导课 如果函数$y=f(x)$在 x 处函数值的改变量

$$\Delta y = f(x+\Delta x) - f(x) = \underline{A\Delta x} + o(\Delta x)$$ 其中 A 与 Δx 无关

称函数在点 x 处可微

$$\mathrm{d}y = f'(x)\Delta x \xrightarrow{\text{应用}} \begin{cases} \text{近似计算} \\ \text{估计误差} \end{cases}$$

对于二元函数$z=f(x,y)$

$$f(x+\Delta x, y) - f(x,y) \approx f_x(x,y)\Delta x$$

偏增量　　　　偏微分

$$f(x, y+\Delta y) - f(x,y) \approx f_y(x,y)\Delta y$$

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) \overset{?}{=} A\Delta x + B\Delta y + o(\sqrt{\Delta x^2 + \Delta y^2})$$

全增量

第九章

第三节

全微分

内容

一 全微分

二 定理与性质

一、全微分

定义: 如果函数 $z=f(x,y)$在定义域 D内的点(x,y)处全增量 $\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$可表示成

$$\Delta z = A\Delta x + B\,\Delta y + o(\rho)\,,\quad \rho = \sqrt{(\Delta x)^2+(\Delta y)^2}$$

其中 A , B 不依赖于Δx , Δy , 仅与 x , y 有关, 则称函数 $f(x,y)$在点(x,y) 可微, $A\Delta x+B\Delta y$ 称为函数 $f(x,y)$ 在点 (x,y) 的全微分, 记作

$$\mathrm{d}z = \mathrm{d}f = A\Delta x + B\Delta y$$

若函数在域 D 内各点都可微, 则称此函数在D 内可微.

二、定理与性质

① $z = f(x, y)$ 在点(x, y) 可微 $\underset{\not\Leftarrow}{\Rightarrow}$ 连续

证: 因函数在点(x, y)可微, 故 $\Delta z = A\Delta x + B\Delta y + o(\rho)$,

则有 $\lim\limits_{\rho\to 0}\Delta z = 0$

考虑 $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} f(x+\Delta x, y+\Delta y)$

$$= \lim_{\rho\to 0}[f(x, y) + \Delta z] = f(x, y)$$

因此函数$z=f(x,y)$在(x,y)连续

二、定理与性质

② $z=f(x,y)$ 在点(x,y) 可微 $\substack{\Longrightarrow \\ \not\Longleftarrow}$ 偏导数存在

证: 因函数在点(x,y)可微, 故 $\Delta z = A\Delta x + B\Delta y + o(\rho)$,

其中 A, B 不依赖于Δx, Δy, 仅与 x, y 有关,

令 $\Delta y = 0$, 上式为 $f(x+\Delta x, y) - f(x,y) = A\Delta x + o(|\Delta x|)$

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = A$$

同样可证 $\dfrac{\partial z}{\partial y} = B$, 因此有 $\mathrm{d}\,z = \dfrac{\partial z}{\partial x}\Delta x + \dfrac{\partial z}{\partial y}\Delta y$

习惯上把自变量的增量用微分表示

说明:

$z=f(x,y)$ $\quad \boxed{\mathrm{d}\,z = \dfrac{\partial z}{\partial x}\mathrm{d}x + \dfrac{\partial z}{\partial y}\mathrm{d}y}$

$u=f(x,y,z)$ $\quad \boxed{\mathrm{d}u = \dfrac{\partial u}{\partial x}\mathrm{d}x + \dfrac{\partial u}{\partial y}\mathrm{d}y + \dfrac{\partial u}{\partial z}\mathrm{d}z}$

反例:

$$函数z= f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

在点(0,0)连续,两个偏导数都存在,但函数在该点不可微.

证明

$$0\le\left|\frac{xy}{\sqrt{x^2+y^2}}\right|\le\frac{\frac{1}{2}(x^2+y^2)}{\sqrt{x^2+y^2}}=\frac{1}{2}\sqrt{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)}\frac{1}{2}\sqrt{x^2+y^2}=0$$ 所以$f(x,y)$在(0,0)连续

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=0$$

同理 $f_y(0,0)=0$ 偏导数存在

反例:

$$\text{函数}z=f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

在点(0,0)连续,两个偏导数都存在,但函数在该点不可微.

证明 $f_x(0,0)=0,\ f_y(0,0)=0$

$$\Delta z=f(0+\Delta x,0+\Delta y)-f(0,0)=\frac{\Delta x\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta z-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\overset{?}{=}o$$

$$=\lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta x\Delta y}{(\Delta x)^2+(\Delta y)^2}$$ 极限不存在

因此,函数在点(0,0)不可微

③ 设$\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}$在$(x, y)$ **连续** $\overset{\Longrightarrow}{\not\Longleftarrow}$ $f(x,y)$在(x,y)**可微**

证: $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$=[f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)]$

$+[f(x, y+\Delta y) - f(x,y)]$

根据拉格朗日中值定理

$= f_x(x+\theta_1\Delta x, y+\Delta y)\Delta x + f_y(x, y+\theta_2\Delta y)\Delta y \quad (0<\theta_1,\theta_2<1)$

$=[f_x(x,y)+\underbrace{f_x(x+\theta_1\Delta x, y+\Delta y)-f_x(x,y)}_{\varepsilon_1}]\Delta x$

$+[f_y(x,y)+\underbrace{f_y(x, y+\theta_2\Delta y)-f_y(x,y)}_{\varepsilon_2}]\Delta y$

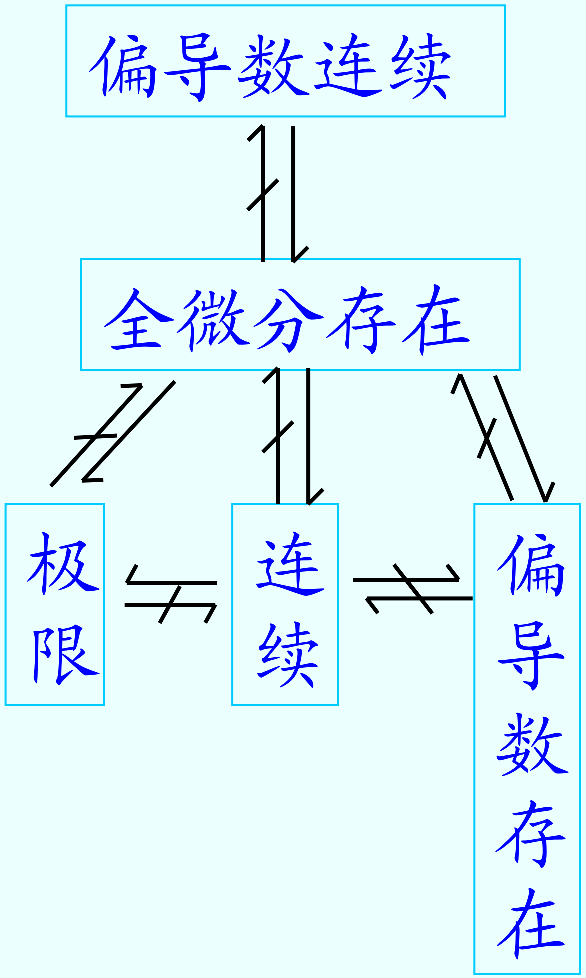
$= f_x(x,y)\Delta x + f_y(x,y)\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y$

偏导数连续

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \varepsilon_1 = 0,$

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \varepsilon_2 = 0$

$0 \le \left|\dfrac{\varepsilon_1\Delta x+\varepsilon_2\Delta y}{\underbrace{\sqrt{(\Delta x)^2+(\Delta y)^2}}_{\to 0}}\right| \le |\varepsilon_1|+|\varepsilon_2| \to 0 \begin{pmatrix}\Delta x\to 0\\ \Delta y\to 0\end{pmatrix}$ 故$f(x,y)$在(x,y)可微



反例:

$$z = f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$$

(i)问$f_x(0,0)$, $f_y(0,0)$是否存在√

(ii)问$f(x,y)$在(0,0)是否连续

(iii)问$f(x,y)$在(0,0)是否可微

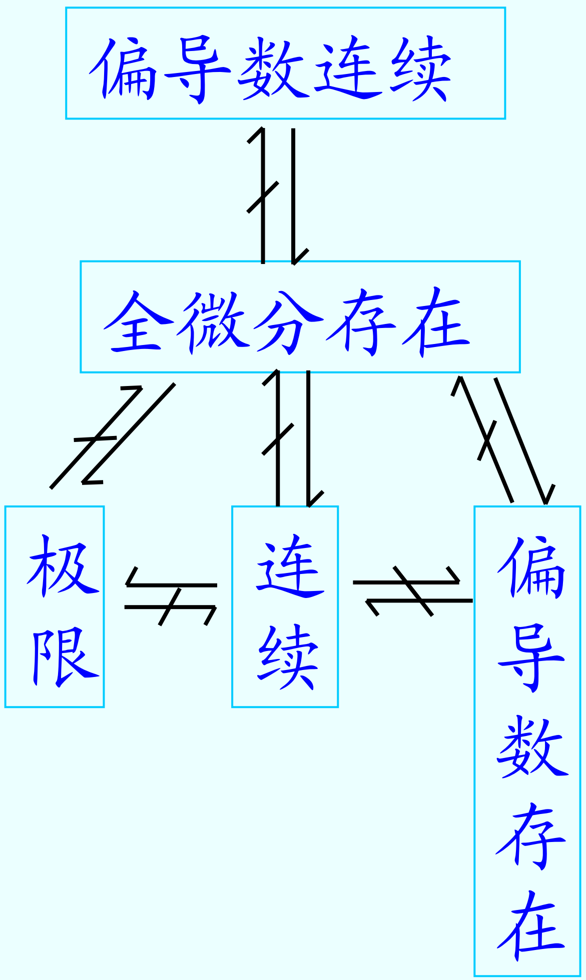
(iv)问$\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}$在(0,0)是否连续

解

(i) $$f_x(0,0) = \lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{(\Delta x)^2\sin\dfrac{1}{(\Delta x)^2}}{\Delta x} = 0$$

同理 $f_y(0,0) = 0$



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(iii)问$f(x,y)$在$(0,0)$是否可微

(iv)问$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$在$(0,0)$是否连续

解 (ii) $\lim\limits_{(x,y)\to(0,0)}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}$

$$= \lim_{t\to 0} t\sin\frac{1}{t} = 0 = f(0,0)$$ 连续

$$
\begin{tikzcd}[column sep=large, row sep=large]
 & \boxed{\text{偏导数连续}} \arrow[d, Rightarrow, shift left] & \\
 & \boxed{\text{全微分存在}} \arrow[u, Rightarrow, shift left, "/" marking] \arrow[dl, Rightarrow, shift left] \arrow[d, Rightarrow, shift left] \arrow[dr, Rightarrow, shift left] & \\
\boxed{\text{极限}} \arrow[ur, Rightarrow, shift left, "/" marking] \arrow[r, Rightarrow, shift left, "/" marking] & \boxed{\text{连续}} \arrow[u, Rightarrow, shift left, "/" marking] \arrow[l, Rightarrow, shift left] \arrow[r, Leftrightarrow, "/" marking] & \boxed{\text{偏导数存在}} \arrow[ul, Rightarrow, shift left, "/" marking]
\end{tikzcd}
$$

反例:

$$
z = f(x,y) = \begin{cases} (x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}
$$

(i)问$f_x(0,0)$, $f_y(0,0)$是否存在✓

(ii)问$f(x,y)$在(0,0)是否连续✓

(iii)问$f(x,y)$在(0,0)是否可微✓

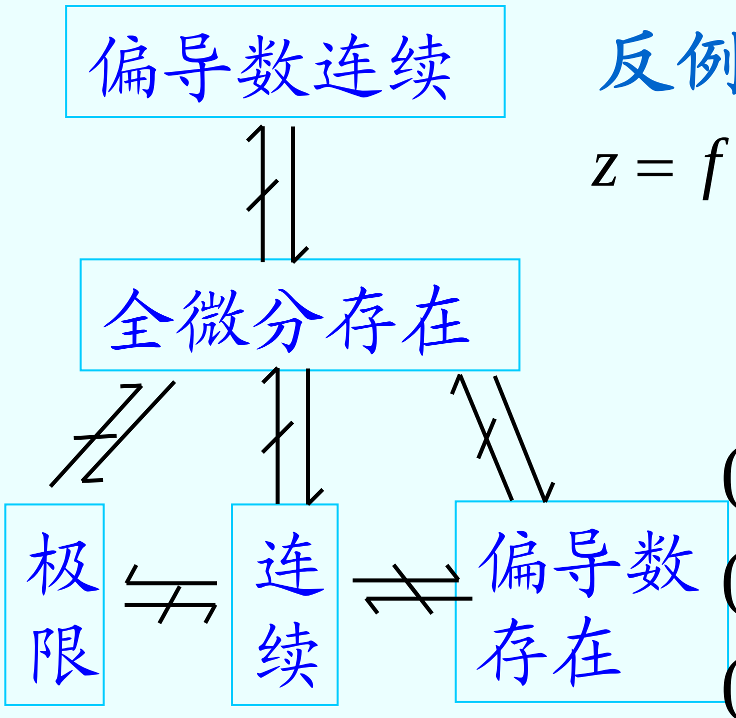
(iv)问$\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}$在(0,0)是否连续

解(iii)

$$
\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta z - \overset{=0}{f_x(0,0)}\Delta x - \overset{=0}{f_y(0,0)}\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}} \overset{?}{=} 0
$$

$$
= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{f(\Delta x,\Delta y) - \overset{=0}{f(0,0)}}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \sqrt{(\Delta x)^2+(\Delta y)^2}\,\sin\frac{1}{(\Delta x)^2+(\Delta y)^2}
$$

$= 0$ 可微



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(i)问$f_x(0,0)$, $f_y(0,0)$是否存在✓

(ii)问$f(x,y)$在$(0,0)$是否连续✓

(iii)问$f(x,y)$在$(0,0)$是否可微✓

(iv)问$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$在$(0,0)$是否连续 ×

解(iv) $$\frac{\partial f}{\partial x} = 2x\sin\frac{1}{x^2+y^2} + (x^2+y^2)\cos\frac{1}{x^2+y^2}\cdot\left(-\frac{1}{(x^2+y^2)^2}\cdot 2x\right)$$

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{\partial f}{\partial x} = 0 - \lim_{\substack{x\to 0\\ y\to 0}}\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2} = -\lim_{\substack{x\to 0\\ y=x}}\frac{1}{x}\cos\frac{1}{2x^2}$$

极限不存在

$\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ 在$(0,0)$不连续

典型题 求全微分

例1 $z = x^y + y^x$ 求 $\mathrm{d}z$

解: $$\frac{\partial z}{\partial x} = y x^{y-1} + y^x \ln y \qquad \frac{\partial z}{\partial y} = x^y \ln x + x y^{x-1}$$

$$\mathrm{d}z = (y x^{y-1} + y^x \ln y)\mathrm{d}x + (x^y \ln x + x y^{x-1})\mathrm{d}y$$

例2 设 $u = \left(\frac{y}{x}\right)^z$，求 $\mathrm{d}u\,|_{(1,1,1)}$

解: $$\frac{\partial u}{\partial x} = z\left(\frac{y}{x}\right)^{z-1} \cdot \left(-\frac{y}{x^2}\right) \qquad \frac{\partial u}{\partial y} = z\left(\frac{y}{x}\right)^{z-1} \cdot \frac{1}{x} \qquad \frac{\partial u}{\partial z} = \left(\frac{y}{x}\right)^{z} \cdot \ln\frac{y}{x}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(1,1,1)} = -1 \qquad \left.\frac{\partial u}{\partial y}\right|_{(1,1,1)} = 1 \qquad \left.\frac{\partial u}{\partial z}\right|_{(1,1,1)} = 0$$

$$\mathrm{d}u\,|_{(1,1,1)} = -\mathrm{d}x + \mathrm{d}y$$