# 第五章

## 定积分

§1　定积分的概念与性质

§2　微积分基本公式

§3　定积分的换元法和分部积分法

§4　反常积分

第五章

# 第三节

# 定积分的换元法和分部积分法


## 内容

一、定积分的换元法

二、定积分的分部积分法

# 一、定积分的换元法

假设 (1) $f(x)$在$[a,b]$上连续；

(2)函数$x=\varphi(t)$在$[\alpha,\beta]$上是单值的且有连续导数；

(3)在$[\alpha,\beta]$上,$a\le\varphi(t)\le b$,且

| $x$ | $a$ | $b$ |
|---|---|---|
| $t$ | $\alpha$ | $\beta$ |

,

则有 $\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$（$\alpha>\beta$仍成立）

**证明** 设$F(x)$是$f(x)$的一个原函数, $\boxed{\int_a^b f(x)\mathrm{d}x=F(b)-F(a)}$

设$\psi(t)=F(\varphi(t))$, $\psi'(t)=F'(\varphi(t))\cdot\varphi'(t)=f(\varphi(t))\varphi'(t)$

故$\psi(t)$是$f(\varphi(t))\varphi'(t)$的一个原函数

$$\boxed{\int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t= \qquad F(b)-F(a)}$$

# 一、定积分的换元法

**假设** (1) $f(x)$在$[a,b]$上连续；

(2)函数$x=\varphi(t)$在$[\alpha,\beta]$上是单值的且有连续导数；

(3)在$[\alpha,\beta]$上，$a\le\varphi(t)\le b$，且 $\begin{array}{c|cc} x & a & b \\ \hline t & \alpha & \beta \end{array}$，

则有 $\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$（$\alpha>\beta$仍成立）

$\longrightarrow$ 对应第二类换元积分法

**说明** 1) 换元公式也可反过来使用，即

对应凑微分法

$$\int_\alpha^\beta f[\varphi(t)]\varphi'(t)\,\mathrm{d}t=\int_\alpha^\beta f[\varphi(t)]\,\mathrm{d}\varphi(t)=\int_a^b f(x)\,\mathrm{d}x \ (令\ x=\varphi(t))$$

2) **换元要换限**，原函数中的变量不必代回

3) 如果原函数中变量没变，则上下限就不变

## 典型题 ①换元积分法

第二类换元

**例1** $\int_0^{\frac{1}{2}} \frac{x^2 \mathrm{d}x}{\sqrt{1-x^2}}$

**解**: 设$x=\sin t, \mathrm{d}x=\cos t\mathrm{d}t$, 

| $x$ | $0$ | $\frac{1}{2}$ |
|---|---|---|
| $t$ | $0$ | $\frac{\pi}{6}$ |

,

$$\text{原式}=\int_0^{\frac{\pi}{6}} \frac{\sin^2 t \cdot \cos t\mathrm{d}t}{\cos t}=\int_0^{\frac{\pi}{6}} \frac{1-\cos 2t}{2}\mathrm{d}t=\left(\tfrac{1}{2}t-\tfrac{1}{4}\sin 2t\right)\Big|_0^{\frac{\pi}{6}}=\frac{\pi}{12}-\frac{\sqrt{3}}{8}$$

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**例2** $\int_1^4 \frac{\mathrm{d}x}{1+\sqrt{x}}$

**解**: 设$\sqrt{x}=t,\ x=t^2,\ \mathrm{d}x=2t\mathrm{d}t$, 

| $x$ | $1$ | $4$ |
|---|---|---|
| $t$ | $1$ | $2$ |

,

$$\text{原式}=\int_1^2 \frac{2t\overset{+2-2}{\mathrm{d}t}}{1+t}=2\int_1^2 \mathrm{d}t-2\int_1^2 \frac{1}{1+t}\mathrm{d}t=2-2\ln(1+t)\Big|_1^2=2-2\ln\frac{3}{2}$$

## 典型题 ①换元积分法

**例3** $\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,\mathrm{d}x$

**解:** $\sqrt{\sin^3 x-\sin^5 x}=\sin^{\frac{3}{2}}x\cdot|\cos x|$

第一类换元

$$原式=\int_0^{\frac{\pi}{2}}\sin^{\frac{3}{2}}x\cos x\mathrm{d}x+\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac{3}{2}}x\cdot(-\cos x)\mathrm{d}x$$

$$=\int_0^{\frac{\pi}{2}}\sin^{\frac{3}{2}}x\mathrm{d}\sin x-\int_{\frac{\pi}{2}}^{\pi}\sin^{\frac{3}{2}}x\mathrm{d}\sin x$$

$$=[\tfrac{2}{5}\sin^{\frac{5}{2}}x]_0^{\frac{\pi}{2}}-[\tfrac{2}{5}\sin^{\frac{5}{2}}x]_{\frac{\pi}{2}}^{\pi}$$

$$=\frac{4}{5}$$

变量没变上
下限就不变

令$t=\sin x$, $\mathrm{d}t=\cos x\mathrm{d}x$

$$原式=\int_0^1 t^{\frac{3}{2}}\mathrm{d}t-\int_1^0 t^{\frac{3}{2}}\mathrm{d}t$$

$$=2\cdot\tfrac{2}{5}t^{\frac{5}{2}}\Big|_0^1$$

$$=\frac{4}{5}$$

## 典型题 ①换元积分法+分段函数

**例4** 设函数$f(x)=\begin{cases}1+x^2 & x\le 0\\ e^{-x} & x>0\end{cases}$ 求 $\int_1^3 f(x-2)\mathrm{d}x$

**法1**: 令$x-2=t$, $\mathrm{d}x=\mathrm{d}t$

$$\begin{array}{c|cc} x & 1 & 3 \\ \hline t & -1 & 1 \end{array},$$

原式$=\int_{-1}^{1} f(t)\mathrm{d}t=\int_{-1}^{1} f(x)\mathrm{d}x$

$$=\int_{-1}^{0}(1+x^2)\mathrm{d}x+\int_0^1 e^{-x}\mathrm{d}x$$

$$=\frac{7}{3}-e^{-1}$$

**法2**:

$$f(x-2)=\begin{cases}1+(x-2)^2 & x-2\le 0 \Rightarrow x\le 2\\ e^{-(x-2)} & x-2>0 \Rightarrow x>2\end{cases}$$

原式$=\int_1^2 1+(x-2)^2 dx+\int_2^3 e^{-(x-2)}\mathrm{d}x$

$$=\frac{7}{3}-e^{-1}$$

## 典型题 ①换元积分法 +积分上限函数

用$u$换$t, x$看作数

$\left(\int_0^x f(x+t)\mathrm{d}t\right)'$ 要领 令$u=x+t$, $\mathrm{d}u=\mathrm{d}t$, $\begin{array}{c|cc} t & 0 & x \\ \hline u & x & 2x \end{array}$,

先求$\int_0^x f(x+t)\mathrm{d}t=\int_x^{2x} f(u)\mathrm{d}u$ 再求导

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**例5** 设 $F(x)=\int_0^x tf(x^2-t^2)\mathrm{d}t$, $f$连续, 求$F'(x)$

**解**: 令$u=x^2-t^2$, $\mathrm{d}u=-2t\mathrm{d}t$, $\begin{array}{c|cc} t & 0 & x \\ \hline u & x^2 & 0 \end{array}$,

$$F(x)=\int_{x^2}^0 f(u)(-\frac{1}{2})\mathrm{d}u=\frac{1}{2}\int_0^{x^2} f(u)\mathrm{d}u$$

$$F'(x)=\frac{1}{2}f(x^2)\cdot 2x=xf(x^2)$$

②**对称区间上的定积分** 设 $f(x)$在$[-a, a]$上连续,则

(i) $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a}[f(x)+f(-x)]\mathrm{d}x$;

(ii) $\int_{-a}^{a} f(x)\mathrm{d}x = \begin{cases} 2\int_{0}^{a} f(x)\,\mathrm{d}x & f(x)\text{为偶函数} \\ 0 & f(x)\text{为奇函数} \end{cases}$

(iii) $f(x)$为偶 $\Leftrightarrow \int_{0}^{x} f(t)\mathrm{d}t$为奇; $f(x)$为奇$\Leftrightarrow \int_{0}^{x} f(t)\mathrm{d}t$为偶

**证** (i) $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{-a}^{0} f(x)\mathrm{d}x + \int_{0}^{a} f(x)\mathrm{d}x.$

令$x$=-$t$，则$\mathrm{d}x$ =-d$t$， $\begin{array}{c|cc} x & -a & 0 \\ \hline t & a & 0 \end{array}$,

$\int_{-a}^{0} f(x)\mathrm{d}x = \int_{a}^{0} f(-t)(-\mathrm{d}t) = \int_{0}^{a} f(-t)\mathrm{d}t = \int_{0}^{a} f(-x)\mathrm{d}x.$ 得证

②**对称区间上的定积分** 设 $f(x)$在$[-a, a]$上连续,则

(i) $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a}[f(x)+f(-x)]\mathrm{d}x;$

对称区间，被积函数非奇非偶时可以应用

(ii) $\int_{-a}^{a} f(x)\mathrm{d}x = \begin{cases} 2\int_{0}^{a} f(x)\mathrm{d}x & f(x)\text{为偶函数} \\ 0 & f(x)\text{为奇函数} \end{cases}$

(iii) $f(x)$为偶 $\Leftrightarrow \int_{0}^{x} f(t)\mathrm{d}t$为奇; $f(x)$为奇$\Leftrightarrow \int_{0}^{x} f(t)\mathrm{d}t$为偶

**例** $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\mathrm{d}x}{1+\sin x} = \int_{0}^{\frac{\pi}{4}}\left[\frac{1}{1+\sin x}+\frac{1}{1-\sin x}\right]\mathrm{d}x$

$$= \int_{0}^{\frac{\pi}{4}} \frac{2}{\cos^2 x}\mathrm{d}x = 2\tan x\Big|_{0}^{\frac{\pi}{4}} = 2$$

**②对称区间上的定积分** 设 $f(x)$在[-$a$, $a$]上连续,则

(i) $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a}\left[f(x)+f(-x)\right]\mathrm{d}x;$

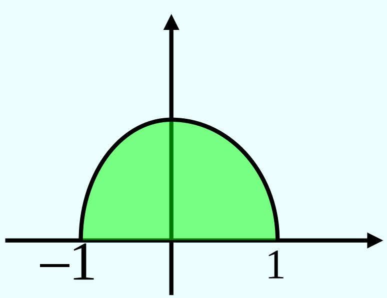
(ii) $\int_{-a}^{a} f(x)\mathrm{d}x = \begin{cases} 2\int_{0}^{a} f(x)\,\mathrm{d}x & f(x)\text{为偶函数} \\ 0 & f(x)\text{为奇函数} \end{cases}$

(iii) $f(x)$为偶 $\Leftrightarrow \int_{0}^{x} f(t)\mathrm{d}t$为奇; $f(x)$为奇$\Leftrightarrow \int_{0}^{x} f(t)\mathrm{d}t$为偶

**证**(ii) 若$f(x)$为偶,$f(-x)=f(x)$;若$f(x)$为奇,$f(-x)=-f(x)$; 由(i)立得

**例** $\int_{-1}^{1}(\cancel{\ln\frac{1+x}{1-x}}+\sqrt{1-x^2})\mathrm{d}x$

$= \frac{\pi}{2}$




**常见奇函数**

$\ln\frac{1-x}{1+x}$　　$\ln(\sec x \pm \tan x)$

$\ln(x+\sqrt{x^2+1})$

**②对称区间上的定积分** 设 $f(x)$在[-$a$, $a$]上连续,则

(i) $\int_{-a}^{a} f(x)\mathrm{d}x = \int_{0}^{a}[f(x)+f(-x)]\mathrm{d}x;$

(ii) $\int_{-a}^{a} f(x)\mathrm{d}x = \begin{cases} 2\int_{0}^{a} f(x)\,\mathrm{d}x & f(x)\text{为偶函数} \\ 0 & f(x)\text{为奇函数} \end{cases}$

(iii) $f(x)$为偶 $\Leftrightarrow \int_{0}^{x} f(t)\mathrm{d}t$为奇; $f(x)$为奇$\Leftrightarrow \int_{0}^{x} f(t)\mathrm{d}t$为偶

证(iii) $\Phi(x) = \int_{0}^{x} f(t)\mathrm{d}t$

$$\Phi(-x) = \int_{0}^{-x} f(t)\mathrm{d}t \overset{\text{令}t=-u}{=\!=\!=} -\int_{0}^{x} f(-u)\mathrm{d}u$$

$$\overset{f(x)\text{偶}}{=\!=\!=} -\int_{0}^{x} f(u)\mathrm{d}u = -\Phi(x)$$

$\therefore \Phi(x)$为奇函数

其它类似可得

**③三角函数的定积分**　设 $f(x)$在[0,1]上连续,则

$$\text{(i)}\int_0^{\pi} f(\sin x)\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$$

$$\text{(ii)}\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x$$

$$\text{(iii)}\int_0^{\frac{\pi}{2}} f(\sin x,\cos x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x,\sin x)\mathrm{d}x$$

证 (i) $\int_0^{\pi} f(\sin x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x + \int_{\frac{\pi}{2}}^{\pi} f(\sin x)\mathrm{d}x$　令$x=\pi-t$

$$= \int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x \cancel{+}- \int_{\frac{\pi}{2}}^{0} f(\sin \cancel{t}x)\mathrm{d}\,\cancel{t}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x$$

$$\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x \xlongequal{x=\frac{\pi}{2}-t} \int_{\frac{\pi}{2}}^{0} f(\cos t)(-\mathrm{d}t) = \int_0^{\frac{\pi}{2}} f(\cos \cancel{t}x)\mathrm{d}\cancel{t}x$$　类似可得(iii)

例 $$\int_0^a \frac{\mathrm{d}x}{x+\sqrt{a^2-x^2}} \xlongequal{x=a\sin t} \int_0^{\frac{\pi}{2}} \frac{\cancel{a}\cos t\mathrm{d}t}{\cancel{a}\sin t+\cancel{a}\cos t} = \int_0^{\frac{\pi}{2}} \frac{\sin t\mathrm{d}t}{\cos t+\sin t} = \frac{1}{2}\int_0^{\frac{\pi}{2}} 1\mathrm{d}t = \frac{\pi}{4}$$

## ③三角函数的定积分 设 $f(x)$在[0,1]上连续,则

$$\text{(i)}\ \int_0^{\pi} f(\sin x)\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x = 2\int_0^{\frac{\pi}{2}} f(\cos x)\mathrm{d}x$$

$$\text{(ii)}\int_0^{\pi} xf(\sin x)\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\mathrm{d}x$$

$$\text{(iii)}\int_0^{\frac{\pi}{2}} f(\sin x,\cos x)\mathrm{d}x = \int_0^{\frac{\pi}{2}} f(\cos x,\sin x)\mathrm{d}x$$

**证** $$\text{(ii)}\ \int_0^{\pi} xf(\sin x)\mathrm{d}x \overset{x=\pi-t}{=\!=\!=\!=} \int_{\pi}^{0} (\pi-t)f(\sin t)(-\mathrm{d}t)$$

$$= \int_0^{\pi} \pi f(\sin t)\mathrm{d}t - \int_0^{\pi} tf(\sin t)\mathrm{d}t$$ 移项即得

**例** $$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\mathrm{d}x = -\frac{\pi}{2}\int_0^{\pi} \frac{\mathrm{d}\cos x}{1+\cos^2 x}$$

$$= -\frac{\pi}{2}[\arctan(\cos x)]\Big|_0^{\pi} = \frac{\pi^2}{4}$$

## ④周期函数的定积分　设$T$是$f(x)$的周期,则

(i) $\int_a^{a+T} f(x)\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x$

(ii) $\int_a^{a+nT} f(x)\mathrm{d}x = n\int_0^T f(x)\mathrm{d}x \quad n\in N, a\in R$

**证** (i) $\int_a^{a+T} f(x)\mathrm{d}x = \cancel{\int_a^0 f(x)\mathrm{d}x} + \int_0^T f(x)\mathrm{d}x + \cancel{\int_T^{a+T} f(x)\mathrm{d}x}$

其中 $\int_T^{a+T} f(x)\mathrm{d}x \xlongequal{x=t+T} \int_0^a f(t+T)\mathrm{d}t = \int_0^a f(t)\mathrm{d}t$

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**例** 设$F(x)=\int_x^{x+2\pi} e^{\sin t}\cdot\sin t\mathrm{d}t$,则$F(x)$为正常数

**证** 由(i)$F(x)=\int_0^{2\pi} e^{\sin t}\cdot\sin t\mathrm{d}t = -\int_0^{2\pi} e^{\sin t}\mathrm{d}\cos t$

$$= \underbrace{-e^{\sin t}\cos t\,\Big|_0^{2\pi}}_{=0} + \underbrace{\int_0^{2\pi} e^{\sin t}\cos^2 t\mathrm{d}t}_{>0}$$

## 二、定积分的分部积分法

设函数$u = u(x)$, $v = v(x)$在$[a, b]$上具有连续导数，

即$u' = u'(x)$,$v' = v'(x)$ 连续，由不定积分的分部积分法，

$$\int u(x)v'(x)\mathrm{d}x = u(x)v(x) - \int v(x)u'(x)\mathrm{d}x,$$

则 $$\int_a^b u(x)v'(x)\mathrm{d}x = \left[u(x)v(x) - \int v(x)u'(x)\mathrm{d}x\right]_a^b$$

即 $$\int_a^b u(x)v'(x)\mathrm{d}x = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\mathrm{d}x$$

**说明** **(i)$u$与$v$的选择方法与不定积分的分部积分法相同**

**(ii)求出原函数后不要忘了代入上下限求出数值**

## 典型题

**例1** $\int_0^1 \frac{\ln(1+x)}{(2-x)^2}\mathrm{d}x$

**解**: 原式

$$= \int_0^1 \ln(1+x)\mathrm{d}\frac{1}{2-x}$$

$$= \frac{\ln(1+x)}{2-x}\Big|_0^1 - \int_0^1 \frac{1}{(2-x)(1+x)}\mathrm{d}x$$

$$= \ln 2 - \frac{1}{3}\int_0^1 \frac{1}{2-x} + \frac{1}{1+x}\mathrm{d}x$$

$$= \frac{1}{3}\ln 2$$

**例2** $\int_0^1 e^{\sqrt{x}}\mathrm{d}x$

换元+分部

**解**: 令 $t=\sqrt{x},\ x=t^2, \mathrm{d}x = 2t\mathrm{d}t,$

原式 $= \int_0^1 e^t \cdot 2t\mathrm{d}t$

$$= 2\int_0^1 t\mathrm{d}e^t$$

$$= 2(te^t\big|_0^1 - \int_0^1 e^t\mathrm{d}t)$$

$$= 2(e - e^t\big|_0^1)$$

$$= 2$$

**例3.** 证明 $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}\cos^n x\,\mathrm{d}x$

$$=\begin{cases}\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}, & n\text{ 为偶数}\\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\cdots\frac{4}{5}\cdot\frac{2}{3}, & n\text{ 为奇数}\end{cases}$$

证: $I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x=-\int_0^{\frac{\pi}{2}}\sin^{n-1}x\,\mathrm{d}\cos x$

$$=\underbrace{\left[-\cos x\cdot\sin^{n-1}x\right]\Big|_0^{\frac{\pi}{2}}}_{0}+(n-1)\int_0^{\frac{\pi}{2}}\cos^2 x\sin^{n-2}x\,\mathrm{d}x$$

$$=(n-1)\int_0^{\frac{\pi}{2}}(\sin^{n-2}x-\sin^n x)\,\mathrm{d}x=(n-1)I_{n-2}-(n-1)I_n$$

由此得递推公式 $I_n=\frac{n-1}{n}I_{n-2}$

**例3.** 证明 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x \,\mathrm{d}x$

$$= \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots\cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}, & n \text{ 为偶数} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots\cdots \dfrac{4}{5} \cdot \dfrac{2}{3}, & n \text{ 为奇数} \end{cases}$$

**证:** 由此得递推公式 $I_n = \frac{n-1}{n} I_{n-2}$

于是 $I_{2m} = \dfrac{2m-1}{2m} \cdot \dfrac{2m-3}{2m-2} \cdots\cdots \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{\pi}{2}$

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$$

而 $I_0 = \int_0^{\frac{\pi}{2}} \mathrm{d}x = \dfrac{\pi}{2}$, $\quad I_1 = \int_0^{\frac{\pi}{2}} \sin x \,\mathrm{d}x = 1$

故所证结论成立 .

**例3.** 证明 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x\,\mathrm{d}x$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ 为偶数} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots\cdots \frac{4}{5} \cdot \frac{2}{3}, & n \text{ 为奇数} \end{cases}$$

**例:** $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \sin^6 x\,\mathrm{d}x$

$$= 2\int_0^{\frac{\pi}{2}} [\sin^6 x - \sin^8 x]\mathrm{d}x$$

$$= 2\left(\frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right)$$

$$= \frac{15}{384}\pi$$

**例4** 设 $f(x)=\int_0^x \frac{\sin t}{\pi - t}\mathrm{d}t$ 求 $\int_0^\pi f(x)\mathrm{d}x$

分部+积分变限函数

**解:** $\int_0^\pi f(x)\mathrm{d}x = xf(x)\big|_0^\pi - \int_0^\pi xf'(x)\mathrm{d}x$

$$= \pi f(\pi) - \int_0^\pi x\cdot \frac{\sin x}{\pi - x}\mathrm{d}x$$

$$= \pi f(\pi) + \int_0^\pi \frac{(\pi - x)\sin x - \pi \sin x}{\pi - x}\mathrm{d}x$$

$$= \pi f(\pi) + \int_0^\pi \sin x\mathrm{d}x - \pi \boxed{\int_0^\pi \frac{\sin x}{\pi - x}\mathrm{d}x}$$

$f(\pi)$

$$= \int_0^\pi \sin x\mathrm{d}x$$

$$= 2$$

**例5** 设 $\int_0^2 f(x)\mathrm{d}x=1,\ f(2)=\frac{1}{2},\ f'(2)=0$，求 $\int_0^1 x^2 f''(2x)\mathrm{d}x$

分部积分+函数符号

**解:** $\int_0^1 x^2 f''(2x)\mathrm{d}x$

$$=\frac{1}{2}\int_0^1 x^2\mathrm{d}f'(2x)$$

$$=\frac{1}{2}x^2 f'(2x)\Big|_0^1-\frac{1}{2}\int_0^1 f'(2x)\cdot 2x\mathrm{d}x$$

$$=\frac{1}{2}\underbrace{f'(2)}_{=0}-\int_0^1 f'(2x)x\mathrm{d}x=-\frac{1}{2}\int_0^1 x\mathrm{d}f(2x)$$

$$=-\frac{1}{2}\left(xf(2x)\Big|_0^1-\int_0^1 f(2x)\mathrm{d}x\right)$$

$$=-\frac{1}{2}\underbrace{f(2)}_{=\frac{1}{2}}+\frac{1}{2}\boxed{\int_0^1 f(2x)\mathrm{d}x}$$

$$=0$$

$$\int_0^1 f(2x)\mathrm{d}x$$

$$\xlongequal{令2x=t}\int_0^2 f(t)\cdot\frac{1}{2}\mathrm{d}t$$

$$=\frac{1}{2}\int_0^2 f(x)\mathrm{d}x$$

$$=\frac{1}{2}$$