第九章

第五节

隐函数的求导法则

内容

一、一个二元方程情形

二、一个三元方程情形

三、两个四元方程的方程组情形

一、一个二元方程情形

隐函数存在定理1 若点 $P(x_0, y_0)$ 的某邻域内满足

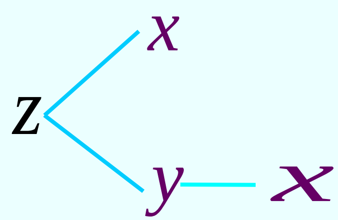
①$F(x,y)$有连续偏导数; ②$F(x_0, y_0) = 0$; ③$F_y(x_0, y_0) \neq 0$

则方程 $F(x, y) = 0$ 在点 x_0 的**某邻域内**可唯一确定一个单值连续函数 $y=f(x)$, 满足条件 $y_0 = f(x_0)$, 并有连续导数

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$ **(隐函数求导公式)**

证明: 略, 仅就求导公式推导如下:

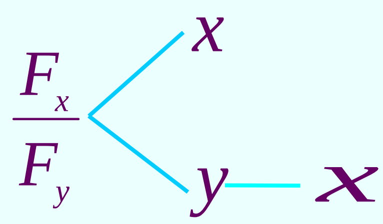
$$F(x, y) = 0 \Longrightarrow F(x, f(x)) \equiv 0$$



$$\frac{\mathrm{d}F}{\mathrm{d}x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Longrightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y}$$

二阶 若$F(x,y)$ 的二阶偏导数连续，

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$$



$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\partial}{\partial x}\left(-\frac{F_x}{F_y}\right)+\frac{\partial}{\partial y}\left(-\frac{F_x}{F_y}\right)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$=-\frac{F_{xx}F_y-F_{yx}F_x}{F_y^2}-\frac{F_{xy}F_y-F_{yy}F_x}{F_y^2}\left(-\frac{F_x}{F_y}\right)$$

$$=-\frac{F_{xx}F_y^2-2F_{xy}F_xF_y+F_{yy}F_x^2}{F_y^3}$$

$$-\frac{F_x}{F_y}$$

或 $$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{\left(F_{xx}+F_{xy}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)\cdot F_y-F_x\left(F_{yx}+F_{yy}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right)}{F_y^2}$$

例 1 验证方程 $x^2+y^2-1=0$ 在点(0,1)的某邻域内能唯一确定一个单值可导,且 $x=0$ 时 $y=1$ 的隐函数 $y=f(x)$ 并求这函数的一阶和二阶导数在 $x=0$ 的值

解 令 $F(x,y)=x^2+y^2-1$ 则① $F_x=2x,\ F_y=2y,$

② $F(0,1)=0,$ ③ $F_y(0,1)=2\neq 0,$ 由隐函数存在定理知

方程 $x^2+y^2-1=0$ 在点(0,1)的某邻域内能唯一确定一个单值可导,且 $x=0$ 时 $y=1$ 的隐函数 $y=f(x)$

公式法 $$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}=-\frac{x}{y},\quad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0}=0,$$

直接法 $$2x+2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}$$

微分形式不变性

$$2x\mathrm{d}x+2y\mathrm{d}y=0$$

$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}$$

例 1 验证方程 $x^2+y^2-1=0$ 在点(0,1)的某邻域内能唯一确定一个单值可导,且x=0时y=1的隐函数y=f(x)并求这函数的一阶和二阶导数在x=0的值

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方程 $x^2+y^2-1=0$ 在点(0,1)的某邻域内能唯一确定一个单值可导,且x=0时y=1的隐函数y=f(x)

直接法

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{y-xy'}{y^2}=-\frac{y-x\cdot(-\frac{x}{y})}{y^2}$$

$$=-\frac{1}{y^3}\Rightarrow\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{\substack{x=0\\y=1}}=-1$$

微分形式不变性

$$2x\mathrm{d}x+2y\mathrm{d}y=0$$

$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{x}{y}$$

二、一个三元方程情形

隐函数存在定理2 若点 $P(x_0,y_0,z_0)$ 的某邻域内满足

① $F(x,y,z)$ 有连续偏导数 ② $F(x_0,y_0,z_0)=0$ ③ $F_z(x_0,y_0,z_0)\neq 0$

则 $F(x,y,z)=0$ 在点 (x_0,y_0) 的**某邻域内**可唯一确定一个单值连续函数 $z=f(x,y)$，满足条件 $z_0=f(x_0,y_0)$，并有连续偏导数

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$ **(隐函数求导公式)**

证明： 略，仅就求导公式推导如下：

$F(x,y,z)=0 \Longrightarrow F(x,y,f(x,y))\equiv 0$ 两端分别对 x,y 求导

$$\begin{cases} F_x+F_z\cdot\dfrac{\partial z}{\partial x}=0 \\ F_y+F_z\cdot\dfrac{\partial z}{\partial y}=0 \end{cases}$$

因为 F_z 连续

$F_z(x_0,y_0,z_0)\neq 0$

在 (x_0,y_0) 的某邻域内 $F_z\neq 0$

$$\Longrightarrow \frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

二、一个三元方程情形

隐函数存在定理2 若点 $P(x_0,y_0,z_0)$ 的某邻域内满足

① $F(x,y,z)$ 有连续偏导数 ② $F(x_0,y_0,z_0)=0$ ③ $F_z(x_0,y_0,z_0)\neq 0$

则 $F(x,y,z)=0$ 在点 (x_0,y_0) 的**某邻域内**可唯一确定一个单值连续函数 $z=f(x,y)$，满足条件 $z_0=f(x_0,y_0)$，并有连续偏导数

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z},\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$ **(隐函数求导公式)**

二阶

$$\frac{\partial^2 z}{\partial x^2}=-\frac{\left(F''_{xx}+F''_{xz}\cdot\frac{\partial z}{\partial x}\right)\cdot F_z-F_x\left(F''_{zx}+F''_{zz}\cdot\frac{\partial z}{\partial x}\right)}{F_z^2}$$ 将 $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}$ 代入

$$=-\frac{F''_{xx}F_z^2-2F''_{zx}F'_xF'_z+F''_{zz}F'^2_x}{F_z^3}$$

其他类似可得

隐函数求偏导

解题思路

求一阶偏导

①公式法 需把方程写成$F(x,y)=0$或$F(x,y,z)=0$形式

求F_x, F_y时，对其中一个变量求偏导，将其他变量当作常数

②直接法 求偏导时，始终要注意函数是什么，自变量是什么

③利用微分形式不变性

求二阶偏导 $\longrightarrow$ 直接法

注：结果允许有因变量

典型题

例1 设 $z = x + ye^z$ 求 $\frac{\partial^2 z}{\partial x \partial y}$

解 **公式法** 令 $F = z - x - ye^z$ 则 $F_x = -1,\ F_y = -e^z,\ F_z = 1 - ye^z$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{1}{1 - ye^z} = \frac{1}{1 + x - z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^z}{1 + x - z}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(1 + x - z)^2} \cdot \left(-\frac{\partial z}{\partial y}\right) = \frac{e^z}{(1 + x - z)^3}$$

直接法 将 $z = x + ye^z$ 两边直接对x或y求偏导

$$z_x = 1 + ye^z \cdot z_x \Longrightarrow z_x = \frac{1}{1 - ye^z}$$

$$z_y = e^z + ye^z \cdot z_y \Longrightarrow z_y = \frac{e^z}{1 - ye^z}$$

$$z_{xy} = e^z z_x + ye^z \cdot z_y \cdot z_x + ye^z \cdot z_{xy}$$

解得z_{xy}

典型题

例1 设 $z = x + ye^z$ 求 $\dfrac{\partial^2 z}{\partial x \partial y}$

解 **公式法** 令 $F = z - x - ye^z$ 则 $F_x = -1,\ F_y = -e^z,\ F_z = 1 - ye^z$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{1}{1-ye^z} = \frac{1}{1+x-z} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{e^z}{1+x-z}$$

直接法 $$\frac{\partial^2 z}{\partial x \partial y} = -\frac{1}{(1+x-z)^2}\cdot\left(-\frac{\partial z}{\partial y}\right) = \frac{e^z}{(1+x-z)^3}$$

全微分法 $\mathrm{d}z = \mathrm{d}x + e^z\,\mathrm{d}y + ye^z\,\mathrm{d}z$

$$\mathrm{d}z = \underbrace{\frac{1}{1-ye^z}}_{=\frac{\partial z}{\partial x}}\mathrm{d}x + \underbrace{\frac{e^z}{1-ye^z}}_{=\frac{\partial z}{\partial y}}\mathrm{d}y$$

再求 $\dfrac{\partial^2 z}{\partial x \partial y}$

典型题

例2 设$z=f(x,y)$是由方程$z-y-x+xe^{z-y-x}=0$确定的隐函数,求dz

解 令$F=z-y-x+xe^{z-y-x}$

$$F_x=-1+e^{z-y-x}+xe^{z-y-x}(-1),\quad F_y=-1-xe^{z-y-x},\quad F_z=1+xe^{z-y-x}$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{1+(x-1)e^{z-y-x}}{1+xe^{z-y-x}}\qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=1$$

$$\mathrm{d}z=\frac{1+(x-1)e^{z-y-x}}{1+xe^{z-y-x}}\mathrm{d}x+\mathrm{d}y$$

全微分法 $\mathrm{d}z-\mathrm{d}y-\mathrm{d}x+e^{z-y-x}\mathrm{d}x+xe^{z-y-x}\mathrm{d}(z-y-x)=0$

$$\mathrm{d}z-\mathrm{d}y-\mathrm{d}x+e^{z-y-x}\mathrm{d}x+xe^{z-y-x}(\mathrm{d}z-\mathrm{d}y-\mathrm{d}x)=0$$

求出dz

典型题

例3 已知 $xy = xf(z) + yg(z), xf'(z) + yg'(z) \neq 0$ 其中 $z=z(x,y)$是x和y的函数,求证$[x-g(z)]\dfrac{\partial z}{\partial x} = [y-f(z)]\dfrac{\partial z}{\partial y}$

解 公式法 设$F(x,y,z) = xf(z) + yg(z) - xy$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{f(z)-y}{xf'(z)+yg'(z)} \qquad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{g(z)-x}{xf'(z)+yg'(z)}$$

$$[x-g(z)]\frac{\partial z}{\partial x} = -\frac{[x-g(z)][f(z)-y]}{xf'(z)+yg'(z)} = [y-f(z)]\frac{\partial z}{\partial y}$$

$$[g(z)-x][y-f(z)]$$

直接法 方程两端对x求偏导

$$y = f(z) + xf'(z)\cdot\frac{\partial z}{\partial x} + yg'(z)\cdot\frac{\partial z}{\partial x} \Longrightarrow \frac{\partial z}{\partial x} = \frac{y-f(z)}{xf'(z)+yg'(z)}$$

同理对y求偏导

典型题

例4 设方程 $F(x+\frac{z}{y}, y+\frac{z}{x})=0$ 确定了隐函数 $z=z(x,y)$

其中 F 为可微函数,求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

解 直接法

方程两端分别对 x 求偏导

$$F_1'(1+\frac{1}{y}\cdot\frac{\partial z}{\partial x})+F_2'(\frac{\frac{\partial z}{\partial x}\cdot x-z}{x^2})=0 \Longrightarrow \frac{\partial z}{\partial x}=\frac{y(zF_2'-x^2F_1')}{x(xF_1'+yF_2')}$$

方程两端分别对 y 求偏导

$$F_1'(\frac{\frac{\partial z}{\partial y}\cdot y-z}{y^2})+F_2'(1+\frac{1}{x}\cdot\frac{\partial z}{\partial y})=0 \Longrightarrow \frac{\partial z}{\partial y}=\frac{y(zF_1'-y^2F_2')}{x(xF_1'+yF_2')}$$

例5 设$z=z(x,y)$由方程$x^2+y^2+z^2=yf(\frac{z}{y})$所确定的隐函数,

其中$f(u)$具有一阶连续偏导数,证明

$$(x^2-y^2-z^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}=2xz$$

解 **直接法** 方程两边对x,y求偏导,有

$$2x+2z\cdot\frac{\partial z}{\partial x}=yf'(\frac{z}{y})\cdot\frac{1}{y}\cdot\frac{\partial z}{\partial x} \Longrightarrow \frac{\partial z}{\partial x}=\frac{2x}{f'(\frac{z}{y})-2z}$$

$$2y+2z\cdot\frac{\partial z}{\partial y}=f(\frac{z}{y})+yf'(\frac{z}{y})\cdot\frac{\frac{\partial z}{\partial y}\cdot y-z}{y^2}$$

$$\Longrightarrow \frac{\partial z}{\partial y}=\frac{2y-f(\frac{z}{y})+\frac{z}{y}f'(\frac{z}{y})}{f'(\frac{z}{y})-2z}$$

例5 设$z=z(x,y)$由方程$x^2+y^2+z^2=yf\left(\dfrac{z}{y}\right)$所确定的隐函数,其中$f(u)$具有一阶连续偏导数,证明

$$(x^2-y^2-z^2)\frac{\partial z}{\partial x}+2xy\frac{\partial z}{\partial y}=2xz$$

解 **直接法** 方程两边对x,y求偏导,有

$$\frac{\partial z}{\partial x}=\frac{2x}{f'\left(\frac{z}{y}\right)-2z}\qquad \frac{\partial z}{\partial y}=\frac{2y-f\left(\frac{z}{y}\right)+\frac{z}{y}f'\left(\frac{z}{y}\right)}{f'\left(\frac{z}{y}\right)-2z}$$

$$[2y^2-(x^2+y^2+z^2)+zf'\left(\frac{z}{y}\right)]$$

$$左边=\frac{\left\{(x^2-y^2-z^2)\cancel{2}x+\cancel{2}xy\left[2y-f\left(\frac{z}{y}\right)+\frac{z}{y}f'\left(\frac{z}{y}\right)\right]\right\}2x}{f'\left(\frac{z}{y}\right)-2z}=\frac{\left[-2z^2+zf'\left(\frac{z}{y}\right)\right]2x}{f'\left(\frac{z}{y}\right)-2z}$$

$$=2xz$$

三、两个四元方程的方程组情形

隐函数存在定理还可以推广到方程组的情形.

以两个方程确定两个隐函数的情况为例，即

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \longrightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

由 F、G 的偏导数组成的行列式

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

称为F、G 的**雅可比行列式**.

雅可比

三、两个四元方程的方程组情形

隐函数存在定理3 若点$P(x_0,y_0,u_0,v_0)$的某邻域内满足

①$F(x,y,u,v)$, $G(x,y,u,v)$有连续偏导数

②$F(x_0,y_0,u_0,v_0)=0, G(x_0,y_0,u_0,v_0)=0$

③P处的Jacobi行列式 $J\Big|_P=\left.\dfrac{\partial(F,G)}{\partial(u,v)}\right|_P=\begin{vmatrix}\dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v}\\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial v}\end{vmatrix}\neq 0$,

则方程组$\begin{cases}F(x,y,u,v)=0\\ G(x,y,u,v)=0\end{cases}$在点$(x_0,y_0)$的某一邻域内可唯一确定一组满足条件$\begin{cases}u_0=u(x_0,y_0)\\ v_0=v(x_0,y_0)\end{cases}$的单值连续函数$\begin{cases}u=u(x,y)\\ v=v(x,y)\end{cases}$

且有偏导数公式如下

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \Rightarrow \begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$$

隐函数求导公式

$$\frac{\partial u}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{x},v)}=-\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}$$

$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(\underline{y},v)}=-\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}$$

$$\frac{\partial v}{\partial x}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{x})}=-\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}$$

$$\frac{\partial v}{\partial y}=-\frac{1}{J}\frac{\partial(F,G)}{\partial(u,\underline{y})}=-\frac{1}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}$$

注: 求F_x,F_y,F_u,F_v及G_x,G_y,G_u,G_v时其它变量当常量

证明:略,仅推导公式

二元线性代数方程组解的公式

$$\begin{cases} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{cases}$$

解:

$$x = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$y = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

证 设方程组 $\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases}$ 有隐函数组 $\begin{cases} u=u(x,y) \\ v=v(x,y) \end{cases}$，

则 $\begin{cases} F(x,y,u(x,y),v(x,y))\equiv 0 \\ G(x,y,u(x,y),v(x,y))\equiv 0 \end{cases}$

两边对 x 求导得 $\begin{cases} \cancel{F_x}+F_u\cdot\dfrac{\partial u}{\partial x}+F_v\cdot\dfrac{\partial v}{\partial x}=0 \quad -F_x \\ \cancel{G_x}+G_u\cdot\dfrac{\partial u}{\partial x}+G_v\cdot\dfrac{\partial v}{\partial x}=0 \quad -G_x \end{cases}$

两边对 y 求导可得 $\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial y}$

这是关于 $\dfrac{\partial u}{\partial x},\dfrac{\partial v}{\partial x}$ 的线性方程组，在点 P 的某邻域内

系数行列式

$$J=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}\neq 0,$$

故得

$$\frac{\partial u}{\partial x}=-\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} \qquad \frac{\partial v}{\partial x}=-\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

例1. 设 $xu - yv = 0,\ yu + xv = 1$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$.

解: 方程组两边对 x 求导，并移项得

$$\begin{cases} u + x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = 0 \\ y\dfrac{\partial u}{\partial x} + v + x\dfrac{\partial v}{\partial x} = 0 \end{cases} \Longrightarrow \begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}$$

由题设 $J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2 \neq 0$

故有
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{1}{J}\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix} = -\dfrac{xu + yv}{x^2 + y^2} \\ \dfrac{\partial v}{\partial x} = \dfrac{1}{J}\begin{vmatrix} x & -u \\ y & -v \end{vmatrix} = -\dfrac{xv - yu}{x^2 + y^2} \end{cases}$$

练习: 求 $\dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial y}$

答案:

$$\begin{cases} \dfrac{\partial u}{\partial y} = -\dfrac{yu - xv}{x^2 + y^2} \\ \dfrac{\partial v}{\partial y} = -\dfrac{xu + yv}{x^2 + y^2} \end{cases}$$

例2. 设$y=y(x), z=z(x)$是由方程$z=x f(x+y)$和$F(x,y,z)=0$所确定的函数,其中f和F分别具有一阶连续导数和一阶连续偏导数,求 $\dfrac{\mathrm{d}z}{\mathrm{d}x}$

解 分别在各方程两端对 x 求导,得

$$\begin{cases} z'=f+x\cdot f'\cdot(1+y') \\ F_x+F_y\cdot y'+F_z\cdot z'=0 \end{cases} \longrightarrow \begin{cases} -x f'\cdot y'+z'=f+x f' \\ F_y\cdot y'+F_z\cdot z'=-F_x \end{cases}$$

$$\therefore \quad \frac{\mathrm{d}z}{\mathrm{d}x}=\frac{\begin{vmatrix} -x f' & f+x f' \\ F_y & -F_x \end{vmatrix}}{\begin{vmatrix} -x f' & 1 \\ F_y & F_z \end{vmatrix}}=\frac{(f+x f')F_y-x f'\cdot F_x}{F_y+x f'\cdot F_z}$$

$$(F_y+x f'\cdot F_z\neq 0)$$

例3. 设$\begin{cases} u = f(ux, v+y) \\ v = g(u-x, v^2 y) \end{cases}$，其中$f$和$g$具有一阶连续偏导数

求$\dfrac{\partial u}{\partial x}, \dfrac{\partial v}{\partial x}$

解 方程两边对 x 求导，

$$\begin{cases} \dfrac{\partial u}{\partial x} = f_1' \cdot \left(u + x\dfrac{\partial u}{\partial x}\right) + f_2' \cdot \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} = g_1' \cdot \left(\dfrac{\partial u}{\partial x} - 1\right) + g_2' \cdot 2vy \cdot \dfrac{\partial v}{\partial x} \end{cases} \longrightarrow \begin{cases} (xf_1' - 1)\dfrac{\partial u}{\partial x} + f_2'\dfrac{\partial v}{\partial x} = -uf_1' \\ g_1'\dfrac{\partial u}{\partial x} + (2yvg_2' - 1)\dfrac{\partial v}{\partial x} = g_1' \end{cases}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -uf_1' & f_2' \\ g_1' & 2yvg_2' - 1 \end{vmatrix}}{\begin{vmatrix} xf_1' - 1 & f_2' \\ g_1' & 2yvg_2' - 1 \end{vmatrix}} \qquad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} xf_1' - 1 & -uf_1' \\ g_1' & g_1' \end{vmatrix}}{\begin{vmatrix} xf_1' - 1 & f_2' \\ g_1' & 2yvg_2' - 1 \end{vmatrix}}$$

$$J = (xf_1' - 1)(2yvg_2' - 1) - f_2'g_1' \neq 0$$

例4. 设函数 $x = x(u,v),\ y = y(u,v)$ 在点 (u,v) 的某一邻域内有连续的偏导数，且 $\dfrac{\partial(x,y)}{\partial(u,v)} \neq 0$

1) 证明函数组 $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$ 在与点 (u,v) 对应的点 (x,y) 的某一邻域内唯一确定一组单值、连续且具有连续偏导数的反函数 $u = u(x,y),\ v = v(x,y)$.

2) 求 $u = u(x,y),\ v = v(x,y)$ 对 x,y 的偏导数.

解: 1) 令 $F(x,y,u,v) \equiv x - x(u,v) = 0$

$$G(x,y,u,v) \equiv y - y(u,v) = 0$$

则有 $J = \dfrac{\partial(F,G)}{\partial(u,v)} = \dfrac{\partial(x,y)}{\partial(u,v)} \neq 0,$ 由定理 3 可知结论 1) 成立.

2) 求反函数的偏导数.

$$\begin{cases} x \equiv x(u(x,y),v(x,y)) \\ y \equiv y(u(x,y),v(x,y)) \end{cases}$$

$$J=\frac{\partial(F,G)}{\partial(u,v)}=\frac{\partial(x,y)}{\partial(u,v)}\neq 0$$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

两边对 x 求导, 得

$$\begin{cases} 1=\dfrac{\partial x}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial x}{\partial v}\cdot\dfrac{\partial v}{\partial x} \\ 0=\dfrac{\partial y}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial y}{\partial v}\cdot\dfrac{\partial v}{\partial x} \end{cases}$$

类似可得

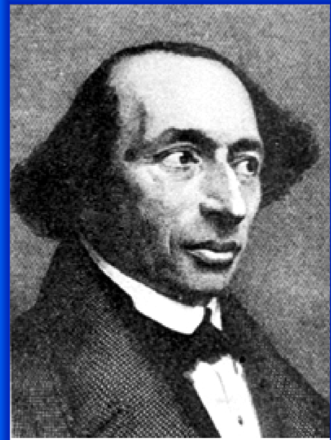
$$\frac{\partial u}{\partial y}=-\frac{1}{J}\frac{\partial x}{\partial v},$$

$$\frac{\partial v}{\partial y}=\frac{1}{J}\frac{\partial x}{\partial u}$$

注意 $J\neq 0$,

$$\frac{\partial u}{\partial x}=\frac{1}{J}\begin{vmatrix} 1 & \dfrac{\partial x}{\partial v} \\ 0 & \dfrac{\partial y}{\partial v} \end{vmatrix}=\frac{1}{J}\frac{\partial y}{\partial v},\qquad \frac{\partial v}{\partial x}=\frac{1}{J}\begin{vmatrix} \dfrac{\partial x}{\partial u} & 1 \\ \dfrac{\partial y}{\partial u} & 0 \end{vmatrix}=-\frac{1}{J}\frac{\partial y}{\partial u}$$

雅可比(1804 – 1851)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德国数学家. 他在数学方面最主要的成就是和挪威数学家阿贝儿相互独地奠定了椭圆函数论的基础. 他对行列式理论也作了奠基性的工作. 在偏微分方程的研究中引进了“雅可比行列式”, 并应用在微积分中. 他的工作还包括代数学, 变分法, 复变函数和微分方程, 在分析力学,动力学及数学物理方面也有贡献 . 他在柯尼斯堡大学任教18年, 形成了以他为首的学派.