第四节

多元复合函数的求导法则

内容

一 多元复合函数求导

二 典型题

三 全微分形式不变性

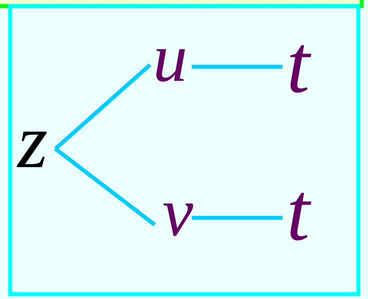
一、多元复合函数求导的链式法则

1. 中间变量为一元函数

求导法则 若 $u=\varphi(t), v=\psi(t)$ 均可导, $z=f(u,v)$在(u,v)具有一阶连续偏导,则复合函数$z=f[\varphi(t),\psi(t)]$可导,且有

链式法则 $$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}=f_1'\cdot\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}+f_2'\cdot\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}$$

证: 设 t 取增量Δt,则相应中间变量增量 $\Delta u, \Delta v$



一、多元复合函数求导的链式法则

1. 中间变量为一元函数

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证: 设 t 取增量 Δt, 则相应中间变量增量 $\Delta u, \Delta v$

u — t

$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+\varepsilon_1\Delta u+\varepsilon_2\Delta v$$

③设 $\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}$ 在 (x,y) **连续** $\rightleftarrows$ $f(x,y)$ 在 (x,y) **可微**

证: $\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$

$=[f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)]$

$\quad+[f(x,y+\Delta y)-f(x,y)]$

根据拉格朗日中值定理

$=f_x(x+\theta_1\Delta x,y+\Delta y)\Delta x+f_y(x,y+\theta_2\Delta y)\Delta y \quad (0<\theta_1,\theta_2<1)$

$=[f_x(x,y)+\underbrace{f_x(x+\theta_1\Delta x,y+\Delta y)-f_x(x,y)}_{\varepsilon_1}]\Delta x$

$\quad+[f_y(x,y)+\underbrace{f_y(x,y+\theta_2\Delta y)-f_y(x,y)}_{\varepsilon_2}]\Delta y$

$=f_x(x,y)\Delta x+f_y(x,y)\Delta y+\varepsilon_1\Delta x+\varepsilon_2\Delta y$

偏导数连续

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\varepsilon_1=0,$

$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\varepsilon_2=0$

$0\le\left|\dfrac{\varepsilon_1\Delta x+\varepsilon_2\Delta y}{\underbrace{\sqrt{(\Delta x)^2+(\Delta y)^2}}_{\to 0}}\right|\le|\varepsilon_1|+|\varepsilon_2|\to 0\begin{pmatrix}\Delta x\to 0\\ \Delta y\to 0\end{pmatrix}$ 故 $f(x,y)$ 在 (x,y) 可微

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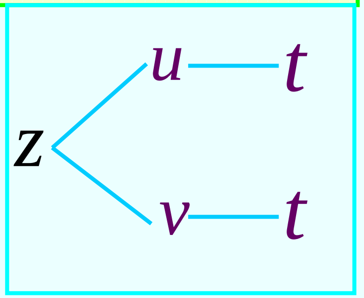
证: 设 t 取增量 Δt, 则相应中间变量增量 $\Delta u, \Delta v$



$$\Delta z=\frac{\partial z}{\partial u}\Delta u+\frac{\partial z}{\partial v}\Delta v+\varepsilon_1\Delta u+\varepsilon_2\Delta v$$

当 $\Delta u\to 0, \Delta v\to 0$ 时

$\varepsilon_1\to 0,\ \ \varepsilon_2\to 0$

$$\frac{\Delta z}{\Delta t}=\frac{\partial z}{\partial u}\boxed{\frac{\Delta u}{\Delta t}}+\frac{\partial z}{\partial v}\boxed{\frac{\Delta v}{\Delta t}}+\varepsilon_1\frac{\Delta u}{\Delta t}+\varepsilon_2\frac{\Delta v}{\Delta t}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}$$

当 $\Delta t\to 0$ 时 $\begin{cases}\Delta u\to 0, \Delta v\to 0, \text{仍有} \varepsilon_1\to 0,\ \ \varepsilon_2\to 0\\ \dfrac{\Delta u}{\Delta t}\to\dfrac{\mathrm{d}u}{\mathrm{d}t},\ \ \dfrac{\Delta v}{\Delta t}\to\dfrac{\mathrm{d}v}{\mathrm{d}t}\end{cases}$

$$\boxed{\lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t}}=\frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t}$$

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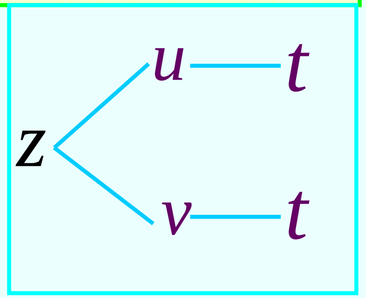
$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}=f_1'\cdot\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t}+f_2'\cdot\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}$$

说明 ①首先根据函数关系画出链式图,根据"**联线相乘,分线相加**"链式法则,最后计算



$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$

全导数 偏导数 导数

一、多元复合函数求导的链式法则

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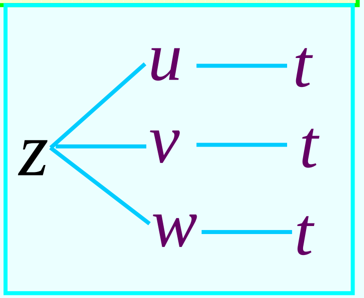
链式法则 $$\frac{dz}{dt}=\frac{\partial z}{\partial u}\cdot\frac{du}{dt}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dt}=f_1'\cdot\frac{d\varphi(t)}{dt}+f_2'\cdot\frac{d\psi(t)}{dt}$$

说明



② 三元函数 $z=f(u,v,w)$, $u=\varphi(t), v=\psi(t), w=w(t)$

或 $z=f(\varphi(t),\psi(t),w(t))$（不引进中间变量）

$$\frac{dz}{dt}=\frac{\partial z}{\partial u}\frac{du}{dt}+\frac{\partial z}{\partial v}\frac{dv}{dt}+\frac{\partial z}{\partial w}\frac{dw}{dt}=f_1'\frac{d\varphi(t)}{dt}+f_2'\frac{d\psi(t)}{dt}+f_3'\frac{dw(t)}{dt}$$

例1 设 $z = f(e^t, \sin t) = e^{e^t \sin t}$ 求 $\dfrac{\mathrm{d}z}{\mathrm{d}t}$

解法一 直接求 $\dfrac{\mathrm{d}z}{\mathrm{d}t} = e^{e^t \sin t}(e^t \sin t + e^t \cos t)$

解法二 设 $z = f(u, v) = e^{uv}, u = e^t, v = \sin t$

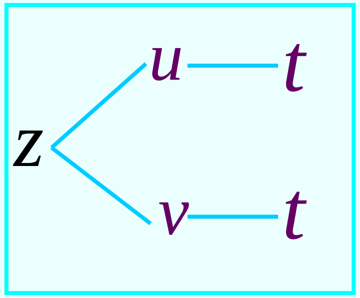
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{\partial z}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}t} = e^{uv} \cdot v \cdot e^t + e^{uv} \cdot u \cdot \cos t$$

$$= e^{e^t \sin t}(e^t \sin t + e^t \cos t)$$

解法三 $\dfrac{\mathrm{d}z}{\mathrm{d}t} = f_1' \cdot e^t + f_2' \cdot \cos t$

一阶

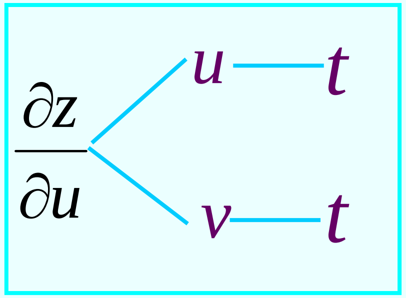
$$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}$$

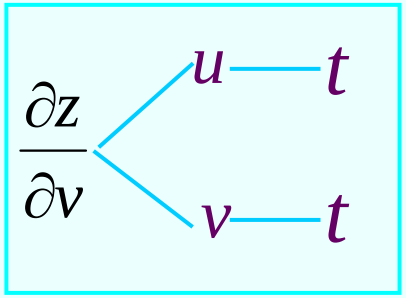


二阶

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2}=\left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}\right)\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$$

$$+\left(\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}\right)\cdot\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}^2 v}{\mathrm{d}t^2}$$





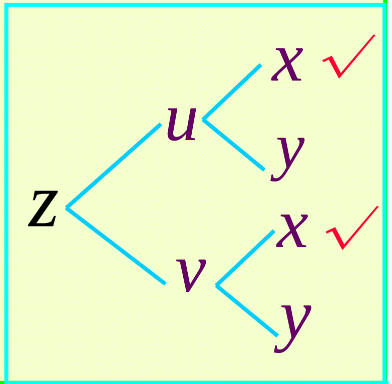
2. 中间变量为二元函数

求导法则　设 $u=\varphi(x,y), v=\psi(x,y)$ 对x,y均有偏导数

$z=f(u,v)$ 在(u,v)有连续偏导数,则复合函数

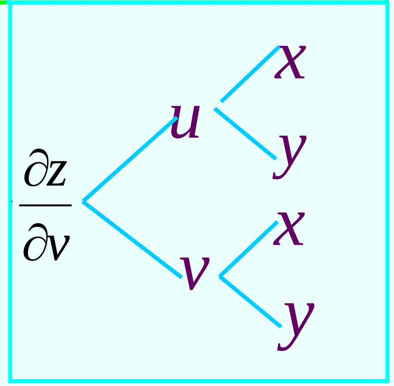
$z=f[\varphi(x,y),\psi(x,y)]$ 对x,y有偏导数,且

$$\begin{cases} \dfrac{\partial z}{\partial x}=\boxed{\dfrac{\partial z}{\partial u}}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}=f_1'\dfrac{\partial \varphi}{\partial x}+f_2'\dfrac{\partial \psi}{\partial x} \\ \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}=f_1'\dfrac{\partial \varphi}{\partial y}+f_2'\dfrac{\partial \psi}{\partial y} \end{cases}$$



$$\frac{\partial^2 z}{\partial x^2}=\left(\frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\partial v}\cdot\frac{\partial v}{\partial x}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial u}\cdot\frac{\partial^2 u}{\partial x^2}$$

$$+\left(\frac{\partial^2 z}{\partial v\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\cdot\frac{\partial v}{\partial x}\right)\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial^2 v}{\partial x^2}$$



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$z=f[\varphi(x,y),\psi(x,y)]$ 对x,y有偏导数,且

$$\begin{cases}\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\boxed{\dfrac{\partial v}{\partial x}}=f_1'\dfrac{\partial\varphi}{\partial x}+f_2'\dfrac{\partial\psi}{\partial x}\\[2ex] \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}=f_1'\dfrac{\partial\varphi}{\partial y}+f_2'\dfrac{\partial\psi}{\partial y}\end{cases}$$

$$\frac{\partial^2 z}{\partial x^2}=\left(f_{11}''\cdot\frac{\partial\varphi}{\partial x}+f_{12}''\cdot\frac{\partial\psi}{\partial x}\right)\cdot\frac{\partial\varphi}{\partial x}+f_1'\cdot\frac{\partial^2\varphi}{\partial x^2}$$

$$+\left(f_{21}''\cdot\frac{\partial\varphi}{\partial x}+f_{22}''\cdot\frac{\partial\psi}{\partial x}\right)\cdot\frac{\partial\psi}{\partial x}+f_2'\cdot\frac{\partial^2\psi}{\partial x^2}$$

$z=f_1'[\varphi(x,y),\psi(x,y)]$

$z=f_2'[\varphi(x,y),\psi(x,y)]$

其它二阶偏导同理可得

2. 中间变量为二元函数

$$\frac{\partial^2 z}{\partial x^2}=\left(f''_{11}\cdot\frac{\partial\varphi}{\partial x}+f''_{12}\cdot\frac{\partial\psi}{\partial x}\right)\cdot\frac{\partial\varphi}{\partial x}+f'_1\cdot\frac{\partial^2\varphi}{\partial x^2}$$

$$+\left(f''_{21}\cdot\frac{\partial\varphi}{\partial x}+f''_{22}\cdot\frac{\partial\psi}{\partial x}\right)\cdot\frac{\partial\psi}{\partial x}+f'_2\cdot\frac{\partial^2\psi}{\partial x^2}$$

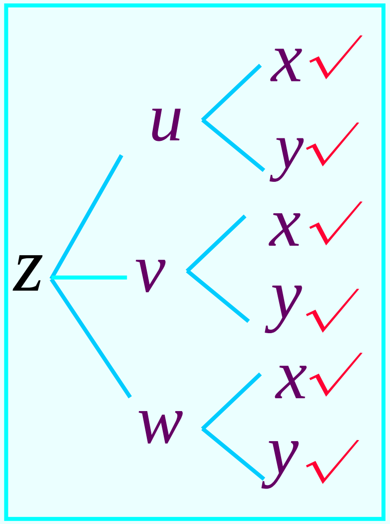
$$z=f_1'[\varphi(x,y),\psi(x,y)]$$

$$z=f_2'[\varphi(x,y),\psi(x,y)]$$

其它二阶偏导同理可得

若$f''_{12}=f''_{21}$

$$=f''_{11}\cdot\left(\frac{\partial\varphi}{\partial x}\right)^2+2f''_{12}\cdot\frac{\partial\varphi}{\partial x}\cdot\frac{\partial\psi}{\partial x}+f''_{22}\cdot\left(\frac{\partial\psi}{\partial x}\right)^2+f'_1\cdot\frac{\partial^2\varphi}{\partial x^2}+f'_2\cdot\frac{\partial^2\psi}{\partial x^2}$$

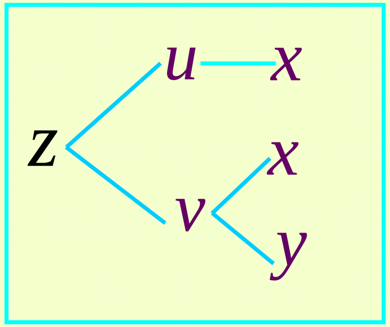


$$\begin{cases}\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}+\dfrac{\partial z}{\partial w}\cdot\dfrac{\partial w}{\partial x}\\[2ex]\dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial y}+\dfrac{\partial z}{\partial w}\cdot\dfrac{\partial w}{\partial y}\end{cases}$$

3. 中间变量既有一元又有二元情形

求导法则 $z = f(u,v), u = \varphi(x), v = \psi(x,y)$

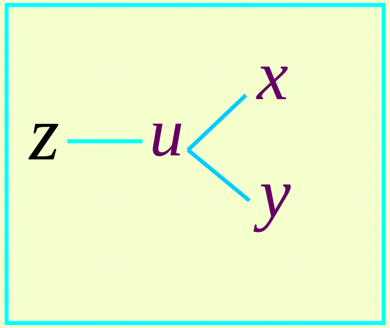
$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x} + \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial x} \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial y} \end{cases}$$



4. 只有一个中间变量

求导法则 $z = f(u), u = \varphi(x,y)$

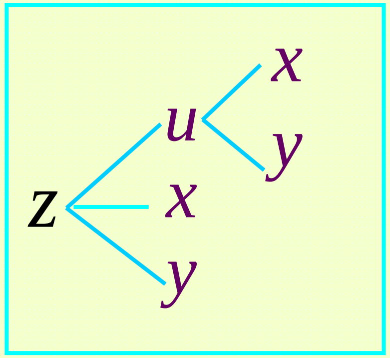
$$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\mathrm{d}f}{\mathrm{d}u} \cdot \dfrac{\partial u}{\partial x} = f'(u)\dfrac{\partial u}{\partial x} \\ \dfrac{\partial z}{\partial y} = \dfrac{\mathrm{d}f}{\mathrm{d}u} \cdot \dfrac{\partial u}{\partial y} = f'(u)\dfrac{\partial u}{\partial y} \end{cases}$$



5. 复合函数的中间变量本身又是复合函数的自变量

求导法则 $z=f(u,x,y),u=u(x,y)$

$$\begin{cases} \dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial f}{\partial x} \\ \dfrac{\partial z}{\partial y}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y}+\dfrac{\partial f}{\partial y} \end{cases}$$



说明:

$\dfrac{\partial z}{\partial x}$ 是把复合函数$z=f(u(x,y),x,y)$中的y看作不变而对x求偏导

$\dfrac{\partial f}{\partial x}$ 是把复合函数$z=f(u,x,y)$中的u及y看作不变而对x求偏导

例$z=u+x+y,u=xy$,求$\dfrac{\partial z}{\partial x}$

法一 $\dfrac{\partial z}{\partial x}=\dfrac{\partial(xy+x+y)}{\partial x}=y+1$

法二 $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial f}{\partial x}=1\cdot y+1$

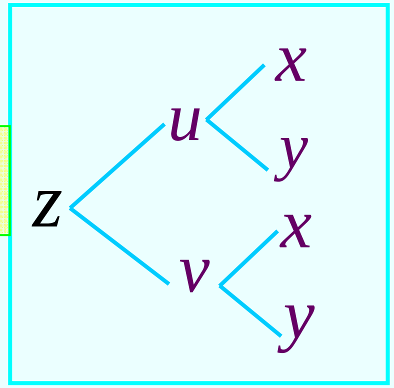
二、典型题

例1. 设 $z=u^{v}$，$u=\ln\sqrt{x^2+y^2}$，$v=\arctan\dfrac{y}{x}$ 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解: $$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=vu^{v-1}\cdot\frac{x}{x^2+y^2}+u^{v}\ln u\cdot\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}=\frac{1}{x^2+y^2}(vu^{v-1}\cdot x-u^{v}\ln u\cdot y)$$

第二种



$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$=vu^{v-1}\cdot\frac{y}{x^2+y^2}+u^{v}\ln u\cdot\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}=\frac{1}{x^2+y^2}(vu^{v-1}\cdot y+u^{v}\ln u\cdot x)$$

二、典型题

注: $f_1'(xy,\frac{y}{x})$和$f_2'(xy,\frac{y}{x})$仍是中间变量的函数

例2. 设 $z=x^3 f(xy,\frac{y}{x})$(f具有二阶连续偏导数)求 $\frac{\partial z}{\partial y},\frac{\partial^2 z}{\partial y^2},\frac{\partial^2 z}{\partial x\partial y}$

解: $$\frac{\partial z}{\partial y}=x^3(\boxed{f_1'}\cdot x+f_2'\cdot\frac{1}{x})=\boxed{x^4 f_1'}+x^2 f_2'$$

$$\frac{\partial^2 z}{\partial y^2}=x^3\left[(f_{11}''\cdot x+f_{12}''\cdot\frac{1}{x})\cdot x+(f_{21}''\cdot x+f_{22}''\cdot\frac{1}{x})\cdot\frac{1}{x}\right]$$

$$\underline{\underline{f_{12}''=f_{21}''}}\ x^5 f_{11}''+2x^3 f_{12}''+xf_{22}''$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=4x^3 f_1'+x^4[f_{11}''\cdot y+f_{12}''\cdot(-\frac{y}{x^2})]+2xf_2'$$

$$+x^2[f_{21}''\cdot y+f_{22}''\cdot(-\frac{y}{x^2})]$$

$$\underline{\underline{f_{12}''=f_{21}''}}\ 4x^3 f_1'+2xf_2'+x^4 yf_{11}''-yf_{22}''$$

二、典型题

练习 $z=f(e^x\cos y, y^2-x^2)$ (f具有二阶连续偏导数)求 $\dfrac{\partial^2 z}{\partial x\partial y}$

解： $\dfrac{\partial z}{\partial x}=f_1'\cdot e^x\cos y+f_2'\cdot(-2x)$

$$\frac{\partial^2 z}{\partial x\partial y}=[f_{11}''\cdot(-e^x\sin y)+f_{12}''\cdot(2y)]\cdot e^x\cos y+f_1'(-e^x\sin y)$$

$$+[f_{21}''\cdot(-e^x\sin y)+f_{22}''\cdot(2y)](-2x)$$

$\underset{=\!=\!=}{f_{12}''=f_{21}''}$ 整理

二、典型题

例3. 设 $z=\sin(xy)+\varphi(x,\frac{x}{y})$, $\varphi(u,v)$二阶偏导数存在,求 $\frac{\partial^2 z}{\partial x\partial y}$(或$z_{xy}$).

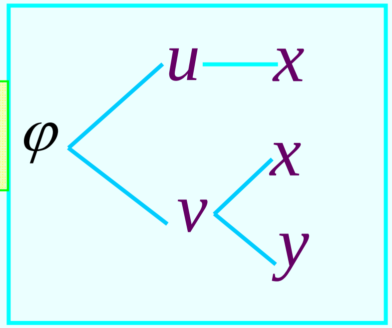
解: $\frac{\partial z}{\partial x}=y\cos xy+\varphi_1'+\varphi_2'\cdot\frac{1}{y}$

$\varphi_1'(x,\frac{x}{y})$, $\varphi_2'(x,\frac{x}{y})$

$$\frac{\partial^2 z}{\partial x\partial y}=\cos xy+y\,(-\sin xy)\cdot x+\varphi_{12}''\cdot(-\frac{x}{y^2})$$

$$+\varphi_{22}''\cdot(-\frac{x}{y^2})\cdot\frac{1}{y}+\varphi_2'\cdot(-\frac{1}{y^2})$$

第三种



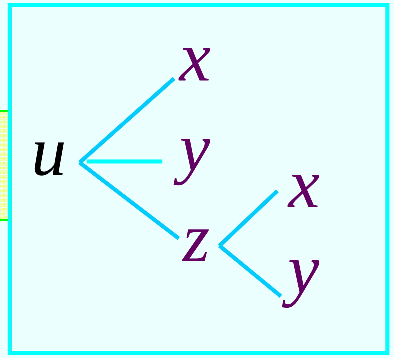
二、典型题

例4. 设 $u=f(x,y,z)=e^{x^2+y^2+z^2}$ 而 $z=x^2\sin y$，求 $\dfrac{\partial u}{\partial x}$

解: $$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$=2x\cdot e^{x^2+y^2+z^2}+2z\cdot e^{x^2+y^2+z^2}\cdot 2x\sin y$$

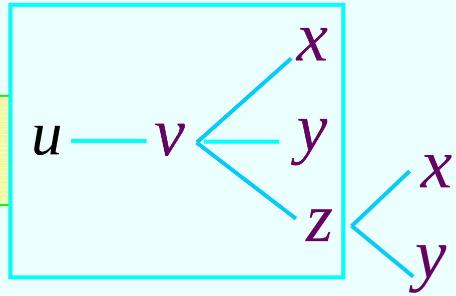
第五种



例5. 设 $u=f(\underbrace{x^2+y^2+z^2}_{v})$, $z=x^2\sin y$，求 $\dfrac{\partial u}{\partial x}$

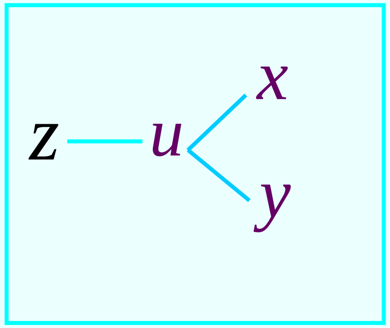
解: $$\frac{\partial u}{\partial x}=f'(x^2+y^2+z^2)\cdot(2x+2z\cdot 2x\sin y)$$

第四种



二、典型题

例6. $z=\int_a^{\sqrt{xy}}\sin t^2\mathrm{d}t$ 求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$



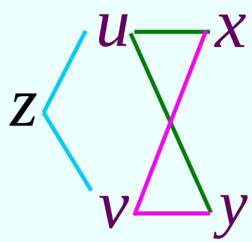
解: 令 $u=\sqrt{xy}$, $z=\int_a^{u}\sin t^2\mathrm{d}t$, $u=\sqrt{xy}$

$$\frac{\partial z}{\partial x}=\frac{\mathrm{d}z}{\mathrm{d}u}\cdot\frac{\partial u}{\partial x}=\sin u^2\cdot\frac{y}{2\sqrt{xy}}=\sin(xy)\cdot\frac{y}{2\sqrt{xy}}$$

$$\frac{\partial z}{\partial y}=\frac{\mathrm{d}z}{\mathrm{d}u}\cdot\frac{\partial u}{\partial y}=\sin u^2\cdot\frac{x}{2\sqrt{xy}}=\sin(xy)\cdot\frac{x}{2\sqrt{xy}}$$

例7. 设 $u = xy, v = \dfrac{x}{y},$ 试以新的自变量变换方程

$$x^3 \frac{\partial z}{\partial x} - xy^2 \frac{\partial z}{\partial y} = 0$$



解: $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}\cdot y + \frac{\partial z}{\partial v}\cdot\frac{1}{y}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}\cdot x + \frac{\partial z}{\partial v}\cdot(-\frac{x}{y^2})$$

代入原方程

$$x^3 y\cdot\frac{\partial z}{\partial u} + \frac{x^3}{y}\cdot\frac{\partial z}{\partial v} - x^2 y^2\cdot\frac{\partial z}{\partial u} + xy^2\cdot\frac{x}{y^2}\cdot\frac{\partial z}{\partial v} = 0$$

$$\Longrightarrow (u^2 v - u^2)\cdot\frac{\partial z}{\partial u} + (uv^2 + uv)\cdot\frac{\partial z}{\partial v} = 0$$

三 全微分形式不变性

设函数 $z=f(u,v),u=\varphi(x,y),v=\psi(x,y)$ 都可微,

则复合函数 $z=f(\varphi(x,y),\psi(x,y))$ 的全微分为

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y$$

$$=(\underline{\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}}+\underline{\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}})\mathrm{d}x+(\underline{\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}}+\underline{\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}})\mathrm{d}y$$

$$=\frac{\partial z}{\partial u}(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y)+\frac{\partial z}{\partial v}(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y)$$

$$=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v$$

可见无论 u,v 是自变量还是中间变量,其全微分表达形式都一样, 这性质叫做全微分形式不变性.

例8: $u=f(x,xy,xyz)$,且f存在一阶连续偏导数,求函数u的全部偏导数

解: 由全微分形式不变性

$$\mathrm{d}u = f_1'\mathrm{d}x + f_2'\mathrm{d}(xy) + f_3'\mathrm{d}(xyz)$$

$$= \underline{f_1'\mathrm{d}x} + \underline{f_2'(y\mathrm{d}x} + \underline{x\mathrm{d}y)} + \underline{f_3'(yz\mathrm{d}x} + \underline{xz\mathrm{d}y} + \underline{xy\mathrm{d}z)}$$

$$= (\underbrace{f_1' + f_2'y + f_3'yz}_{\frac{\partial u}{\partial x}})\mathrm{d}x + (\underbrace{f_2'x + f_3'xz}_{\frac{\partial u}{\partial y}})\mathrm{d}y + \underbrace{f_3'xy}_{\frac{\partial u}{\partial z}}\mathrm{d}z$$