The Incomplete Codex of Mathematics for Computer Scientists
From Programmers to Hackers: Mathematical Basis to Computer Science

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Chapter 1

Introduction

Let's face it: mathematics is hard.

But as a computer scientist, you need to know the principle of mathematics. And we know, it's not easy. The many mathematical principles are dispersed throughout many areas of mathematics, whether it is number theory, calculus, analysis, or statistics.

This book aims to give some help to computer scientists who are tired of searching the highly dispersed information on the net or in the books. This includes the theoretical parts of computer science, such as graph, language, and complexity theories.

In the first part, mathematical preliminaries, we see the important parts from many parts of mathematics as mentioned above. This may not be directly related to any algorithms, but this will serve as a basis for many theoretical parts of computer science.

In the second part, theory-heavy parts of computer science are described as mathematically precise as possible.

Contents

Part I

Mathematical Preliminaries

Chapter 2

Logic

There wouldn't be math or any branch of science if there weren't logic. In this section, basic mathematical proofs and the methods of proof will be discussed.

2.1 Boolean Algebra

Most branches of mathematics use propositions; that is, mathematical statements that can be determined to be either true or false. In Boolean algebra, variables and constants can take on two values: true(1) or false(0). By taking the statements to be the variables in Boolean algebra, we can think of mathematical statements as formulas of Boolean algebra.

In Boolean algebra, there are only two values, true(1) and false(0), and three basic operators, two of which are binary and one unary.

AND operator(conjunction), often denoted as $p \cdot q$ or $p \wedge q$, has the value true iff p and q are both true; false if either p or q are false. The truth-table for the AND operator is as follows:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

OR operator(disjunction), often denoted as $p + q$ or $p \vee q$, has the value false iff p and q are both false; true if either p or q are true. The truth-table for the OR operator is as follows:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

NOT operator(negation), often denoted as p', ~p, or $\neg p$, is a unary operator. The operator switched the state of the variable, that is, if it is true its value is false; if false the value is true. The truth-table for the NOT operator is as follows:

p	$\neg p$
0	1
1	0

Derived by composition of the basic operators, there are many secondary operators: to name the most important operators, implication(→), exclusive-or(XOR, $\oplus$), and equivalence(=, ≡). The truth-table for the operators are as follows:

p	q	$p \to q$	$p \oplus q$	$p \equiv q$
0	0	1	0	1
0	1	1	1	0
1	0	0	1	0
1	1	1	0	1

The operators are derived as follows:

$$
\begin{array}{lcllcl}
p \to q & = & \neg p \vee y & & & \\
p \oplus q & = & (p \vee q) \wedge \neg(p \wedge q) & = & (p \wedge \neg q) \vee (\neg p \wedge q) \\
p \equiv q & = & \neg(p \oplus q) & = & (p \wedge q) \vee (\neg p \wedge \neg q)
\end{array}
$$

2.2 Proof Techniques

There are many methods of proof. In this section, the common methods of proof used in mathematics will be discussed.

2.2.1 Direct Proof

2.2.2 Proof by Mathematical Induction

2.2.3 Proof by Contraposition

2.2.4 Proof by Construction

2.2.5 Proof by Exhaustion

2.2.6 Computer-assisted Proof

Chapter 3

Algebraic Structures

3.1 Algebraic Structures

3.1.1 Sets

Definition 1 (Set)
A <u>set</u> is a collection of distinct objects.

To see some traits on sets, we literally start from nothing:

Axiom 2 (Empty Set Axiom)
There is a set containing no members, that is:

$$\exists B \text{ such that } \forall x, (x \notin B)$$

We call this set the <u>empty set</u>, and denote it by the symbol $\emptyset$.

We now have $\emptyset$; we now write down a few rules for how to manipulate sets.

Axiom 3 (Axiom of Extensionality)
Two sets are equal if and only if they share the same elements, that is:

$$\forall A, B[\forall z, ((z \in A) \Leftrightarrow (z \in B)) \Rightarrow (A = B)]$$

Axiom 4 (Axiom of Pairing)
Given any two sets A and B, there is a set which have the members just A and B, that is:
$$\forall A, B \exists C \forall x[x \in C \Leftrightarrow ((x = A) \vee (x = B))]$$

If A and B are distinct sets, we write this set C as $\{A, B\}$; if $A = B$, we write it as $\{A\}$.

Axiom 5 (Axiom of Union, simple version)
Given any two sets A and B, there is a set whose members are those sets belonging to either A or B, that is:

$$\forall A, B \exists C \forall x[x \in C \Leftrightarrow ((x \in A) \vee (x \in B))]$$

We write this set C as $A \cup B$.

In the simplified version of Axiom of Union, we take union of only two things, but we sometimes we want to take unions of more than two things or even more than finitely many things. This is given by the full version of the axiom:

Axiom 6 (Axiom of Union, full version)
Given any set A, there is a set C whose elements are exactly the members of the members of A, that is:

$$\forall A \exists C[x \in C \Leftrightarrow (\exists A'(A' \in A) \wedge (x \in A'))]$$

We denote this set C as

$$\bigcup_{A' \in A} A'$$

Axiom 7 (Axiom of Intersection, simple version)
Given any two sets A and B, there is a set whose members are member of both A and B, that is:

$$\forall A, B \exists C \forall x[(x \in C) \Leftrightarrow ((x \in A) \wedge (x \in B))]$$

Sometimes as union, we would want to take intersection of more than finitely many things. This is given by the full version of the axiom:

Axiom 8 (Axiom of Intersection, full version)
Given any set A, there is a set C whose elements are exactly the members of all members of A, that is:

$$\forall A \exists C \forall x[(x \in C) \Leftrightarrow (\forall A'((A' \in A) \Rightarrow (x \in A')))]$$

We denote this set C as

$$\bigcap_{A' \in A} A'$$

Axiom 9 (Axiom of Subset)
For any two sets A and B, we say that $B \subset A$ if and only if every member of B is a member of A, that is:

$$(B \subseteq A) \Leftrightarrow (\forall x(x \in B) \Rightarrow (x \in A))$$

By the Axiom of Subset we can define the power set of an any given set:

Definition 10 (Power Set)
For any set A, the <u>power set</u> of the set A, denoted $P(A)$, whose members are precisely the collection of all possible subsets of A, that is:

$$\forall A \exists P(A) \forall B((B \subseteq A) \Leftrightarrow (B \in P(A)))$$

Definition 11 (Equivalence Relation)
Let S be a set. An <u>Equivalence Relation</u> on S is a relation, denoted by ~, with the following properties, $\forall a, b, c \in S$:

- **Reflexivity** $a \sim a$

- **Symmetry** $a \sim b \Leftrightarrow b \sim a$

- **Transitivity** $(a \sim b) \wedge (b \sim c) \Rightarrow (a \sim c)$

Definition 12 (Setoid)
A <u>setoid</u> is a set in which an equivalence relation is defined, denoted $(S, \sim)$.

Definition 13 (Equivalence Class)
The equivalence class of $a \in S$ under ~, denoted $[a]$, is defined as $[a] = \{b \in S | a \sim b\}$.

Definition 14 (Order)
Let S be a set. An <u>order</u> on S is a relation, denoted by $<$, with the following properties:

- If $x \in S$ and $y \in S$ then one and only one of the following statements is true:
$$x < y, x = y, y < x$$

- For $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Remark

- It is possible to write $x > y$ in place of $y < x$

- The notation $x \leq y$ indicates that $x < y$ or $x = y$.

Definition 15 (Ordered Set)
An <u>ordered set</u> is a set in which an order is defined, denoted $(S, <)$.

Definition 16 (Bound)
Suppose S is an ordered set, and $E \subset S$.
If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is <u>bounded above</u>, and call β an <u>upper bound</u> of E. If there exists $\alpha \in S$ such that $x \geq \alpha$ for every $x \in E$, we say that E is <u>bounded below</u>, and call α a <u>lower bound</u> of E.

Definition 17 (Least Upper Bound)
Suppose that S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ with the following properties:

- β is an upper bound of E

- If $\gamma < \beta$, then γ is not an upper bound of E

Then β is called the <u>Least Upper Bound</u> of E or the <u>supremum</u> of E, denoted
$$\beta = sup(E)$$

Definition 18 (Greatest Lower Bound)
Suppose that S is an ordered set, and $E \subset S$. If there exists a $\alpha \in S$ with the following properties:

- α is a lower bound of E

- If $\gamma < \alpha$, then γ is not an lower bound of E

Then α is called the <u>Greatest Lower Bound</u> of E or the <u>infimum</u> of E, denoted
$$\beta = inf(E)$$

Definition 19 (least-upper-bound property)
An ordered set S is said to have the <u>least-upper-bound property</u> if the following is true:
if $E \subset S$, E is not empty, and E is bounded above, then $sup(E)$ exists in S.

Definition 20 (greatest-lower-bound property)
An ordered set S is said to have the <u>greatest-lower-bound property</u> if the following is true:
if $E \subset S$, E is not empty, and E is bounded below, then $inf(E)$ exists in S.

Theorem 21
Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below.
Let L be the set of all lower bounds of B. Then
$$\alpha = sup(L)$$

exists in S, and $\alpha = inf(B)$.

Proof. Note that $\forall x \in L, y \in B, x \leq y$.
L is nonempty as B is bounded below.
L is bounded above since $\forall x \in S \backslash L, \forall y \in L, x > y$.
Since S has the least-upper-bound property and $L \subset S$, $\exists \alpha = sup(L)$.
The followings hold:

- α is a lower bound of B.
  $(\because)$ $\forall \gamma \in B, \gamma > \alpha$

- β with $\beta > \alpha$ is not a lower bound of B
  $(\because)$ Since α is an upper bound of L, $\beta \notin L$.

Hence $\alpha = inf(B)$. $\hspace{2cm}\square$

Corollary 22
For all ordered sets, the Least Upper Bound property and the Greatest Lower Bound Porperty are equivalent.

3.1.2 Group

Definition 23 (Group)
A <u>group</u> is a set G with a binary operation $\cdot$, denoted $(G, \cdot)$, which satisfies the following conditions:

- **Closure:** $\forall a, b \in G, a \cdot b \in G$

- **Associativity:** $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

- **Identity:** $\exists e \in G, \forall a \in G, a \cdot e = e \cdot a = a$

- **Inverse:** $\forall a \in G, \exists a^{-1} \in G, a \cdot a^{-1} = a^{-1} \cdot a = e$

Definition 24 (Semigroup)
A <u>semigroup</u> is $(G, \cdot)$, which satisfies Closure and Associativity.

Definition 25 (Monoid)
A <u>monoid</u> is a semigroup $(G, \cdot)$ which also has identity.

Definition 26 (Abelian Group)
An <u>Abelian Group</u> or <u>Commutative Group</u> is a group $(G, \cdot)$ with the following property:

- **Commutativity:** $\forall a, b \in G, a \cdot b = b \cdot a$

3.1.3 Ring

Definition 27 (Ring)
A <u>Ring</u> is a set R with two binary operations $+$ and $\cdot$, often called the addition and multiplication of the ring, denoted $(R, +, \cdot)$, which satisfies the following conditions:

- $(R, +)$ is an abelian group

- $(R, \cdot)$ is a semigroup

- **Distribution:** $\cdot$ is distributive with respect to $+$, that is, $\forall a, b, c \in R$:

  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
  - $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

The identity element of + is often noted 0.

Definition 28 (Ring with identity(1))
A <u>Ring with identity</u> is a ring $(R, +, \cdot)$ of which $(R, \cdot)$ is a monoid. The identity element of $\cdot$ is often noted 1.

Definition 29 (Commutative Ring)
A <u>commutative ring</u> is a ring $(R, +, \cdot)$ of which $\cdot$ is commutative.

Definition 30 (Zero Divisor)
For a ring $(R, +, \cdot)$, let 0 be the identity of $+$.
$a, b \in R$, $a \neq 0$ and $b \neq 0$, if $a \cdot b = 0$, a, b are called the zero divisors of the ring.

Definition 31 (Integral Domain)
An <u>integral domain</u> is a commutative ring $(R, +, \cdot)$ with 1 which does not have zero divisors.

3.1.4 Field

Definition 32 (Field)
A <u>Field</u> is a set F with two binary operations $+$ and $\cdot$, often called the addition and multiplication of the field, denoted $(R, +, \cdot)$, which satisfies the following conditions:

- $(F, +, \cdot)$ is a ring

- $(F \backslash \{0\}, \cdot)$ is a group

Alternatively, a Field may be defined with a set of <u>Field Axioms</u> listed below:

(A) **Axioms for Addition**

    (A1) **Closed under Addition**
        $\forall a, b \in F, a + b \in F$

    (A2) **Addition is Commutative**
        $\forall a, b \in F, a + b = b + a$

    (A3) **Addition is Associative**
        $\forall a, b, c \in F, (a + b) + c = a + (b + c)$

    (A4) **Identity of Addition**
        $\exists 0 \in F, \forall a \in F, 0 + a = a$

    (A5) **Inverse of Addition**
        $\forall a \in F, \exists -a \in F, a + (-a) = 0$

(M) **Axioms for Multiplication**

    (M1) **Closed under Multiplication**
        $\forall a, b \in F, a \cdot b \in F$

    (M2) **Multiplication is Commutative**
        $\forall a, b \in F, a \cdot b = b \cdot a$

    (M3) **Multiplication is Associative**
        $\forall a, b, c \in F, (a \cdot b) \cdot c = a \cdot (b \cdot c)$

    (M4) **Identity of Multiplication**
        $\exists 1 \in F, \forall a \in F, 1 \cdot a = a$

(M5) **Inverse of Multiplication**
$\forall a \in F \backslash \{0\}, \exists a^{-1} \in F, a \cdot a^{-1} = 1$

(D) **Distributive Law**
$\forall a, b, c \in F, (a + b) \cdot c = a \cdot c + b \cdot c$
where $\cdot$ takes precedence over $+$.

Theorem 33
Let F be a field. Let 0 be the additive identity of F. Then, $\forall a \in F, 0 \cdot a = 0$

Definition 34 (Ordered Field)
An <u>ordered field</u> is a field F which is an ordered set, such that the order is compatible with the field operations, that is:

- $x + y < x + z$ if $x, y, z \in F$ and $y < z$

- $xy > 0$ if $x, y \in F$, $x > 0$ and $y > 0$

3.1.5 Polynomial Ring

Definition 35 (Polynomial over a Ring)
A polynomial $f(x)$ over the ring $(R, +, \cdot)$ is defined as

$$f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x^1 + \cdots, a_i \in R$$

where $a_i = 0$ for all but finitely many values of i.
The <u>degree</u> of the polynomial $\deg(f)$ is defined as $\deg(f) = \max\{n | n \in \mathbb{N}, a_n \neq 0\}$.
The <u>leading coefficient</u> of the polynomial is defined as $a_{\deg(f)}$.

Definition 36 (Addition and Multiplication of Polynomials)
Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, $a_i, b_i \in R$ be a polynomial over the ring $(R, +, \cdot)$. Define:

$$f(x) + g(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

$$f(x)g(x) = \sum_{k=0}^{\infty} (c_k) x^k \text{ where } c_k = \sum_{i+j=k} a_i b_j$$

Definition 37 (Polynomial Ring)
The set of polynomials over the ring $(R, +, \cdot)$, $R[x] = \{f(x) | f(x) \text{ is a polynomial over } R\}$ is called the <u>Polynomial Ring(or Polynomials) over R</u>.

Theorem 38 (Degree of Polynomial on Addition and Multiplication)
Let $f(x), g(x) \in R[x]$ with $\deg(f) = n$, $\deg(g) = m$.

- $0 \leq \deg(f + g) \leq \max(\deg(f), \deg(g))$

- $\deg(fg) \leq \deg(f) + \deg(g)$.
    If $(R, +, \cdot)$ is an integral domain, $\deg(fg) = \deg(f) + \deg(g)$

Theorem 39 (Relationship between a Ring and its Polynomial Ring)
Let $(R, +, \cdot)$ be a ring and $R[x]$ the polynomials over R.

1. If $(R, +, \cdot)$ is a commutative ring with 1, then $(R[x], +, \cdot)$ is a commutative ring with 1.

2. If $(R, +, \cdot)$ is a integral domain, then $(R[x], +, \cdot)$ is a integral domain.

Theorem 40 (Division Algorithm for Polynomials over a Ring)
Let $(R, +, \cdot)$ be a commutative ring with 1.
Let $f(x), g(x) \in R[x]$, $g(x) \neq 0$ with the leading coefficient of $g(x)$ being invertible.
Then, $\exists! q(x), r(x) \in R[x]$ such that

$$f(x) = q(x)g(x) + r(x)$$

where either $r(x) = 0$ or $\deg(r) < \deg(g)$.

Proof. Use induction on $\deg(f)$.
1. $f(x) = 0$ or $\deg(f) < \deg(g)$: $q(x) = 0, r(x) = f(x)$
2. $\deg(f) = \deg(g) = 0$: $q(x) = f(x) \cdot g(x)^{-1}, r(x) = 0$
3. $\deg(f) \geq \deg(g)$:

   1) Existence
Let $\deg(f) = n$, $\deg(g) = m$, $n > m$.
Suppose the theorem holds for $\deg(f) < n$.
Let $f(x) = a_0 + a_1 x^1 + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x^1 + \cdots + b_m x^m$.
Choose $f_1(x) = f(x) - (a_n b_m^{-1}) x^{n-m} g(x) \in R[x]$.
Since $\deg(f_1) < n$, $\exists q(x), r(x) \in R[x]$ so that $f_1(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg(r) < \deg(g)$.
$f_1(x) = f(x) - (a_n b_m^{-1}) x^{n-m} g(x) = g(x)q(x) + r(x)$
$f(x) = g(x)((a_n b_m^{-1}) x^{n-m} + q(x)) + r(x)$
Hence such pair exists.

   2) Uniqueness
Suppose $f(x) = g(x)q_1(x) + r_1(x) = g(x)q_2(x) + r_2(x)$.
$g(x)(q_1(x) - q_2(x)) = r_2(x) - r_1(x)$
If $r_1 \neq r_2$, $\deg(g) > \deg(r_2 - r_1) = \deg(g(q_1 - q_2))$.
Since $\deg(g(q_1 - q_2)) \geq \deg(g)$ if $q_1 - q_2 \neq 0$, $q_1 = q_2$, but if so, $r_1 = r_2$.
If $r_1 = r_2$, trivially $q_1 = q_2$.
Hence they exist uniquely. □

3.1.6 Vector Space

Definition 41 (Vector Space)
A <u>vector space</u> over a field F is a set V together with two operations, addition $(+ : V \times V \to V)$ and scalar multiplication $(\cdot : F \times V \to V)$, satisfying the following axioms:

(A) **Axioms for Addition**

   (A1) **Closed under Addition**
      $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} \in V$

   (A2) **Addition is Commutative**
      $\forall \vec{u}, \vec{v} \in V, \vec{u} + \vec{v} = \vec{v} + \vec{u}$

   (A3) **Addition is Associative**
      $\forall \vec{u}, \vec{v}, \vec{w} \in v, (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

   (A4) **Identity of Addition(Zero vector)**
      $\exists \vec{0} \in V, \forall \vec{u} \in F, \vec{0} + \vec{u} = \vec{u} + \vec{0} = \vec{u}$

   (A5) **Inverse of Addition(Negative)**
      $\forall \vec{u} \in V, \exists - \vec{u} \in V, \vec{u} + (-\vec{u}) = 0$

(M) **Axioms for Scalar Multiplication**

(M1) **Closed under Scalar Multiplication**
$\forall k \in F, \vec{u} \in V, k \cdot \vec{u} \in V$

(M2) **Scalar Multiplication is Distributive(1)**
$\forall k \in F, \vec{u}, \vec{v} \in V, k \cdot (\vec{u} + \vec{v}) = k \cdot \vec{u} + k \cdot \vec{v}$

(M3) **Scalar Multiplication is Distributive(2)**
$\forall k, m \in F, \vec{u} \in V, (k + m) \cdot \vec{u} = k \cdot \vec{u} + m \cdot \vec{u}$

(M4) **Scalar Multiplication is Associative**
$\forall k, m \in F, \vec{u} \in V, (km) \cdot \vec{u} = k \cdot (m \cdot \vec{u})$

(M5) **Identity of Scalar Multiplication**
$\exists 1 \in F, \forall \vec{u} \in V, 1 \cdot \vec{u} = \vec{u}$

Theorem 42
Let V be a vector space over a field F. $\vec{u} \in V$, $k \in F$, 0 the additive identity of F, 1 the multiplicative identity of F, $\vec{0}$ the additive identity of V. Then, the followings hold:

- $0 \cdot \vec{u} = \vec{0}$

- $k \cdot \vec{0} = \vec{0}$

- $-1 \cdot \vec{u} = -\vec{u}$

- If $k \cdot \vec{u} = \vec{0}$, then $k = 0$ or $\vec{u} = \vec{0}$.

Definition 43 (Subspace of a Vector Space)
A subset W of a vector space V is called a <u>subspace</u> of V if W is a vector space under the addition and scalar multiplication defined on V.

Theorem 44
If W is a set of one or more vectors in a vector space V over the field F, then W is a subspace of V iff the following conditions hold:

- $\forall \vec{u}, \vec{v} \in W, \vec{u} + \vec{v} \in W$

- $\forall k \in F, \vec{u} \in W, k \cdot \vec{u} \in W$

Theorem 45
If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then $\cap_{i=1}^{r} W_i$ is also a subspace of V.

Definition 46 (Linear Combination)
If $\vec{w}$ is a vector in a vector space V over the field F, then $\vec{w}$ is said to be a <u>Linear Combination</u> of the vectors $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r} \in V$ if $\vec{w}$ can be expressed in the form $\vec{w} = \sum_{i=1}^{r} k_i \vec{v_i}$, where $k_1, k_2, \ldots, k_r \in F$. These scalars are called the <u>coefficients</u> of the linear combination.

Definition 47 (Span)
The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the <u>Span</u> of S, and we say that the vectors in S <u>span</u> that subspace.

Theorem 48
If $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ and $S' = \{\vec{w_1}, \vec{w_2}, \ldots, \vec{w_k}\}$ are nonempty sets of vectors in a vector space V, then $\mathbf{span}(S) = \mathbf{span}(S')$ iff each vector in S is a linear combination of those in S' and vice versa.

Definition 49 (Basis)
If V is any vector space and $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ is a finite set of linearly independent vectors in V which spans V, then S is called a <u>basis</u> for V.

Theorem 50
All bases for a finite-dimensional vector space have the same number of vectors.

Definition 51 (Dimension)
The <u>dimension</u> of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

Theorem 52
If $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ is a basis for a vector space V, then every vector $\vec{v}$ in V can be expressed in the form $\vec{v} = c_1 \vec{v_1} + c_2 \vec{v_2} + \cdots + c_r \vec{v_r}$ in exactly one way.

Definition 53 (Coordinate)
Let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ be a basis for a vector space V over the field F, and $\vec{v} = c_1 \vec{v_1} + c_2 \vec{v_2} + \cdots + c_r \vec{v_r}$ is the expression for a vector $\vec{V}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the <u>coordinates</u> of $\vec{v}$ relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in F^n constructed from these coordinates is called the <u>coordinate vector of $\vec{v}$ relative to S</u>, denoted by $(\vec{v})_S = (c_1, c_2, \ldots, c_n)$.

Chapter 4

Number Theory

4.1 Arithmetic

4.1.1 Integer Arithmetic

Theorem 54 (Division Algorithm)

Definition 55 (Divisibility)

Theorem 56 (Euclidean Algorithm)

Theorem 57 (Extended Euclidean Algorithm)

Definition 58 (Linear Diophantine Equation)

Theorem 59 (Solutions for Linear Diophantine Equation)

4.1.2 Modular Arithmetic

Definition 60 (Modulus)

Chapter 5

Analysis

Chapter 6

Linear Algebra

The target of Linear Algebra is to solve a system of homogenous linear equations. To do so, we deal with vectors and matrices.

6.1 Vector Spaces

For the definitions on vector spaces, subspaces, and bases, refer to the chapter 3.1.6.

6.1.1 Linear Independence

We now define linear independence, one of the most important concepts utilized in linear algebra.

Definition 61 (Linear Independence)
if $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ is a nonempty set of vectors in a vector space V, then the vector equation $k_1\vec{v_1} + k_2\vec{v_2} + \cdots + k_r\vec{v_r} = \vec{0}$ has at least one solution, namely, $k_1 = 0, k_2 = 0, \ldots, k_r = 0$, the <u>trivial solution</u>. If this is the only solution, then S is said to be a <u>linearly independent set</u>. If there are solutions in addition to the trivial solution, then S is said to be <u>linearly dependent</u>.

Theorem 62
Let $S = \{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_r}\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

6.1.2 Orthogonality

Definition 63 (Euclidean Inner Product)
Let $\vec{u} = (u_1, u_2, \ldots, u_n)$ and $\vec{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$. The inner product of the two vectors $\vec{u}$ and $\vec{v}$ is defined as

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Definition 64 (Norm)
The <u>norm</u> of $\vec{u} = (u_1, u_2, \ldots, u_n)$ in $\mathbb{R}^n$, denoted $\|\vec{u}\|$, is defined by

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = \sqrt{\sum_{i=1}^{n} u_i^2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Definition 65 (Unit vectors)
A vector $\vec{u}$ in $\mathbb{R}^n$ is said to be a unit vector iff $\|\vec{u}\| = 1$.

Definition 66 (Angle)
The <u>angle</u> between two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is defined by

$$\theta = \cos^{-1}(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|})$$

Definition 67 (Orthogonal Vectors)
Two nonzero vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ are said to be <u>orthogonal</u> or <u>perpendicular</u> if $\vec{u} \cdot \vec{v} = 0$.

Definition 68 (Orthogonal set)
A nonempty set of vectors in $\mathbb{R}^n$ is called an <u>orthogonal set</u> if all pairs of distinct vectors in the set are orthogonal. If they are also all unit vectors, it is called an <u>orthonormal set</u>.
In other words, for a set $\{u_1, u_2, \ldots, u_n\}$ to be orthogonal:

$$u_i \cdot u_j \begin{cases} \|u_i\|^2 & i = j \\ 0 & i \neq j \end{cases}$$

And for the set to be orthonormal, in addition to above, $\forall i \in \{1, 2, \ldots n\}, \|u_i\| = 1$.

6.2 Matrix

6.2.1 Matrices and its operations

Definition 69 (Matrix)
A <u>matrix</u> is a rectangular array of numbers. The numbers in the array are called the <u>entries</u> in the matrix.

Equality, addition, and subtraction can only be defined on same-sized matrices, and is defined elementwise; scalar multiplication is also defined elementwise.

Definition 70 (Matrix Multiplication)
If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the <u>product</u> AB is the $m \times n$ matrix whose entries are determined as follows: The entry of AB on row i and column j, multiply the corresponding entries from the row i from A and column j from B, then add them all together.

Matrices of the same size may be used in a linear combination, just like vectors[46].

Definition 71 (Linear Combination of a Matrix)
If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \cdots + c_r A_r$$

is called a <u>linear combination</u> of $A_1, A_2, \ldots, A_r$ with coefficients $c_1, c_2, \ldots, c_r$.

Theorem 72
If A is an $m \times n$ matrix and if $\vec{x}$ is an $n \times 1$ column vector, then the product $A\vec{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\vec{x}$.

Definition 73 (Transpose)
For any $m \times n$ matrix, then the <u>transpose</u> of A, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A and so forth.

Definition 74 (Trace)
For a square matrix A, the <u>trace</u> of A, denoted $tr(A)$, is defined to be the sum of the entries on the main diagonal of A.

6.3 Inverse

6.3.1

6.4 Determinants

6.4.1

6.5 Eigenvalues and Eigenvectors

6.5.1

6.6 Solving Linear Equations

We come to this final section, the ultimate target of linear algebra: solving a system of linear equations.

6.6.1 Linear Equations to Matrices

Definition 75 (Homogeneous Linear Equation)
A <u>homogeneous linear equation</u> in the variables $x_1, x_2, \ldots, x_n$ has the form $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = 0$, where $a_1 a_2 \ldots a_n \neq 0$.

A finite set of linear equations is called a <u>system of linear equations</u>, or more briefly, a <u>linear system</u>. The variables are called <u>unknowns</u>. A <u>solution</u> of a linear system in $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution $x_i = s_i$ makes each equation a true statement.

The system of m linear equations in n unknowns,

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

can be represented in matrices as follows:

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ \vdots \\ x_n
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ \vdots \\ b_m
\end{bmatrix}
$$

By designating the three matrices A, $\vec{x}$ and $\vec{b}$ respectively, we can say that $A\vec{x} = \vec{B}$. In this equation, A is called the <u>coefficient matrix</u> of the system.

The <u>augmented matrix</u> for the system is obtained by adjoining $\vec{b}$ to A as the last column as follows:

$$
\left[
\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m1} & a_{m2} & \cdots & a_{mn} & b_m
\end{array}
\right]
$$

Chapter 7

Calculus

7.1 Limits

You may have seen an equation of the form $\lim_{x \to a} f(x) = L$. Intuitively, it means that as x *approaches* a, $f(x)$ goes *arbitrarily close* to L. But no, this "intuition" is not how mathematics works. What do you mean by "approaches?" What do you mean by "arbitrarily close?" How are you going to prove any theorem with this "definition?"

Let's give a precise definition of a limit. $f(x)$ goes arbitrarily close to L, but how close does that mean? It can go closer than any positive number. That means for any $\epsilon > 0$, $f(x)$ can go closer to L than ϵ. That is, $|f(x) - L| < \epsilon$.

Next, x approaches a, but how close does it approach a? How much should x approach a so that $f(x)$ goes arbitrarily close to L, in other words, $|f(x) - L| < \epsilon$? Well, close enough. When x is closer to a than some threshold, say δ, we would have $|f(x) - L| < \epsilon$. But it doesn't need to exactly be a. Expressing this mathematically, we get $0 < |x - a| < \delta$.

Combine those two inequalities, and presto! We have this definition of a limit.

Definition 76 (Limit at a)
Let f be a function defined on some open interval that contains a, except possibly at a itself. Then we say $\lim_{x \to a} f(x) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Similarly, we can define left-hand limits, right-hand limits, and limits at infinity.

Definition 77
.

This allows us to prove the theorems involving limits.

Theorem 78
.

Definition 79 (Continuous function)
.

7.2 Differentiation

Definition 80 (Derivative)
The <u>derivative of a function f at a</u>, denoted $f'(a)$, is $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$,

if this limit exists. f is <u>differentiable at a</u> if $f'(a)$ exists.

Theorem 81
If f is differentiable at a, then f is continuous at a.

Proof. . □

7.3 Derivative Formulae

Theorem 82
Let f and g be differentiable functions and c be a constant.

1. $c' = 0$.

2. $(cf)' = c(f')$.

3. $(f + g)' = f' + g'$.

4. $(f - g)' = f' - g'$.

5. $(fg)' = fg' + gf'$.

6. $(\frac{f}{g})' = \frac{gf' - fg'}{g^2}$, where $g(x) \neq 0$.

7. $(x^c)' = cx^{c-1}$, where c is a rational number. (It also holds for real numbers, but we won't prove it here.)

Proof. . □

Theorem 83

1. $(\sin x)' = \cos x$.

2. $(\cos x)' = -\sin x$.

3. $(\tan x)' = \sec^2 x$.

4. $(\csc x)' = -\csc x \cot x$.

5. $(\sec x)' = \sec x \tan x$.

6. $(\cot x)' = -\csc^2 x$.

Proof. . □

Theorem 84 (Chain rule)
If g is differentiable at x and f is differentiable at $g(x)$, then $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and $F'(x) = f'(g(x))g'(x)$.

Proof. . □

7.4 Integration

Chapter 8

Statistics

Chapter 9

From $\mathbb{N}$ to $\mathbb{R}$

9.1 $\mathbb{N}$: The set of Natural Numbers

9.1.1 Construction of $\mathbb{N}$

We start from the Axioms of Set[2,3,4,5,6,9], the definition of power set[10], the definition of equivalence relation and class[11,13] and the following definitions:

Definition 85 (Successor)
For any set x, the <u>successor of x</u>, denoted $\sigma(x)$, is defined as the following set:

$$\sigma(x) = x \cup \{x\}$$

Let us define $0 = \emptyset$, $1 = \sigma(\emptyset) = \sigma(0)$. Using the definition of successors, and following the pattern, $2 = \sigma(1)$, $3 = \sigma(2)$, and so on. Basically we can make any finite number using the definition of successor and the Axioms of Set, but actually getting all of the natural numbers at once(or any infinitely large set, since only the empty set is guaranteed to exist by the axioms) is not possible with our axioms. We define the concept of Inductive Sets and make another Axiom for this purpose:

Definition 86 (Inductive Set)
A set A is called <u>inductive</u> if it satisfies the following two properties:

- $\emptyset \in A$

- $(x \in A) \Rightarrow (\sigma(x) \in A)$

Axiom 87 (Axiom of Infinity)
There is an inductive set, that is:

$$\exists A (\emptyset \in A) \wedge (\forall x \in A, \sigma(x) \in A)$$

Theorem 88
Take any two inductive sets, S and T. Then, $S \cap T$ is also an inductive set.

Proof. Let $U = S \cap T$.

1. $\emptyset \in U$

    $\emptyset \in S$ and $\emptyset \in T$ since S and T are both inductive.

2. $(x \in U) \Rightarrow (\sigma(x) \in U)$

    $\forall x \in U, (x \in S) \wedge (x \in T)$.
    Since S and T are both inductive, $(\sigma(x) \in S) \wedge (\sigma(x) \in T)$
    Therefore $\sigma(x) \in U$.

Therefore U is inductive.   □

Corollary 89
An intersection of any number of inductive sets is inductive.

Theorem 90
For any inductive set S, define N_S as follows:

$$N_S = \bigcap_{\substack{A \subseteq S \\ A \text{ is inductive}}} A$$

Take any two inductive sets, S and T. Then $N_S = N_T$.

Proof. Suppose not; WLOG, $\exists x$ such that $x \in N_S$ and $x \notin N_T$.
Let $X = N_S \cap N_T$. Then X is inductive, $X \subset N_S$, and $x \notin X$.
Since by the definition of N_S, $N_S = X \cap N_S$, but $x \notin X \cap N_S$ hence the RHS and the LHS are different.
Therefore the assumption is wrong; therefore $N_S = N_T$.   □

 Using this theorem, we can finally define the set of natural numbers:

Definition 91 (The Set (N) of natural numbers)
Take any inductive set S, and let

$$N = \bigcap_{\substack{A \subseteq S \\ A \text{ is inductive}}} A$$

This set is the natural numbers, which we denote as $\mathbb{N}$.

9.1.2 Operations on $\mathbb{N}$

We now define two operations on $\mathbb{N}$, addition(+) and multiplication($\cdot$).

Definition 92 (Addition and Multiplication on $\mathbb{N}$)
The operation of addition, denoted by +, is defined by following two recursive rules:

1. $\forall n \in \mathbb{N}, n + 0 = n$

2. $\forall n, m \in \mathbb{N}, n + \sigma(m) = \sigma(n + m)$

Similarly the operation of multiplication, denoted by $\cdot$, is defined by following two recursive rules:

1. $\forall n \in \mathbb{N}, n \cdot 0 = 0$

2. $\forall n, m \in \mathbb{N}, n \cdot \sigma(m) = n \cdot m + n$

Lemma 93 (Operations on 0)
$\forall x \in \mathbb{N}$

- $x + 0 = 0 + x$

- $x \cdot 0 = 0 \cdot x$

Proposition 94 (Properties of + and $\cdot$)
$\forall x, y, z \in \mathbb{N}$,

- **Associativity of Addition** $x + (y + z) = (x + y) + z$

- **Commutativity of Addition** $x + y = y + x$

- **Associativity of Multiplication** $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

- **Commutativity of Multiplication** $x \cdot y = y \cdot x$

- **Distributive Law** $x \cdot (y + z) = x \cdot y + x \cdot z$

- **Cancellation Law for Addition** $x + z = y + z \Rightarrow x = y$

9.1.3 Ordering on $\mathbb{N}$

Definition 95 (Ordering on $\mathbb{N}$)
For $n, m \in \mathbb{N}$, we say that n is less than m, written $n \geq m$, if there exists a $k \in \mathbb{N}$ such that $m = n + k$. We also write $n < m$ if $k \neq 0$.

Theorem 96
$(N, <)$ is an ordered set[15].

Proposition 97
The followings are true:

- If $n \neq 0$, then $0 < n$.

- Let $x, y, z \in \mathbb{N}$. Then the followings are true:

    - $(x \leq y) \wedge (y < z) \Rightarrow (x < z)$
    - $(x < y) \wedge (y \leq z) \Rightarrow (x < z)$
    - $(x \leq y) \wedge (y \leq z) \Rightarrow (x \leq z)$
    - $(x < y) \Rightarrow (x + z < y + z)$
    - $(x < y) \Rightarrow (xz < yz)$

- $\forall n \in \mathbb{N}, n \neq n + 1$

- $\forall n, k \in \mathbb{N}, k \neq 0, n \neq n + k$

Definition 98 (Least Element)
Let $S \subset \mathbb{N}$. An element $n \in S$ is called a <u>least element</u> if $\forall m \in S, n \leq m$

Proposition 99 (Uniqueness of the Least Element)
Let $S \subset \mathbb{N}$. Then if S has a least element, then it is unique.

Theorem 100 (Well-Ordering Property)
Let S be a nonempty subset of $\mathbb{N}$. Then S has a least element.

Note
The well-ordering property states that the set of natural numbers $\mathbb{N}$ has the greatest lower bound property[20] and thereby theorem 21, has the least upper bound property[19].

9.1.4 Properties of $\mathbb{N}$

Many of the mathematics book defines the set of Natural Numbers as the set satisfying the <u>Peano Axioms</u>.

Proposition 101 (Peano Axioms)

1. 0, which we defined as the empty set $\emptyset$, is a natural number.

2. There exist a distinguished set map $\sigma : \mathbb{N} \to \mathbb{N}$

3. σ is injective

4. There does not exist an element $n \in \mathbb{N}$ such that $\sigma(n) = 0$

5. (Principle of Induction) If $S \in N$ is inductive, then $S = N$.

Proposition 102
Suppose that a is a natural number, and that $b \in a$. Then $b \subseteq a$, $a \nsubseteq b$.

Proposition 103
For any two natural numbers $a, b \in \mathbb{N}$, if $\sigma(a) = \sigma(b)$, then $a = b$.

Lemma 104
If $n \in \mathbb{N}$ and $n \neq 0$, then there exists $m \in \mathbb{N}$ such that $\sigma(m) = n$.

9.2 $\mathbb{Z}$: The set of Integers

9.2.1 Construction of $\mathbb{Z}$

We now have the set of natural numbers, and starting there, we construct the set of integers.

Proposition 105
Define a relation $\equiv$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \equiv (c, d)$ iff $a + d = b + c$. This relation is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

Let $\mathbb{Z}$ be the set of equivalence classes under this relation, and the equivalence class containing (a, b) be denoted by $[a, b]$.

9.2.2 Operations on $\mathbb{Z}$

Definition 106 (Addition and Multiplication on $\mathbb{Z}$)
Addition and multiplication on $\mathbb{Z}$ are defined by:

- $[a, b] + [c, d] = [a + c, b + d]$

- $[a, b] \cdot [c, d] = [ac + bd, ad + bc]$

Definition 107 (Subtraction on $\mathbb{Z}$)
Subtraction on $\mathbb{Z}$ is defined by:

$$[a, b] - [c, d] = [a, b] + [d, c]$$

9.2.3 Ordering on $\mathbb{Z}$

Definition 108 (Ordering on $\mathbb{Z}$)
Let $[a, b], [c, d] \in \mathbb{Z}$. $[a, b] < [c, d]$ iff $a + d < b + c$.

9.2.4 Property of $\mathbb{Z}$

Theorem 109 (Arithmetic Properties of $\mathbb{Z}$)

1. Addition and multiplication are well-defined.

2. Addition and multiplication have identity elements $[n, n]$ and $[n, n + 1]$, respectively.

3. Addition and multiplication are commutative and associative.

4. The distributive law holds.

5. Each element $[a, b]$ has an additive inverse $[b, a]$.

We can treat $\mathbb{N}$ to be a subset of $\mathbb{Z}$ by identifying the number n with the class $[0, n]$. Since $[0, a] + [0, b] = [0, a + b]$ and $[0, a] \cdot [0, b] = [0, ab]$, these operations mirror the corresponding operation in $\mathbb{N}$.

Given $n \in \mathbb{N}$, we write $-n$ for $[n, 0]$, 0 for $[n, n]$, and 1 for $[n, n+1]$. By the fifth arithmetic property of $\mathbb{Z}$[109], this defines $-n$ to be the additive inverse of n. We also use the minus sign for subtraction; it is therefore natural to write $[a, b]$ as $b - a$.

Proposition 110
For $a, b \in \mathbb{N}$, let $-b$, a, and b be defined in $\mathbb{Z}$ as above. Then

$$a - b = a + (-b) \text{ and } -(-b) = b$$

9.3 $\mathbb{Q}$: **The set of Rational Numbers**

We construct the set of rational numbers from the set of integers as follows:

9.3.1 Construction of $\mathbb{Q}$

Proposition 111
Define a relation $\equiv$ on $\mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$ by $(a, b) \equiv (c, d)$ iff $ad = bc$. This relation is an equivalence relation on $\mathbb{Z} \times (\mathbb{Z} \backslash \{0\})$.

Let $\mathbb{Q}$ be the set of equivalence classes under this relation, and the equivalence class containing (a, b) is denoted by a/b or $\frac{a}{b}$, and $\frac{a}{b} = \frac{c}{d}$ mean that (a, b) and (c, d) belong to the same equivalence class. Especially we write 0 and 1 to denote $\frac{0}{1}$ and $\frac{1}{1}$, respectively.

9.3.2 Operations on $\mathbb{Q}$

Definition 112 (Addition and Multiplication on $\mathbb{Q}$)
The <u>sum</u> and <u>product</u> of $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ are defined by

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \text{ and } \frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$$

Definition 113 (Subtraction on $\mathbb{Q}$)
Subtraction on $\mathbb{Z}$ is defined by:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

Definition 114 (Division on $\mathbb{Q}$)
Division on $\mathbb{Z}$ is defined by:

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

9.3.3 Ordering on $\mathbb{Q}$

Definition 115 (Ordering on $\mathbb{Q}$)
Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$. $\frac{a}{b} < \frac{c}{d}$ iff $(bd > 0 \wedge ad < bc) \vee (bd < 0 \wedge ad > bc)$.

9.3.4 Property of $\mathbb{Q}$

Theorem 116 (Arithmetic Properties of $\mathbb{Q}$)

1. Addition and multiplication are well-defined.

2. Addition and multiplication have identity elements 0 and 1, respectively.

3. Addition and multiplication are commutative and associative.

4. The distributive law holds.

Theorem 117
$(\mathbb{Q}, +, \cdot)$ forms an ordered field.

9.4 $\mathbb{R}$: **The set of Real Numbers**

9.4.1 Construction of $\mathbb{R}$

One simple way to construct $\mathbb{R}$ is by proving the following theorem:

Theorem 118 (Existence of $\mathbb{R}$)
There exists an ordered field $\mathbb{R}$ containing $\mathbb{Q}$ as a subfield which has the least-upper-bound property.

But where's the fun in that? We will be constructing the field of real numbers using Cauchy sequences[**??**], starting with the following proposition:

Theorem 119
Define a relation $\equiv$ on the set S of Cauchy sequences of rational numbers as follows:
$$\{a_n\} \equiv \{b_n\} \text{ iff } (a_n - b_n) \to 0$$

This relation is an equivalence relation.

Now let us define $\mathbb{R}$ as the set of equivalence classes of S under the relation $\equiv$.

9.4.2 Operations on $\mathbb{R}$

Before the definition of operations on $\mathbb{R}$, we need to find out whether if the Cauchy sequences of rational numbers are closed under addition and multiplication, and it turns out they do, as stated in the following proposition:

Proposition 120
The set S of Cauchy sequences of rational numbers is closed under addition, multiplication, and scalar multiplication, that is:

1. If $\{a_n\} \in S$ and $\{b_n\} \in S$, then $\{a_n + b_n\} \in S$

2. If $\{a_n\} \in S$ and $\{b_n\} \in S$, then $\{a_n b_n\} \in S$

3. If $\{a_n\} \in S$ and $c \in \mathbb{Q}$, then $\{c a_n\} \in S$

We can finally go on to defining the operations on $\mathbb{R}$.

Definition 121 (Addition and Multiplication on $\mathbb{R}$)
Let $\{a_n\}$ and $\{b_n\}$ be sequences contained in the real numbers α, β, respectively. Then the <u>sum</u> and <u>product</u> of α and β are defined by:

$$\alpha + \beta = \{a_n + b_n\} \text{ and } \alpha\beta = \{a_n b_n\}$$

We can define subtraction and division on $\mathbb{R}$ similar to addition and multiplication, by term-by-term calculation on each term of the Cauchy sequence.

9.4.3 Ordering on $\mathbb{R}$

Definition 122 (Ordering on $\mathbb{R}$)
Let $\alpha = \{a_n\}, \beta = \{b_n\} \in \mathbb{R}$. $\alpha < \beta$ iff $\exists N \in \mathbb{N}, \forall n \geq N, a_n < b_n$.

9.4.4 Property of $\mathbb{R}$

Theorem 123 (Arithmetic Properties of $\mathbb{R}$)

1. Addition and multiplication are well-defined.

2. Addition and multiplication have identity elements $\{0\}$ and $\{1\}$, respectively.

3. Addition and multiplication are commutative and associative.

4. The distributive law holds.

5. Each element $\{a_n\}$ has an additive inverse $\{-a_n\}$.

Theorem 124
$(\mathbb{R}, +, \cdot)$ forms an ordered field.

We now define an extension to $\mathbb{R}$ as follows:

Definition 125 (Extended Real Number System)
The extended real number system, denoted $\mathbb{R}^+$, $[-\infty, \infty]$, or $\mathbb{R} \cup \{-\infty, \infty\}$, consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define $\forall x \in \mathbb{R}$,

$$-\infty < x < \infty$$

Remark
The extended real number system does not form a field.

9.5 $\mathbb{C}$: The set of Complex Numbers

We construct the set of complex numbers from $\mathbb{R}$. Unlike the previous constructions, we do not construct it using equivalence class. Instead the construction is done by considering the quotient ring of polynomial ring over $\mathbb{R}$ modulo $i^2 + 1$.

Definition 126
Complex number is defined as the quotient ring $\mathbb{R}[i]/(i^2 + 1)$, with operations defined as normal.

Theorem 127
$(\mathbb{C}, +, \cdot)$ forms a field.

Part II

Advanced Topics

Chapter 10

Abstract Algebra

10.1 Group Basics

10.1.1 Groups

The first thing we would encounter in abstract algebra is a group... but you already encountered it. Refer to the chapter "Algebraic Structures" for the definition of a group and an abelian group.

If the context is obvious, we will skip $\cdot$ and write ab instead of $a \cdot b$. The identity of a group will be denoted e or 1.

G will always denote group in this chapter, unless stated otherwise.

Definition 128
The product of n occurrences of x is denoted x^n. The product of n occurrences of x^{-1} is denoted x^{-n}. Also $x^0 = 1$.

Proposition 129
If $a, b, c \in G$, then

1. the identity of G is unique.

2. the inverse a^{-1} is unique.

3. $(a^{-1})^{-1} = a$.

4. $(ab)^{-1} = b^{-1}a^{-1}$.

5. if $ab = ac$, then $b = c$. Also, if $ba = ca$, then $b = c$.

6. For $n, m \in \mathbb{Z}$, $x^n x^m = x^{n+m}$ and $(x^n)^{-1} = x^{-n}$.

Proof.

1. Let e_1 and e_2 be the identities of G. Then $e_1 e_2 = e_2 e_1 = e_1$, and $e_2 e_1 = e_1 e_2 = e_2$, from the definition of the identity. Therefore $e_1 = e_2$.

2. Let b_1 and b_2 the inverses of a. Then $b_1 = b_1(ab_2) = (b_1 a)b_2 = b_2$.

3. The definition of an inverse shows that a is an inverse of a^{-1}. From (ii), such an inverse is unique.

4. $(ab)b^{-1}a^{-1} = a(bb^{-1})a^{-1} = aa^{-1} = e$. Similarly $b^{-1}a^{-1}(ab) = e$. From (ii), the inverse of ab is unique.

5. $ab = ac \implies a^{-1}ab = a^{-1}ac \implies b = c$. Similar argument for $ba = ca$.

6. TODO

$\square$

Definition 130

- $a, b \in G$ <u>commute</u> if $ab = ba$.

- The <u>order</u> of $x \in G$, denoted $|x|$, is the smallest positive integer n such that $x^n = 1$. If no such n exists, then $|x| = \infty$.

- The <u>order</u> of G, denoted $|G|$, is the cardinality of G as a set.

Definition 131

- TODO: Z/nZ

- TODO: Sn

10.1.2 Isomorphism

Now, we want to tell whether two groups are "same," in the sense that there is a bijection between them preserving the relations.

Definition 132 (Homomorphisms and Isomorphisms)
Let $(G, \star)$, $(H, \diamond)$ be two groups. Then a map $\varphi : G \to H$ is a <u>homomorphism</u> if for all $x, y \in G$, $\varphi(x \star y) = \varphi(x) \diamond \varphi(y)$. An <u>isomorphism</u> is a bijective homomorphism. If there is an isomorphism between G and H, we say they are <u>isomorphic</u> and denote $G \cong H$.

We may skip $\star$ and $\diamond$ here too if the context is clear, but you have to understand which operations are used at each positions.

Theorem 133
If $(G, \star)$ and $(H, \diamond)$ are isomorphic, with the isomorphism φ, then

1. $|G| = |H|$.

2. G is abelian iff H is abelian.

3. For any $x \in G$, $|x| = |\varphi(x)|$.

Proof. (1) φ is bijective.
   (2) Suppose G is abelian. Take $c, d \in H$. Since φ is surjective, there are $a, b \in G$ such that $\varphi(a) = c$ and $\varphi(b) = d$. Then $cd = \varphi(a)\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(b)\varphi(a) = dc$. Therefore H is abelian.
   Suppose H is abelian. Take $a, b \in G$. Then $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a) = \varphi(ba)$. Since φ is injective, $ab = ba$. Therefore G is abelian.
   (3) TODO

$\square$

10.1.3 Group Actions

Definition 134 (Group Action)
A <u>group action</u> of G on a set A is a map $G \times A \to A$, mapping $g \times a$ to $g \cdot a$, such that for all $g_1, g_2 \in G$ and $a \in A$,

1. $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, and

2. $1 \cdot a = a$.

We say that G <u>acts</u> on A.

Again, we may skip $\cdot$.
TODO: permutation representation

10.1.4 Subgroups

Definition 135 (Subgroups)
A subset H of G is a subgroup of G if H is nonempty, and for all $x, y \in H$, we have $xy \in H$ and $x^{-1} \in H$. We denote $H \leq G$.

A subgroup is also a group. To see why, there is an element x in H since H is nonempty, then $x^{-1} \in H$, and finally $xx^{-1} = 1 \in H$.

Proposition 136 (The Subgroup Criterion)
A subset H of G is a subgroup if and only if H is nonempty and for all $x, y \in H$, $xy^{-1} \in H$. Also, a finite subset H is a subgroup if and only if H is nonempty and closed under multiplication.

Proof. ($\Rightarrow$) Trivial.
($\Leftarrow$) Since H is nonempty, take an element $x \in H$. Then $xx^{-1} = 1 \in H$. This gives $1x^{-1} = x^{-1} \in H$. Finally, $x(y^{-1})^{-1} = xy \in H$ since $y^{-1} \in H$.
If H is finite, then $|x| = n < \infty$. Therefore $x^{-1} = x^{n-1}$. Now we can use the first part of this proposition. $\square$

10.1.5 Cyclic Groups

Definition 137 (Cyclic Group)
G is cyclic if $G = \{x^n | n \in \mathbb{Z}\}$ for some $x \in G$. For such x, we denote $G = \langle x \rangle$ and say x is a generator of G.

Proposition 138
$|\langle x \rangle| = |x|$ in $\langle x \rangle$. (These values can be infinite.)

Proof. TODO. Isn't it, like, trivial?? Right?? $\square$

Theorem 139
Any two cyclic groups with the same order are isomorphic.

Proof. Suppose $|\langle x \rangle| = |\langle y \rangle| = n < \infty$. We will show that $\varphi : \langle x \rangle \to \langle y \rangle$ defined by $\varphi(x^k) = y^k$ is well-defined and an isomorphism.
Suppose $x^a = x^b$. Then $x^{a-b} = 1$, and so $n|(a-b)$. Therefore $\varphi(x^a) = y^a = y^b = \varphi(x^b)$. This shows that φ is well-defined.
Next, φ is clearly a homomorphism and a surjection. Since the two groups are finite, φ is a bijection. Therefore φ is an isomorphism.
Next, if $|\langle x \rangle| = \infty$, then $\varphi : \mathbb{Z} \to \langle x \rangle$ defined by $\varphi(n) = x^n$ is an isomorphism. $\square$

Chapter 11

Topology

11.1 Topological Space

11.1.1 Topological Space

In analysis, we've dealt with functions in metric spaces and their proper-
ties. What we will do in this chapter is extend this notion to the spaces
without metrics. But without metrics, our definition of open sets no longer
makes sense. We need a new definition.

Remember the theorem [**??**] stating that a union of open sets is open, and
a finite intersection of open sets is also open? Well...

Definition 140 (Topological Space)
A <u>topological space</u> is a set X together with a collection $\mathcal{T}$ of subsetes of
X such that

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

2. A union of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

3. A finite intersection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

$\mathcal{T}$ is a <u>topology</u> on X, and the sets in T are called <u>open</u> sets. The
complement of an open set is a <u>closed</u> set.

Definition 141

- Given a set X, the power set $\mathcal{P}(X)$ is the <u>discrete topology</u>. This space
  is called the <u>discrete space</u>. The set $\{\emptyset, X\}$ is the <u>indiscrete topology</u>.

- A subset is <u>cofinite</u>, and <u>cocountable</u>, if its complement is finite, and
  countable, respectively. The set of $\emptyset$, X, and all cofinite subsets
  of X, together forms the <u>cofinite topology</u>. Replacing cofinite with
  cocountable, we get the <u>cocountable topology</u>.

From now on, we will assume X and Y are topological spaces, unless stated
otherwise.

Definition 142
Let $A \subseteq X$.

- A point $x \in A$ is an <u>interior point</u> of A if some open neighborhood of x
  is contained in A. The set of all interior points of A is the <u>interior</u>
  of A, denoted $int(A)$.

- A point $x \in X$ is an <u>adherent point</u> of A if every open neighborhood of x intersects A. The set of all adherent points of A is the <u>closure</u> of A, denoted $\bar{A}$.

- The <u>boundary</u> of A is $\partial A = \bar{A} \cap \overline{(X \backslash A)}$.

- $x \in X$ is a <u>limit point</u> of A if every open neighborhood of x contains at least one point in A different from x. The set of all limit points of A is denoted A'.

- $x \in A$ is an <u>isolated point</u> of A if some open neighborhood of x does not contain any point in A different from x. The set of all isolated points of A is denoted $A^{\cdot}$.

11.1.2 Base

11.1.3 Continuity and Convergence

11.1.4 Subspaces

11.2 Connected Spaces

11.2.1 Connectedness

11.2.2 Total Disconnectedness

11.2.3 Path Connectedness

11.3 Separation Axioms

11.4 Countability Axioms

11.5 Compact Spaces

11.5.1 Compactness

11.5.2 Other Types of Compactness

11.5.3 Boundedness

11.6 Metrization

11.7 Sequence of Functions

11.8 Paracompact Spaces

Part III

Applications to Computer Science

Chapter 12

Language Theory

Automaton is defined as a machine or control mechanism designed to automatically follow a predetermined sequence of operations, or respond to predetermined instructions. Theoretically, they all can be considered as the simplest form of algorithm, whether it is finite state automaton, push down automaton, or Turing machine. They all accept an input, and produce output; usually the output is *accept* or *reject*, but in the case of Turing machines, the output may be something different.

Before we start talking about the machines however we need to define what "Language" is.

Definition 143 (Language)
A (formal) language L over an alphabet Σ is a subset of Σ^*, that is, a set of words over that alphabet.

In this section, we explore Regular Language[145], Context-free Language[151], Decidable Language[158], and Recognizable Language[159] and the mechanisms, or machines, that are related those languages.

12.1 Regular Language

12.1.1 Regular Expression

If you have studied regular expression using some programming languages, then you might have easier time understanding the following definition. The regular expression used in real life is much more powerful than the regular expression mentioned below, as more special characters and syntaxes are allowed. However the following regular expression consists of the "basics" of regular expression, and is used in language theories as the regular expression:

Definition 144 (Regular Expression(RE))
Given a finite alphabet Σ, the following constants are defined as regular expressions:

- **Empty set**: $\emptyset$, denoting the set $\emptyset$.

- **Empty string**: ϵ, denoting the set containing only the "empty" string, which has no characters at all.

- **Literal character**: $a \in \Sigma$, denoting the only character a.

And when given regular expressions R and S, the following operations over them produce regular expressions:

- **Concatenation**: RS, denoting the concatenation of strings in R and S, in that order.

  R^n denotes the concatenation of R, n times: Specifically, $R^0 = \{\epsilon\}$.

- **Alternation**: $R|S$, denoting the set union of the strings in R and S.

- **Kleene star**: R^*, denoting $\bigcup_{i \in \mathbb{N}} R^i$.

Definition 145 (Regular Languages)
<u>Regular Languages</u> are languages that can be represented with regular expressions.

Theorem 146 (Pumping Lemma for Regular Languages)
Let L be a regular language. Then, there exists an integer $p \geq 1$, depending only on L, such that every string $w \in L$ of length at least p, called the <u>pumping length</u>, can be written as $w = xyz$ (i.e. w can be divided into three substrings), satisfying the following conditions:

- $|y| \geq 1$

- $|xy| \leq p$

- $\forall n \geq 0, xy^n z \in L$

12.1.2 Deterministic Finite State Automaton

Definition 147 (Deterministic Finite Automaton(DFA))
A DFA M is a 5-tuple, $(Q, \Sigma, \delta, q_0, F)$, consisting of:

- Finite set of states Q;

- Finite set of input symbols called the alphabet Σ;

- Transition function $\delta : Q \times \Sigma \to Q$;

- Initial state $q_0 \in Q$;

- Set of accepting states $F \subseteq Q$.

Let $w = a_1 a_2 \ldots a_n$ be a string over the alphabet Σ. DFA M accepts the string w if a sequence of states, $r_0, r_1, \ldots, r_n \in Q$ exists with the following conditions:

- $r_0 = q_0$

- $r_{i+1} = \delta(r_i, a_{i+1}), i = 0, \ldots, n-1$

- $r_n \in F$

Theorem 148
DFAs recognize exactly the set of regular languages.

12.1.3 Nondeterministic Finite Automaton

Definition 149 (Nondeterministic Finite Automaton(NFA))
A NFA M is a 5-tuple, $(Q, \Sigma, \Delta, q_0, F)$, consisting of:

- Finite set of states Q;

- Finite set of input symbols called the alphabet Σ;

- Transition function $\Delta : Q \times \Sigma \to P(Q)$ where P is the powerset function;

- Initial state $q_0 \in Q$;

- Set of accepting states $F \subseteq Q$.

Sometimes the transition function Δ is represented as the transition relation, $\Delta \subseteq Q \times \Sigma \times Q$.

Let $w = a_1 a_2 \ldots a_m$ be a string over the alphabet Σ, where $a_i \in \Sigma_\epsilon$. NFA M accepts the string w if a sequence of states, $r_0, r_1, \ldots, r_n \in Q$ exists with the following conditions:

- $r_0 = q_0$

- $r_{i+1} \in \Delta(r_i, a_{i+1})$, or in relation form, $(r_i, a_{i+1}, r_{i+1}) \in \Delta$, $i = 0, \ldots, n-1$

- $r_n \in F$

Theorem 150
NFAs recognize exactly the set of regular languages.

12.2 Context-Free Language

12.2.1 Context-free Grammar

Definition 151 (Context-Free Grammar(CFG))
A CFL is a 4-tuple (V, Σ, R, S) where:

- V is the set of nonterminal variables;

- Σ is the set of terminal characters;

- R is the set of rules, where each rules are in the form of $A \to w, A \in V, w \in (\Sigma \cup V)^*$

- S is the starting variable.

Definition 152 (Context-free Languages)
Context-free Languages are languages that can be represented with context-free grammars.

Theorem 153 (Pumping Lemma for Context-free Languages)
Let L be a regular language. Then, there exists an integer $p \geq 1$, depending only on L, such that every string $s \in L$ of length at least p, called the pumping length, can be written as $s = uvwxy$(i.e. w can be divided into five substrings), satisfying the following conditions:

- $|vx| \geq 1$

- $|vwx| \leq p$

- $\forall n \geq 0, uv^n wx^n y \in L$

12.2.2 Push-down Automaton

Similar to Finite Automatons, Push-down automatons have deterministic version and nondeterministic version; Only the nondeterministic version is shown here as similar method can be used to convert it into a deterministic version.

Definition 154 (Push-down Automaton(PDA))
A PDA is a 6-tuple $(Q, \Sigma, \Gamma, q_0, \Delta, F)$ where:

- Q is the set of states;

- Σ is the set of input alphabet;

- Γ is the set of stack alphabet;

- q_0 is the starting state;

- Δ is the transition relation of $Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times Q \times \Gamma_\epsilon$

- F is the set of accepting states

Δ is often written as a transition function of $Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times \to P(Q \times \Gamma_\epsilon)$ where P is the powerset function.

Sometimes the last element of the relation is extended to Γ_ϵ^*; in that case, when inserting into the stack, insert the last element first. Let $w = w_1 w_2 \ldots w_m$ be a string over the alphabet Σ, where $w_i \in \Sigma_\epsilon$. NFA M accepts the string w if a sequence of states, $r_0, r_1, \ldots, r_n \in Q$, and a sequence of stack strings $s_0, s_1, \ldots, s_n \in \Gamma^*$ exists with the following conditions:

- $r_0 = q_0$, $s_0 = \epsilon$

- $(r_i, w_{i+1}, a, r_{i+1}, b) \in \Delta$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\epsilon$, and $t \in \Gamma^*$.

  If $b = \epsilon$, then it is a pop-operation. If $a = \epsilon$, then it is a push-operation.

- $r_m \in F$, $s_m = \epsilon$

Theorem 155
PDAs recognize exactly the set of CFLs.

Proof. The proof is quite tedious; so only a partial proof is given: we are going to convert any given CFG into PDA. Suppose a CFG $(V_0, \Sigma_0, R_0, S_0)$ is given.

We can construct a new PDA $(Q, \Sigma, \Gamma, q_0, \Delta, F)$ from the given CFL s.t.

- $Q = \{Q_S, Q_M, Q_F\}$

- $\Sigma = \Sigma_0$

- $\Gamma = V_0 \cup \Sigma_0$

- $q_0 = Q_S$

- $F = Q_F$

- $\Delta = $
  $\{(Q_S, \epsilon, \epsilon, Q_M, S\$)\} \cup$
  $\{(Q_M, \epsilon, \epsilon, X, Q_M, W) | X-> W \in R\} \cup$
  $\{(Q_M, a, a, Q_M, \epsilon) | a \in \Sigma_0\} \cup$
  $\{(Q_M, \epsilon, \$, Q_F, \epsilon)\}$

This exactly simulates the parse tree of the CFL. $\qquad\square$

12.3 Turing Machines

Definition 156 (Turing Machine)
A Turing machine consists of:

- A <u>tape</u> divided into consecutive cells. Each cell contains a symbol from the tape alphabet, which contains a blank symbol and one or more other symbols. The tape is assumed to be infinitely long to the left; cells that have not been written before are assumed to be filled with the blank symbol.

- A <u>head</u> which can read a single symbol on the tape at a time, and is able to move one(and only one at once) cell to the right or the left.

- A <u>state register</u> which stores the state of the TM, starting from the starting state(defined below) and following the transition function's rule(also defined below).

Formally, a TM is a 7 tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where:

- Q is the set of states;

- Γ is the set of tape alphabet;

- $b \in \Gamma$ is the blank symbol, the only symbol allowed to occur infinitely often at any step of the computation;

- $\Sigma \subseteq \Gamma \backslash \{b\}$ is the set of input symbols, that is, the set of symbols allowed to appear in the initial tape contents;

- $q_0 \in Q$ is the starting state;

- $F \subseteq Q$ is the set of accepting states, and the initial tape contents is said to be accepted by M if it eventually halts in a state from F;

- δ is a partial function called the transition function of $(Q \backslash F) \times \Gamma \to Q \times \Gamma \times \{L, R\}$, where L and R signifies left and right shifts of the tape. If δ is undefined on the current state and the current tape symbol, then the machine halts.

Using the components of TM and the formal definition, the Turing machine accepts iff it halts on the set of accepting states, and it rejects iff it halts on the set of rejecting states. It may loop infinitely, of which it neither accepts nor rejects the tape.

   The definition of a Turing Machine is not unique. Some definitions use multiple tapes, using one of them as the input tape that can't be modified and another as the output tape. Some has more than one halting states. Some include N in the final output of the transition function, indicating no movement of the head. But in general, a Turing machine starts from one state, follows the decision function every step, and halts at the halting state. Some of the many variations on the Turing machine are mentioned in 12.5.2.
   In fact, the different definitions of a Turing machine turns out to be the same, in the sense that a function $f : \{0,1\}^* \to \{0,1\}$ is computable using one definition of a Turing machine iff it is computable using another definition of a Turing Machine.
   We now give the following thesis from the creator of the λ-calculus, Alonzo Church and Alan Turing.

Thesis 157 (Church-Turing Thesis)
A function on Natural Numbers which is computable by a human being following
an algorithm, ignoring resource limitations, if and only if it is computable
by a Turing Machine.

12.4 Decidable and Recognizable Languages

Definition 158 (Decidable Languages)
Decidable Languages are languages that can be represented with decidable
Turing machines; that is, the set of Turing machines that always accepts
accepting words and rejects others.

Definition 159 (Recognizable Languages)
Recognizable Languages are languages that can be represented with recog-
nizable Turing machines; that is, the set of Turing machines that always
accepts accepting words.

Decidable and Recognizable Turing Machines seem similar; however rec-
ognizable machines does not have to reject a non-accepting word; it may
instead loop infinitely.

12.5 Equivalences to Turing Machine

The followings can be shown to be computationally equivalent to a Turing
machine; however no proofs are given since they are usually long and arduous.

12.5.1 Push-down Automaton with Two Stacks

The simplest version that is equivalent to a Turing Machine would be a PDA
which has two stacks. The two stacks can simulate the tape of the Turing
machine by pushing and popping.

12.5.2 Variations on the Turing Machine

The following variations on the Turing machine are equivalent to the original
Turing machine:

- Variations on the Definition

    - Allowing N, "no shift", in the movement rules;
    - Having a single accepting state, say q_{accept} and a single rejecting
      state, say q_{reject}, and forcing the transition function δ to be a
      function. In this variant, the machine accepts iff it ends in
      q_{accept}, and rejects iff it ends in q_{reject}.

- Variations on the Form of the Machine

    - Tape is infinite only in one direction;
    - Tape is infinite in both directions;
    - Tape is 2-dimensional;
    - There exists multiple tapes that the machine can access concurrently.

There are many more variations other than these.

12.5.3 General Recursive Functions

Definition 160 (General Recursive Functions)
General Recursive Functions, otherwise known as μ-recursive functions, is a set of functions $\forall n \in \mathbb{N}, f : \mathbb{N}^n \to \mathbb{N}$ that includes the three "Initial", or "Basic" functions, and closed under three operators:

- Initial Functions

    - **Constant Function:** $\forall n, k \in \mathbb{N}, f(x_1, \ldots, x_k) = n$
        Alternative definition use a Zero function: $\forall k \in \mathbb{N}, Z(x_1, \ldots, x_k) = 0$
    - **Successor Function** S: $S(x) = x + 1$
    - **Projection Function** P_i^k:
        This is also called the Identity Function I_i^k

- Operators

    - **Composition Operator** $\circ$: Given an m-ary function $h(x_1, \ldots, x_m)$ and m k-ary functions $g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)$:

        $$h \circ (g_1, \ldots, g_m) = f \text{ where } f(x_1, \ldots, x_k) = h(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k))$$

        This is also called the Substitution Operator.

    - **Primitive Recursion Operator** ρ: Given the k-ary function $g(x_1, \ldots, x_k)$ and (k+2)-ary function $h(y, z, x_1, \ldots, x_k)$:

        $$\rho(g, h) = f \text{ where}$$
        $$f(0, x_1, \ldots, x_k) = g(x_1, \ldots, x_k)$$
        $$f(y + 1, x_1, \ldots, x_k) = h(y, f(y, x_1, \ldots, x_k), x_1, \ldots, x_k)$$

    - **Minimization Operator** μ: Given a (k+1)-ary total function $f(y, x_1, \ldots, x_k)$:

        $$\mu(f)(x_1, \ldots, x_k) = z \Leftrightarrow f(z, x_1, \ldots, x_k) = 0 \text{ and}$$
        $$f(i, x_1, \ldots, x_k) > 0 \text{ for } i = 0, \ldots, z - 1$$

        Intuitively, this operator seeks the smallest argument that causes the function to return 0; if none exists, the search never ends and therefore cannot return.

12.5.4 Lambda Calculus

Lambda Calculus, first defined by Alonzo Church, is a formal system of mathematical logic for expressing computation based on function-like objects.

Definition 161 (Lambda Expression)
Lambda expressions are composed of:

- Variables, $v_1, \ldots, v_n, \ldots$

- The abstraction symbols lambda λ and dot .

- Parentheses ()

For some applications, terms for logical and mathematical constants and operation may be included.

The set of lambda expressions, Λ, can be defined inductively:

- If x is a variable, then $x \in \Lambda$

- If x is a variable and $M \in \Lambda$, then $(\lambda x.M) \in \Lambda$

    This rule is also known as Abstractions.

- If $M, N \in \Lambda$, then $(MN) \in \Lambda$

    This rule is also known as Application.

Though only the definition is given, This Wikipedia article can be helpful to understand how lambda calculus works.

Chapter 13

Theory of Computation

13.1 Computability

Turing machine was already defined in [156], but let's write down the definition here for convenience:

Definition 162 (Turing Machine)
A Turing machine consists of:

- A <u>tape</u> divided into consecutive cells. Each cell contains a symbol from the tape alphabet, which contains a blank symbol and one or more other symbols. The tape is assumed to be infinitely long to the left; cells that have not been written before are assumed to be filled with the blank symbol.

- A <u>head</u> which can read a single symbol on the tape at a time, and is able to move one(and only one at once) cell to the right or the left.

- A <u>state register</u> which stores the state of the TM, starting from the starting state(defined below) and following the transition function's rule(also defined below).

Formally, a TM is a 7 tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where:

- Q is the set of states;

- Γ is the set of tape alphabet;

- $b \in \Gamma$ is the blank symbol, the only symbol allowed to occur infinitely often at any step of the computation;

- $\Sigma \subseteq \Gamma \backslash \{b\}$ is the set of input symbols, that is, the set of symbols allowed to appear in the initial tape contents;

- $q_0 \in Q$ is the starting state;

- $F \subseteq Q$ is the set of accepting states, and the initial tape contents is said to be accepted by M if it eventually halts in a state from F;

- δ is a partial function called the transition function of $(Q \backslash F) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$, where L and R signifies left and right shifts of the tape. If δ is undefined on the current state and the current tape symbol, then the machine halts.

Using the components of TM and the formal definition, the Turing machine accepts iff it halts on the set of accepting states, and it rejects iff it halts on the set of rejecting states. It may loop infinitely, of which it neither accepts nor rejects the tape.

We will abbreviate "Turing Machine" as TM, or DTM when necessary.

Usually, the proofs involving TMs do not give a formal construction of the machines because it is an extremely tedious process. Instead the proof describes what the machine does. It would be intuitive to see that such a machine can indeed be constructed.

Definition 163 (Decision Problem)

- A TM M <u>runs</u> in time $T(n)$ if it halts in at most $T(n)$ steps for every input with length n.

- A <u>decision problem</u> is a subset of the set of natural numbers $\mathbb{N}$. We assume $0 \in \mathbb{N}$.

- An <u>input</u> is a natural number that will be written on the initial tape in binary. The <u>length</u> of the input is the number of cells required to represent it, which is $|x| = \lceil \log_2(x+1) \rceil$.

- A TM M <u>accepts</u> an input x if it accepts with the given input x. Similarly, M <u>rejects</u> x if it rejects with the given input x.

- A TM M <u>decides</u> a decision problem L if, for all $x \in \mathbb{N}$, M accepts x if $x \in L$ and M rejects x otherwise. In that case, L is <u>decidable</u>.

- A TM M <u>semi-decides</u> a decision problem L if, for all $x \in \mathbb{N}$, M accepts x if $x \in L$ and M runs infinitely otherwise. In that case, L is <u>semi-decidable</u>.

What if we want other types of inputs such as two natural numbers, rational numbers, ASCII strings, graphs, and so on? In that case, we can <u>encode</u> them as natural numbers. For example, there are some easy-to-compute injection from $\mathbb{N}^2$ to $\mathbb{N}$, and we assume they are given as the encoded forms. But since there is no injection from $\mathbb{R}$ to $\mathbb{N}$, we cannot give real numbers as inputs.

Note also that since a TM itself is finite, the set of all TMs is countable. Therefore TMs can also be encoded, and even be given as an input to other TMs! Yo dawg, I heard you like TMs...

Now, a natural question is whether undecidable problems exist at all. The answer is yes, because there are countably many Turing machines but uncountably many subsets of $\mathbb{N}$. We will also define one of the most important undecidable problems:

Definition 164 (Halting Problem)
The <u>halting problem</u> **HALT** is the set of pairs $(M, x) \in \mathbb{N}$ such that a TM M halts on input x. (Remind that (M, x) is encoded into a single natural number.)

Theorem 165
HALT is undecidable.

Proof. Suppose there is a machine N that decides **HALT**. Construct another machine N' that does the following: given input x, simulate N on input (x, x). If N accepts it, run an infinite loop. Otherwise, accept.

Now consider what happens with N' is given the input N'. If N' accepts itself, then it means N rejects the input (N', N'), i.e. N' on input N' never halts. But this is a contradiction to the assumption that N' accepts

itself. On the other hand, if N' runs an infinite loop, then it means N accepts the input (N', N'), i.e. N' on input N' eventually halts. This is again a contradiction. Therefore no such N exists. $\square$

However, it should be noted that **HALT** is semi-decidable, since we can just simulate M on the input x and accept if M halts. If M never halts, then our simulation would not halt either.

13.2 Nondeterministic Turing Machine

Definition 166 (Nondeterministic Turing Machine)
A <u>nondeterministic Turing Machine</u> (NDTM) is the same as a DTM except that δ is a relation instead of a partial function. The next state of the state $(q, c) \in (Q \backslash F) \times \Gamma$ can be any of $(q', c', d') \in Q \times \Gamma \times \{L, R\}$ such that $(q, c)\delta(q', c', d')$. An NDTM <u>accepts</u> an input x if it always halts and it accepts x in at least one sequence of execution. An NDTM <u>rejects</u> x if it always rejects x.

Although an NDTM looks stronger than a DTM, it is actually possible to simulate an NDTM using a DTM. To check whether an NDTM accepts x, generate all execution sequences of the machine and check whether one of them leads to acceptance. However, the power of an NDTM is that by randomly "guessing" the next state of execution, it can sometimes easily compute what would take exponential time for a DTM, in polynomial time. Details will be introduced soon.

13.3 Relations Between Decidabilities

Theorem 167

1. A finite problem L is decidable.

2. A problem L is decidable iff its complement is decidable.

3. A problem L is decidable iff L and its complement are both semi-decidable.

Proof. (1) Let $L = \{x_1, \cdots, x_k\}$. Construct a TM that takes an input x and decides whether $(x = x_1) \vee (x = x_2) \vee \cdots \vee (x = x_k)$. Even if we don't know the contents of L, it is still true that such a TM exists.

(2) Suppose there is a TM M that decides the complement of L. Construct another TM that takes an input x, simulates M on x, then accepts iff M rejects. The other direction is the same.

(3) If L is decidable, then clearly L is semi-decidable. From (2), the complement of L is decidable, so it is semi-decidable. Conversely, let M be a TM that decides L and M' be a TM that decides the complement of L. Construct another TM that takes an input x, and simulates each step of M and M' on x one by one. If M halts, then accept. If M' halts, then reject. $\square$

13.4 Computational Complexity

(TODO: Write something about asymptotic notation here)

Definition 168 (Asymptotic notation)
Let f and g be two functions from $\mathbb{N}$ to $\mathbb{N}$. Then we say:

- $f = O(g)$ if there is a constant c such that $f(n) \leq c \cdot g(n)$ for every sufficiently large n. That is, $n > N$ for some N.

- $f = \Omega(g)$ if $g = O(f)$.

- $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

- $f = o(g)$ if for every constant $c > 0$, $f(n) < c \cdot g(n)$ for every sufficiently large n.

- $f = \omega(g)$ if $g = o(f)$.

Definition 169 (P, NP, EXP)

- **P** is the set of boolean functions computable with a deterministic Turing machine in time $O(n^c)$ for some constant $c > 0$.

- **NP** is the set of boolean functions computable with a non-deterministic Turing machine in time $O(n^c)$ for some constant $c > 0$.

- **EXP** is the set of boolean functions computable with a deterministic Turing machine in time $O(2^{n^c})$ for some constant $c > 0$.

Theorem 170
P $\subseteq$ **NP** $\subseteq$ **EXP**.

Proof. A DTM is automatically an NDTM, so **P** $\subseteq$ **NP**. To show **NP** $\subseteq$ **EXP**, let M be an NDTM that runs in time $p(n)$ where p is a polynomial. Then since there are at most $2^{p(n)}$ execution sequences of M, we can simulate all executions in exponential time. Accept the input x iff M accepts for at least one execution sequence. $\square$

13.5 Reduction

Is there a polynomial-time algorithm for a given decision problem? Computer scientists are interested in this question because if there is one, it is usually a small-degree polynomial like $O(n^2)$ or $O(n^5)$. Some problems have a special property that if the problem has a polynomial-time algorithm, then several other problems do.

Definition 171 (Polynomial-time Karp reduction)
A problem $A \subseteq \{0,1\}^*$ is <u>polynomial-time Karp reducible</u> to $B \subseteq \{0,1\}^*$, denoted $A \leq_p B$, if there is a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$, $x \in A$ iff $f(x) \in B$.

The intuitive meaning is that a problem of A can be "reduced" to a problem of B, and if we can solve B in polynomial-time, then we can solve A in polynomial-time too.

Definition 172 (NP-complete)
A problem A is <u>NP-hard</u> if every problem in **NP** is polynomial-time reducible to A, and <u>NP-complete</u> if A is NP-hard and NP.

Theorem 173

1. If $A \leq_p B$ and $B \leq_p C$, then $A \leq_p C$.

2. An NP-complete problem A is in **P** iff **P** = **NP**.

3. If $A \leq_p B$ and A is NP-hard, then B is NP-hard.

Proof. (1) Let f be a reduction from A to B with polynomial time $p(n)$, and g from B to C with $q(n)$. Then $g \circ f$ is a reduction from A to C with polynomial time $q(p(n))$.

(2) Suppose A is NP-complete and in **P**. Then any problem B in **NP** can be polynomial-time reduced to A, so transitivity implies that B is polynomial-time computable. The converse is trivial.

(3) Any problem C in **NP** can be polynomial-time reduced to A. Transitivity implies that C can be polynomial-time reduced to B. □

Now the obvious question is, does such a strong problem actually exist? The answer is yes, and a lot of important problems are NP-complete.

(TODO: SAT)

Having proven that SAT is NP-hard, more problems can be proven NP-hard if we can reduce SAT to those problems in polynomial-time. Here are only a tiny fraction of the NP-complete problems:

Definition 174 (NP-complete problems)

- The <u>3-SAT problem</u> is a SAT problem where each clause contains exactly 3 variables.

- Given a graph G and an integer $0 \le k \le |V(G)|$, the <u>clique problem</u> asks whether there is a complete induced subgraph of G with size at least k.

- The <u>independent set problem</u> asks whether there is a subset S of $V(G)$ with size at least k such that no two vertices in S are adjacent, and 0 otherwise.

- The <u>vertex cover problem</u> asks whether there is a subset S of $V(G)$ with size at most k such that each edge is adjacent to at least one vertex in S.

- The <u>chromatic number problem</u> asks whether G is 3-colorable.

- Given a set S of integers and an integer k, the <u>subset sum problem</u> asks whether there is a subset of S whose sum of elements equals k.

- Given an $n \times m$ matrix A and an $n \times 1$ matrix b of integers, the <u>integer programming problem</u> asks whether there is an $m \times 1$ matrix x of integers such that each element of $Ax + b$ is non-negative.

Theorem 175
All problems in [174] are NP-complete.

Proof. Clearly all problems described are NP. We will only show that they are all NP-hard.

If we can reduce SAT to 3-SAT in polynomial time, then [173] will show that 3-SAT is NP-hard. To do this, note that

- x is equivalent to $x \vee x \vee x$,

- $x_1 \vee x_2$ is equivalent to $x_1 \vee x_2 \vee x_2$,

- $x_1 \vee \cdots \vee x_n$ is equivalent to $(x_1 \vee x_2 \vee y_1) \wedge (\neg y_1 \vee x_3 \vee y_2) \wedge \cdots (\neg y_{n-4} \vee x_{n-2} \vee y_{n-3}) \wedge (\neg y_{n-3} \vee x_{n-1} \vee x_n)$, where $n \ge 4$ and $y_1, \cdots, y_{n-3}$ are new variables unused in the original SAT formula.

Next, we reduce 3-SAT to a clique problem. (TODO)

G has a clique of size k iff $\bar{G}$ has an independent set of size k. This shows that clique and independent set are polynomial-time reducible to each other.

G has an independent set of size k iff G has a vertex cover of size $|V(G)|-k$, by taking the complement of the independent set. Therefore independent set and vertex cover are polynomial-time reducible to each other.

We reduce 3-SAT to a chromatic number problem. (TODO)

We reduce 3-SAT to a subset sum problem. (TODO)

Finally, we reduce 3-SAT to an integer programming problem. Given a 3-SAT formula with n variables, set $0 \leq x_i \leq 1$ for $i = 1, \cdots, n$, and convert the clause $(x_a \vee x_b \vee x_c)$ into $x_a + x_b + x_c \geq 1$. If the clause contains $\neg x_a$, convert it to $1 - x_a$. This system of inequalities can easily be converted to the matrix form.

$\square$

Chapter 14

Graph Theory

14.1 Basic Graph Definitions

Definition 176 (Graph)
A <u>graph</u> G is represented by a pair of sets $(V(G), E(G))$, and a relation $\sim_G \subseteq V(G) \times E(G)$ such that for each $e \in E(G)$, there are exactly one or two $v \in V(G)$ such that $v \sim_G e$. An element of $V(G)$ is a <u>vertex</u>, and an element of $E(G)$ is an <u>edge</u>. If $v \sim_G e$, we say v is <u>incident</u> with e, and v is an <u>end</u> of e.
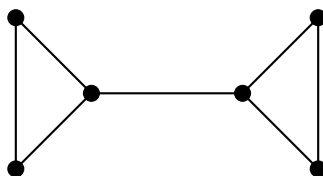

Figure 14.1: A graph with 6 vertices and 7 edges.

From now on, we will skip (G) and just write V and E if the context is obvious. Similarly we will skip $_G$ and just write $\sim$. Also, for simple graphs, we may write an edge as vw where v and w are the ends of the edge.

Definition 177

- A vertex v is <u>adjacent</u> to another vertex w if there is an edge e such that $v \sim e$ and $w \sim e$. We also say that v is a <u>neighbor</u> of w.

- A <u>loop</u> is an edge with exactly one end.

- Two edges e_1 and e_2 are <u>parallel</u> if $e_1 \neq e_2$ and the set of ends of e_1 equals that of e_2.

- A graph G is <u>simple</u> if it has no loops or parallel edges.

- Two graphs G and H are <u>isomorphic</u> if there are two bijections $f_V : V(G) \to V(H)$ and $f_E : E(G) \to E(H)$ such that for all $v \in V(G)$ and $e \in E(G)$, $v \sim_G e$ iff $f_V(v) \sim_H f_E(e)$.

Note that some texts might use a different definition of graphs. One common definition is that $E(G)$ is a set of two-element subsets of $V(G)$. With this definition, our definition of a simple graph is just called a graph, and our definition of a graph is called a multigraph (and you need to change "set" to "multiset").

Definition 178 (Subgraph)

- A graph G is a <u>subgraph</u> of a graph H if $V(G) \subseteq V(H)$, $E(G) \subseteq E(H)$, with the same incidence relation, i.e. the set of ends of any edge e in G equals that of e in H.

- For $e \in E$, $G \backslash e$ is $(V(G), E(G) \backslash \{e\})$ with the same incidence relation.

- For $v \in V$, $G \backslash v$ is $(V(G) \backslash \{v\}, E')$, where E' is the set of edges in G not incident with v, with the same incidence relation.

- A subgraph H of G is <u>spanning</u> if $V(H) = V(G)$.

- A subgraph H of G is <u>induced</u> if $E(H)$ equals the set of edges in G whose set of ends is contained in $V(H)$. We say H is <u>induced by</u> $V(H)$.

- For $X \subseteq V$, $G[X]$ is a subgraph of G induced by X.

Definition 179

- A <u>complete graph</u> with n vertices, denoted K_n, is a simple graph in which for any pair of different vertices there is an edge connecting them.

- A <u>cycle graph</u> with n vertices, denoted C_n, is a simple graph whose edge set is $\{v_1 v_2, \cdots, v_{n-1} v_n, v_n v_1\}$, where $V = \{v_1, \cdots, v_n\}$.

- A graph G is <u>bipartite</u> if V can be partitioned into non-empty subsets A and B such that no edges connect two vertices in A or two vertices in B.

- A <u>complete bipartite graph</u> with $n+m$ vertices, denoted $K_{n,m}$, is a simple bipartite graph with $|A| = n$, $|B| = m$ in which for any vertex in A and in B, there is an edge connecting them.

- For a simple graph G, the <u>complement</u> $\bar{G}$ of G is a simple graph on $V(G)$ such that any two different vertices v and w are adjacent in G iff they are not adjacent in $\bar{G}$.
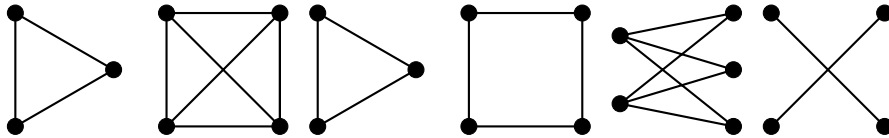
Figure 14.2: From left to right: K_3, K_4, C_3, C_4, $K_{2,3}$, and $\bar{C}_4$. Note that K_3 is isomorphic to C_3.

Definition 180

- A <u>walk</u> from $v \in V$ and $w \in V$ is an alternating sequence $v_0 e_1 v_1 e_2 \cdots e_k v_k$ of vertices and edges such that $v_0 = v$, $v_k = w$, and the set of ends of e_i equals $\{v_{i-1}, v_i\}$. k is the <u>length</u> of the walk.

- A <u>trail</u> is a walk with distinct edges.

- A <u>closed walk</u> is a walk with $v = w$ and $k > 0$.

- A <u>circuit</u> is a trail that is also a closed walk.

- A <u>path</u> is a walk with distinct vertices.

- A <u>cycle</u> is a circuit with distinct $\{v_0, \cdots, v_{k-1}\}$.

Definition 181 (Connectivity)

- A graph is <u>connected</u> if for any two vertices in V there is a path connecting them.

- A <u>connected component</u> of a graph G is $G[X]$ such that $G[X]$ is connected, and for any $Y \subseteq V$ such that $X \subsetneq Y$, $G[Y]$ is not connected.

14.2 Degrees

Definition 182 (Degree)
The <u>degree</u> of $v \in V(G)$, denoted $deg_G(v)$, is the number of non-loop edges incident with v, plus two times the number of loops incident with v.

Again, we might skip G and write $deg(v)$. As we progress, it will be clear why it is convenient to count a loop twice.

Lemma 183 (Degree Sum Formula)
$\sum_{v \in V} deg(v) = 2|E|$.

Proof. Induction on $|E|$, with the trivial base case $|E| = 0$. Suppose $|E| > 0$. Let $\sum_{v \in V} deg(v) = A$ and $2|E| = B$. Take any edge e, and the induction with $G \backslash e$ shows that $A - 2 = B - 2$. Therefore $A = B$. □

Lemma 184 (Handshaking Lemma)
A graph has an even number of odd-degree vertices.

Proof. $2|E|$ is an even number. From [183], exactly even number of the terms $deg(v)$ must be odd. □

The degree sum formula is sometimes also called the handshaking lemma.

Definition 185 (Degree Sequence)
The <u>degree sequence</u> of a graph G, or the <u>score</u> of G, is the sequence of degrees $(deg(v_1), \cdots, deg(v_{|V|}))$.

Now, how can we figure out if a sequence is a degree sequence of some graph? The following theorem gives a simple $O(\sum d_i)$-time algorithm to answer the question:

Theorem 186 (Havel-Hakimi Algorithm)
Let $(d_1, \cdots, d_n)$ be a sequence of integers such that $0 \le d_1 \le \cdots \le d_n$ and $n > 1$. It is a degree sequence of some simple graph iff $(d_1, \cdots, d_{z-1}, d_z - 1, \cdots, d_{n-1} - 1)$ is a degree sequence of some simple graph, where $z = n - d_n$.

Proof. ($\Leftarrow$) If $(d_1, \cdots, d_{z-1}, d_z - 1, \cdots, d_{n-1} - 1)$ is a degree sequence of some simple graph, then we can make $(d_1, \cdots, d_n)$ by adding a vertex and connecting to the vertices with degrees $d_z - 1, \cdots, d_{n-1} - 1$.
($\Rightarrow$) Let G be a simple graph such that $deg_G(v_i) = d_i$ for all $v_i \in V(G)$. We will construct a simple graph H with $deg_H(u_i) = d_i$ for all $u_i \in V(H)$ such that v_n is connected to $v_{n-d_n}, \cdots, v_n - 1$. Then the conclusion follows by taking $H - v_n$.
If $d_n = n - 1$, then simply take $H = G$. Otherwise, define $j(G)$ as the largest index j such that v_n is not adjacent to v_j. Among all graphs with $deg_H(u_i) = d_i$,

take one graph with the smallest $j(H)$. (Note that such H exists because at least one graph, namely G, satisfies the degree sequence condition.)

Suppose $j = j(H) \geq n - d_n$. Then there is an index $i < j$ such that u_n is adjacent to u_i. Since $deg_H(u_i) \leq deg_H(u_j)$, there is a vertex w adjacent to u_j but not to u_i. Now, consider a new graph H' derived from H by removing $u_i u_n$ and $u_j u_k$, and adding $u_j u_n$ and $u_i u_k$. Then $deg_{H'}(u_i) = d_i$ and $j(H') < j(H)$, contradicting the minimality of H. Therefore $j(H) = n - d_n$. $\square$

14.3 Trees

One of the important classes of graphs is a tree. There are many ways to define a tree. First we will state one definition, and then prove that other definitions are equivalent.

Definition 187 (Tree)
A <u>forest</u> is a simple graph without any cycle. A <u>tree</u> is a connected forest. A <u>leaf</u> of a forest is a vertex with degree 1.

Before moving on to the equivalence, we introduce two useful lemmas:

Lemma 188
A tree with at least two vertices has at least two leaves.

Proof. Let P be a path with maximum length, and x and y be its end-vertices. Then $x \neq y$, and we claim that x and y are leaves. Suppose x is not a leaf, so x has an edge e not used by P. Let z be the other end of e. Since $P + e$ has no cycles, z is not used by P. Therefore $e + P$ is a path longer than P, contradicting the maximality. Repeat this proof to show that y is also a leaf. $\square$

Lemma 189
Let G be a graph, $v \in V$, and $deg(v) = 1$. Then G is a tree iff $G - v$ is a tree.

Proof. Suppose G is a tree. Since G is connected, for any two vertices $x \neq y$, there is a path from x to y. Since v has only one incident edge, this path does not contain v. Therefore this path is also a path on $G - v$. Clearly $G - v$ has no cycles. Therefore $G - v$ is a tree.

Suppose G is not a tree. If G is disconnected, then $G - v$ is clearly disconnected. If G has a cycle, then since this cycle does not contain v, it is also a cycle on $G - v$. Therefore $G - v$ is not a tree in either cases. $\square$

This enables us to apply induction on the number of vertices of a tree. Check the base case where there are ≤ 2 vertices. Then, let v be a leaf of a tree, and remove v. Since the resulting graph is also a tree, we can apply the inductive hypothesis. Then we can use this to prove the statement for the original tree.

Theorem 190
The following statements are equivalent for a simple graph G:

1. G is a tree.

2. For any two vertices u and v of G, there is exactly one path connecting them.

3. G is connected, and for any edge e of G, $G \backslash e$ is disconnected.

4. G has no cycle, and for any two vertices u and v not having an edge between them, $G + uv$ has a cycle.

5. G is connected, and $|E| = |V| - 1$.

Proof. (1 $\Rightarrow$ 2) Since G is connected, there is a path. Next, induction on $|V|$. The base case $|V| \leq 2$ is trivial. From [188] and [189], there is a leaf v and $G - v$ is a tree. Any two vertices a, b of $G - v$ has a unique path. Since a path from a to b in G cannot contain v, there is a unique path in G as well. Since v is adjacent to a unique vertex u, and there is a unique path from a to u, there is a unique path from a to v which is the path to u plus the edge uv.

(2 $\Rightarrow$ 3) Since every pair of vertices has a path, G is connected. For any e, its two endpoints have a unique path which is e itself. Therefore there is no path between them in $G \backslash e$.

(3 $\Rightarrow$ 4) If G has a cycle, then take any edge e in the cycle C. In any walk that contains e, this e can be replaced with walking the opposite direction on C to form another valid walk. (The rigorous argument is left to the reader for exercise.) Therefore $G \backslash e$ is connected, contradiction. Next, since G is connected, taking a path from u to v and then taking uv gives a cycle in $G + uv$.

(4 $\Rightarrow$ 1) If G is disconnected, then connecting two vertices in different connected components does not give a cycle, contradiction. This is because once you take that edge, there is no way to come back to the starting vertex. Therefore G is a connected graph with no cycle, i.e. a tree.

(1 $\Rightarrow$ 5) G is connected from the definition. Next, induction on $|V|$. The base case $|V| \leq 2$ is trivial. There is a leaf v and $G - v$ is a tree. $G - v$ has $|E(G)| - 1$ edges and $|V(G)| - 1$, so $|E(G)| - 1 = |V(G)| - 2$.

(5 $\Rightarrow$ 3) Adding an edge decreases the number of connected components by at most one. A graph with no edges has $|V|$ connected components. Therefore a graph with $|V| - 2$ edges is disconnected. $\square$

How can we figure out if a sequence is a degree sequence of some tree? It turns out to be a lot simpler than [186] and basically anything that makes sense can be a degree sequence of a tree:

Theorem 191 (Degree Sequence of a Tree)
A sequence $(d_1, \cdots, d_n)$ is a degree sequence of some tree iff all d_i are positive and $\sum d_i = 2n - 2$.

Proof. ($\Rightarrow$) Clear from [190](5) and [183].

($\Leftarrow$) Induction on n, with trivial base cases $n \leq 2$. Now suppose $n \geq 3$. There exists i and j such that $d_i = 1$ and $d_j > 1$; WLOG assume $i = 1$ and $j = 2$. From induction, $(d_2 - 1, d_3, \cdots, d_n)$ is a degree sequence of some tree. Take any vertex v in the tree with degree $d_2 - 1$, and add a leaf adjacent to v. This constructs a tree with the degree sequence $(d_1, \cdots, d_n)$. $\square$

14.3.1 Spanning Trees

Definition 192 (Spanning Subgraph)
A <u>spanning subgraph</u> of a graph G is a subgraph of G such that its vertex set equals $V(G)$. A <u>spanning tree</u> is a spanning graph that is a tree.

Theorem 193
A connected graph G has a spanning tree.

Proof. Let $m = |E|$, and label the edges as e_0, $\cdots$, e_m, arbitrarily. Define

the subsets $E_0, \cdots, E_m$ of E, as

$$
\begin{cases}
E_0 = \varnothing \\
E_i = E_{i-1} \cup \{e_i\} \quad \text{if the spanning subgraph of } G \\
\qquad\qquad\qquad\quad \text{with } E = E_{i-1} \cup \{e_i\} \text{ has no cycle} \\
E_i = E_{i-1} \qquad\quad \text{otherwise.}
\end{cases}
$$

Let H be the spanning subgraph of G with $E = E_m$. Clearly, H has no cycle. If $e_i \notin E_m$ and $H + e_i$ has no cycle, then E_i would contain e_i, contradiction. From [190], H is a tree. □

(TODO: minimum spanning tree)
There are other minimum spanning tree algorithms like Prim's algorithm or Borůvka's algorithm.

14.4 Planar Graphs

Definition 194 (Planar Graph)
A <u>plane graph</u> is a graph G where:

- $V \subseteq \mathbb{R}^2$;

- every edge is an arc between two endpoints;

- the interior of each edge contains no vertex and no point of any other edge.

The connected components of $\mathbb{R}^2 \setminus G$ are called <u>faces</u> of G. Since G is contained in a sufficiently large disc, exactly one face is unbounded; that face is called the <u>outer face</u> of G. All other faces are called <u>inner faces</u> of G. A graph H is <u>planar</u> if it is isomorphic to some plane graph.

Theorem 195 (Euler's Formula)
If G is a connected plane graph, and the number of faces of G is F, then

$$|V| - |E| + F = 2.$$

Proof. Induction on $|E|$. The base case is when G has no edges, one vertex, and one face; the formula clearly holds.
Pick any edge e. If e is a loop, removing it reduces $|E|$ and F by one. Otherwise, contracting it reduces $|V|$ and $|E|$ by one. Either way the result follows by induction. □

Theorem 196
If G is simple and planar, and $|V| \geq 3$, then $|E| \leq 3|V| - 6$. If in addition G has no triangles (i.e. K_3 as a subgraph), then $|E| \leq 2|V| - 4$.

Proof. Count the number N of pairs (f, e) where the face f and the edge e are incident. For each face, there are at least 3 edges incident to it, for otherwise there would be parallel edges or loops. Therefore $N \geq 3F$. On the other hand, each edge is incident to exactly two faces, so $N = 2|E|$. This gives $3F \leq 2|E|$. From [195], $3F = 6 - 3|V| + 3|E| \leq 2|E|$, and the first result follows.
The second result can be proved in the exactly same way, using $N \geq 4F$. □

Corollary 197
K_5 and $K_{3,3}$ are not planar.

Proof. K_5 has 5 vertices and 10 edges. $K_{3,3}$ has no triangles, 6 vertices, and 9 edges. The result follows from [196]. □

    TODO: add a figure of K5 and K33.
    It clearly follows that any subdivision of K_5 or $K_{3,3}$ are not planar. Surprisingly, those two graphs are the only graphs that "need to be checked" to determine if a given graph is planar. The proof requires several more lemmas and theorems, so we have moved the proof to the appendix.

Theorem 198 (Kuratowski's Theorem)
A graph G is planar if and only if it does not have K_5 or $K_{3,3}$ as a topological minor.

14.5 Coloring

Definition 199 (Coloring)
A k-coloring of a graph G is a function $c : V(G) \to \{1, 2, \cdots, k\}$ such that if u and v are adjacent vertices, then $c(u) \neq c(v)$. G is k-colorable if there is a k-coloring of G. The chromatic number $\chi(G)$ of G is the smallest integer k such that G is k-colorable.

    Perhaps the most famous theorem about graph coloring is the four-color theorem. (TODO: write something)

Theorem 200 (Four-color Theorem)
If G is planar, then $\chi(G) \leq 4$.

    Unfortunately, the proof is too long and complicated to contain in the codex. We prove a weaker result:

Theorem 201 (Five-color Theorem)
If G is planar, then $\chi(G) \leq 5$.

Proof. Induction on $|V|$. For $|V| \leq 5$, the theorem is trivial.
    From [196], G has a vertex v of degree at most 5. If $deg_G(v) < 5$, then inductively find a 5-coloring of $G - v$, and color v by some color in $\{1, 2, 3, 4, 5\}$ not appearing in the neighbors of v. If $deg_G(v) = 5$ and not all colors are used in the neighbors of v, then the same argument applies.
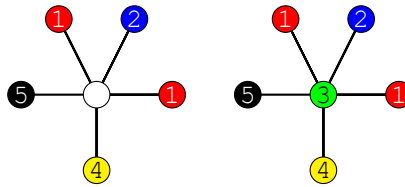
    Now suppose all 5 colors are used. Denote the neighbors of v as u_1, u_2, u_3, u_4, u_5, in clockwise order. Without loss of generality, we will assume that $c(u_i) = i$.
    The main idea of the rest of the proof is that we want to change the color of one of the neighbors, say change $c(u_i)$ to k. This is impossible if u_i has a neighbor of color k, in which case we want to also change the color of that neighbor, to k'. But then that neighbor might have yet another neighbor of color k', and this continues to form a chain. Hence we introduce the Kempe chain, named after Alfred Kempe.
    Let V_{ij} be the set of vertices w in G such that there is a path from u_i to w consisting of vertices of color i or j. Note that if we switch the colors

of the vertices in V_{ij} (i.e. change i to j and j to i), and leave everything else the same, then the result is still a coloring.

If V_{13} does not contain u_3, then switch the colors of the vertices in V_{13} and color v by 1.

(TODO: picture)

Otherwise, V_{24} does not contain u_4; switch the colors of the vertices in V_{24} and color v by 2. This gives a 5-coloring of G.

(TODO: picture) □

Fun fact: In 1879, the Kempe chain method was used to "prove" the four-color theorem by Alfred Kempe. No one noticed that this "proof" had an error until eleven years later when Percy Heawood found the error. What we saw above is the modification of the incorrect proof to prove the weaker theorem. The correct proof of four-color theorem was completed in 1976 by Kenneth Appel and Wolfgang Haken.

Here is his "proof." Argue similarly as above with induction. If $deg_G(v) = 4$ and all 4 colors are used, then apply the Kempe chain method. Now suppose $deg_G(v) = 5$ and all 4 colors are used. Then one color is used exactly twice.

There are two cases: the two neighbors with that color are next to each other in clockwise order, or they are not. The first case is easy, just use the Kempe chain method. The second case is where the fun starts.

(TODO: picture. u5-u4-u1-u2-u3 clockwise; u1 and u5 has the same color. Cetner is noted v.)

WLOG, u_k has color k. For convenience, color 5 is the same as color 1.

If V_{42} does not contain u_2, then switch the colors of the vertices in V_{25} and color v by 4. Otherwise, if V_{43} does not contain u_3, then switch and color v by 4. Otherwise, V_{13} does not contain u_3 and V_{52} does not contain u_2. Switch each chain and color v by 1.

(TODO: second case picture.)

Can you find a critical error in this argument? If you want to know, refer to the appendix.

Chapter 15

Cryptosystem

Cryptography is one of the most advanced area of applied mathematics. It uses many terms not used in many other branches of mathematics or applied mathematics, and is often called "state-of-the-art"-est part of mathematics.

15.1 Basic Terminology

Definition 202 (Basic Terminology on Cryptosystems)

- **Plaintext**: The text before encryption

- **Ciphertext**: The text after encryption

- **Cryptosystems**: Encryption and decryption algorithms, see definition below for more

    Encryption: Using some sort of algorithm to change the content of a message so that it is unrecognizable.

    Decryption: Processing the encrypted message to change it back to the message.

- **Key**: A value required to encrypt or decrypt.

    Encryption Key: The key for encryption.

    Decryption Key: The key for decryption.

- **Cryptanalysis**: Decrypting the ciphertext without any prior knowledge(i.e. key).

Now that the basic terminologies are defined, we can go on with defining "Cryptosystem":

Definition 203 (Cryptosystem)
A cryptosystem is defined as a tuple of three algorithms, (G, E, D);

G The key generation algorithm, sometimes abbreviated as <u>KeyGen</u>, chooses the encryption key k_1 and the decryption key k_2 from the set of possible keys. The set of possible keys is called the <u>key space</u>. Usually each key from the key space is chosen at uniformly random probability.

E The Encryption Algorithm, sometimes abbreviated as <u>Enc</u>, uses the encryption key k_1, takes the plaintext m as an input, and produces the ciphertext c. This is usually denoted as follows:

$$E_{k_1}(m) = c$$

D The Decryption Algorithm, sometimes abbreviated as <u>Dec</u>, uses the decryption key k_2, takes the ciphertext c as an input, and gains the plaintext m. This is usually denoted as follows:

$$D_{k_2}(c) = m$$

For a cryptosystem to be valid, by encrypting the plaintext m and decrypting the ciphertext, we must be able to get m, that is;

$$D_{k_2}(E_{k_1}(m)) = m$$

A cryptosystem is classified into two categories; if the encryption key is the same as the decryption key, it is called a <u>Symmetric Key Algorithm</u>; if not, it is called an <u>Asymmetric Key Algorithm</u> or a <u>Public Key Algorithm</u>. A symmetric key algorithm is again classified into two categories; <u>Block Cipher</u> and <u>Stream Cipher</u>.

Definition 204 (Kerckhoffs' Principle)
Kerckhoffs' Principle states that a cryptosystem must be secure even if everything about the cryptosystem except for the key is exposed.

Kerckhoffs' Principle says that the cryptosystem's security must depend only on the secrecy of the key. Its core comes from the idea that "The enemy knows the system". In some, "Security through obscurity"(i.e. hiding the cryptosystem itself) holds but Kerckhoffs' Principle has its value for the following reasons:

1. Storing a smaller sized key is easier than hiding the entire cryptosystem. Also the cryptosystem is not safe from reverse engineering, but keys are, as they are usually a random number.

2. If the key is exposed, it is easier to change only the key, not the entire cryptosystem.

3. A cryptosystem is often used for many users, and everybody using the same cryptosystem allows for more efficient usage of space.

4. If the cryptosystem itself is kept a secret, if a problem arises(i.e. reverse engineering) to expose the cryptosystem, then the entire thing must be redesigned. This takes a lot of knowledge and time.

5. A cryptosystem is made weak by a small mistake; these mistakes are not found before the cryptosystems are analyzed fully, which is most easily done by making the system public. If they are indeed made public, the cryptosystem can be checked for security, allowing for a more secure system.

15.2 Encryption of Arbitrary Length Message

15.2.1 Padding

When using a block cipher, we need the length of the message to be an exact multiple of the length of the block used in the block cipher. If not, we use <u>padding</u> to make the message longer to make it an exact multiple. There are many ways to do so, but the following paddings are the most prominent:

- Zero Padding, otherwise known as Null Padding

    Pad the message with zero(00) bytes to make the length be an exact multiple of the cipher block length. This may cause a problem if the last bytes of the message are 00.

- Bit Padding

    Pad the message with $10|00^n$, so that we can know the start of padding. In this case, the message must be padded even if its length is a multiple of the cipher block length.

- Byte Padding

    Same as zero padding, except the last byte is equal to the length of padding, that is; if we require four more bytes, the padding is 00|00|00|04. The message must also be padded even if its length is a multiple of the cipher block length.

- PKCS#7 Padding

    Similar to byte padding, except every byte of the padding is equal to the length of padding, that is; if we require four more bytes, the padding is 04|04|04|04.

15.2.2 Modes of Operation

Sometimes we are required to encrypt a longer message than the length of the block. The plaintext are first padded using one of the techniques above, and the padded plaintext P is separated into blocks of padding length, $P_1, P_2, \cdots, P_N$. They are then encrypted using the key K, sometimes with the help of the initialization vector IV, and produces the ciphertexts $C_1, C_2, \cdots, C_N$. There are five major ways(or "modes") to do this; ECB, CBC, CFB, OFB, and CTR.

Electronic Code Book(ECB)

ECB mode is the simplest mode of them all. They simply take each blocks and encrypt them separately. In equation:

- **Encryption** $C_i = E_K(P_i)$

- **Decryption** $P_i = D_K(C_i)$

Electronic Codebook (ECB) mode encryption   Electronic Codebook (ECB) mode decryption
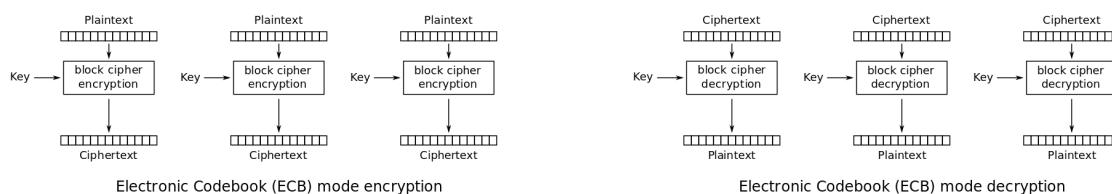
Figure 15.1: ECB Mode

Since same plaintext blocks are encrypted into same ciphertext block, the blocks can be copied, or replayed, to change the message easily. This is called the Block Replay Attack.

Cipher Block Chaining(CBC)

CBC takes the previous ciphertext block and XOR($\bigoplus$) it with the plaintext before encryption. The first block has no previous ciphertext block, hence it is XOR-ed with the IV. In equation:

- **Encryption** $C_0 = IV$, $C_i = E_K(P_i \bigoplus C_{i-1}), i = 1, 2, 3, \cdots, N$

- **Decryption** $C_0 = IV$, $C_i = D_K(C_i) \bigoplus C_{i-1}, i = 1, 2, 3, \cdots, N$

Cipher Block Chaining (CBC) mode encryption   Cipher Block Chaining (CBC) mode decryption
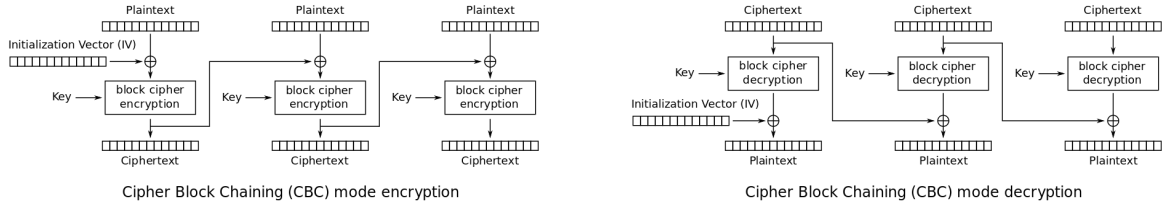
Figure 15.2: CBC Mode

Cipher Feedback(CFB)

CFB can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher. In the diagram given below, original block sizes are used. In equation:

- **Encryption** $C_0 = IV$, $C_i = E_K(P_i \bigoplus C_{i-1}), i = 1, 2, 3, \cdots, N$

- **Decryption** $C_0 = IV$, $C_i = D_K(C_i) \bigoplus C_{i-1}, i = 1, 2, 3, \cdots, N$

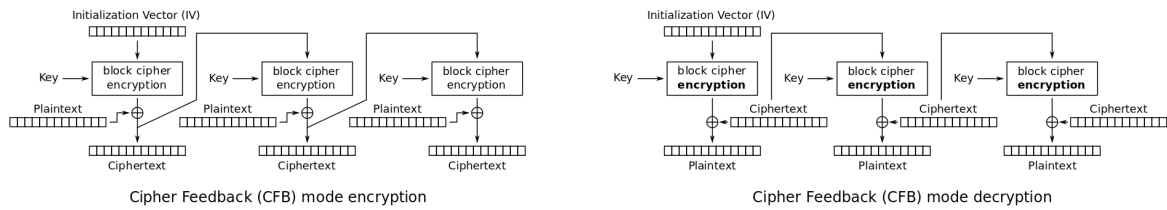Cipher Feedback (CFB) mode encryption   Cipher Feedback (CFB) mode decryption

Figure 15.3: CFB Mode

By altering the equation to the following we have the "stream cipherized" version, where $<<$ is the shift operation, $head(a, x)$ is the first x bits of a, and n is the size of the IV:

- **Shift Register** $S_0 = IV$, $S_i = ((S_i << x) + C_i) \mod 2^n$

- **Encryption** $C_i = head(E_K(S_{i-1}), x) \bigoplus P_i$

- **Decryption** $P_i = head(E_K(S_{i-1}), x) \bigoplus C_i$

Output Feedback(OFB)

OFB can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher.

- **Input and Output** $I_0 = IV$, $I_j = E_K(I_{j-1}), j = 1, 2, 3, \cdots, N$

- **Encryption** $C_j = P_j \bigoplus I_j, i = 1, 2, 3, \cdots, N$

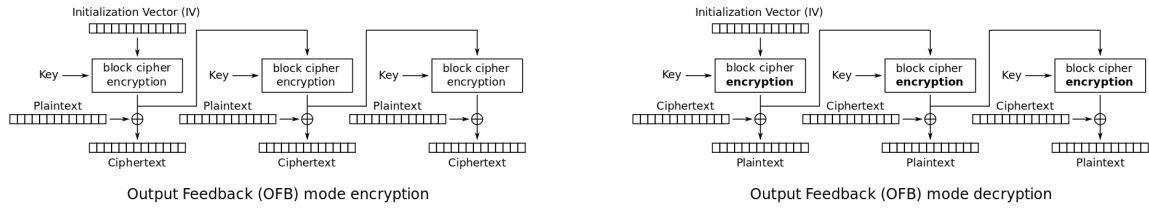- **Decryption** $P_j = C_j \bigoplus I_j, i = 1, 2, 3, \cdots, N$

Figure 15.4: OFB Mode

We can similarly alter the equation as OFB so that it can be used as a stream cipher.

Counter(CTR)

CTR can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher. It encrypts the counter value instead of the plaintext, and XORs the value to gain the ciphertext.

- **Encryption** $C_i = P_i \bigoplus E_K(Counter), i = 1, 2, 3, \cdots, N$

- **Decryption** $P_i = C_i \bigoplus E_K(Counter), i = 1, 2, 3, \cdots, N$
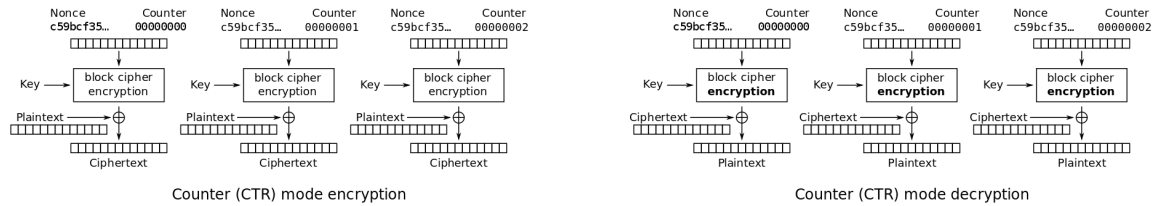

Figure 15.5: CTR Mode

We can similarly alter the equation as OFB so that it can be used as a stream cipher.

Characteristics

Table 15.1 shows the characteristics for each modes of operation.

- Block Pattern: Whether if the overall pattern is kept after encryption

		ECB	CBC	CFB	OFB	CTR
Block Pattern		O	X	X	X	X
Preprocessing		X	X	X	O	O
Parallel Processing	Encryption	O	X	X	O	O
	Decryption	O	O	O	O	O
Error Propagation		X	(P_i, P_{i+1})	$\lceil \frac{n}{r} \rceil$ blocks	X	X
Encryption Unit		n	n	$r(\leq n)$	$r(\leq n)$	$r(\leq n)$

Table 15.1: Characteristics for Each Modes of Operation

- Preprocessing: Whether if preprocessing is possible on encryption and decryption

- Parallel Processing: Whether if parallel processing is possible on encryption or decryption

- Error Propagation: If there is an error in the encryption/decryption process, whether if the error spreads through other blocks

- Encryption Unit: The minimum requirement byte for encryption

15.3 Types of Attack

15.3.1 Attacking Classical Cryptosystems

Classical Cryptosystems are typically a substitution cipher and/or a transposition cipher. Since most, if not all, the classical cryptosystems are broken, the two valid ways to attack any classical cryptosystems is given here.

- Brute Force Attack

    When the attacker gains the ciphertext c, the attacker uses every key possibility to try to gain m. This is otherwise known as the Exhaustive Key Search Attack. Theoretically this can be done to any symmetric-key cipher; but this is inapplicable to most modern cryptosystems as they have an extremely large key space.

- Frequency Analysis

    The plaintext having some pattern, such as the alphabet 'e' appearing with the most frequency, will help the attacker gain knowledge on the plaintext just by seeing the ciphertext.

15.4 Cryptographic Hash Functions

A general hash function has the following properties:

- They take an arbitrary size of data as input, and;

- They produce a constant and fixed length data as output.

A cryptographic hash function, in addition to the properties above, must have the following properties:

- Preimage Resistance

    If the hash value y is given, it must be hard to find an x such that $h(x) = y$, that is, the hash function must have one-wayness.

- Second Preimage Resistance

    If the message x is given, it must be hard to find an $x' \neq x$ such that $h(x) = h(x')$.

- Collision Resistance

    It must be hard to find $x \neq x'$ such that $h(x) = h(x')$. The pair (x, x') is called the collision pair.

15.5 Attacking the Cyrptosystems

Attacks on cryptosystems are classified into <u>passive</u> and <u>active</u> attack. Passive attacks simply eavesdrops the transmission, and gains what the attacker wants without modification of the message. This type of attacking includes <u>eavesdropping</u>, of which the attacker intercepts the message in the middle to check the plaintext. This type of attacker is often referred to as "Eve"(as in eavesdropping) in theories. Active attackers will modify the message, which includes Modification, Deletion, Impersonation, and Replay. This type of attacker can also be referred to as "Eve", but sometimes is referred to as "Mallory", for malicious user.

- Modification: Changes the order of the message or changes a part of it to alter the meaning.

- Deletion: Intercepts the message and does not send it, interrupting the communication.

- Impersonation: Fakes their own identity to be identified as a correct user.

- Replay: Send a message again after eavesdropping, expecting some kind of result.

The four methods above are just the general ways to attack. We need to attack the system itself to know how to attack it. There are four methods of attack on system:

- **Ciphertext Only Attack**

    The attacker knows only the ciphertext.

- **Known Plaintext Attack**

    The attacker knows a list of (message, ciphertext) pair, and attempts to crack a ciphertext not in the list.

- **Chosen Plaintext Attack**

    The attacker has access to an oracle that can encrypt the message, and attempts to crack a ciphertext.

- **Chosen Ciphertext Attack**

    The attacker has access to an oracle that can decrypt a ciphertext, except for the target ciphertext.

There are three important properties to encryption schemes:

- Semantic Security

    A semantically secure encryption scheme is infeasible for any computationally bounded adversary to derive a significant information about the original plaintext when given only its ciphertext and the corresponding public key if any. This can be represented as a game between the oracle and the adversary, as below:

  1. The oracle generates a key for the challenge.

  2. The adversary is given the encryption oracle(or the public key, in the case of public key cryptosystem).

  3. The adversary can perform any number of polynomially bounded number of encryptions or operations.

  4. The adversary generates two equal-length messages m_0 and m_1, and transmits it to the oracle.

  5. The oracle randomly chooses $b \in \{0,1\}$ to encrypt the message m_b to C.

  6. The adversary, upon receiving C, guesses b.

  If the adversary cannot guess b correctly with significantly greater than 50% probability, then the scheme is said to be semantically secure under CPA.

- Indistinguishability

    If a cryptosystem is indistinguishable, then an adversary would not be able to distinguish pairs of ciphertexts based on the message they encrypt. There are three types: IND-CPA, IND-CCA1, and IND-CCA2. They can be represented as a game between the oracle and the adversary. In both cases, they are said to be secure if the adversary does not have a clear advantage. Note that for any nonzero α, if the adversary has access to the LR-oracle multiple times, and if the probability of advantage of the adversary is $0.5 \pm \alpha$, a repetitive trial is capable of bringing the odds up close to 1(for $0.5+\alpha$) or down close to 0(for $0.5-\alpha$, in which the adversary may simply chooses the opposite).

  - **IND-CPA**

    1. The oracle generates a key for the challenge.
    2. The adversary is given the encryption oracle(or the public key, in the case of public key cryptosystem).
    3. The adversary can perform any number of polynomially bounded number of encryptions or operations.
    4. The adversary generates two distinct equal-length messages m_0 and m_1, and transmits it to the oracle.
    5. The oracle randomly chooses $b \in \{0,1\}$ to encrypt the message m_b to C.
    6. The adversary, upon receiving C, performs polynomially bounded encryptions or operations, and guesses b.

  - **IND-CCA**

    1. The oracle generates a key for the challenge.
    2. The adversary is given the decryption oracle and the public key, in the case of public key cryptosystem.
        Note that in the case of the public key cryptosystem, the encryption oracle is also given.

3. The adversary can perform any number of polynomially bounded number of decryptions or operations.

4. The adversary generates two distinct equal-length messages m_0 and m_1, and tramsmits it to the oracle.

5. The oracle randomly chooses $b \in \{0,1\}$ to encrypt the message m_b to C.

6. The adversary, upon receiving C, performs polynomially bounded operations.

   In the case of IND-CCA1, the adversary may not make further calls to the decryption oracle.

   In the case of IND-CCA2, the adversary may make further calls to the decryption oracle, but may not submit C.

7. The adversary guesses b.

This can be said with a random oracle. In that case, the adversary submits only one message and the oracle returns the encryption of the message or the random string equal to the length of the encryption with a fair chance. The adversary then guesses whether if the message is randomly generated or encrypted.

• Non-malleability

   Cryptosystems are called "malleable" if it is possible to transform a ciphertext into another ciphertext which decrypts to a related plaintext. Cryptosystems that are not malleable are called non-malleable. These, similar to indistinguishability, can be represented as a game between the oracle and the adversary, and are called NM-CPA, NM-CCA1, NM-CCA2. Some cryptosystems, however, are malleable by design(i.e. RSA cryptosystem), but has low probability that it would be abused.

Theorem 205
The following relations for each security properties hold:

• IND-CPA $\Leftrightarrow$ Semantic security under CPA

• NM-CPA $\Rightarrow$ IND-CPA

• NM-CCA2 $\Leftrightarrow$ IND-CCA2

• NM-CPA does not necessarily imply IND-CCA2.

15.6 Digital Signatures

Digital signatures are used in pair with the public key cryptosystems to verify the sender of the messages. When attacking, there are three major methods:

• **Key-Only Attack**

   The attacker only has access to the digital signature algorithm and the public key of the signer, pk_A. This is similar to the Ciphertext Only attack.

• **Known Message Attack**

   The attacker has access to the digital signature algorithm, the public key of the signer, and a list of (message, signature) pairs. This is similar to the Known Plaintext attack.

- **Chosen Message Attack**

     The attacker has access to the digital signature algorithm, the public key of the signer, and an oracle that takes a message as an input and returns signature as an output.

The attacker can have three different purposes:

- **Total Break**

     The attacker wants to gain the private key of the signer.

- **Selective Forgery**

     The attacker wants to generate a valid signature for a message the attacker wants(i.e. any message for that matter).

- **Existential Forgery**

     The attacker wants to generate a valid (message, signature) pair for any message.

It is said that an attack is valid if the attack succeeds with a non-negligible probability.

15.7 Zero-Knowledge Authentication

Three major ways to authenticate a user is using password, challenge-response, and zero-knowledge authentication. Passwords must be sent through network, thereby they are susceptible to interception. Challenge-response can be abused by malicious users to crack the secret key. That is where the concept of zero-knowledge interactive proof comes in.

   An interactive proof system can be described as a communication between the verifier and the prover. They exchange messages to check whether if the statement is true or false. In here, the prover is assumed to have unlimited calculating power but cannot be trusted; the verifier has bounded computation power but is assumed to be always honest. Messages are sent between the prover and the verifier until the verifier has an answer to the problem and has convinces itself that the answer is correct.

   Any interactive proof system must have the following properties:

- **Completeness:** If the statement is true, the honest verifier will be convinced of this fact by an honest prover.

- **Soundness:** If the statement is false, no cheating prover can convince the honest verifier that it is true, except with some small probability.

In authentication, if the proof is only interactive, a malicious verifier may abuse the protocol to reveal the "knowledge"(in the case for cryptosystems, private keys) only the prover knows. This is where the concept of "Zero-knowledgeness" comes in.

- **Zero-knowledgeness:** If the statement is true, no verifier can learn anything apart from the fact that the statement is true.

The best way to describe this is by an analogy of a colorblind person. Suppose the person has two balls that looks the exactly same for them. Their friend, as a non-colorblind person, want to prove that the two balls are of different color. The colorblind person resumes the role of verifier and the non-colorblind friend the prover. Here is an example protocol on how the fact can be proven:

1. Verifier shows you a ball.

2. Prover memorize it.

3. Verifier then hides both balls, and choose to keep the ball shown before or change the ball.

4. Verifier shows the newly chosen ball.

5. Prover tell verifier whether if the ball has been changed or not.

6. If the prover is wrong, the prover has told a lie; end the protocol.

7. If the prover is right, the prover may be telling the truth; continue the protocol until convinced.

If the statement('The two balls are of different color') is false, then the prover(in this case, cheating) cannot tell whether if the ball has been changed; therefore their guess is right for 50% of the time. n consecutive application of the protocol gives $\frac{1}{2^n}$ chance of success, and as the number of trials increase, the less the cheating prover will be able to pass the protocol.

If the statement, on the other hand, is indeed true, then the prover can tell whether if the ball has been switched every time. In the verifier's point of view, the prover's n-th consecutive success for verification proves that they are lying at $\frac{1}{2^n}$ probability; their improbable probability of success at lying will thereby prove their honesty.

15.8 RSA Cryptosystem and Signature

15.8.1 Keygen

1. Choose two primes p and q.

2. Let $n = p \cdot q$.

3. Choose e such that $(e, \phi(n)) = 1$

4. Find d such that $e \cdot d \equiv 1 \mod \phi(n)$

Public Key: (n, e)
Private Key: d or (p, q, d), depending on the method.

15.8.2 Cryptosystem

Encryption

$C \equiv M^e \mod n$

Decryption

- **Basic Method**
  $C^d \equiv (M^e)^d \equiv M^{\phi(n) \cdot k + 1} \equiv M \mod n$

- **Chinese Remainder Theorem**
  Split $C^d \mod n$ into two congruences: $C^d \mod p$ and $C^d \mod q$.
  Using Euler's Theorem(If $(a, n) = 1$, $a^{\phi(n)} \mod n$.), reduce d to reduce the number of multiplication. There is a more formularized version of this, which will not be mentioned in here.

15.8.3 Signature

Signing

$S \equiv M^d \bmod n$

Verifying

Compare $S^e \bmod n$ to M. If equal, accept; otherwise reject.

15.8.4 Attacking the Cryptosystem

On the Case of Exposed Private Key e

Total break is possible.
For the public key, $n = pq$ where p and q are primes.
Then, $\phi(n) = (p-1)(q-1)$.
We know that $ed \equiv 1 \bmod \phi(n)$.
By the definition of modular, $ed - 1 = k\phi(n)$ for some k.
For a large enough $n = pq$, $\frac{\phi(n)}{n} = \frac{(p-1)(q-1)}{pq} = 1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq} \simeq 1$.
We can find k by dividing both sides of the equation $ed - 1 = k\phi(n)$ by n, since $\frac{ed-1}{n} = k\frac{\phi(n)}{n} \simeq k$.
We can then find $\phi(n) = \frac{ed-1}{k}$.
Since $n = pq$ and $\phi(n) = (p-1)(q-1) = pq - (p+q) + 1 = n - (p+q) + 1$, $p+q = n - \phi(n) + 1$.
Then the quadratic equation $(x-p)(x-q) = x^2 - (p+q)x + pq = x^2 - (n - \phi(n) + 1)x + n = 0$
can be solved to yield p and q.

Chosen Ciphertext Attack

1. Alice sends $C \equiv M^e \bmod n$ to Bob

2. Eve intercepts Alice's transmission; chooses x s.t. $(x,n) = 1$(and therefore $x^{-1} \bmod n$ exists) to send $C' = Cx^e \bmod n$ to Bob.

3. Bob decrypts C' as $(C')^d \equiv (Cx^e)^d \equiv C^d x^{ed} \equiv Mx \bmod n$

4. Eve intercepts Bob's decryption result, Mx, and multiplies x^{-1} modulo n to gain M.

Coppersmith Attack

Theorem 206 (Coppersmith)
Let $n \in \mathbb{Z}$ and $f \in \mathbb{Z}[x]$ be a monic polynomial(i.e. leading coefficient of f is 1) of degree d over integer.
Set $X = n^{1/d - \epsilon}$ for $1/d > epsilon > 0$. Then given n and f, the attacker, using the LLL Algorithm, can efficiently find all integer $x_0 < X$ such that $f(x_0) \equiv 0 \bmod n$.

Note
In the case of RSA, Finding M when given $C \equiv M^e \bmod n$ can be interpreted as finding the solution of the equation $f(x) \equiv x^e - C \bmod n$. This attack's strength is the ability to find all small roots of the polynomials modulo a composite N.

Håstad's Broadcast Attack

This attack is viable if the value of e is fixed and is small, and the same message is broadcast without padding.

Suppose the same plaintext M is encrypted to multiple people, each using same e and different moduli, say N_i. If Eve successfully intercepts e or more messages, say $C_1, C_2, \cdots, C_e$, $C_i \equiv M^e \bmod N_i$. We may assume $(N_i, N_j) = 1$ for $i \neq j$, otherwise the attacker will be able to factorize some N_i by finding their GCD. Using the Chinese Remainder Theorem on the e congruences, the attacker may compute $C \in Z^*_{\Pi N_i}$ such that $C_i \equiv C \bmod N_i$. Then, $C \equiv M^e \bmod \Pi N_i$; however since $M < N_i$ for each i, $M^e < \Pi N_i$; thus $C = M^e$ holds over the integers, and the attacker can easily find the message M.

For more generalized version, the following theorem is available:

Theorem 207 (Håstad)
Suppose $N_1, \cdots, N_k$ are relatively prime integers and set $N_{\min} = \min_i \{N_i\}$. Let $g_i(x) \in \mathbb{Z}/N_i[x]$ be k polynomials of maximum degree q. Suppose there exists a unique $M < N_{\min}$ satisfying $g_i(M) \equiv 0 \bmod N_i \forall i \in \{1, \cdots, k\}$. Furthermore, suppose $k > q$. Then there is an efficient algorithm which, given $\langle N_i, g_i(x) \rangle \forall i$, computes M.

This theorem can be used in the following way:
Suppose the i-th plaintext is padded with the polynomial $f_i(x)$. Let $g_i(x) = (f_i(x))^{e_i} - C_i \bmod N_i$. Then $g_i(M) \equiv 0 \bmod N_i$ is true, and the Coppersmith's Attack[15.8.4] can be used.

Franklin-Reiter Related Message Attack

This attack is viable if the value of e is fixed and is small, and the same message is broadcast with padding.

Theorem 208
Let (n, e) be the public key of RSA, and e is small. Let $f(x) = ax + b \in Z_n[x]$, $b \neq 0$; i.e. f is the padding function.
Suppose that $M_1 \neq M_2$ and $M_1 \equiv f(M_2) \bmod n$.
Then, given the quintuplet (n, e, C_1, C_2, f), M_1 and M_2 can be recovered in $O((log_2 n)^2)$

Proof.
$C_1 \equiv M_1^e \bmod n$
$C_2 \equiv M_2^e \bmod n$
$M_1 \equiv f(M_2) \equiv a M_2 + b \bmod n$

Let $g_2(x) = x^e - C_2 \bmod n$, and $g_1(x) = (ax + b)^e - C_1 \bmod n$

$g_1(x) = (ax + b)^e - C_1$
$= (ax + b)^e - M_1^e$
$= (ax + b)^e - (aM_2 + b)^e$
$= ((ax + b) - (aM_2 + b))Q(x)$
$= a(x - M_2)Q(x)$

$g_2(x) = x^e - C_2$
$= x^e - M_2^e$
$= (x - M_2)Q'(x)$

$\to (x - M_2) | (g_1(x), g_2(x))$
Using the euclidean algorithm on the two polynomials g_1 and g_2, M_2 can be recovered. $\qquad \square$

15.8.5 Forgeries of the Signature

Known Message Attack

Suppose (M_1, S_1) and (M_2, S_2) are both valid signatures.
Then, $(M_1 M_2, S_1 S_2)$ is also a valid signature.

Chosen Message Attack

Eve chooses M_1 and M_2 s.t. $M = M_1 M_2$.
Eve asks Alice to sign M_1 and M_2; let them be S_1 and S_2.
Then $S_1 S_2$ is a valid signature for M.

15.9 ElGamal Cryptosystem

15.9.1 Keygen

Choose a prime p. Note that $(Z_p^*, \times)$ is a cyclic group.

- Choose e_1 to be the primitive root of $(Z_p^*, \times)$

- Choose $d \in Z_p^*$ and compute $e_2 \equiv e_1^d \mod p$

In theory, p and e_1 can be shared as long as e_2 are kept distinct.
Public Key: (e_1, e_2, p)
Private Key: d

15.9.2 Cryptosystem

Encryption

Randomly choose $r \in Z_p^*$. M is the message.

- $C_1 \equiv e_1^r \mod p$

- $C_2 \equiv M e_2^r \mod p$

Ciphertext: (C_1, C_2)

Decryption

$C_2(C_1^d)^{-1} \equiv M e_2^r (e_1^{rd})^{-1} \equiv M(e_1^d)^r (e_1^{rd})^{-1} \equiv M \mod p$

15.9.3 Signature

Signing

Randomly choose $r \in Z_p^*$. M is the message.

- $S_1 \equiv e_1^r \mod p$

- $S_2 \equiv (M - dS_1)r^{-1} \mod (p-1)$

Signature: (S_1, S_2)

Verifying

Calculate:

- $V_1 \equiv e_1^M \mod p$

- $V_2 \equiv e_2^{S_1} S_1^{S_2} \mod p$

Verify with:

- Check $0 < S_1 < p, \ 0 < S_2 < p - 1$.

- Check $V_1 = V_2$

$V_2 \equiv e_2^{S_1} S_1^{S_2} \equiv (e_1^d)^{S_1}(e_1^r)^{S_2} \equiv e_1^{dS_1 + rS_2} \equiv e_1^M \equiv V_1 \mod p$

15.9.4 Attacking the Cryptosystem

Exposure of r

Since (C_1, C_2) and r are exposed, $M = C_2(e_2^r)^{-1} \mod p$.

Baby step, Giant step

When the random number r is small, then the following meet-in-the-middle attack is possible:
$y = e_1^x \mod p$.
Let $m = \lceil \sqrt{p} \rceil$.
Then, $\exists q, r \in \mathbb{Z}$ such that $x = mq + r, 0 \le r \le m - 1$
$\Rightarrow y = e_1^x \equiv e_1^{mq+r} \mod p$
$\Rightarrow y(e_1^{-m})^q \equiv g^r \mod p$
Hence we can find r using the following protocol:

1. Construct the table with entries $(r, e_1^r \mod p), 0 \le r \le m - 1$: (Baby step table)

2. Compute the value $g^{-m} \mod p$: (Giant step value)

3. For q from 0 to $m - 1$, find q such that $y(g^{-m})^q \equiv g^r \mod p$ in the table.

Known Plaintext Attack

Suppose the random number r is reused to encrypt two distinct messages, M and M'.
Suppose M encrypted to (C_1, C_2); M' encrypted to (C_1', C_2').
Note that $C_1 = C_1' = e_1^r, \ C_2 = Me_2^r, \ C_2' = M'e_2^r$.
If we know M', then $\frac{C_2 \times M'}{C_2'} = \frac{Me_2^r \times M'}{M'e_2^r} = M$

15.9.5 Forgeries of the Signature

Constructing from Scratch: One Variable

Choose $1 < x < p - 1$.

- $S_1 \equiv e_1^x e_2 \mod p$

- $S_2 \equiv -S_1 \mod (p - 1)$

- $M \equiv xS_2 \mod (p - 1)$

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Constructing from Scratch: Two Variables

Choose $u, v \in Z_p^*$ such that $(v, p-1) = 1$ so that $\exists v^{-1} \mod (p-1)$

- $S_1 \equiv e_1^u e_2^v \mod p$

- $S_2 \equiv -S_1 v^{-1} \mod (p-1)$

- $M \equiv S_2 u \mod (p-1)$

Known Plaintext Attack

This method can be used if the range conditions are not checked properly.
A valid signature $(M, (S_1, S_2))$ is given for $(M, p-1) = 1$ so that $\exists M^{-1} \mod (p-1)$.
Choose a message M'.
Set $u = M' M^{-1} \mod (p-1)$.
Compute $S_2 \equiv S_2 u \mod (p-1)$.
Solve the following set of linear congruences using CRT:

$$\begin{cases} S_1' = S_1 u \mod (p-1) \\ S_1' = S_1 \mod p \end{cases}$$

Then, $(M', (S_1', S_2'))$ is also a valid signature, if the range conditions are not checked.

15.10 Schnorr Digital Signature

Signatures based on cryptosystems have a weakness: they pose a threat to expose the secret key, or makes it easier to forge a specific message. Schnorr Digital Signature is a signature-only algorithm that helps solve this.

15.10.1 Keygen

- Choose a cryptographic hash function h.

- Choose a prime p.

- Choose a prime q such that:

    $q | p - 1$, and;

    The size of q is the same as the hash output.

- Choose e_0 such that it is a generator in Z_p^*.

- Set $e_1 \equiv e_0^{(p-1)/q} \not\equiv 1 \mod p$.

- Choose d.

- Set $e_2 \equiv e_1^d \mod p$.

Public Key: (h, e_1, e_2, p, q)
Private Key: d

15.10.2 Signature

Signing

Choose $r \in Z_q^*$ at random.

- $S_1 = h(M \| e_1^r \mod p)$ where $\|$ is concatenation.
- $S_2 = r + dS_1 \mod q$

Signature: (S_1, S_2)

Verifying

Calculate $V = h(M \| e_1^{S_2} e_2^{-S_1} \mod p)$.
If $V = S_1$, then M is accepted.
$e_1^{S_2} e_2^{-S_1} = e_1^{r+dS_1} e_1^{-dS_1} = e_1^r \mod p$

Part IV

Appendix

Chapter 16

Appendix

16.1 Cook-Levin Theorem

In this section

16.2 Kuratowski Theorem

In this section we prove [198].

16.2.1 The Preparation

First, we show that a planar graph can be drawn so that an arbitrary vertex or an edge is incident to the outer face.

Lemma 209
If G is planar and $v \in V(G)$, then there is a planar embedding of G such that v is on the boundary of the outer face. The same can be done for $e \in E(G)$.

Proof. We use the <u>stereographic projection</u>. In $\mathbb{R}^3$, let $z = -1$ be the plane P and $x^2 + y^2 + z^2 = 1$ be the sphere S. $(0,0,1)$ is the "north pole" of S. Define the projection $\rho : S \backslash \{(0,0,1)\} \to P$ as follows: given (x,y,z) on S which is not the north pole, draw a straight line through $(0,0,1)$ and (x,y,z). There is a unique intersection of this line with P, denoted as $(X,Y,-1)$. Then $\rho(x,y,z) = (X,Y,-1)$. Clearly ρ is bijective.

Given an embedding of a planar graph G on P, ρ^{-1} gives an embedding of G on S. Rotate the embedding so that a face incident to v or e contains the north pole. ρ gives an embedding of G on P such that the face is the outer face. $\square$

Next, we introduce the notion of connectivity. Although connectivity is a crucial part of graph theory, we didn't put this into the main part of the codex because of the length concerns.

Definition 210 (Connectivity)
A graph G is <u>k-connected</u> if $|V| > k$ and, for every $S \subset V$ with $|S| < k$, $G \backslash S$ is connected.

Theorem 211
If G is 3-connected with $|V(G)| \geq 5$, then there is an edge e such that G/e is 3-connected.

Proof. Let $e = xy$ and suppose G/e is not 3-connected. Then G/e has a cut set $\{v,z\}$. Since G is 3-connected, this set has a vertex, say v, which is the new vertex made by contracting e. That is, $\{x,y,z\}$ is a cut set of G.

Suppose that for every e, G/e is not 3-connected, so to every e corresponds a vertex z_e. Among all edges, take $e = xy$ and z_e such that $G - x - y - z$ has the largest component C, and denote another component as D. Each of x,y,z has neighbors in C and in D since G is 3-connected. Take a neighbor u of z in D and let $v = z_{zu}$.

If $v \in V(C) \cup \{x,y\}$, then $G - z - v$ is disconnected, contradicting the connectivity of G. Otherwise, $G - z - u - v$ has a component that contains all vertices in C and x and y in addition, contradicting the choice of C.

(TODO: picture) □

Then, we show the connection between minors and topological minors.

Lemma 212

$K_{3,3}$ is a topological minor of G iff $K_{3,3}$ is a minor of G.

Proof. A topological minor of G is also a minor of G. We just need to prove the other direction of the lemma.

□

Lemma 213

If K_5 is a minor of G, then $K_{3,3}$ or K_5 is a topological minor of G.

Proof. . □

16.2.2 The Proof

The last step is closely related to the Kuratowski's theorem.

Definition 214 (Convex Embedding)
A <u>convex embedding</u> of a planar graph G is a plane graph in which all edges are straight line segments and all face boundaries are convex polygons.

Lemma 215
If G is simple, 3-connected, and has no K_5 or $K_{3,3}$ as a minor, then G has a convex embedding on a plane, with no three vertices on a line.

Proof. TODO □

We are finally ready to prove the Kuratowski's theorem. For convenience, we will restate the theorem:

A graph G is planar if and only if it does not have K_5 or $K_{3,3}$ as a topological minor.

Proof. Induction on $|V|$, with trivial base case $|V| \leq 4$.

If G is disconnected, from induction there is a planar embedding of each component. Since each embedding is bounded by a finite disc, their union can be drawn on a plane.

If G is connected but not 2-connected, then take a cut-vertex v. Let G_1, $\cdots$, G_n be the connected components of $G - v$, and H_i be the subgraph induced by $V(G_i) \cup \{v\}$. Take an embedding of each H_i such that v is in the outer face [209] and squeeze it into an angle $< 2\pi/n$ at the vertex v. Joining those embeddings together forms an embedding of G.

If G is 2-connected but not 3-connected, TODO

If G is 3-connected, the conclusion immediately follows from [215]. □

16.3 What's Wrong With Kempe's Proof?

Kempe argued that switching V_{13} and V_{52} allows v to be colored by 1, but consider the following graph:

(TODO: counterexample picture)

Both chains cannot be switched because then the vertices a and b would have the same color!

In this graph, such a problem could be avoided by deliberately changing the order of vertices to be selected for induction. However, there are graphs on which such a workaround is not possible. The following is the smallest counterexample possible, and is called the Soifer graph:

(TODO: Soifer graph)