第十章

第二节

二重积分的计算法

内容

一、利用直角坐标计算二重积分

二、利用极坐标计算二重积分

一、利用直角坐标计算二重积分

$z = f(x,y)$

预备知识：

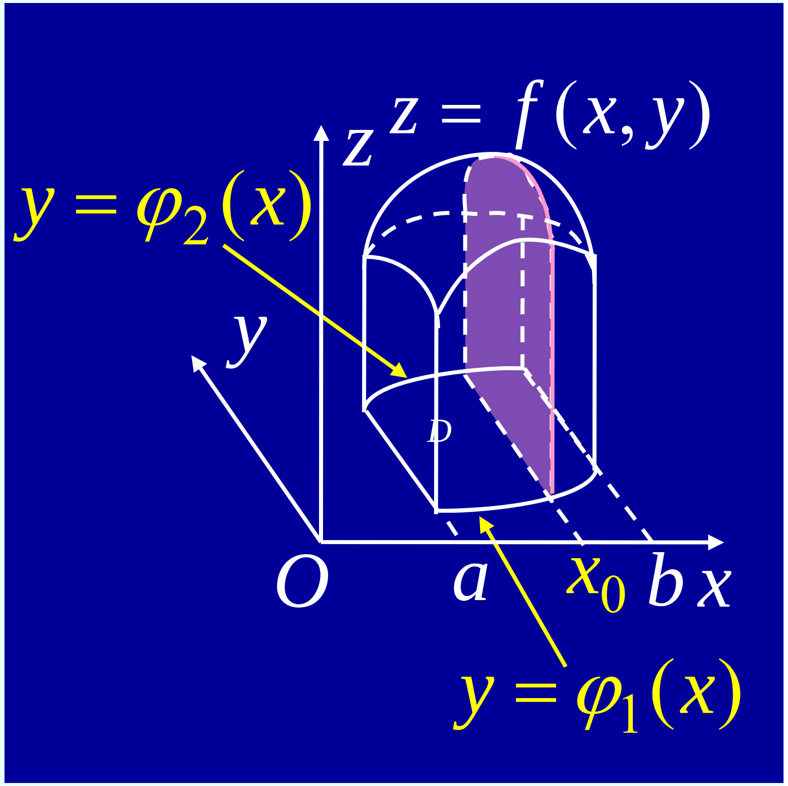
1.曲边梯形的面积

2.截面面积为已知的立体的体积

一、利用直角坐标计算二重积分

设积分区域(曲顶柱体的底)为

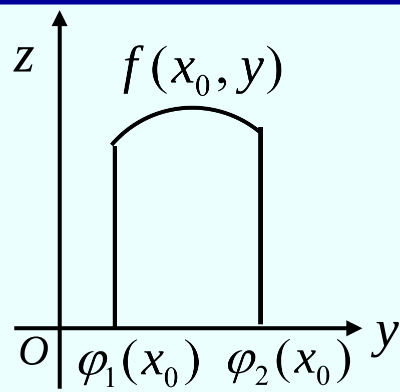
X型 $D=\left\{(x,y)\left|\begin{array}{c}\varphi_1(x)\le y\le\varphi_2(x)\\ a\le x\le b\end{array}\right.\right\}$



任取 $x_0\in[a,b]$, 平面 $x=x_0$

截柱体的截面积为

$$A(x_0)=\int_{\varphi_1(x_0)}^{\varphi_2(x_0)} f(x_0,y)\,\mathrm{d}y$$



故曲顶柱体体积为

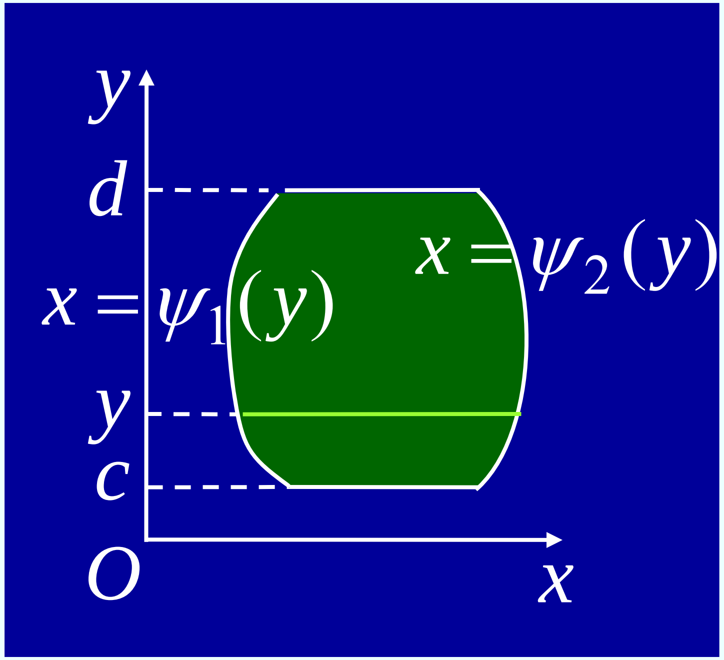
二重积分 $V=\iint\limits_D f(x,y)\,\mathrm{d}\sigma=\int_a^b A(x)\,\mathrm{d}x$

$$=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y\right]\mathrm{d}x\overset{\text{记作}}{=}\int_a^b\mathrm{d}x\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y$$ 二次积分

同样, 曲顶柱体的底为

Y型 $D=\{(x,y)\ |\psi_1(y)\le x\le\psi_2(y),\ c\le y\le d\ \}$

则其体积可按如下两次积分计算

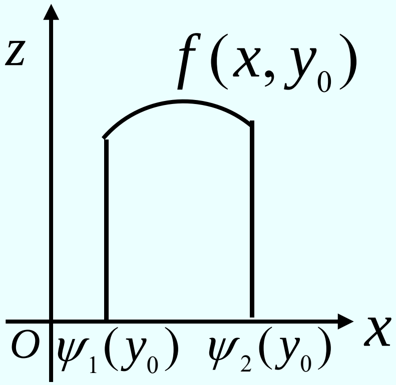


$$V=\iint_D f(x,y)\,\mathrm{d}\sigma$$

$$=\int_c^d A(y)\,\mathrm{d}y$$

$$=\int_c^d\left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x\ \right]\mathrm{d}y$$

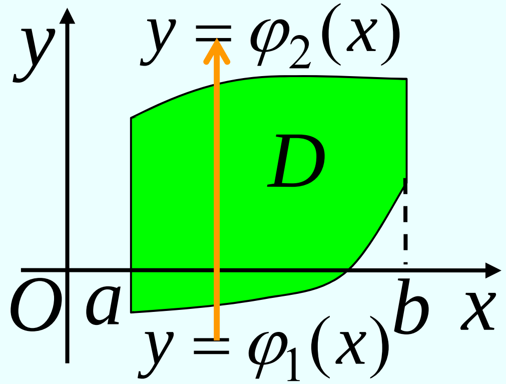
$$\overset{\text{记作}}{=}\int_c^d \mathrm{d}y\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x$$



一、利用直角坐标计算二重积分

若D属于上下曲边型(X-型)

$$D:\begin{cases}\varphi_1(x)\le y\le\varphi_2(x)\\ a\le x\le b\end{cases}$$

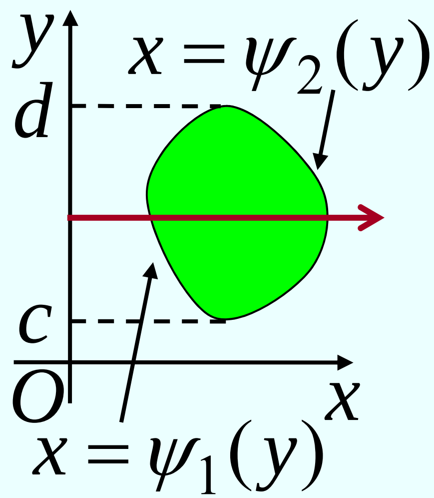


$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b \mathrm{d}x\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y$$

先对**y**后对**x**积分

若D属于左右曲边型(Y-型)

$$D:\begin{cases}\psi_1(y)\le x\le\psi_2(y)\\ c\le y\le d\end{cases}$$



$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y=\int_c^d \mathrm{d}y\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x$$

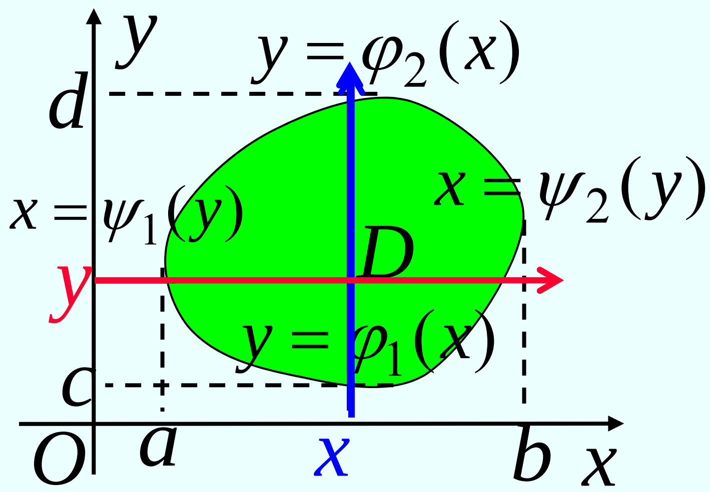
先对**x**后对**y**积分

说明: (1) 若积分区域既是 X - 型区域又是 Y - 型区域 ,

则有 $\iint\limits_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

$$= \int_a^b \mathrm{d}x \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y$$

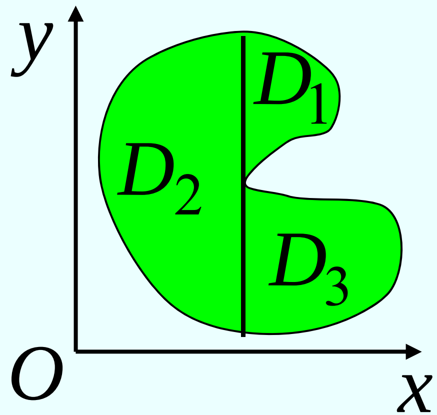
$$= \int_c^d \mathrm{d}y \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x$$



为计算方便, 可**选择积分次序**, 必要时还可**交换积分次序**

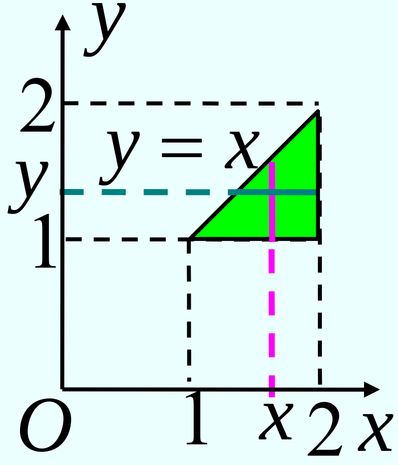
(2) 若积分域较复杂, 可将它分成若干 X - 型域或 Y - 型域 ,则

$$\iint_D = \iint_{D_1} + \iint_{D_2} + \iint_{D_3}$$



例1. 计算$I=\iint_D xy\,\mathrm{d}\sigma$, 其中$D$ 是直线 $y=1$, $x=2$, 及 $y=x$ 所围的闭区域.

解法1. 将D看作X - 型区域, 则D: $\begin{cases} 1\le y\le x \\ 1\le x\le 2 \end{cases}$



$$I=\int_1^2 \mathrm{d}x\int_1^x xy\,\mathrm{d}y=\int_1^2\left[\tfrac{1}{2}xy^2\right]_1^x \mathrm{d}x$$

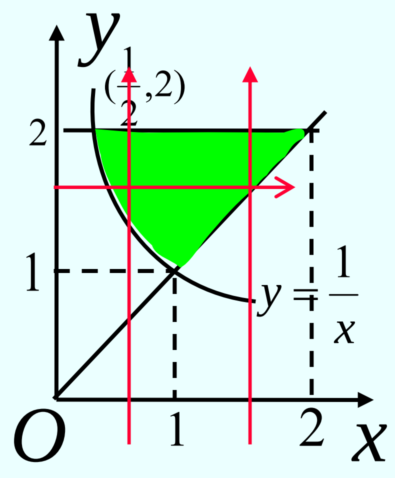
$$=\int_1^2\left[\tfrac{1}{2}x^3-\tfrac{1}{2}x\right]\mathrm{d}x=\frac{9}{8}$$

解法2. 将D看作Y - 型区域, 则D: $\begin{cases} y\le x\le 2 \\ 1\le y\le 2 \end{cases}$

$$I=\int_1^2 \mathrm{d}y\int_y^2 xy\,\mathrm{d}x=\int_1^2\left[\tfrac{1}{2}x^2y\right]_y^2\mathrm{d}y=\int_1^2\left[2y-\tfrac{1}{2}y^3\right]\mathrm{d}y=\frac{9}{8}$$

例2. 化二重积分$\iint\limits_D f(x,y)\mathrm{d}\sigma$为直角坐标系下的二次积分,其中$D$由直线$y=x$,$y=2$及 $y=\dfrac{1}{x}(x>0)$所围成的区域

解法1. Y - 型 $I=\int_1^2 \mathrm{d}y\int_{\frac{1}{y}}^{y} f(x,y)\mathrm{d}x$



解法2. X - 型

$$I=\int_{\frac{1}{2}}^{1}\mathrm{d}x\int_{\frac{1}{x}}^{2} f(x,y)\mathrm{d}y+\int_1^2\mathrm{d}x\int_x^2 f(x,y)\mathrm{d}y$$

解题步骤

1)画出积分区域图形,求出交点

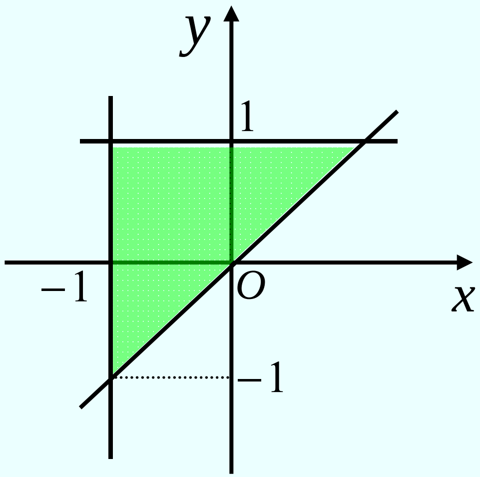
2)选择适当的积分次序原则有: ①被积函数容易积出

②尽可能使积分区域少分块,以简化计算

例3. 计算 $\iint_D y\sqrt{1+x^2-y^2}\,\mathrm{d}\sigma$ D由$y=x, x=-1$和$y=1$所围成的闭区域

选择容易积出的积分次序

解: Y - 型 $I=\int_{-1}^{1}\mathrm{d}y\int_{-1}^{y} y\sqrt{1+x^2-y^2}\,\mathrm{d}x$ 很麻烦



X - 型 $I=\int_{-1}^{1}\mathrm{d}x\int_{x}^{1} y\sqrt{1+x^2-y^2}\,\mathrm{d}y$

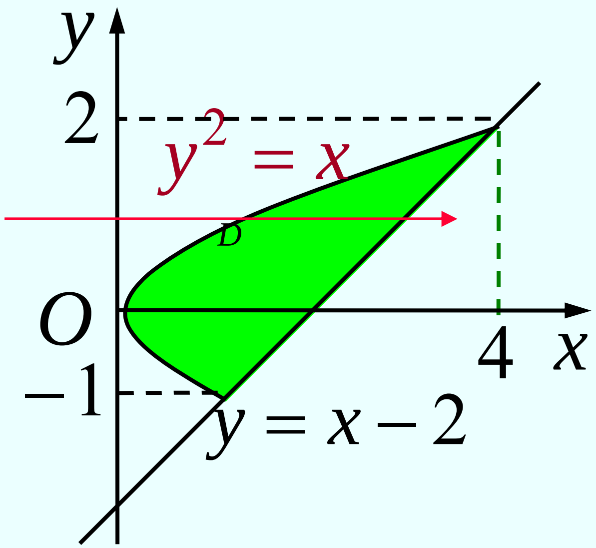
$$=-\frac{1}{3}\int_{-1}^{1}(|x|^3-1)\,\mathrm{d}x=-\frac{2}{3}\int_{0}^{1}(x^3-1)\,\mathrm{d}x$$

$$=\frac{1}{2}$$

其中$\int_{x}^{1} y\sqrt{1+x^2-y^2}\,\mathrm{d}y=-\frac{1}{2}\int_{x}^{1}\sqrt{1+x^2-y^2}\,\mathrm{d}(1+x^2-y^2)$

$$=-\frac{1}{2}\times\frac{2}{3}(1+x^2-y^2)^{\frac{3}{2}}\Big|_{y=x}^{y=1}=-\frac{1}{3}(|x|^3-1)$$

例4. 计算 $\iint_D xy\,\mathrm{d}\sigma$, 其中$D$ 是抛物线 $y^2=x$ 及直线 $y=x-2$ 所围成的闭区域.



解: 为计算简便, Y-型

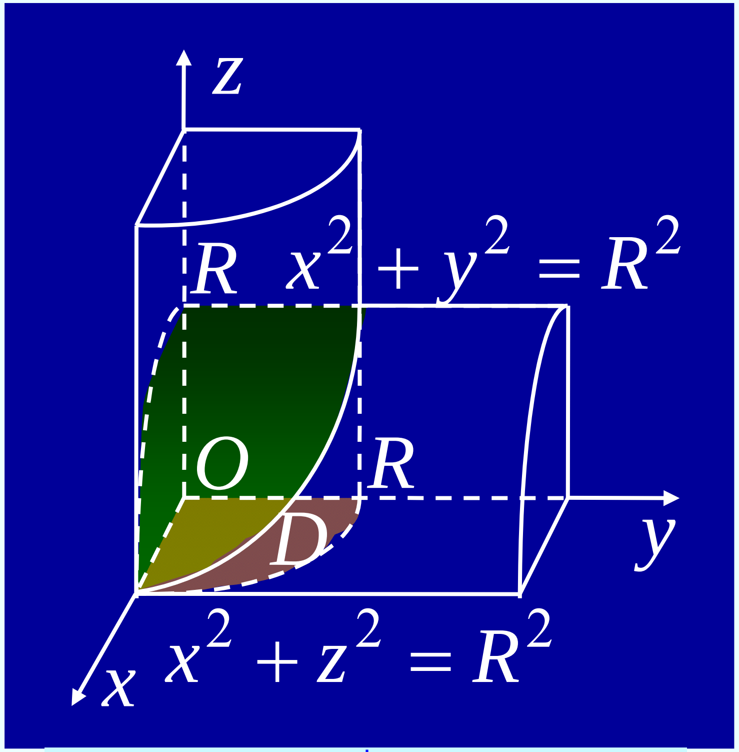
$$\iint_D xy\,\mathrm{d}\sigma = \int_{-1}^{2}\mathrm{d}y\int_{y^2}^{y+2} xy\,\mathrm{d}x$$

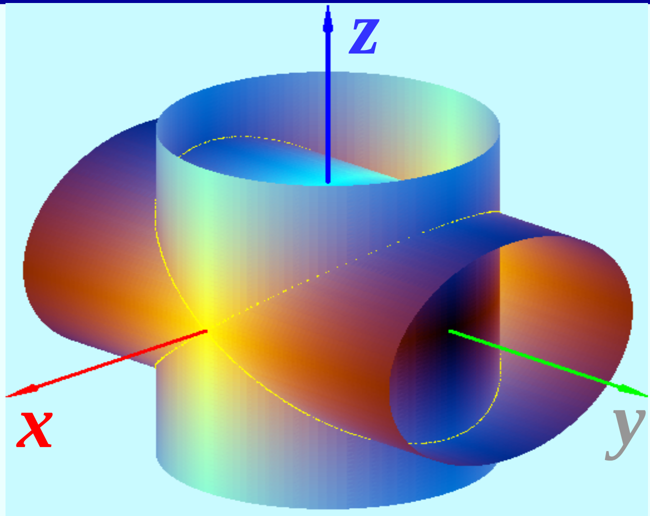
$$=\int_{-1}^{2}\left[\frac{1}{2}x^2y\right]_{y^2}^{y+2}\mathrm{d}y=\frac{1}{2}\int_{-1}^{2}[y(y+2)^2-y^5]\,\mathrm{d}y$$

$$=\frac{1}{2}\left[\frac{y^4}{4}+\frac{4}{3}y^3+2y^2-\frac{1}{6}y^6\right]_{-1}^{2}=\frac{45}{8}$$

选择少分块
的积分次序

例5. 求两个底圆半径为R 的直交圆柱面所围的体积.





解: 设两个直圆柱方程为

$$x^2+y^2=R^2,\quad x^2+z^2=R^2$$

利用对称性, 考虑第一卦限部分,

其曲顶柱体的顶为 $z=\sqrt{R^2-x^2}$

$$(x,y)\in D:\begin{cases}0\le y\le\sqrt{R^2-x^2}\\ 0\le x\le R\end{cases}$$

$$V=8\iint_D\sqrt{R^2-x^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$=8\int_0^R\mathrm{d}x\int_0^{\sqrt{R^2-x^2}}\sqrt{R^2-x^2}\,\mathrm{d}y$$

$$=8\int_0^R(R^2-x^2)\,\mathrm{d}x=\frac{16}{3}R^3$$

主要题型 (1)利用交换积分次序计算二重积分

例1. 计算 $\int_0^1 \mathrm{d}x \boxed{\int_1^x} \frac{\sin y}{y} \mathrm{d}y$

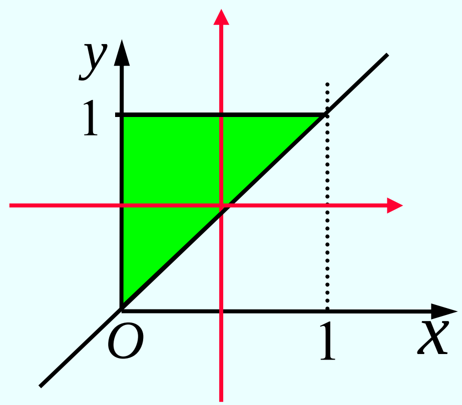
分析 由于 $\int \frac{\sin y}{y} \mathrm{d}y$ 不能用有限形式给出,可以交换积分次序

解: 画图 $I = -\int_0^1 \mathrm{d}x \int_x^1 \frac{\sin y}{y} \mathrm{d}y$

$$= -\int_0^1 \mathrm{d}y \int_0^y \frac{\sin y}{y} \mathrm{d}x$$

$$= -\int_0^1 \sin y \, \mathrm{d}y$$

$$= \cos 1 - 1$$



说明: 有些二次积分为了积分方便, 还需交换积分顺序.

主要题型 (1)利用交换积分次序计算二重积分

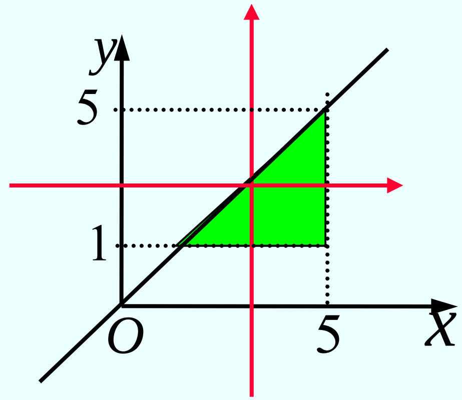
例2. 计算 $\int_1^5 \mathrm{d}y \int_{\boxed{y}}^{5} \frac{\mathrm{d}x}{y\ln x}$

注： 对应的积分区域：$1<y<5,\ y<x<5$

解: 画图 $I = \int_1^5 \mathrm{d}x \int_1^x \frac{1}{y\ln x}\mathrm{d}y$

$$= \int_1^5 \frac{1}{\ln x}\cdot \ln y\Big|_{y=1}^{y=x}\ \mathrm{d}x$$

$$= \int_1^5 \mathrm{d}x\ = 4$$



一般形如 $\int_a^b \mathrm{d}x \int_x^c \frac{\sin y}{y}\mathrm{d}y,\ \int_a^b \mathrm{d}x \int_x^c \sin y^2\,\mathrm{d}y,\ \int_a^b \mathrm{d}x \int_x^c \cos y^2\,\mathrm{d}y,$

$\int_a^b \mathrm{d}x \int_x^c e^{-y^2}\,\mathrm{d}y,\ \int_a^b \mathrm{d}x \int_x^c e^{y^2}\,\mathrm{d}y,\ \int_a^b \mathrm{d}x \int_x^c e^{\frac{x}{y}}\,\mathrm{d}y,\ \int_a^b \mathrm{d}x \int_x^c \frac{1}{\ln y}\mathrm{d}y$

等二次积分一定要交换积分次序才能计算

主要题型 (2)仅交换积分次序

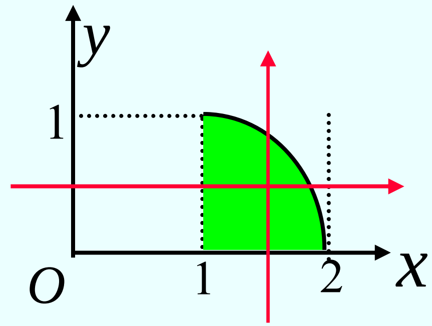
①由积分上下限画出**D**的草图,并求出交点,观察是否上限大于下限;若不是,则将上下限颠倒

②根据新的积分次序,写出新的二次积分

例1 $I=\int_1^2 \mathrm{d}x\int_0^{\sqrt{2x-x^2}} f(x,y)\mathrm{d}y$ 改变积分次序

解:

$$I=\int_0^1 \mathrm{d}y\int_1^{1+\sqrt{1-y^2}} f(x,y)\mathrm{d}x$$



$$y=\sqrt{2x-x^2}$$

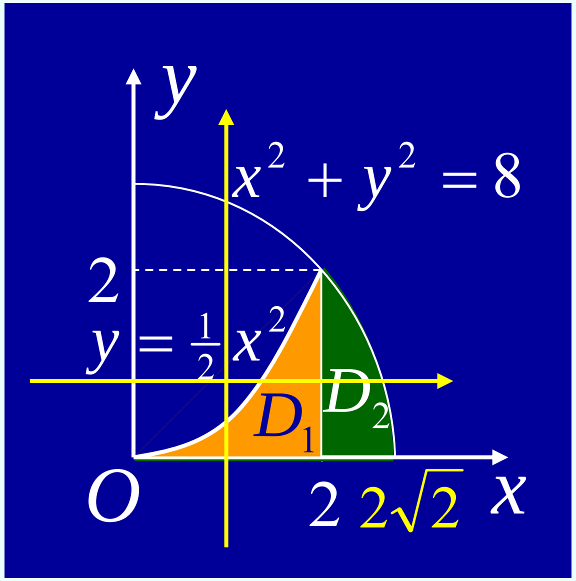
$$\Rightarrow (x-1)^2+y^2=1$$

例2. 交换下列积分顺序

$$I=\int_0^2 \mathrm{d}x\int_0^{\frac{x^2}{2}} f(x,y)\mathrm{d}y+\int_2^{2\sqrt{2}} \mathrm{d}x\int_0^{\sqrt{8-x^2}} f(x,y)\mathrm{d}y$$

解: 积分域由两部分组成:

$$D_1:\begin{cases}0\le y\le \frac{1}{2}x^2\\ 0\le x\le 2\end{cases},\quad D_2:\begin{cases}0\le y\le \sqrt{8-x^2}\\ 2\le x\le 2\sqrt{2}\end{cases}$$



视为Y-型区域，则

$$I=\int_0^2 \mathrm{d}y\int_{\sqrt{2y}}^{\sqrt{8-y^2}} f(x,y)\mathrm{d}x$$

主要题型 (3)分段函数$\iint_D f(x,y)d\sigma$

解题关键：确定分段函数的分段域,从而确定每一分段支

①求$\iint_D \max\{\varphi_1,\varphi_2\}\mathrm{d}\sigma, \iint_D \min\{\varphi_1,\varphi_2\}\mathrm{d}\sigma$

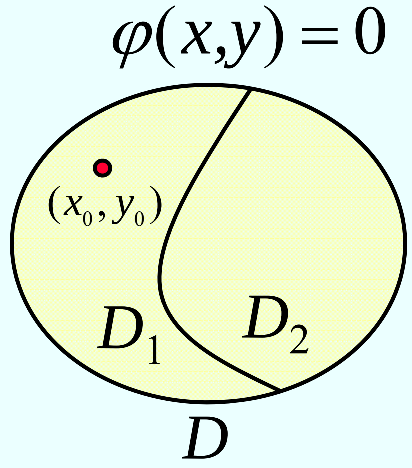
Step1 令$\varphi_1(x,y)=\varphi_2(x,y)$,得一曲线$\varphi(x,y)=0$

Step2 此曲线将D分成两部分D_1,D_2,

如$(x_0,y_0)\in D_1$时,$\varphi_1(x_0,y_0)>\varphi_2(x_0,y_0)$

则对$(x,y)\in D_1$时,$\varphi_1(x,y)>\varphi_2(x,y)$;

$(x,y)\in D_2$时,$\varphi_1(x,y)<\varphi_2(x,y)$



Step3 $\iint_D \max\{\varphi_1,\varphi_2\}\mathrm{d}\sigma=\iint_{D_1}\varphi_1\mathrm{d}\sigma+\iint_{D_2}\varphi_2\mathrm{d}\sigma$

$\iint_D \min\{\varphi_1,\varphi_2\}\mathrm{d}\sigma=\iint_{D_1}\varphi_2\mathrm{d}\sigma+\iint_{D_2}\varphi_1\mathrm{d}\sigma$

主要题型 (3)分段函数$\iint\limits_D f(x,y)d\sigma$

解题关键：确定分段函数的分段域,从而确定每一分段支

② $\iint\limits_D \operatorname{sgn}\varphi(x,y)\mathrm{d}\sigma$

令$\varphi(x,y)=0$,这样曲线$\varphi(x,y)=0$将积分域分成D_1,D_2,

如$(x_0,y_0)\in D_1$时,$\varphi(x_0,y_0)>0$; 则$(x,y)\in D_1,\varphi(x,y)>0$;

$(x,y)\in D_2,\varphi(x,y)<0$ 故$\iint\limits_D \operatorname{sgn}\varphi(x,y)\mathrm{d}\sigma=\iint\limits_{D_1}1\,\mathrm{d}x\mathrm{d}y-\iint\limits_{D_2}1\,\mathrm{d}x\mathrm{d}y$

③ $\iint\limits_D |\varphi(x,y)|\,\mathrm{d}x\mathrm{d}y$

令$\varphi(x,y)=0$, 此曲线将D分成D_1,D_2,

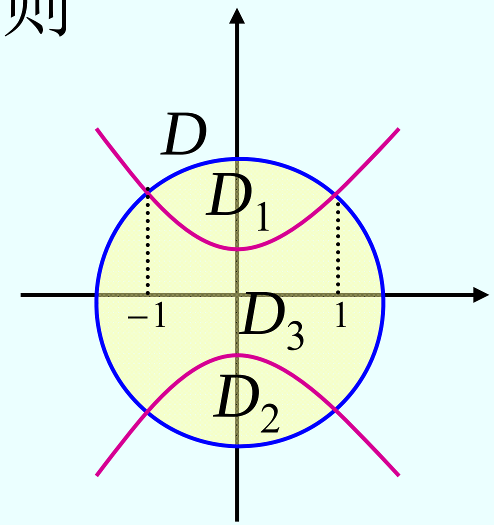
如$(x_0,y_0)\in D_1$时,$\varphi(x_0,y_0)>0$, 则$(x,y)\in D_1,\varphi(x,y)>0$;

在D_2内,$\varphi(x,y)<0$ 故$\iint\limits_D |\varphi|\,\mathrm{d}x\mathrm{d}y=\iint\limits_{D_1}\varphi\,\mathrm{d}x\mathrm{d}y-\iint\limits_{D_2}\varphi\,\mathrm{d}x\mathrm{d}y$

例1 计算二重积分 $\iint\limits_{x^2+y^2\le 4} \operatorname{sgn}(x^2-y^2+2)\,dxdy$ 其中 $\operatorname{sgn} x=\begin{cases}1 & x>0\\ 0 & x=0\\ -1 & x<0\end{cases}$

解 $x^2-y^2+2=0$ 将区域D分成D_1,D_2,D_3，则

$$\operatorname{sgn}(x^2-y^2+2)=\begin{cases}1 & (x,y)\in D_3\\ -1 & (x,y)\in D_1\cup D_2\end{cases}$$



$$\text{原式}=\iint\limits_{D_3}dxdy-\iint\limits_{D_1}dxdy-\iint\limits_{D_2}dxdy$$

$$=4\pi-4\iint\limits_{D_1}dxdy \quad =4\left[\frac{\pi}{3}+\ln(2+\sqrt{3})\right]$$

联立 $\begin{cases}x^2-y^2+2=0\\ x^2+y^2=4\end{cases}$

$\Rightarrow x^2=1$

$$\iint\limits_{D_1}dxdy=\int_{-1}^{1}dx\int_{\sqrt{x^2+2}}^{\sqrt{4-x^2}}dy$$

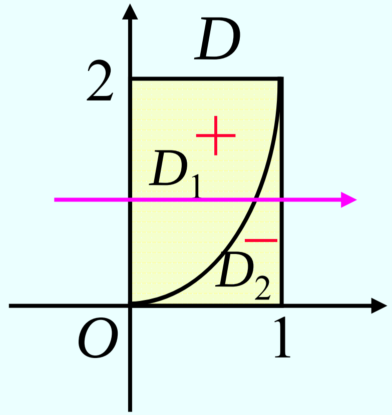
$$=\int_{-1}^{1}\left(\sqrt{4-x^2}-\sqrt{x^2+2}\right)dx$$

$$=\frac{2}{3}\pi-2\ln\frac{1+\sqrt{3}}{\sqrt{2}}$$

$$\boxed{\int\sqrt{a^2-x^2}\,dx=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C}$$

$$\boxed{\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C}$$

例2 若 $D=\{(x,y)\,|\,0\le x\le 1, 0\le y\le 2\}$, 则 $\iint\limits_D |y^2-4x|\,\mathrm{d}x\mathrm{d}y$ 的二次积分表达式



解 令 $y^2-4x=0$ 将区域 D 分成 D_1, D_2, 则

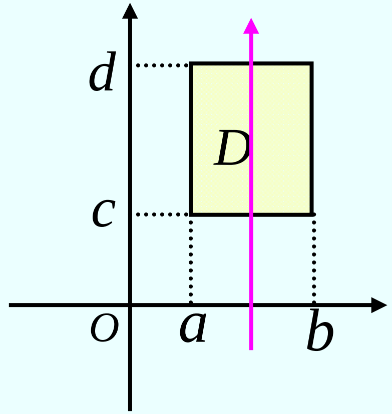
$$\text{原式}=\iint\limits_{D_1} y^2-4x\,\mathrm{d}x\mathrm{d}y+\iint\limits_{D_2} 4x-y^2\,\mathrm{d}x\mathrm{d}y$$

$$=\int_0^2 \mathrm{d}y\int_0^{\frac{y^2}{4}}(y^2-4x)\mathrm{d}x+\int_0^2 \mathrm{d}y\int_{\frac{y^2}{4}}^1(4x-y^2)\mathrm{d}x$$

主要题型 (4)证明题 $\iint\limits_D f(x,y)d\sigma$

例1 若$f(x,y)=f_1(x)f_2(y)$, $D=\{(x,y)\mid a\le x\le b, c\le y\le d\}$

求证 $\iint\limits_D f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y=\int_a^b f_1(x)\mathrm{d}x\cdot\int_c^d f_2(x)\mathrm{d}x$



证明 $\iint\limits_D f_1(x)f_2(y)\mathrm{d}x\mathrm{d}y$

$$=\int_a^b \mathrm{d}x\int_c^d f_1(x)f_2(y)\mathrm{d}y$$

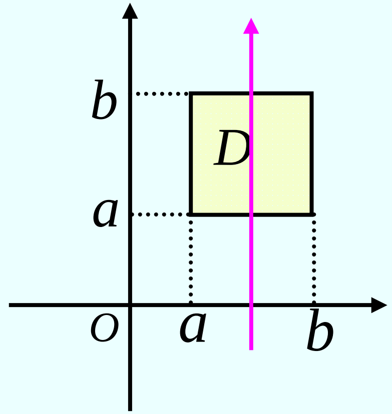
$$=\int_a^b f_1(x)\int_c^d f_2(y)\mathrm{d}y\,\mathrm{d}x$$

$$=\int_c^d f_2(y)\mathrm{d}y\cdot\int_a^b f_1(x)\mathrm{d}x$$

主要题型 (4)证明题 $\iint\limits_D f(x,y)d\sigma$

例2 设$f(x)$在$[a,b]$上连续且恒大于0,证明

求证 $\int_a^b f(x)\mathrm{d}x\cdot\int_a^b \frac{1}{f(x)}\mathrm{d}x \ge (b-a)^2$



证明 $\int_a^b f(x)\mathrm{d}x\cdot\int_a^b \frac{1}{f(x)}\mathrm{d}x$

$$= \int_a^b f(x)\mathrm{d}x\cdot\int_a^b \frac{1}{f(y)}\mathrm{d}y$$

$$= \iint\limits_D \frac{f(x)}{f(y)}\mathrm{d}x\mathrm{d}y \quad D=\{(x,y)\,|\,a\le x\le b, a\le y\le b\}$$ 轮换对称性

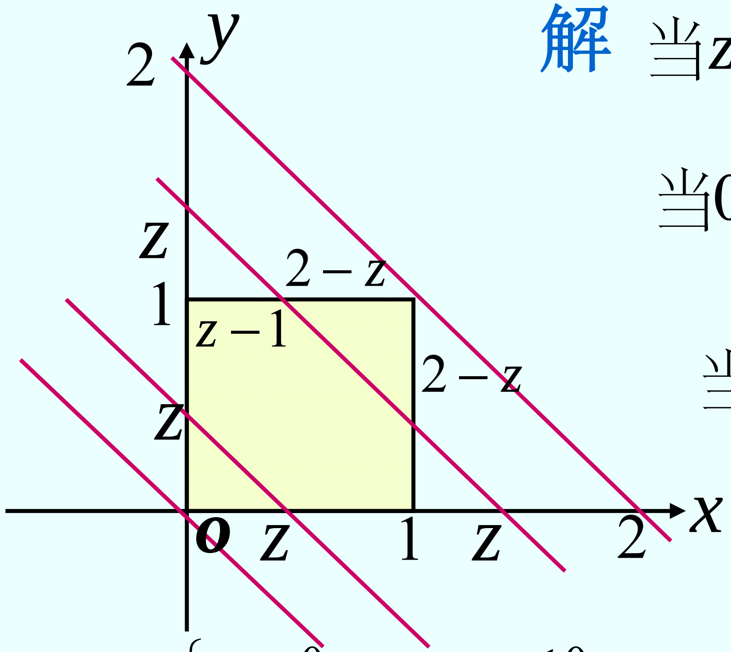
$$= \iint\limits_D \frac{f(y)}{f(x)}\mathrm{d}x\mathrm{d}y = \frac{1}{2}\iint\limits_D \frac{[f(x)]^2+[f(y)]^2}{f(x)f(y)}\mathrm{d}x\mathrm{d}y$$

$$\ge \frac{1}{2}\iint\limits_D \frac{2f(x)\cdot f(y)}{f(x)f(y)}\mathrm{d}x\mathrm{d}y = \iint\limits_D \mathrm{d}x\mathrm{d}y = (b-a)^2$$

主要题型 (5)在含参区域D(变动)上求二重积分

例 设$F(z)=\iint\limits_D f(x,y)d\sigma$ 其中$f(x,y)=\begin{cases}1 & 0\le x\le 1, 0\le y\le 1\\ 0 & 其它\end{cases}$

$D=\{(x,y)\mid x+y\le z\}$,求$F(z)$



解 当$z\le 0$时 $f(x,y)=0$, $F(z)=\iint\limits_{x+y\le z} f(x,y)\mathrm{d}x\mathrm{d}y=0$

当$0<z<1$时 $F(z)=\iint\limits_{x+y\le z} f(x,y)\mathrm{d}x\mathrm{d}y=\dfrac{z^2}{2}$

当$1\le z<2$时 $F(z)=\iint\limits_{x+y\le z} f(x,y)\mathrm{d}x\mathrm{d}y$

$$=1-\frac{1}{2}(2-z)^2=-\frac{z^2}{2}+2z-1$$

当$z\ge 2$时 $F(z)=\iint\limits_{x+y\le z} f(x,y)\mathrm{d}x\mathrm{d}y=1$

$$F(z)=\begin{cases}0 & z\le 0\\ \frac{z^2}{2} & 0<z<1\\ -\frac{z^2}{2}+2z-1 & 1\le z<2\\ 1 & z\ge 2\end{cases}$$

二、利用极坐标计算二重积分

当区域D为中心在原点的圆形扇形或圆环形等，被积函数为x^2+y^2的函数时选用极坐标系

在极坐标系下，用同心圆ρ=常数及射线 θ =常数，分划区域D 为

$$\Delta\sigma_i \quad (i=1,2,\cdots,n)$$

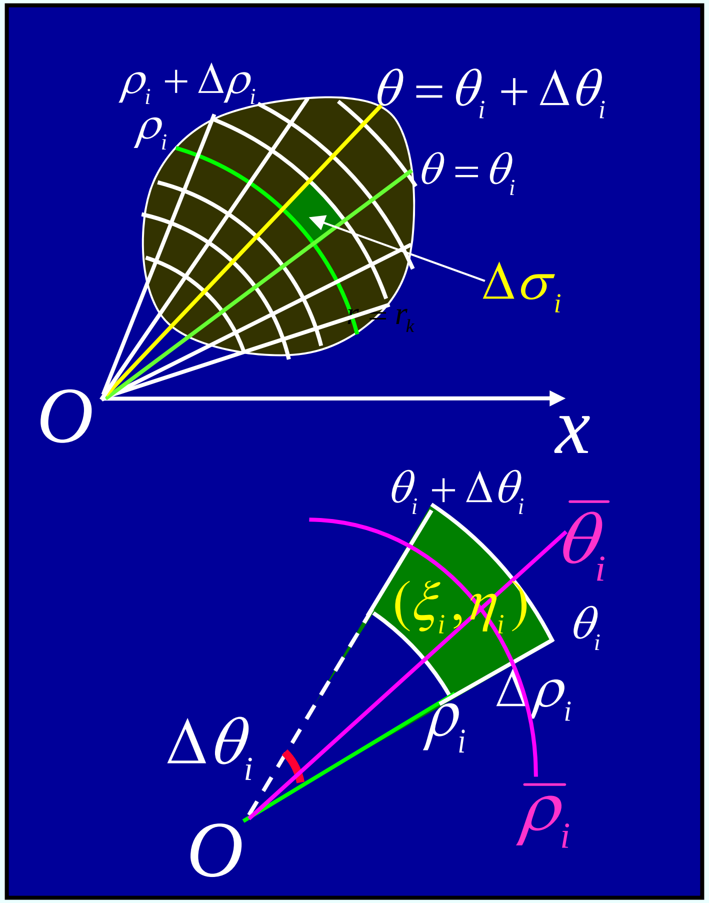
则除包含边界点的小区域外，

$$\Delta\sigma_i = \tfrac{1}{2}(\rho_i+\Delta\rho_i)^2\cdot\Delta\theta_i - \tfrac{1}{2}\rho_i^2\cdot\Delta\theta_i$$

$$= \tfrac{1}{2}[\rho_i+(\rho_i+\Delta\rho_i)]\Delta\rho_i\cdot\Delta\theta_i$$

$$= \bar{\rho}_i\cdot\Delta\rho_i\cdot\Delta\theta_i$$

$\bar{\rho}_i$ 为相邻两圆弧的半径的平均值

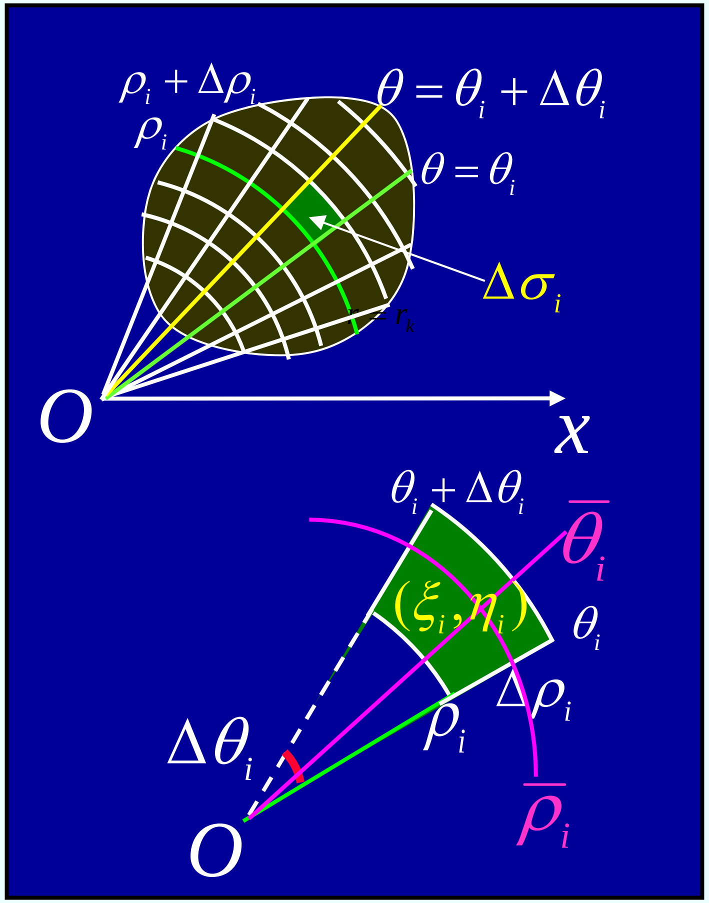


二、利用极坐标计算二重积分

$\Delta\sigma_i = \overline{\rho}_i \cdot \Delta\rho_i \cdot \Delta\theta_i$　　$\overline{\rho}_i$ 为相邻两圆弧的半径的平均值

$\xi_i = \overline{\rho}_i \cos\overline{\theta}_i,\ \eta_k = \overline{\rho}_i \sin\overline{\theta}_i$

$$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta\sigma_i = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\overline{\rho}_i\cos\overline{\theta}_i, \overline{\rho}_i\sin\overline{\theta}_i)\overline{\rho}_i\Delta\rho_i\Delta\theta_i$$



二、利用极坐标计算二重积分

$$\Delta\sigma_i = \overline{\rho}_i \cdot \Delta\rho_i \cdot \Delta\theta_i$$

$\overline{\rho}_i$ 为相邻两圆弧的半径的平均值

$$\xi_i = \overline{\rho}_i \cos\overline{\theta}_i,\ \eta_k = \overline{\rho}_i \sin\overline{\theta}_i$$

$$\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i, \eta_i)\Delta\sigma_i = \lim_{\lambda\to 0}\sum_{i=1}^{n} f(\overline{\rho}_i \cos\overline{\theta}_i, \overline{\rho}_i \sin\overline{\theta}_i)\overline{\rho}_i\Delta\rho_i\Delta\theta_i$$

$$\iint_D f(x,y)\,\mathrm{d}\sigma = \iint_D f(\rho\cos\theta, \rho\sin\theta)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

$$\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint_D f(\rho\cos\theta, \rho\sin\theta)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

从直角坐标系变换为极坐标系的变换公式

利用极坐标计算

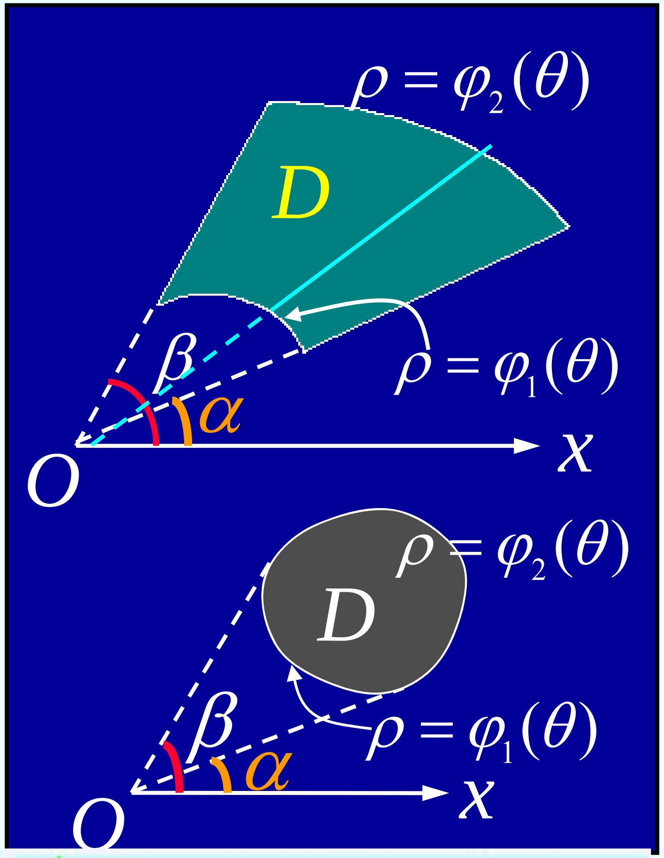
①若极点在域*D*的边界曲线外

$$D:\begin{cases}\varphi_1(\theta)\le\rho\le\varphi_2(\theta)\\ \alpha\le\theta\le\beta\end{cases},$$

$$\iint_D f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

先对ρ后对θ

$$=\int_\alpha^\beta \mathrm{d}\theta\int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho$$

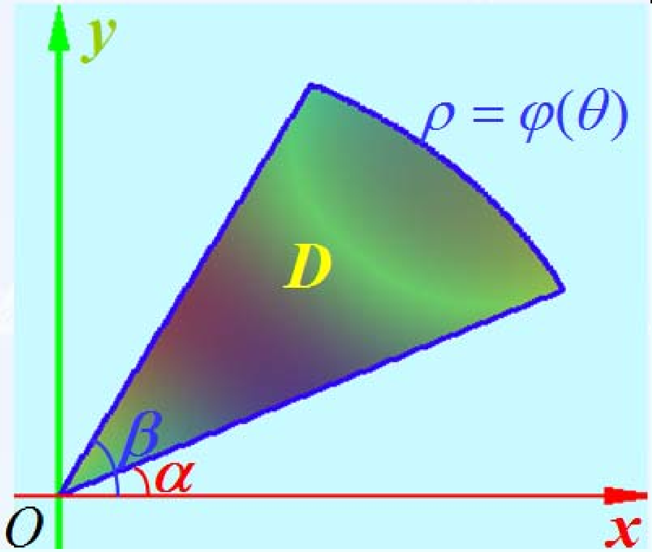


②若极点在域*D*的边界曲线上

$$D:\begin{cases}0\le\rho\le\varphi(\theta)\\ \alpha\le\theta\le\beta\end{cases}$$

$$\iint_D f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

$$=\int_\alpha^\beta \mathrm{d}\theta\int_0^{\varphi(\theta)} f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho$$



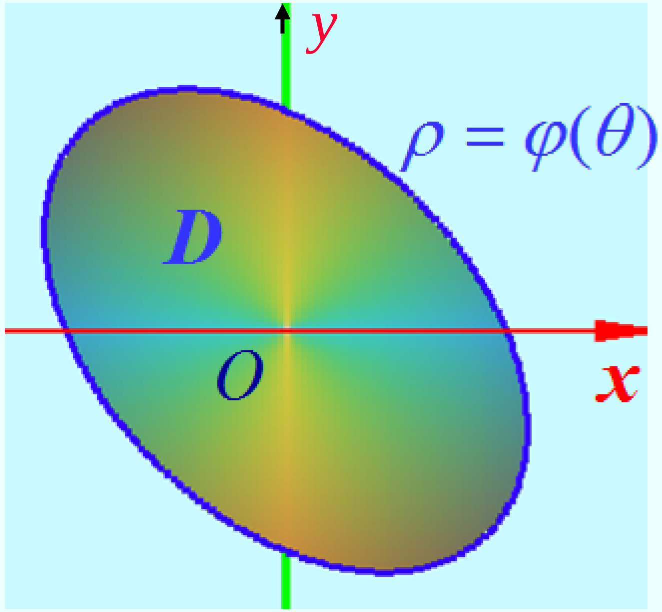
利用极坐标计算

③若极点在域D内

$$D:\begin{cases}0\le\rho\le\varphi(\theta)\\0\le\theta\le 2\pi\end{cases}$$

$$\iint_D f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\varphi(\theta)}f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho$$

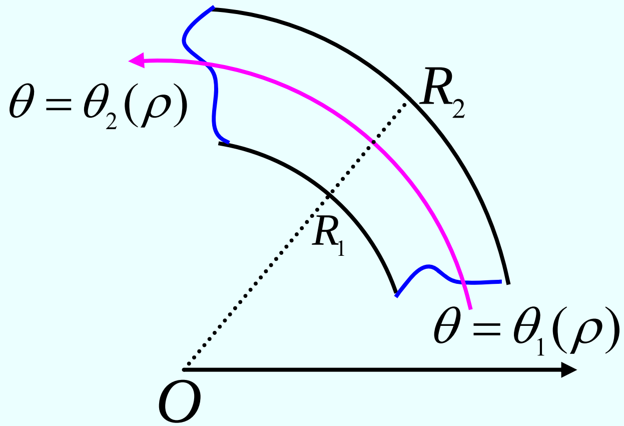


④若极点在域D的边界曲线外

$$D:\begin{cases}\theta_1(\rho)\le\theta\le\theta_2(\rho)\\R_1\le\rho\le R_2\end{cases},$$ 先对θ后对ρ

$$\iint_D f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

$$=\int_{R_1}^{R_2}\mathrm{d}\rho\int_{\theta_1(\rho)}^{\theta_2(\rho)}f(\rho\cos\theta,\rho\sin\theta)\rho\,\mathrm{d}\theta$$



注: 箭头要穿透变化多的曲线,以极点O为圆心,画同心圆弧

解题思路

①选择极坐标系从两方面考虑,即D为圆形,扇形,环形

或$f(x,y)$呈现$f(x^2+y^2), f(\frac{y}{x})$等形状

②选定极坐标系后,一般化为先对ρ后对θ的积分

确定极角θ的范围$\alpha \le \theta \le \beta$

在此范围从极点O出发作一射线,该射线穿入D的边界线,

进入$\varphi_1(\theta)$为积分下限;穿出$\varphi_2(\theta)$为积分上限

确定ρ,θ范围的方法

一画图　二代入法 将极坐标变换代入到积分域中

解出关于ρ,θ的不等式,即可得ρ,θ的范围

典型题 (1) 利用极坐标计算二重积分

例1. 计算 $\iint\limits_D e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y$ 其中 $D: x^2+y^2\le R^2$.

由于 e^{-x^2} 的原函数不是初等函数，故本题无法用直角坐标计算

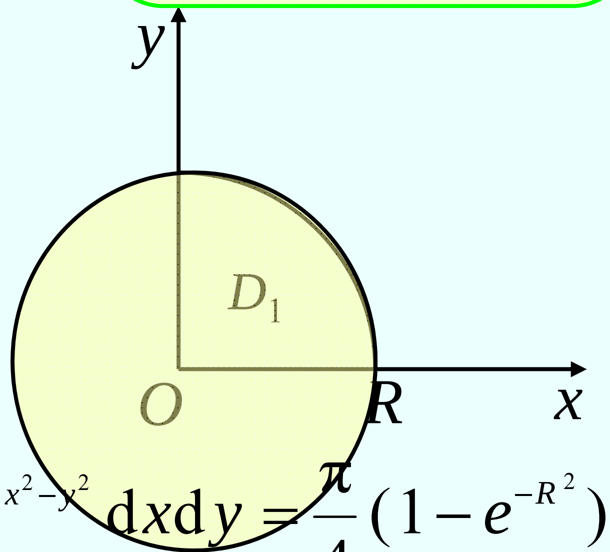
解: 在极坐标系下 $D:\begin{cases}0\le\rho\le R\\ 0\le\theta\le 2\pi\end{cases}$,

$$\text{原式}=\iint\limits_D e^{-\rho^2}\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^R \rho e^{-\rho^2}\,\mathrm{d}\rho$$

$$=2\pi\left[-\frac{1}{2}e^{-\rho^2}\right]_0^R$$

$$=\pi(1-e^{-R^2})$$



$$\iint\limits_{D_1} e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y=\frac{\pi}{4}(1-e^{-R^2})$$

注: 利用上题可得一个在概率论与数理统计及工程上非常有用的反常积分公式 $\int_0^{+\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x=\frac{\sqrt{\pi}}{2}$

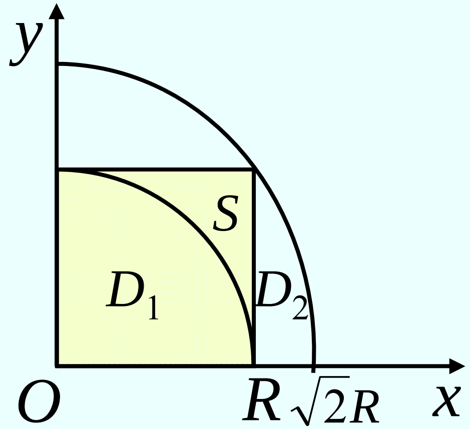
$$D_1 \subset S \subset D_2$$

$$\iint_{D_1} e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y \le \iint_{S} e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y \le \iint_{D_2} e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y$$

由于e^{-x^2}的原函数不是初等函数，故本题无法用直角坐标计算

$$\frac{\pi}{4}(1-e^{-R^2}) \le \int_0^R e^{-x^2}\mathrm{d}x \cdot \int_0^R e^{-y^2}\mathrm{d}y \le \frac{\pi}{4}(1-e^{-2R^2})$$

$$\frac{\pi}{4}(1-e^{-R^2}) \le \left(\int_0^R e^{-x^2}\mathrm{d}x\right)^2 \le \frac{\pi}{4}(1-e^{-2R^2})$$



令$R \to +\infty$, 上式两端趋于同一极限 $\dfrac{\pi}{4}$

从而 $\boxed{\displaystyle\int_0^{+\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}}$

$$\boxed{\iint_{D_1} e^{-x^2-y^2}\,\mathrm{d}x\mathrm{d}y = \frac{\pi}{4}(1-e^{-R^2})}$$

注: 利用上题可得一个在概率论与数理统计及工程上非常有用的反常积分公式 $\displaystyle\int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}$

例2. 求球体 $x^2+y^2+z^2\le 4a^2$ 被圆柱面 $x^2+y^2=2ax$ $(a>0)$ 所截得的(含在柱面内的)立体的体积.

解: 设 $D: 0\le\rho\le 2a\cos\theta,\ 0\le\theta\le\dfrac{\pi}{2}$

由对称性

$$V=4\iint_D\sqrt{4a^2-\rho^2}\,\rho\,\mathrm{d}\rho\,\mathrm{d}\theta$$

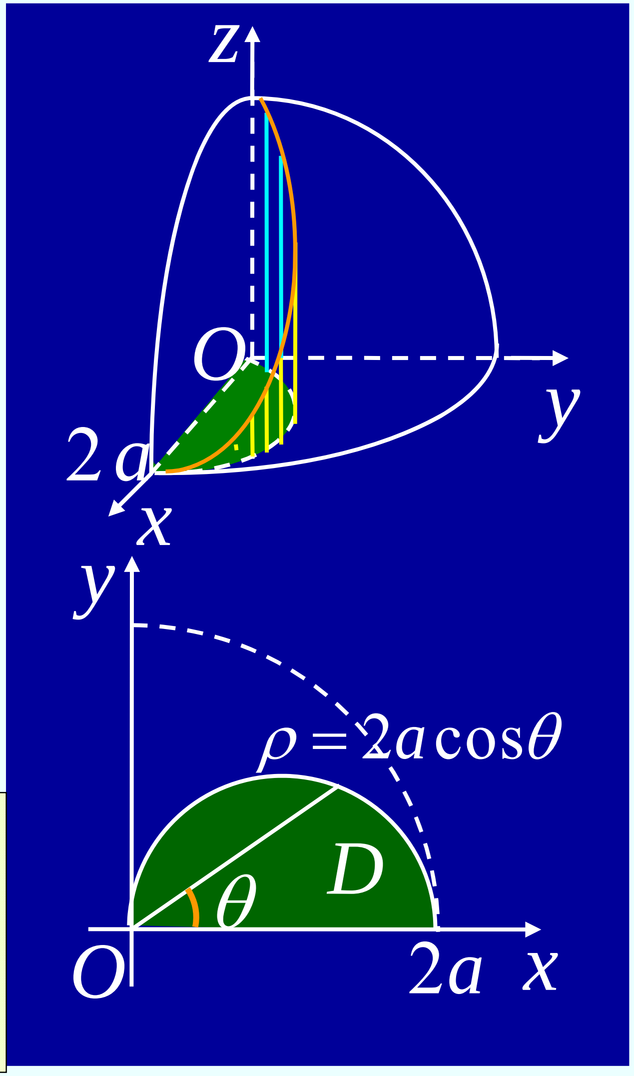
$$=4\int_0^{\pi/2}\mathrm{d}\theta\int_0^{2a\cos\theta}\sqrt{4a^2-\rho^2}\,\rho\,\mathrm{d}\rho$$

$$=\frac{32}{3}a^3\int_0^{\pi/2}(1-\sin^3\theta)\,\mathrm{d}\theta$$

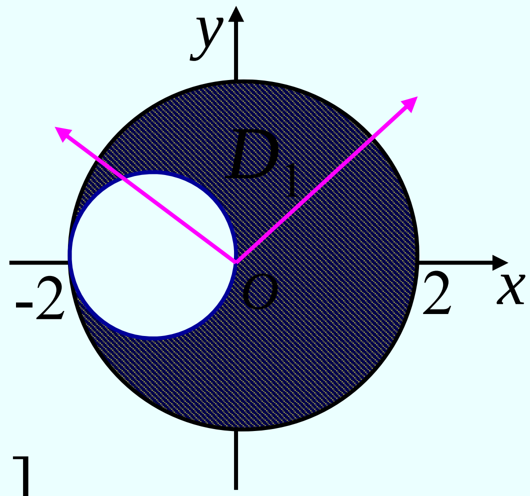
$$=\frac{32}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$$

$I_n=\int_0^{\frac{\pi}{2}}\sin^n x\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}\cos^n x\,\mathrm{d}x$　　n 为正整数

$$=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & n\text{ 为偶数}\\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{4}{5}\cdot\dfrac{2}{3} & n\text{为奇数（}n>1\text{）}\end{cases}$$



例3 求$\iint\limits_D(\sqrt{x^2+y^2}+y)\mathrm{d}\sigma$，其中$D$是由圆 $x^2+y^2=4$ 和 $(x+1)^2+y^2=1$ 所围成的平面区域



解 由对称性$\iint\limits_D y\mathrm{d}\sigma=0$

$$I=2\iint\limits_{D_1}\sqrt{x^2+y^2}\mathrm{d}x\mathrm{d}y$$

$$=2\left[\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^2\rho\cdot\rho\mathrm{d}\rho+\int_{\frac{\pi}{2}}^{\pi}\mathrm{d}\theta\int_{-2\cos\theta}^2\rho\cdot\rho\mathrm{d}\rho\right]$$

$$\rho^2+2\rho\cos\theta=0$$

$$\Rightarrow\rho=-2\cos\theta$$

$$=\frac{16}{9}(3\pi-2)$$

典型题 (2) 直角坐标系与极坐标系互化

①直角坐标系化为极坐标系的二次积分

解题步骤

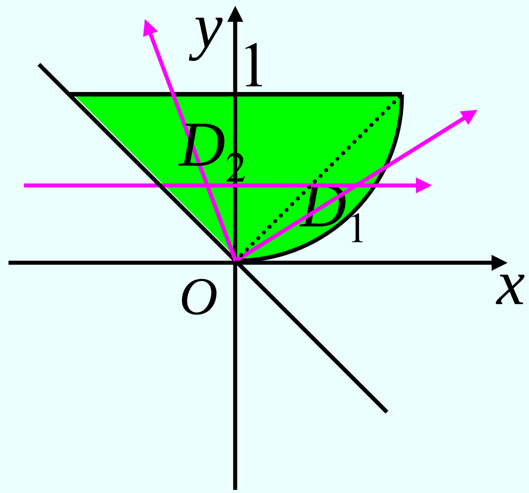
Step1 由直角坐标系下的二次积分上下限写出积分区域D的不等式,并画出D的图形

Step2 将D的边界曲线表示为极坐标系下方程,写出积分区域不等式

Step3 写出极坐标系下二次积分的上下限

例1 将$\int_0^1 \mathrm{d}y\int_{-y}^{\sqrt{y}} f(x,y)\mathrm{d}x$化为极坐标系下的二次积分

解 $D:\{(x,y)\,|\,0\le y\le 1, -y\le x\le \sqrt{y}\}$



$$D_1:\{(\rho,\theta)\,|\,0\le\theta\le\frac{\pi}{4},\ 0\le\rho\le\frac{\sin\theta}{\cos^2\theta}\}$$

$$D_2:\{(\rho,\theta)\,|\,\frac{\pi}{4}\le\theta\le\frac{3\pi}{4},\ 0\le\rho\le\frac{1}{\sin\theta}\}$$

$$原式=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_0^{\frac{\sin\theta}{\cos^2\theta}} f(\rho\cos\theta,\rho\sin\theta)\rho\mathrm{d}\rho$$

$$+\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\mathrm{d}\theta\int_0^{\frac{1}{\sin\theta}} f(\rho\cos\theta,\rho\sin\theta)\rho\mathrm{d}\rho$$

$$y=x^2$$

$$\Rightarrow \rho\sin\theta=\rho^2\cos^2\theta$$

$$\Rightarrow \rho=\frac{\sin\theta}{\cos^2\theta}$$

$$y=1\Rightarrow\rho\sin\theta=1$$

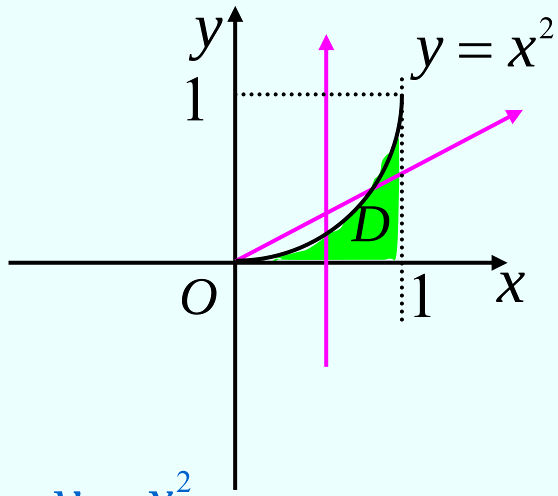
$$\Rightarrow\rho=\frac{1}{\sin\theta}$$

例2 将$\int_0^1 \mathrm{d}x\int_0^{x^2} f(x,y)\mathrm{d}y$化为极坐标系下的二次积分

解 $D:\{(x,y)\mid 0\le x\le 1, 0\le y\le x^2\}$

$$D:\{(\rho,\theta)\mid 0\le\theta\le\frac{\pi}{4},\ \frac{\sin\theta}{\cos^2\theta}\le\rho\le\frac{1}{\cos\theta}\}$$

$$\text{原式}=\int_0^{\frac{\pi}{4}}\mathrm{d}\theta\int_{\frac{\sin\theta}{\cos^2\theta}}^{\frac{1}{\cos\theta}} f(\rho\cos\theta,\rho\sin\theta)\rho\mathrm{d}\rho$$



$y=x^2$

$\Rightarrow \rho\sin\theta=\rho^2\cos^2\theta$

$\Rightarrow \rho=\dfrac{\sin\theta}{\cos^2\theta}$

$x=1\Rightarrow \rho\cos\theta=1$

$\Rightarrow \rho=\dfrac{1}{\cos\theta}$

典型题 (2) 直角坐标系与极坐标系互化

②极坐标系下二次积分化为直角坐标系下二次积分

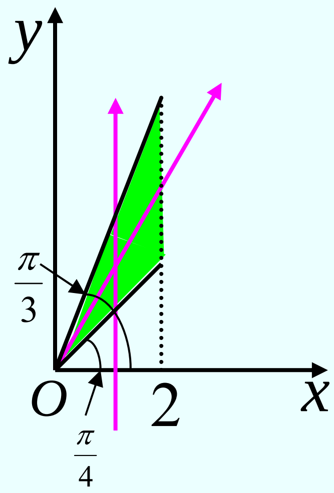
解题步骤

Step1 由极坐标系下的二次积分上下限写出积分区域D的不等式,并画出D的图形

Step2 将D的边界曲线表示为直角坐标系下方程,写出积分区域不等式

Step3 写出直角坐标系下二次积分的上下限

例 将积分变换$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\mathrm{d}\theta\int_0^{\frac{2}{\cos\theta}}\rho f(\rho,\theta)\mathrm{d}\rho$化为直角坐标系下的二次积分



解 $D:\{(\rho,\theta)\mid \frac{\pi}{4}\le\theta\le\frac{\pi}{3},\ 0\le\rho\le\frac{2}{\cos\theta}\ \}$

$$\rho=\frac{2}{\cos\theta}\Rightarrow\rho\cos\theta=2\Rightarrow x=2$$

$D:\{(x,y)\mid 0\le x\le 2,\ x\le y\le\sqrt{3}x\}$

$$\text{原式}=\int_0^2\mathrm{d}x\int_x^{\sqrt{3}x} f(\sqrt{x^2+y^2},\arctan\frac{y}{x})\mathrm{d}y$$