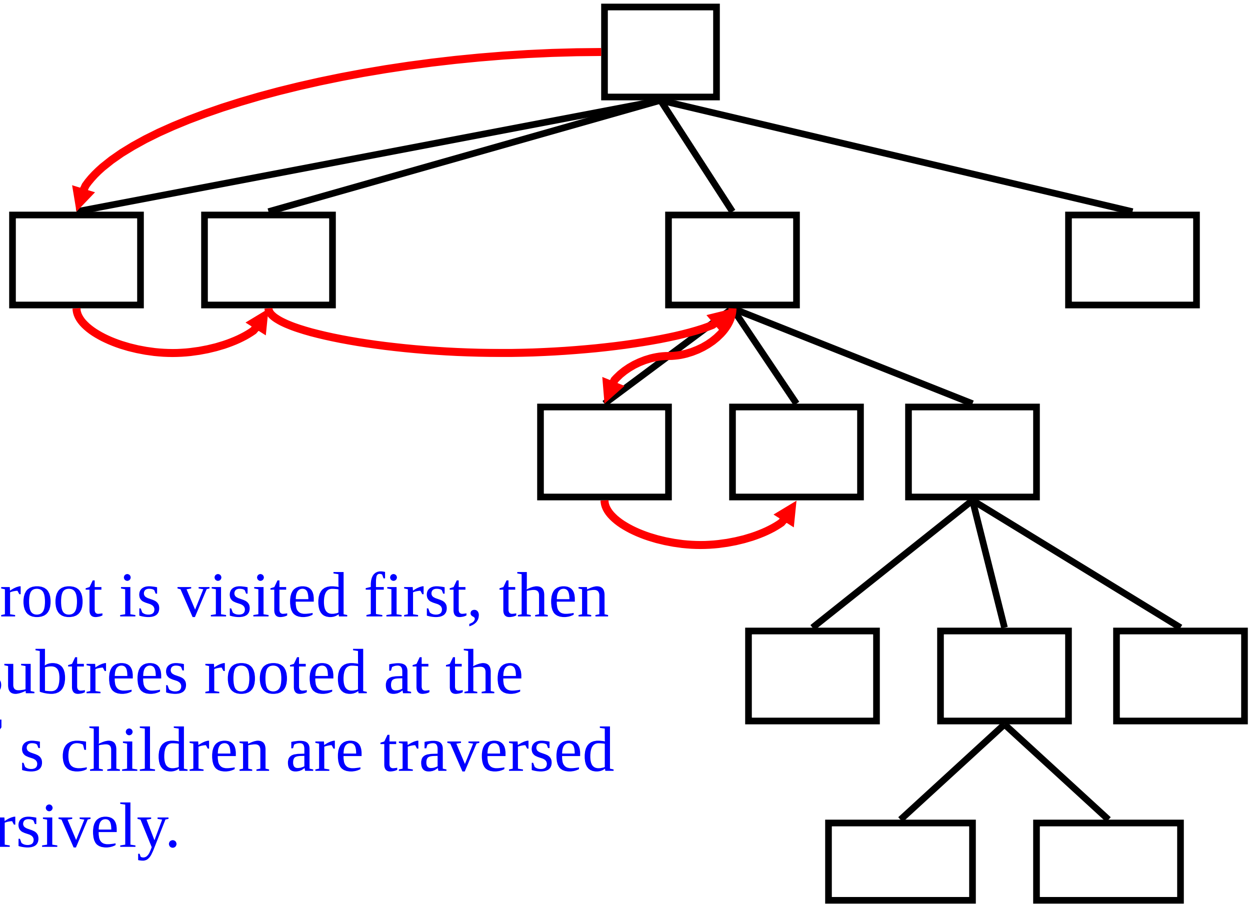

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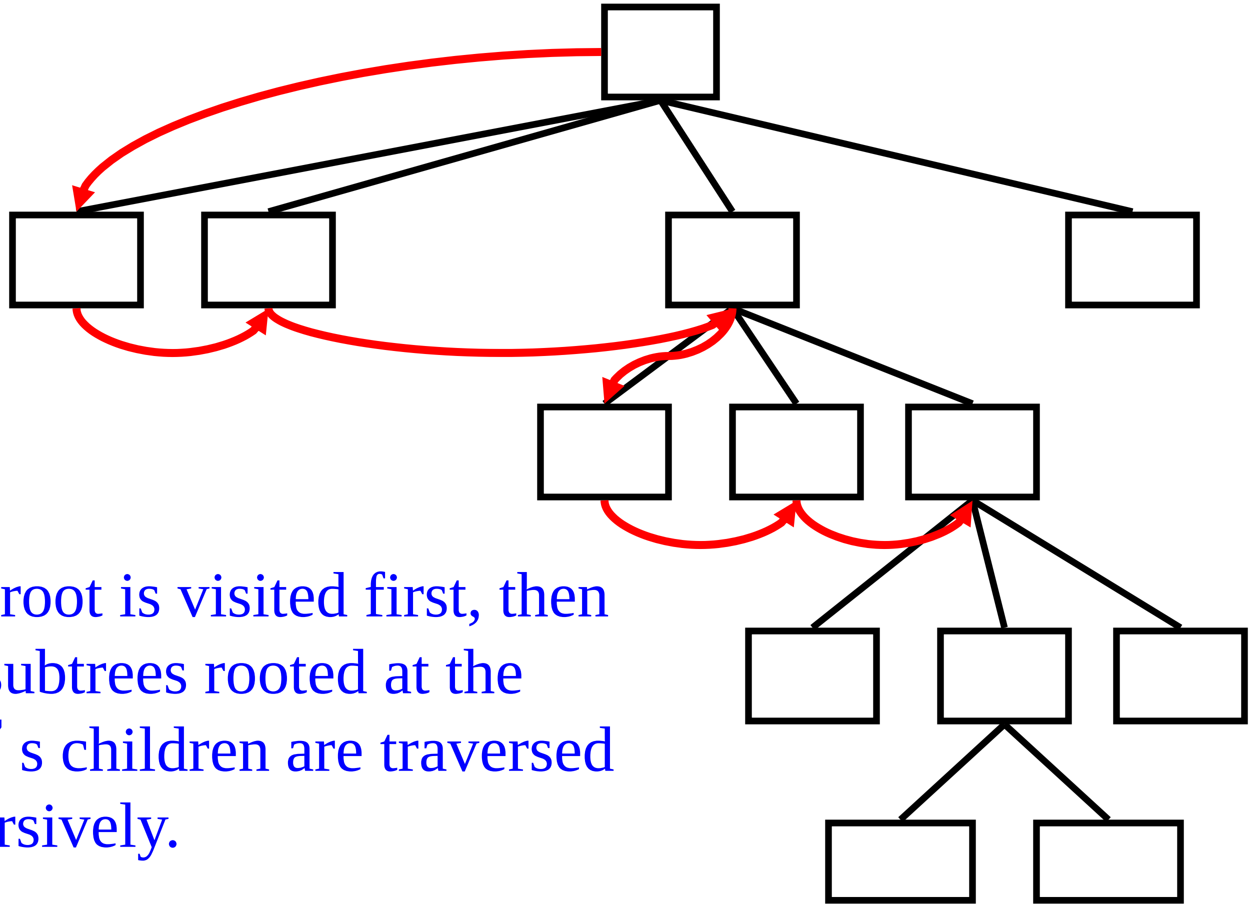

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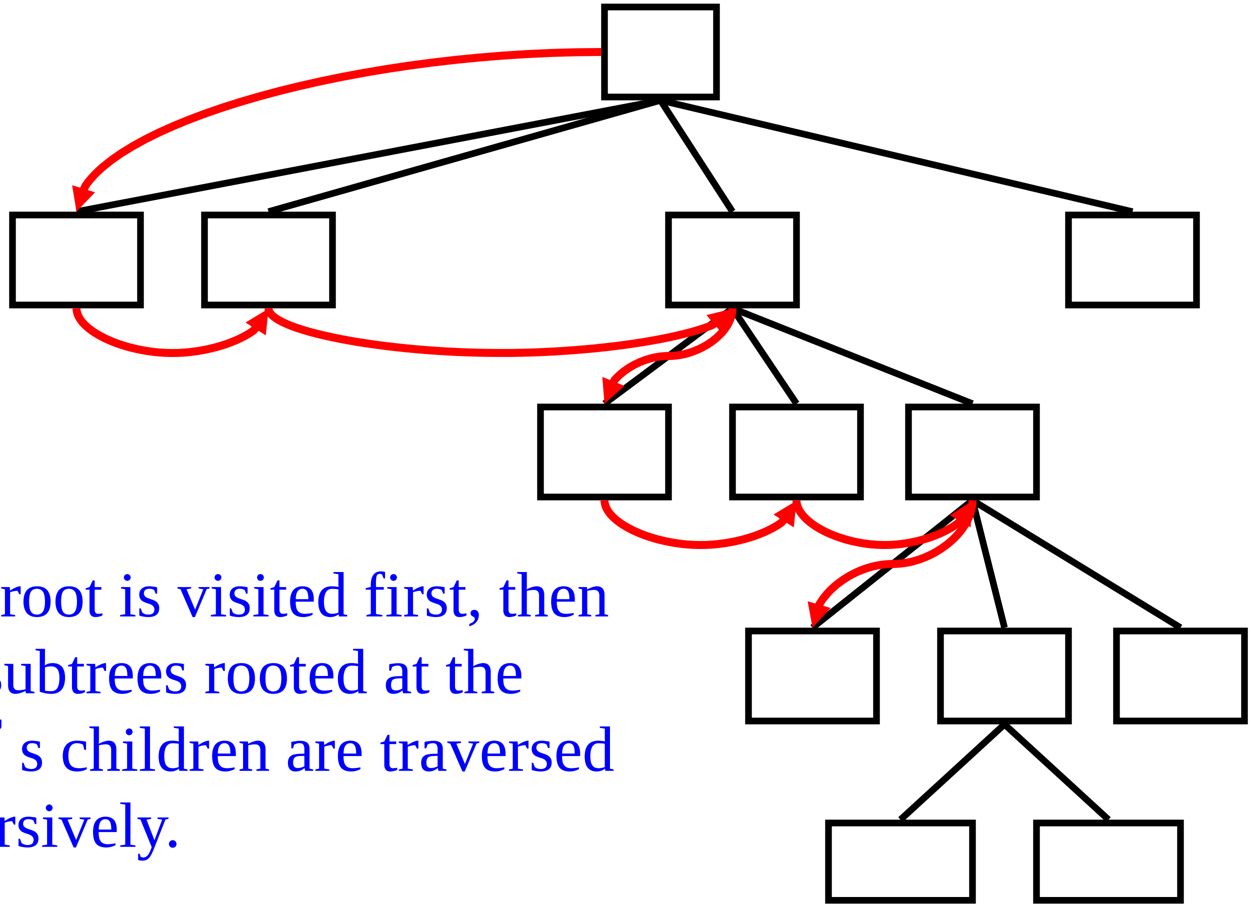

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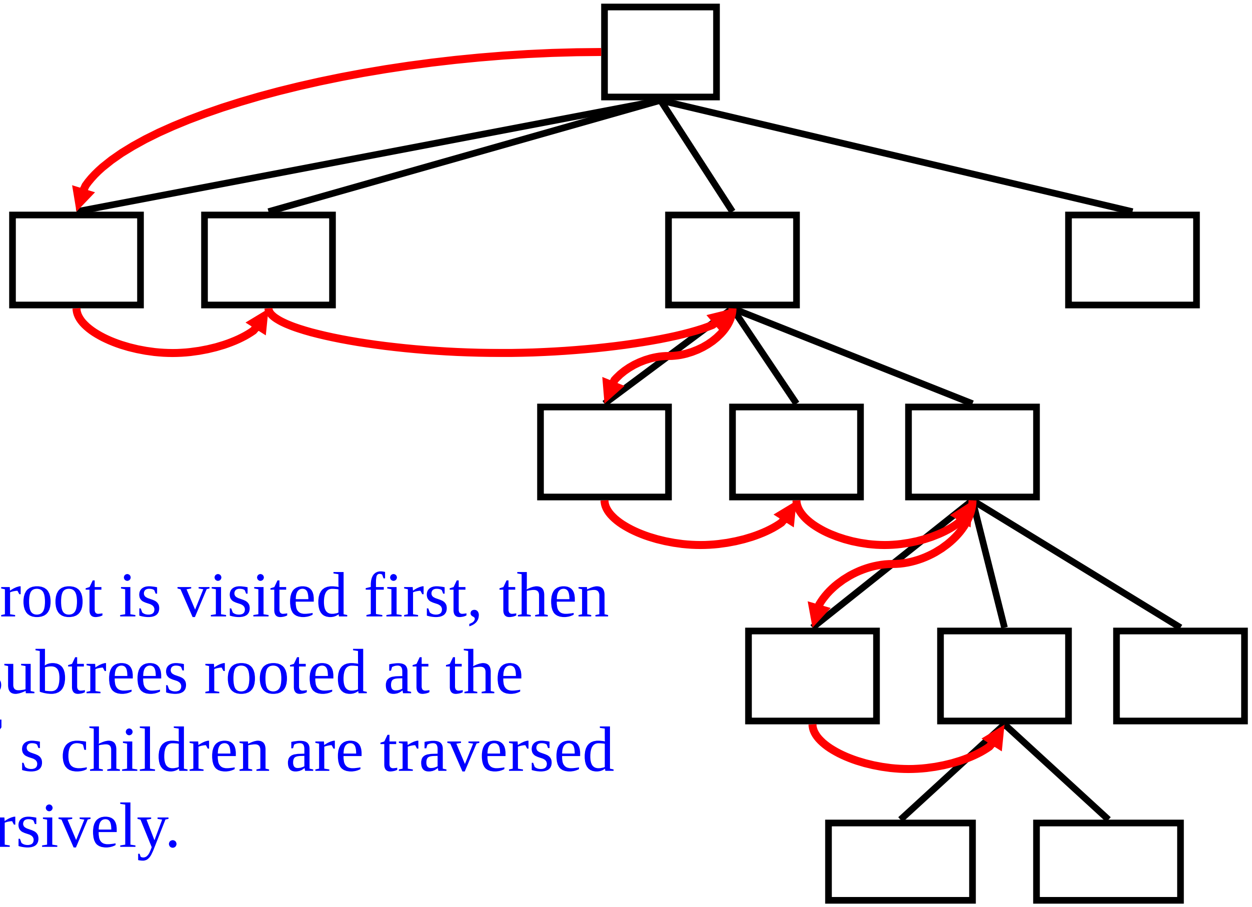

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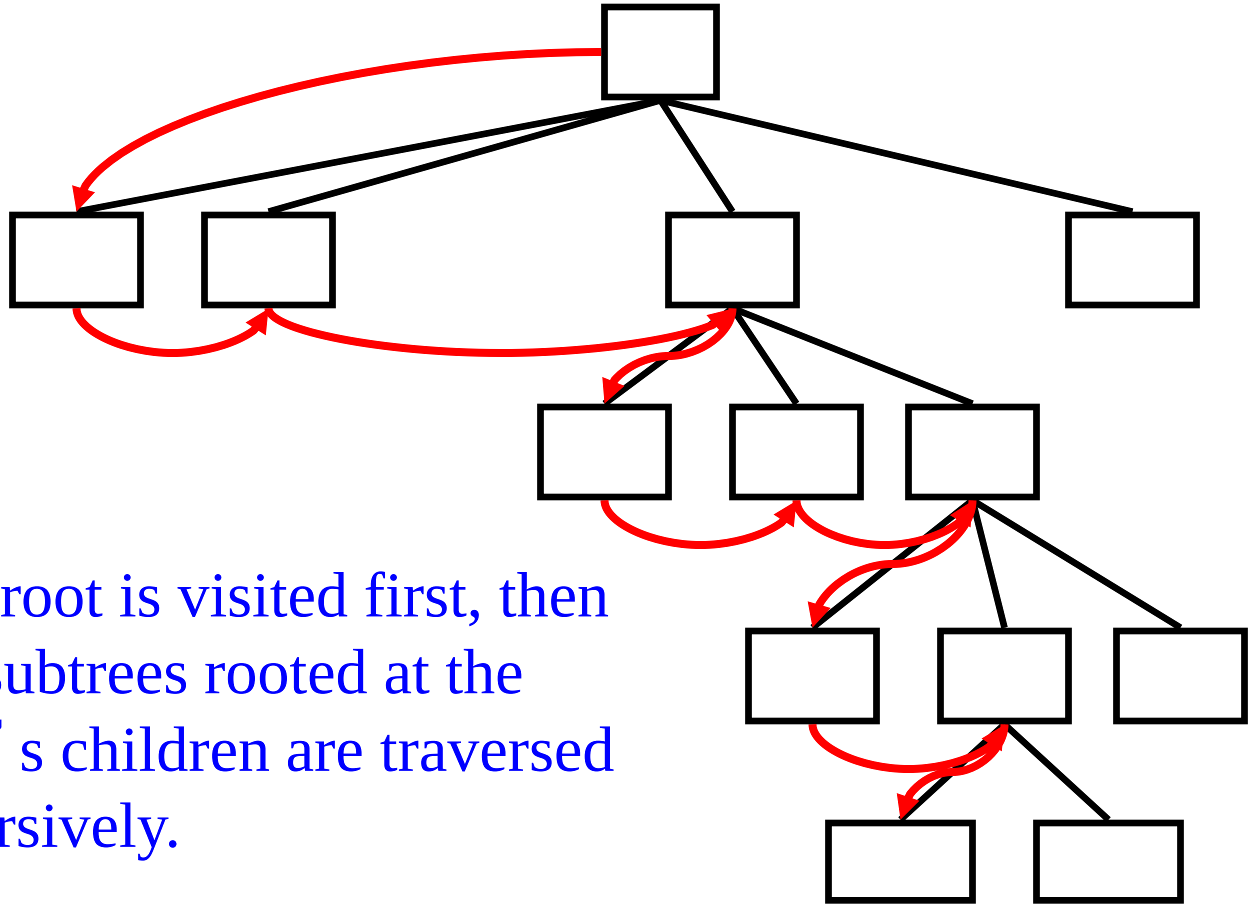

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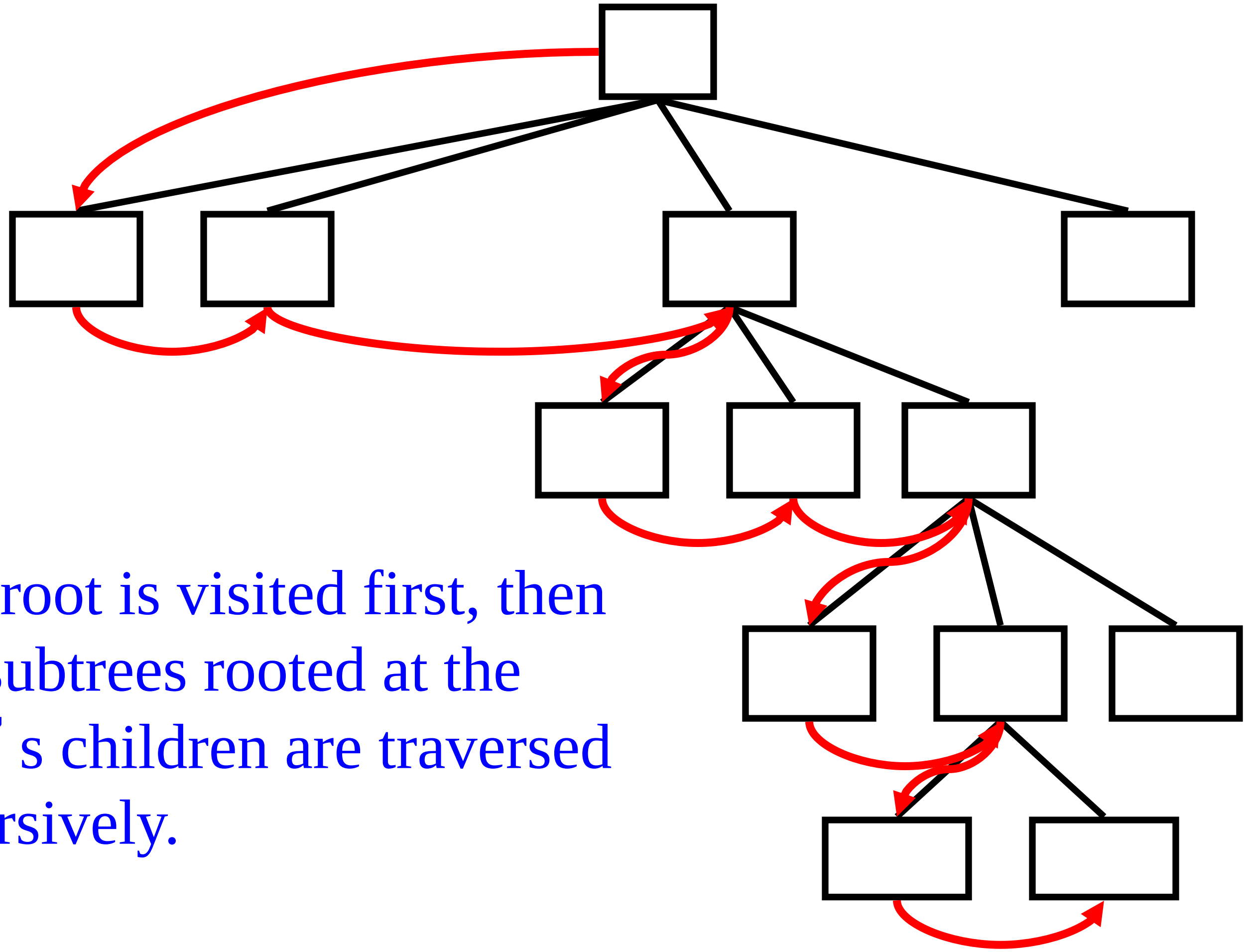

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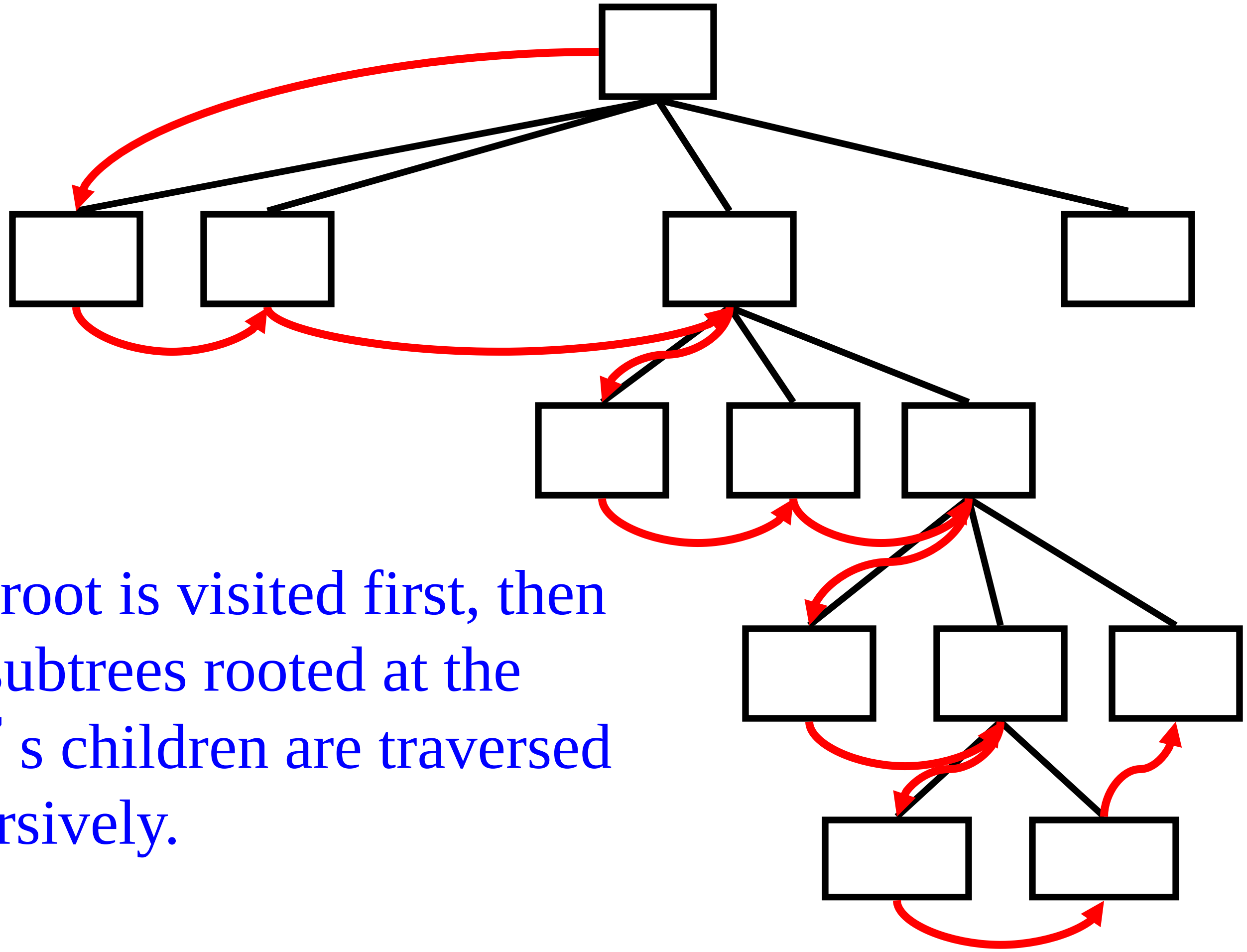

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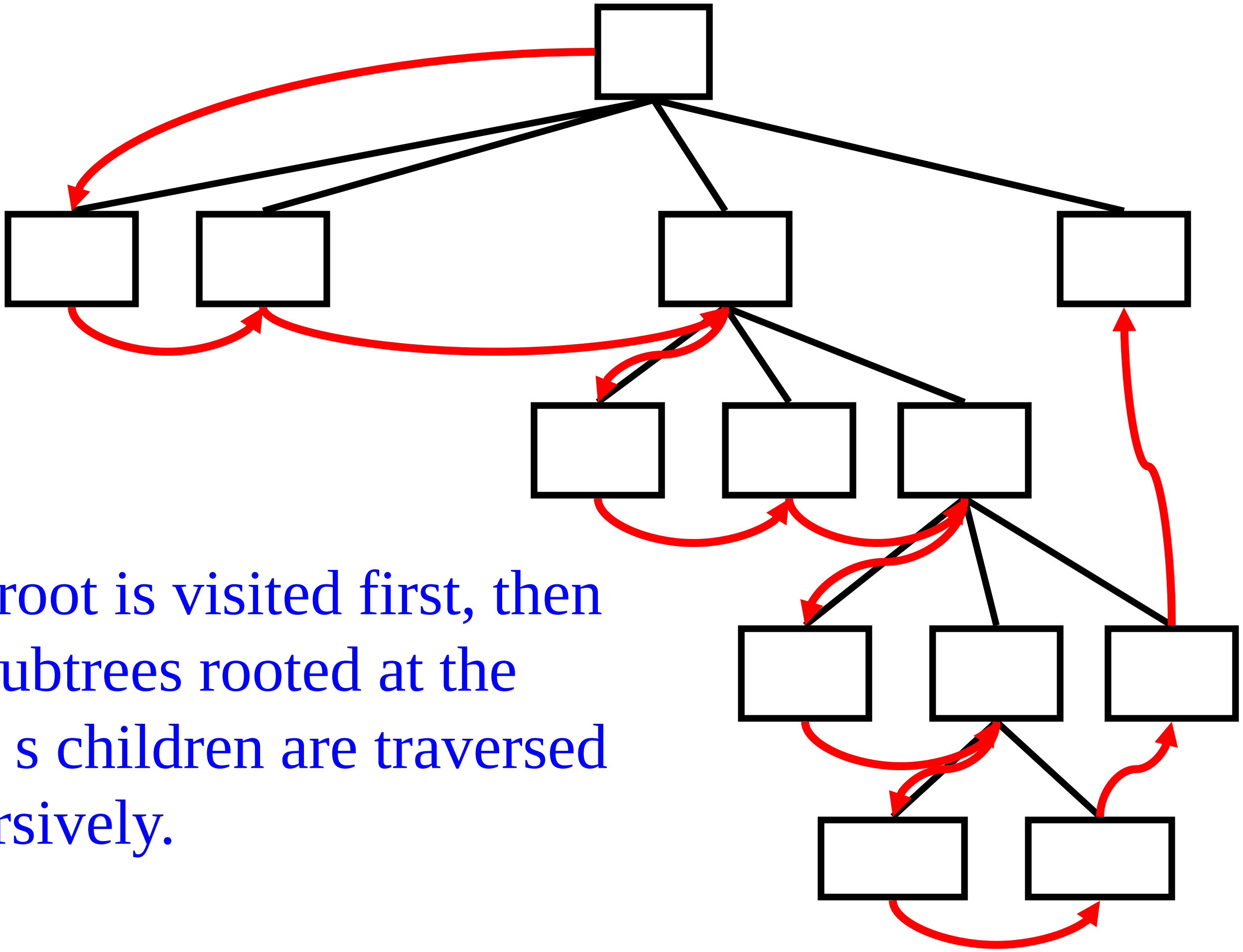

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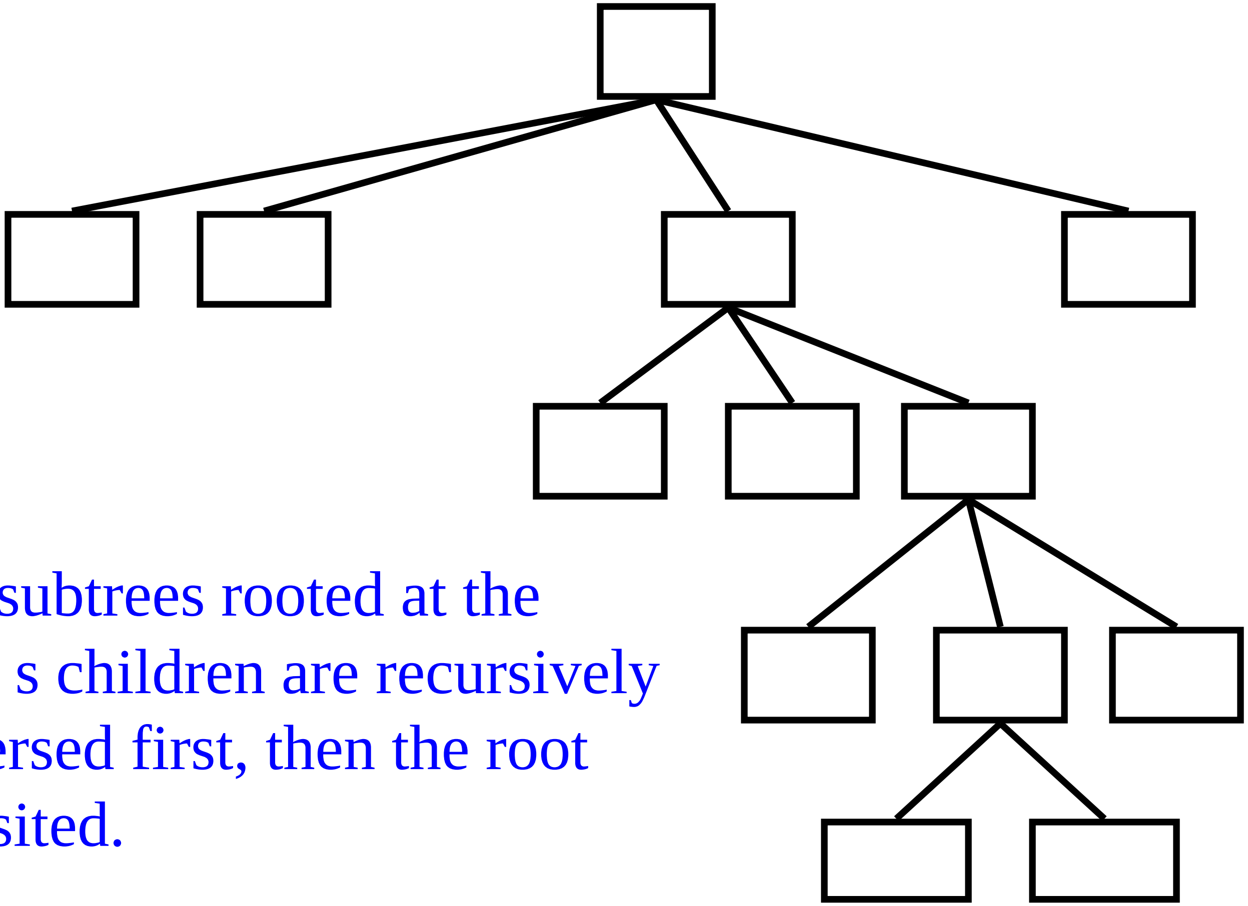
Postorder Traversal



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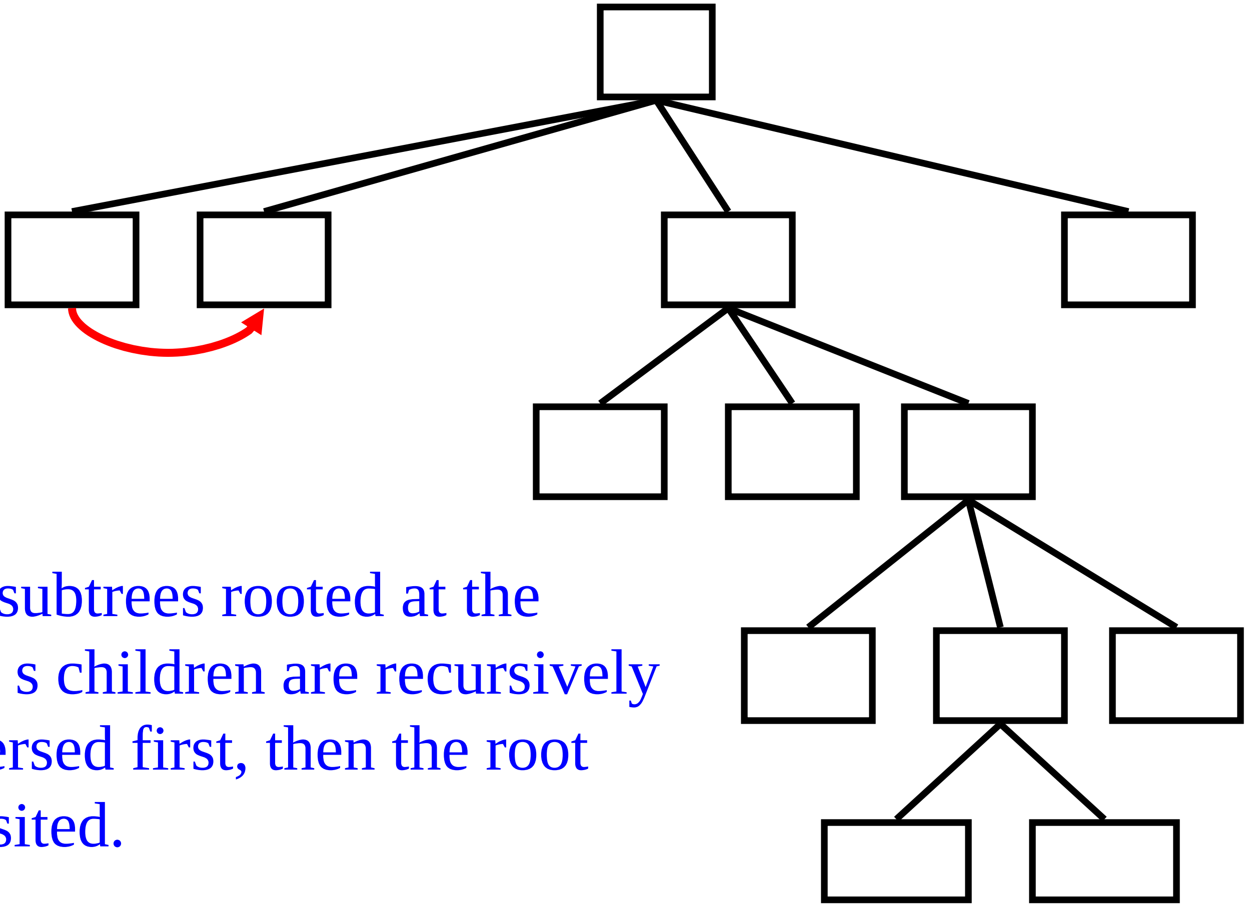

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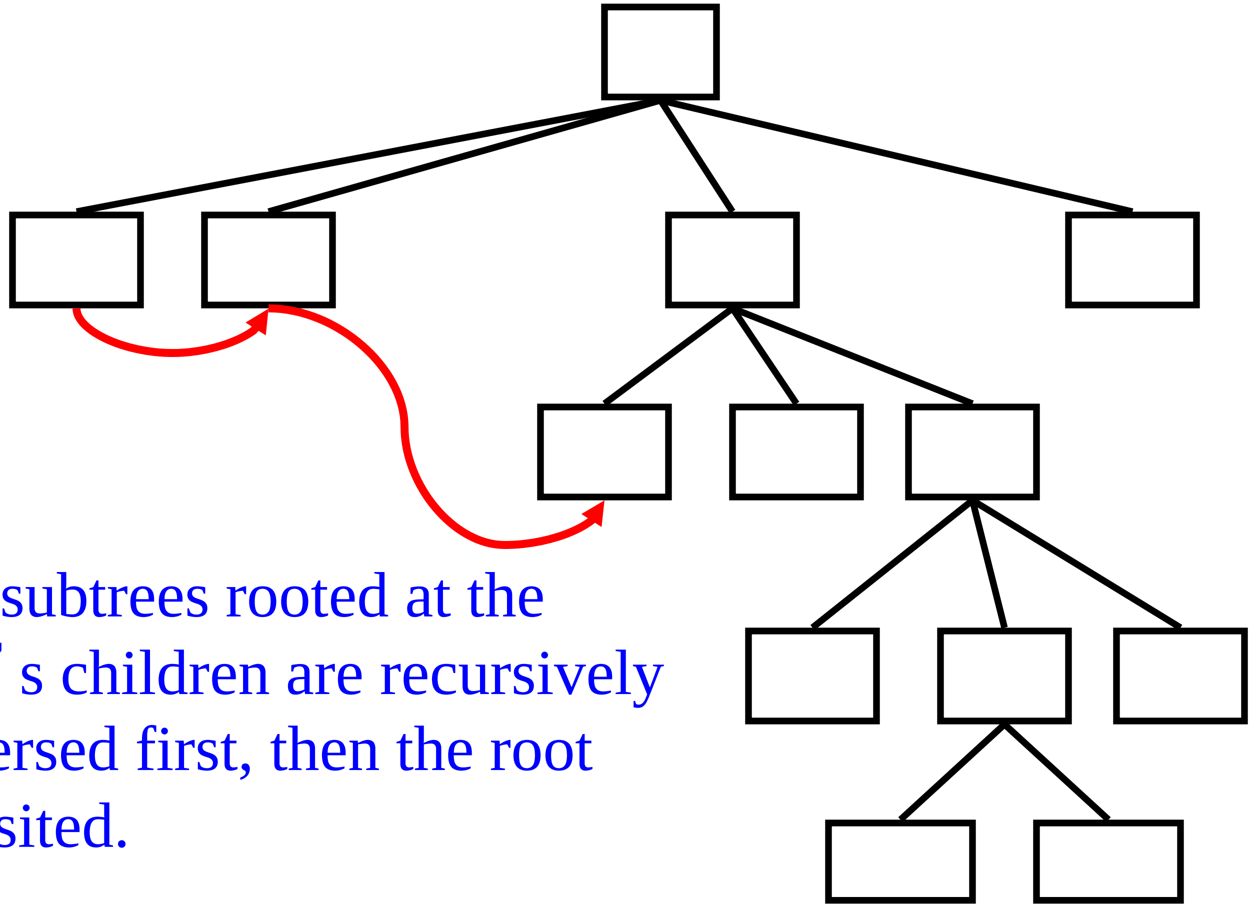

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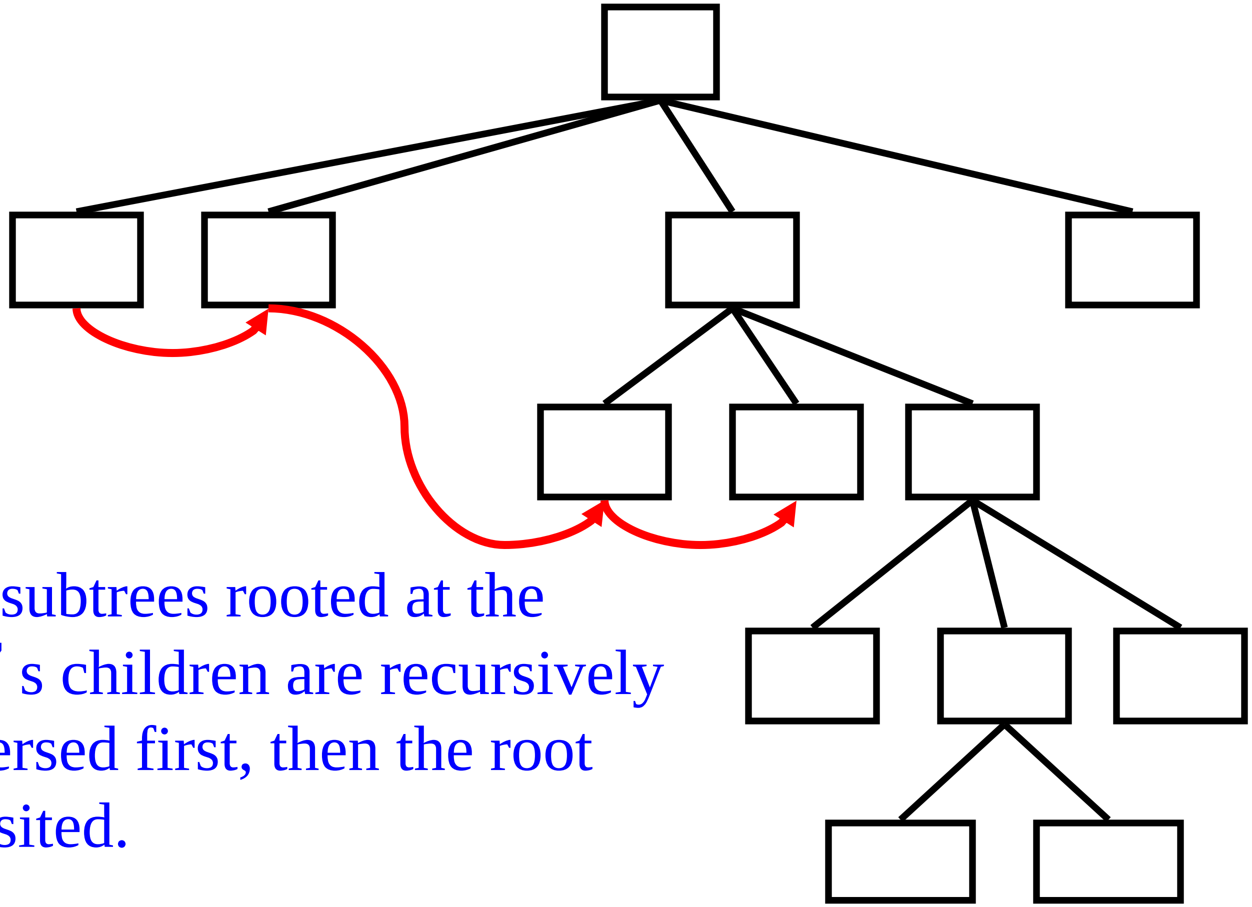

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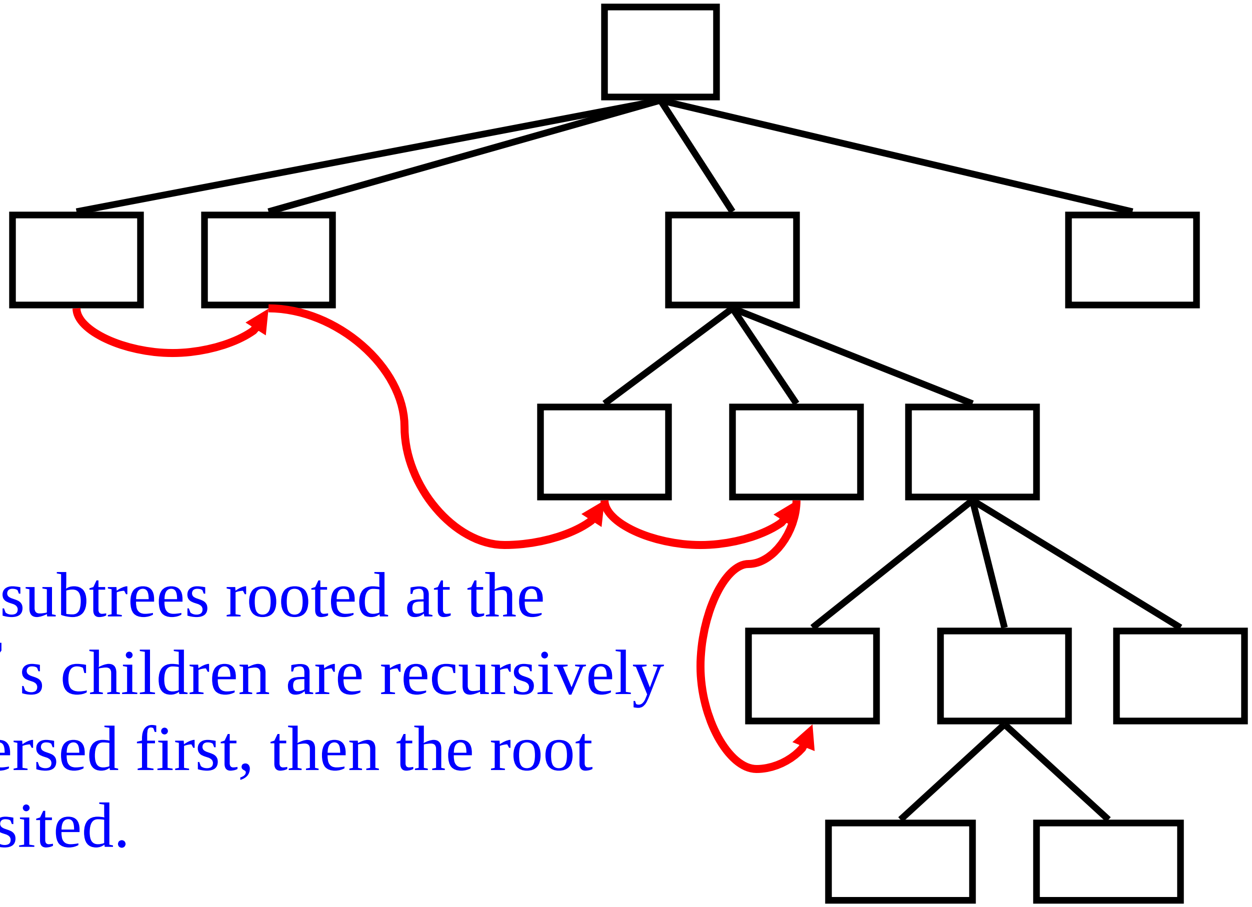

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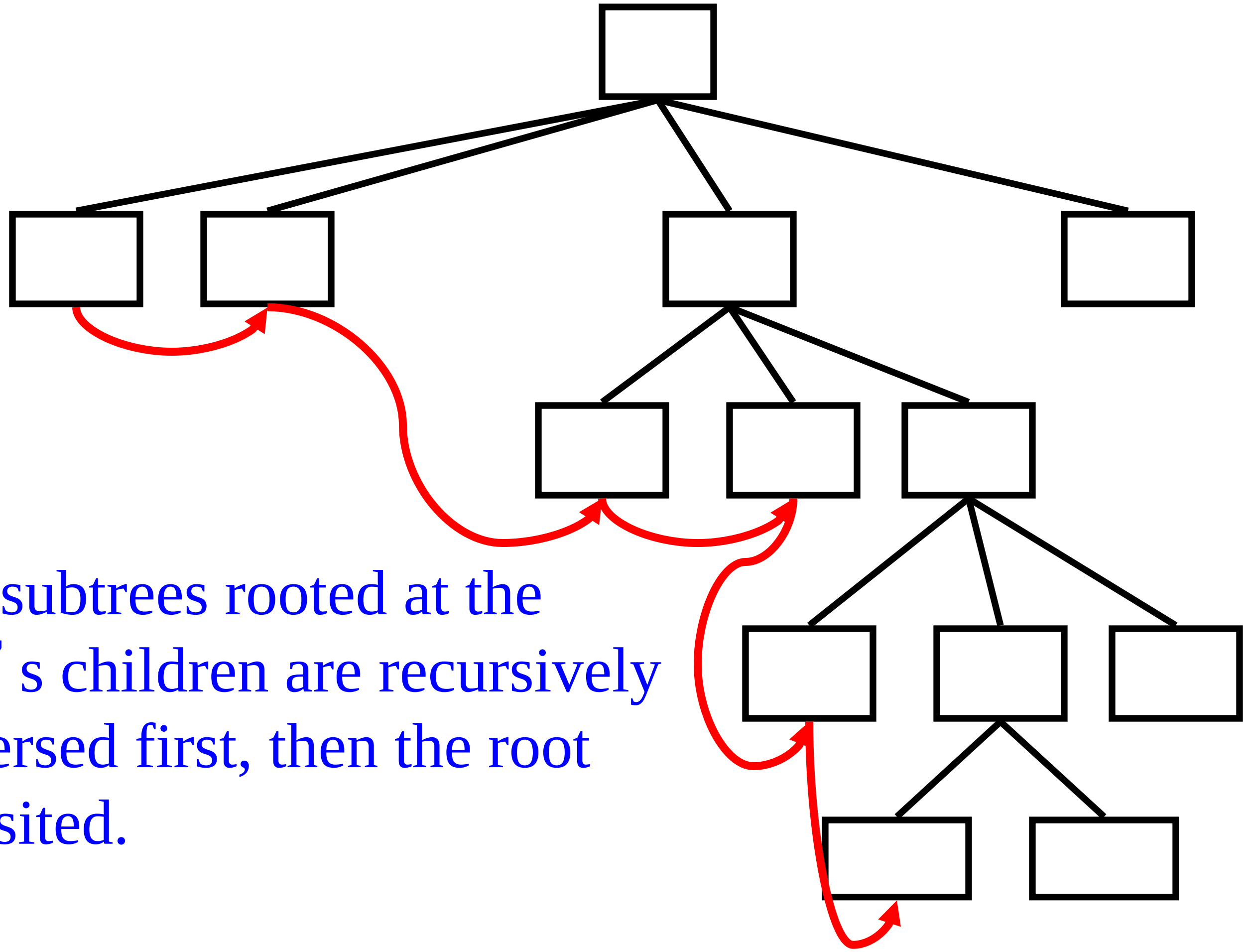

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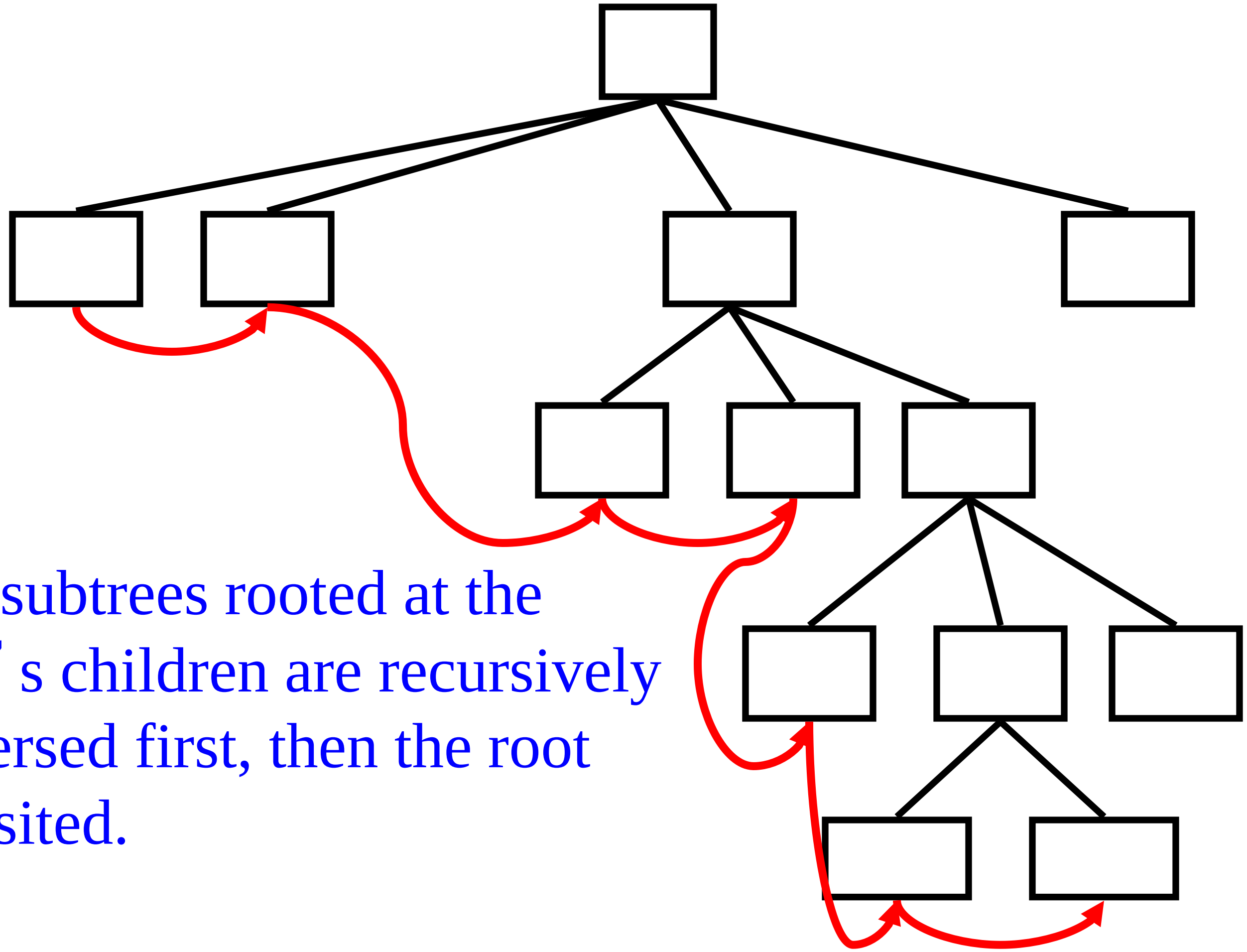

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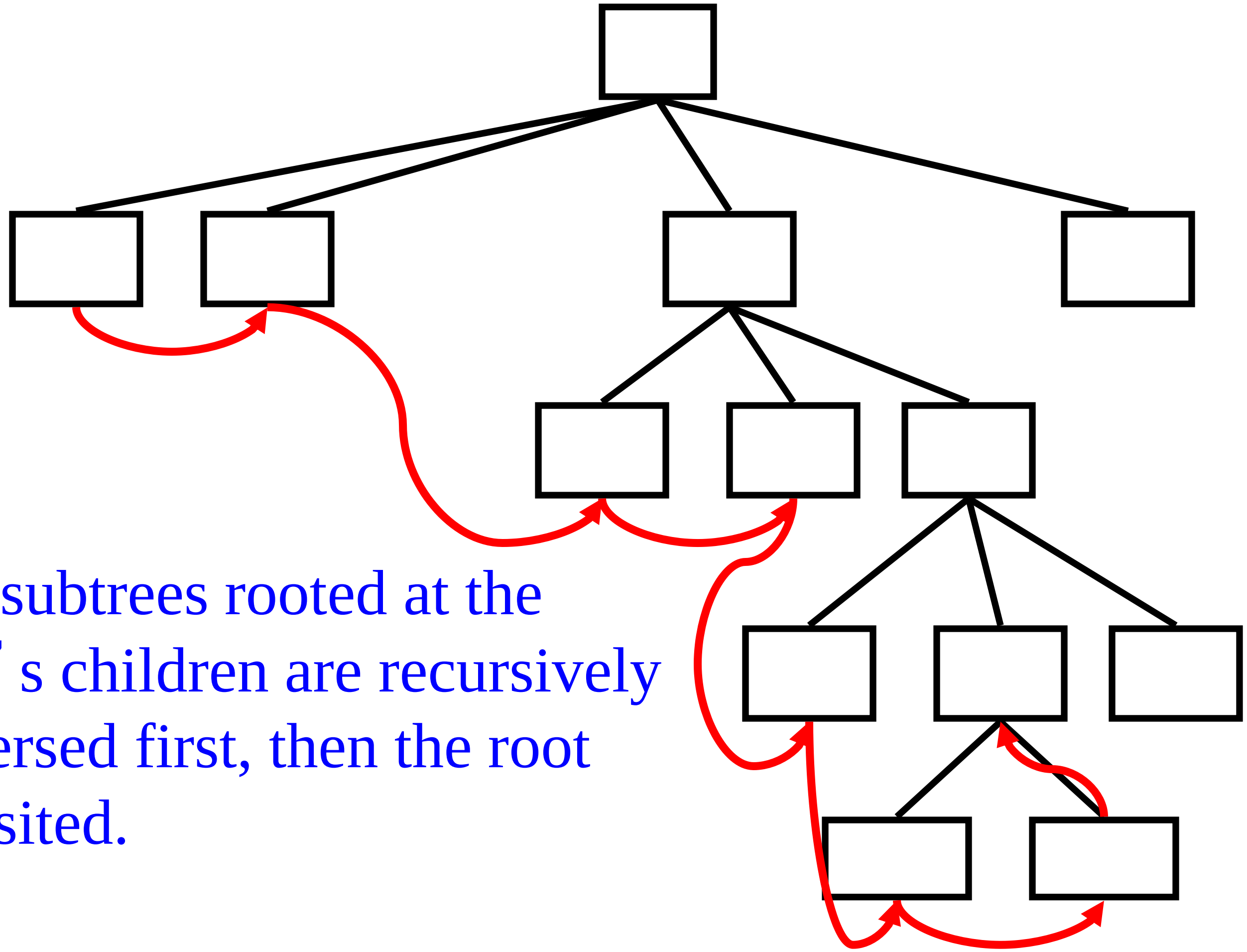

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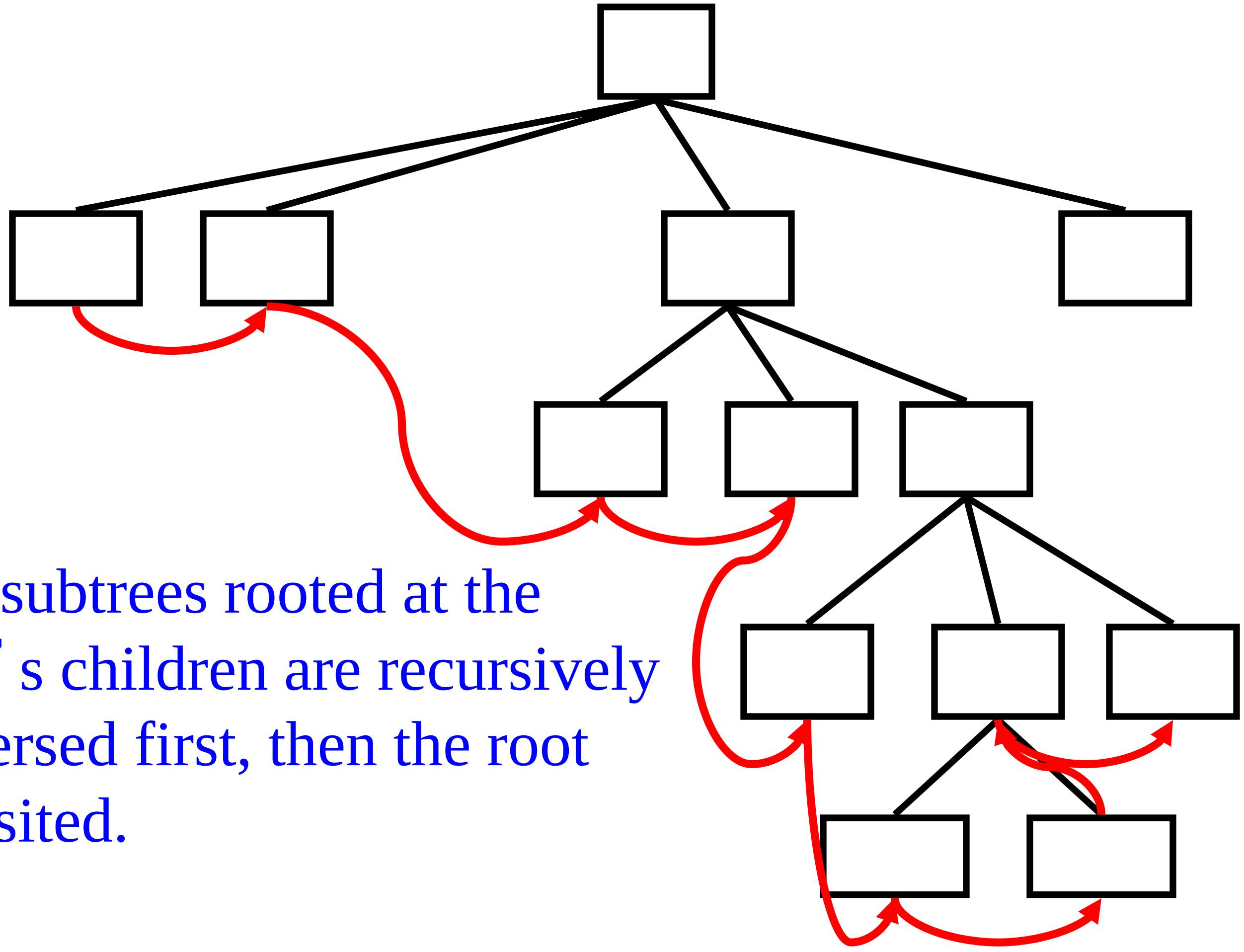

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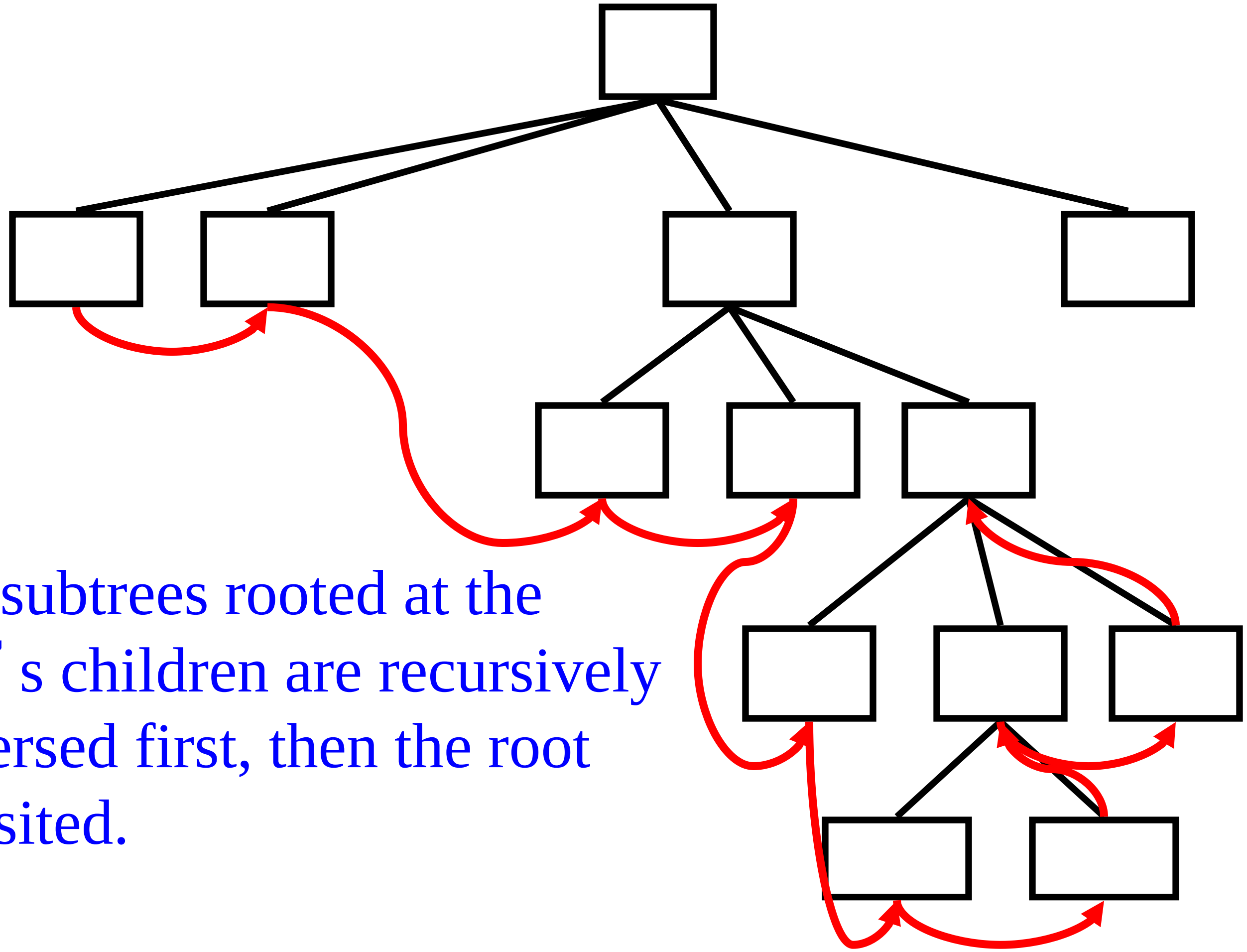

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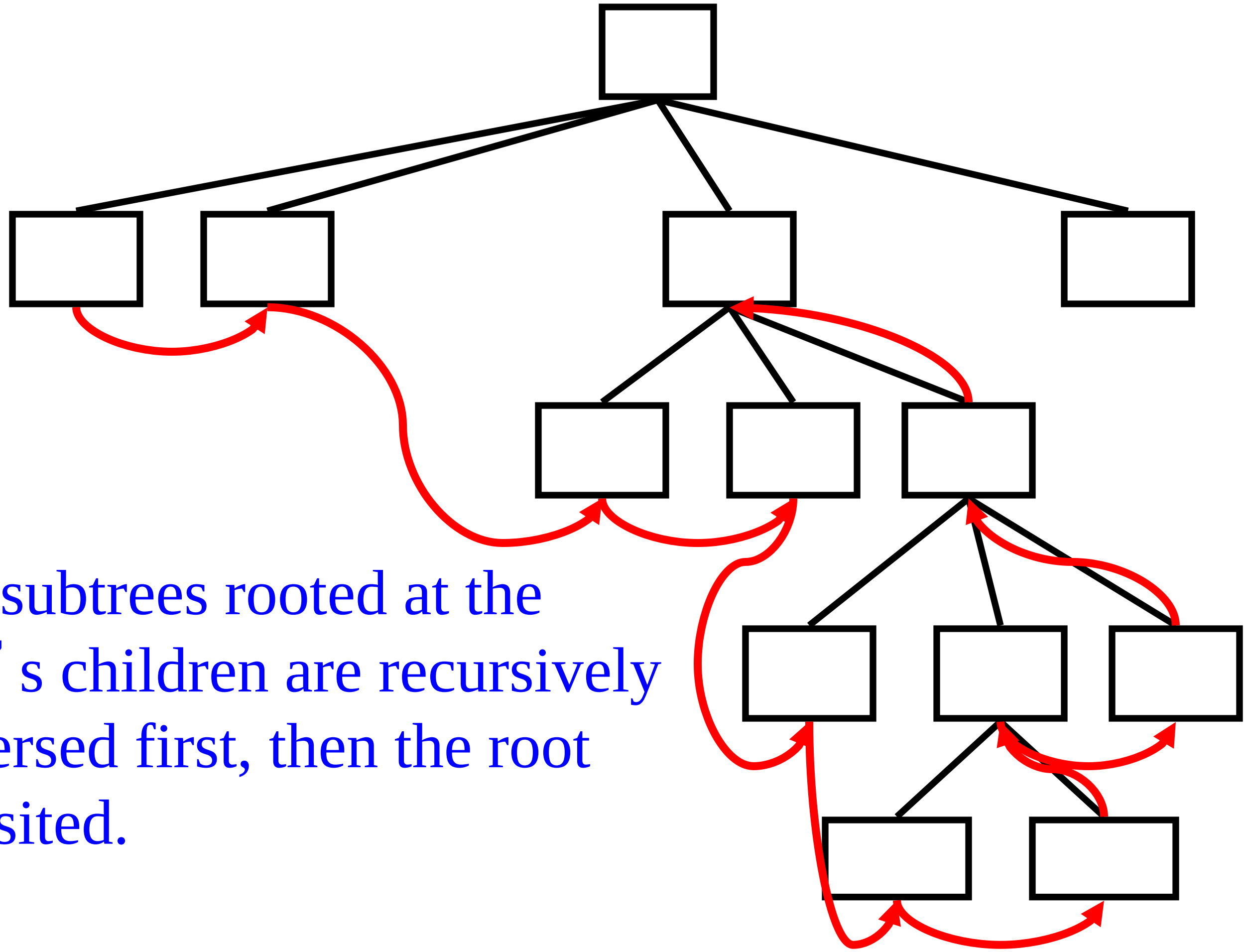

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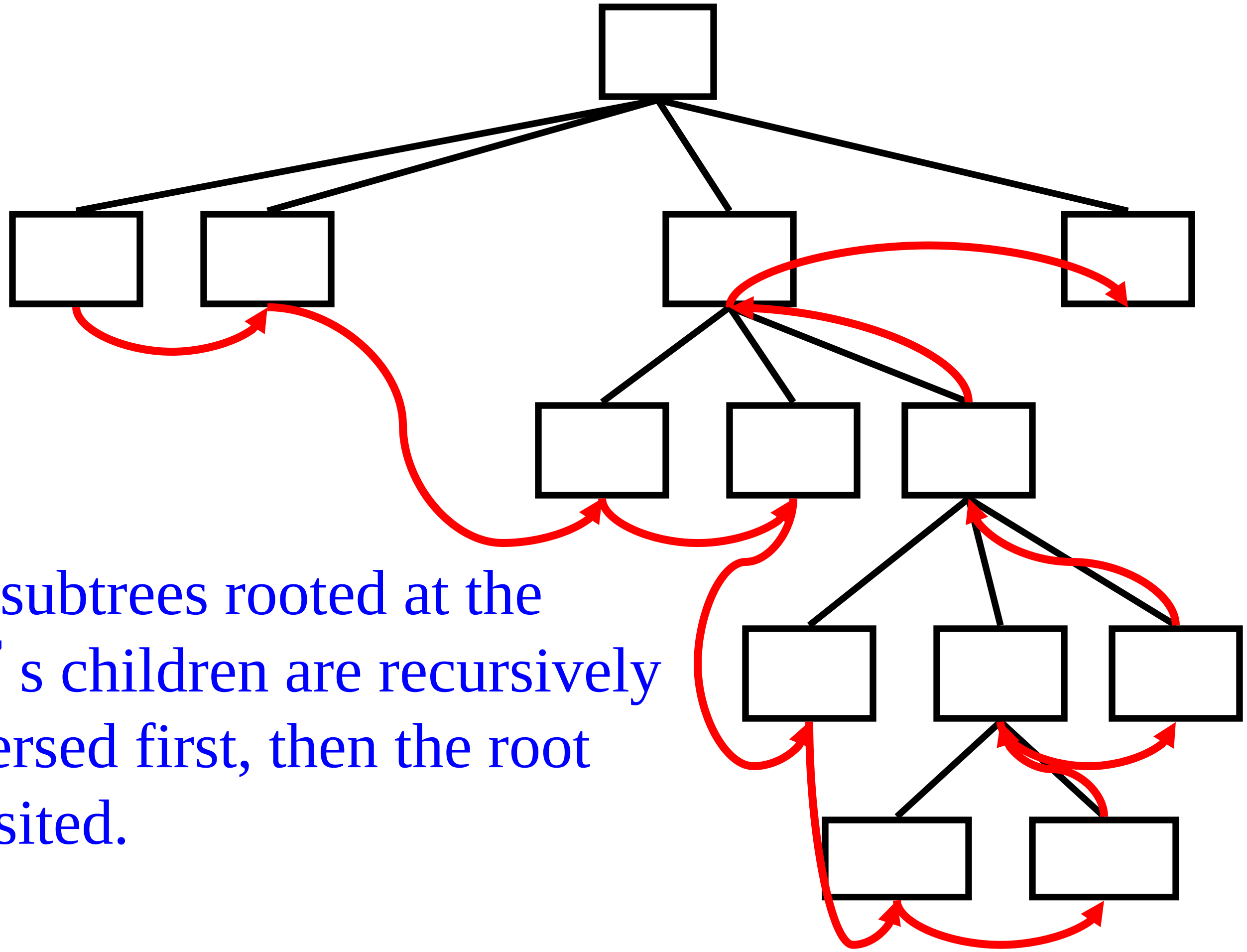

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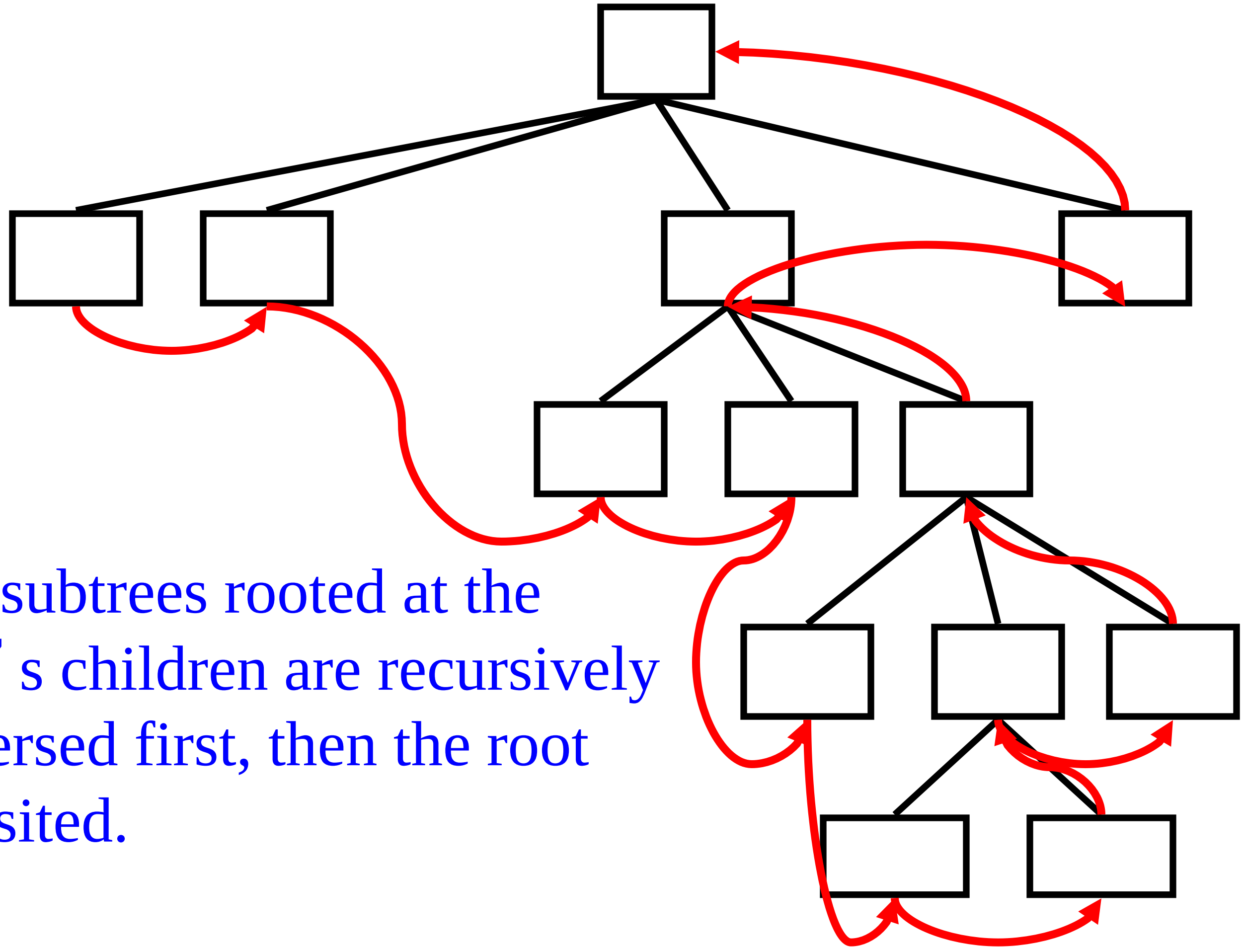

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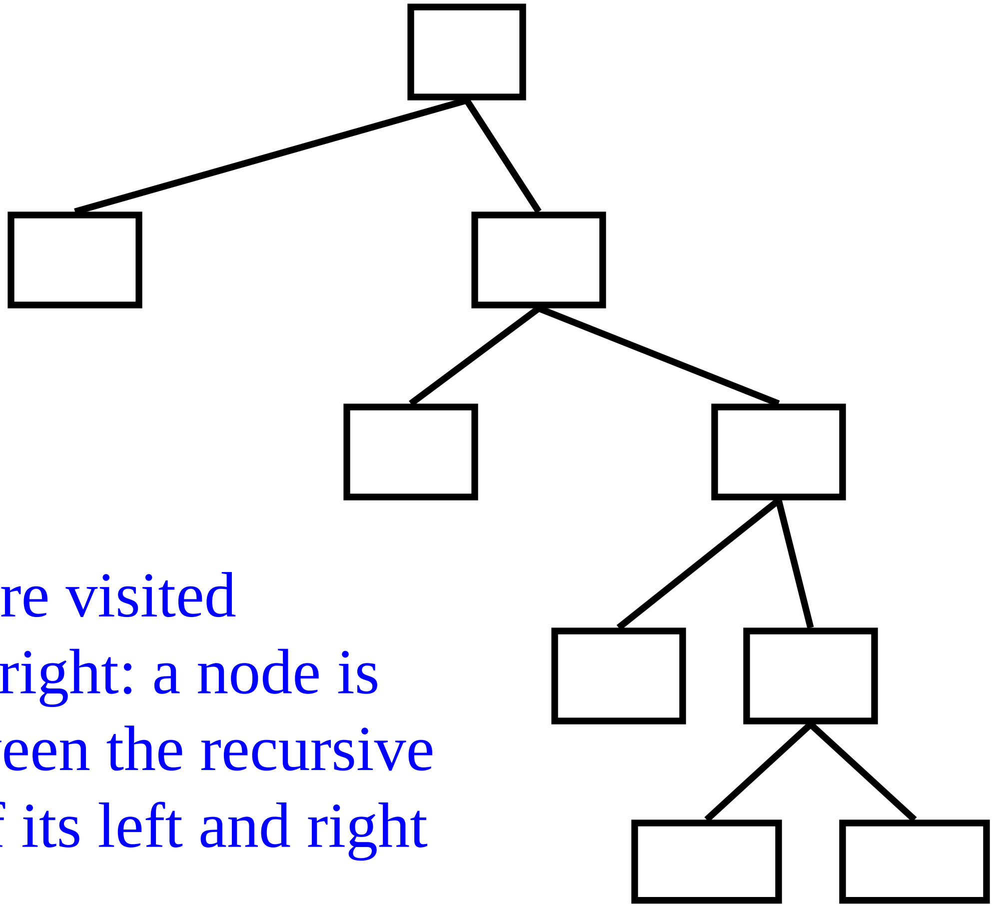
Inorder Traversal for Binary Trees



The nodes are visited from left to right: a node is visited between the recursive traversals of its left and right subtrees.

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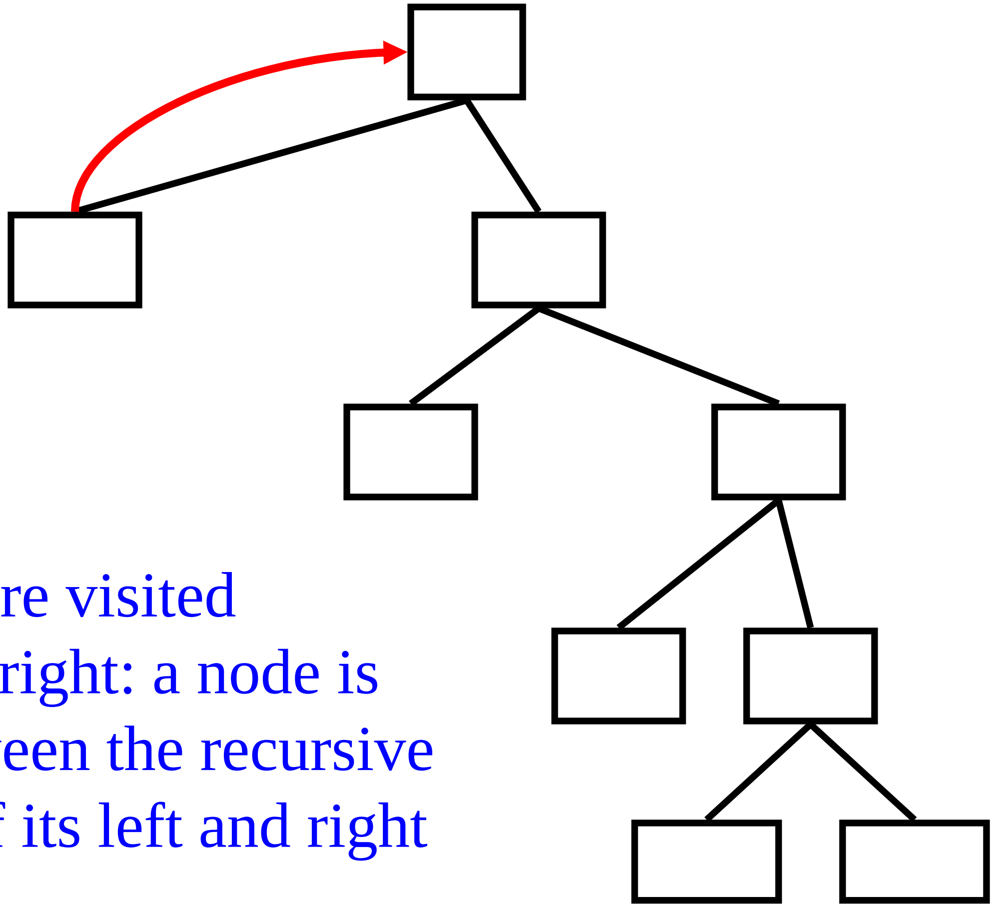

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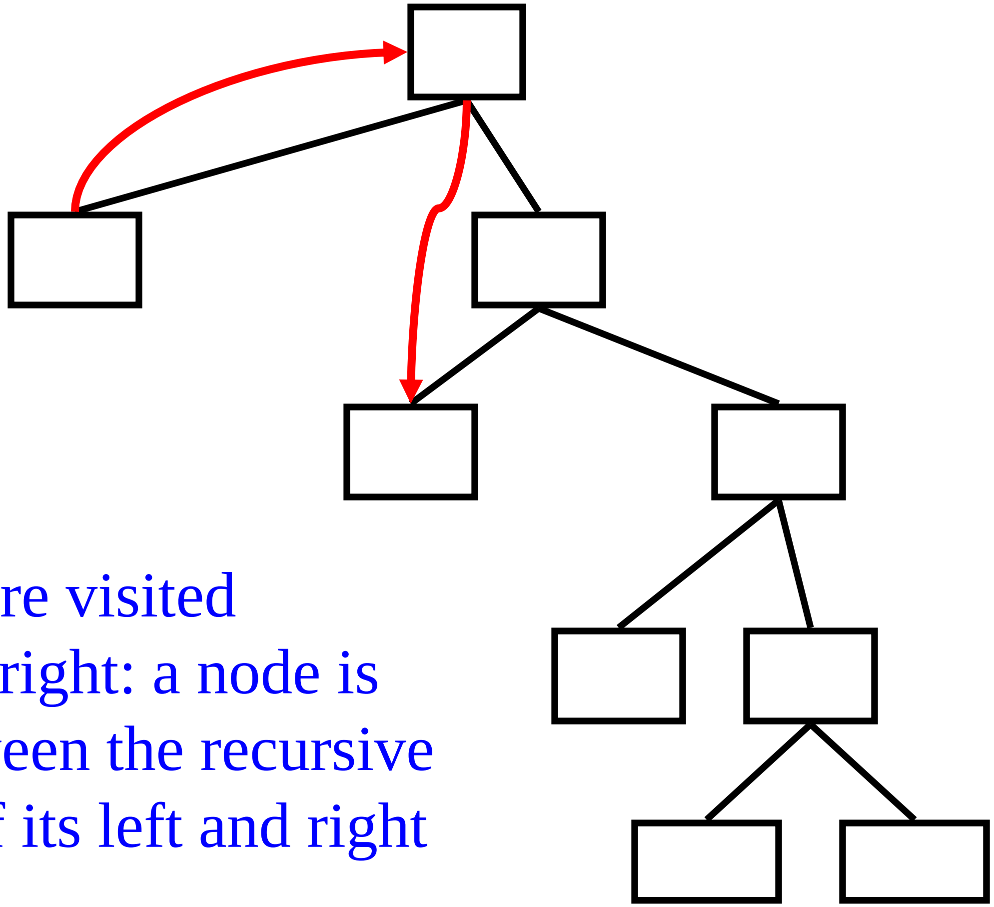

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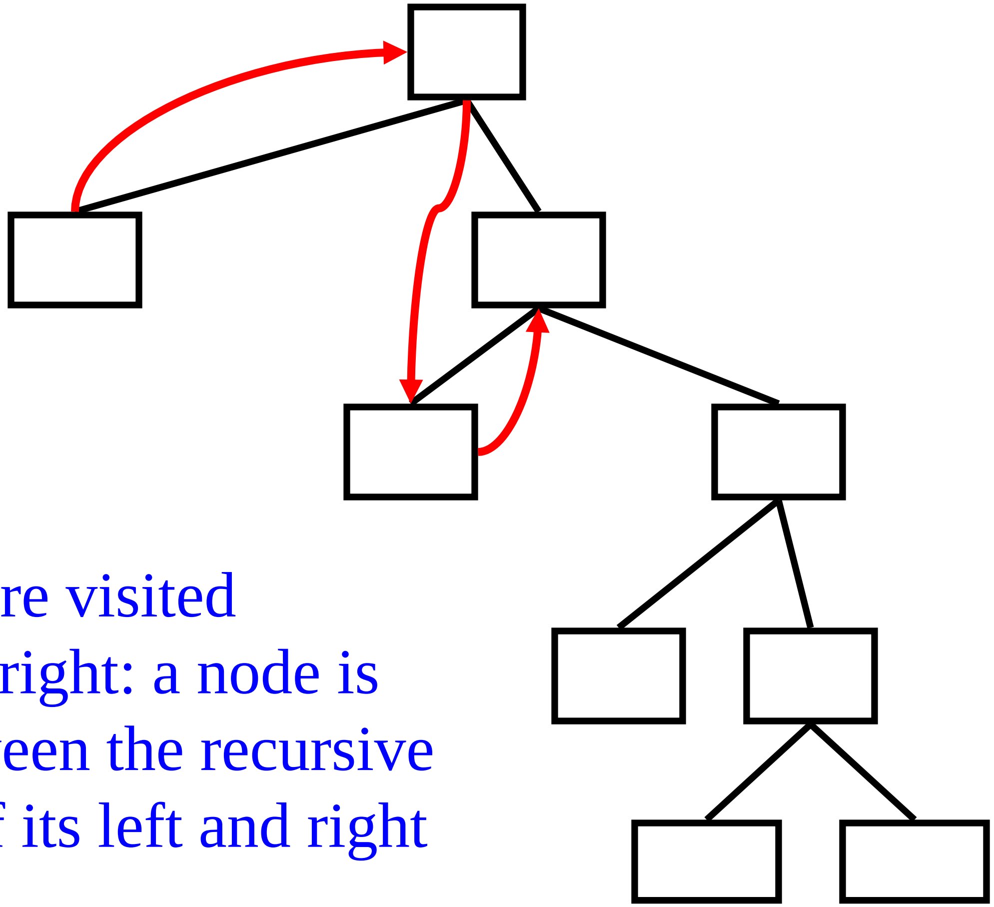

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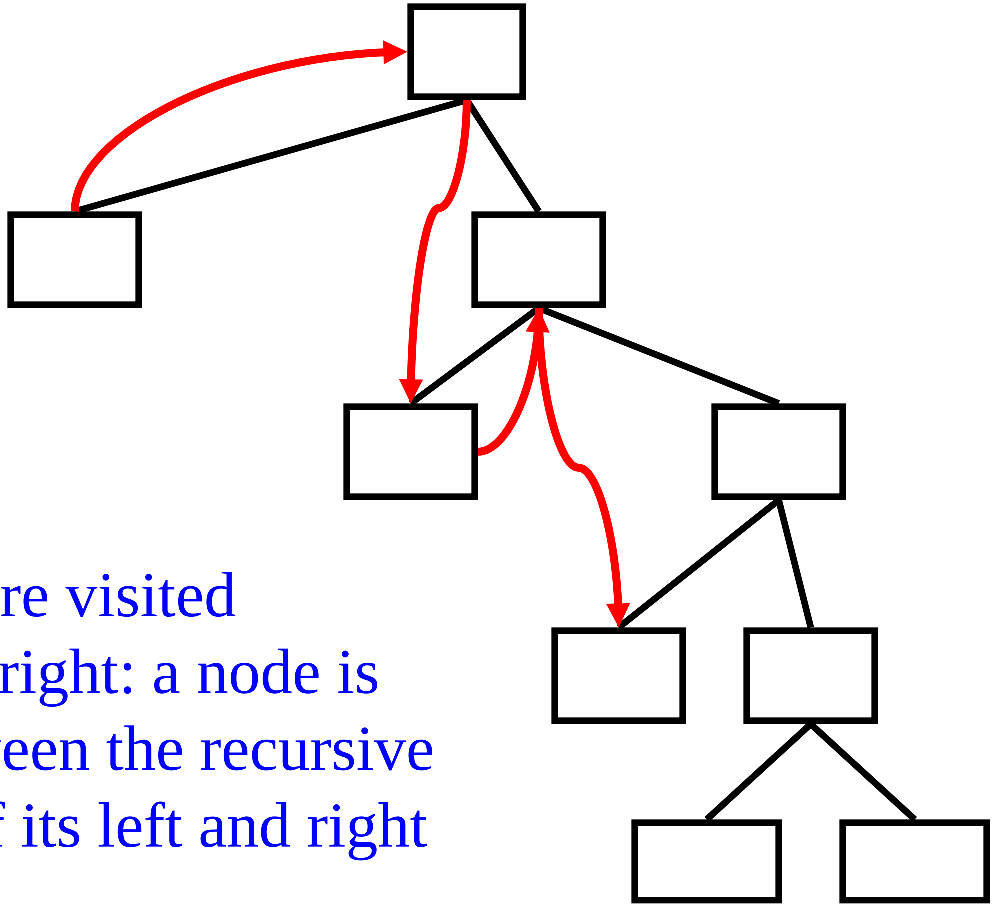

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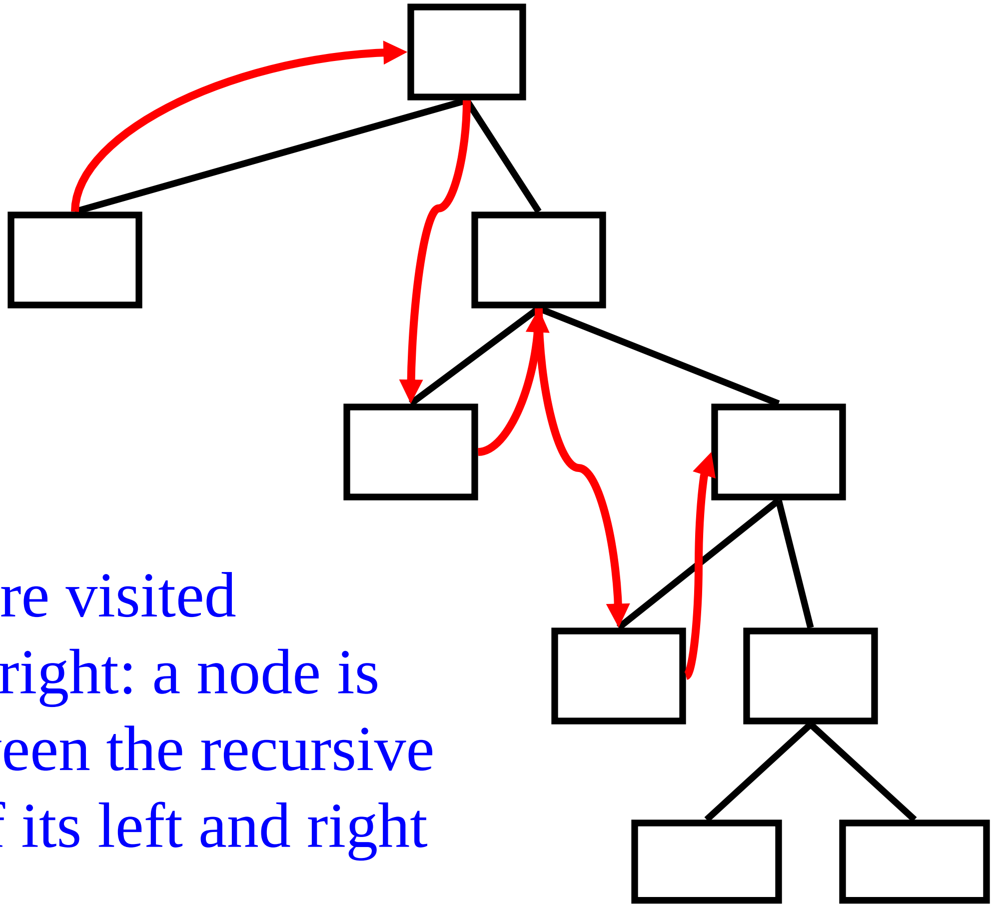

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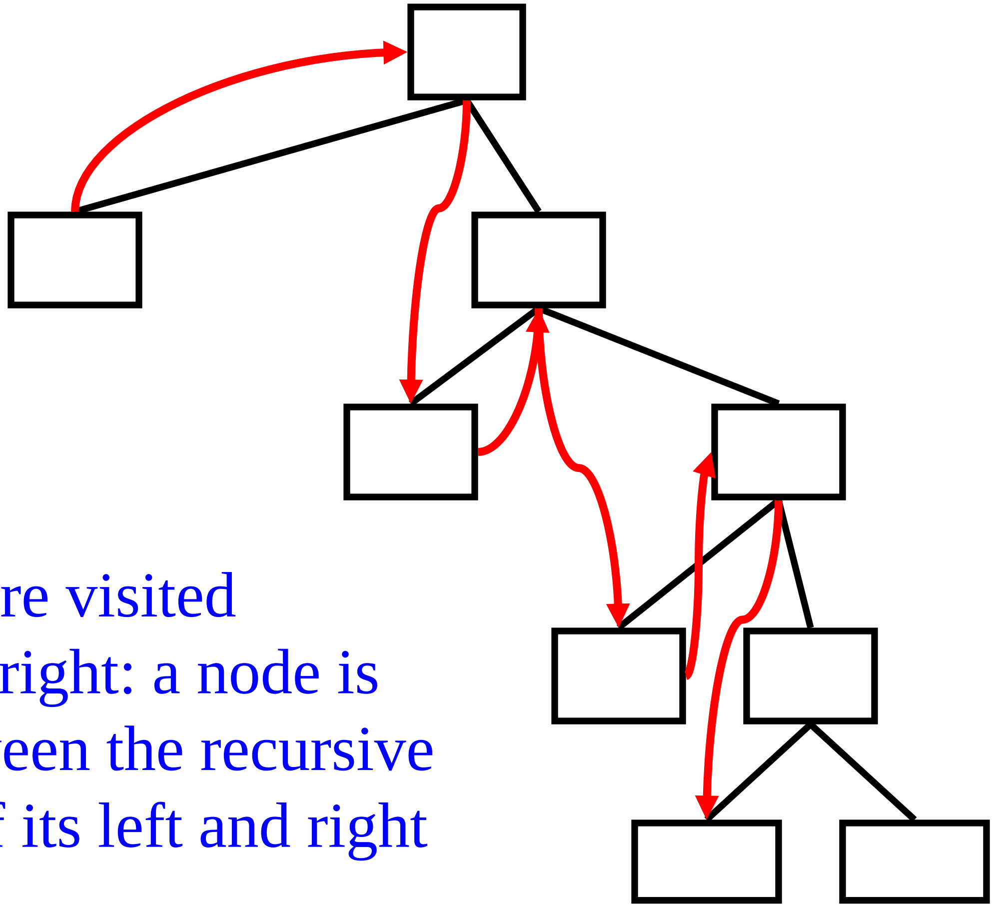

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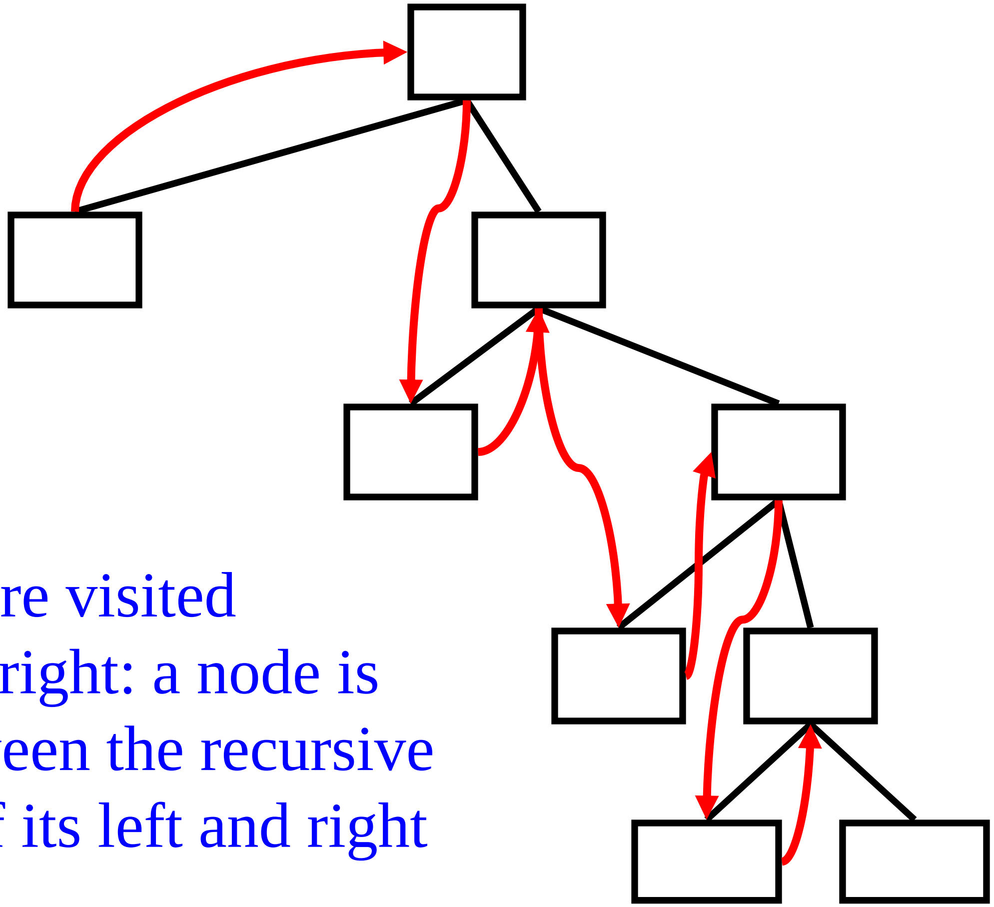

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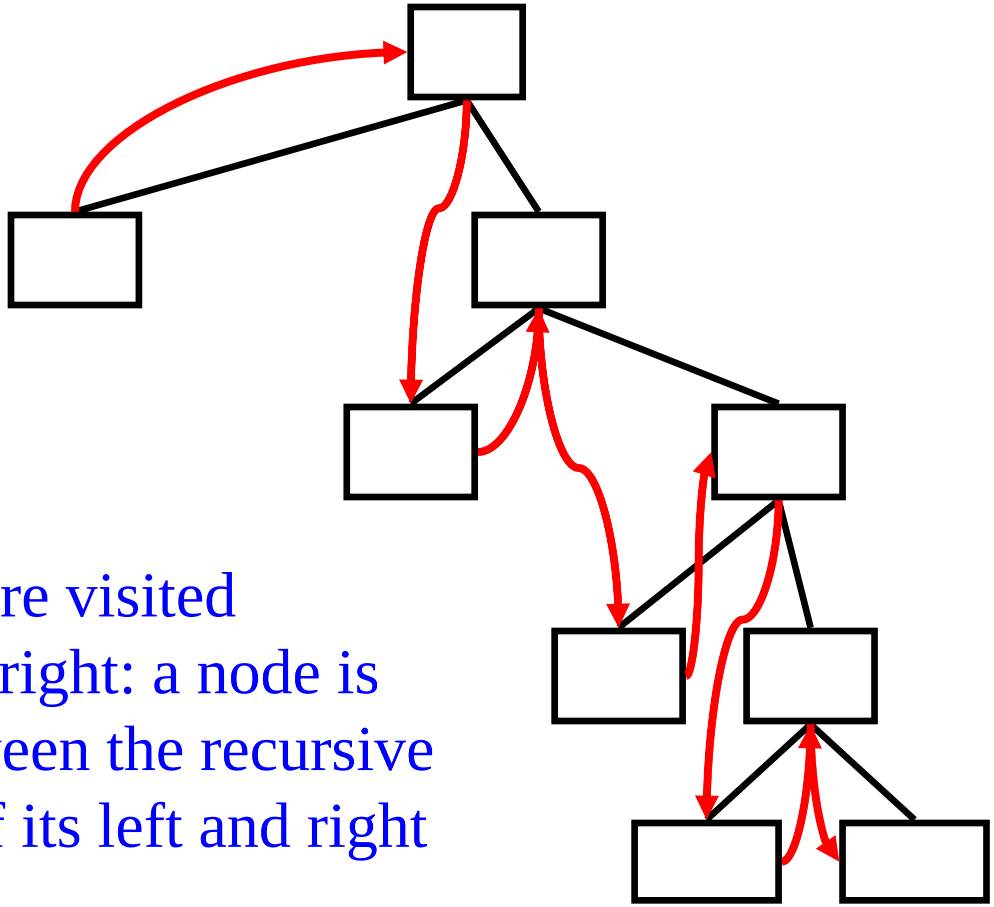

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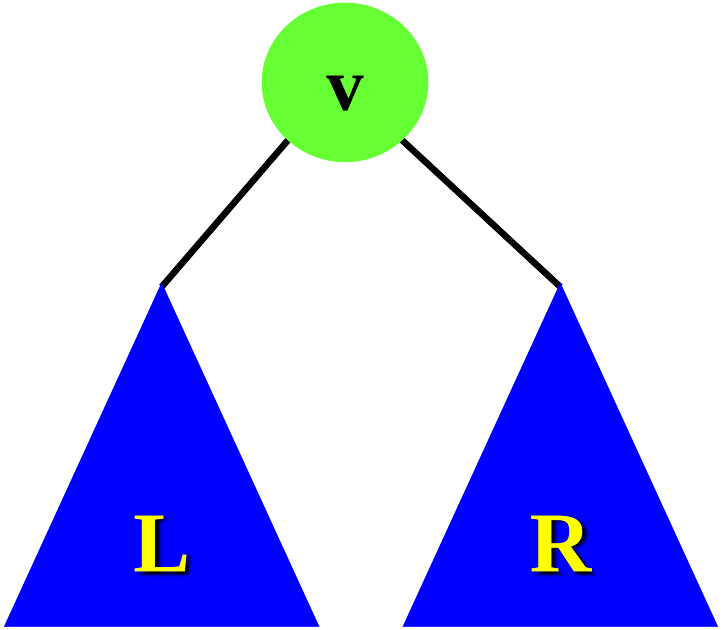
Tree Traversals



- Preorder
  - v, L, R
- Postorder
  - L, R, v
- Inorder
  - L, v, R
- Levelorder
  - v, vL, vR, L, R

Tree Representations

- Dynamic data structures
  - Using references or pointers
- Heap encoding
  - Using an array

Dynamic Tree Data Structure

```
public class TreeNode {
        private TreeNode left;
        private TreeNode right;
        private TreeNode parent;
        private int data;

        void setLeft(TreeNode t) { left = t;}
        TreeNode getLeft() { return left; }

        void setRight(TreeNode t) { right = t;}
        TreeNode getRight() { return right; }

        void setParent(TreeNode p) { parent = p;}
        TreeNode getParent() { return parent; }

        void setData(int d) { data = d;}
        int getData() { return data; }
}
```

<table>
<tr><td colspan="2">data</td></tr>
<tr><td colspan="2">parent</td></tr>
<tr><td>left</td><td>right</td></tr>
</table>

Preorder Traversal
Depth First Search (DFS)

```
algorithm preorder(Tree t)
   if t <> null then
         processNode(t.v)
         preorder(t.left)
         preorder(t.right)
    end
end
```

Inorder Traversal

```
algorithm inorder(Tree t)
  if t <> null then
      inorder(t.left)
      processNode(t.v)
      inorder(t.right)
   end
end
```

Postorder Traversal

```
algorithm postorder(Tree t)
  if t <> null then
      postorder(t.left)
      postorder(t.right)
      processNode(t.v)
   end
end
```

Levelorder Traversal
Breadth First Search (BFS)

```
algorithm levelorder(Tree t)
    Queue q
    q.enqueue(t)
    while not q.empty() do
        t = q.dequeue()
        processNode(t)
        if t.left <> null then q.enqueue(t.left) end
        if t.right <> null then q.enqueue(t.right) end
    end
end
```

Non-recursive Preorder Traversal Depth First Search (DFS)

```
algorithm dfs(Tree t)
    if t = null then return end
    Stack s
    s. push(t)
    while not s.empty() do
        t = s.pop()
        processNode(t)
        if t.right <> null then s.push(t.right) end
        if t.left <> null then s.push(t.left) end
    end
end
```

Running Time of Tree Traversals

- Each node is visited a fixed number of times (i.e., 3 times)
- **Theorem**
  - The time complexity of Preorder, Inorder, Postorder is $O(n)$

Tree Traversal Summary

Pre-order	Depth first	Stack
Inorder	Symmetric	Stack
Postorder	Bottom up	Stack
Levelorder	Breadth first	Queue

Tree Traversal Numberings

```
pre = 1;   in = 1;   post = 1
algorithm eulerNumberings(Tree t)
   if t <> null then
         t.pre = pre;   pre = pre + 1
         eulerNumberings(t.left)
         t.in = in;   in = in + 1
         eulerNumberings(t.right)
         t.post = post;   post = post + 1
    end
end
```

Heap Encoding

Parent of A[k] is at A[k/2]

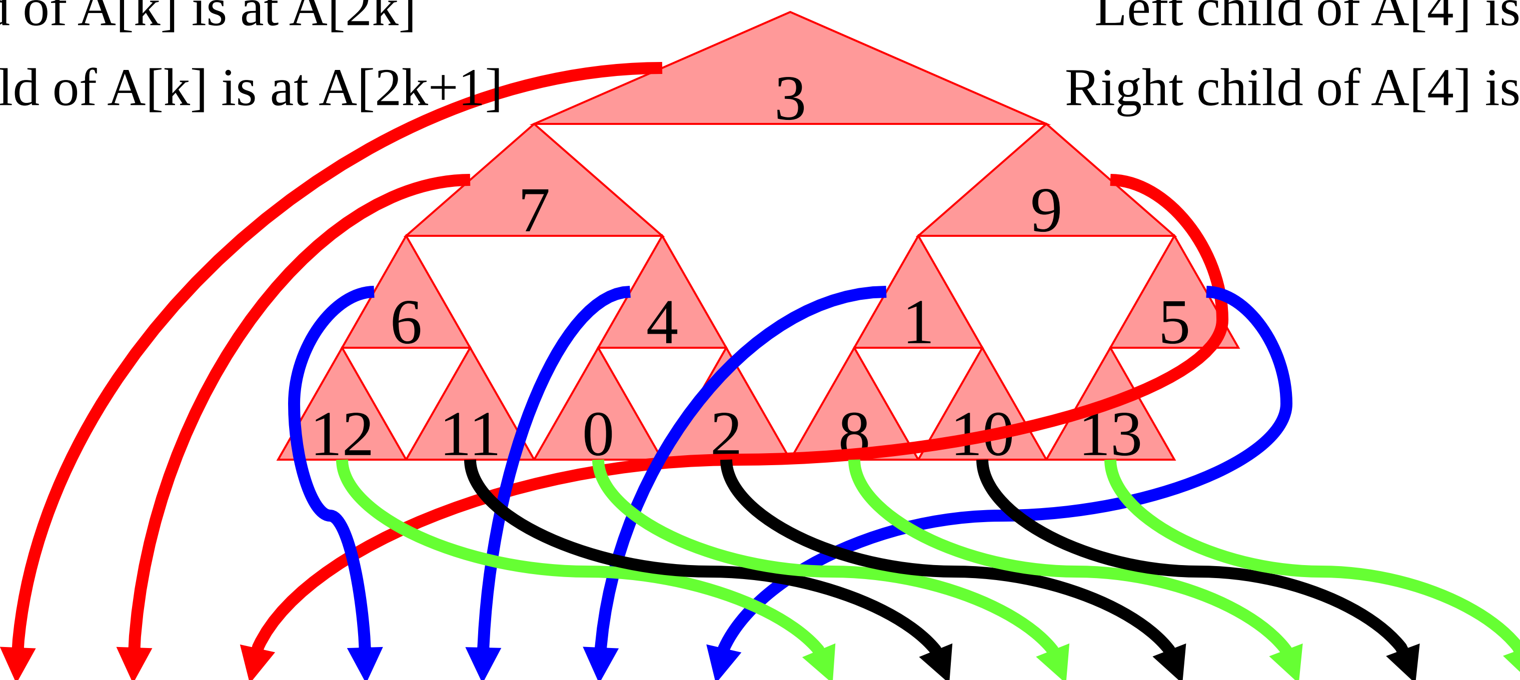
Left child of A[k] is at A[2k]

Right child of A[k] is at A[2k+1]

Parent of A[6] is at A[3]

Left child of A[4] is at A[8]

Right child of A[4] is at A[9]



A	3	7	9	6	4	1	5	12	11	0	2	8	10	13
index	1	2	3	4	5	6	7	8	9	10	11	12	13	14

Note index starts at 1

Data Structures for *binary* Trees

Operation	Time with vector-based structure	Time with linked structure
positions, elements traversals (iterators): pre-, in-, post-, level-order	$O(n)$	$O(n)$
size, isEmpty	O(1)	O(1)
swapElements, replaceElement	O(1)	O(1)
leftChild, rightChild, sibling	O(1)	O(1)
isInternal, isExternal, isRoot	O(1)	O(1)
root, parent, child	O(1)	O(1)