The Incomplete Codex of Mathematics for Computer Scientists

From Programmers to Hackers: Mathematical Basis to Computer Science

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Chapter 1

Introduction

Let’s face it: mathematics is hard.

But as a computer scientist, you need to know the principle of mathematics. And we know, it’s not easy. The many mathematical principles are dispersed throughout many areas of mathematics, whether it is number theory, calculus, analysis, or statistics.

This book aims to give some help to computer scientists who are tired of searching the highly dispersed information on the net or in the books. This includes the theoretical parts of computer science, such as graph, language, and complexity theories.

In the first part, mathematical preliminaries, we see the important parts from many parts of mathematics as mentioned above. This may not be directly related to any algorithms, but this will serve as a basis for many theoretical parts of computer science.

In the second part, theory-heavy parts of computer science are described as mathematically precise as possible.

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Part I

Mathematical Preliminaries

Chapter 2

Logic

There wouldn't be math or any branch of science if there weren't logic. In this section, basic mathematical proofs and the methods of proof will be discussed.

2.1 Boolean Algebra

Most branches of mathematics use propositions; that is, mathematical statements that can be determined to be either true or false. In Boolean algebra, variables and constants can take on two values: true(1) or false(0). By taking the statements to be the variables in Boolean algebra, we can think of mathematical statements as formulas of Boolean algebra.

In Boolean algebra, there are only two values, true(1) and false(0), and three basic operators, two of which are binary and one unary.

AND operator(conjunction), often denoted as $p \cdot q$ or $p \wedge q$, has the value true iff p and q are both true; false if either p or q are false. The truth-table for the AND operator is as follows:

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

OR operator(disjunction), often denoted as $p + q$ or $p \vee q$, has the value false iff p and q are both false; true if either p or q are true. The truth-table for the OR operator is as follows:

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

NOT operator(negation), often denoted as p', ~p, or $\neg p$, is a unary operator. The operator switched the state of the variable, that is, if it is true its value is false; if false the value is true. The truth-table for the NOT operator is as follows:

p	$\neg p$
0	1
1	0

Derived by composition of the basic operators, there are many secondary operators: to name the most important operators, implication($\rightarrow$), exclusive-or(XOR, $\bigoplus$), and equivalence($=$, $\equiv$). The truth-table for the operators are as follows:

p	q	$p \rightarrow q$	$p \bigoplus q$	$p \equiv q$
0	0	1	0	1
0	1	1	1	0
1	0	0	1	0
1	1	1	0	1

The operators are derived as follows:

$$
\begin{array}{lllll}
p \rightarrow q & = & \neg p \vee y & & \\
p \bigoplus q & = & (p \vee q) \wedge \neg(p \wedge q) & = & (p \wedge \neg q) \vee (\neg p \wedge q) \\
p \equiv q & = & \neg(p \bigoplus q) & = & (p \wedge q) \vee (\neg p \wedge \neg q)
\end{array}
$$

2.2 Proof Techniques

There are many methods of proof. In this section, the common methods of proof used in mathematics will be discussed.

2.2.1 Direct Proof

2.2.2 Proof by Mathematical Induction

2.2.3 Proof by Contraposition

2.2.4 Proof by Construction

2.2.5 Proof by Exhaustion

2.2.6 Computer-assisted Proof

Chapter 3

Algebraic Structures

3.1 Algebraic Structures

3.1.1 Sets

Definition 1 (Set)
A set is a collection of distinct objects.

To see some traits on sets, we literally start from nothing:

Axiom 2 (Empty Set Axiom)
There is a set containing no members, that is:

$$\exists B \text{ such that } \forall x, (x \notin B)$$

We call this set the empty set, and denote it by the symbol $\emptyset$.

We now have $\emptyset$; we now write down a few rules for how to manipulate sets.

Axiom 3 (Axiom of Extensionality)
Two sets are equal if and only if they share the same elements, that is:

$$\forall A, B[\forall z, ((z \in A) \Leftrightarrow (z \in B)) \Rightarrow (A = B)]$$

Axiom 4 (Axiom of Pairing)
Given any two sets A and B, there is a set which have the members just A and B, that is:

$$\forall A, B \exists C \forall x[x \in C \Leftrightarrow ((x = A) \vee (x = B))]$$

If A and B are distinct sets, we write this set C as $\{A, B\}$; if $A = B$, we write it as $\{A\}$.

Axiom 5 (Axiom of Union, simple version)
Given any two sets A and B, there is a set whose members are those sets belonging to either A or B, that is:

$$\forall A, B \exists C \forall x[x \in C \Leftrightarrow ((x \in A) \vee (x \in B))]$$

We write this set C as $A \cup B$.

In the simplified version of Axiom of Union, we take union of only two things, but we sometimes we want to take unions of more than two things or even more than finitely many things. This is given by the full version of the axiom:

Axiom 6 (Axiom of Union, full version)
Given any set A, there is a set C whose elements are exactly the members of the members of A, that is:

$$\forall A \exists C[x \in C \Leftrightarrow (\exists A'(A' \in A) \wedge (x \in A'))]$$

We denote this set C as

$$\bigcup_{A' \in A} A'$$

Axiom 7 (Axiom of Intersection, simple version)
Given any two sets A and B, there is a set whose members are member of both A and B, that is:

$$\forall A, B \exists C \forall x[(x \in C) \Leftrightarrow ((x \in A) \wedge (x \in B))]$$

Sometimes as union, we would want to take intersection of more than finitely many things. This is given by the full version of the axiom:

Axiom 8 (Axiom of Intersection, full version)
Given any set A, there is a set C whose elements are exactly the members of all members of A, that is:

$$\forall A \exists C \forall x[(x \in C) \Leftrightarrow (\forall A'((A' \in A) \Rightarrow (x \in A')))]$$

We denote this set C as

$$\bigcap_{A' \in A} A'$$

Axiom 9 (Axiom of Subset)
For any two sets A and B, we say that $B \subset A$ if and only if every member of B is a member of A, that is:

$$(B \subseteq A) \Leftrightarrow (\forall x(x \in B) \Rightarrow (x \in A))$$

By the Axiom of Subset we can define the power set of an any given set:

Definition 10 (Power Set)
For any set A, the power set of the set A, denoted $P(A)$, whose members are precisely the collection of all possible subsets of A, that is:

$$\forall A \exists P(A) \forall B((B \subseteq A) \Leftrightarrow (B \in P(A)))$$

Definition 11 (Equivalence Relation)
Let S be a set. An Equivalence Relation on S is a relation, denoted by ~, with the following properties, $\forall a, b, c \in S$:

- **Reflexivity** $a \sim a$
- **Symmetry** $a \sim b \Leftrightarrow b \sim a$
- **Transitivity** $(a \sim b) \wedge (b \sim c) \Rightarrow (a \sim c)$

Definition 12 (Setoid)
A setoid is a set in which an equivalence relation is defined, denoted $(S, \sim)$.

Definition 13 (Equivalence Class)
The equivalence class of $a \in S$ under ~, denoted $[a]$, is defined as $[a] = \{b \in S | a \sim b\}$.

Definition 14 (Order)
Let S be a set. An order on S is a relation, denoted by $<$, with the following properties:

- If $x \in S$ and $y \in S$ then one and only one of the following statements is true:

$$x < y, x = y, y < x$$

- For $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

Remark

- It is possible to write $x > y$ in place of $y < x$
- The notation $x \leq y$ indicates that $x < y$ or $x = y$.

Definition 15 (Ordered Set)
An ordered set is a set in which an order is defined, denoted $(S, <)$.

Definition 16 (Bound)
Suppose S is an ordered set, and $E \subset S$.
If there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is bounded above, and call β an upper bound of E. If there exists $\alpha \in S$ such that $x \geq \alpha$ for every $x \in E$, we say that E is bounded below, and call α a lower bound of E.

Definition 17 (Least Upper Bound)
Suppose that S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ with the following properties:

- β is an upper bound of E
- If $\gamma < \beta$, then γ is not an upper bound of E

Then β is called the Least Upper Bound of E or the supremum of E, denoted

$$\beta = sup(E)$$

Definition 18 (Greatest Lower Bound)
Suppose that S is an ordered set, and $E \subset S$. If there exists a $\alpha \in S$ with the following properties:

- α is a lower bound of E
- If $\gamma < \alpha$, then γ is not an lower bound of E

Then α is called the Greatest Lower Bound of E or the infimum of E, denoted

$$\beta = inf(E)$$

Definition 19 (least-upper-bound property)
An ordered set S is said to have the least-upper-bound property if the following is true:
if $E \subset S$, E is not empty, and E is bounded above, then $sup(E)$ exists in S.

Definition 20 (greatest-lower-bound property)
An ordered set S is said to have the greatest-lower-bound property if the following is true:
if $E \subset S$, E is not empty, and E is bounded below, then $inf(E)$ exists in S.

Theorem 21
Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below.
Let L be the set of all lower bounds of B. Then

$$\alpha = sup(L)$$

exists in S, and $\alpha = inf(B)$.

Proof. Note that $\forall x \in L, y \in B, x \leq y$.
L is nonempty as B is bounded below.
L is bounded above since $\forall x \in S \backslash L, \forall y \in L, x > y$.
Since S has the least-upper-bound property and $L \subset S$, $\exists \alpha = sup(L)$.
The followings hold:

- α is a lower bound of B.
  $(\because)$ $\forall \gamma \in B, \gamma > \alpha$
- β with $\beta > \alpha$ is not a lower bound of B
  $(\because)$ Since α is an upper bound of L, $\beta \notin L$.

Hence $\alpha = inf(B)$. □

Corollary 22
For all ordered sets, the Least Upper Bound property and the Greatest Lower Bound Porperty are equivalent.

3.1.2 Group

Definition 23 (Group)
A <u>group</u> is a set G with a binary operation $\cdot$, denoted $(G, \cdot)$, which satisfies the following conditions:

- **Closure:** $\forall a, b \in G, a \cdot b \in G$
- **Associativity:** $\forall a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- **Identity:** $\exists e \in G, \forall a \in G, a \cdot e = e \cdot a = a$
- **Inverse:** $\forall a \in G, \exists a^{-1} \in G, a \cdot a^{-1} = a^{-1} \cdot a = e$

Definition 24 (Semigroup)
A <u>semigroup</u> is $(G, \cdot)$, which satisfies Closure and Associativity.

Definition 25 (Monoid)
A <u>monoid</u> is a semigroup $(G, \cdot)$ which also has identity.

Definition 26 (Abelian Group)
An <u>Abelian Group</u> or <u>Commutative Group</u> is a group $(G, \cdot)$ with the following property:

- **Commutativity:** $\forall a, b \in G, a \cdot b = b \cdot a$

3.1.3 Ring

Definition 27 (Ring)
A <u>Ring</u> is a set R with two binary operations $+$ and $\cdot$, often called the addition and multiplication of the ring, denoted $(R, +, \cdot)$, which satisfies the following conditions:

- $(R, +)$ is an abelian group
- $(R, \cdot)$ is a semigroup
- **Distribution:** $\cdot$ is distributive with respect to $+$, that is, $\forall a, b, c \in R$:
  - $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
  - $(a + b) \cdot c = (a \cdot c) + (b \cdot c)$

The identity element of $+$ is often noted 0.

Definition 28 (Ring with identity(1))
A Ring with identity is a ring $(R, +, \cdot)$ of which $(R, \cdot)$ is a monoid. The identity element of $\cdot$ is often noted 1.

Definition 29 (Commutative Ring)
A commutative ring is a ring $(R, +, \cdot)$ of which $\cdot$ is commutative.

Definition 30 (Zero Divisor)
For a ring $(R, +, \cdot)$, let 0 be the identity of $+$.
$a, b \in R$, $a \neq 0$ and $b \neq 0$, if $a \cdot b = 0$, a, b are called the zero divisors of the ring.

Definition 31 (Integral Domain)
An integral domain is a commutative ring $(R, +, \cdot)$ with 1 which does not have zero divisors.

3.1.4 Field

Definition 32 (Field)
A Field is a set F with two binary operations $+$ and $\cdot$, often called the addition and multiplication of the field, denoted $(R, +, \cdot)$, which satisfies the following conditions:

- $(F, +, \cdot)$ is a ring
- $(F \backslash \{0\}, \cdot)$ is a group

Alternatively, a Field may be defined with a set of Field Axioms listed below:

(A) **Axioms for Addition**

- (A1) **Closed under Addition**
  $\forall a, b \in F, a + b \in F$
- (A2) **Addition is Commutative**
  $\forall a, b \in F, a + b = b + a$
- (A3) **Addition is Associative**
  $\forall a, b, c \in F, (a + b) + c = a + (b + c)$
- (A4) **Identity of Addition**
  $\exists 0 \in F, \forall a \in F, 0 + a = a$
- (A5) **Inverse of Addition**
  $\forall a \in F, \exists -a \in F, a + (-a) = 0$

(M) **Axioms for Multiplication**

- (M1) **Closed under Multiplication**
  $\forall a, b \in F, a \cdot b \in F$
- (M2) **Multiplication is Commutative**
  $\forall a, b \in F, a \cdot b = b \cdot a$
- (M3) **Multiplication is Associative**
  $\forall a, b, c \in F, (a \cdot b) \cdot c = a \cdot (b \cdot c)$
- (M4) **Identity of Multiplication**
  $\exists 1 \in F, \forall a \in F, 1 \cdot a = a$

(M5) **Inverse of Multiplication**
$\forall a \in F \backslash \{0\}, \exists a^{-1} \in F, a \cdot a^{-1} = 1$

(D) **Distributive Law**
$\forall a, b, c \in F, (a + b) \cdot c = a \cdot c + b \cdot c$
where $\cdot$ takes precedence over $+$.

Theorem 33
Let F be a field. Let 0 be the additive identity of F. Then, $\forall a \in F, 0 \cdot a = 0$

Definition 34 (Ordered Field)
An ordered field is a field F which is an ordered set, such that the order is compatible with the field operations, that is:

- $x + y < x + z$ if $x, y, z \in F$ and $y < z$
- $xy > 0$ if $x, y \in F$, $x > 0$ and $y > 0$

3.1.5 Polynomial Ring

Definition 35 (Polynomial over a Ring)
A polynomial $f(x)$ over the ring $(R, +, \cdot)$ is defined as

$$f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x^1 + \cdots, a_i \in R$$

where $a_i = 0$ for all but finitely many values of i.
The degree of the polynomial $\deg(f)$ is defined as $\deg(f) = \max\{n | n \in \mathbb{N}, a_n \neq 0\}$.
The leading coefficient of the polynomial is defined as $a_{\deg(f)}$.

Definition 36 (Addition and Multiplication of Polynomials)
Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{i=0}^{\infty} b_i x^i$, $a_i, b_i \in R$ be a polynomial over the ring $(R, +, \cdot)$. Define:

$$f(x) + g(x) = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

$$f(x) g(x) = \sum_{k=0}^{\infty} (c_k) x^k \text{ where } c_k = \sum_{i+j=k} a_i b_j$$

Definition 37 (Polynomial Ring)
The set of polynomials over the ring $(R, +, \cdot)$, $R[x] = \{f(x) | f(x)$ is a polynomial over $R\}$ is called the Polynomial Ring(or Polynomials) over R.

Theorem 38 (Degree of Polynomial on Addition and Multiplication)
Let $f(x), g(x) \in R[x]$ with $\deg(f) = n$, $\deg(g) = m$.

- $0 \leq \deg(f + g) \leq \max(\deg(f), \deg(g))$
- $\deg(fg) \leq \deg(f) + \deg(g)$.
  If $(R, +, \cdot)$ is an integral domain, $\deg(fg) = \deg(f) + \deg(g)$

Theorem 39 (Relationship between a Ring and its Polynomial Ring)
Let $(R, +, \cdot)$ be a ring and $R[x]$ the polynomials over R.

1. If $(R, +, \cdot)$ is a commutative ring with 1, then $(R[x], +, \cdot)$ is a commutative ring with 1.
2. If $(R, +, \cdot)$ is a integral domain, then $(R[x], +, \cdot)$ is a integral domain.

Theorem 40 (Division Algorithm for Polynomials over a Ring)
Let $(R, +, \cdot)$ be a commutative ring with 1.
Let $f(x), g(x) \in R[x]$, $g(x) \neq 0$ with the leading coefficient of $g(x)$ being invertible.
Then, $\exists! q(x), r(x) \in R[x]$ such that

$$f(x) = q(x)g(x) + r(x)$$

where either $r(x) = 0$ or $\deg(r) < \deg(g)$.

Proof. Use induction on $\deg(f)$.
1. $f(x) = 0$ or $\deg(f) < \deg(g)$: $q(x) = 0, r(x) = f(x)$
2. $\deg(f) = \deg(g) = 0$: $q(x) = f(x) \cdot g(x)^{-1}, r(x) = 0$
3. $\deg(f) \geq \deg(g)$:

1) Existence
Let $\deg(f) = n$, $\deg(g) = m$, $n > m$.
Suppose the theorem holds for $\deg(f) < n$.
Let $f(x) = a_0 + a_1 x^1 + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x^1 + \cdots + b_m x^m$.
Choose $f_1(x) = f(x) - (a_n b_m^{-1}) x^{n-m} g(x) \in R[x]$.
Since $\deg(f_1) < n$, $\exists q(x), r(x) \in R[x]$ so that $f_1(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg(r) < \deg(g)$.
$f_1(x) = f(x) - (a_n b_m^{-1}) x^{n-m} g(x) = g(x)q(x) + r(x)$
$f(x) = g(x)((a_n b_m^{-1}) x^{n-m} + q(x)) + r(x)$
Hence such pair exists.

2) Uniqueness
Suppose $f(x) = g(x)q_1(x) + r_1(x) = g(x)q_2(x) + r_2(x)$.
$g(x)(q_1(x) - q_2(x)) = r_2(x) - r_1(x)$
If $r_1 \neq r_2$, $\deg(g) > \deg(r_2 - r_1) = \deg(g(q_1 - q_2))$.
Since $\deg(g(q_1 - q_2)) \geq \deg(g)$ if $q_1 - q_2 \neq 0$, $q_1 = q_2$, but if so, $r_1 = r_2$.
If $r_1 = r_2$, trivially $q_1 = q_2$.
Hence they exist uniquely. □

3.1.6 Vector Space

Definition 41 (Vector Space)
A vector space over a field(sometimes called the scalar of the vector space) F is a set V together with two operations, addition($+ : V \times V \to V$) and scalar multiplication($\cdot : F \times V \to V$), satisfying the following axioms:

(A) **Axioms for Addition**

- (A1) **Closed under Addition**
  $\forall \boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{u} + \boldsymbol{v} \in V$
- (A2) **Addition is Commutative**
  $\forall \boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
- (A3) **Addition is Associative**
  $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in v, (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$
- (A4) **Identity of Addition(Zero vector)**
  $\exists \boldsymbol{0} \in V, \forall \boldsymbol{u} \in F, \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u} + \boldsymbol{0} = \boldsymbol{u}$
- (A5) **Inverse of Addition(Negative)**
  $\forall \boldsymbol{u} \in V, \exists -\boldsymbol{u} \in V, \boldsymbol{u} + (-\boldsymbol{u}) = 0$

(M) **Axioms for Scalar Multiplication**

(M1) **Closed under Scalar Multiplication**
$\forall k \in F, \boldsymbol{u} \in V, k \cdot \boldsymbol{u} \in V$

(M2) **Scalar Multiplication is Distributive(1)**
$\forall k \in F, \boldsymbol{u}, \boldsymbol{v} \in V, k \cdot (\boldsymbol{u} + \boldsymbol{v}) = k \cdot \boldsymbol{u} + k \cdot \boldsymbol{v}$

(M3) **Scalar Multiplication is Distributive(2)**
$\forall k, m \in F, \boldsymbol{u} \in V, (k + m) \cdot \boldsymbol{u} = k \cdot \boldsymbol{u} + m \cdot \boldsymbol{u}$

(M4) **Scalar Multiplication is Associative**
$\forall k, m \in F, \boldsymbol{u} \in V, (km) \cdot \boldsymbol{u} = k \cdot (m \cdot \boldsymbol{u})$

(M5) **Identity of Scalar Multiplication**
$\exists 1 \in F, \forall \boldsymbol{u} \in V, 1 \cdot \boldsymbol{u} = \boldsymbol{u}$

A vector space over $\mathbb{R}$ is called a real vector space.

Theorem 42
Let V be a vector space over a field F. $\boldsymbol{u} \in V$, $k \in F$, 0 the additive identity of F, 1 the multiplicative identity of F, $\mathbf{0}$ the additive identity of V. Then, the followings hold:

- $0 \cdot \boldsymbol{u} = \mathbf{0}$
- $k \cdot \mathbf{0} = \mathbf{0}$
- $-1 \cdot \boldsymbol{u} = -\boldsymbol{u}$
- If $k \cdot \boldsymbol{u} = \mathbf{0}$, then $k = 0$ or $\boldsymbol{u} = \mathbf{0}$.

Definition 43 (Subspace of a Vector Space)
A subset W of a vector space V is called a subspace of V if W is a vector space under the addition and scalar multiplication defined on V.

Theorem 44
If W is a set of one or more vectors in a vector space V over the field F, then W is a subspace of V iff the following conditions hold:

- $\forall \boldsymbol{u}, \boldsymbol{v} \in W, \boldsymbol{u} + \boldsymbol{v} \in W$
- $\forall k \in F, \boldsymbol{u} \in W, k \cdot \boldsymbol{u} \in W$

Theorem 45
If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then $\cap_{i=1}^{r} W_i$ is also a subspace of V.

Definition 46 (Linear Combination)
If $\boldsymbol{w}$ is a vector in a vector space V over the field F, then $\boldsymbol{w}$ is said to be a Linear Combination of the vectors $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r} \in V$ if $\boldsymbol{w}$ can be expressed in the form $\boldsymbol{w} = \sum_{i=1}^{r} k_i \boldsymbol{v_i}$, where $k_1, k_2, \ldots, k_r \in F$. These scalars are called the coefficients of the linear combination.

Definition 47 (Span)
The subspace of a vector space V that is formed from all possible linear combinations of the vectors in a nonempty set S is called the Span of S, and we say that the vectors in S span that subspace.

Theorem 48
If $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}\}$ and $S' = \{\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_k}\}$ are nonempty sets of vectors in a vector space V, then $\mathtt{span}(S) = \mathtt{span}(S')$ iff each vector in S is a linear combination of those in S' and vice versa.

Definition 49 (Basis)
If V is any vector space and $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}\}$ is a finite set of linearly independent vectors in V which spans V, then S is called a basis for V.

Theorem 50
All bases for a finite-dimensional vector space have the same number of vectors.

Definition 51 (Dimension)
The dimension of a finite-dimensional vector space V, denoted by $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, the zero vector space is defined to have dimension zero.

Theorem 52 (Plus/Minus Theorem)
Let S be a nonempty set of vectors in a vector space V.

- If S is a linearly independent set, and if $\boldsymbol{v}$ is a vector in V that is outside of $span(S)$, then the set $S \cup \{\boldsymbol{v}\}$ that results by inserting $\boldsymbol{v}$ into S is still linearly independent.
- If $\boldsymbol{v} \in S$ is expressible as a linear combination of the vectors in $S - \{\boldsymbol{v}\}$, then $span(S) = span(S - \{\boldsymbol{v}\})$.

Theorem 53
Let V be an n dimensional vector space, and let S be a set in V with exactly n vectors. Then S is a basis for V iff $span(S) = V$ or S is linearly independent.

Theorem 54
Let S be a finite set of vectors in a finite dimensional vector space V.

- If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.
- If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Theorem 55
If W is a subspace of a finite-dimensional vector space V, then:

- W is finite dimensional
- $dim(W) \leq dim(V)$
- $W = V$ iff $dim(W) = dim(V)$.
- A is positive definite iff all eivenvalues of A are positive.

Theorem 56 (Uniqueness of Basis Representation)
If $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}\}$ is a basis for a vector space V, then every vector $\boldsymbol{v}$ in V can be expressed in the form $\boldsymbol{v} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_r\boldsymbol{v_r}$ in exactly one way.

Definition 57 (Coordinate)
Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}\}$ be a basis for a vector space V over the field F, and $\boldsymbol{v} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_r\boldsymbol{v_r}$ is the expression for a vector $\boldsymbol{V}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the coordinates of $\boldsymbol{v}$ relative to the basis S. The vector $(c_1, c_2, \ldots, c_n)$ in F^n constructed from these coordinates is called the coordinate vector of $\boldsymbol{v}$ relative to S, denoted by $(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_n)$.

3.1.7 Inner Product Space

Definition 58 (Inner Product Space)
An inner product on a real vector space V is a function that associates a real number $\langle \boldsymbol{u}, \boldsymbol{v} \rangle$ with each pair of vectors in V in a such way that the following axioms are satisfied for all vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and all scalars k.

1. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{u} \rangle$ [Symmetry Axiom]
2. $\langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle + \langle \boldsymbol{v}, \boldsymbol{w} \rangle$ [Additivity Axiom]
3. $\langle k\boldsymbol{u}, \boldsymbol{v} \rangle = k \langle \boldsymbol{u}, \boldsymbol{v} \rangle$ [Homogeneity Axiom]
4. $\langle \boldsymbol{v}, \boldsymbol{v} \rangle \geq 0$ and $\langle \boldsymbol{v}, \boldsymbol{v} \rangle = 0$ iff $\boldsymbol{v} = \boldsymbol{0}$. [Positivity Axiom]

A real vector space with an inner product is called a real inner product space.

Definition 59 (Norm and Distance)
If V is a real inner product space, then the norm or length of a vector $\boldsymbol{v}$ in V, denoted by $\|\boldsymbol{v}\|$, is defined by

$$\|\boldsymbol{v}\| = \sqrt{\langle \boldsymbol{v}, \boldsymbol{v} \rangle}$$

and the distance between two vectors, denoted by $d(\boldsymbol{u}, \boldsymbol{v})$, is defined by

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{\langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{u} - \boldsymbol{v} \rangle}$$

A vector of norm 1 is called a unit vector.

If V is an inner product space, then the set of points in V that satisfy $\|\boldsymbol{u}\| = 1$ is called the unit sphere or sometimes the unit circle in V.

Theorem 60
If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in a real inner place V and if k is a scalar, then:

- $\|\boldsymbol{v}\| \geq 0$ with equality iff $\boldsymbol{v} = \boldsymbol{0}$
- $\|k\boldsymbol{v}\| = |k|\|\boldsymbol{v}\|$
- $d(\boldsymbol{u}, \boldsymbol{v}) = d(\boldsymbol{v}, \boldsymbol{u})$
- $d(\boldsymbol{u}, \boldsymbol{v}) \geq 0$ with equality iff $\boldsymbol{u} = \boldsymbol{v}$

Definition 61
If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and if k is a scalar, then:

- $\langle \boldsymbol{0}, \boldsymbol{v} \rangle = \langle \boldsymbol{v}, \boldsymbol{0} \rangle = 0$
- $\langle \boldsymbol{u}, \boldsymbol{v} + \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle + \langle \boldsymbol{u}, \boldsymbol{w} \rangle$
- $\langle \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{v} \rangle - \langle \boldsymbol{u}, \boldsymbol{w} \rangle$
- $\langle \boldsymbol{u} - \boldsymbol{v}, \boldsymbol{w} \rangle = \langle \boldsymbol{u}, \boldsymbol{w} \rangle - \langle \boldsymbol{v}, \boldsymbol{w} \rangle$
- $k \langle \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, k\boldsymbol{v} \rangle$

Chapter 4

Number Theory

4.1 Arithmetic

4.1.1 Integer Arithmetic

Theorem 62 (Division Algorithm)

Definition 63 (Divisibility)

Theorem 64 (Euclidean Algorithm)

Theorem 65 (Extended Euclidean Algorithm)

Definition 66 (Linear Diophantine Equation)

Theorem 67 (Solutions for Linear Diophantine Equation)

4.1.2 Modular Arithmetic

Definition 68 (Modulus)

Chapter 5

Analysis

5.1 Metric Spaces

5.1.1 Topology of Metric Spaces

Definition 69 (Metric Space)
A set X equipped with a function $d : X \times X \to \mathbb{R}$ is a metric space if d satisfies, for all $p, q, r \in X$:

1. $d(p,q) > 0$ for $p \neq q$, and $d(p,p) = 0$.
2. $d(p,q) = d(q,p)$.
3. $d(p,q) \leq d(p,r) + d(r,q)$. This inequality is called the triangle inequality.

The elements of X are called points. The function d is called a metric.

Definition 70
Let X be a metric space, $E \subseteq X$, and $p \in X$.

- A neighborhood of p, denoted $N_r(p)$, is $\{q \in X | d(p,q) \leq r\}$, where $r > 0$.
- p is a limit point of E if every neighborhood of p contains $q \in E$ different from p. The set of all limit points of E is denoted E'.
- The boundary of E is (TODO)
- p is an interior point of E if there is a neighborhood of p that is contained in E.
- p is an isolated point of E if $p \in E$ and p is not a limit point of E.
- E is open if every point in E is an interior point.
- E is closed if if every limit point of E is in E.
- E is bounded if there is a neighborhood of some p that contains E.
- E is dense if every point of X is a limit point of E or a point of E.

Here is a figure demonstrating these notions in the space $\mathbb{R}$ with the metric $d(x,y) = |x-y|$ and $E = [0,1) \cap \{2\}$:
(TODO)
Note that a set can be both open and closed. For example, an empty set is (vacuously) both open and closed. X itself is also both open and closed.

The notions in topology will be covered in greater detail in the Topology chapter.

IMPORTANT: From now on in this chapter, assume X is always a metric space with the metric d, and $E \subseteq X$, unless stated otherwise.

Proposition 71

1. A neighborhood is open.
2. If p is a limit point of E, then every neighborhood contains infinitely many points of E.
3. E is open iff E^C is closed.
4. E is closed iff E^C is open.

Proof. 1. Let $q \in N_r(p)$. Then $N_{r-d(p,q)}(q) \subseteq N_r(p)$ because, if $x \in N_{r-d(p,q)}(q)$, then $d(p,x) \leq d(p,q) + d(q,r) < d(p,q) + r - d(p,q) = r$ so $x \in N_r(p)$.

2. Suppose some neighborhood $N_r(p)$ contains only finitely many points of E, namely x_1, $\cdots$, x_k. Let $r = min_{i=1}^{k} d(p,x_i)$. Then $N_r(p)$ contains no points of E, contradiction.

3. Suppose E is open and x is a limit point of E^C. Then since every neighborhood of x intersects E^C, x is not an interior point of E. Therefore $x \in E^C$. Conversely, suppose E^C is closed and $x \in E$. Since $x \notin E^C$, x is not a limit point of E^C. Therefore there is a neighborhood of x which does not intersect E^C, and that is contained in E. Therefore x is an interior point.

4. $E = (E^C)^C$. □

Proposition 72

1. TODO

Proof. TODO □

5.1.2 Compact Sets

Definition 73 (Compact Set)
An open cover of E is a collection of open subsets of X whose union contains E. A finite subcover of an open cover is a finite subset whose union still contains E. E is compact if every open cover of E contains a finite subcover.

TODO

5.2 Sequences

Definition 74 (Convergence)
A sequence $\{p_n\}$ in X converges to $p \in X$ if, for every $\epsilon > 0$, there is an integer N such that $n \geq N$ implies $d(p_n, p) < \epsilon$. We also write $p_n \to p$, or $lim_{n \to p} p_n = p$. A sequence diverges if it does not converge.

Proposition 75
Let $\{p_n\}$ be a sequence in X.

1. $\{p_n\} \to p \in X$ iff for every $N_r(p)$, there are only finitely many terms of $\{p_n\}$ that are not in $N_r(p)$.
2. If $\{p_n\}$ converges to both $p, q \in X$, then $p = q$.

3. If $\{p_n\}$ converges, then it is bounded.

4. If p is a limit point of E, then there is a sequence in E that converges to p.

Proof. 1. Suppose $\{p_n\} \to p \in X$. Then for every $r > 0$, there is N such that $n \geq N$ implies $d(p_n, p) < r$, i.e. $p_n \in N_r(p)$. Conversely, given $\epsilon > 0$, suppose there are only finitely many terms $p_{n_1}, p_{n_2}, \cdots, p_{n_k}$ that are not in $N_\epsilon(p)$. Then $n \geq n_k + 1$ implies $p_n \in N_\epsilon(p)$, i.e. $d(p_n, p) < \epsilon$.

2. Given any $\epsilon > 0$, take N, M such that $n \geq N$ implies $d(p_n, p) < \epsilon/2$ and $n \geq M$ implies $d(p_n, q) < \epsilon/2$. Then $n \geq max(N, M)$ implies $d(p, q) \leq d(p_n, p) + d(p_n, q) < \epsilon$. Since ϵ is arbitrary, $p = q$.

3. Let $p_n \to p$. Take N such that $n \geq N$ implies $d(p_n, p) < 1$. Then every p_n satisfies $d(p_n, p) \leq max(1, d(p_1, p), \cdots, d(p_N, p))$.

4. Take each p_n as any point in $E \cap N_{1/n}(p)$. Then for any $\epsilon > 0$, there is $N > 1/\epsilon$, and $n > N$ implies $d(p_n, p) < \epsilon$. Therefore $p_n \to p$. □

Definition 76 (Cauchy Sequence)
A sequence $\{p_n\}$ in X is Cauchy if for every $\epsilon > 0$ there is an integer N such that $n, m \geq N$ implies $d(p_n, p_m) < \epsilon$.

Every convergent sequence is Cauchy, as we will show, but not every Cauchy sequence converges. For example, $\{1/n\}$ in the metric space $(0, 1]$ does not converge.

Proposition 77
Every convergent sequence is Cauchy.

Proof. Let $p_n \to p$. Given $\epsilon > 0$, take N such that $n \geq N$ implies $d(p_n, p) < \epsilon/2$. Then $n, m \geq N$ implies $d(p_n, p_m) \leq d(p_n, p) + d(p_m, p) < \epsilon$. □

5.3 Series

5.4 Continuity

5.5 Differentiation

5.6 Integral

5.7 Sequences and Series of Functions

Chapter 6

Linear Algebra

The target of Linear Algebra is to solve a system of homogenous linear equations. To do so, we deal with vectors and matrices.

6.1 Vector Spaces

For the definitions on vector spaces, subspaces, and bases, refer to the chapter 3.1.6.

6.1.1 Linear Independence

We now define linear independence, one of the most important concepts utilized in linear algebra.

Definition 78 (Linear Independence)
if $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}\}$ is a nonempty set of vectors in a vector space V, then the vector equation $k_1\boldsymbol{v_1} + k_2\boldsymbol{v_2} + \cdots + k_r\boldsymbol{v_r} = \boldsymbol{0}$ has at least one solution, namely, $k_1 = 0, k_2 = 0, \ldots, k_r = 0$, the <u>trivial solution</u>. If this is the only solution, then S is said to be a <u>linearly independent set</u>. If there are solutions in addition to the trivial solution, then S is said to be <u>linearly dependent</u>.

Theorem 79
Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r}\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

6.1.2 Orthogonality

Refer to Chapter 3.1.7 on information on general inner product spaces and definition on the real inner product space.

Definition 80 (Euclidean Inner Product)
Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$. The inner product of the two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as

$$\boldsymbol{u} \cdot \boldsymbol{v} = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

is called the <u>Euclidean inner product</u> or <u>standard inner product</u>.

We call R^n with the Euclidean inner product <u>Euclidean n-space</u>.

Definition 81 (Euclidean Norm)
The norm of $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ in $\mathbb{R}^n$, denoted $\|\boldsymbol{u}\|$, is defined by

$$\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}} = \sqrt{\sum_{i=1}^{n} u_i^2} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$$

Definition 82 (Euclidean Distance)
If $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$, then the distance between $\boldsymbol{u}$ and $\boldsymbol{v}$, denoted $d(\boldsymbol{u}, \boldsymbol{v})$, and define it to be:

$$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

Definition 83 (Unit vectors)
A vector $\boldsymbol{u}$ in $\mathbb{R}^n$ is said to be a unit vector iff $\|\boldsymbol{u}\| = 1$.

Definition 84 (Angle)
The angle between two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ is defined by

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$$

Theorem 85 (Cauchy-Schwarz Inequality)
If $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ are vectors in $\mathbb{R}^n$, then $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq \|u\|\|v\|$.
In terms of components:

$$|u_1v_1 + u_2v_2 + \cdots + u_nv_n| \leq \left(u_1^2 + u_2^2 + \cdots + u_n^2\right)^{1/2} \left(v_1^2 + v_2^2 + \cdots + v_n^2\right)^{1/2}$$

Theorem 86 (Triangle Inequality)
If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are vectors in $\mathbb{R}^n$, then:

- $\|\boldsymbol{u} + \boldsymbol{v}\| \geq \|\boldsymbol{u}\| + \|\boldsymbol{v}\|$: Triangle Inequality for Vectors
- $d(\boldsymbol{u}, \boldsymbol{v}) \geq d(\boldsymbol{u}, \boldsymbol{v}) + d(\boldsymbol{w}, \boldsymbol{v})$: Triangle Inequality for Distances

Theorem 87 (Equations for Vectors within the Euclidean Space)
If $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors in $\mathbb{R}^n$

- $\|\boldsymbol{u} + \boldsymbol{v}\|^2 + \|\boldsymbol{u} - \boldsymbol{v}\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$: Parallelogram Equation for Vectors
- $\boldsymbol{u} \cdot \boldsymbol{v} = \frac{1}{4}\|\boldsymbol{u} + \boldsymbol{v}\|^2 - \frac{1}{4}\|\boldsymbol{u} - \boldsymbol{v}\|^2$

Definition 88 (Orthogonal Vectors)
Two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in $\mathbb{R}^n$ are said to be orthogonal or perpendicular if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

Definition 89 (Orthogonal set)
A nonempty set of vectors in $\mathbb{R}^n$ is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal. If they are also all unit vectors, it is called an orthonormal set.
In other words, for a set $\{u_1, u_2, \ldots, u_n\}$ to be orthogonal:

$$u_i \cdot u_j \begin{cases} \|u_i\|^2 & i = j \\ 0 & i \neq j \end{cases}$$

And for the set to be orthonormal, in addition to above, $\forall i \in \{1, 2, \ldots n\}, \|u_i\| = 1$.

Definition 90
An orthogonal set of nonzero vectors is linearly independent.

Definition 91 (Orthogonal Complement)
The orthogonal complement of a subspace W of an inner product space V, denoted $W^{\perp}$, is defined to be the set of all vectors in V that are orthogonal to every vector of W.

Theorem 92
Suppose W is a subspace of an inner product space V.

- $W^{\perp}$ is a subspace of V.
- $W \cap W^{\perp} = \{\mathbf{0}\}$

Theorem 93
Suppose W is a subspace of an inner product space V. Then, $(W^{\perp})^{\perp} = W$.

Theorem 94
Let $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ be a basis for an inner product space V.

- If S is an orthogonal basis for V, and $\boldsymbol{u} \in V$, then

$$\boldsymbol{u} = \frac{\langle \boldsymbol{u}, \boldsymbol{v_1} \rangle}{\|v_1\|^2}\boldsymbol{v_1} + \frac{\langle \boldsymbol{u}, \boldsymbol{v_2} \rangle}{\|v_2\|^2}\boldsymbol{v_2} + \cdots + \frac{\langle \boldsymbol{u}, \boldsymbol{v_n} \rangle}{\|v_n\|^2}\boldsymbol{v_n}$$

  And thus

$$(\boldsymbol{u})_S = \left(\frac{\langle \boldsymbol{u}, \boldsymbol{v_1} \rangle}{\|v_1\|^2}, \frac{\langle \boldsymbol{u}, \boldsymbol{v_2} \rangle}{\|v_2\|^2}, \cdots, \frac{\langle \boldsymbol{u}, \boldsymbol{v_n} \rangle}{\|v_n\|^2} \right)$$

- If S is an orthonormal basis for V, and $\boldsymbol{u} \in V$, then

$$\boldsymbol{u} = \langle \boldsymbol{u}, \boldsymbol{v_1} \rangle \boldsymbol{v_1} + \langle \boldsymbol{u}, \boldsymbol{v_2} \rangle \boldsymbol{v_2} + \cdots + \langle \boldsymbol{u}, \boldsymbol{v_n} \rangle \boldsymbol{v_n}$$

  And thus

$$(\boldsymbol{u})_S = (\langle \boldsymbol{u}, \boldsymbol{v_1} \rangle, \langle \boldsymbol{u}, \boldsymbol{v_2} \rangle, \cdots, \langle \boldsymbol{u}, \boldsymbol{v_n} \rangle)$$

Theorem 95 (Projection Theorem)
If W is a finite-dimensional subspace of an inner product space V, then every $\boldsymbol{u} \in V$ can be expressed in exactly one way in the form $\boldsymbol{u} = \boldsymbol{w_1} + \boldsymbol{w_2}$, where $\boldsymbol{w_1} \in W$ and $\boldsymbol{w_2} \in W^{\perp}$.

In the theorem above, the vector $\boldsymbol{w_1}$ is called the orthogonal projection of $\boldsymbol{u}$ on W or vector component of $\boldsymbol{u}$ along W, and the vector $\boldsymbol{w_2}$ is called the vector component of $\boldsymbol{u}$ orthogonal to W. Calculating this can be done using the orthogonal or orthonormal basis of W, as given in theorem [94]. We also give a method to use any basis on appendix [16.2], although often times using orthonormal basis will yield a more comprehensive understanding of $\boldsymbol{u}$ through its coordinates. We now give the following theorem:

Theorem 96
Every nonzero finite-dimensional inner product space has an orthonormal basis.

We now give a method to convert any given basis of a vector space to an orthogonal(or orthonormal) basis. This process is called the Gram-Schmidt Process.

Method 97 (Gram-Schmidt Process)
To convert a basis $\{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$ into an orthogonal basis $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$, perform the following computations, where $W_i = span(\{u_k | k \leq i\})$:

1. $\boldsymbol{v_1} = \boldsymbol{u_1}$

2. $\boldsymbol{v_2} = \boldsymbol{u_2} - \text{proj}_{W_1}\boldsymbol{u_2} = \boldsymbol{u_2} - \frac{\langle \boldsymbol{u_2}, \boldsymbol{v_1} \rangle}{\|v_1\|^2}\boldsymbol{v_1}$

3. $\boldsymbol{v_3} = \boldsymbol{u_3} - \text{proj}_{W_1}\boldsymbol{u_3} = \boldsymbol{u_3} - \frac{\langle \boldsymbol{u_3}, \boldsymbol{v_1} \rangle}{\|v_1\|^2}\boldsymbol{v_1} - \frac{\langle \boldsymbol{u_3}, \boldsymbol{v_2} \rangle}{\|v_2\|^2}\boldsymbol{v_2}$

   $\vdots$

And continue for n steps.
Note that $W_i = span(\{u_k | k \leq i\}) = span(\{v_k | k \leq i\})$.
Optionally, normalize to get the orthonormal basis.

Theorem 98
If W is a finite-dimensional inner product space, then:

- Every orthogonal set of nonzero vectors in W can be enlarged to an orthogonal basis for W.
- Every orthonormal set in W can be enlarged to an orthonormal basis for W.

6.2 Matrix

6.2.1 Matrices and its operations

Definition 99 (Matrix)
A matrix is a rectangular array of numbers. The numbers in the array are called the entries in the matrix.

Equality, addition, and subtraction can only be defined on same-sized matrices, and is defined elementwise; scalar multiplication is also defined elementwise.

Definition 100 (Matrix Multiplication)
If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the product AB is the $m \times n$ matrix whose entries are determined as follows: The entry of AB on row i and column j, multiply the corresponding entries from the row i from A and column j from B, then add them all together.

Matrices of the same size may be used in a linear combination, just like vectors[46].

Definition 101 (Linear Combination of a Matrix)
If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form

$$c_1A_1 + c_2A_2 + \cdots + c_rA_r$$

is called a linear combination of $A_1, A_2, \ldots, A_r$ with coefficients $c_1, c_2, \ldots, c_r$.

Theorem 102
If A is an $m \times n$ matrix and if $\boldsymbol{x}$ is an $n \times 1$ column vector, then the product $A\boldsymbol{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\boldsymbol{x}$.

Definition 103 (Transpose)
For any $m \times n$ matrix, then the transpose of A, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A and so forth.

Definition 104 (Trace)
For a square matrix A, the trace of A, denoted $tr(A)$, is defined to be the sum of the entries on the main diagonal of A.

6.3 Matrices and Vector Spaces

6.3.1 Fundamental Spaces of a Matrix

There are four important vector spaces on any given matrix, which are row, column, null, and left null spaces.

Definition 105 (Fundamental Spaces of a Matrix)

- A Column space of a matrix, denoted $im(A)$(image of A), $range(A)$(range of A), $col(A)$ or $C(A)$, is the vectors spanned by the column vectors of the matrix. $dim(col(A))$ is often called the rank of A, denoted $rank(A)$.
- A Row space of a matrix, denoted $col(A^T)$, and sometimes called the coimage, is the vectors spanned by the row vectors of the matrix.
- A Null space of a matrix, denoted $ker(A)$(kernel of A), $null(A)$ or $N(A)$, is the vector space of the solution vectors of the equation $A\boldsymbol{x} = \mathbf{0}$. $dim(null(A))$ is often called the nullity of A, denoted $nullity(A)$.
- A Left Null space of a matrix, denoted $null(A^T)$, and sometimes called the cokernel, is the vector space of the solutions vectors of the equation $A^T\boldsymbol{y} = \mathbf{0}$.

The four spaces together are called the fundamental spaces of a matrix.

Definition 106 (Rank of a Matrix)
The rank of a matrix A, denoted $rank(A)$, is defined to be the dimension of the column space.

Definition 107 (Full Rank)
A matrix is said to have full rank if its rank is largest possible among the matrices of the same dimensions, which is the minimum of the number of rows and columns.

Theorem 108
$rank(A)$ equals the number of nonzero rows in $rref(A)$.

From the definitions above, we gain the fundamental theorem of linear algebra.

Theorem 109 (Fundamental Theorem of Linear Algebra, Pt. 1)
Suppose a matrix A is $m \times n$. Let $r = rank(A)$. Fundamental subspaces of the matrix A has the following dimensions:

Name of Subspace	Containing Space	Dimension
Column Space ($C(A)$)	$\mathbb{R}^m$	$rank(A) = r$
Null Space ($N(A)$)	$\mathbb{R}^n$	$nullity(A) = n - r$
Row Space ($C(A^T)$)	$\mathbb{R}^n$	$rank(A) = r$
Left Nullspace ($N(A^T)$)	$\mathbb{R}^m$	$corank(A) = m - r$

Theorem 110 (Fundamental Theorem of Linear Algebra, Pt. 2)

- $N(A)^\perp = C(A^T)$ in $\mathbb{R}^n$, that is, nullspace and row space are orthogonal complements.
- $C(A)^\perp = N(A^T)$ in $\mathbb{R}^m$, that is, column space and left null space are orthogonal complements.

6.3.2 Change of Basis

We start from the definition of basis[49] and the concept of coordinates[57]. We assume that the scalar is $\mathbb{R}$ for simplicity, although any other field may be used as a scalar.

Say that we are talking about a general vector space V over a scalar F which has $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$ as its basis, by theorem [56], any vector $\boldsymbol{v} \in V$ can be represented uniquely as $\boldsymbol{v} = c_1\boldsymbol{v_1} + c_2\boldsymbol{v_2} + \cdots + c_n\boldsymbol{v_n}$, where $c_i \in F$. Observe that the vector $(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_n) \in \mathbb{F}^n$, and hence once basis S is given for a vector space V, theorem [56] ensures that this correspondence between vectors in V and $\mathbb{F}^n$ is one-to-one.

However this is not that simple. Suppose the ordering of the basis vectors is switched via a permutation σ, so that $S = \{u_i | u_i = v_{\sigma(i)}\}$. Now, the set of the basis stays the same, but $(\boldsymbol{v})_S = (c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(n)})$. In this exact reason, when we determine the coordinates, an ordered set(i.e. a set in which ordering matters) is used. Some authors call a set of basis vectors of which changing the order is restricted an ordered basis. We simply opt to the solution that when discussing a vector space and its basis S, the order of the vectors in S remain fixed unless stated otherwise.

In the special case where $V = \mathbb{R}^n$ and S is the standard basis, i.e. $S = \{e_1, e_2, \ldots, e_n\}$ where e_i has zeroes as all of its components except for the i-th component, the coordinate vector $(\boldsymbol{v})_S$ and the vector $\boldsymbol{v}$ are the same.

If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a finite-dimensional vector space V, and if $(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_n)$ is the coordinate vector of $\boldsymbol{v}$ relative to S, then the mapping $\boldsymbol{v} \to (\boldsymbol{v})_S$ creates a connection between vectors in the general vector space V and vectors in the vector space $\mathbb{R}^n$, which is more familiar to handle. We call this mapping the coordinate map from V to $\mathbb{R}^n$. Since we have all the tools to analyze this when we represent this vector as a matrix, we will be representing this mapping in the matrix form,

$$[\boldsymbol{v}]_S = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where the square brackets simply emphasize the fact that this is in a matrix of a column vector form.

The Change-of-Basis Problem states the following:

If $\boldsymbol{v}$ is a vector in a finite-dimensional vector space V, and if we change the basis for V from a basis B to basis B', how are the coordinate vectors $[\boldsymbol{b}]_B$ and $[\boldsymbol{b}]_{B'}$ related?

To solve this problem, let:

- $B = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_n}\}$
- $M = \begin{bmatrix} \boldsymbol{u_1} & \boldsymbol{u_2} & \ldots & \boldsymbol{u_n} \end{bmatrix}$
- $B' = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_n}\}$
- $N = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \ldots & \boldsymbol{v_n} \end{bmatrix}$
- $(\boldsymbol{b})_B = (c_1, c_2, \ldots, c_n)$
- $(\boldsymbol{b})_{B'} = (d_1, d_2, \ldots, d_n)$

We can see from this that $M[\boldsymbol{b}]_B = N[\boldsymbol{b}]_{B'} = \boldsymbol{b}$.

Since the vectors of B' are all in the vector space V, we can calculate their coordinates with respect to B, so say that $\boldsymbol{p_i} = [\boldsymbol{v_i}]_B$.

If we consider a matrix given by $P = \begin{bmatrix} \boldsymbol{p_1} & \boldsymbol{p_2} & \ldots & \boldsymbol{p_n} \end{bmatrix}$, we can clearly see that $M = NP$. Substitution yields $NP[\boldsymbol{b}]_B = N[\boldsymbol{b}]_{B'} = \boldsymbol{b}$, which indicates, by the uniqueness of coordinates(theorem [56]), $[\boldsymbol{b}]_{B'} = P[\boldsymbol{b}]_B$. The matrix P, often denoted $P_{B \to B'}$ is called the transition matrix from B to B'.

In words, this can be represented as follows: *The columns of the transition matrix from an old basis to a new basis are the coordinate vectors of the old basis relative to the new basis.*

The following theorem is about the invertibility[119] of the transition matrix, which is written in a future section.

Theorem 111
If P is the transition matrix from a basis B' to a basis B for a finite-dimensional vector space V, then P is invertible and P^{-1} is the transition matrix from B to B'.

We now conclude this section by introducing a procedure for computing $P_{B\to B'}$:

Method 112 (Computing a Transition Matrix $P_{B\to B'}$)
B and B' are basis for a finite-dimensional vector space V.

Step 1. Form the matrix $[B'|B]$.

Step 2. Use elementary row operations to reduce the matrix to its rref.

Step 3. The resulting matrix is $[I|P_{B\to B'}]$.

6.4 Inverse

6.4.1 Elementary Row Operations and Matrices

Definition 113 (Elementary Row Operations)
The following three operations are said to be the elementary row operations on a matrix:

1. Multiply a row through by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

Definition 114 (Elementary Row Matrices)
An $n\times n$ matrix is called an elementary matrix if it can be obtained from the $n\times n$ identity matrix I_n by performing a single elementary row operation.

Theorem 115 (Elementary Row Operations and Elementary Row Matrices)
If the elementary matrix E results from performing a certain row operation on I_m and A is an $m\times n$ matrix, then the product EA is the matrix that results when this same row operation is performed on A.

Definition 116 (Reduced-row Echelon Form)
A matrix that is in its reduced-row echelon form(rref) has the following properties:

1. If a row does not consist entirely of zeroes, then the first nonzero number in the row is a 1. We call this a leading 1.
2. If there are any rows that consist entirely of zeroes, then they are grouped together at the bottom of the matrix.
3. In any two successive rows that do not consist entirely of zeroes, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

4. Each column that contains a leading 1 has zeroes everywhere else in that column

A matrix that has the first three properties is said to be in row echelon form.

Theorem 117
If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has a row of zeroes or R is the identity matrix I_n.

There are two important facts on echelon forms:

1. Every matrix has a unique rref.
2. Row echelon forms are not unique, but, they have the same:
   - number of zero rows
   - positions of leading 1's

     the positions are called the pivot positions of A

     the columns are called the pivot column of A

Method 118 (Gauss-Jordan Elimination)
This method will use elementary row operations and through two phases, forward and backward phases, reduces a matrix into its reduced row echelon form.

Phase 1. Forward Phase[1]

Step 1. Locate the leftmost column that does not consist entirely of zeroes.

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

Step 3. Multiply the first row by a constant so that it has a leading 1.

Step 4. Add suitable multiples of the top row to the rows below so that all entries below the leading 1 become zeroes.

Step 5. Restart from Step 1, ignoring the upper rows until the entire matrix is in row echelon form.

Phase 2. Backward Phase

Step 7. Beginning with the last nonzero row and working upward, add suitable multiples of each row to the rows above to make the entries above the leading 1's to 0.

6.4.2 Finding the Inverse for a Matrix

Definition 119 (Inverse)
If A is a square matrix, and if a matrix B of the same size can be found so that $AB = BA = I$, then A is said to be invertible or nonsingular and B is called an inverse of A, denoted by A^{-1}. If no such matrix B can be found, then A is said to be singular or non-invertible.

Theorem 120
If B and C are both inverses of the matrix A, then $B = C$.

[1] If only this phase is used to produce a row echelon form, this is called the Gaussian elimination.

Theorem 121 (Inverse of a 2-by-2 matrix)
The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible iff $ad - bc \neq 0$, in which case the inverse is given by:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 122
If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
In general, a product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

Theorem 123
If A is invertible, then A^T is also invertible, and $(A^T)^{-1} = (A^{-1})^T$.

Theorem 124
A^TA is invertible iff the column vectors of A are linearly independent.

Theorem 125
Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Method 126 (Inversion Algorithm)
To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.
For easier approach, simply use Gauss-Jordan Elimination[118] to the augmented matrix $[A|I_n]$ so that it becomes $\left[I_n|A^{-1}\right]$.

6.4.3 Matrix Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Recall that a function is a rule that associates with each element of a set A one and only one element in a set B. If f associates the element $b \in B$ with $a \in A$, we write $b = f(a)$ and we say that b is the image of a under f or that $f(a)$ is the value of f at a. The set A is called the domain of f and the set B the codomain of f. The set $f(A) = \{f(a)|a \in A\}$ is called the range of f.

Definition 127 (Transformation)
If V and W are vector spaces, and if f is a function with domain V and codomain W, then we say that f is a transformation from V to W or that f maps V to W, which we denote with $f : V \to W$.
In the special case where $V = W$, f is also called an operator on V.

Since we are talking about matrices, we are going to consider the transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ which can be represented as a matrix multiplication as follows:

$$\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or more briefly as $\boldsymbol{w} = A\boldsymbol{x}$.

This can be viewed as a linear system, but if we consider $\boldsymbol{w}$ as a vector in $\mathbb{R}^m$ and $\boldsymbol{x}$ as a vector in $\mathbb{R}^n$, we can see this as a transformation. We call this matrix transformation (or matrix operator if $m = n$), denoted by $T_A : \mathbb{R}^n \to \mathbb{R}^m$, and thereby $\boldsymbol{w} = T_A(\boldsymbol{x})$, or sometimes $\boldsymbol{x} \xrightarrow{T_A} \boldsymbol{w}$. T_A is called multiplication by A, and the matrix A is called the standard matrix for the transformation.

Theorem 128
For every matrix A, the matrix transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ has the following properties for all vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ and for every scalar k:

1. $T_A(\boldsymbol{0}) = \boldsymbol{0}$
2. $T_A(k\boldsymbol{u}) = kT_A(\boldsymbol{u})$ [Homogeniety Property]
3. $T_A(\boldsymbol{u}+\boldsymbol{v}) = T_A(\boldsymbol{u}) + T_A(\boldsymbol{v})$ [Additivity Property]
4. $T_A(\boldsymbol{u}-\boldsymbol{v}) = T_A(\boldsymbol{u}) - T_A(\boldsymbol{v})$

Theorem 129
If $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ are matrix transformations, and if $\forall \boldsymbol{x} \in \mathbb{R}, T_A(\boldsymbol{x}) = T_B(\boldsymbol{x})$, then $A = B$.

Definition 130
If $T_A : \mathbb{R}^n \to \mathbb{R}^k$ and $T_B : \mathbb{R}^k \to \mathbb{R}^m$ are matrix transformations, the composition of T_B with T_A, denoted by $T_B \circ T_A$, is defined by $\boldsymbol{x} \xrightarrow{T_B \circ T_A} T_B(T_A(\boldsymbol{x}))$.

Theorem 131
If $T_A : \mathbb{R}^n \to \mathbb{R}^k$ and $T_B : \mathbb{R}^k \to \mathbb{R}^m$ are matrix transformations, $T_B \circ T_A = T_{BA}$.

Theorem 132
$T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation iff the following relationships hold $\forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$ and for every scalar k:

1. $T_A(\boldsymbol{u}+\boldsymbol{v}) = T_A(\boldsymbol{u}) + T_A(\boldsymbol{v})$ [Additivity Property]
2. $T_A(k\boldsymbol{u}) = kT_A(\boldsymbol{u})$ [Homogeniety Property]

The additivity and homogeneity properties in the theorem above are called linearity conditions, and a transformation that satisfies these conditions is called a linear transformation. Restating the theorem above gives the following:

Theorem 133
Every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation and vice versa.

6.5 Determinants

Recall from [121] that the 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $ad - bc \neq 0$. The term $ad - bc$ is the determinant of the matrix A. Determinant is a scalar value that can be computed from the elements of a square matrix which encodes certain properties of the matrix.

6.5.1 Calculating Determinants

There are two methods to calculate the determinant.

Method of Cofactor Expansion

Definition 134 (Minors and Cofactors)
Let A be a square matrix. Then the minor of entry a_{ij}, denoted by Mij, is defined to be the determinant of the submatrix that remains after the i-th row and j-th column are deleted from A. The number $C_{ij} = (-1)^{i+j}M_{ij}$ is called the cofactor of entry a_{ij}.

Definition 135 (Adjoint)
If A is $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the matrix of cofactors from A. The transpose of this matrix is called the adjoint of A, denoted by $\texttt{adj}(A)$.

Definition 136 (Determinant)
If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the determinant of A, and the sums themselves are called cofactor expansions of A.

The cofactor expansion along the j-th column is as follows:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

and the cofactor expansion along the i-th row is as follows:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

Theorem 137
If A is an $n \times n$ triangular matrix, then $\det(A)$ is the product of entries on the main diagonal of the matrix; that is, $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

Method of Elementary Row Operations

This section presents a series of theorems that can be proven with the cofactor expansion formula that will suffice by themselves, paired with the theorem for determinants for triangular matrices[137](or simply the fact that $\forall n, \det(I_n) = 1$), to find the determinant by continuously applying elementary row operations to the target matrix.

Theorem 138
Let A be a square matrix. If A has a row of zeroes or a column of zeroes, then $\det(A) = 0$.

Theorem 139
Let A be a square matrix. Then $\det(A) = \det(A^T)$.

Theorem 140
Let A be an $n \times n$ matrix.

1. If B is the matrix that results when a single row or single column or A is multiplies by a scalar k, then $\det(B) = k\det(A)$.
2. If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.

3. If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added another column, then $\det(B) = \det(A)$.

Corollary 141
Let E be an $n \times n$ matrix.

1. If E results from multiplying a row of I_n by a nonzero number k, then $\det(E) = k$.
2. If E results from interchanging two rows of I_n, then $det(E) = -1$.
3. If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

Theorem 142
If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

Often times, the method of elementary row operations may be applied partially to assist with cofactor expansion formula.

6.5.2 Properties of Determinants

Theorem 143
Let A, B, and C be $n\times n$ matrices that differ only in a single row, say the r-th row, and assume that the r-th row of C can be obtained by adding corresponding entries in the r-th row of A and B. Then, $\det(C) = \det(A) + \det(B)$.
The same result holds for columns.

Pairing the theorem above with theorem 140's first fact, we can say that the determinant is a linear function of each row separately.

Theorem 144
A square matrix A is invertible iff $\det(A) \neq 0$.

Theorem 145
If A and B are square matrices of the same size, then $\det(AB) = \det(A)\det(B)$.

Theorem 146
If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Theorem 147 (Inverse of a Matrix using its Adjoint)
If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\texttt{adj}(A)$$

Theorem 148 (Cramer's Rule)
If $A\boldsymbol{x} = \boldsymbol{b}$ is a system of n linear equations such that $det(A) \neq 0$, then the system has a unique solution. The solution is:

$$\forall i, x_i = \frac{\det(A_i)}{\det(A)}$$

where A_i is the matrix obtained by replacing the entries in the j-th column of A by the entries in the matrix $\boldsymbol{b}$.

6.6 Eigenvalues and Eigenvectors

6.6.1 Characteristic Polynomial

Definition 149 (Characteristic Polynomial)
The <u>characteristic polynomial</u> of an $n \times n$ square matrix A is defined to be

$$p_A(t) = \det(tI - A)$$

To name some of the properties of any given characteristic polynomial themselves, they are monic(meaning that the leading coefficient is 1), and has degree n.

The characteristic polynomial encodes many properties of the matrix. The most obvious property would be $p_A(t) = \det(A)$, and a slightly less obvious one would be that the coefficient of the term t^{n-1} equals tr(A).

You might be wondering, "Hey, why is this book talking about some kind of polynomial in a chapter about eigenvalues and eigenvectors?" Well, that'll become obvious once you see the definition of eigenvalues and eigenvectors.

6.6.2 Eigenvalues and Eigenvectors

Definition 150 (Eigenvalues and Eigenvectors)
If A is an $n \times n$ matrix, then a nonzero vector $\boldsymbol{x}$ in $\mathbb{R}^n$ is called an <u>eigenvector</u> of A if $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some $\lambda \in \mathbb{R}$. λ is called an <u>eigenvalue</u> of A, and $\boldsymbol{x}$ is said to be an <u>eigenvector corresponding to λ</u>.

Now since $A\boldsymbol{x} = \lambda\boldsymbol{x} = \lambda I_n \boldsymbol{x}$, it follows that $(\lambda I_n - A)\boldsymbol{x} = \mathbf{0}$. Hey, haven't we seen that equation before for the characteristic polynomial[149]? We can change this equation into $p_A(\lambda) = 0$, therefore the following theorem:

Theorem 151
For an $n \times n$ matrix A, λ is an eigenvalue of A iff $p_A(\lambda) = \det(\lambda I - A) = 0$.

Now, visiting the Fundamental Theorem of Algebra, we can see that there are exactly n (possibly complex and multiple) roots of a characteristic polynomial of $n \times n$ matrix, and therefore can have up to n distinct eigenvalues.

Theorem 152 (TFAE for eigenvalues)
If A is an $n \times n$ matrix, the following statements are equivalent:

1. λ is an eigenvalue of A
2. The system of equations $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$ has nontrivial solutions
3. There is a nonzero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$
4. λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$

Since the eigenvectors corresponding to an eigenvalue λ of a matrix A are the nonzero vectors that satisfy the equation $(\lambda I - A)\boldsymbol{x} = \mathbf{0}$, these eigenvectors are the nonzero vectors in the null space of the matrix $\lambda I - A$. We call this null space the <u>eigenspace</u> of A corresponding to λ.

Theorem 153
If k is a positive integer, λ is an eigenvalue of a matrix A, and $\boldsymbol{x}$ is a corresponding eigenvector, then λ^k is an eigenvalue of A^k and $\boldsymbol{x}$ is a corresponding eigenvector.

Theorem 154
A square matrix A is invertible iff $\lambda = 0$ is not an eigenvalue of A.

6.7 Special Matrices

6.7.1 Diagonal Matrices

A diagonal matrix is a square matrix in which all entries off the main diagonal are zero. They can be represented in the following form:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

A diagonal matrix is invertible iff all of its diagonal entries are nonzero, and its inverse is:

$$D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & \cdots & 0 \\ 0 & d_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^{-1} \end{bmatrix}$$

It is easy to calculate powers of diagonal matrices. More specifically,

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

6.7.2 Triangular Matrices

A lower triangular matrix is a matrix in which all the entries above the main diagonal are zero; an upper triangular matrix is a matrix in which all the entries below the main diagonal are zero. Either of they are called triangular. They can be represented in the following form:

$$L = \begin{bmatrix} l_{11} & 0 & \cdots & 0 & 0 \\ l_{21} & l_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ l_{(n-1)1} & l_{(n-1)2} & \cdots & l_{(n-1)(n-1)} & 0 \\ l_{n1} & l_{n2} & \cdots & l_{n(n-1)} & l_{nn} \end{bmatrix}, U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1(n-1)} & u_{1n} \\ 0 & u_{22} & \cdots & u_{2(n-1)} & u_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & u_{(n-1)(n-1)} & u_{(n-1)n} \\ 0 & 0 & \cdots & 0 & u_{nn} \end{bmatrix}$$

Theorem 155 1. The transpose of a lower triangular matrix is upper triangular, and vice versa.

2. The product of lower triangular matrices is lower triangular, and same for the upper triangular matrices.

3. A triangular matrix is invertible iff its diagonal entries are all nonzero.

4. The inverse of an invertible lower triangular matrix is lower triangular, and same for the invertible upper triangular matrices.

6.7.3 Symmetric Matrices

A symmetric matrix is a square matrix such that $S = S^T$. Specifically, S is symmetric iff $\forall 1 \leq i, j \leq n, S_{ij} = S_{ji}$.

It is important to note that for two symmetric matrices A and B, $(AB)^T = B^T A^T = BA$, and therefore their product is not guaranteed to be symmetric unless $AB = BA$, that is, A and B commute.

Theorem 156
The product of two symmetric matrices is symmetric iff the matrices commute.

In general, a symmetric matrix may not be invertible. However if they are, the following theorem shows an interesting fact:

Theorem 157
If A is an invertible symmetric matrix, then A^{-1} is symmetric.

As a side note, a skew-symmetric matrix is a square matrix such that $S = -S^T$.

6.7.4 Orthogonal Matrix

Definition 158 (Orthogonal Matrix)
?? A square matrix A is said to be orthogonal if its transpose is the same as its invers, that is, if $A^{-1} = A^T$, or equivalently, $AA^T = A^T A = I$.

Theorem 159 (TFAE for Orthogonal Matrices)
The followings are equivalent for an $n \times n$ matrix A

1. A is orthogonal
2. The row vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product
3. The column vectors of A form an orthonormal set in $\mathbb{R}^n$ with the Euclidean inner product
4. $\|A\mathbf{x}\| = \|\mathbf{x}\|$ for $\mathbf{x} \in \mathbb{R}^{\mathbf{n}}$
5. $A\mathbf{x} \cdot \mathbf{Ay} = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbf{n}}$

Theorem 160 1. The inverse of an orthogonal matrix is orthogonal

2. A product of orthogonal matrices is orthogonal
3. If A is orthogonal, then $\det(A) = \pm 1$.

Theorem 161
If S is an orthonormal basis for an n-dimensional inner product space V, and if:

$$(\boldsymbol{u})_S = (u_1, u_2, \ldots, u_n), (\boldsymbol{v})_S = (v_1, v_2, \ldots, v_n)$$

then:

1. $\|\boldsymbol{u}\| = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$
2. $d(\boldsymbol{u}, \boldsymbol{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$
3. $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$

Theorem 162
Let V be a finite-dimensional inner product space. If P is the transition matrix from one orthonormal basis for V to another orthonormal basis for V, then P is an orthogonal matrix.

6.7.5 Similar Matrices

Definition 163 (Similar Matrices)
If A and B are square matrices, then we say that B is similar to A or A and B are similar matrices if there is an invertible matrix P such that $B = P^{-1}AP$

Definition 164 (Orthogonally Similar Matrices)
If A and B are square matrices, then we say that A and B are orthogonally similar if there is an orthogonal matrix P such that $P^TAP = B$.

Theorem 165 (Invariants of Similar Matrices)
Suppose A and B are similar matrices. The following properties are the same for A and B:

Property	**Description**
Determinant	$\det(A) = \det(B)$
Invertibility	$\exists A^{-1}$ iff $\exists B^{-1}$
Rank	$rank(A) = rank(B)$
Nullity	$nullity(A) = nullity(B)$
Trace	$tr(A) = tr(B)$
Characteristic Polynomial	$p_A(t) = p_B(t)$
Eigenvalues	$p_A(\lambda) = 0$ iff $p_B(\lambda) = 0$
Eigenspace dimension	$dim(null(\lambda I - A)) = dim(null(\lambda I - A))$

6.8 Preprocessing Matrices for Easier Computation

In this section we see methods for to preprocess matrices to enable easier and faster computation, including solving linear equations and calculating the power of a square matrix. This is often called a decomposition or factorization of a matrix.

6.8.1 LU-decomposition

LU-decomposition is a process of which we factorize a matrix A into two matrices, $A = LU$, of which L is a lower triangular matrix and U is an upper triangular matrix.

We modify Gaussian Elimination[118]'s Forward Phase, of which we do not reorder the rows while we eliminate the matrix into its echelon form(or in this case, upper triangular matrix). Then we gain the following form:

$$(E_r \dots E_2 E_1)A = U$$

Now using theorem 125, we know that the elementary matrices are invertible, and hence we gain the following:

$$A = (E_r \dots E_2 E_1)^{-1}U = E_1^{-1}E_2^{-1} \dots E_r^{-1}U$$

In the forward phase of the Gaussian Elimination, since we eliminate the possibility of permutation, all the elementary matrices will either be a constant multiple of a row, or subtraction of a multiple of an upper row from a lower row. We now give the following theorem:

Theorem 166
If A is a square matrix that can be reduced to a row echelon form U by Gaussian Elimination without row interchanges, then A can be factored as $A = LU$, where L is a lower triangular matrix.

There are two major variants of LU-decomposition.

First, note that LU-decomposition is not unique, since multiplying a nonzero k to column i in L and dividing by k to the row i in U will still give the multiplication result to be A. To solve this problem, we restrict the diagonal entries of the matrices L and U to all be ones, and introduce a diagonal matrix D to the mix. By making this a three-matrix factorization, i.e. $A = LDU$, this is a unique decomposition. This is called the LDU-decomposition.

Second, we did not allow any row exchanges, but row exchanges are sometimes performed in computer algorithms to reduce roundoff errors that occur due to floating-point arithmetic. By allowing row exchanges, we can alter the decomposition to be $QA = LU$, where Q is a permutation matrix. It is common to express this as $A = PLU$, where $P = Q^{-1}$, which is called the PLU-decomposition of A.

6.8.2 QR-Decomposition

We start with a matrix with linearly independent columns(hence, invertible) A. Since A's columns are linearly independent, we can apply the Gram-Schmidt Process[97] to orthonormalize its column vectors to, say a matrix Q.

What we need to think about is how the columns of A(say $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}$) relate with columns of Q(say $\boldsymbol{q_1}, \boldsymbol{q_2}, \ldots, \boldsymbol{q_n}$). If we follow the Gram-Schmidt Process, $\boldsymbol{a_k}$ can be represented as a linear combination of the vectors in the set $Q_k\{\boldsymbol{q_i}|1 \le i \le k\}$. Considering the fact that the projection of $\boldsymbol{a_i}$ onto the space $span(Q_k)$, according to the projection formula[94], is:

$$\sum_{i=1}^{n} \langle \boldsymbol{q_i}, \boldsymbol{a_k} \rangle q_i = \sum_{i=1}^{k} \langle \boldsymbol{q_i}, \boldsymbol{a_k} \rangle q_i$$

Equality holds as $\boldsymbol{q_\alpha}$ and $\boldsymbol{a_\beta}$ are orthogonal if $\alpha > \beta$. Therefore the relationship between A and Q can be represented as follows:

$$A = Q \begin{bmatrix} \langle \boldsymbol{q_1}, \boldsymbol{a_1} \rangle & \langle \boldsymbol{q_1}, \boldsymbol{a_2} \rangle & \cdots & \langle \boldsymbol{q_1}, \boldsymbol{a_n} \rangle \\ \langle \boldsymbol{q_2}, \boldsymbol{a_1} \rangle & \langle \boldsymbol{q_2}, \boldsymbol{a_2} \rangle & \cdots & \langle \boldsymbol{q_2}, \boldsymbol{a_n} \rangle \\ \vdots & \vdots & & \vdots \\ \langle \boldsymbol{q_n}, \boldsymbol{a_1} \rangle & \langle \boldsymbol{q_n}, \boldsymbol{a_1} \rangle & \cdots & \langle \boldsymbol{q_n}, \boldsymbol{a_n} \rangle \end{bmatrix} = Q \begin{bmatrix} \langle \boldsymbol{q_1}, \boldsymbol{a_1} \rangle & \langle \boldsymbol{q_1}, \boldsymbol{a_2} \rangle & \cdots & \langle \boldsymbol{q_1}, \boldsymbol{a_n} \rangle \\ 0 & \langle \boldsymbol{q_2}, \boldsymbol{a_2} \rangle & \cdots & \langle \boldsymbol{q_2}, \boldsymbol{a_n} \rangle \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \langle \boldsymbol{q_n}, \boldsymbol{a_n} \rangle \end{bmatrix}$$

or, $A = QR$. Note that Q is an orthogonal matrix[**??**] derived from Gram-Schmidt process and R is an upper triangular matrix.

6.8.3 Diagonalization of a Matrix

Definition 167 (Diagonalizablility)
A square matrix A is said to be diagonalizable if it is similar to some diagonal matrix; that is, if there exists an invertible matrix P such that $P^{-1}AP$ is diagonal. In this case the matrix P is said to diagonalize A.

Theorem 168
If A is an $n \times n$ matrix, A is diagonalizable iff A has n linearly independent eigenvectors.

Method 169 (Procedure for Diagonalizing a Matrix)
We assume that A is an $n \times n$ matrix with n linearly independent eigenvectors.

1. Find the eigenvalues λ_i with their corresponding eigenvectors $\boldsymbol{p_i}$. This is the step where you can verify that this matrix is indeed diagonalizable.

2. Form the matrix $P = [\boldsymbol{p_1}|\boldsymbol{p_2}|\dots|\boldsymbol{p_n}]$.

3. The matrix $P^{-1}AP$ will be diagonal and have the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ corresponding to the eigenvectors $\boldsymbol{p_1}, \boldsymbol{p_2}, \dots, \boldsymbol{p_n}$ as its successive diagonal entries.

To help verify that the matrix is indeed diagonalizable, we give the following theorem:

Theorem 170
If $\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, then $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_n}\}$ is a linearly independent set.

and therefore follows the following theorem:

Theorem 171
If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

6.8.4 Orthogonal Diagonalization

Definition 172 (Orthogonal Diagonalizability)
If A is orthogonally similar to some diagonal matrix, i.e. for some diagonal matrix D, there exists an orthogonal matrix P s.t. $P^TAP = D$, then we say that A is <u>orthogonally diagonalizable</u> and P <u>orthogonally diagonalizes</u> A.

The first goal in this section is to determine what conditions a matrix must satisfy to be orthogonally diagonalizable. Suppose $P^TAP = D$, where P is orthogonal and D is diagonal. Since $PP^T = P^TP = I$, $PP^TAPP^T = A = PDP^T$. Transpose both sides, we gain $A^T = (PDP^T)^T = PD^TP^T = PDP^T = A$, hence A is symmetric.

Theorem 173 (TFAE for Orthogonally Diagonalizable Matrix)
If A is an $n \times n$ matrix, then the followings are equivalent:

1. A is orthogonally diagonalizable

2. A has an orthonormal set of n eigenvectors

3. A is symmetric.

Theorem 174
If A is a symmetric matrix, then:

- The eigenvalues of A are all real.

- Eigenvectors from different eigenspaces are orthogonal.

Method 175 (Orthogonally Diagonalizing an $n \times n$ Symmetric Matrix)
To apply this, the matrix must be symmetric.

1. Find a basis for each eigenspace of A.

2. Apply the Gram-Schmidt process[97] to each of these bases to obtain an orthonormal basis for each eigenspace.

3. Form the matrix P whose columns are the vectors constructed in step 2. This matrix will orthogonally diagonalize A, and the eigenvalues on the diagonal of $D = P^TAP$ will be in the same order as their corresponding eigenvectors in P.

Some matrices may not be orthogonally diagonalizable. However it may still be possible to achieve simplification through P^TAP by choosing an appropriate orthogonal matrix P. We now give two theorems that illustrate this:

Theorem 176 (Schur's Theorem)
If A is an $n \times n$ matrix with real entries and real eigenvalues, then there is an orthogonal matrix P such that P^TAP is an upper triangular matrix of the form:

$$P^TAP = \begin{bmatrix} \lambda_1 & \times & \times & \cdots & \times \\ 0 & \lambda_2 & \times & \cdots & \times \\ 0 & 0 & \lambda_3 & \cdots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

which is called a Schur decomposition of A.

Theorem 177 (Hessenberg's Theorem)
If A is an $n \times n$ matrix with real entries, then there is an orthogonal matrix P such that P^TAP is a matrix of the form:

$$P^TAP = \begin{bmatrix} \times & \times & \cdots & \times & \times & \times \\ \times & \times & \cdots & \times & \times & \times \\ 0 & \times & \cdots & \times & \times & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \times & \times & \times \\ 0 & 0 & \cdots & 0 & \times & \times \end{bmatrix}$$

which is a matrix in which each entry below the subdiagonal is zero. This form of matrices are said to be in upper Hessenberg form, and such decomposition is called an upper Hessenberg decomposition of A.

Quadratic Forms and Definite Matrices

The expressions of which we have been studying has the form $\sum_{i=1}^{n} a_i x_i$. They are called a linear form on $\mathbb{R}^n$. In this chapter however we will be looking at equations of in the form of $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$. This is called the quadratic forms on $\mathbb{R}^n$, and the terms of the form $a_{ij}x_ix_j$ are called the cross product terms. It is common to combine the cross product terms involving x_ix_j with x_jx_i to avoid duplicate term, hence when we put $a_{ij} = a_{ji}$, we get the following form:

$$\sum_{i=1}^{n} a_i i x_i^2 + \sum_{i=1}^{n}\sum_{j=i}^{n} 2a_{ij}x_ix_j$$

If we let $\boldsymbol{x}$ be the column vector of the variables x_i, then the form above may be represented as following:

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \boldsymbol{x}^T A \boldsymbol{x}$$

Since we assumed $a_{ij} = a_{ji}$, note that A is symmetric. In general, if A is a symmetric $n \times n$ matrix and $\boldsymbol{x}$ is an $n \times 1$ column vector of variables, then we call the function $Q_A(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ the quadratic form associated with A. This can also be represented in a dot product notation as $Q_A(\boldsymbol{x}) = \boldsymbol{x} \cdot A\boldsymbol{x} = A\boldsymbol{x} \cdot \boldsymbol{x}$

To solve many questions arising from a quadratic form, many can be solved by simplifying the quadratic form $\boldsymbol{x}^T A\boldsymbol{x}$ by making the substitution $\boldsymbol{x} = P\boldsymbol{y}$, which will express the variables $x_1, x_2, \ldots, x_n$ in terms of new variables $y_1, y_2, \ldots, y_n$. If P is invertible, we call this substitution a change of variable, and if P is orthogonal, then we call this an orthogonal change of variable. By making this substitution:

$$\boldsymbol{x}^T A\boldsymbol{x} = (P\boldsymbol{y})^T A(P\boldsymbol{y}) = \boldsymbol{y}^T P^T AP\boldsymbol{y} = \boldsymbol{y}^T (P^T AP)\boldsymbol{y}$$

of which $B = P^T AP$ is symmetric. Hence this produces a new quadratic form $Q_B(\boldsymbol{y})$ in the variables $y_1, y_2, \ldots, y_n$. If we somehow manage to choose an orthogonal P to orthogonally diagonalize A, then $Q_B(\boldsymbol{y}) = \sum_{i=1}^{n} \lambda_i y_i^2$. We have the following theorem:

Theorem 178 (The Principal Axes Theorem)
If A is a symmetric $n \times n$ matrix, then there is an orthogonal change of variable that transforms the quadratic form $Q_A(\boldsymbol{x}) = \boldsymbol{x}^T A\boldsymbol{x}$ into a quadratic form $Q_D(\boldsymbol{y}) = \boldsymbol{y}^T D\boldsymbol{y}$ with no cross product terms. Specifically if P orthogonally diagonalized A, then making the change of variable $\boldsymbol{x} = P\boldsymbol{y}$ will yield the quadratic form $\boldsymbol{x}^T A\boldsymbol{x} = \boldsymbol{y}^T D\boldsymbol{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$, in which λ_i are the eigenvalues of A corresponding to the eigenvectors that form the successive columns of P.

Definition 179
A quadratic form $\boldsymbol{x}^T A\boldsymbol{x}$ is said to be:

- positive definite if $\boldsymbol{x}^T A\boldsymbol{x} > 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$
- negative definite if $\boldsymbol{x}^T A\boldsymbol{x} < 0$ for $\boldsymbol{x} \neq \boldsymbol{0}$
- indefinite if otherwise.

Theorem 180
If A is symmetric matrix, then:

- A is positive definite iff all eivenvalues of A are positive.
- A is negative definite iff all eivenvalues of A are negative.
- A is indefinite iff there are positive and negative eivenvalues of A.

For the next theorem, we define the following:

Definition 181
The k-th principal submatrix of an $n \times n$ matrix A to be the $k \times k$ submatrix consisting of the first k rows and columns of A.

Theorem 182 (TFAE for positive definite matrices)
When a symmetric matrix A has one of the following five properties, it has them all:

1. All n pivots are positive.
2. All n determinants of principal submatrices are positive.
3. All n eigenvalues are positive.
4. $\boldsymbol{x}^T A\boldsymbol{x}$ is positive except at $\boldsymbol{x} = 0$. This is sometimes called the energy-based definition.
5. $A = R^T R$ for a matrix R with independent columns.

6.8.5 Singular Value Decomposition(SVD)

We saw, in diagonalization, that every symmetric matrix A can be expressed as $A = PDP^T$ where P is an $n \times n$ orthogonal matrix of eigenvectors of A. In this chapter, this will be referred to as an eigenvalue decomposition. If it is not symmetric, it has a Hessenberg decomposition, and if it has real eigenvalues, it has a Schur decomposition.

There are two alternate ways to decompose a general square matrix A.

The first is $A = PJP^{-1}$, where P is invertible but not necessarily orthogonal, and J is "nearly diagonal", or in a Jordan Form. Since using this decomposition is not that much popular in computations, we are simply going to refer to this Wikipedia article.

The next is $A = U\Sigma V^T$ in which U and V are orthogonal but not necessarily the same. This is called a Singular Value Decomposition.

The matrix products of the form $A^T A$ will play an important role in SVD, so we begin with the theorems:

Theorem 183
If A is an $m \times n$ matrix, then:

1. A and $A^T A$ have the same null space
2. A and $A^T A$ have the same row space
3. A^T and $A^T A$ have the same columns space
4. A and $A^T A$ have the same rank

Theorem 184
If A is an $m \times n$ matrix, then:

1. $A^T A$ is orthogonally diagonalizable.
2. The eigenvalues of $A^T A$ are nonnegative.

We now give the definition of "singular value"(as in SVD):

Definition 185 (Singular Value)
If A is an $m \times n$ matrix, and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A^T A$, then the numbers $\sigma_i = \sqrt{\lambda_i}$ are called the singular values of A.

Additionally, we define the main diagonal of an $m \times n$ matrix to be the entries in the position $a_{ii}, 1 \leq i \leq \min(m,n)$. Now we can finally define SVD. The first given theorem is the brief form that captures the main idea, second theorem for helping the method for SVD, and the last method for an expanded form that spells out the details.

Theorem 186 (Singular Value Decomposition)
If A is an $m \times n$ matrix, then A can be expressed in the form

$$A = U\Sigma V^T = \boldsymbol{u_1}\sigma_1\boldsymbol{v_1^T} + \boldsymbol{u_2}\sigma_2\boldsymbol{v_2^T} + \cdots + \boldsymbol{u_n}\sigma_n\boldsymbol{v_n^T}$$

where U and V are orthogonal matrices and Σ is an $m \times n$ matrix whose diagonal entries are the singular values of A and whose other entries are zero.

Theorem 187
Suppose $A = U\Sigma V^T = \boldsymbol{u_1}\sigma_1\boldsymbol{v_1^T} + \boldsymbol{u_2}\sigma_2\boldsymbol{v_2^T} + \cdots + \boldsymbol{u_n}\sigma_n\boldsymbol{v_n^T}$ is an SVD of A. Then, $A\boldsymbol{v_i} = \sigma_i\boldsymbol{u_i}$.

Method 188 (Singular Value Decomposition(Expanded Form))
If A is an $m \times n$ matrix of rank k, then A can be factored as $A = U\Sigma V^T$, in which U, Σ, V have sizes $m \times m$, $m \times n$, and $n \times n$, respectively, using the following method:

1. Find $V = \begin{bmatrix} \boldsymbol{v_1} & \boldsymbol{v_2} & \cdots & \boldsymbol{v_n} \end{bmatrix}$ which orthogonally diagonalizes A^TA. Optionally, since $\boldsymbol{v_i}$ is an eigenvector of A^TA, sort it so that the eigenvector corresponding to the larger eigenvalue comes first.

   This may be done in the following way: Find the eigenvalue and eigenvector pairs, $(\boldsymbol{v_i}, \lambda_i)$ pairs of A^TA, and optionally sort them as above. This will have $r = rank(A)$ vectors. Now, if there are any empty spots, use find the basis of the nullspace of A(which will contain $n-r$ vectors), orthonormalize them using the Gram-Schmidt Process along with the already established eigenvalues to an orthonormal basis for $\mathbb{R}^n$.

2. For each $\boldsymbol{v_i}$, calculate $A\boldsymbol{v_i} = \sigma_i \boldsymbol{u_i}$. Since $\boldsymbol{u_i}$ must be a unit vector, we can get:

   $\sigma_i = \|A\boldsymbol{v_i}\|$

   $\boldsymbol{u_i} = A\boldsymbol{v_i}/\sigma_i$, which is a unit eigenvector of AA^T.

3. If necessary, extend U using the nullspace of A^T using the Gram-Schmidt Process along with the already established $\boldsymbol{u_i}$ to an orthonormal basis for $\mathbb{R}^m$.

6.9 Solving Linear Equations

We come to this final section, the ultimate target of linear algebra: solving a system of linear equations.

6.9.1 Linear Equations to Matrices

A finite set of linear equations is called a system of linear equations, or more briefly, a linear system. The variables are called unknowns.

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

A solution of a linear system in $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution $x_i = s_i$ makes each equation a true statement.

We say that a linear system is consistent if it has at least one solution and inconsistent if it has no solutions.

Theorem 189
A system of linear equations has zero, one, or infinitely many solutions. There are no possibilities.

If a linear system has infinitely many solutions, then a set of parametric equations from which all solutions can be obtained by assigning numerical values to the parameters is called a general solution of the system.

If all constant terms are zero, that is, $\forall i, b_i = 0$, it is said to be homogeneous. A homogeneous system of linear equations always is consistent since it has $\forall i, x_i = 0$ as its solution: this is called the trivial solution. If there are other solutions, they are called the nontrivial solution.

The system of linear equations above can be represented in a matrix multiplication as shown below:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

By designating the three matrices A, $\boldsymbol{x}$ and $\boldsymbol{b}$ respectively, we can say that $A\boldsymbol{x} = \boldsymbol{b}$. In this equation, A is called the coefficient matrix of the system.

The augmented matrix for the system is obtained by adjoining $\boldsymbol{b}$ to A as the last column as follows:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Note the correspondence between basic algebraic operations on a given set of linear systems and elementary row operations on the augmented matrix of the said systems. In the order in the definition [113], the correspondences are:

1. Multiply an equation through by a nonzero constant
2. Interchange two equations
3. Add a constant times one equation to another.

By applying elementary row operations to the augmented matrix, we can get to the point where the augmented matrix is reduced to its reduced row echelon form. The variables corresponding to the leading 1's in the augmented matrix is called the leading variables. The remaining variables are called free variables.

There is an important theorem regarding the number of free variables and homogeneous systems:

Theorem 190 (Free Variable Theorem for Homogeneous Systems)
If a homogeneous linear system has n unknowns, and if the rref of its augmented matrix has r nonzero rows, then the system has $n-r$ free variables.

Corollary 191
A homogeneous linear system with more unknowns than equations has infinitely many solutions.

In the following sections on finding solutions or parametric equation for solutions where it applies, the coefficient matrix will be noted as A, the vector of variables will be noted as $\boldsymbol{x}$ and the variables as $x_1, x_2, \ldots, x_n$, and the vector for the constants as $\boldsymbol{b}$.

6.9.2 Method of Inverses

This can be used iff A is an invertible matrix.

Find the inverse of A, A^{-1}. The only possible solution is $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

6.9.3 Method of LU-decomposition

This can be used for matrices which are LU-factorizable, i.e. we can use this if we can apply Gaussian Elimination without any row exchanges.

1. We first decompose $A = LU$. The system in question then becomes $LU\boldsymbol{x} = \boldsymbol{b}$.
2. Let $\boldsymbol{y} = U\boldsymbol{x}$, where $\boldsymbol{y} = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix}^T$. The system in question then becomes $L\boldsymbol{y} = \boldsymbol{b}$.
3. We know that L is a lower triangular matrix; hence we can simply use front-substitution to find y_i in order.
4. Now we have the equation $U\boldsymbol{x} = \boldsymbol{y}$, of which $\boldsymbol{y}$ is known. Since U is an upper triangular matrix, we can use back-substitution to find x_i in reverse order.

6.9.4 Method of RREF

This can be used for any matrix A.

1. Reduce the augmented matrix $[A|\boldsymbol{b}]$ to its RREF $[R|\boldsymbol{c}]$
2. See if R has a zero row. If any of the value of $\boldsymbol{c}$ corresponding to the zero row is nonzero, the system is inconsistent.
3. Exchange the free variables with parametric variables.
4. Transpose the free variables to RHS so the leading variables(the pivots) are the only ones left on the LHS.
5. The resulting expressions are the parametric equation for solutions.

6.9.5 Method of Particular and Special Special Solutions

This can be used for any matrix A.

This method is extremely similar to Method of RREF[6.9.4].

In this method, we first find the nullspace of A.

Theorem 192
If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\boldsymbol{x} = \boldsymbol{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.

Theorem 193
The general solution of a consistent linear system $A\boldsymbol{x} = \boldsymbol{b}$ can be obtained by adding any specific solution of $A\boldsymbol{x} = \boldsymbol{b}$ to the general solution of $A\boldsymbol{x} = \boldsymbol{0}$.

The theorem above indicates that we need to find the nullspace of A along with a specific solution of $A\boldsymbol{x} = \boldsymbol{b}$ to find the whole, general solution of $A\boldsymbol{x} = \boldsymbol{b}$.

Using Gauss-Jordan Elimination[118], we reduce the augmented matrix $[A|\boldsymbol{b}]$ to its rref, and detect if there are any inconsistencies. This corresponds to the first step on Method of RREF.

First, we find the nullspace for A. We get the basis vectors from the rref of A. This corresponds to the second and third steps on Method of RREF.

Next, we find the specific solution of $A\boldsymbol{x} = \boldsymbol{b}$. From the rref of $[A|\boldsymbol{b}]$ say $[R|\boldsymbol{c}]$, solve the equation by setting all free variables to 0. In doing so, since all leading variables have coefficient 1, the values of c immediately correspond to the specific solution of the leading variables. This process is therefore almost automatic.

Now we have the nullspace of A and the specific solution of $A\boldsymbol{x} = \boldsymbol{b}$; add those two together to gain the whole solution. This corresponds to the fourth and final steps on Method of RREF.

6.9.6 Least Squares Approximation

Sometimes there might not exist any solution for a given linear system. In this case, we have no choice but to find the best approximation of the linear system.

Before ending this chapter, we summarize this chapter by gathering all the facts on invertible matrices, written in the appendix[16.1].

Chapter 7

Calculus

7.1 Limits

You may have seen an equation of the form $\lim_{x \to a} f(x) = L$. Intuitively, it means that as x *approaches* a, $f(x)$ goes *arbitrarily close* to L. But no, this "intuition" is not how mathematics works. What do you mean by "approaches?" What do you mean by "arbitrarily close?" How are you going to prove any theorem with this "definition?"

Let's give a precise definition of a limit. $f(x)$ goes arbitrarily close to L, but how close does that mean? It can go closer than any positive number. That means for any $\epsilon > 0$, $f(x)$ can go closer to L than ϵ. That is, $|f(x) - L| < \epsilon$.

Next, x approaches a, but how close does it approach a? How much should x approach a so that $f(x)$ goes arbitrarily close to L, in other words, $|f(x) - L| < \epsilon$? Well, close enough. When x is closer to a than some threshold, say δ, we would have $|f(x) - L| < \epsilon$. But it doesn't need to exactly be a. Expressing this mathematically, we get $0 < |x - a| < \delta$.

Combine those two inequalities, and presto! We have this definition of a limit.

Definition 194 (Limit at a)
Let f be a function defined on some open interval that contains a, except possibly at a itself. Then we say $\lim\limits_{x \to a} f(x) = L$ if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Similarly, we can define left-hand limits, right-hand limits, and limits at infinity.

Definition 195
.

This allows us to prove the theorems involving limits.

Theorem 196
.

Definition 197 (Continuous function)
.

7.2 Differentiation

Definition 198 (Derivative)
The <u>derivative of a function f at a</u>, denoted $f'(a)$, is $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$,

if this limit exists. f is differentiable at a if $f'(a)$ exists.

Theorem 199
If f is differentiable at a, then f is continuous at a.

Proof. . □

7.3 Derivative Formulae

Theorem 200
Let f and g be differentiable functions and c be a constant.

1. $c' = 0$.
2. $(cf)' = c(f')$.
3. $(f+g)' = f' + g'$.
4. $(f-g)' = f' - g'$.
5. $(fg)' = fg' + gf'$.
6. $(\frac{f}{g})' = \frac{gf'-fg'}{g^2}$, where $g(x) \neq 0$.
7. $(x^c)' = cx^{c-1}$, where c is a rational number. (It also holds for real numbers, but we won't prove it here.)

Proof. . □

Theorem 201

1. $(\sin x)' = \cos x$.
2. $(\cos x)' = -\sin x$.
3. $(\tan x)' = \sec^2 x$.
4. $(\csc x)' = -\csc x \cot x$.
5. $(\sec x)' = \sec x \tan x$.
6. $(\cot x)' = -\csc^2 x$.

Proof. . □

Theorem 202 (Chain rule)
If g is differentiable at x and f is differentiable at $g(x)$, then $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and $F'(x) = f'(g(x))g'(x)$.

Proof. . □

7.4 Integration

Chapter 8

Statistics

Chapter 9

From $\mathbb{N}$ to $\mathbb{R}$

9.1 $\mathbb{N}$: The set of Natural Numbers

9.1.1 Construction of $\mathbb{N}$

We start from the Axioms of Set[2,3,4,5,6,9], the definition of power set[10], the definition of equivalence relation and class[11,13] and the following definitions:

Definition 203 (Successor)
For any set x, the successor of x, denoted $\sigma(x)$, is defined as the following set:

$$\sigma(x) = x \cup \{x\}$$

Let us define $0 = \emptyset$, $1 = \sigma(\emptyset) = \sigma(0)$. Using the definition of successors, and following the pattern, $2 = \sigma(1)$, $3 = \sigma(2)$, and so on. Basically we can make any finite number using the definition of successor and the Axioms of Set, but actually getting all of the natural numbers at once(or any infinitely large set, since only the empty set is guaranteed to exist by the axioms) is not possible with our axioms. We define the concept of Inductive Sets and make another Axiom for this purpose:

Definition 204 (Inductive Set)
A set A is called inductive if it satisfies the following two properties:

- $\emptyset \in A$
- $(x \in A) \Rightarrow (\sigma(x) \in A)$

Axiom 205 (Axiom of Infinity)
There is an inductive set, that is:

$$\exists A(\emptyset \in A) \wedge (\forall x \in A, \sigma(x) \in A)$$

Theorem 206
Take any two inductive sets, S and T. Then, $S \cap T$ is also an inductive set.

Proof. Let $U = S \cap T$.

1. $\emptyset \in U$
   $\emptyset \in S$ and $\emptyset \in T$ since S and T are both inductive.
2. $(x \in U) \Rightarrow (\sigma(x) \in U)$
   $\forall x \in U, (x \in S) \wedge (x \in T)$.
   Since S and T are both inductive, $(\sigma(x) \in S) \wedge (\sigma(x) \in T)$
   Therefore $\sigma(x) \in U$.

Therefore U is inductive. $\square$

Corollary 207
An intersection of any number of inductive sets is inductive.

Theorem 208
For any inductive set S, define N_S as follows:

$$N_S = \bigcap_{\substack{A \subseteq S \\ A \text{ is inductive}}} A$$

Take any two inductive sets, S and T. Then $N_S = N_T$.

Proof. Suppose not; WLOG, $\exists x$ such that $x \in N_S$ and $x \notin N_T$.
Let $X = N_S \cap N_T$. Then X is inductive, $X \subset N_S$, and $x \notin X$.
Since by the definition of N_S, $N_S = X \cap N_S$, but $x \notin X \cap N_S$ hence the RHS and the LHS are different.
Therefore the assumption is wrong; therefore $N_S = N_T$. $\square$

Using this theorem, we can finally define the set of natural numbers:

Definition 209 (The Set (N) of natural numbers)
Take any inductive set S, and let

$$N = \bigcap_{\substack{A \subseteq S \\ A \text{ is inductive}}} A$$

This set is the natural numbers, which we denote as $\mathbb{N}$.

9.1.2 Operations on $\mathbb{N}$

We now define two operations on $\mathbb{N}$, addition$(+)$ and multiplication$(\cdot)$.

Definition 210 (Addition and Multiplication on $\mathbb{N}$)
The operation of addition, denoted by $+$, is defined by following two recursive rules:

1. $\forall n \in \mathbb{N}, n + 0 = n$
2. $\forall n, m \in \mathbb{N}, n + \sigma(m) = \sigma(n + m)$

Similarly the operation of multiplication, denoted by $\cdot$, is defined by following two recursive rules:

1. $\forall n \in \mathbb{N}, n \cdot 0 = 0$
2. $\forall n, m \in \mathbb{N}, n \cdot \sigma(m) = n \cdot m + n$

Lemma 211 (Operations on 0)
$\forall x \in \mathbb{N}$

- $x + 0 = 0 + x$
- $x \cdot 0 = 0 \cdot x$

Proposition 212 (Properties of $+$ and $\cdot$)
$\forall x, y, z \in \mathbb{N}$,

- **Associativity of Addition** $x + (y + z) = (x + y) + z$
- **Commutativity of Addition** $x + y = y + x$

- **Associativity of Multiplication** $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
- **Commutativity of Multiplication** $x \cdot y = y \cdot x$
- **Distributive Law** $x \cdot (y + z) = x \cdot y + x \cdot z$
- **Cancellation Law for Addition** $x + z = y + z \Rightarrow x = y$

9.1.3 Ordering on $\mathbb{N}$

Definition 213 (Ordering on $\mathbb{N}$)
For $n, m \in \mathbb{N}$, we say that n is less than m, written $n \geq m$, if there exists a $k \in \mathbb{N}$ such that $m = n + k$. We also write $n < m$ if $k \neq 0$.

Theorem 214
$(N, <)$ is an ordered set[15].

Proposition 215
The followings are true:

- If $n \neq 0$, then $0 < n$.
- Let $x, y, z \in \mathbb{N}$. Then the followings are true:
  - $(x \leq y) \wedge (y < z) \Rightarrow (x < z)$
  - $(x < y) \wedge (y \leq z) \Rightarrow (x < z)$
  - $(x \leq y) \wedge (y \leq z) \Rightarrow (x \leq z)$
  - $(x < y) \Rightarrow (x + z < y + z)$
  - $(x < y) \Rightarrow (xz < yz)$
- $\forall n \in \mathbb{N}, n \neq n + 1$
- $\forall n, k \in \mathbb{N}, k \neq 0, n \neq n + k$

Definition 216 (Least Element)
Let $S \subset \mathbb{N}$. An element $n \in S$ is called a least element if $\forall m \in S, n \leq m$

Proposition 217 (Uniqueness of the Least Element)
Let $S \subset \mathbb{N}$. Then if S has a least element, then it is unique.

Theorem 218 (Well-Ordering Property)
Let S be a nonempty subset of $\mathbb{N}$. Then S has a least element.

Note
The well-ordering property states that the set of natural numbers $\mathbb{N}$ has the greatest lower bound property[20] and thereby theorem 21, has the least upper bound property[19].

9.1.4 Properties of $\mathbb{N}$

Many of the mathematics book defines the set of Natural Numbers as the set satisfying the Peano Axioms.

Proposition 219 (Peano Axioms)

1. 0, which we defined as the empty set $\emptyset$, is a natural number.
2. There exist a distinguished set map $\sigma : \mathbb{N} \to \mathbb{N}$
3. σ is injective

4. There does not exist an element $n \in \mathbb{N}$ such that $\sigma(n) = 0$

5. (Principle of Induction) If $S \in N$ is inductive, then $S = N$.

Proposition 220
Suppose that a is a natural number, and that $b \in a$. Then $b \subseteq a$, $a \nsubseteq b$.

Proposition 221
For any two natural numbers $a, b \in \mathbb{N}$, if $\sigma(a) = \sigma(b)$, then $a = b$.

Lemma 222
If $n \in \mathbb{N}$ and $n \neq 0$, then there exists $m \in \mathbb{N}$ such that $\sigma(m) = n$.

9.2 $\mathbb{Z}$: The set of Integers

9.2.1 Construction of $\mathbb{Z}$

We now have the set of natural numbers, and starting there, we construct the set of integers.

Proposition 223
Define a relation $\equiv$ on $\mathbb{N} \times \mathbb{N}$ by $(a, b) \equiv (c, d)$ iff $a + d = b + c$. This relation is an equivalence relation on $\mathbb{N} \times \mathbb{N}$.

Let $\mathbb{Z}$ be the set of equivalence classes under this relation, and the equivalence class containing (a, b) be denoted by $[a, b]$.

9.2.2 Operations on $\mathbb{Z}$

Definition 224 (Addition and Multiplication on $\mathbb{Z}$)
Addition and multiplication on $\mathbb{Z}$ are defined by:

- $[a, b] + [c, d] = [a + c, b + d]$
- $[a, b] \cdot [c, d] = [ac + bd, ad + bc]$

Definition 225 (Subtraction on $\mathbb{Z}$)
Subtraction on $\mathbb{Z}$ is defined by:

$$[a, b] - [c, d] = [a, b] + [d, c]$$

9.2.3 Ordering on $\mathbb{Z}$

Definition 226 (Ordering on $\mathbb{Z}$)
Let $[a, b], [c, d] \in \mathbb{Z}$. $[a, b] < [c, d]$ iff $a + d < b + c$.

9.2.4 Property of $\mathbb{Z}$

Theorem 227 (Arithmetic Properties of $\mathbb{Z}$)

1. Addition and multiplication are well-defined.

2. Addition and multiplication have identity elements $[n, n]$ and $[n, n + 1]$, respectively.

3. Addition and multiplication are commutative and associative.

4. The distributive law holds.

5. Each element $[a, b]$ has an additive inverse $[b, a]$.

We can treat $\mathbb{N}$ to be a subset of $\mathbb{Z}$ by identifying the number n with the class $[0,n]$. Since $[0,a]+[0,b]=[0,a+b]$ and $[0,a]\cdot[0,b]=[0,ab]$, these operations mirror the corresponding operation in $\mathbb{N}$.

Given $n\in\mathbb{N}$, we write $-n$ for $[n,0]$, 0 for $[n,n]$, and 1 for $[n,n+1]$. By the fifth arithmetic property of $\mathbb{Z}$[227], this defines $-n$ to be the additive inverse of n. We also use the minus sign for subtraction; it is therefore natural to write $[a,b]$ as $b-a$.

Proposition 228
For $a,b\in\mathbb{N}$, let $-b$, a, and b be defined in $\mathbb{Z}$ as above. Then

$$a-b=a+(-b) \text{ and } -(-b)=b$$

9.3 $\mathbb{Q}$: The set of Rational Numbers

We construct the set of rational numbers from the set of integers as follows:

9.3.1 Construction of $\mathbb{Q}$

Proposition 229
Define a relation $\equiv$ on $\mathbb{Z}\times(\mathbb{Z}\backslash\{0\})$ by $(a,b)\equiv(c,d)$ iff $ad=bc$. This relation is an equivalence relation on $\mathbb{Z}\times(\mathbb{Z}\backslash\{0\})$.

Let $\mathbb{Q}$ be the set of equivalence classes under this relation, and the equivalence class containing (a,b) is denoted by a/b or $\frac{a}{b}$, and $\frac{a}{b}=\frac{c}{d}$ mean that (a,b) and (c,d) belong to the same equivalence class. Especially we write 0 and 1 to denote $\frac{0}{1}$ and $\frac{1}{1}$, respectively.

9.3.2 Operations on $\mathbb{Q}$

Definition 230 (Addition and Multiplication on $\mathbb{Q}$)
The <u>sum</u> and <u>product</u> of $\frac{a}{b},\frac{c}{d}\in\mathbb{Q}$ are defined by

$$\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd} \text{ and } \frac{a}{b}\frac{c}{d}=\frac{ac}{bd}$$

Definition 231 (Subtraction on $\mathbb{Q}$)
Subtraction on $\mathbb{Z}$ is defined by:

$$\frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd}$$

Definition 232 (Division on $\mathbb{Q}$)
Division on $\mathbb{Z}$ is defined by:

$$\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$$

9.3.3 Ordering on $\mathbb{Q}$

Definition 233 (Ordering on $\mathbb{Q}$)
Let $\frac{a}{b},\frac{c}{d}\in\mathbb{Q}$. $\frac{a}{b}<\frac{c}{d}$ iff $(bd>0\wedge ad<bc)\vee(bd<0\wedge ad>bc)$.

9.3.4 Property of $\mathbb{Q}$

Theorem 234 (Arithmetic Properties of $\mathbb{Q}$)

1. Addition and multiplication are well-defined.

2. Addition and multiplication have identity elements 0 and 1, respectively.

3. Addition and multiplication are commutative and associative.

4. The distributive law holds.

Theorem 235
$(\mathbb{Q}, +, \cdot)$ forms an ordered field.

9.4 $\mathbb{R}$: The set of Real Numbers

9.4.1 Construction of $\mathbb{R}$

One simple way to construct $\mathbb{R}$ is by proving the following theorem:

Theorem 236 (Existence of $\mathbb{R}$)
There exists an ordered field $\mathbb{R}$ containing $\mathbb{Q}$ as a subfield which has the least-upper-bound property.

But where's the fun in that? We will be constructing the field of real numbers using Cauchy sequences[**??**], starting with the following proposition:

Theorem 237
Define a relation $\equiv$ on the set S of Cauchy sequences of rational numbers as follows:

$$\{a_n\} \equiv \{b_n\} \text{ iff } (a_n - b_n) \to 0$$

This relation is an equivalence relation.

Now let us define $\mathbb{R}$ as the set of equivalence classes of S under the relation $\equiv$.

9.4.2 Operations on $\mathbb{R}$

Before the definition of operations on $\mathbb{R}$, we need to find out whether if the Cauchy sequences of rational numbers are closed under addition and multiplication, and it turns out they do, as stated in the following proposition:

Proposition 238
The set S of Cauchy sequences of rational numbers is closed under addition, multiplication, and scalar multiplication, that is:

1. If $\{a_n\} \in S$ and $\{b_n\} \in S$, then $\{a_n + b_n\} \in S$

2. If $\{a_n\} \in S$ and $\{b_n\} \in S$, then $\{a_n b_n\} \in S$

3. If $\{a_n\} \in S$ and $c \in \mathbb{Q}$, then $\{ca_n\} \in S$

We can finally go on to defining the operations on $\mathbb{R}$.

Definition 239 (Addition and Multiplication on $\mathbb{R}$)
Let $\{a_n\}$ and $\{b_n\}$ be sequences contained in the real numbers α, β, respectively. Then the <u>sum</u> and <u>product</u> of α and β are defined by:

$$\alpha + \beta = \{a_n + b_n\} \text{ and } \alpha\beta = \{a_n b_n\}$$

We can define subtraction and division on $\mathbb{R}$ similar to addition and multiplication, by term-by-term calculation on each term of the Cauchy sequence.

9.4.3 Ordering on $\mathbb{R}$

Definition 240 (Ordering on $\mathbb{R}$)
Let $\alpha = \{a_n\}, \beta = \{b_n\} \in \mathbb{R}$. $\alpha < \beta$ iff $\exists N \in \mathbb{N}, \forall n \geq N, a_n < b_n$.

9.4.4 Property of $\mathbb{R}$

Theorem 241 (Arithmetic Properties of $\mathbb{R}$)

1. Addition and multiplication are well-defined.
2. Addition and multiplication have identity elements $\{0\}$ and $\{1\}$, respectively.
3. Addition and multiplication are commutative and associative.
4. The distributive law holds.
5. Each element $\{a_n\}$ has an additive inverse $\{-a_n\}$.

Theorem 242
$(\mathbb{R}, +, \cdot)$ forms an ordered field.

We now define an extension to $\mathbb{R}$ as follows:

Definition 243 (Extended Real Number System)
The <u>extended real number system</u>, denoted $\mathbb{R}^+$, $[-\infty, \infty]$, or $\mathbb{R} \cup \{-\infty, \infty\}$, consists of the real field $\mathbb{R}$ and two symbols, $+\infty$ and $-\infty$. We preserve the original order in $\mathbb{R}$, and define $\forall x \in \mathbb{R}$,

$$-\infty < x < \infty$$

Remark
The extended real number system does not form a field.

9.5 $\mathbb{C}$: The set of Complex Numbers

We construct the set of complex numbers from $\mathbb{R}$. Unlike the previous constructions, we do not construct it using equivalence class. Instead the construction is done by considering the quotient ring of polynomial ring over $\mathbb{R}$ modulo $i^2 + 1$.

Definition 244
Complex number is defined as the quotient ring $\mathbb{R}[i]/(i^2 + 1)$, with operations defined as normal.

Theorem 245
$(\mathbb{C}, +, \cdot)$ forms a field.

Part II

Advanced Topics

Chapter 10

Abstract Algebra

10.1 Group Basics

10.1.1 Groups

The first thing we would encounter in abstract algebra is a group... but you already encountered it. Refer to the chapter "Algebraic Structures" for the definition of a group and an abelian group.

If the context is obvious, we will skip $\cdot$ and write ab instead of $a \cdot b$. The identity of a group will be denoted e or 1.

G will always denote group in this chapter, unless stated otherwise.

Definition 246
The product of n occurrences of x is denoted x^n. The product of n occurrences of x^{-1} is denoted x^{-n}. Also $x^0 = 1$.

Proposition 247
If $a, b, c \in G$, then

1. the identity of G is unique.
2. the inverse a^{-1} is unique.
3. $(a^{-1})^{-1} = a$.
4. $(ab)^{-1} = b^{-1}a^{-1}$.
5. if $ab = ac$, then $b = c$. Also, if $ba = ca$, then $b = c$.
6. For $n, m \in \mathbb{Z}$, $x^n x^m = x^{n+m}$ and $(x^n)^{-1} = x^{-n}$.

Proof.

1. Let e_1 and e_2 be the identities of G. Then $e_1e_2 = e_2e_1 = e_1$, and $e_2e_1 = e_1e_2 = e_2$, from the definition of the identity. Therefore $e_1 = e_2$.
2. Let b_1 and b_2 the inverses of a. Then $b_1 = b_1(ab_2) = (b_1a)b_2 = b_2$.
3. The definition of an inverse shows that a is an inverse of a^{-1}. From (ii), such an inverse is unique.
4. $(ab)b^{-1}a^{-1} = a(bb^{-1})a^{-1} = aa^{-1} = e$. Similarly $b^{-1}a^{-1}(ab) = e$. From (ii), the inverse of ab is unique.
5. $ab = ac \implies a^{-1}ab = a^{-1}ac \implies b = c$. Similar argument for $ba = ca$.

6. TODO

$\square$

Definition 248

- $a, b \in G$ commute if $ab = ba$.
- The order of $x \in G$, denoted $|x|$, is the smallest positive integer n such that $x^n = 1$. If no such n exists, then $|x| = \infty$.
- The order of G, denoted $|G|$, is the cardinality of G as a set.

Definition 249

- TODO: Z/nZ
- TODO: Sn

10.1.2 Isomorphism

Now, we want to tell whether two groups are "same," in the sense that there is a bijection between them preserving the relations.

Definition 250 (Homomorphisms and Isomorphisms)
Let $(G, \star)$, $(H, \diamond)$ be two groups. Then a map $\varphi : G \to H$ is a homomorphism if for all $x, y \in G$, $\varphi(x \star y) = \varphi(x) \diamond \varphi(y)$. An isomorphism is a bijective homomorphism. If there is an isomorphism between G and H, we say they are isomorphic and denote $G \cong H$.

We may skip $\star$ and $\diamond$ here too if the context is clear, but you have to understand which operations are used at each positions.

Theorem 251
If $(G, \star)$ and $(H, \diamond)$ are isomorphic, with the isomorphism φ, then

1. $|G| = |H|$.
2. G is abelian iff H is abelian.
3. For any $x \in G$, $|x| = |\varphi(x)|$.

Proof. (1) φ is bijective.
(2) Suppose G is abelian. Take $c, d \in H$. Since φ is surjective, there are $a, b \in G$ such that $\varphi(a) = c$ and $\varphi(b) = d$. Then $cd = \varphi(a)\varphi(b) = \varphi(ab) = \varphi(ba) = \varphi(b)\varphi(a) = dc$. Therefore H is abelian.
Suppose H is abelian. Take $a, b \in G$. Then $\varphi(ab) = \varphi(a)\varphi(b) = \varphi(b)\varphi(a) = \varphi(ba)$. Since φ is injective, $ab = ba$. Therefore G is abelian.
(3) TODO $\square$

10.1.3 Group Actions

Definition 252 (Group Action)
A group action of G on a set A is a map $G \times A \to A$, mapping $g \times a$ to $g \cdot a$, such that for all $g_1, g_2 \in G$ and $a \in A$,

1. $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, and
2. $1 \cdot a = a$.

We say that G acts on A.

Again, we may skip $\cdot$.
TODO: permutation representation

10.1.4 Subgroups

Definition 253 (Subgroups)
A subset H of G is a <u>subgroup</u> of G if H is nonempty, and for all $x,y \in H$, we have $xy \in H$ and $x^{-1} \in H$. We denote $H \leq G$.

A subgroup is also a group. To see why, there is an element x in H since H is nonempty, then $x^{-1} \in H$, and finally $xx^{-1} = 1 \in H$.

Proposition 254 (The Subgroup Criterion)
A subset H of G is a subgroup if and only if H is nonempty and for all $x,y \in H$, $xy^{-1} \in H$. Also, a finite subset H is a subgroup if and only if H is nonempty and closed under multiplication.

Proof. ($\Rightarrow$) Trivial.
($\Leftarrow$) Since H is nonempty, take an element $x \in H$. Then $xx^{-1} = 1 \in H$. This gives $1x^{-1} = x^{-1} \in H$. Finally, $x(y^{-1})^{-1} = xy \in H$ since $y^{-1} \in H$.
If H is finite, then $|x| = n < \infty$. Therefore $x^{-1} = x^{n-1}$. Now we can use the first part of this proposition. $\square$

10.1.5 Cyclic Groups

Definition 255 (Cyclic Group)
G is <u>cyclic</u> if $G = \{x^n | n \in \mathbb{Z}\}$ for some $x \in G$. For such x, we denote $G = \langle x \rangle$ and say x is a <u>generator</u> of G.

Proposition 256
$|\langle x \rangle| = |x|$ in $\langle x \rangle$. (These values can be infinite.)

Proof. TODO. Isn't it, like, trivial?? Right?? $\square$

Theorem 257
Any two cyclic groups with the same order are isomorphic.

Proof. Suppose $|\langle x \rangle| = |\langle y \rangle| = n < \infty$. We will show that $\varphi : \langle x \rangle \to \langle y \rangle$ defined by $\varphi(x^k) = y^k$ is well-defined and an isomorphism.
Suppose $x^a = x^b$. Then $x^{a-b} = 1$, and so $n|(a-b)$. Therefore $\varphi(x^a) = y^a = y^b = \varphi(x^b)$. This shows that φ is well-defined.
Next, φ is clearly a homomorphism and a surjection. Since the two groups are finite, φ is a bijection. Therefore φ is an isomorphism.
Next, if $|\langle x \rangle| = \infty$, then $\varphi : \mathbb{Z} \to \langle x \rangle$ defined by $\varphi(n) = x^n$ is an isomorphism. $\square$

Chapter 11

Topology

11.1 Topological Space

11.1.1 Topological Space

In analysis, we've dealt with functions in metric spaces and their properties. What we will do in this chapter is extend this notion to the spaces without metrics. But without metrics, our definition of open sets no longer makes sense. We need a new definition.

Remember the theorem [**??**] stating that a union of open sets is open, and a finite intersection of open sets is also open? Well...

Definition 258 (Topological Space)
A <u>topological space</u> is a set X together with a collection $\mathcal{T}$ of subsetes of X such that

1. $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
2. A union of sets in $\mathcal{T}$ is also in $\mathcal{T}$.
3. A finite intersection of sets in $\mathcal{T}$ is also in $\mathcal{T}$.

$\mathcal{T}$ is a <u>topology</u> on X, and the sets in T are called <u>open</u> sets. The complement of an open set is a <u>closed</u> set.

Definition 259

- Given a set X, the power set $\mathcal{P}(X)$ is the <u>discrete topology</u>. This space is called the <u>discrete space</u>. The set $\{\emptyset, X\}$ is the <u>indiscrete topology</u>.
- A subset is <u>cofinite</u>, and <u>cocountable</u>, if its complement is finite, and countable, respectively. The set of $\emptyset$, X, and all cofinite subsets of X, together forms the <u>cofinite topology</u>. Replacing cofinite with cocountable, we get the <u>cocountable topology</u>.

From now on, we will assume X and Y are topological spaces, unless stated otherwise.

Definition 260
Let $A \subseteq X$.

- A point $x \in A$ is an <u>interior point</u> of A if some open neighborhood of x is contained in A. The set of all interior points of A is the <u>interior</u> of A, denoted $int(A)$.

- A point $x \in X$ is an adherent point of A if every open neighborhood of x intersects A. The set of all adherent points of A is the closure of A, denoted $\bar{A}$.
- The boundary of A is $\partial A = \bar{A} \cap \overline{(X \setminus A)}$.
- $x \in X$ is a limit point of A if every open neighborhood of x contains at least one point in A different from x. The set of all limit points of A is denoted A'.
- $x \in A$ is an isolated point of A if some open neighborhood of x does not contain any point in A different from x. The set of all isolated points of A is denoted $A^{\cdot}$.

11.1.2 Base

11.1.3 Continuity and Convergence

11.1.4 Subspaces

11.2 Connected Spaces

11.2.1 Connectedness

11.2.2 Total Disconnectedness

11.2.3 Path Connectedness

11.3 Separation Axioms

11.4 Countability Axioms

11.5 Compact Spaces

11.5.1 Compactness

11.5.2 Other Types of Compactness

11.5.3 Boundedness

11.6 Metrization

11.7 Sequence of Functions

11.8 Paracompact Spaces

Part III

Applications to Computer Science

Chapter 12

Language Theory

Automaton is defined as a machine or control mechanism designed to automatically follow a predetermined sequence of operations, or respond to predetermined instructions. Theoretically, they all can be considered as the simplest form of algorithm, whether it is finite state automaton, push down automaton, or Turing machine. They all accept an input, and produce output; usually the output is *accept* or *reject*, but in the case of Turing machines, the output may be something different.

Before we start talking about the machines however we need to define what "Language" is.

Definition 261 (Language)
A (formal) language L over an alphabet Σ is a subset of Σ^*, that is, a set of words over that alphabet.

In this section, we explore Regular Language[263], Context-free Language[269], Decidable Language[276], and Recognizable Language[277] and the mechanisms, or machines, that are related those languages.

12.1 Regular Language

12.1.1 Regular Expression

If you have studied regular expression using some programming languages, then you might have easier time understanding the following definition. The regular expression used in real life is much more powerful than the regular expression mentioned below, as more special characters and syntaxes are allowed. However the following regular expression consists of the "basics" of regular expression, and is used in language theories as the regular expression:

Definition 262 (Regular Expression(RE))
Given a finite alphabet Σ, the following constants are defined as regular expressions:

- **Empty set:** $\emptyset$, denoting the set $\emptyset$.
- **Empty string:** ϵ, denoting the set containing only the "empty" string, which has no characters at all.
- **Literal character:** $a \in \Sigma$, denoting the only character a.

And when given regular expressions R and S, the following operations over them produce regular expressions:

- **Concatenation:** RS, denoting the concatenation of strings in R and S, in that order.

  R^n denotes the concatenation of R, n times: Specifically, $R^0 = \{\epsilon\}$.

- **Alternation:** $R|S$, denoting the set union of the strings in R and S.

- **Kleene star:** R^*, denoting $\bigcup_{i\in\mathbb{N}} R^i$.

Definition 263 (Regular Languages)
Regular Languages are languages that can be represented with regular expressions.

Theorem 264 (Pumping Lemma for Regular Languages)
Let L be a regular language. Then, there exists an integer $p \geq 1$, depending only on L, such that every string $w \in L$ of length at least p, called the pumping length, can be written as $w = xyz$(i.e. w can be divided into three substrings), satisfying the following conditions:

- $|y| \geq 1$
- $|xy| \leq p$
- $\forall n \geq 0, xy^n z \in L$

12.1.2 Deterministic Finite State Automaton

Definition 265 (Deterministic Finite Automaton(DFA))
A DFA M is a 5-tuple, $(Q, \Sigma, \delta, q_0, F)$, consisting of:

- Finite set of states Q;
- Finite set of input symbols called the alphabet Σ;
- Transition function $\delta : Q \times \Sigma \to Q$;
- Initial state $q_0 \in Q$;
- Set of accepting states $F \subseteq Q$.

Let $w = a_1 a_2 \ldots a_n$ be a string over the alphabet Σ. DFA M accepts the string w if a sequence of states, $r_0, r_1, \ldots, r_n \in Q$ exists with the following conditions:

- $r_0 = q_0$
- $r_{i+1} = \delta(r_i, a_{i+1}), i = 0, \ldots, n-1$
- $r_n \in F$

Theorem 266
DFAs recognize exactly the set of regular languages.

12.1.3 Nondeterministic Finite Automaton

Definition 267 (Nondeterministic Finite Automaton(NFA))
A NFA M is a 5-tuple, $(Q, \Sigma, \Delta, q_0, F)$, consisting of:

- Finite set of states Q;
- Finite set of input symbols called the alphabet Σ;
- Transition function $\Delta : Q \times \Sigma \to P(Q)$ where P is the powerset function;

- Initial state $q_0 \in Q$;
- Set of accepting states $F \subseteq Q$.

Sometimes the transition function Δ is represented as the transition relation, $\Delta \subseteq Q \times \Sigma \times Q$.

Let $w = a_1 a_2 \dots a_m$ be a string over the alphabet Σ, where $a_i \in \Sigma_\epsilon$. NFA M accepts the string w if a sequence of states, $r_0, r_1, \dots, r_n \in Q$ exists with the following conditions:

- $r_0 = q_0$
- $r_{i+1} \in \Delta(r_i, a_{i+1})$, or in relation form, $(r_i, a_{i+1}, r_{i+1}) \in \Delta$, $i = 0, \dots, n-1$
- $r_n \in F$

Theorem 268
NFAs recognize exactly the set of regular languages.

12.2 Context-Free Language

12.2.1 Context-free Grammar

Definition 269 (Context-Free Grammar(CFG))
A CFL is a 4-tuple (V, Σ, R, S) where:

- V is the set of nonterminal variables;
- Σ is the set of terminal characters;
- R is the set of rules, where each rules are in the form of $A \to w, A \in V, w \in (\Sigma \cup V)^*$
- S is the starting variable.

Definition 270 (Context-free Languages)
Context-free Languages are languages that can be represented with context-free grammars.

Theorem 271 (Pumping Lemma for Context-free Languages)
Let L be a regular language. Then, there exists an integer $p \geq 1$, depending only on L, such that every string $s \in L$ of length at least p, called the pumping length, can be written as $s = uvwxy$(i.e. w can be divided into five substrings), satisfying the following conditions:

- $|vx| \geq 1$
- $|vwx| \leq p$
- $\forall n \geq 0, uv^n wx^n y \in L$

12.2.2 Push-down Automaton

Similar to Finite Automatons, Push-down automatons have deterministic version and nondeterministic version; Only the nondeterministic version is shown here as similar method can be used to convert it into a deterministic version.

Definition 272 (Push-down Automaton(PDA))
A PDA is a 6-tuple $(Q, \Sigma, \Gamma, q_0, \Delta, F)$ where:

- Q is the set of states;
- Σ is the set of input alphabet;
- Γ is the set of stack alphabet;
- q_0 is the starting state;
- Δ is the transition relation of $Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times Q \times \Gamma_\epsilon$
- F is the set of accepting states

Δ is often written as a transition function of $Q \times \Sigma_\epsilon \times \Gamma_\epsilon \times \to P(Q \times \Gamma_\epsilon)$ where P is the powerset function.
Sometimes the last element of the relation is extended to Γ_ϵ^*; in that case, when inserting into the stack, insert the last element first. Let $w = w_1 w_2 \dots w_m$ be a string over the alphabet Σ, where $w_i \in \Sigma_\epsilon$. NFA M accepts the string w if a sequence of states, $r_0, r_1, \dots, r_n \in Q$, and a sequence of stack strings $s_0, s_1, \dots, s_n \in \Gamma^*$ exists with the following conditions:

- $r_0 = q_0$, $s_0 = \epsilon$
- $(r_i, w_{i+1}, a, r_{i+1}, b) \in \Delta$, where $s_i = at$ and $s_{i+1} = bt$ for some $a, b \in \Gamma_\epsilon$, and $t \in \Gamma^*$.

  If $b = \epsilon$, then it is a pop-operation. If $a = \epsilon$, then it is a push-operation.
- $r_m \in F$, $s_m = \epsilon$

Theorem 273
PDAs recognize exactly the set of CFLs.

Proof. The proof is quite tedious; so only a partial proof is given: we are going to convert any given CFG into PDA. Suppose a CFG $(V_0, \Sigma_0, R_0, S_0)$ is given.
We can construct a new PDA $(Q, \Sigma, \Gamma, q_0, \Delta, F)$ from the given CFL s.t.

- $Q = \{Q_S, Q_M, Q_F\}$
- $\Sigma = \Sigma_0$
- $\Gamma = V_0 \cup \Sigma_0$
- $q_0 = Q_S$
- $F = Q_F$
- $\Delta =$
  $\{(Q_S, \epsilon, \epsilon, Q_M, S\$)\} \cup$
  $\{(Q_M, \epsilon, \epsilon, X, Q_M, W) | X -> W \in R\} \cup$
  $\{(Q_M, a, a, Q_M, \epsilon) | a \in \Sigma_0\} \cup$
  $\{(Q_M, \epsilon, \$, Q_F, \epsilon)\}$

This exactly simulates the parse tree of the CFL. $\square$

12.3 Turing Machines

Definition 274 (Turing Machine)
A Turing machine consists of:

- A tape divided into consecutive cells. Each cell contains a symbol from the tape alphabet, which contains a blank symbol and one or more other symbols. The tape is assumed to be infinitely long to the left; cells that have not been written before are assumed to be filled with the blank symbol.
- A head which can read a single symbol on the tape at a time, and is able to move one(and only one at once) cell to the right or the left.
- A state register which stores the state of the TM, starting from the starting state(defined below) and following the transition function's rule(also defined below).

Formally, a TM is a 7 tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where:

- Q is the set of states;
- Γ is the set of tape alphabet;
- $b \in \Gamma$ is the blank symbol, the only symbol allowed to occur infinitely often at any step of the computation;
- $\Sigma \subseteq \Gamma \backslash \{b\}$ is the set of input symbols, that is, the set of symbols allowed to appear in the initial tape contents;
- $q_0 \in Q$ is the starting state;
- $F \subseteq Q$ is the set of accepting states, and the initial tape contents is said to be accepted by M if it eventually halts in a state from F;
- δ is a partial function called the transition function of $(Q\backslash F) \times \Gamma \to Q \times \Gamma \times \{L, R\}$, where L and R signifies left and right shifts of the tape. If δ is undefined on the current state and the current tape symbol, then the machine halts.

Using the components of TM and the formal definition, the Turing machine accepts iff it halts on the set of accepting states, and it rejects iff it halts on the set of rejecting states. It may loop infinitely, of which it neither accepts nor rejects the tape.

The definition of a Turing Machine is not unique. Some definitions use multiple tapes, using one of them as the input tape that can't be modified and another as the output tape. Some has more than one halting states. Some include N in the final output of the transition function, indicating no movement of the head. But in general, a Turing machine starts from one state, follows the decision function every step, and halts at the halting state. Some of the many variations on the Turing machine are mentioned in 12.5.2.

In fact, the different definitions of a Turing machine turns out to be the same, in the sense that a function $f : \{0,1\}^* \to \{0,1\}$ is computable using one definition of a Turing machine iff it is computable using another definition of a Turing Machine.

We now give the following thesis from the creator of the λ-calculus, Alonzo Church and Alan Turing.

Thesis 275 (Church-Turing Thesis)
A function on Natural Numbers which is computable by a human being following an algorithm, ignoring resource limitations, if and only if it is computable by a Turing Machine.

12.4 Decidable and Recognizable Languages

Definition 276 (Decidable Languages)
Decidable Languages are languages that can be represented with decidable Turing machines; that is, the set of Turing machines that always accepts accepting words and rejects others.

Definition 277 (Recognizable Languages)
Recognizable Languages are languages that can be represented with recognizable Turing machines; that is, the set of Turing machines that always accepts accepting words.

Decidable and Recognizable Turing Machines seem similar; however recognizable machines does not have to reject a non-accepting word; it may instead loop infinitely.

12.5 Equivalences to Turing Machine

The followings can be shown to be computationally equivalent to a Turing machine; however no proofs are given since they are usually long and arduous.

12.5.1 Push-down Automaton with Two Stacks

The simplest version that is equivalent to a Turing Machine would be a PDA which has two stacks. The two stacks can simulate the tape of the Turing machine by pushing and popping.

12.5.2 Variations on the Turing Machine

The following variations on the Turing machine are equivalent to the original Turing machine:

- Variations on the Definition
  - Allowing N, "no shift", in the movement rules;
  - Having a single accepting state, say q_{accept} and a single rejecting state, say q_{reject}, and forcing the transition function δ to be a function. In this variant, the machine accepts iff it ends in q_{accept}, and rejects iff it ends in q_{reject}.
- Variations on the Form of the Machine
  - Tape is infinite only in one direction;
  - Tape is infinite in both directions;
  - Tape is 2-dimensional;
  - There exists multiple tapes that the machine can access concurrently.

There are many more variations other than these.

12.5.3 General Recursive Functions

Definition 278 (General Recursive Functions)
General Recursive Functions, otherwise known as μ-recursive functions, is a set of functions $\forall n \in \mathbb{N}, f : \mathbb{N}^n \to \mathbb{N}$ that includes the three "Initial", or "Basic" functions, and closed under three operators:

- Initial Functions
  - **Constant Function:** $\forall n, k \in \mathbb{N}, f(x_1, \ldots, x_k) = n$
    Alternative definition use a Zero function: $\forall k \in \mathbb{N}, Z(x_1, \ldots, x_k) = 0$
  - **Successor Function** S: $S(x) = x + 1$
  - **Projection Function** P_i^k:
    This is also called the Identity Function I_i^k
- Operators
  - **Composition Operator** $\circ$: Given an m-ary function $h(x_1, \ldots, x_m)$ and m k-ary functions $g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k)$:

    $$h \circ (g_1, \ldots, g_m) = f \text{ where } f(x_1, \ldots, x_k) = h(g_1(x_1, \ldots, x_k), \ldots, g_m(x_1, \ldots, x_k))$$

    This is also called the Substitution Operator.
  - **Primitive Recursion Operator** ρ: Given the k-ary function $g(x_1, \ldots, x_k)$ and (k+2)-ary function $h(y, z, x_1, \ldots, x_k)$:

    $$\begin{aligned} \rho(g, h) = f &\text{ where} \\ f(0, x_1, \ldots, x_k) &= g(x_1, \ldots, x_k) \\ f(y+1, x_1, \ldots, x_k) &= h(y, f(y, x_1, \ldots, x_k), x_1, \ldots, x_k) \end{aligned}$$

  - **Minimization Operator** μ: Given a (k+1)-ary total function $f(y, x_1, \ldots, x_k)$:

    $$\begin{aligned} \mu(f)(x_1, \ldots, x_k) = z \Leftrightarrow & f(z, x_1, \ldots, x_k) = 0 \text{ and} \\ & f(i, x_1, \ldots, x_k) > 0 \text{ for } i = 0, \ldots, z-1 \end{aligned}$$

    Intuitively, this operator seeks the smallest argument that causes the function to return 0; if none exists, the search never ends and therefore cannot return.

12.5.4 Lambda Calculus

Lambda Calculus, first defined by Alonzo Church, is a formal system of mathematical logic for expressing computation based on function-like objects.

Definition 279 (Lambda Expression)
Lambda expressions are composed of:

- Variables, $v_1, \ldots, v_n, \ldots$
- The abstraction symbols lambda λ and dot .
- Parentheses ()

For some applications, terms for logical and mathematical constants and operation may be included.

The set of lambda expressions, Λ, can be defined inductively:

- If x is a variable, then $x \in \Lambda$
- If x is a variable and $M \in \Lambda$, then $(\lambda x.M) \in \Lambda$

  This rule is also known as Abstractions.
- If $M, N \in \Lambda$, then $(MN) \in \Lambda$

  This rule is also known as Application.

Though only the definition is given, This Wikipedia article can be helpful to understand how lambda calculus works.

Chapter 13

Theory of Computation

13.1 Computability

Turing machine was already defined in [274], but let's write down the definition here for convenience:

Definition 280 (Turing Machine)
A Turing machine consists of:

- A tape divided into consecutive cells. Each cell contains a symbol from the tape alphabet, which contains a blank symbol and one or more other symbols. The tape is assumed to be infinitely long to the left; cells that have not been written before are assumed to be filled with the blank symbol.
- A head which can read a single symbol on the tape at a time, and is able to move one(and only one at once) cell to the right or the left.
- A state register which stores the state of the TM, starting from the starting state(defined below) and following the transition function's rule(also defined below).

Formally, a TM is a 7 tuple $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where:

- Q is the set of states;
- Γ is the set of tape alphabet;
- $b \in \Gamma$ is the blank symbol, the only symbol allowed to occur infinitely often at any step of the computation;
- $\Sigma \subseteq \Gamma \backslash \{b\}$ is the set of input symbols, that is, the set of symbols allowed to appear in the initial tape contents;
- $q_0 \in Q$ is the starting state;
- $F \subseteq Q$ is the set of accepting states, and the initial tape contents is said to be accepted by M if it eventually halts in a state from F;
- δ is a partial function called the transition function of $(Q \backslash F) \times \Gamma \to Q \times \Gamma \times \{L, R\}$, where L and R signifies left and right shifts of the tape. If δ is undefined on the current state and the current tape symbol, then the machine halts.

Using the components of TM and the formal definition, the Turing machine accepts iff it halts on the set of accepting states, and it rejects iff it halts on the set of rejecting states. It may loop infinitely, of which it neither accepts nor rejects the tape.

We will abbreviate "Turing Machine" as TM, or DTM when necessary.

Usually, the proofs involving TMs do not give a formal construction of the machines because it is an extremely tedious process. Instead the proof describes what the machine does. It would be intuitive to see that such a machine can indeed be constructed.

Definition 281 (Decision Problem)

- A TM M runs in time $T(n)$ if it halts in at most $T(n)$ steps for every input with length n.
- A decision problem is a subset of the set of natural numbers $\mathbb{N}$. We assume $0 \in \mathbb{N}$.
- An input is a natural number that will be written on the initial tape in binary. The length of the input is the number of cells required to represent it, which is $|x| = \lceil \log_2(x+1) \rceil$.
- A TM M accepts an input x if it accepts with the given input x. Similarly, M rejects x if it rejects with the given input x.
- A TM M decides a decision problem L if, for all $x \in \mathbb{N}$, M accepts x if $x \in L$ and M rejects x otherwise. In that case, L is decidable.
- A TM M semi-decides a decision problem L if, for all $x \in \mathbb{N}$, M accepts x if $x \in L$ and M runs infinitely otherwise. In that case, L is semi-decidable.

What if we want other types of inputs such as two natural numbers, rational numbers, ASCII strings, graphs, and so on? In that case, we can encode them as natural numbers. For example, there are some easy-to-compute injection from $\mathbb{N}^2$ to $\mathbb{N}$, and we assume they are given as the encoded forms. But since there is no injection from $\mathbb{R}$ to $\mathbb{N}$, we cannot give real numbers as inputs.

Note also that since a TM itself is finite, the set of all TMs is countable. Therefore TMs can also be encoded, and even be given as an input to other TMs! Yo dawg, I heard you like TMs...

Now, a natural question is whether undecidable problems exist at all. The answer is yes, because there are countably many Turing machines but uncountably many subsets of $\mathbb{N}$. We will also define one of the most important undecidable problems:

Definition 282 (Halting Problem)
The halting problem **HALT** is the set of pairs $(M,x) \in \mathbb{N}$ such that a TM M halts on input x. (Remind that (M,x) is encoded into a single natural number.)

Theorem 283
HALT is undecidable.

Proof. Suppose there is a machine N that decides **HALT**. Construct another machine N' that does the following: given input x, simulate N on input (x,x). If N accepts it, run an infinite loop. Otherwise, accept.

Now consider what happens with N' is given the input N'. If N' accepts itself, then it means N rejects the input (N',N'), i.e. N' on input N' never halts. But this is a contradiction to the assumption that N' accepts itself. On the other hand, if N' runs an infinite loop, then it means N accepts the input (N', N'), i.e. N' on input N' eventually halts. This is again a contradiction. Therefore no such N exists. $\square$

However, it should be noted that **HALT** is semi-decidable, since we can just simulate M on the input x and accept if M halts. If M never halts, then our simulation would not halt either.

13.2 Nondeterministic Turing Machine

Definition 284 (Nondeterministic Turing Machine)
A nondeterministic Turing Machine (NDTM) is the same as a DTM except that δ is a relation instead of a partial function. The next state of the state $(q, c) \in (Q \backslash F) \times \Gamma$ can be any of $(q', c', d') \in Q \times \Gamma \times \{L, R\}$ such that $(q, c)\delta(q', c', d')$. An NDTM accepts an input x if it always halts and it accepts x in at least one sequence of execution. An NDTM rejects x if it always rejects x.

Although an NDTM looks stronger than a DTM, it is actually possible to simulate an NDTM using a DTM. To check whether an NDTM accepts x, generate all execution sequences of the machine and check whether one of them leads to acceptance. However, the power of an NDTM is that by randomly "guessing" the next state of execution, it can sometimes easily compute what would take exponential time for a DTM, in polynomial time. Details will be introduced soon.

13.3 Relations Between Decidabilities

Theorem 285

1. A finite problem L is decidable.
2. A problem L is decidable iff its complement is decidable.
3. A problem L is decidable iff L and its complement are both semi-decidable.

Proof. (1) Let $L = \{x_1, \cdots, x_k\}$. Construct a TM that takes an input x and decides whether $(x = x_1) \vee (x = x_2) \vee \cdots \vee (x = x_k)$. Even if we don't know the contents of L, it is still true that such a TM exists.

(2) Suppose there is a TM M that decides the complement of L. Construct another TM that takes an input x, simulates M on x, then accepts iff M rejects. The other direction is the same.

(3) If L is decidable, then clearly L is semi-decidable. From (2), the complement of L is decidable, so it is semi-decidable. Conversely, let M be a TM that decides L and M' be a TM that decides the complement of L. Construct another TM that takes an input x, and simulates each step of M and M' on x one by one. If M halts, then accept. If M' halts, then reject. $\square$

13.4 Computational Complexity

(TODO: Write something about asymptotic notation here)

Definition 286 (Asymptotic notation)
Let f and g be two functions from $\mathbb{N}$ to $\mathbb{N}$. Then we say:

- $f = O(g)$ if there is a constant c such that $f(n) \le c \cdot g(n)$ for every sufficiently large n. That is, $n > N$ for some N.
- $f = \Omega(g)$ if $g = O(f)$.
- $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.
- $f = o(g)$ if for every constant $c > 0$, $f(n) < c \cdot g(n)$ for every sufficiently large n.
- $f = \omega(g)$ if $g = o(f)$.

Definition 287 (P, NP, EXP)

- **P** is the set of boolean functions computable with a deterministic Turing machine in time $O(n^c)$ for some constant $c > 0$.
- **NP** is the set of boolean functions computable with a non-deterministic Turing machine in time $O(n^c)$ for some constant $c > 0$.
- **EXP** is the set of boolean functions computable with a deterministic Turing machine in time $O(2^{n^c})$ for some constant $c > 0$.

Theorem 288
$\mathbf{P} \subseteq \mathbf{NP} \subseteq \mathbf{EXP}$.

Proof. A DTM is automatically an NDTM, so $\mathbf{P} \subseteq \mathbf{NP}$. To show $\mathbf{NP} \subseteq \mathbf{EXP}$, let M be an NDTM that runs in time $p(n)$ where p is a polynomial. Then since there are at most $2^{p(n)}$ execution sequences of M, we can simulate all executions in exponential time. Accept the input x iff M accepts for at least one execution sequence. □

13.5 Reduction

Is there a polynomial-time algorithm for a given decision problem? Computer scientists are interested in this question because if there is one, it is usually a small-degree polynomial like $O(n^2)$ or $O(n^5)$. Some problems have a special property that if the problem has a polynomial-time algorithm, then several other problems do.

Definition 289 (Polynomial-time Karp reduction)
A problem $A \subseteq \{0,1\}^*$ is polynomial-time Karp reducible to $B \subseteq \{0,1\}^*$, denoted $A \le_p B$, if there is a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that for every $x \in \{0,1\}^*$, $x \in A$ iff $f(x) \in B$.

The intuitive meaning is that a problem of A can be "reduced" to a problem of B, and if we can solve B in polynomial-time, then we can solve A in polynomial-time too.

Definition 290 (NP-complete)
A problem A is NP-hard if every problem in **NP** is polynomial-time reducible to A, and NP-complete if A is NP-hard and NP.

Theorem 291

1. If $A \le_p B$ and $B \le_p C$, then $A \le_p C$.
2. An NP-complete problem A is in **P** iff $\mathbf{P} = \mathbf{NP}$.
3. If $A \le_p B$ and A is NP-hard, then B is NP-hard.

Proof. (1) Let f be a reduction from A to B with polynomial time $p(n)$, and g from B to C with $q(n)$. Then $g \circ f$ is a reduction from A to C with polynomial time $q(p(n))$.

(2) Suppose A is NP-complete and in **P**. Then any problem B in **NP** can be polynomial-time reduced to A, so transitivity implies that B is polynomial-time computable. The converse is trivial.

(3) Any problem C in **NP** can be polynomial-time reduced to A. Transitivity implies that C can be polynomial-time reduced to B. □

Now the obvious question is, does such a strong problem actually exist? The answer is yes, and a lot of important problems are NP-complete.

(TODO: SAT)

Having proven that SAT is NP-hard, more problems can be proven NP-hard if we can reduce SAT to those problems in polynomial-time. Here are only a tiny fraction of the NP-complete problems:

Definition 292 (NP-complete problems)

- The 3-SAT problem is a SAT problem where each clause contains exactly 3 variables.
- Given a graph G and an integer $0 \leq k \leq |V(G)|$, the clique problem asks whether there is a complete induced subgraph of G with size at least k.
- The independent set problem asks whether there is a subset S of $V(G)$ with size at least k such that no two vertices in S are adjacent, and 0 otherwise.
- The vertex cover problem asks whether there is a subset S of $V(G)$ with size at most k such that each edge is adjacent to at least one vertex in S.
- The chromatic number problem asks whether G is 3-colorable.
- Given a set S of integers and an integer k, the subset sum problem asks whether there is a subset of S whose sum of elements equals k.
- Given an $n \times m$ matrix A and an $n \times 1$ matrix b of integers, the integer programming problem asks whether there is an $m \times 1$ matrix x of integers such that each element of $Ax + b$ is non-negative.

Theorem 293
All problems in [292] are NP-complete.

Proof. Clearly all problems described are NP. We will only show that they are all NP-hard.

If we can reduce SAT to 3-SAT in polynomial time, then [291] will show that 3-SAT is NP-hard. To do this, note that

- x is equivalent to $x \vee x \vee x$,
- $x_1 \vee x_2$ is equivalent to $x_1 \vee x_2 \vee x_2$,
- $x_1 \vee \cdots \vee x_n$ is equivalent to $(x_1 \vee x_2 \vee y_1) \wedge (\neg y_1 \vee x_3 \vee y_2) \wedge \cdots (\neg y_{n-4} \vee x_{n-2} \vee y_{n-3}) \wedge (\neg y_{n-3} \vee x_{n-1} \vee x_n)$, where $n \geq 4$ and $y_1, \cdots, y_{n-3}$ are new variables unused in the original SAT formula.

Next, we reduce 3-SAT to a clique problem. (TODO)

G has a clique of size k iff $\bar{G}$ has an independent set of size k. This shows that clique and independent set are polynomial-time reducible to each other.

G has an independent set of size k iff G has a vertex cover of size $|V(G)|-k$, by taking the complement of the independent set. Therefore independent set and vertex cover are polynomial-time reducible to each other.

We reduce 3-SAT to a chromatic number problem. (TODO)

We reduce 3-SAT to a subset sum problem. (TODO)

Finally, we reduce 3-SAT to an integer programming problem. Given a 3-SAT formula with n variables, set $0 \le x_i \le 1$ for $i = 1, \cdots, n$, and convert the clause $(x_a \vee x_b \vee x_c)$ into $x_a + x_b + x_c \ge 1$. If the clause contains $\neg x_a$, convert it to $1 - x_a$. This system of inequalities can easily be converted to the matrix form.

□

Chapter 14

Graph Theory

14.1 Basic Graph Definitions

Definition 294 (Graph)
A graph G is represented by a pair of sets $(V(G), E(G))$, and a relation $\sim_G \subseteq V(G) \times E(G)$ such that for each $e \in E(G)$, there are exactly one or two $v \in V(G)$ such that $v \sim_G e$. An element of $V(G)$ is a vertex, and an element of $E(G)$ is an edge. If $v \sim_G e$, we say v is incident with e, and v is an end of e.

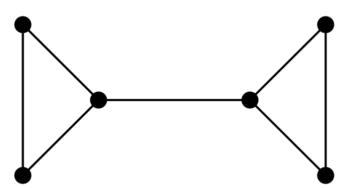


Figure 14.1: A graph with 6 vertices and 7 edges.

From now on, we will skip (G) and just write V and E if the context is obvious. Similarly we will skip $_G$ and just write $\sim$. Also, for simple graphs, we may write an edge as vw where v and w are the ends of the edge.

Definition 295

- A vertex v is adjacent to another vertex w if there is an edge e such that $v \sim e$ and $w \sim e$. We also say that v is a neighbor of w.
- A loop is an edge with exactly one end.
- Two edges e_1 and e_2 are parallel if $e_1 \neq e_2$ and the set of ends of e_1 equals that of e_2.
- A graph G is simple if it has no loops or parallel edges.
- Two graphs G and H are isomorphic if there are two bijections $f_V : V(G) \to V(H)$ and $f_E : E(G) \to E(H)$ such that for all $v \in V(G)$ and $e \in E(G)$, $v \sim_G e$ iff $f_V(v) \sim_H f_E(e)$.

Note that some texts might use a different definition of graphs. One common definition is that $E(G)$ is a set of two-element subsets of $V(G)$. With this definition, our definition of a simple graph is just called a graph, and our definition of a graph is called a multigraph (and you need to change "set" to "multiset").

Definition 296 (Subgraph)

- A graph G is a subgraph of a graph H if $V(G) \subseteq V(H)$, $E(G) \subseteq E(H)$, with the same incidence relation, i.e. the set of ends of any edge e in G equals that of e in H.
- For $e \in E$, $G \backslash e$ is $(V(G), E(G) \backslash \{e\})$ with the same incidence relation.
- For $v \in V$, $G \backslash v$ is $(V(G) \backslash \{v\}, E')$, where E' is the set of edges in G not incident with v, with the same incidence relation.
- A subgraph H of G is spanning if $V(H) = V(G)$.
- A subgraph H of G is induced if $E(H)$ equals the set of edges in G whose set of ends is contained in $V(H)$. We say H is induced by $V(H)$.
- For $X \subseteq V$, $G[X]$ is a subgraph of G induced by X.

Definition 297

- A complete graph with n vertices, denoted K_n, is a simple graph in which for any pair of different vertices there is an edge connecting them.
- A cycle graph with n vertices, denoted C_n, is a simple graph whose edge set is $\{v_1v_2, \cdots, v_{n-1}v_n, v_nv_1\}$, where $V = \{v_1, \cdots, v_n\}$.
- A graph G is bipartite if V can be partitioned into non-empty subsets A and B such that no edges connect two vertices in A or two vertices in B.
- A complete bipartite graph with $n+m$ vertices, denoted $K_{n,m}$, is a simple bipartite graph with $|A| = n$, $|B| = m$ in which for any vertex in A and in B, there is an edge connecting them.
- For a simple graph G, the complement $\bar{G}$ of G is a simple graph on $V(G)$ such that any two different vertices v and w are adjacent in G iff they are not adjacent in $\bar{G}$.

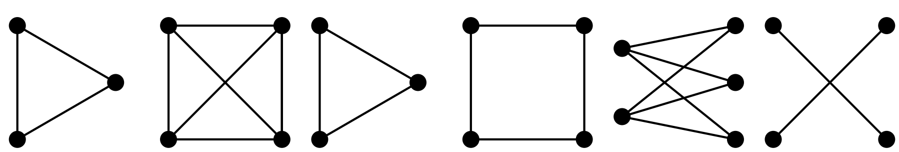


Figure 14.2: From left to right: K_3, K_4, C_3, C_4, $K_{2,3}$, and $\bar{C}_4$. Note that K_3 is isomorphic to C_3.

Definition 298

- A walk from $v \in V$ and $w \in V$ is an alternating sequence $v_0e_1v_1e_2 \cdots e_kv_k$ of vertices and edges such that $v_0 = v$, $v_k = w$, and the set of ends of e_i equals $\{v_{i-1}, v_i\}$. k is the length of the walk.
- A trail is a walk with distinct edges.
- A closed walk is a walk with $v = w$ and $k > 0$.
- A circuit is a trail that is also a closed walk.

- A path is a walk with distinct vertices.
- A cycle is a circuit with distinct $\{v_0, \cdots, v_{k-1}\}$.

Definition 299 (Connectivity)

- A graph is connected if for any two vertices in V there is a path connecting them.
- A connected component of a graph G is $G[X]$ such that $G[X]$ is connected, and for any $Y \subseteq V$ such that $X \subsetneq Y$, $G[Y]$ is not connected.

14.2 Degrees

Definition 300 (Degree)
The degree of $v \in V(G)$, denoted $deg_G(v)$, is the number of non-loop edges incident with v, plus two times the number of loops incident with v.

Again, we might skip G and write $deg(v)$. As we progress, it will be clear why it is convenient to count a loop twice.

Lemma 301 (Degree Sum Formula)
$\sum_{v \in V} deg(v) = 2|E|$.

Proof. Induction on $|E|$, with the trivial base case $|E| = 0$. Suppose $|E| > 0$. Let $\sum_{v \in V} deg(v) = A$ and $2|E| = B$. Take any edge e, and the induction with $G \backslash e$ shows that $A - 2 = B - 2$. Therefore $A = B$. □

Lemma 302 (Handshaking Lemma)
A graph has an even number of odd-degree vertices.

Proof. $2|E|$ is an even number. From [301], exactly even number of the terms $deg(v)$ must be odd. □

The degree sum formula is sometimes also called the handshaking lemma.

Definition 303 (Degree Sequence)
The degree sequence of a graph G, or the score of G, is the sequence of degrees $(deg(v_1), \cdots, deg(v_{|V|}))$.

Now, how can we figure out if a sequence is a degree sequence of some graph? The following theorem gives a simple $O(\sum d_i)$-time algorithm to answer the question:

Theorem 304 (Havel-Hakimi Algorithm)
Let $(d_1, \cdots, d_n)$ be a sequence of integers such that $0 \le d_1 \le \cdots \le d_n$ and $n > 1$. It is a degree sequence of some simple graph iff $(d_1, \cdots, d_{z-1}, d_z - 1, \cdots, d_{n-1} - 1)$ is a degree sequence of some simple graph, where $z = n - d_n$.

Proof. ($\Leftarrow$) If $(d_1, \cdots, d_{z-1}, d_z - 1, \cdots, d_{n-1} - 1)$ is a degree sequence of some simple graph, then we can make $(d_1, \cdots, d_n)$ by adding a vertex and connecting to the vertices with degrees $d_z - 1, \cdots, d_{n-1} - 1$.

($\Rightarrow$) Let G be a simple graph such that $deg_G(v_i) = d_i$ for all $v_i \in V(G)$. We will construct a simple graph H with $deg_H(u_i) = d_i$ for all $u_i \in V(H)$ such that v_n is connected to $v_{n-d_n}, \cdots, v_n - 1$. Then the conclusion follows by taking $H - v_n$.

If $d_n = n - 1$, then simply take $H = G$. Otherwise, define $j(G)$ as the largest index j such that v_n is not adjacent to v_j. Among all graphs with $deg_H(u_i) = d_i$, take one graph with the smallest $j(H)$. (Note that such H exists because at least one graph, namely G, satisfies the degree sequence condition.)

Suppose $j = j(H) \geq n - d_n$. Then there is an index $i < j$ such that u_n is adjacent to u_i. Since $deg_H(u_i) \leq deg_H(u_j)$, there is a vertex w adjacent to u_j but not to u_i. Now, consider a new graph H' derived from H by removing $u_i u_n$ and $u_j u_k$, and adding $u_j u_n$ and $u_i u_k$. Then $deg_{H'}(u_i) = d_i$ and $j(H') < j(H)$, contradicting the minimality of H. Therefore $j(H) = n - d_n$. □

14.3 Trees

One of the important classes of graphs is a tree. There are many ways to define a tree. First we will state one definition, and then prove that other definitions are equivalent.

Definition 305 (Tree)
A <u>forest</u> is a simple graph without any cycle. A <u>tree</u> is a connected forest. A <u>leaf</u> of a forest is a vertex with degree 1.

Before moving on to the equivalence, we introduce two useful lemmas:

Lemma 306
A tree with at least two vertices has at least two leaves.

Proof. Let P be a path with maximum length, and x and y be its end-vertices. Then $x \neq y$, and we claim that x and y are leaves. Suppose x is not a leaf, so x has an edge e not used by P. Let z be the other end of e. Since $P + e$ has no cycles, z is not used by P. Therefore $e + P$ is a path longer than P, contradicting the maximality. Repeat this proof to show that y is also a leaf. □

Lemma 307
Let G be a graph, $v \in V$, and $deg(v) = 1$. Then G is a tree iff $G - v$ is a tree.

Proof. Suppose G is a tree. Since G is connected, for any two vertices $x \neq y$, there is a path from x to y. Since v has only one incident edge, this path does not contain v. Therefore this path is also a path on $G - v$. Clearly $G - v$ has no cycles. Therefore $G - v$ is a tree.

Suppose G is not a tree. If G is disconnected, then $G - v$ is clearly disconnected. If G has a cycle, then since this cycle does not contain v, it is also a cycle on $G - v$. Therefore $G - v$ is not a tree in either cases. □

This enables us to apply induction on the number of vertices of a tree. Check the base case where there are ≤ 2 vertices. Then, let v be a leaf of a tree, and remove v. Since the resulting graph is also a tree, we can apply the inductive hypothesis. Then we can use this to prove the statement for the original tree.

Theorem 308
The following statements are equivalent for a simple graph G:

1. G is a tree.
2. For any two vertices u and v of G, there is exactly one path connecting them.
3. G is connected, and for any edge e of G, $G \backslash e$ is disconnected.
4. G has no cycle, and for any two vertices u and v not having an edge between them, $G + uv$ has a cycle.

5. G is connected, and $|E| = |V| - 1$.

Proof. ($1 \Rightarrow 2$) Since G is connected, there is a path. Next, induction on $|V|$. The base case $|V| \leq 2$ is trivial. From [306] and [307], there is a leaf v and $G - v$ is a tree. Any two vertices a, b of $G - v$ has a unique path. Since a path from a to b in G cannot contain v, there is a unique path in G as well. Since v is adjacent to a unique vertex u, and there is a unique path from a to u, there is a unique path from a to v which is the path to u plus the edge uv.

($2 \Rightarrow 3$) Since every pair of vertices has a path, G is connected. For any e, its two endpoints have a unique path which is e itself. Therefore there is no path between them in $G \backslash e$.

($3 \Rightarrow 4$) If G has a cycle, then take any edge e in the cycle C. In any walk that contains e, this e can be replaced with walking the opposite direction on C to form another valid walk. (The rigorous argument is left to the reader for exercise.) Therefore $G \backslash e$ is connected, contradiction. Next, since G is connected, taking a path from u to v and then taking uv gives a cycle in $G + uv$.

($4 \Rightarrow 1$) If G is disconnected, then connecting two vertices in different connected components does not give a cycle, contradiction. This is because once you take that edge, there is no way to come back to the starting vertex. Therefore G is a connected graph with no cycle, i.e. a tree.

($1 \Rightarrow 5$) G is connected from the definition. Next, induction on $|V|$. The base case $|V| \leq 2$ is trivial. There is a leaf v and $G - v$ is a tree. $G - v$ has $|E(G)| - 1$ edges and $|V(G)| - 1$, so $|E(G)| - 1 = |V(G)| - 2$.

($5 \Rightarrow 3$) Adding an edge decreases the number of connected components by at most one. A graph with no edges has $|V|$ connected components. Therefore a graph with $|V| - 2$ edges is disconnected. $\square$

How can we figure out if a sequence is a degree sequence of some tree? It turns out to be a lot simpler than [304] and basically anything that makes sense can be a degree sequence of a tree:

Theorem 309 (Degree Sequence of a Tree)
A sequence $(d_1, \cdots, d_n)$ is a degree sequence of some tree iff all d_i are positive and $\sum d_i = 2n - 2$.

Proof. ($\Rightarrow$) Clear from [308](5) and [301].

($\Leftarrow$) Induction on n, with trivial base cases $n \leq 2$. Now suppose $n \geq 3$. There exists i and j such that $d_i = 1$ and $d_j > 1$; WLOG assume $i = 1$ and $j = 2$. From induction, $(d_2 - 1, d_3, \cdots, d_n)$ is a degree sequence of some tree. Take any vertex v in the tree with degree $d_2 - 1$, and add a leaf adjacent to v. This constructs a tree with the degree sequence $(d_1, \cdots, d_n)$. $\square$

14.3.1 Spanning Trees

Definition 310 (Spanning Subgraph)
A spanning subgraph of a graph G is a subgraph of G such that its vertex set equals $V(G)$. A spanning tree is a spanning graph that is a tree.

Theorem 311
A connected graph G has a spanning tree.

Proof. Let $m = |E|$, and label the edges as e_0, $\cdots$, e_m, arbitrarily. Define the subsets E_0, $\cdots$, E_m of E, as

$$\begin{cases} E_0 = \varnothing \\ E_i = E_{i-1} \cup \{e_i\} & \text{if the spanning subgraph of } G \\ & \text{with } E = E_{i-1} \cup \{e_i\} \text{ has no cycle} \\ E_i = E_{i-1} & \text{otherwise.} \end{cases}$$

Let H be the spanning subgraph of G with $E = E_m$. Clearly, H has no cycle. If $e_i \notin E_m$ and $H + e_i$ has no cycle, then E_i would contain e_i, contradiction. From [308], H is a tree. □

(TODO: minimum spanning tree)

There are other minimum spanning tree algorithms like Prim's algorithm or Borůvka's algorithm.

14.4 Planar Graphs

Definition 312 (Planar Graph)
A plane graph is a graph G where:

- $V \subseteq \mathbb{R}^2$;
- every edge is an arc between two endpoints;
- the interior of each edge contains no vertex and no point of any other edge.

The connected components of $\mathbb{R}^2 \backslash G$ are called faces of G. Since G is contained in a sufficiently large disc, exactly one face is unbounded; that face is called the outer face of G. All other faces are called inner faces of G. A graph H is planar if it is isomorphic to some plane graph.

Theorem 313 (Euler's Formula)
If G is a connected plane graph, and the number of faces of G is F, then

$$|V| - |E| + F = 2.$$

Proof. Induction on $|E|$. The base case is when G has no edges, one vertex, and one face; the formula clearly holds.

Pick any edge e. If e is a loop, removing it reduces $|E|$ and F by one. Otherwise, contracting it reduces $|V|$ and $|E|$ by one. Either way the result follows by induction. □

Theorem 314
If G is simple and planar, and $|V| \geq 3$, then $|E| \leq 3|V| - 6$. If in addition G has no triangles (i.e. K_3 as a subgraph), then $|E| \leq 2|V| - 4$.

Proof. Count the number N of pairs (f, e) where the face f and the edge e are incident. For each face, there are at least 3 edges incident to it, for otherwise there would be parallel edges or loops. Therefore $N \geq 3F$. On the other hand, each edge is incident to exactly two faces, so $N = 2|E|$. This gives $3F \leq 2|E|$. From [313], $3F = 6 - 3|V| + 3|E| \leq 2|E|$, and the first result follows.

The second result can be proved in the exactly same way, using $N \geq 4F$. □

Corollary 315
K_5 and $K_{3,3}$ are not planar.

Proof. K_5 has 5 vertices and 10 edges. $K_{3,3}$ has no triangles, 6 vertices, and 9 edges. The result follows from [314]. □

TODO: add a figure of K5 and K33.

It clearly follows that any subdivision of K_5 or $K_{3,3}$ are not planar. Surprisingly, those two graphs are the only graphs that "need to be checked" to determine if a given graph is planar. The proof requires several more lemmas and theorems, so we have moved the proof to the appendix.

Theorem 316 (Kuratowski's Theorem)
A graph G is planar if and only if it does not have K_5 or $K_{3,3}$ as a topological minor.

14.5 Coloring

Definition 317 (Coloring)
A k-coloring of a graph G is a function $c : V(G) \to \{1, 2, \cdots, k\}$ such that if u and v are adjacent vertices, then $c(u) \neq c(v)$. G is k-colorable if there is a k-coloring of G. The chromatic number $\chi(G)$ of G is the smallest integer k such that G is k-colorable.

Perhaps the most famous theorem about graph coloring is the four-color theorem. (TODO: write something)

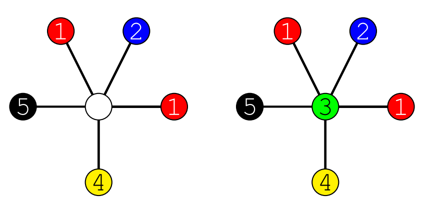
Theorem 318 (Four-color Theorem)
If G is planar, then $\chi(G) \leq 4$.

Unfortunately, the proof is too long and complicated to contain in the codex. We prove a weaker result:

Theorem 319 (Five-color Theorem)
If G is planar, then $\chi(G) \leq 5$.

Proof. Induction on $|V|$. For $|V| \leq 5$, the theorem is trivial.

From [314], G has a vertex v of degree at most 5. If $deg_G(v) < 5$, then inductively find a 5-coloring of $G - v$, and color v by some color in $\{1, 2, 3, 4, 5\}$ not appearing in the neighbors of v. If $deg_G(v) = 5$ and not all colors are used in the neighbors of v, then the same argument applies.



Now suppose all 5 colors are used. Denote the neighbors of v as u_1, u_2, u_3, u_4, u_5, in clockwise order. Without loss of generality, we will assume that $c(u_i) = i$.

The main idea of the rest of the proof is that we want to change the color of one of the neighbors, say change $c(u_i)$ to k. This is impossible if u_i has a neighbor of color k, in which case we want to also change the color of that neighbor, to k'. But then that neighbor might have yet another neighbor of color k', and this continues to form a chain. Hence we introduce the Kempe chain, named after Alfred Kempe.

Let V_{ij} be the set of vertices w in G such that there is a path from u_i to w consisting of vertices of color i or j. Note that if we switch the colors of the vertices in V_{ij} (i.e. change i to j and j to i), and leave everything else the same, then the result is still a coloring.

If V_{13} does not contain u_3, then switch the colors of the vertices in V_{13} and color v by 1.

(TODO: picture)

Otherwise, V_{24} does not contain u_4; switch the colors of the vertices in V_{24} and color v by 2. This gives a 5-coloring of G.

(TODO: picture) $\square$

Fun fact: In 1879, the Kempe chain method was used to "prove" the four-color theorem by Alfred Kempe. No one noticed that this "proof" had an error until eleven years later when Percy Heawood found the error. What we saw above is the modification of the incorrect proof to prove the weaker theorem. The correct proof of four-color theorem was completed in 1976 by Kenneth Appel and Wolfgang Haken.

Here is his "proof." Argue similarly as above with induction. If $deg_G(v) = 4$ and all 4 colors are used, then apply the Kempe chain method. Now suppose $deg_G(v) = 5$ and all 4 colors are used. Then one color is used exactly twice.

There are two cases: the two neighbors with that color are next to each other in clockwise order, or they are not. The first case is easy, just use the Kempe chain method. The second case is where the fun starts.

(TODO: picture. u5-u4-u1-u2-u3 clockwise; u1 and u5 has the same color. Cetner is noted v.)

WLOG, u_k has color k. For convenience, color 5 is the same as color 1.

If V_{42} does not contain u_2, then switch the colors of the vertices in V_{25} and color v by 4. Otherwise, if V_{43} does not contain u_3, then switch and color v by 4. Otherwise, V_{13} does not contain u_3 and V_{52} does not contain u_2. Switch each chain and color v by 1.

(TODO: second case picture.)

Can you find a critical error in this argument? If you want to know, refer to the appendix.

Chapter 15

Cryptosystem

Cryptography is one of the most advanced area of applied mathematics. It uses many terms not used in many other branches of mathematics or applied mathematics, and is often called ”state-of-the-art”-est part of mathematics.

15.1 Basic Terminology

Definition 320 (Basic Terminology on Cryptosystems)

- **Plaintext:** The text before encryption
- **Ciphertext:** The text after encryption
- **Cryptosystems:** Encryption and decryption algorithms, see definition below for more

  Encryption: Using some sort of algorithm to change the content of a message so that it is unrecognizable.

  Decryption: Processing the encrypted message to change it back to the message.
- **Key**: A value required to encrypt or decrypt.

  Encryption Key: The key for encryption.

  Decryption Key: The key for decryption.
- **Cryptanalysis**: Decrypting the ciphertext without any prior knowledge(i.e. key).

Now that the basic terminologies are defined, we can go on with defining ”Cryptosystem”:

Definition 321 (Cryptosystem)
A cryptosystem is defined as a tuple of three algorithms, (G, E, D);

G The key generation algorithm, sometimes abbreviated as KeyGen, chooses the encryption key k_1 and the decryption key k_2 from the set of possible keys. The set of possible keys is called the key space. Usually each key from the key space is chosen at uniformly random probability.

E The Encryption Algorithm, sometimes abbreviated as Enc, uses the encryption key k_1, takes the plaintext m as an input, and produces the ciphertext c. This is usually denoted as follows:

$$E_{k_1}(m) = c$$

D The Decryption Algorithm, sometimes abbreviated as Dec, uses the decryption key k_2, takes the ciphertext c as an input, and gains the plaintext m. This is usually denoted as follows:

$$D_{k_2}(c) = m$$

For a cryptosystem to be valid, by encrypting the plaintext m and decrypting the ciphertext, we must be able to get m, that is;

$$D_{k_2}(E_{k_1}(m)) = m$$

A cryptosystem is classified into two categories; if the encryption key is the same as the decryption key, it is called a Symmetric Key Algorithm; if not, it is called an Asymmetric Key Algorithm or a Public Key Algorithm. A symmetric key algorithm is again classified into two categories; Block Cipher and Stream Cipher.

Definition 322 (Kerckhoffs' Principle)
Kerckhoffs' Principle states that a cryptosystem must be secure even if everything about the cryptosystem except for the key is exposed.

Kerckhoffs' Principle says that the cryptosystem's security must depend only on the secrecy of the key. Its core comes from the idea that "The enemy knows the system". In some, "Security through obscurity"(i.e. hiding the cryptosystem itself) holds but Kerckhoffs' Principle has its value for the following reasons:

1. Storing a smaller sized key is easier than hiding the entire cryptosystem. Also the cryptosystem is not safe from reverse engineering, but keys are, as they are usually a random number.

2. If the key is exposed, it is easier to change only the key, not the entire cryptosystem.

3. A cryptosystem is often used for many users, and everybody using the same cryptosystem allows for more efficient usage of space.

4. If the cryptosystem itself is kept a secret, if a problem arises(i.e. reverse engineering) to expose the cryptosystem, then the entire thing must be redesigned. This takes a lot of knowledge and time.

5. A cryptosystem is made weak by a small mistake; these mistakes are not found before the cryptosystems are analyzed fully, which is most easily done by making the system public. If they are indeed made public, the cryptosystem can be checked for security, allowing for a more secure system.

15.2 Encryption of Arbitrary Length Message

15.2.1 Padding

When using a block cipher, we need the length of the message to be an exact multiple of the length of the block used in the block cipher. If not, we use padding to make the message longer to make it an exact multiple. There are many ways to do so, but the following paddings are the most prominent:

- Zero Padding, otherwise known as Null Padding

  Pad the message with zero(00) bytes to make the length be an exact multiple of the cipher block length. This may cause a problem if the last bytes of the message are 00.

- Bit Padding

  Pad the message with $10|00^n$, so that we can know the start of padding. In this case, the message must be padded even if its length is a multiple of the cipher block length.

- Byte Padding

  Same as zero padding, except the last byte is equal to the length of padding, that is; if we require four more bytes, the padding is $00|00|00|04$. The message must also be padded even if its length is a multiple of the cipher block length.

- PKCS#7 Padding

  Similar to byte padding, except every byte of the padding is equal to the length of padding, that is; if we require four more bytes, the padding is $04|04|04|04$.

15.2.2 Modes of Operation

Sometimes we are required to encrypt a longer message than the length of the block. The plaintext are first padded using one of the techniques above, and the padded plaintext P is separated into blocks of padding length, $P_1, P_2, \cdots, P_N$. They are then encrypted using the key K, sometimes with the help of the initialization vector IV, and produces the ciphertexts $C_1, C_2, \cdots, C_N$. There are five major ways(or "modes") to do this; ECB, CBC, CFB, OFB, and CTR.

Electronic Code Book(ECB)

ECB mode is the simplest mode of them all. They simply take each blocks and encrypt them separately. In equation:

- **Encryption** $C_i = E_K(P_i)$
- **Decryption** $P_i = D_K(C_i)$

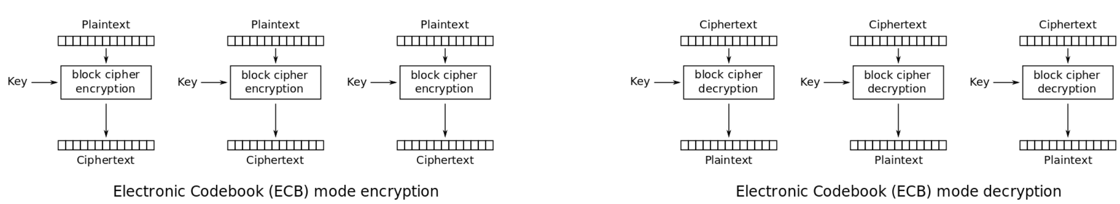


Figure 15.1: ECB Mode

Since same plaintext blocks are encrypted into same ciphertext block, the blocks can be copied, or replayed, to change the message easily. This is called the Block Replay Attack.

Cipher Block Chaining(CBC)

CBC takes the previous ciphertext block and XOR($\bigoplus$) it with the plaintext before encryption. The first block has no previous ciphertext block, hence it is XOR-ed with the IV. In equation:

- **Encryption** $C_0 = IV$, $C_i = E_K(P_i \bigoplus C_{i-1}), i = 1, 2, 3, \cdots, N$
- **Decryption** $C_0 = IV$, $C_i = D_K(C_i) \bigoplus C_{i-1}, i = 1, 2, 3, \cdots, N$

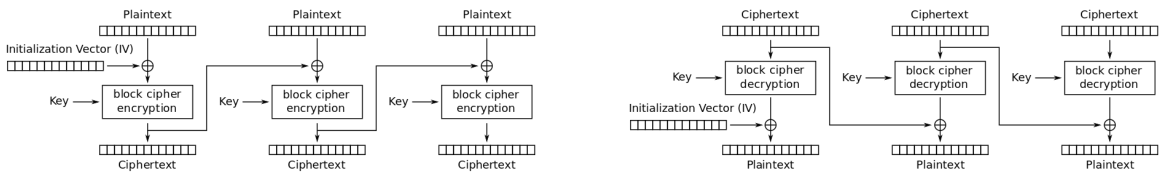


Figure 15.2: CBC Mode

Cipher Feedback(CFB)

CFB can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher. In the diagram given below, original block sizes are used. In equation:

- **Encryption** $C_0 = IV$, $C_i = E_K(P_i \bigoplus C_{i-1}), i = 1, 2, 3, \cdots, N$
- **Decryption** $C_0 = IV$, $C_i = D_K(C_i) \bigoplus C_{i-1}, i = 1, 2, 3, \cdots, N$

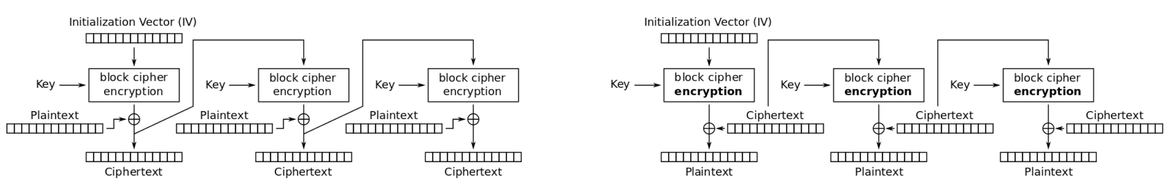


Figure 15.3: CFB Mode

By altering the equation to the following we have the "stream cipherized" version, where $<<$ is the shift operation, $head(a, x)$ is the first x bits of a, and n is the size of the IV:

- **Shift Register** $S_0 = IV$, $S_i = ((S_i << x) + C_i) \mod 2^n$
- **Encryption** $C_i = head(E_K(S_{i-1}), x) \bigoplus P_i$
- **Decryption** $P_i = head(E_K(S_{i-1}), x) \bigoplus C_i$

Output Feedback(OFB)

OFB can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher.

- **Input and Output** $I_0 = IV$, $I_j = E_K(I_{j-1}), j = 1, 2, 3, \cdots, N$
- **Encryption** $C_j = P_j \bigoplus I_j, i = 1, 2, 3, \cdots, N$
- **Decryption** $P_j = C_j \bigoplus I_j, i = 1, 2, 3, \cdots, N$

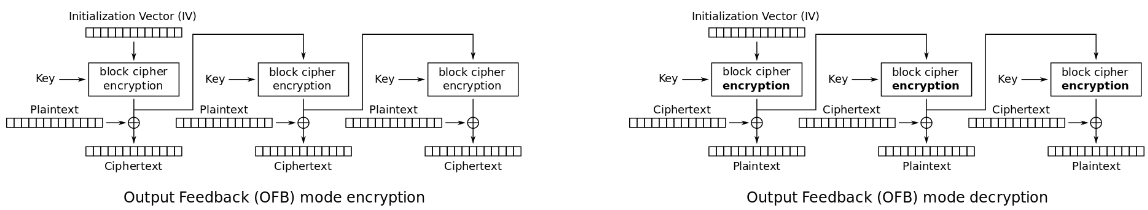


Figure 15.4: OFB Mode

We can similarly alter the equation as OFB so that it can be used as a stream cipher.

Counter(CTR)

CTR can be used to encrypt a block even smaller than the size of the encryption block, and can be used to make a stream cipher out of block cipher. It encrypts the counter value instead of the plaintext, and XORs the value to gain the ciphertext.

- **Encryption** $C_i = P_i \bigoplus E_K(Counter), i = 1, 2, 3, \cdots, N$
- **Decryption** $P_i = C_i \bigoplus E_K(Counter), i = 1, 2, 3, \cdots, N$

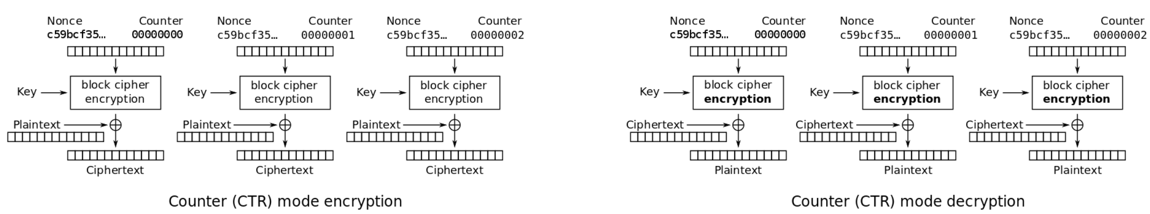


Figure 15.5: CTR Mode

We can similarly alter the equation as OFB so that it can be used as a stream cipher.

Characteristics

Table 15.1 shows the characteristics for each modes of operation.

- Block Pattern: Whether if the overall pattern is kept after encryption

		ECB	CBC	CFB	OFB	CTR
Block Pattern		O	X	X	X	X
Preprocessing		X	X	X	O	O
Parallel Processing	Encryption	O	X	X	O	O
	Decryption	O	O	O	O	O
Error Propagation		X	(P_i, P_{i+1})	$\lceil \frac{n}{r} \rceil$ blocks	X	X
Encryption Unit		n	n	$r(\leq n)$	$r(\leq n)$	$r(\leq n)$

Table 15.1: Characteristics for Each Modes of Operation

- Preprocessing: Whether if preprocessing is possible on encryption and decryption
- Parallel Processing: Whether if parallel processing is possible on encryption or decryption
- Error Propagation: If there is an error in the encryption/decryption process, whether if the error spreads through other blocks
- Encryption Unit: The minimum requirement byte for encryption

15.3 Types of Attack

15.3.1 Attacking Classical Cryptosystems

Classical Cryptosystems are typically a substitution cipher and/or a transposition cipher. Since most, if not all, the classical cryptosystems are broken, the two valid ways to attack any classical cryptosystems is given here.

- Brute Force Attack

  When the attacker gains the ciphertext c, the attacker uses every key possibility to try to gain m. This is otherwise known as the Exhaustive Key Search Attack. Theoretically this can be done to any symmetric-key cipher; but this is inapplicable to most modern cryptosystems as they have an extremely large key space.

- Frequency Analysis

  The plaintext having some pattern, such as the alphabet 'e' appearing with the most frequency, will help the attacker gain knowledge on the plaintext just by seeing the ciphertext.

15.4 Cryptographic Hash Functions

A general hash function has the following properties:

- They take an arbitrary size of data as input, and;
- They produce a constant and fixed length data as output.

A <u>cryptographic hash function</u>, in addition to the properties above, must have the following properties:

- Preimage Resistance

  If the hash value y is given, it must be hard to find an x such that $h(x) = y$, that is, the hash function must have one-wayness.

- Second Preimage Resistance

  If the message x is given, it must be hard to find an $x' \neq x$ such that $h(x) = h(x')$.

- Collision Resistance

  It must be hard to find $x \neq x'$ such that $h(x) = h(x')$. The pair (x, x') is called the collision pair.

15.5 Attacking the Cyrptosystems

Attacks on cryptosystems are classified into passive and active attack. Passive attacks simply eavesdrops the transmission, and gains what the attacker wants without modification of the message. This type of attacking includes eavesdropping, of which the attacker intercepts the message in the middle to check the plaintext. This type of attacker is often referred to as ”Eve”(as in eavesdropping) in theories. Active attackers will modify the message, which includes Modification, Deletion, Impersonation, and Replay. This type of attacker can also be referred to as ”Eve”, but sometimes is referred to as ”Mallory”, for malicious user.

- Modification: Changes the order of the message or changes a part of it to alter the meaning.
- Deletion: Intercepts the message and does not send it, interrupting the communication.
- Impersonation: Fakes their own identity to be identified as a correct user.
- Replay: Send a message again after eavesdropping, expecting some kind of result.

The four methods above are just the general ways to attack. We need to attack the system itself to know how to attack it. There are four methods of attack on system:

- **Ciphertext Only Attack**

  The attacker knows only the ciphertext.

- **Known Plaintext Attack**

  The attacker knows a list of (message, ciphertext) pair, and attempts to crack a ciphertext not in the list.

- **Chosen Plaintext Attack**

  The attacker has access to an oracle that can encrypt the message, and attempts to crack a ciphertext.

- **Chosen Ciphertext Attack**

  The attacker has access to an oracle that can decrypt a ciphertext, except for the target ciphertext.

There are three important properties to encryption schemes:

- Semantic Security

  A semantically secure encryption scheme is infeasible for any computationally bounded adversary to derive a significant information about the original plaintext when given only its ciphertext and the corresponding public key if any. This can be represented as a game between the oracle and the adversary, as below:

  1. The oracle generates a key for the challenge.
  2. The adversary is given the encryption oracle(or the public key, in the case of public key cryptosystem).
  3. The adversary can perform any number of polynomially bounded number of encryptions or operations.
  4. The adversary generates two equal-length messages m_0 and m_1, and transmits it to the oracle.
  5. The oracle randomly chooses $b \in \{0,1\}$ to encrypt the message m_b to C.
  6. The adversary, upon receiving C, guesses b.

  If the adversary cannot guess b correctly with significantly greater than 50% probability, then the scheme is said to be semantically secure under CPA.

- Indistinguishability

  If a cryptosystem is indistinguishable, then an adversary would not be able to distinguish pairs of ciphertexts based on the message they encrypt. There are three types: IND-CPA, IND-CCA1, and IND-CCA2. They can be represented as a game between the oracle and the adversary. In both cases, they are said to be secure if the adversary does not have a clear advantage. Note that for any nonzero α, if the adversary has access to the LR-oracle multiple times, and if the probability of advantage of the adversary is $0.5 \pm \alpha$, a repetitive trial is capable of bringing the odds up close to 1(for $0.5+\alpha$) or down close to 0(for $0.5-\alpha$, in which the adversary may simply chooses the opposite).

  - **IND-CPA**
    1. The oracle generates a key for the challenge.
    2. The adversary is given the encryption oracle(or the public key, in the case of public key cryptosystem).
    3. The adversary can perform any number of polynomially bounded number of encryptions or operations.
    4. The adversary generates two distinct equal-length messages m_0 and m_1, and transmits it to the oracle.
    5. The oracle randomly chooses $b \in \{0,1\}$ to encrypt the message m_b to C.
    6. The adversary, upon receiving C, performs polynomially bounded encryptions or operations, and guesses b.
  - **IND-CCA**
    1. The oracle generates a key for the challenge.
    2. The adversary is given the decryption oracle and the public key, in the case of public key cryptosystem.

       Note that in the case of the public key cryptosystem, the encryption oracle is also given.

3. The adversary can perform any number of polynomially bounded number of decryptions or operations.
4. The adversary generates two distinct equal-length messages m_0 and m_1, and tramsmits it to the oracle.
5. The oracle randomly chooses $b \in \{0, 1\}$ to encrypt the message m_b to C.
6. The adversary, upon receiving C, performs polynomially bounded operations.

   In the case of IND-CCA1, the adversary may not make further calls to the decryption oracle.

   In the case of IND-CCA2, the adversary may make further calls to the decryption oracle, but may not submit C.
7. The adversary guesses b.

This can be said with a random oracle. In that case, the adversary submits only one message and the oracle returns the encryption of the message or the random string equal to the length of the encryption with a fair chance. The adversary then guesses whether if the message is randomly generated or encrypted.

- Non-malleability

  Cryptosystems are called ”malleable” if it is possible to transform a ciphertext into another ciphertext which decrypts to a related plaintext. Cryptosystems that are not malleable are called non-malleable. These, similar to indistinguishability, can be represented as a game between the oracle and the adversary, and are called NM-CPA, NM-CCA1, NM-CCA2. Some cryptosystems, however, are malleable by design(i.e. RSA cryptosystem), but has low probability that it would be abused.

Theorem 323
The following relations for each security properties hold:

- IND-CPA $\Leftrightarrow$ Semantic security under CPA
- NM-CPA $\Rightarrow$ IND-CPA
- NM-CCA2 $\Leftrightarrow$ IND-CCA2
- NM-CPA does not necessarily imply IND-CCA2.

15.6 Digital Signatures

Digital signatures are used in pair with the public key cryptosystems to verify the sender of the messages. When attacking, there are three major methods:

- **Key-Only Attack**

  The attacker only has access to the digital signature algorithm and the public key of the signer, pk_A. This is similar to the Ciphertext Only attack.

- **Known Message Attack**

  The attacker has access to the digital signature algorithm, the public key of the signer, and a list of (message, signature) pairs. This is similar to the Known Plaintext attack.

- **Chosen Message Attack**

  The attacker has access to the digital signature algorithm, the public key of the signer, and an oracle that takes a message as an input and returns signature as an output.

The attacker can have three different purposes:

- **Total Break**

  The attacker wants to gain the private key of the signer.

- **Selective Forgery**

  The attacker wants to generate a valid signature for a message the attacker wants(i.e. any message for that matter).

- **Existential Forgery**

  The attacker wants to generate a valid (message, signature) pair for any message.

It is said that an attack is valid if the attack succeeds with a non-negligible probability.

15.7 Zero-Knowledge Authentication

Three major ways to authenticate a user is using password, challenge-response, and zero-knowledge authentication. Passwords must be sent through network, thereby they are susceptible to interception. Challenge-response can be abused by malicious users to crack the secret key. That is where the concept of zero-knowledge interactive proof comes in.

An interactive proof system can be described as a communication between the verifier and the prover. They exchange messages to check whether if the statement is true or false. In here, the prover is assumed to have unlimited calculating power but cannot be trusted; the verifier has bounded computation power but is assumed to be always honest. Messages are sent between the prover and the verifier until the verifier has an answer to the problem and has convinces itself that the answer is correct.

Any interactive proof system must have the following properties:

- **Completeness**: If the statement is true, the honest verifier will be convinced of this fact by an honest prover.

- **Soundness**: If the statement is false, no cheating prover can convince the honest verifier that it is true, except with some small probability.

In authentication, if the proof is only interactive, a malicious verifier may abuse the protocol to reveal the "knowledge"(in the case for cryptosystems, private keys) only the prover knows. This is where the concept of "Zero-knowledgeness" comes in.

- **Zero-knowledgeness**: If the statement is true, no verifier can learn anything apart from the fact that the statement is true.

The best way to describe this is by an analogy of a colorblind person. Suppose the person has two balls that looks the exactly same for them. Their friend, as a non-colorblind person, want to prove that the two balls are of different color. The colorblind person resumes the role of verifier and the non-colorblind friend the prover. Here is an example protocol on how the fact can be proven:

1. Verifier shows you a ball.
2. Prover memorize it.
3. Verifier then hides both balls, and choose to keep the ball shown before or change the ball.
4. Verifier shows the newly chosen ball.
5. Prover tell verifier whether if the ball has been changed or not.
6. If the prover is wrong, the prover has told a lie; end the protocol.
7. If the prover is right, the prover may be telling the truth; continue the protocol until convinced.

If the statement('The two balls are of different color') is false, then the prover(in this case, cheating) cannot tell whether if the ball has been changed; therefore their guess is right for 50% of the time. n consecutive application of the protocol gives $\frac{1}{2^n}$ chance of success, and as the number of trials increase, the less the cheating prover will be able to pass the protocol.

If the statement, on the other hand, is indeed true, then the prover can tell whether if the ball has been switched every time. In the verifier's point of view, the prover's n-th consecutive success for verification proves that they are lying at $\frac{1}{2^n}$ probability; their improbable probability of success at lying will thereby prove their honesty.

15.8 RSA Cryptosystem and Signature

15.8.1 Keygen

1. Choose two primes p and q.
2. Let $n = p \cdot q$.
3. Choose e such that $(e, \phi(n)) = 1$
4. Find d such that $e \cdot d \equiv 1 \mod \phi(n)$

Public Key: (n, e)
Private Key: d or (p, q, d), depending on the method.

15.8.2 Cryptosystem

Encryption

$C \equiv M^e \mod n$

Decryption

- **Basic Method**
  $C^d \equiv (M^e)^d \equiv M^{\phi(n)\cdot k+1} \equiv M \mod n$
- **Chinese Remainder Theorem**
  Split $C^d \bmod n$ into two congruences: $C^d \mod p$ and $C^d \mod q$.
  Using Euler's Theorem(If $(a, n) = 1$, $a^{\phi(n)} \mod n$.), reduce d to reduce the number of multiplication. There is a more formularized version of this, which will not be mentioned in here.

15.8.3 Signature

Signing

$S \equiv M^d \mod n$

Verifying

Compare $S^e \mod n$ to M. If equal, accept; otherwise reject.

15.8.4 Attacking the Cryptosystem

On the Case of Exposed Private Key e

Total break is possible.
For the public key, $n = pq$ where p and q are primes.
Then, $\phi(n) = (p-1)(q-1)$.
We know that $ed \equiv 1 \mod \phi(n)$.
By the definition of modular, $ed - 1 = k\phi(n)$ for some k.
For a large enough $n = pq$, $\frac{\phi(n)}{n} = \frac{(p-1)(q-1)}{pq} = 1 - \frac{1}{p} - \frac{1}{q} + \frac{1}{pq} \simeq 1$.
We can find k by dividing both sides of the equation $ed-1 = k\phi(n)$ by n, since $\frac{ed-1}{n} = k\frac{\phi(n)}{n} \simeq k$.
We can then find $\phi(n) = \frac{ed-1}{k}$.
Since $n = pq$ and $\phi(n) = (p-1)(q-1) = pq - (p+q) + 1 = n - (p+q) + 1$, $p+q = n - \phi(n) + 1$.
Then the quadratic equation $(x-p)(x-q) = x^2 - (p+q)x + pq = x^2 - (n - \phi(n) + 1)x + n = 0$ can be solved to yield p and q.

Chosen Ciphertext Attack

1. Alice sends $C \equiv M^e \mod n$ to Bob
2. Eve intercepts Alice's transmission; chooses x s.t. $(x, n) = 1$(and therefore $x^{-1} \mod n$ exists) to send $C' = Cx^e \mod n$ to Bob.
3. Bob decrypts C' as $(C')^d \equiv (Cx^e)^d \equiv C^d x^{ed} \equiv Mx \mod n$
4. Eve intercepts Bob's decryption result, Mx, and multiplies x^{-1} modulo n to gain M.

Coppersmith Attack

Theorem 324 (Coppersmith)
Let $n \in \mathbb{Z}$ and $f \in \mathbb{Z}[x]$ be a monic polynomial(i.e. leading coefficient of f is 1) of degree d over integer.
Set $X = n^{1/d-\epsilon}$ for $1/d > epsilon > 0$. Then given n and f, the attacker, using the LLL Algorithm, can efficiently find all integer $x_0 < X$ such that $f(x_0) \equiv 0 \mod n$.

Note
In the case of RSA, Finding M when given $C \equiv M^e \mod n$ can be interpreted as finding the solution of the equation $f(x) \equiv x^e - C \mod n$. This attack's strength is the ability to find all small roots of the polynomials modulo a composite N.

Håstad's Broadcast Attack

This attack is viable if the value of e is fixed and is small, and the same message is broadcast without padding.

Suppose the same plaintext M is encrypted to multiple people, each using same e and different moduli, say N_i. If Eve successfully intercepts e or more messages, say $C_1, C_2, \cdots, C_e$, $C_i \equiv M^e \mod N_i$. We may assume $(N_i, N_j) = 1$ for $i \neq j$, otherwise the attacker will be able to factorize some N_i by finding their GCD. Using the Chinese Remainder Theorem on the e congruences, the attacker may compute $C \in Z^*_{\Pi N_i}$ such that $C_i \equiv C \mod N_i$. Then, $C \equiv M^e \mod \Pi N_i$; however since $M < N_i$ for each i, $M^e < \Pi N_i$; thus $C = M^e$ holds over the integers, and the attacker can easily find the message M.

For more generalized version, the following theorem is available:

Theorem 325 (Håstad)
Suppose $N_1, \cdots, N_k$ are relatively prime integers and set $N_{\texttt{min}} = \min_i\{N_i\}$. Let $g_i(x) \in \mathbb{Z}/N_i[x]$ be k polynomials of maximum degree q. Suppose there exists a unique $M < N_{\texttt{min}}$ satisfying $g_i(M) \equiv 0 \mod N_i \forall i \in \{1, \cdots, k\}$. Furthermore, suppose $k > q$. Then there is an efficient algorithm which, given $\langle N_i, g_i(x) \rangle \forall i$, computes M.

This theorem can be used in the following way:
Suppose the i-th plaintext is padded with the polynomial $f_i(x)$. Let $g_i(x) = (f_i(x))^{e_i} - C_i \mod N_i$. Then $g_i(M) \equiv 0 \mod N_i$ is true, and the Coppersmith's Attack[15.8.4] can be used.

Franklin-Reiter Related Message Attack

This attack is viable if the value of e is fixed and is small, and the same message is broadcast with padding.

Theorem 326
Let (n, e) be the public key of RSA, and e is small. Let $f(x) = ax + b \in Z_n[x]$, $b \neq 0$; i.e. f is the padding function.
Suppose that $M_1 \neq M_2$ and $M_1 \equiv f(M_2) \mod n$.
Then, given the quintuplet (n, e, C_1, C_2, f), M_1 and M_2 can be recovered in $O((log_2 n)^2)$

Proof.
$C_1 \equiv M_1^e \mod n$
$C_2 \equiv M_2^e \mod n$
$M_1 \equiv f(M_2) \equiv aM_2 + b \mod n$

Let $g_2(x) = x^e - C_2 \mod n$, and $g_1(x) = (ax+b)^e - C_1 \mod n$

$g_1(x) = (ax+b)^e - C_1$
$= (ax+b)^e - M_1^e$
$= (ax+b)^e - (aM_2+b)^e$
$= ((ax+b) - (aM_2+b))Q(x)$
$= a(x - M_2)Q(x)$

$g_2(x) = x^e - C_2$
$= x^e - M_2^e$
$= (x - M_2)Q'(x)$

$\rightarrow (x - M_2)|(g_1(x), g_2(x)$
Using the euclidean algorithm on the two polynomials g_1 and g_2, M_2 can be recovered. ∎

15.8.5 Forgeries of the Signature

Known Message Attack

Suppose (M_1, S_1) and (M_2, S_2) are both valid signatures.
Then, (M_1M_2, S_1S_2) is also a valid signature.

Chosen Message Attack

Eve chooses M_1 and M_2 s.t. $M = M_1M_2$.
Eve asks Alice to sign M_1 and M_2; let them be S_1 and S_2.
Then S_1S_2 is a valid signature for M.

15.9 ElGamal Cryptosystem

15.9.1 Keygen

Choose a prime p. Note that $(Z_p^*, \times)$ is a cyclic group.

- Choose e_1 to be the primitive root of $(Z_p^*, \times)$
- Choose $d \in Z_p^*$ and compute $e_2 \equiv e_1^d \mod p$

In theory, p and e_1 can be shared as long as e_2 are kept distinct.
Public Key: (e_1, e_2, p)
Private Key: d

15.9.2 Cryptosystem

Encryption

Randomly choose $r \in Z_p^*$. M is the message.

- $C_1 \equiv e_1^r \mod p$
- $C_2 \equiv Me_2^r \mod p$

Ciphertext: (C_1, C_2)

Decryption

$C_2(C_1^d)^{-1} \equiv Me_2^r(e_1^{rd})^{-1} \equiv M(e_1^d)^r(e_1^{rd})^{-1} \equiv M \mod p$

15.9.3 Signature

Signing

Randomly choose $r \in Z_p^*$. M is the message.

- $S_1 \equiv e_1^r \mod p$
- $S_2 \equiv (M - dS_1)r^{-1} \mod (p-1)$

Signature: (S_1, S_2)

Verifying

Calculate:

- $V_1 \equiv e_1^M \mod p$
- $V_2 \equiv e_2^{S_1} S_1^{S_2} \mod p$

Verify with:

- Check $0 < S_1 < p$, $0 < S_2 < p-1$.
- Check $V_1 = V_2$

$V_2 \equiv e_2^{S_1} S_1^{S_2} \equiv (e_1^d)^{S_1}(e_1^r)^{S_2} \equiv e_1^{dS_1+rS_2} \equiv e_1^M \equiv V_1 \mod p$

15.9.4 Attacking the Cryptosystem

Exposure of r

Since (C_1, C_2) and r are exposed, $M = C_2(e_2^r)^{-1} \mod p$.

Baby step, Giant step

When the random number r is small, then the following meet-in-the-middle attack is possible:
$y = e_1^x \mod p$.
Let $m = \lceil \sqrt{p} \rceil$.
Then, $\exists q, r \in \mathbb{Z}$ such that $x = mq + r, 0 \le r \le m-1$
$\Rightarrow y = e_1^x \equiv e_1^{mq+r} \mod p$
$\Rightarrow y(e_1^{-m})^q \equiv g^r \mod p$
Hence we can find r using the following protocol:

1. Construct the table with entries $(r, e_1^r \mod p), 0 \le r \le m-1$: (Baby step table)
2. Compute the value $g^{-m} \mod p$: (Giant step value)
3. For q from 0 to $m-1$, find q such that $y(g^{-m})^q \equiv g^r \mod p$ in the table.

Known Plaintext Attack

Suppose the random number r is reused to encrypt two distinct messages, M and M'.
Suppose M encrypted to (C_1, C_2); M' encrypted to (C_1', C_2').
Note that $C_1 = C_1' = e_1^r$, $C_2 = Me_2^r$, $C_2' = M'e_2^r$.
If we know M', then $\frac{C_2 \times M'}{C_2'} = \frac{Me_2^r \times M'}{M'e_2^r} = M$

15.9.5 Forgeries of the Signature

Constructing from Scratch: One Variable

Choose $1 < x < p-1$.

- $S_1 \equiv e_1^x e_2 \mod p$
- $S_2 \equiv -S_1 \mod (p-1)$
- $M \equiv xS_2 \mod (p-1)$

Constructing from Scratch: Two Variables

Choose $u, v \in Z_p^*$ such that $(v, p-1) = 1$ so that $\exists v^{-1} \mod (p-1)$

- $S_1 \equiv e_1^u e_2^v \mod p$
- $S_2 \equiv -S_1 v^{-1} \mod (p-1)$
- $M \equiv S_2 u \mod (p-1)$

Known Plaintext Attack

This method can be used if the range conditions are not checked properly.
A valid signature $(M, (S_1, S_2))$ is given for $(M, p-1) = 1$ so that $\exists M^{-1} \mod (p-1)$.
Choose a message M'.
Set $u = M'M^{-1} \mod (p-1)$.
Compute $S_2 \equiv S_2 u \mod (p-1)$.
Solve the following set of linear congruences using CRT:

$$\begin{cases} S_1' = S_1 u \mod (p-1) \\ S_1' = S_1 \mod p \end{cases}$$

Then, $(M', (S_1', S_2'))$ is also a valid signature, if the range conditions are not checked.

15.10 Schnorr Digital Signature

Signatures based on cryptosystems have a weakness: they pose a threat to expose the secret key, or makes it easier to forge a specific message. Schnorr Digital Signature is a signature-only algorithm that helps solve this.

15.10.1 Keygen

- Choose a cryptographic hash function h.
- Choose a prime p.
- Choose a prime q such that:

    $q | p-1$, and;

    The size of q is the same as the hash output.
- Choose e_0 such that it is a generator in Z_p^*.
- Set $e_1 \equiv e_0^{(p-1)/q} \not\equiv 1 \mod p$.
- Choose d.
- Set $e_2 \equiv e_1^d \mod p$.

Public Key: (h, e_1, e_2, p, q)
Private Key: d

15.10.2 Signature

Signing

Choose $r \in Z_q^*$ at random.

- $S_1 = h(M || e_1^r \mod p)$ where $||$ is concatenation.
- $S_2 = r + dS_1 \mod q$

Signature: (S_1, S_2)

Verifying

Calculate $V = h(M || e_1^{S_2} e_2^{-S_1} \mod p)$.
If $V = S_1$, then M is accepted.
$e_1^{S_2} e_2^{-S_1} = e_1^{r+dS_1} e_1^{-dS_1} = e_1^r \mod p$

Part IV

Appendix

Chapter 16

Appendix

16.1 Equivalent Statements for Invertible Matrices

For $n \times n$ matrix A, the followings are equivalent:

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ only has the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A can be represented as a product of elementary matrices.

(e) $A\mathbf{x} = \mathbf{b}$ is consistent $\forall n \times 1$ matrix $\mathbf{b}$.

(f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution $\forall n \times 1$ matrix $\mathbf{b}$.

(g) $\det(A) \neq 0$.

(h) column vectors of A are linearly independent.

(i) row vectors of A are linearly independent.

(j) column vectors of A span $\mathbb{R}^n$.

(k) row vectors of A span $\mathbb{R}^n$.

(l) column vectors of A form a basis for $\mathbb{R}^n$.

(m) row vectors of A form a basis for $\mathbb{R}^n$.

(n) $\text{rank}(A) = n$

(o) $\text{nullity}(A) = 0$

(p) $(\text{Null}(A))^\perp = \mathbb{R}^n$

(q) $(\text{Row}(A))^\perp = \{\mathbb{O}\}$

(r) range of T_A is $\mathbb{R}^n$

(s) T_A is one-to-one.

(t) $\lambda = 0$ is not an eigenvalue of A.

(u) $A^T A$ is invertible.

16.2 Formula for Projection Onto a Subspace

We are projecting the vector $\mathbf{b}$ onto a subspace of $\mathbb{R}^n$ with basis $S = \{\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}\}$. Let $A = [\mathbf{a_1}|\mathbf{a_2}|\ldots|\mathbf{a_n}]$. Then A has linearly independent columns. We now come to theorem [124].

Proof for Theorem 124. Consider the following equation:

$$A^T A\mathbf{x} = \mathbf{0}$$

$A\mathbf{x}$ is an element in the column space of A and also the null space of A^T. However since the two spaces are orthogonal complements, implying $A\mathbf{x} = \mathbf{0}$.

If A has linearly independent columns, then $A\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$, hence $null(A^T A) = \{\mathbf{0}\}$. Since $A^T A$ is square, by theorem 16.1, it is invertible. □

The combination $\mathbf{p} = \hat{\mathbf{x}}_1 \mathbf{a_1} + \cdots + \hat{\mathbf{x}}_n \mathbf{a_n} = \mathbf{A}\hat{\mathbf{x}}$ that is closest to $\mathbf{b}$ is derived by the equation $\mathbf{b} = \mathbf{p} + \mathbf{e}$, and since $\|\mathbf{e}\|$ must be minimized, it must pe perpendicular to $\mathbf{span}(S)$, and therefore $A^T\mathbf{e} = \mathbf{0}$. Rewriting the equation to $\mathbf{b} - \mathbf{p} = \mathbf{e}$ and multiplying A^T to the left side, we get:

$$A^T(\mathbf{b} - \mathbf{p}) = \mathbf{A^T}(\mathbf{b} - \mathbf{A}\hat{\mathbf{x}}) = \mathbf{0}\mathbf{A^T}\mathbf{A}\hat{\mathbf{x}} = \mathbf{A^T}\mathbf{b}$$

And now by theorem [124], $A^T A$ is invertible. Therefore $\hat{x} = (A^T A)^{-1} A^T \mathbf{b}$ and $\mathbf{p} = \mathbf{A}\hat{\mathbf{x}} = \mathbf{A}(\mathbf{A^T}\mathbf{A})^{-1}\mathbf{A^T}\mathbf{b}$.

This formula yields the $n \times n$ projection matrix of A that produces $\mathbf{p} = \mathbf{Pb}$:

$$P_A = A(A^T A)^{-1} A^T$$

16.3 Cook-Levin Theorem

In this section

16.4 Kuratowski Theorem

In this section we prove [316].

16.4.1 The Preparation

First, we show that a planar graph can be drawn so that an arbitrary vertex or an edge is incident to the outer face.

Lemma 327
If G is planar and $v \in V(G)$, then there is a planar embedding of G such that v is on the boundary of the outer face. The same can be done for $e \in E(G)$.

Proof. We use the stereographic projection. In $\mathbb{R}^3$, let $z = -1$ be the plane P and $x^2 + y^2 + z^2 = 1$ be the sphere S. $(0,0,1)$ is the "north pole" of S. Define the projection $\rho : S \backslash \{(0,0,1)\} \to P$ as follows: given (x,y,z) on S which is not the north pole, draw a straight line through $(0,0,1)$ and (x,y,z). There is a unique intersection of this line with P, denoted as $(X,Y,-1)$. Then $\rho(x,y,z) = (X,Y,-1)$. Clearly ρ is bijective.

Given an embedding of a planar graph G on P, ρ^{-1} gives an embedding of G on S. Rotate the embedding so that a face incident to v or e contains the north pole. ρ gives an embedding of G on P such that the face is the outer face. □

Next, we introduce the notion of connectivity. Although connectivity is a crucial part of graph theory, we didn't put this into the main part of the codex because of the length concerns.

Definition 328 (Connectivity)
A graph G is <u>k-connected</u> if $|V| > k$ and, for every $S \subset V$ with $|S| < k$, $G \backslash S$ is connected.

Theorem 329
If G is 3-connected with $|V(G)| \geq 5$, then there is an edge e such that G/e is 3-connected.

Proof. Let $e = xy$ and suppose G/e is not 3-connected. Then G/e has a cut set $\{v, z\}$. Since G is 3-connected, this set has a vertex, say v, which is the new vertex made by contracting e. That is, $\{x, y, z\}$ is a cut set of G.

Suppose that for every e, G/e is not 3-connected, so to every e corresponds a vertex z_e. Among all edges, take $e = xy$ and z_e such that $G - x - y - z$ has the largest component C, and denote another component as D. Each of x, y, z has neighbors in C and in D since G is 3-connected. Take a neighbor u of z in D and let $v = z_{zu}$.

If $v \in V(C) \cup \{x, y\}$, then $G - z - v$ is disconnected, contradicting the connectivity of G. Otherwise, $G - z - u - v$ has a component that contains all vertices in C and x and y in addition, contradicting the choice of C.

(TODO: picture) □

Then, we show the connection between minors and topological minors.

Lemma 330
$K_{3,3}$ is a topological minor of G iff $K_{3,3}$ is a minor of G.

Proof. A topological minor of G is also a minor of G. We just need to prove the other direction of the lemma.

□

Lemma 331
If K_5 is a minor of G, then $K_{3,3}$ or K_5 is a topological minor of G.

Proof. . □

16.4.2 The Proof

The last step is closely related to the Kuratowski's theorem.

Definition 332 (Convex Embedding)
A <u>convex embedding</u> of a planar graph G is a plane graph in which all edges are straight line segments and all face boundaries are convex polygons.

Lemma 333
If G is simple, 3-connected, and has no K_5 or $K_{3,3}$ as a minor, then G has a convex embedding on a plane, with no three vertices on a line.

Proof. TODO □

We are finally ready to prove the Kuratowski's theorem. For convenience, we will restate the theorem:

> A graph G is planar if and only if it does not have K_5 or $K_{3,3}$ as a topological minor.

Proof. Induction on $|V|$, with trivial base case $|V| \leq 4$.

If G is disconnected, from induction there is a planar embedding of each component. Since each embedding is bounded by a finite disc, their union can be drawn on a plane.

If G is connected but not 2-connected, then take a cut-vertex v. Let G_1, $\cdots$, G_n be the connected components of $G - v$, and H_i be the subgraph induced by $V(G_i) \cup \{v\}$. Take an embedding of each H_i such that v is in the outer face [327] and squeeze it into an angle $< 2\pi/n$ at the vertex v. Joining those embeddings together forms an embedding of G.

If G is 2-connected but not 3-connected, TODO

If G is 3-connected, the conclusion immediately follows from [333]. □

16.5 What's Wrong With Kempe's Proof?

Kempe argued that switching V_{13} and V_{52} allows v to be colored by 1, but consider the following graph:

(TODO: counterexample picture)

Both chains cannot be switched because then the vertices a and b would have the same color!

In this graph, such a problem could be avoided by deliberately changing the order of vertices to be selected for induction. However, there are graphs on which such a workaround is not possible. The following is the smallest counterexample possible, and is called the Soifer graph:

(TODO: Soifer graph)