第四章

不定积分

第四章

第二节

换元积分法

内容

一、第一类换元积分法

二、第二类换元积分法

一、第一类换元积分法

由上一讲我们已知道，对应于一个求导公式，就有一个积分公式，那么，对应于复合函数的求导法则，有没有一个相应的积分法则呢？这就是下面的定理.

设$f(u)$具有原函数,即$F'(u)=f(u)$, $\int f(u)du=F(u)+C$

如果u是中间变量$u=\varphi(x)$且可微,那么复合函数微分法有

$dF(\varphi(x))=f[\varphi(x)]\varphi'(x)dx$ 由不定积分定义

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=F(\varphi(x))+C=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}$$

定理 1(第一类换元法)凑微分法

设$f(u)$具有原函数, $u=\varphi(x)$ 可导,则有换元公式

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}$$

怎么用

$$\int g(x)\mathrm{d}x=\int f[\varphi(x)]\varphi'(x)\mathrm{d}x=\int f[\varphi(x)]\mathrm{d}\varphi(x)=[\int f(u)\mathrm{d}u]_{u=\varphi(x)}$$

分解　　凑微分　　要容易积出

$$(1)\ \int f(ax+b)\mathrm{d}x=\frac{1}{a}\int f(ax+b)\mathrm{d}(ax+b)$$

例1 求 $\int \sin(3x+2)\,\mathrm{d}x.$

解 原式 $=\frac{1}{3}\int \sin(3x+2)\,\mathrm{d}(3x+2)$.令 $3x+2=u$ 则

$$=\frac{1}{3}\int \sin u\,\mathrm{d}u$$

$$=-\frac{1}{3}\cos u+C$$

$$=-\frac{1}{3}\cos(3x+2)+C.$$

$$\boxed{(1)\ \int f(ax+b)\mathrm{d}x = \frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)}$$

例2 求$\displaystyle\int \frac{1}{3+2x}\mathrm{d}x$

解 原式$\displaystyle = \frac{1}{2}\int \frac{1}{3+2x}\mathrm{d}(3+2x)$ 令 $3+2x=u$ 则

$$= \frac{1}{2}\int \frac{1}{u}\mathrm{d}u$$

$$= \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|3+2x| + C$$

$$\boxed{(2)\ \int f(ax^n+b)x^{n-1}\,\mathrm{d}x=\frac{1}{na}\int f(ax^n+b)\,\mathrm{d}(ax^n+b)}$$

例3 求 $\int x\cos x^2\,\mathrm{d}x$

熟练以后不必写中间变量

解 原式 $=\frac{1}{2}\int\cos x^2\,\mathrm{d}x^2$ 令 $x^2=u$ 则

$$=\frac{1}{2}\int\cos u\,\mathrm{d}u$$

$$=\frac{1}{2}\sin u+C$$

$$=\frac{1}{2}\sin x^2+C$$

$$\boxed{(2)\ \int f(ax^n+b)x^{n-1}\,\mathrm{d}x=\frac{1}{na}\int f(ax^n+b)\,\mathrm{d}(ax^n+b)}$$

例4 设 $\int xf(x)\,\mathrm{d}x=\arcsin x+C$，求 $\int\frac{1}{f(x)}\,\mathrm{d}x$

解 $xf(x)=\frac{1}{\sqrt{1-x}}$ 从而 $\frac{1}{f(x)}=x\sqrt{1-x^2}$

于是 $\int\frac{1}{f(x)}\,\mathrm{d}x=\int x\sqrt{1-x^2}\,\mathrm{d}x$

$$=-\frac{1}{2}\int\sqrt{1-x^2}\,\mathrm{d}(1-x^2)$$

$$=-\frac{1}{2}\cdot\frac{2}{3}(1-x^2)^{\frac{3}{2}}+C=-\frac{1}{3}(1-x^2)^{\frac{3}{2}}+C$$

$$(3)\ \int f(\sqrt{x})\frac{1}{\sqrt{x}}\mathrm{d}x = 2\int f(\sqrt{x})\,\mathrm{d}\sqrt{x}$$

例5 求 $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\mathrm{d}x$

解 原式 $= 2\int e^{\sqrt{x}}\mathrm{d}\sqrt{x} \ = 2e^{\sqrt{x}} + C$

$$(4)\ \int f(\frac{1}{x})\frac{1}{x^2}\mathrm{d}x = -\int f(\frac{1}{x})\,\mathrm{d}\frac{1}{x}$$

例6 求 $\int \frac{1}{x^2}e^{\frac{1}{x}}\mathrm{d}x$

解 原式 $= -\int e^{\frac{1}{x}}\mathrm{d}\frac{1}{x} = -e^{\frac{1}{x}} + C$

$$(5)\ \int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\,\mathrm{d}\ln x$$

例7 求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$

解 原式 $= \frac{1}{2}\int \frac{1}{1+2\ln x}\mathrm{d}(1+2\ln x)$

$$= \frac{1}{2}\ln|1+2\ln x| + C$$

$$(6)\ \int f(\sin x)\cos x\mathrm{d}x = \int f(\sin x)\mathrm{d}\sin x$$

$$\int f(\cos x)\sin x\mathrm{d}x = -\int f(\cos x)\mathrm{d}\cos x$$

例8 求$\int \tan x\mathrm{d}x$

解 原式$=\int \frac{\sin x}{\cos x}\mathrm{d}x = -\int \frac{1}{\cos x}\mathrm{d}\cos x = -\ln|\cos x| + C$

例9 求$\int \sin^5 x\cdot\cos^3 x\mathrm{d}x$

若$\int \sin^5 x\cdot\cos^2 x\mathrm{d}x$

凑$\mathrm{d}\cos x$

解 原式$=\int \sin^5 x\cdot\cos^2 x\mathrm{d}\sin x$

$$=\int \sin^5 x\cdot(1-\sin^2 x)\mathrm{d}\sin x$$

$$=\tfrac{1}{6}\sin^6 x - \tfrac{1}{8}\sin^8 x + C$$

$$(7)\ \int f(\tan x)\sec^2 x\,dx = \int f(\tan x)\,d\tan x$$

$$\int f(\cot x)\csc^2 x\,dx = -\int f(\cot x)\,d\cot x$$

$$\int f(\sec x)\sec x\tan x\,dx = \int f(\sec x)\,d\sec x$$

$$\int f(\csc x)\csc x\cot x\,dx = -\int f(\csc x)\,d\csc x$$

例10 求 $\int \tan^{10} x \cdot \sec^2 x dx$

解 原式 $= \int \tan^{10} x d\tan x$

$$= \tfrac{1}{11}\tan^{11} x + C$$

例11 求 $\int \sec^6 x dx$

解 原式 $= \int (\tan^2 x + 1)^2 d\tan x$

$$= \int (u^2 + 1)^2 du$$

$$= \int u^4 + 2u^2 + 1 du$$

$$= \tfrac{1}{5}u^5 + \tfrac{2}{3}u^3 + u + C$$

$$= \tfrac{1}{5}\tan^5 x + \tfrac{2}{3}\tan^3 x + \tan x + C$$

$$(7)\quad \int f(\tan x)\sec^2 x\,\mathrm{d}x = \int f(\tan x)\,\mathrm{d}\tan x$$

$$\int f(\cot x)\csc^2 x\,\mathrm{d}x = -\int f(\cot x)\,\mathrm{d}\cot x$$

$$\int f(\sec x)\sec x\tan x\,\mathrm{d}x = \int f(\sec x)\,\mathrm{d}\sec x$$

$$\int f(\csc x)\csc x\cot x\,\mathrm{d}x = -\int f(\csc x)\,\mathrm{d}\csc x$$

例12 求 $\int \tan^5 x \cdot \sec^3 x\,\mathrm{d}x$

解 原式 $= \int \tan^4 x \cdot \sec^2 x\,\mathrm{d}\sec x$

$$= \int (\sec^2 x - 1)^2 \cdot \sec^2 x\,\mathrm{d}\sec x$$

$$= \int (u^2 - 1)^2 \cdot u^2\,\mathrm{d}u \quad = \int u^6 - 2u^4 + u^2\,\mathrm{d}u$$

$$= \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C = \tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C$$

恒等变形

例13. 求 $\int \csc x \mathrm{d}x$

解: 原式 $= \int \frac{\mathrm{d}x}{\sin x}$

$$= \int \frac{\mathrm{d}x}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}$$

$$= \int \frac{\mathrm{d}\frac{x}{2}}{\tan\frac{x}{2}\cdot\cos^2\frac{x}{2}}$$

$$= \int \frac{\mathrm{d}\tan\frac{x}{2}}{\tan\frac{x}{2}}$$

$$= \ln|\tan\tfrac{x}{2}| + C$$

$$\tan\tfrac{x}{2} = \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \frac{1-\cos x}{\sin x}$$

$$= \csc x - \cot x$$

$$\boxed{\therefore \int \csc x \mathrm{d}x = \ln|\csc x - \cot x| + C}$$

$$\boxed{\int \sec x \mathrm{d}x} = \int \frac{\mathrm{d}x}{\cos x} = \int \frac{\mathrm{d}(x+\frac{\pi}{2})}{\sin(x+\frac{\pi}{2})}$$

$$= \ln|\csc(x+\tfrac{\pi}{2}) - \cot(x+\tfrac{\pi}{2})| + C$$

$$\boxed{\therefore \int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C}$$

恒等变形

例14. 求 $\int \cos 3x \cdot \cos 2x \mathrm{d}x$

$\sin\alpha \cdot \cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$

$\cos\alpha \cdot \sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$\cos\alpha \cdot \cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$

$\sin\alpha \cdot \sin\beta = -\frac{1}{2}[\cos(\alpha+\beta)-\cos(\alpha-\beta)]$

解: 原式 $= \dfrac{1}{2}\int(\cos 5x + \cos x)\mathrm{d}x$

$$= \tfrac{1}{10}\sin 5x + \tfrac{1}{2}\sin x + C$$

例15. 求 $\int \cos^4 x \mathrm{d}x$

$\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2\sin^2 x$

$= 2\cos^2 x - 1$

解: 原式 $= \displaystyle\int\left(\frac{1+\cos 2x}{2}\right)^2 \mathrm{d}x$

$$= \frac{1}{4}\int(1 + 2\cos 2x + \cos^2 2x)\mathrm{d}x = \frac{1}{4}\int\left(1+2\cos 2x + \frac{1+\cos 4x}{2}\right)\mathrm{d}x$$

$$= \frac{1}{4}x + \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

$$(8)\ \int f(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}\mathrm{d}x=\int f(\arcsin x)\mathrm{d}\arcsin x$$

例16 求 $\int\sqrt{\arcsin x}\cdot\frac{1}{\sqrt{1-x^2}}\mathrm{d}x$

解 原式 $=\int\sqrt{\arcsin x}\,\mathrm{d}\arcsin x\ =\frac{2}{3}(\arcsin x)^{\frac{3}{2}}+C$

$$(8)'\ \int\frac{1}{\sqrt{a^2-x^2}}\mathrm{d}x=\int\frac{1}{\sqrt{1-(\frac{x}{a})^2}}\mathrm{d}\tfrac{x}{a}=\arcsin(\tfrac{x}{a})+C\quad(a>0)$$

例17 求 $\int\frac{2x+3}{\sqrt{1+2x-x^2}}\mathrm{d}x$

解 原式 $=\int\frac{-(2-2x)+5}{\sqrt{1+2x-x^2}}\mathrm{d}x=-\int\frac{2-2x}{\sqrt{1+2x-x^2}}\mathrm{d}x\ +5\int\frac{\mathrm{d}(x-1)}{\sqrt{2-(x-1)^2}}$

$$=-\int\frac{\mathrm{d}(1+2x-x^2)}{\sqrt{1+2x-x^2}}+5\int\frac{\mathrm{d}\frac{x\text{-}1}{\sqrt{2}}}{\sqrt{1-(\frac{x\text{-}1}{\sqrt{2}})^2}}=-2\sqrt{1+2x-x^2}+5\arcsin\frac{x\text{-}1}{\sqrt{2}}+C$$

$$\boxed{(9)\ \int f(\arctan x)\cdot\frac{1}{1+x^2}\mathrm{d}x=\int f(\arctan x)\mathrm{d}\arctan x}$$

例18 求$\int\frac{(\arctan x)^2}{1+x^2}\mathrm{d}x$

解 原式 $=\int(\arctan x)^2\mathrm{d}\arctan x=\frac{1}{3}(\arctan x)^3+C$

$$\boxed{(9)'\ \int\frac{1}{a^2+x^2}\mathrm{d}x=\frac{1}{a}\int\frac{1}{1+(\frac{x}{a})^2}\mathrm{d}\tfrac{x}{a}=\frac{1}{a}\arctan(\tfrac{x}{a})+C\quad(a>0)}$$

例19 求$\int\frac{\mathrm{d}x}{x^2+2x+3}$

解 原式$=\int\frac{\mathrm{d}x}{(x+1)^2+2}=\int\frac{\mathrm{d}(x+1)}{(x+1)^2+2}$

$$=\frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C$$

$$(10)\ \int f(e^x)\cdot e^x\mathrm{d}x=\int f(e^x)\mathrm{d}e^x$$

例20 求 $\int\frac{1}{1+e^x}\mathrm{d}x$

解 原式 $=\int\frac{1+e^x-e^x}{1+e^x}\mathrm{d}x=\int 1-\frac{e^x}{1+e^x}\mathrm{d}x$

$$=x-\int\frac{d(1+e^x)}{1+e^x}=x-\ln(1+e^x)+C$$

$$(11)\ \int f(\sqrt{1+x^2})\cdot\frac{x}{\sqrt{1+x^2}}\mathrm{d}x=\int f(\sqrt{1+x^2})\mathrm{d}\sqrt{1+x^2}$$

例21 求 $\int\frac{x\tan\sqrt{1+x^2}\mathrm{d}x}{\sqrt{1+x^2}}$

由例8

解 原式 $=\int\tan\sqrt{1+x^2}\mathrm{d}\sqrt{1+x^2}=-\ln|\cos\sqrt{1+x^2}|+C$

$$(12)\ \int f(\ln\frac{1+x}{1-x})\cdot\frac{1}{1-x^2}\mathrm{d}x=\frac{1}{2}\int f(\ln\frac{1+x}{1-x})\mathrm{d}\ln\frac{1+x}{1-x}$$

例22 求 $\int\frac{1}{1-x^2}\ln\frac{1+x}{1-x}\mathrm{d}x$

解 原式 $=\frac{1}{2}\int\ln\frac{1+x}{1-x}\mathrm{d}\ln\frac{1+x}{1-x}=\frac{1}{4}\left(\ln\frac{1+x}{1-x}\right)^2+C$

$$(12)'\ \int\frac{1}{x^2-a^2}\mathrm{d}x=\frac{1}{2a}\int\frac{1}{x-a}-\frac{1}{x+a}\mathrm{d}x=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C\quad(a>0)$$

$$=\frac{1}{2a}(\ln|x-a|-\ln|x+a|)+C$$

例23 求 $\int\frac{x^2-1}{x^4+1}\mathrm{d}x$

解 原式 $=\int\frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\mathrm{d}x=\int\frac{\mathrm{d}(x+\frac{1}{x})}{(x+\frac{1}{x})^2-2}=\frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|$

思考 $\int\frac{x^2+1}{x^4+1}\mathrm{d}x,\ \int\frac{1}{x^4+1}\mathrm{d}x$

二、第二类换元积分法

前面介绍的第一换元法是将积分凑成

$\int f[\varphi(x)]\mathrm{d}\varphi(x)$ 通过变量代换 $\varphi(x)=u$ 转化为 $\int f(u)du$

在求出 $\int f(u)du$ 之后，再用 $u=\varphi(x)$ 代回原变量

通过变量代换 $x=\varphi(t)$

将积分 $\int f(x)dx=\int f[\varphi(t)]\varphi'(t)dt=\int g(t)dt$

在求出 $\int g(t)dt$ 之后，再用 $t=\varphi^{-1}(x)$ 代回原变量

定理2（第二换元法）设 $x=\varphi(t)$ 是单调可导函数,且 $\varphi'(t)\neq 0$ 又设 $f[\varphi(t)]\varphi'(t)$ 具有原函数，则有换元公式

$$\int f(x)dx=\int f[\varphi(t)]\varphi'(t)dt\,\Big|_{t=\varphi^{-1}(x)}$$

其中 $t=\varphi^{-1}(x)$ 是 $x=\varphi(t)$ 的反函数

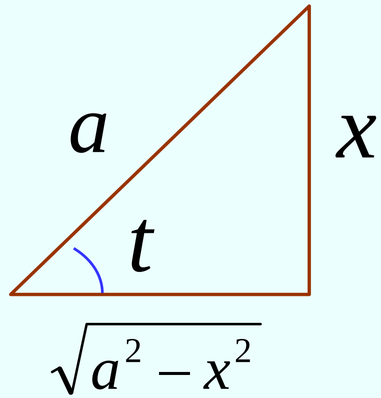
证：设 $f[\varphi(t)]\varphi'(t)$ 的原函数为 $\Phi(t)$，则右式 $=\Phi[\varphi^{-1}(x)]$

两边求导 $\quad \dfrac{d}{dx}\left(\int f(x)dx\right)=f(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi[\varphi^{-1}(x)]=\frac{d\Phi}{dt}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=f[\varphi(t)]\cancel{\varphi'(t)}\cdot\frac{1}{\cancel{\varphi'(t)}}=f(x)$$

例1. 求 $\int\sqrt{a^2-x^2}\,\mathrm{d}x\ \ (a>0)$.

辅助三角形



解: 令 $x=a\sin t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则

$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=a\cos t$$

$$\mathrm{d}x=a\cos t\,\mathrm{d}t$$

$$\therefore\ \text{原式}=\int a\cos t\cdot a\cos t\,\mathrm{d}t=a^2\int\cos^2 t\,\mathrm{d}t=a^2\int\frac{1+\cos 2t}{2}\mathrm{d}t$$

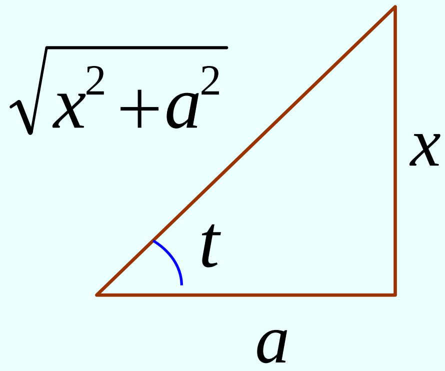
$$=a^2\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)+C$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{\cancel{a^2}}{4}\cdot 2\cdot\frac{x}{\cancel{a}}\cdot\frac{\sqrt{a^2-x^2}}{\cancel{a}}+C$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

例2. 求 $\boxed{\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}}\ (a>0)}$

辅助三角形



解: 令 $x = a\tan t,\ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sec t$$

$$\mathrm{d}x = a\sec^2 t\,\mathrm{d}t$$

$\therefore$ 原式 $= \int \frac{a\sec^2 t}{a\sec t}\mathrm{d}t = \int \sec t\,\mathrm{d}t$

$$= \ln\left|\sec t + \tan t\right| + C_1$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$\boxed{= \ln\left|x + \sqrt{x^2+a^2}\right| + C} \qquad (C = C_1 - \ln a)$$

$$\boxed{\int \frac{dx}{\sqrt{x^2+a^2}} = \ln|x+\sqrt{x^2+a^2}|+C}$$

例3. 求 $\int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$

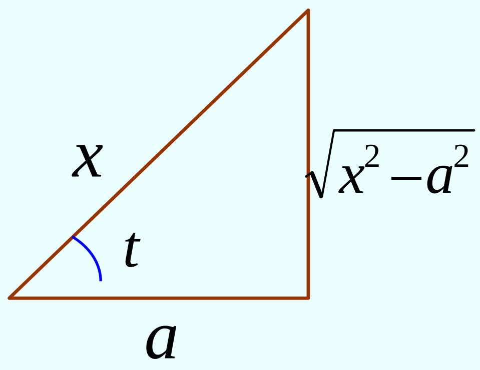
解: 原式 $= \frac{1}{2}\int \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right|+C$

$$\boxed{\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}|+C}$$

解: 当 $x>a$, 令 $x=a\sec t$ $,t\in(0,\frac{\pi}{2})$,

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \int \frac{a\sec t\cdot\tan t\,\mathrm{d}\,t}{a\tan t} = \int \sec t\,\mathrm{d}\,t$$

$$= \ln|\sec t+\tan t|+C_1$$

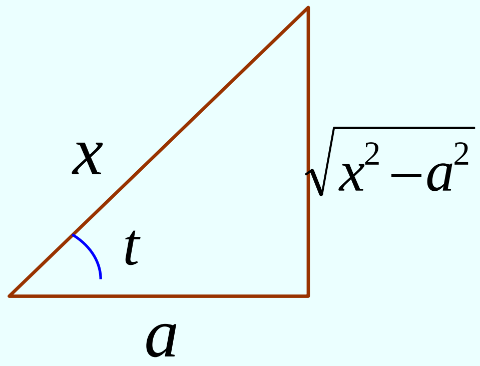


$$= \ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1 = \ln\left|x+\sqrt{x^2-a^2}\right|+C$$

$$\boxed{\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}|+C}$$

解: 当$x>a$, 令$x = a\sec t$,$t \in (0, \frac{\pi}{2})$,

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \int \frac{a\sec t\tan t\,\mathrm{d}\,t}{a\tan t} = \int \sec t\,\mathrm{d}\,t$$



$$= \ln|\sec t + \tan t| + C_1$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C_1 = \boxed{\ln\left|x+\sqrt{x^2-a^2}\right|+C}$$

当$x<-a$,令u=-x, 此时$u>a$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = -\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}} = -\ln\left|u+\sqrt{u^2-a^2}\right|+C_2$$

$$= -\ln\left|-x+\sqrt{x^2-a^2}\right|+C_2 = \ln\left|\frac{1}{\sqrt{x^2-a^2}-x}\right|+C_2$$

$$= \boxed{\ln\left|x+\sqrt{x^2-a^2}\right|+C}$$

$$\boxed{\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}\,|+C}$$

例4. 求 $\displaystyle\int \frac{1-x}{\sqrt{x^2-x-1}}\mathrm{d}x$

解: 原式 $\displaystyle=\int \frac{-\frac{1}{2}(2x-1)+\frac{1}{2}}{\sqrt{x^2-x-1}}\mathrm{d}x = -\frac{1}{2}\int \frac{\mathrm{d}(x^2-x-1)}{\sqrt{x^2-x-1}} + \frac{1}{2}\int \frac{\mathrm{d}(x-\frac{1}{2})}{\sqrt{(x-\frac{1}{2})^2-\frac{5}{4}}}$

$$=-\sqrt{x^2-x+1}+\tfrac{1}{2}\ln|x-\tfrac{1}{2}+\sqrt{x^2-x-1}\,|+C$$

例5. 求 $\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$

解法1: 利用三角代换

当$x>1$时 $x=\sec t,\ t\in(0,\frac{\pi}{2})$,

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}=\int \frac{\sec t\cdot\tan t\mathrm{d}t}{\sec t\cdot\tan t}=t+C=\arccos\frac{1}{x}+C$$

当$x<$-1时 $u=-x>1$

$u>1$

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}=\int \frac{-\mathrm{d}u}{-u\sqrt{u^2-1}}=\int \frac{\mathrm{d}u}{u\sqrt{u^2-1}}=\arccos\frac{1}{u}+C$$

$$=\arccos-\frac{1}{x}+C$$

合并结果 $\arccos\dfrac{1}{|x|}+C$

例5. 求 $\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$

解法2: 利用倒代换 令$x=\frac{1}{t}$ 则$\mathrm{d}x=-\frac{1}{t^2}\mathrm{d}t$

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \int \frac{-\frac{1}{t^2}\mathrm{d}t}{\frac{1}{t}\cdot\sqrt{\frac{1}{t^2}-1}}$$

$$\begin{cases} \text{当}t>0\text{时} \quad \int -\frac{1}{\sqrt{1-t^2}}\mathrm{d}t = -\arcsin t + C = -\arcsin\frac{1}{x} + C \\ \text{当}t<0\text{时} \quad \int \frac{1}{\sqrt{1-t^2}}\mathrm{d}t = \arcsin t + C = \arcsin\frac{1}{x} + C \end{cases}$$

合并结果 $-\arcsin\frac{1}{|x|}+C$

常数C的原因积出的结果形式不唯一

例5. 求 $\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$

解法3: 根式代换

令 $\sqrt{x^2-1}=t \quad \Rightarrow x^2-1=t^2$

两边取微分 $2xdx=2tdt$

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} = \int \frac{\mathrm{d}t}{1+t^2} = \arctan t + C = \arctan\sqrt{x^2-1}+C$$

$$= \int \frac{\cancel{t}\,\mathrm{d}t}{x \cdot x \cdot \cancel{t}}$$

$1+t^2$

例6. 求 $\int\sqrt{1+e^{2x}}\,dx$

解: 根式代换 令$\sqrt{1+e^{2x}}=t$ 则$x=\frac{1}{2}\ln(t^2-1)$

两边取微分 $dx=\dfrac{t}{t^2-1}\mathrm{d}t$

$$\int\sqrt{1+e^{2x}}\,dx=\int t\cdot\frac{t}{t^2-1}\mathrm{d}t=\int 1+\frac{1}{t^2-1}\mathrm{d}t=t+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C$$

$$=\sqrt{1+e^{2x}}+\frac{1}{2}\ln\left|\frac{\sqrt{1+e^{2x}}-1}{\sqrt{1+e^{2x}}+1}\right|+C$$

$$=\int\frac{t^2-1+1}{t^2-1}\mathrm{d}t$$

$$\frac{1}{t^2-1}=\frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)$$

$$\int\frac{1}{t^2-1}dt=\int\frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\tfrac{1}{2}(\ln|t-1|-\ln|t+1|)$$

例7 求 $\int \frac{x^3}{(x^2-2x+2)^2}\mathrm{d}x$

分析分母化成 $[(x-1)^2+1]^2$

$\cos 2x = \cos^2 x - \sin^2 x$

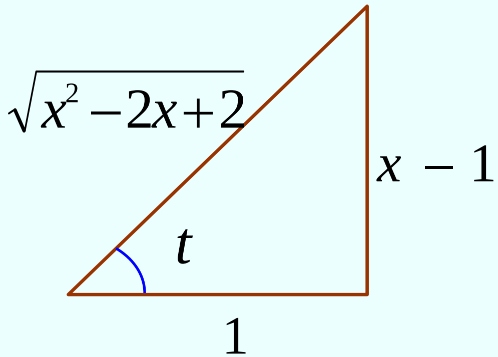
$= 1-2\sin^2 x$

$= 2\cos^2 x - 1$

解 令 x-1= tan$t$, $t\in(-\frac{\pi}{2},\frac{\pi}{2})$,

原式= $\int \frac{(\tan t+1)^3}{\sec^4 t}\cdot \sec^2 t\mathrm{d}t$

辅助三角形



$$= \int \left(\underline{\sin^3 t\sec t} + 3\sin^2 t + \underline{3\sin t\cos t} + \cos^2 t\right)\mathrm{d}t$$

$$= -\int \left(\frac{1-\cos^2 t}{\cos t} + 3\cos t\right)\mathrm{d}(\cos t) + \int (2-\cos 2t)\mathrm{d}t$$

$$= -\ln\cos t - \cos^2 t \quad + 2t - \tfrac{1}{2}\sin 2t + C$$

$$= -\ln\frac{1}{\sqrt{x^2-2x+2}} - \frac{1}{x^2-2x+2} + 2\arctan(x-1) - \frac{x-1}{x^2-2x+2} + C$$

$$= \frac{1}{2}\ln(x^2-2x+2) + 2\arctan(x-1) - \frac{x}{x^2-2x+2} + C$$

被积函数$f(x)$的所含的形式　　所作的代换

三角代换

$f(\sqrt{a^2-x^2})\ a>0 \longrightarrow x=a\sin t\ \left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$

$f(\sqrt{a^2+x^2})\ a>0 \longrightarrow x=a\tan t\ \left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$

$f(\sqrt{x^2-a^2})\ a>0 \longrightarrow$ 当$x>a$,令$x=a\sec t\ \left(0<t<\frac{\pi}{2}\right)$

当$x<-a$,令$u=-x$, $-\int f(\sqrt{u^2-a^2})du$

此时$u>a$,应用上面结果

倒数代换$\frac{1}{x^n\varphi(x)}$(n为自然数)$\longrightarrow x=\frac{1}{t}$ 适合于“头轻脚重”

含根式 $\longrightarrow$ t=根式