CS 278 – Chapter 6 Induction & Recursion

Sequences

Domain - A set of consecutive integers.

Terms in a sequence specified with function and subscript (index).

For example,

In this sequence, the function is g and the domain is {1, 2, 3, 4}.

$g_1 = 3.67$ $g_2 = 2.88$ $g_3 = 3.25$ $g_4 = 3.75$

Finite sequence – domain is a finite set

Infinite sequence – domain is an infinite set

Increasing and decreasing sequences

Increasing: ∀ consecutive indexes k and k+1, $a_k < a_{k+1}$

Non-decreasing: ∀ consecutive indexes k and k+1, $a_k \leq a_{k+1}$

(There's a minor difference between increasing and non-decreasing but it's important.)

Decreasing: ∀ consecutive indexes k and k+1, $a_k > a_{k+1}$

Non-increasing: ∀ consecutive indexes k and k+1, $a_k \geq a_{k+1}$

(There's a minor difference between decreasing and non-increasing but it's important.)

Geometric Sequence

Every term after the initial term is found by multiplying the previous term by a fixed number.

The multiplier is called the common ratio.

Example: Initial term $a_0 = 1$. Common Ratio $r = ½$.

Sequence is 1, ½, ¼, 1/8, 1/16, ...

Example: Initial term $a_0 = 5$. Common Ratio $r = -1$. Final index = 6.

Sequence is 5, -5, 5, -5, 5, -5, 5

Arithmetic Sequence

Each term after the initial term is found by adding a fixed number to the previous term. The addend is called the common difference.

Example: Initial term $a_0 = 2$. Common Difference $d = 3$.

Sequence is 2, 5, 8, 11, ...

Example: Initial term $a_0 = 3$. Common Difference $d = -2$. Final index = 5.

Sequence is 3, 1, -1, -3, -5, -7

Example Problem

Baseball card collection

Start: 500 cards This is a_0.

Accumulate: 10 cards per week This is the common difference, d.

a_n = number of cards in the collection after n weeks

$a_n = 500 + 10n$

Worked Exercise 6.1.1 (a)

Each sequence starts with index 1.

a_1 = ceil(sqrt(1)) = 1

a_2 = ceil(sqrt(2)) = 2

a_3 = ceil(sqrt(3)) = 2

a_3 = ceil(sqrt(4)) = 2

a_3 = ceil(sqrt(5)) = 3

a_6 = ceil(sqrt(6)) = 3

a_7 = ceil(sqrt(7)) = 3

a_8 = ceil(sqrt(8)) = 3

$a_{9\,=}$ ceil(sqrt(9)) = 3

$a_{10\,=}$ ceil(sqrt(10)) = 4

The first 10 terms are 1, 2, 2, 2, 3, 3, 3, 3, 3, 4.

The sequence is non-decreasing. It is not increasing, non-increasing, or decreasing.

Worked Exercise 6.1.1 (b)

Each sequence starts with index 1.

$a_1 = 1$ (given in the problem)

$a_2 = 1$ (given in the problem)

$a_3 = a_1 + a_2 = 2$

$a_4 = a_2 + a_3 = 3$

$a_5 = a_3 + a_4 = 5$

$a_6 = a_4 + a_5 = 8$

$a_7 = a_5 + a_6 = 13$

$a_8 = a_6 + a_7 = 21$

$a_9 = a_7 + a_8 = 34$

$a_{10} = a_8 + a_9 = 55$

The first 10 terms are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55.

The sequence is non-decreasing. It is not increasing, non-increasing, or decreasing.

Worked Exercise 6.1.2 (a)

$a_n = n^2 - 2n$, for $n \geq 1$

Write the first few (5 or 6) terms. -1, 0, 3, 8, 15, ... (Gives you a good idea...)

Use algebra to reason about the pattern. (Gives a better picture...)

$a_{n+1} - a_n = (n + 1)^2 - 2(n + 1) - (n^2 - 2n)$

$= n^2 + 2n + 1 - 2n - 2 - n^2 + 2n = 2n - 1.$ If $n \geq 1$, $2n - 1 > 0$.

Increasing? Yes

Non-decreasing? Yes

Decreasing? No

Non-increasing? No

Worked Exercise 6.1.2 (b)

$a_n = n^2 - 3n$, for $n \geq 1$

Write the first few (5 or 6) terms. -2, -2, 0, 4, 10, ... (Gives you a good idea...)

Use algebra to reason about the pattern. (Gives a better picture...)

$a_{n+1} - a_n = (n + 1)^2 - 3(n + 1) - (n^2 - 3n)$

$= n^2 + 2n + 1 - 3n - 3 - n^2 + 3n = 2n - 2.$ If $n \geq 2$, $2n - 2 > 0$.

Increasing? No

Non-decreasing? Yes

Decreasing? No

Non-increasing? No

Worked Exercise 6.1.3 (a)

Geometric series

$a_1 = 2$ Common ratio $r = 3$

The first 6 terms are 2, 6, 18, 54, 162, 486

Worked Exercise 6.1.3 (b)

Arithmetic series

$a_1 = 2$ Common difference $d = 3$

The first 6 terms are 2, 5, 8, 11, 14, 17

Recurrence relations

Recurrence relation : a rule that defines a_n as a function of previous terms in the sequence.

Arithmetic sequences are naturally defined by a recurrence relation.

$a_n = d + a_{n-1}$

Geometric sequences are naturally defined by a recurrence relation.

$a_n = r \cdot a_{n-1}$

Fibonacci Sequence

Developed by Leonardo Fibonacci in the middle ages.

Used to describe a formula for population of rabbits.

$f_n = f_{n-1} + f_{n-2}$

Initial values of both f_0 and f_1 must be given.

https://www.storyofmathematics.com/medieval_fibonacci.html

Dynamical System

A system that changes over time. The state of the system must be defined by well-defined rules.

Fibonacci's population of rabbits is a dynamical system.

Example: Outstanding balance of a car loan (or any type of loan).

an = outstanding debt after n months.

We need to know 1) interest rate 2) payment amount.

a_n = (1 + monthly interest rate) · a_{n-1} - payment amount for $n \geq 1$

Worked Exercise 6.2.1 (a)

Give the first 6 terms of the sequence.

$a_1 = 1 \quad a_2 = 2 \quad a_n = a_{n-1} \cdot a_{n-2}$

The first 6 terms are 1, 2, 2, 4, 8, 32

Worked Exercise 6.2.1 (b)

Give the first 6 terms of the sequence.

$a_1 = 1$ $a_2 = 5$ $a_n = 2 \cdot a_{n-1} + 3 \cdot a_{n-2}$ for $n \geq 3$

The first 6 terms are 1, 5, 13, 41, 121, 365

Summations

$$\sum_{i=s}^{t} a_i$$

Σ is the Greek capital sigma.   i is the index.   s is the lower limit.   t is the upper limit.

When a summation is written using Σ this is called summation notation.

When a summation is written out $a_1 + a_2 + a_3 + \ldots$, it is called expanded notation.

Writing summations in text

Summations can be written in text (without using the sigma character).

Examples:

sum i, i = 1 to 5 (1 + 2 + 3 + 4 + 5)

sum n^2, n = 4 to 6 (16 + 25 + 36)

Pulling out the final term

Watch your parentheses!

sum a_k, k = m to n = (sum a_k, k = m to n – 1) + a_n for n > m

Example:

sum (3/4)^j, j=0 to 6 = (sum (3/4)^j, j = 0 to 5) + (3/4)^6

Changing variables in summations

1. Determine the new lower limit.
2. Determine the new upper limit.
3. Replace the variable in the terms.

Closed form

Some summations can be expressed without summation notation. This is called **closed form**.

The summation of any arithmetic sequence can be written in closed form.

$$\sum_{k=0}^{n-1}(a+kd) = an + \frac{d(n-1)n}{2}$$

Closed form

The sum of any geometric sequence can be written in closed form.

$$\sum_{k=0}^{n-1} a \cdot r^k = \frac{a(r^n - 1)}{r - 1}$$

Worked Exercise 6.3.1 (a)

Evaluate the summation:

sum k^2, k = -1 to 4

1 + 0 + 1 + 4 + 9 + 16 = 31

Worked Exercise 6.3.1 (c)

Evaluate the summation:

sum k^3, k = -3 to 2

(-27) + (-8) + (-1) + 0 + 1 + 8 = -27

Worked Exercise 6.3.1 (d)

Evaluate the summation:

sum 3^k, k = 0 to 3

1 + 3 + 9 + 27 = 40

Worked Exercise 6.3.2 (a)

$(-2)^5 + (-1)^5 + \ldots + 7^5$

$$\sum_{k=-2}^{7} k^5$$

Worked Exercise 6.3.2 (b)

(-2) + (-1) + 0 + 1 + 2 + 3 + 4 + 5

$$\sum_{k=-2}^{5} k$$

Note: It doesn't matter what variable name you use for the index.

Worked Exercise 6.3.3 (a)

$$\sum_{j=-2}^{18} 2^j \quad = \quad \sum_{j=-2}^{17} 2^j \quad + 2^{18}$$

Worked Exercise 6.3.3 (d)

$$\sum_{k=0}^{m+2} (k^2 - 4k + 1)$$

$$= \sum_{k=0}^{m+1} (k^2 - 4k + 1) \;\; + (m+2)^2 - 4(m+2) + 1$$

Mathematical Induction

Inductive Proof – 2 steps

1. base case – establish that the theorem is true for the first value in the sequence
2. inductive step – if the theorem is true for k, then the theorem also holds for k + 1.

Mathematical Induction

Principle of Mathematical Induction

Let S(n) be a statement parameterized by a positive integer n.

Then S(n) is true for all positive integers n, if:

1. S(1) is true – this is the base case

2. For all k ∈ Z+, S(k) implies S(k+1) – This is the inductive step.

The Inductive Step

The inductive step has two parts:

For all k ∈ Z+, S(k) implies S(k+1).

Let's rewrite this:

For all k ∈ Z+, if S(k) then S(k + 1).

if S(k) is called the inductive hypothesis.

Worked Exercise 6.4.1 (a)

Prove by induction

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

a) Verify P(3) is true. Calculate the left hand side. For n = 3, $1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$
Calculate the right hand side. For n = 3, $3(3 + 1)(2 \cdot 3 + 1) / 6 = (3 \cdot 4 \cdot 7) / 6 = 14$
The left side matches the right side so P(3) is true.

Worked Exercise 6.4.1 (b)

Prove by induction $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

b) Express P(k).

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

Worked Exercise 6.4.1 (c)

Prove by induction $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

c) Express P(k + 1).

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$