3. *Assume that $\mathcal{U} : \mathfrak{A} \longrightarrow \mathscr{M}$ is a right Quillen functor that preserves and reflects the weak equivalences. Then the adjunction*

$$|-|^{\mathbf{c}} : \mathscr{M}_{\mathcal{U}}[\mathfrak{A}]_e^{\mathbf{c}} \leftrightarrows \mathfrak{A} : \iota,$$

is a Quillen equivalence.

4. *In particular we have a diagram of equivalences between the homotopy categories.*

$$\mathbf{ho}[\mathscr{M}_{\mathcal{U}}[\mathfrak{A}]_{e+}^{\mathbf{c}}] \xleftarrow{\simeq} \mathbf{ho}[\mathscr{M}_{\mathcal{U}}[\mathfrak{A}]_e^{\mathbf{c}}] \xrightarrow{\simeq} \mathbf{ho}[\mathfrak{A}].$$

Note. As outline before, in [1, Theorem 8.2], we stated this theorem under the hypothesis that $\mathcal{U}$ is faithful but this is not necessary and is too restrictive. This hypothesis was suggested by the case of algebras and categories. We can also show that if $\mathcal{U}$ is just part of a Quillen equivalence then we still have a Quillen equivalence between $|-|^{\mathbf{c}} : \mathscr{M}_{\mathcal{U}}[\mathfrak{A}]_e^{\mathbf{c}} \leftrightarrows \mathfrak{A} : \iota$, and by closure, the other functor $\mathscr{M} \longrightarrow \mathscr{M}_{\mathcal{U}}[\mathfrak{A}]_e^{\mathbf{c}}$ is also part of a Quillen equivalence. We should warn the reader that if $\mathcal{U}$ is not faithful then the left adjoint $|-|$ is not the target functor anymore.

We remind the reader that both model categories $\mathbf{sSet}_{\mathrm{Q}}$ and $\mathbf{sSet}_{\mathrm{J}}$ are combinatorial and left proper.

2 Applications and Interpretations

2.1 Understanding the Univalence axiom

The discussion that follows is an attempt to understand the univalence axiom, by a non-expert.

Let us consider the identity functor $\mathcal{U} = \mathrm{Id} : \mathbf{sSet}_{\mathrm{Q}} \longrightarrow \mathbf{sSet}_{\mathrm{Q}}$ with the Quillen model structure. Then the category $\mathscr{M}_{\mathcal{U}}[\mathfrak{A}]$ is just the category of morphisms in $\mathbf{sSet}$:

$$\mathrm{Arr}(\mathbf{sSet}) = \mathbf{sSet}^{\mathbb{I}} = \mathrm{Hom}([0 \longrightarrow 1], \mathbf{sSet}) = Path(\mathbf{sSet}).$$

The left adjoint $|-| : \mathbf{sSet}^{\mathbb{I}} \longrightarrow \mathbf{sSet}$ is the target-functor whose right adjoint is the embedding $\iota : \mathbf{sSet} \longrightarrow \mathbf{sSet}^{\mathbb{I}}$ that takes X to the identity (type) morphism Id_X. As far as we understand, the Quillen-Segal type in this context is precisely the equivalence type.

By the second assertion of Theorem 1.2, there exists a model structure on $\mathbf{sSet}_{\mathrm{Q}}{}^{\mathbb{I}} = Path(\mathbf{sSet}_{\mathrm{Q}})$ such that:

1. Every fibrant $\mathcal{F}$ object is a trivial fibration $\mathcal{F} : s(\mathcal{F}) \xrightarrow{\sim} t(\mathcal{F})$, with $s(\mathcal{F})$ fibrant in $\mathbf{sSet}_{\mathrm{Q}}$, which means that $s(\mathcal{F})$ is Kan.