Quantum Speedup on Exhaustive-search Attacks on Cryptosystems

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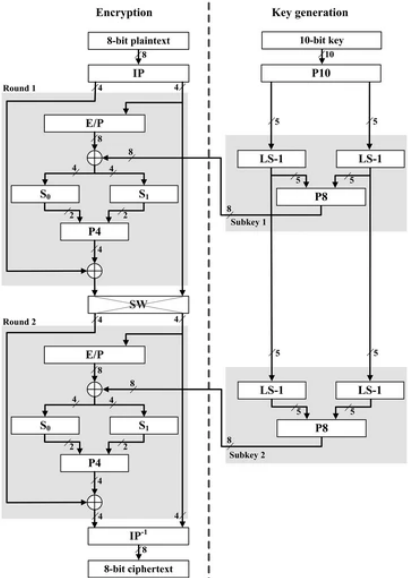
S-DES

S-DES

- Simplified DES with 2 rounds
- Structure similar to DES but simplified with 10-bit key and 8-bit plaintext.
- Quantum oracle needs to be reversible, of which S-DES is (normally) not.

Structure of S-DES

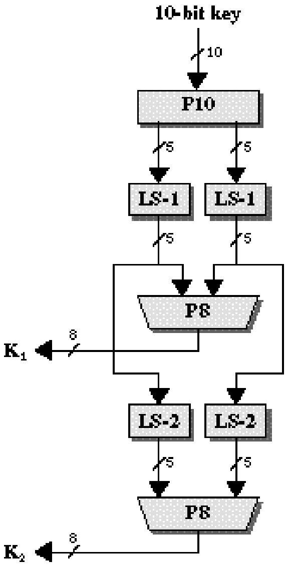


Structure of S-DES

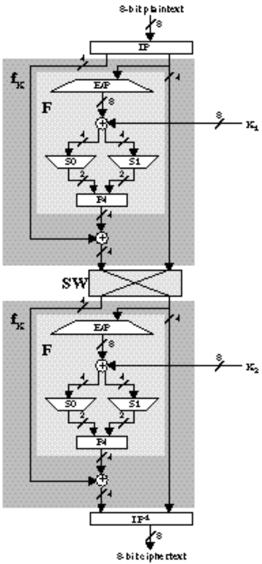
Consists of two main parts:

- Key Scheduling, where we generate subkeys from the given key
- Feistel function, where we encrypt the plaintext using the scheduled keys

Structure of S-DES: Key Scheduling



Structure of S-DES: Feistel function



Grover's Algorithm

Overview

- Grover's algorithm can find a specific state satisfying some condition among $N = 2^n$ candidates in $O(\sqrt{N})$ time, compared to classical runtime complexity $O(N)$.
- Grover's algorithm exploits qualities of quantum amplitudes to gain advantage of probability seperation.
- It can brute-force 128-bit symmetric cryptographic key in roughly 2^{64} iterations.

Algorithm

Input:

- A quantum oracle $\mathcal{O}$ which performs the operation $\mathcal{O}\left|x\right\rangle = (-1)^{f(x)}\left|x\right\rangle$, where $f(x) = 0$ for all $0 \leq x < 2^n$ except x_0, for which $f(x_0) = 1$.
  - Such quantum oracle is viable, and takes $O(1)$ time.
- n qubits initialized to the state $\left|0\right\rangle$

Output: x_0, in runtime $O(\sqrt{2^n})$ with error rate $O(\frac{1}{2^n})$

Algorithm

Procedure:

1. $|0\rangle^{\otimes n}$ (initial state)
2. $H^{\otimes n}|0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle = |\psi\rangle$ (Hadamard transform)
3. $[(2|\psi\rangle\langle\psi| - I)\mathcal{O}]^R|\psi\rangle \approx |x_0\rangle$ (Grover iteration for $R \approx \frac{\pi}{4}\sqrt{2^n}$ times)
4. x_0 (measure)

Grover iteration in a nutshell: negate the amplitude of the desired state, followed by 'diffusion transform' which increases the amplitude of the desired state and lower the others.

Quantum S-DES Oracle

Quantumizing S-DES

- Most parts are permutations(or direct compressions); no problems here.
- Two parts pose a challenge:
    1. Expansion
    2. S-Boxes

Dealing with Expansions

- Due to No-cloning theorem, directly copying a qubit is not possible.
- Use CNOT gate to perform XOR operation to copy the information.

Dealing with S-boxes

Consists of Lookup tables: Not ideal for quantum computing

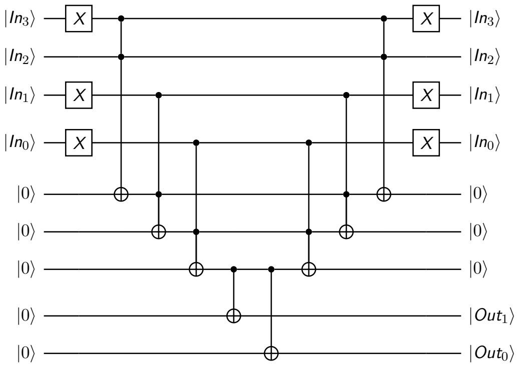
$$S_0 = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 3 & 2 & 1 & 0 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 3 & 2 \end{bmatrix} \quad S_1 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 3 \\ 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 3 \end{bmatrix}$$

Dealing with S-boxes: Quine-McCluskey Method

- Break the S-Boxes into fundamental and/or/not-gates
- Letting the input bits into S-Boxes $ABCD$:
  - S1 bit 1: $AB'C' + A'B'C + BCD' + ABD + BC'D$
  - S1 bit 2: $AD' + B'D' + A'BC + ABC'$
  - S2 bit 1: $AB'C' + ACD + A'CD' + A'BD'$
  - S2 bit 2: $B'C'D + AB'C + BCD + BC'D'$
- Directly calculating this would cause too many ancilla bits to use, i.e. for each and/or operation, one ancilla bit is required.

Dealing with S-boxes: The Brute-force Method

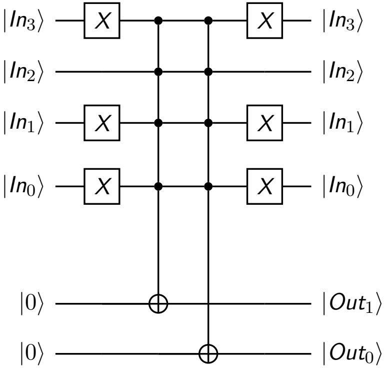
For each possibility in the S-box input, use a circuit similar to this:



...which requires 3 ancilla qubits.

Dealing with S-boxes: The Brute-force Method

Thanks to the Microsoft Q# library, we don't need that much ancilla qubits.



No (explicit) ancilla qubits are required!

Dealing with S-boxes: Combining the two methods?

- Still a hypothetical yet.
- The structure of Quine-McCluskey simplified S-Box is similar to Brute-force
- The number of terms may decrease significantly, and one ancilla bit may be removed compared to the first circuit.
- Needs further testing...

Quantum S-DES Implementation

Quantum S-DES Implementation

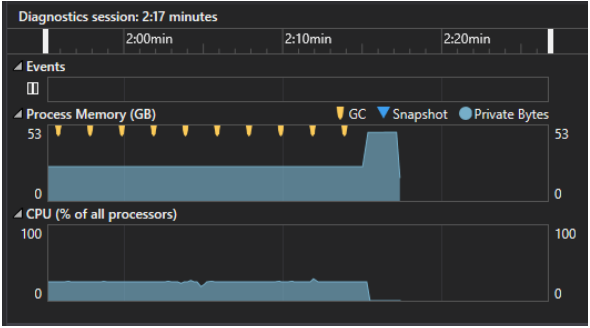
Initially, 34 explicit qubits were used in total.

- 10 qubits for input key
- 1 qubit for making an encryption oracle
- 8 qubits for (intermediate) plaintext
- 8 qubits for expanded plaintext
- 4 qubits for storing S-Box result
- 3 qubit for S-Box ancilla

S-DES implementation is written in Microsoft Q#. Will it run?

Quantum S-DES Implementation

Yes, and no.



After allocating 48GB of RAM, `System.Runtime.InteropServices.SEHException` was raised, indicating an out-of-memory error.

Quantum S-DES Implementation

Reducing the number of qubits was necessary...!

- 10 qubits for input key
- 1 qubit for making an encryption oracle
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We can directly use plaintext qubits to create result qubits from S-Box. After then we can undo all the operations (permutation, *etc.*) which altered plaintext.

Quantum S-DES Implementation

Reducing the number of qubits was necessary...!

- 10 qubits for input key
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- ~~4~~ 2 qubits for storing S-Box result
- 3 qubits for S-Box ancilla

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Additionally, each S-Box operation is independent. Thus we can reuse S-box result qubits after re-initialization.

Quantum S-DES Implementation

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- ~~3 qubits for S-Box ancilla~~

We can directly use plaintext qubits to create result qubits from S-Box. After then we can undo all the operations (permutation, *etc.*) which altered plaintext.
Additionally, each S-Box operation is independent. Thus we can reuse S-box result qubits after re-initialization.
Finally, redundant S-Box ancilla qubits were removed.

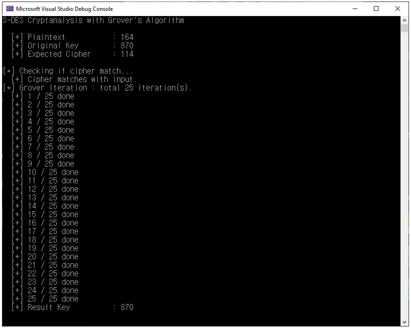
Quantum S-DES Implementation

However, applying Grover's was not simple as it looked

- The oracle should be adjoint, but original version includes measurement (in result ciphertext and S-Box)
- How to create an oracle qubit rather than measuring the result ciphertext?
  - perform CNOT gate 8 times (with 7 more qubits) : infeasible
  - Microsoft Q#'s `Controlled` functor : doable
- Similar to S-Box Controlling
- S-Box also required measurement initially, but change to forementioned circuit

Result : reversible S-DES oracle.

Quantum S-DES Implementation



```
Microsoft Visual Studio Debug Console
S-DES Cryptanalysis with Grover's Algorithm

  [+] Plaintext          : 164
  [+] Original Key       : 870
  [+] Expected Cipher    : 114

[*] Checking if cipher match...
  [+] Cipher matches with input.
[*] Grover Iteration : total 25 iteration(s).
  [+] 1 / 25 done
  [+] 2 / 25 done
  [+] 3 / 25 done
  [+] 4 / 25 done
  [+] 5 / 25 done
  [+] 6 / 25 done
  [+] 7 / 25 done
  [+] 8 / 25 done
  [+] 9 / 25 done
  [+] 10 / 25 done
  [+] 11 / 25 done
  [+] 12 / 25 done
  [+] 13 / 25 done
  [+] 14 / 25 done
  [+] 15 / 25 done
  [+] 16 / 25 done
  [+] 17 / 25 done
  [+] 18 / 25 done
  [+] 19 / 25 done
  [+] 20 / 25 done
  [+] 21 / 25 done
  [+] 22 / 25 done
  [+] 23 / 25 done
  [+] 24 / 25 done
  [+] 25 / 25 done
  [+] Result Key         : 870
```

Expected key matched with original key (with runtime 19 min 31 sec, performing each S-DES encryption within 45s on average).

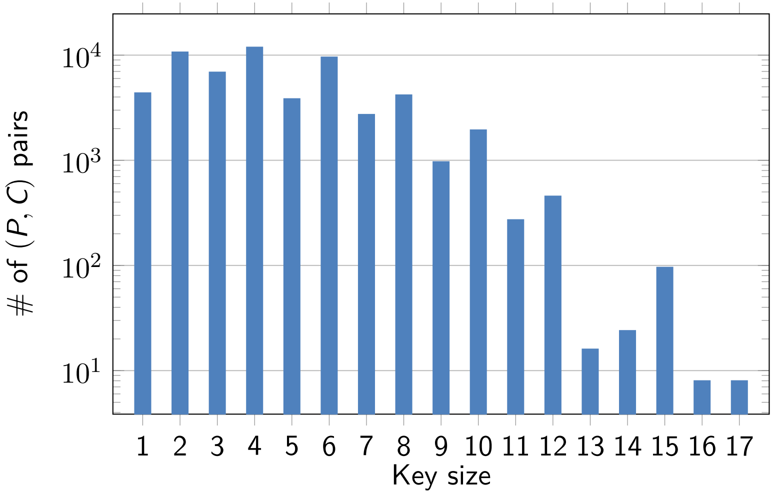
Quantum S-DES Testing

Prerequisites

- Let $SDES(P, K) = C$ be a function s.t. encrypting plaintext P with S-DES and key K yields C.
- Let $f\colon \{0,1\}^8 \times \{0,1\}^8 \to \mathcal{P}(\{0,1\}^{10})$ be a function such that $f(P, C) = \{K \in \{0,1\}^{10} \mid SDES(P, K) = C\}$.
- Since key size is 4x of cipher size, we guessed that for arbitrary P and C, $|f(P, C)|$ is likely to be 4.

Prerequisites : Key Size

...and we were wrong!



Quantum S-DES Testing : Testing with #Key = 1

If $(P, C) = (10100100, 00110011)$. Then $f(P, C) = \{1101100110\}$.

- Program yielded correct key.
- By dumping key qubits, we can observe the probability of each basis.
  - At first, the probability of answer basis is 0.008766 while the others are 0.000969.
  - The gap widens next turn: 0.024224 and 0.000954.
  - Ultimately it becomes 0.999461 and 0.000001 (at 25th iteration).

Quantum S-DES Testing : Testing with #Key $= 4$

If $(P, C) = (11000111, 00010100)$. Then $|f(P, C)| = 4$.

- Program yielded wrong key.
- By dumping key qubits, we can observe the probability of each basis.
  - At first, the probability of answer basis is 0.008698 while the others are 0.000946.
  - The gap widens next turn: 0.023659 and 0.000888.
  - It peaks at 12th iteration: 0.249987 and 0.000000.
  - However it goes back next turn: 0.246547 and 0.000014
  - Ultimately the probability of answer basis becomes lower than the others: 0.000575 and 0.000978.
- Although the algorithm and the implementation is correct, what happened here?

Quantum S-DES Testing: What Went Wrong?

- When running Grover's Algorithm, there is a number of desired steps, say n.
- If we overshoot that number of step, the probability of basis decreases!
- On the $2n$-th step, the probability becomes all the same for each of the cases again, forming a cycle.

Quantum S-DES Gate Analysis

Overview

- Most part can be expressed in permutations, but there are parts where we need to use quantum gates.
- In S-Box, we inevitably need to use Pauli-X and Toffoli gates.
- In EP, we need to copy the bits, and thereby requires the use of CNOT gates.
- In round application, we need to XOR the subkey with some data or S-box result, requiring us to use CNOT gates.

Number of Gates: Brute-forcing S-Box

- Both S-box requires certain number of Pauli-X and Toffoli(CCNOT) gates.
- The sum of gates required for each of the 16 inputs possible:

	Pauli-X	Toffoli
S-box 1	50	18
S-box 2	44	15

- On average, ~3 Pauli-X gates and ~1 Toffoli gate is used per case.

Number of Gates: Quine-McCluskey S-Box

- What about Quine-McCluskey Method?
- Looking at S-boxes:
  - S1 bit 1: $AB'C' + A'B'C + BCD' + ABD + BC'D$
  - S1 bit 2: $AD' + B'D' + A'BC + ABC'$
  - S2 bit 1: $AB'C' + ACD + A'CD' + A'BD'$
  - S2 bit 2: $B'C'D + AB'C + BCD + BC'D'$
- Only OR operations need to be done with Toffoli gates; AND operations can be done the same way as Brute-force.
- For any given input, exactly the following number of operations are required in naïve approach:

	NOT	OR
S-box 1	11	7
S-box 2	10	6

Number of Gates: Quine-McCluskey S-Box

- Using De Morgan's Law($A + B = (A'B')'$), we can implement the OR gate with 3 Pauli-X and one Toffoli(for CCNOT).
- However since we need to revert A and B back to the original state, 2 more Pauli-X are required, totaling 5 usages.
- The total number of gates required theoretically are therefore:

	Pauli-X	Toffoli
S-box 1	46	7
S-box 2	40	6

- Comparing with the Brute-force method, the required number of Toffoli gates are less than halved, while Pauli-X gates shows a small decrease.

Number of Gates: S-Box Input and Output

- For input, 4 CNOT gates are required.
- To copy the output, 2 more CNOT gates are required.
- To make this reversible, S-box application and input processes must be reversed, requiring 2 S-Box applications and 8 CNOT gates.

$\Rightarrow$ Total of 10 CNOT gates, along with two S-box applications.

Number of Gates: Apply Rounds

Assuming Brute-force method:

- Both S-boxes are applied per round, scoring in for 20 CNOT gates, 94 Pauli-X gates, and 33 Toffoli gates.
  - The remaining operations are all permutations.
- There are two rounds in total.

⇒ Total of 40 CNOT gates, 188 Pauli-X gates, and 66 Toffoli gates per single oracle call!

If we apply the Quine-McCluskey method:

- CNOT gate does not change.
- The number of Pauli-X drops to 86, and Toffoli to 13!

⇒ Dropping the total to 40 CNOT, 172 Pauli-X, and 26 Toffoli.