Kinetic Reverse k-Nearest Neighbor Problem *

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Abstract

This paper provides the first solution to the kinetic reverse k-nearest neighbor (RkNN) problem in $\mathbb{R}^d$, which is defined as follows: Given a set P of n moving points in arbitrary but fixed dimension d, an integer k, and a query point $q \notin P$ at any time t, report all the points $p \in P$ for which q is one of the k-nearest neighbors of p.

1 Introduction

The *reverse k-nearest neighbor* (RkNN) problem is a popular variant of the k-nearest neighbor (kNN) problem and asks for the influence of a query point on a point set. Unlike the kNN problem, the exact number of reverse k-nearest neighbors of a query point is not known in advance. The RkNN problem is formally defined as follows: Given a set P of n points in $\mathbb{R}^d$, an integer k, $1 \leq k \leq n-1$, and a query point $q \notin P$, find the set RkNN(q) of all p in P for which q is one of k-nearest neighbors of p. Thus RkNN$(q) = \{p \in P : \ |pq| \leq |pp_k|\}$, where $|.|$ denotes Euclidean distance, and p_k is the k^{th} nearest neighbor of p among the points in P. The *kinetic RkNN* problem is to answer RkNN queries on a set P of moving points, where the trajectory of each point $p \in P$ is a function of time. Here, we assume the trajectories are polynomial functions of maximum degree bounded by some constant s.

Related work. The reverse k-nearest neighbor problem was first posed by Korn and Muthukrishnan [14] in the database community, and then considered extensively in this community due to its many applications, *e.g.*, decision support systems, profile-based marketing, traffic networks, business location planning, clustering and outlier detection, and molecular biology [14, 15, 16]. The reverse k-nearest neighbor queries for a set of continuously moving objects has also attracted the attention of the database community; see [9] and references therein. Examples of moving objects

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include players in multi-player game environments, soldiers in a battlefield, tourists in dangerous environments, and mobile devices in wireless ad-hoc networks.

To our knowledge, in computational geometry, there exist two data structures [17, 10] that give solutions to the RkNN problem. Both of these solutions answer RkNN queries for a set P of stationary points and both only work for $k = 1$. Maheshwari *et al.* (2002) [17] gave a data structure to solve the R1NN problem in $\mathbb{R}^2$. Their data structure, which supports insertions and deletions of points, creates an arrangement of largest empty circles centered at the points of P and answers R1NN queries by point location in the arrangement. Their data structure uses $O(n)$ space and $O(n \log n)$ preprocessing time, and an R1NN query can be answered in time $O(\log n)$. Cheong *et al.* (2011) [10] considered the R1NN problem in fixed dimension $\mathbb{R}^d$, where $d = O(1)$. Their method, which uses a compressed quadtree, partitions space into cells such that each cell contains a small number of candidate points. To answer an R1NN query, their solution finds a cell that contains the query point and then checks all the points in the cell. Their approach uses $O(n)$ space and $O(n \log n)$ preprocessing time, and can answer an R1NN query in $O(\log n)$ time; it seems that the approach by Cheong *et al.* can be extended to answer RkNN queries with preprocessing time $O(kn \log n)$, space $O(kn)$, and query time $O(\log n + k)$.

For a set P of n stationary points, one can report all the 1-nearest neighbors in time $O(n \log n)$ [20], and all the k-nearest neighbors, for any $k \geq 1$, in time $O(kn \log n)$ [13], where the neighbors are reported in order of increasing distance from each point; reporting the unordered set takes time $O(n \log n + kn)$ [6, 11, 13].

For a set of moving points, there are two kinetic data structures [2, 19] to maintain all the k-nearest neighbors, but they only work for $k = 1$.

Our contribution. We provide the *first* solution to the kinetic RkNN problem for *any* $k \geq 1$ in *any* fixed dimension d. To answer an RkNN query for a query point $q \notin P$ at any time t, we partition the d-dimensional space into a constant number of cones around q, and then among the points of P in each cone, we examine the k points having shortest projections on the cone axis. We obtain $O(k)$ candidate points for q such that q might be one of their k-nearest neighbors at time t. To check which if any of these candidate points is a reverse k-nearest neighbor of q, we maintain the k^{th} nearest neighbor p_k of each point $p \in P$ over time. By checking whether $|pq| \leq |pp_k|$ we can easily check whether a candidate point p is one of the reverse k-nearest neighbors of q at time t.

For a set P of n continuously moving points in $\mathbb{R}^d$, where the trajectory of each point is a polynomial function of at most constant degree s, we provide a simple kinetic approach to answer RkNN queries on the moving points. In the preprocessing step, we introduce a method for reporting all the k-nearest neighbors for all the points $p \in P$ in order of increasing distance from p. For $k = \Omega(\log^{d-1} n)$, both our method and the method of Dickerson and Eppstein [13] give the same complexity, but in our view, our method is simpler in practice.

In order to answer RkNN queries, our kinetic approach maintains all the k-nearest neighbors over time. This is the first KDS for maintenance of all the k-nearest neighbors in $\mathbb{R}^d$, for any $k \geq 1$. Our KDS uses $O(n \log^d n + kn)$ space and $O(n \log^d n + kn \log n)$ preprocessing time, and processes $O(\phi(s,n) * n^2)$ events, each in amortized time $O(\log n)$. Here, $\phi(s,n)$ is the complexity of the k-level of a set of n partially-defined polynomial functions, such that each pair of them intersects at most s times. The current bounds on $\phi(s,n)$ are as follows.

$$\phi(s,n) = \begin{cases} O(n^{3/2}\log n), & \text{for } s = 2 \text{ [8];} \\ O(n^{5/3}\text{poly}\log n), & \text{for } s = 3 \text{ [7];} \\ O(n^{31/18}\text{poly}\log n), & \text{for } s = 4 \text{ [7];} \\ O(n^{161/90-\delta}), & \text{for } s = 5, \text{ for some constant } \delta > 0 \text{ [8];} \\ O(n^{2-1/2s}), & \text{for odd } s \text{ [7];} \\ O(n^{2-1/2(s-1)}), & \text{for even } s \text{ [7].} \end{cases}$$

At any time t, an RkNN query can be answered in time $O(\log^d n + k \log\log n)$. Note that if an event occurs at the same time t, we first spend amortized time $O(\log n)$ to update all the k-nearest neighbors, and then we answer the query.

Outline. Section 2 provides two key lemmas, and in fact introduces a new supergraph, namely the *k-Semi-Yao graph*, of the k-nearest neighbor graph. In Section 3, we show how to report all the k-nearest neighbors. Section 4 gives a (kinetic) data structure for answering RkNN queries on moving points, where the trajectory of each point is a bounded-degree polynomial. Also included in this section is an analysis of our kinetic data structure in terms of the kinetic data structure performance criteria. Section 5 concludes.

2 Key Lemmas

Partition the plane around the origin o into six wedges, $W_0, ..., W_5$, each of angle $\pi/3$ (see Figure 1(a)). Denote by $W_l(p)$ the translation of wedge W_l, $0 \leq l \leq 5$, such that its apex moves from o to point p (see Figure 1(b)). Denote by x_l (resp. $x_l(p)$) the vector along the bisector of W_l (resp. $W_l(p)$) directed outward from the apex at o (resp. p). Denote the reflection of $W_l(p)$ through p by $W_{l'}(p)$. Note that $l' = (l+3) \bmod 6$; see Figure 1(b).

Consider the i^{th} nearest neighbor p_i of p. Denote by $L(P \cap W_l(p_i))$ the list of the points in $P \cap W_l(p_i)$, sorted by increasing order of their x_l-coordinates (projections). The following lemma provides a key insight.

Lemma 1 *Let p_i be the i^{th} nearest neighbor of p among a set P of points in $\mathbb{R}^2$, and let $W_l(p_i)$ be the wedge of p_i that contains p. Then point p is among the first i points in $L(P \cap W_l(p_i))$.*

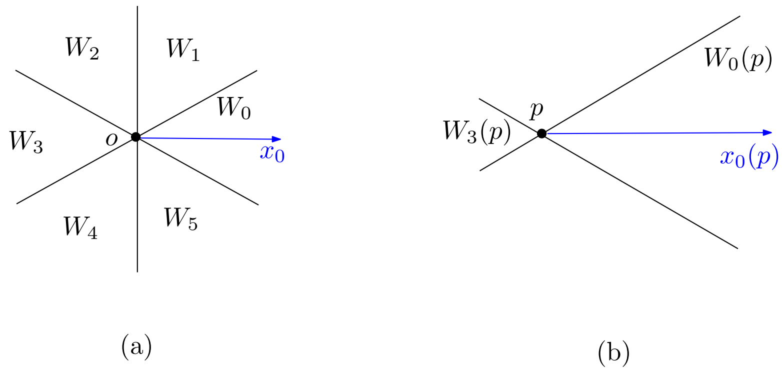


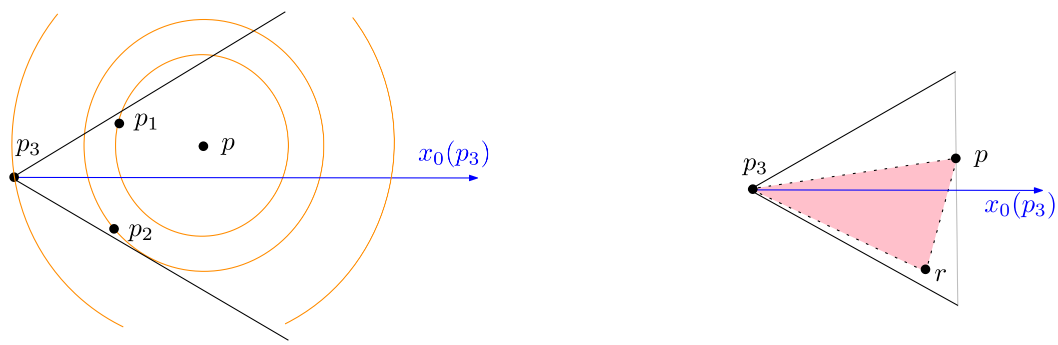
Figure 1: (a) A Partition of the plane into six wedges with common apex at o. (b) A translation of W_0 that moves apex to p. The wedge $W_0(p)$ is the reflection through p of $W_3(p)$ and vise-versa.

Proof. Let $P' = P \backslash \{p_1, ..., p_{i-1}\}$. Then the point p_i is the closest point to p among the points in P'; see Figure 2(a) below. We now prove by contradiction that the point p has the minimum x_l-coordinate among the points in $P' \cap W_l(p_i)$: Assume there is a point $r \in P$ inside the wedge $W_l(p_i)$ whose x_l-coordinate is less than the x_l-coordinate of p; see Figure 2(b) for an example where $i = 3$. Consider the triangle pp_ir. Since p_i is the closest point to p among the points in P', $|pp_i| < |pr|$ which implies that the angle $\angle pp_ir > \angle prp_i$. This is a contradiction, because $\angle pp_ir \leq \pi/3$ and $\angle prp_i > \pi/3$.

Now we add the points $p_1, ..., p_{i-2}$, and p_{i-1} to the point set P'. Consider the worst case scenario that all these $i-1$ points insert inside the wedge $W_l(p_i)$, and that the x_l-coordinates of all these points are less than the x_l-coordinate of p. Then the point p is still among the first i points in the sorted list $L(P \cap W_l(p_i))$. □

The *k-nearest neighbor graph* (k-NNG) of a point set P is constructed by connecting each point in P to all its k-nearest neighbors. If we connect each point $p \in P$ to the first k points in the sorted list $L(P \cap W_l(p))$, for $l = 0, ..., 5$, we obtain what we call the *k-Semi-Yao graph* (k-SYG). Lemma 1 gives a necessary condition for p_i to be the i^{th} nearest neighbor of p: the point p is among the first i points in $L(P \cap W_l(p_i))$, where l is such that $p \in W_l(p_i)$. Therefore, the edge set of the k-SYG covers the edges of the k-NNG. In summary, we have the following.

Lemma 2 *The k-NNG of a set P of points in* $\mathbb{R}^2$ *is a subgraph of the k-SYG of the set P.*



(a) (b)

Figure 2: Point p_3 is the 3rd nearest neighbor of p. After deleting the points p_1 and p_2, point p_3 is the closest point to p; among the points in $W_0(p_3)$, p has the minimum length projection on the bisector $x_0(p_3)$.

3 Reporting All k-Nearest Neighbors

Here we give a simple method for reporting all the k-nearest neighbors via a construction of the k-SYG.

Let C be a *right circular cone* in $\mathbb{R}^d$ with opening angle θ with respect to some given unit vector v. Thus C is the set of points $x \in \mathbb{R}^d$ such that the angle between $\overrightarrow{ox}$ and $\overrightarrow{v}$ is at most $\theta/2$. The angle between any two rays inside C emanating from the apex o is at most θ. From now on, we assume $\theta = \pi/3$.

Now consider a *polyhedral cone* inscribed in the right circular cone C where the polyhedral cone is formed by the intersection of d distinct half-spaces, bounded by $f_1, ..., f_d$, passing through the apex of C. Assuming d is arbitrary but fixed, the d-dimensional space around the origin o can be tiled by a constant number of polyhedral cones $W_0, ..., W_{c-1}$ [1, 2]. Denote by C_l the associated right circular cone of the polyhedral cone W_l. Let x_l be the vector in the direction of the symmetry of C_l. Denote by $W_l(p)$ the translation of the wedge (polyhedral cone) W_l where o moves to p.

A similar approach and analysis as that in Section 2 can be easily used to state (key) Lemmas 1 and 2 for a set of points in $\mathbb{R}^d$.

To construct the k-SYG efficiently, we need a data structure to perform the following operation efficiently: For each $p \in P$ and any of its wedges $W_l(p)$, $0 \leq l \leq c-1$, find the first k points in $L(P \cap W_l(p))$. Such an operation can be performed by using *range tree* data structures. For each wedge W_l with apex at origin o, we construct an associated d-dimensional range tree $\mathcal{T}_l$ as follows.

Consider a particular wedge W_l with apex at o. The wedge W_l is the intersection of d half-spaces $f_1^+, ..., f_d^+$ bounded by $f_1, ..., f_d$ (see Figure 3). Let $\hat{u}_j$ denote the

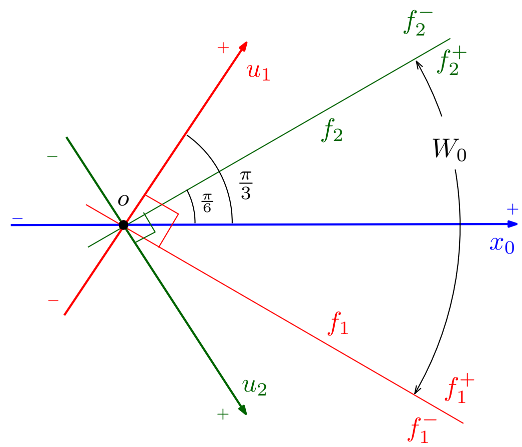


Figure 3: The wedge W_0 in $\mathbb{R}^2$ is bounded by f_1 and f_2. The coordinate axes u_1 and u_2 are orthogonal to f_1 and f_2.

normal to f_j pointing to f_j^+. We define d coordinate axes u_j, $j = 1, ..., d$, through $\hat{u_j}$, where $\hat{u_j}$ gives the respective directions of increasing u_j-coordinate values.

The range tree $\mathcal{T}_l$ is a regular d-dimensional range tree based on the u_j-coordinates, $j = 1, ..., d$. The points at level j are sorted at the leaves according to their u_j-coordinates (for more details about range trees, see Chapter 5 of [5]). From Theorem 5.8 in [5], any d-dimensional range tree, *e.g.*, $\mathcal{T}_l$, uses $O(n \log^{d-1} n)$ space and can be constructed in time $O(n \log^{d-1} n)$; for any point $r \in \mathbb{R}^d$, the points of P inside the query wedge $W_l(r)$ whose sides are parallel to f_j, $j = 1, ..., d$, can be reported in time $O(\log^d n + z)$, where z is the cardinality of the set $P \cap W_l(r)$. In particular, in time $O(\log^d n)$ one can determine a set of $O(\log^d n)$ internal nodes v at level d of $\mathcal{T}_l$, such that $P \cap W_l(r) = \bigcup_v P(v)$, where $P(v)$ is the set of points at the leaves of subtree rooted at v.

Now we add a new level to $\mathcal{T}_l$, based on the coordinate x_l. Let $\mathcal{C}_l(p)$ be the set of the first k points in $L(P \cap W_l(p))$. To find $\mathcal{C}_l(p)$ in an efficient time, we use the level $d+1$ of $\mathcal{T}_l$, which is constructed as follows: For each internal node v at level d of $\mathcal{T}_l$, we create a list $L(P(v))$ sorted by increasing order of x_l-coordinates of the points in $P(v)$. For the set P of n points in $\mathbb{R}^d$, the range tree $\mathcal{T}_l$, which now is a $(d+1)$-dimensional range tree, uses $O(n \log^d n)$ space and can be constructed in time $O(n \log^d n)$.

The following lemma establishes the processing time for obtaining a $\mathcal{C}_l(p)$.

Lemma 3 *Given $\mathcal{T}_l$, the set $\mathcal{C}_l(p)$ can be found in time $O(\log^d n + k \log\log n)$.*

Proof. The proof is by construction. Recall that the set $P \cap W_l(p)$ is the union of $O(\log^d n)$ sets $P(v)$, where v ranges over internal nodes at level d of $\mathcal{T}_l$. Consider the associated sorted lists $L(P(v))$.

We construct a priority queue on the first elements of these $O(\log^d n)$ sorted lists $L(P(v))$ in time $O(\log^d n)$.

By repeating the following two steps k times we can find $\mathcal{C}_l(p)$:

- Delete the element $\hat{p}$ with highest priority from the priority queue, and
- insert the next element into the priority queue from the sorted list $L(P(v_j))$, where v_j is such that $\hat{p} \in P(v_j)$.

Since d is fixed and the size of the priority queue is $O(\log^d n)$, all together these k iterations take $O(k \log\log n)$ time. □

By Lemma 3, we can find all the $\mathcal{C}_l(p)$, for all the points $p \in P$. This gives the following lemma.

Lemma 4 *Using a data structure of size $O(n \log^d n)$, the edges of the k-SYG of a set of n points in fixed dimension d can be reported in time $O(n \log^d n + kn \log\log n)$.*

Next, suppose we are given the k-SYG and we want to report all the k-nearest neighbors. Let E_p be the set of edges incident to the point p in the k-SYG. By sorting these edges in non-decreasing order according to their Euclidean lengths, which can be done in time $O(|E_p| \log |E_p|)$, we can find the k-nearest neighbors of p ordered by increasing distance from p. Since the number of edges in the k-SYG is $O(kn)$ and each edge pp' belongs to exactly two sets E_p and $E_{p'}$, the time to find all the k-nearest neighbors, for all the points $p \in P$, is $\sum_p O(|E_p| \log |E_p|) = O(kn \log n)$.

From the above discussion and Lemmas 2 and 4, the following results.

Theorem 1 *For a set of n points in fixed dimension d, our data structure can report all the k-nearest neighbors, in order of increasing distance from each point, in time $O(n \log^d n + kn \log n)$. The data structure uses $O(n \log^d n + kn)$ space.*

4 RkNN Queries on Moving Points

We are given a set P of n continuously moving points, where the trajectory of each point in P is a polynomial function of bounded degree s. To answer RkNN queries on the moving points, we must keep a valid range tree and track all the k-nearest neighbors during the motion. This section first shows how to maintain a (ranked-based) range tree, and then provides a KDS for maintenance of the k-SYG, which in fact gives a supergraph of the k-NNG over time. Using the kinetic k-SYG, we can easily maintain all the k-nearest neighbors over time. Finally we show how to answer RkNN queries on the moving points.

Kinetic RBRT. Let u_j, $1 \leq j \leq d$, be the coordinate axis orthogonal to the half-space f_j of the wedge W_l, $0 \leq l \leq c-1$ (see Figure 3). Abam and de Berg [1] introduced a variant of the range tree, namely the *ranked-based range tree* (RBRT), which has the following properties. Denote by $\mathcal{T}_l$ the RBRT corresponding to the wedge W_l.

- $\mathcal{T}_l$ can be described as a set of pairs $\Psi_l = \{(B_1, R_1), ..., (B_m, R_m)\}$ such that:
  - For any two points p and q in P where $q \in W_l(p)$, there is a unique pair $(B_i, R_i) \in \Psi_l$ such that $p \in B_i$ and $q \in R_i$.
  - For any pair $(B_i, R_i) \in \Psi_l$, if $p \in B_i$ and $q \in R_i$, then $q \in W_l(p)$ and $p \in W_{l'}(q)$; here $W_{l'}(q)$ is the reflection of $W_l(q)$ through q.

  The Ψ_l is called a *cone separated pair decomposition* (CSPD) for P with respect to W_l. Each pair (B_i, R_i) is generated from an internal node v at level d of the RBRT $\mathcal{T}_l$.

- Each point $p \in P$ is in $O(\log^d n)$ pairs of (B_i, R_i), which means that the number of elements of all the pairs (R_i, B_i) is $O(n \log^d n)$.

- For any point $p \in P$, all the sets B_i (resp. R_i) where $p \in B_i$ (resp. $p \in R_i$) can be found in time $O(\log^d n)$.

- The set $P \cap W_l(p)$ is the union of $O(\log^d n)$ sets R_i, where $p \in B_i$.

- When the points are moving, $\mathcal{T}_l$ remains unchanged as long as the order of the points along axes u_j, $1 \leq j \leq d$, remains unchanged.

- When a u-swap event occurs, meaning that two points exchange their u_j-order, the RBRT $\mathcal{T}_l$ can be updated in worst-case time $O(\log^d n)$ without rebalancing operations.

4.1 Kinetic k-SYG

Here we give a KDS for the k-SYG, for any $k \geq 1$, extending [18].

To maintain the k-SYG, we must track the set $\mathcal{C}_l(p)$ for each point $p \in P$. So, for each $1 \leq i \leq m$, we need to maintain a sorted list $L(R_i)$ of the points in R_i in ascending order according to their x_l-coordinates over time. Note that each set R_i is some $P(v)$, the set of points at the leaves of the subtree rooted at some internal node v at level d of $\mathcal{T}_l$. To maintain these sorted lists $L(R_i)$, we add a new level to the RBRT $\mathcal{T}_l$; the points at the new level are sorted at the leaves in ascending order according to their x_l-coordinates. Therefore, in the modified RBRT $\mathcal{T}_l$, in addition to the u-swap events, we handle new events, called *x-swap events*, when two points exchange their x_l-order. The modified RBRT $\mathcal{T}_l$ behaves like a $(d+1)$-dimensional RBRT. From the last property of an RBRT above, when a u-swap event or an x-swap event occurs, the RBRT $\mathcal{T}_l$ can be updated in worst-case time $O(\log^{d+1} n)$.

Denote by $\ddot{p}_{l,k}$ the k^{th} point in $L(P \cap W_l(p))$. To track the sets $\mathcal{C}_l(p)$, for all the points $p \in P$, we need to maintain the following over time.

- A set of $d+1$ *kinetic sorted lists* $L_j(P)$, $j = 1, ..., d$, and the $L_l(P)$ of the point set P. We use these kinetic sorted lists to track the order of the points in the coordinates u_j and x_l, respectively.

- For each B_i, a sorted list $L(B'_i)$ of the points in B'_i, where $B'_i = \{(p, \ddot{p}_{l,k})|\ p \in B_i\}$. The order of the points in $L(B'_i)$ is according to a *label* of the second points $\ddot{p}_{l,k}$. This sorted list $L(B'_i)$ is used to answer the following query efficiently: Given a query point q and a B_i, find all the points $p \in B_i$ such that $\ddot{p}_{l,k} = q$.

- The k^{th} point $r_{i,k}$ in the sorted list $L(R_i)$. We track the values $r_{i,k}$ in order to make necessary changes to the k-SYG when an x-swap event occurs.

Handling u-swap events. W.l.o.g., let $q \in W_l(p)$ before the event. When a u-swap event between p and q occurs, the point q moves outside the wedge $W_l(p)$; after the event, $q \notin W_l(p)$. Note that the changes that occur in the k-SYG are the deletions and insertions of the edges incident to p inside the wedge $W_l(p)$.

Whenever two points p and q exchange their u_j-order, we do the following updates.

- We update the kinetic sorted list $L_j(P)$. Each swap event in a kinetic sorted list can be handled in time $O(\log n)$.

- We update the RBRT $\mathcal{T}_l$ and if a point is deleted or inserted into a B_i, we update the sorted list $L(B'_i)$. Since each insertion/deletion to $L(B'_i)$ takes $O(\log n)$ time, and since each point is in $O(\log^d n)$ sets B_i, this takes $O(\log^{d+1} n)$ time.

- We update the values of $r_{i,k}$. After updating the RBRT $\mathcal{T}_l$, point q might be inserted or deleted from some R_i and change the values of $r_{i,k}$. So, for all R_i where $q \in R_i$, before and after the event, we do the following. We check whether the x_l-coordinate of q is less than or equal to the x_l-coordinate of $r_{i,k}$; if so, we take the successor or predecessor point of $r_{i,k}$ in $L(R_i)$ as the new value for $r_{i,k}$. This takes $O(\log^{d+1} n)$ time.

- We query to find $\mathcal{C}(p)$. By Lemma 3, this takes $O(\log^d n + k \log\log n)$ time.

- If we get a new value for $\ddot{p}_{l,k}$, we update all the sorted lists $L(B'_i)$ such that $p \in B_i$. This takes $O(\log^{d+1} n)$ time.

Considering the complexity of each step above, and assuming the trajectory of each point is a bounded degree polynomial, the following results.

Lemma 5 *Our KDS for maintenance of the k-SYG handles $O(n^2)$ u-swap events, each in worst-case time $O(\log^{d+1} n + k \log\log n)$.*

Handling x-swap events. When an x-swap event between two consecutive points p and q with p preceding q occurs, it does not change the elements of the pairs (B_i, R_i) of the CSPD Ψ_l. Such an event changes the k-SYG if both p and q are in the same $W_l(w)$, for some $w \in P$, and $w_{l,k} = p$.

We apply the following updates to our KDS when two points p and q exchange their x_l-order.

1. We update the kinetic sorted list $L_l(P)$; this takes $O(\log n)$ time.
2. We update the RBRT $\mathcal{T}_l$, which takes $O(\log^{d+1} n)$ time.
3. We find all the sets R_i where both p and q belong to R_i and such that $r_{i,k} = p$. Also, we find all the sets R_i where $r_{i,k} = q$. This takes $O(\log^d n)$ time.
4. For each R_i, we extract all the pairs $(w, \ddot{w}_{l,k})$ from the sorted lists $L(B'_i)$ such that $\ddot{w}_{l,k} = p$. Note that each change to the pair $(w, \ddot{w}_{l,k})$ is a change to the k-SYG.
5. For each w, we update all the sorted lists $L(B'_i)$ where $(w, \ddot{w}_{l,k}) \in B'_i$: we replace the previous value of $\ddot{w}_{l,k}$, which is p, by the new value q.

Denote by χ_k the number of exact changes to the k-SYG of a set of moving points over time. For each found R_i, the fourth step takes $O(\log n + \xi_i)$ time, where ξ_i is the number of pairs $(w, \ddot{w}_{l,k})$ such that $\ddot{w}_{l,k} = p$. For all these $O(\log^d n)$ sets R_i, this step takes $O(\log^{d+1} n + \sum_i \xi_i)$ time, where $\sum_i \xi_i$ is the number of exact changes to the k-SYG when an x-swap event occurs. Therefore, for all the $O(n^2)$ x-swap events, the total processing time for this step is $O(n^2 \log^{d+1} n + \chi_k)$.

The processing time for the fifth step is a function of χ_k. For each change to the k-SYG, this step spends $O(\log^{d+1} n)$ time to update the sorted lists $L(B'_i)$. Therefore, the total processing time for all the x-swap events in this step is $O(\chi_k * \log^{d+1} n)$.

From the above discussion and an upper bound for χ_k in Lemma 6, Lemma 7 results.

Lemma 6 *The number of changes to the k-SYG of a set of n moving points, where the trajectory of each point is a polynomial function of at most constant degree s, is $\chi_k = O(\phi(s, n) * n)$.*

Proof. Fix a point $p \in P$ and one of its wedges $W_l(p)$. There are $O(n)$ insertions/deletions into the wedge $W_l(p)$ over time. The x_l-coordinates of these points create $O(n)$ partial functions.

The k-SYG changes if a change to $\ddot{p}_{l,k}$ occurs. The number of all changes to $\ddot{p}_{l,k}$ is equal to $\phi(s,n)$, the complexity of the k-level of partially-defined polynomial functions of bounded degree s.

Therefore, considering all the $n = |P|$ points, the number of changes to the k-SYG is within a linear factor of $\phi(s,n)$: $\chi_k = O(\phi(s,n) * n)$. $\square$

Lemma 7 *Our KDS for maintenance of the k-SYG handles $O(n^2)$ x-swap events with a total cost of $O(\phi(s,n) * n \log^{d+1} n)$.*

From Lemmas 5 and 7, the following theorem results.

Theorem 2 *For a set of n moving points in $\mathbb{R}^d$, where the trajectory of each point is a polynomial function of at most constant degree s, our k-SYG KDS uses $O(n \log^d n)$ space and handles $O(n^2)$ events with a total cost of $O(kn^2 \log\log n + \phi(s,n) * n \log^{d+1} n)$.*

4.2 Kinetic All k-Nearest Neighbors

Given a KDS for maintenance of the k-SYG (from Theorem 2), a supergraph of the k-NNG, this section shows how to maintain all the k-nearest neighbors over time. For maintenance of the k-nearest neighbors of each point $p \in P$, we only need to track the order of the edges incident to p in the k-SYG according to their Euclidean lengths. This can easily be done by using a kinetic sorted list. The following theorem summarizes the complexity of our kinetic approach.

Theorem 3 *For a set of n moving points in $\mathbb{R}^d$, where the trajectory of each point is a polynomial of at most constant degree s, our KDS for maintenance of all the k-nearest neighbors, ordered by distance from each point, uses $O(n \log^d n + kn)$ space and $O(n \log^d n + kn \log n)$ preprocessing time. Our KDS handles $O(\phi(s,n) * n^2)$ events, each in $O(\log n)$ amortized time.*

Proof. Let $E_p(t)$ be the set of edges incident to point $p \in P$ in the k-SYG at time t. Let $L(E_p(t))$ denote a kinetic sorted list that maintains the edges in $E_p(t)$ sorted by their Euclidean lengths.

Let m_p be the number of insertions/deletions to the set $E_p(t)$ over time. Since the cardinality of $E_p(t)$ is $O(n)$, each insertion into a kinetic sorted list $L(E_p(t))$ can cause $O(n)$ swaps. Each change, *e.g.*, inserting/deleting an edge pq, to the k-SYG creates two insertions/deletions in the kinetic sorted lists $L(E_p(t))$ and $L(E_q(t))$; this implies that $\sum_p m_p = O(\chi_k)$. By Lemma 6, the kinetic sorted lists handle a total of $O(n \sum_p m_p) = O(\phi(s,n) * n^2)$ events. Each event in a kinetic sorted list is handled in time $O(\log n)$. Thus from this and Theorem 2, the total processing time for swap events is $O(kn^2 \log\log n + \phi(s,n) * n \log^{d+1} n + \phi(s,n) * n^2 \log n) = O(\phi(s,n) * n^2 \log n)$. $\square$

KDS performance criteria. The KDS framework [4] measures the performance of a KDS by four standard criteria, which we now apply to our KDS for maintenance of all the k-nearest neighbors in $\mathbb{R}^d$.

- **Efficiency:** This is the ratio of the number of events that a KDS processes to the number of exact changes to the attribute of interest over time. The exact number of changes for maintenance of all the k-nearest neighbors can be computed as follows. Fix a point $p \in P$. The distances of the $n-1$ points of $P \backslash \{p\}$ to p as functions of time create $2s$-intersecting curves, meaning that each pair intersects at most $2s$ times. The number of changes to the i^{th} nearest neighbor p_i of p equals $\Phi(2s, n-1)$, the complexity of the i-level of the $n-1$ $2s$-intersecting curves. Thus the number of changes to the k-nearest neighbors $p_1, ..., p_k$ of p is $O(\Phi(2s, n) * k)$. The total for all points $p \in P$ is $O(\Phi(2s, n) * kn)$. Since the number of events in our KDS is $O(\phi(s, n) * n^2)$, the efficiency of our KDS is $O(\frac{n}{k})$.

- **Responsiveness:** This is the cost of updating the KDS when an event occurs. In our KDS each event can be handled in amortized time $O(\log n)$. Thus the responsiveness of our KDS is $O(\log n)$ on average.

- **Locality:** The number of updates to a KDS when a point changes its trajectory gives the locality of the KDS. In our KDS, for each two consecutive elements in each of the kinetic sorted lists $L_j(P)$, $L_l(P)$, and $L(E_p(t))$, we have a boolean function of time, called a *certificate*. Each certificate has a failure time, the time when the two consecutive elements exchange their order. If a point changes its trajectory, we update a constant number of these certificates in the kinetic sorted lists $L_j(P)$ and $L_l(P)$. Since the number of edges in the k-SYG is $O(kn)$, if a point changes its trajectory, the number of updates to the certificates in the kinetic sorted lists $L(E_p(t))$ is $O(k)$ on average. Therefore, the locality of our KDS is $O(k)$ on average.

- **Compactness:** This is the number of certificates in the KDS. Since the number of certificates of the kinetic sorted lists $L_j(P)$ and $L_l(P)$ is $O(n)$, and the number of certificates of the kinetic sorted lists $L(E_p(t))$ is $O(kn)$, the compactness of our KDS is $O(kn)$.

Therefore, we can obtain the following.

Lemma 8 *In terms of the KDS performance criteria, the "efficiency", "responsiveness", "locality", and "compactness" of our KDS are $O(nk)$, $O(\log n)$ on average, $O(k)$ on average, and $O(kn)$, respectively.*

4.3 RkNN Queries

Suppose we are given a query point $q \notin P$ at some time t. To find the reverse k-nearest neighbors of q, we seek the points in $P \cap W_l(q)$ and find $\mathcal{C}_l(q)$, the set of the first k points in $L(P \cap W_l(q))$. The set $\cup_l \mathcal{C}_l(q)$ contains $O(k)$ candidate points for q such that q might be one of their k-nearest neighbors. In time $O(\log^d n)$ we can find a set of R_i where $P \cap W_l(q) = \sum_i R_i$. From Lemma 3, and since we have sorted lists $L(R_i)$ at level $d+1$ of $\mathcal{T}_l$, the $O(k)$ candidate points for the query point q can be found in worst-case time $O(\log^d n + k \log\log n)$. Now we check whether these candidate points are the reverse k-nearest neighbors of the query point q at time t or not; this can be easily done by application of Theorem 3, which in fact maintain the k^{th} nearest neighbor p_k of each $p \in P$. Therefore, checking a candidate point can be done in $O(1)$ time by comparing distance $|pq|$ to distance $|pp_k|$. This implies that checking which elements of $\mathcal{C}_l(q)$, for $l = 0, ..., c-1$, are reverse k-nearest neighbors of the query point q takes time $O(k)$.

If a query arrives at a time t that is simultaneous with the time when one of the $O(\phi(s,n) * n^2)$ events occurs, our KDS first spends time $O(\log n)$ in an amortized sense to handle the event, and then spends time $O(\log^d n + k \log\log n)$ to answer the query. Thus we have the following.

Theorem 4 *Consider a set P of n moving points in $\mathbb{R}^d$, where the trajectory of each one is a bounded-degree polynomial. The number of reverse k-nearest neighbors for a query point $q \notin P$ is $O(k)$. Our (kinetic) data structure uses $O(n \log^d n + kn)$ space and $O(n \log^d n + kn \log n)$ preprocessing time. At any time t, an RkNN query can be answered in time $O(\log^d n + k \log\log n)$. If an event occurs at time t, the KDS spends $O(\log n)$ time in an amortized sense on updating itself.*

5 Discussion and Conclusion

In the kinetic setting, where the trajectories of the points are polynomials of bounded degree, to answer the RkNN queries over time we have provided a KDS for maintenance of all the k-nearest neighbors. Our KDS is the first KDS for maintenance of all the k-nearest neighbors in $\mathbb{R}^d$, for any $k \geq 1$. It processes $O(\phi(s,n) * n^2)$ events, each in time $O(\log n)$ in an amortized sense. An open problem is to design a KDS that processes less than $O(\phi(s,n) * n^2)$ events.

Arya *et al.* [3] have a kd-tree implementation to approximate the nearest neighbors of a query point that is in use by practitioners [12] who have found challenging to implement the theoretical algorithms [6, 11, 13, 20]. Since to report all the k-nearest neighbors ordered by distance from each point our method uses multidimensional range trees, which can be easily implemented, we believe our method may be useful in practice.

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