On the Complexity of Clustered-Level Planarity and T-Level Planarity ⋆

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Abstract. In this paper we study two problems related to the drawing of level graphs, that is, T-Level Planarity and Clustered-Level Planarity. We show that both problems are $\mathcal{NP}$-complete in the general case and that they become polynomial-time solvable when restricted to proper instances.

1 Introduction and Overview

A level graph is *proper* if any of its edges spans just two consecutive levels. Several papers about constructing level drawings of level graphs assume that the input graph is proper. Otherwise, they suggest to make it proper by "simply adding dummy vertices" along the edges spanning more than two levels. In this paper we show that this apparently innocent augmentation has dramatic consequences if, instead of constructing just a level drawing, we are also interested in representing additional constraints, like clustering of vertices or consecutivity constraints on the ordering of vertices on levels.

A *level graph* $G = (V, E, \gamma)$ is a graph with a function $\gamma : V \rightarrow \{1, 2, ..., k\}$, with $1 \leq k \leq |V|$ such that $\gamma(u) \neq \gamma(v)$ for each edge $(u, v) \in E$. The set $V_i = \{v|\gamma(v) = i\}$ is the i-th *level* of G. A level graph $G = (V, E, \gamma)$ is *proper* if for every edge $(u, v) \in E$, it holds $\gamma(u) = \gamma(v) \pm 1$. A *level planar drawing* of (V, E, γ) maps each vertex v of each level V_i to a point on line $y = i$, denoted by L_i, and each edge to a y-monotone curve between its endpoints so that no two edges intersect. A level graph is *level planar* if it admits a level planar drawing. A linear-time algorithm for testing level planarity was presented by Jünger and Leipert in [10].

A *clustered-level graph* (*cl-graph*) (V, E, γ, T) is a level graph (V, E, γ) equipped with a *cluster hierarchy* T, that is, a rooted tree where each leaf is an element of V and each internal node μ, called *cluster*, represents the subset V_μ of V composed of the leaves of the subtree of T rooted at μ. A *clustered-level planar drawing* (*cl-planar* drawing) of (V, E, γ, T) is a level planar drawing of level graph (V, E, γ) such that: (1) each cluster μ is represented by a simple region enclosing all and only the vertices in V_μ; (2) no edge intersects the boundary of a cluster more than once; (3) no two cluster boundaries intersect each other; and (4) the intersection of L_i with any cluster μ is a straight-line segment, that is, the vertices of V_i that belong to μ are consecutive along L_i. A cl-graph is *clustered-level planar* (*cl-planar*) if it admits a cl-planar drawing. Clustered-Level Planarity (CL-Planarity) is the problem of testing whether a given cl-graph is cl-planar. The CL-Planarity problem was introduced by Forster and Bachmaier [9], who showed a polynomial-time testing algorithm for the case in which the level graph is a proper hierarchy and the clusters are level-connected.

A $\mathcal{T}$*-level graph* (also known as *generalized k-ary tanglegram*) $(V, E, \gamma, \mathcal{T})$ is a level graph (V, E, γ) equipped with a set $\mathcal{T} = T_1, \ldots, T_k$ of trees such that the leaves of T_i are the vertices of level V_i of (V, E, γ), for $1 \leq i \leq k$. A $\mathcal{T}$*-level planar drawing* of $(V, E, \gamma, \mathcal{T})$ is a level planar drawing of (V, E, γ) such that, for $i = 1, \ldots, k$, the order in which the vertices of V_i appear along L_i is *compatible* with T_i, that is, for each node w of T_i, the leaves of the subtree of T_i rooted at w appear consecutively along L_i. A $\mathcal{T}$-level graph is $\mathcal{T}$*-level planar* if it admits a $\mathcal{T}$-level planar drawing. T-Level Planarity is the problem of testing whether a given $\mathcal{T}$-level graph is $\mathcal{T}$-level planar. The

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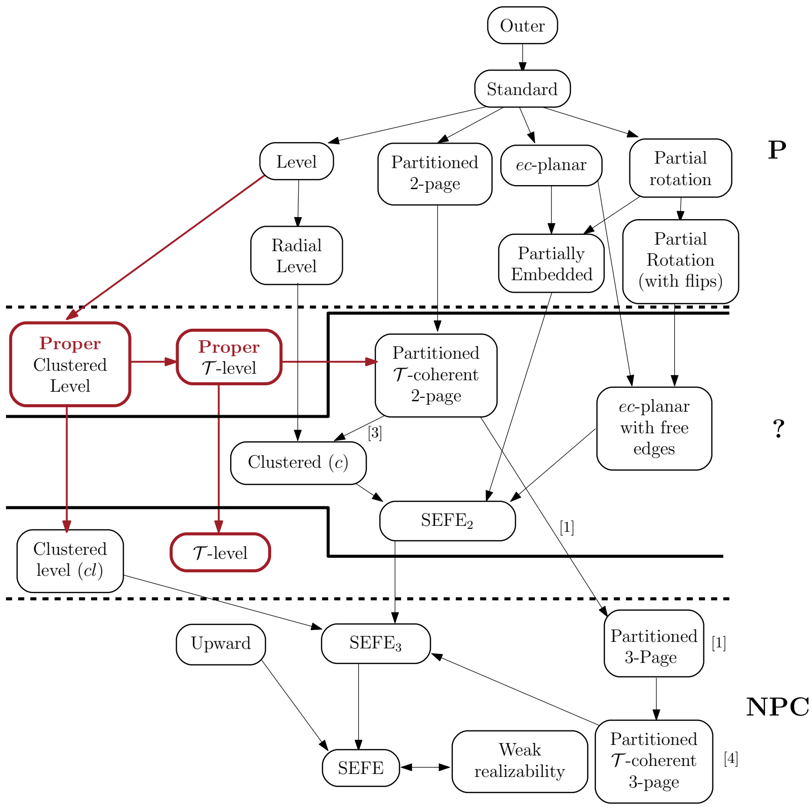
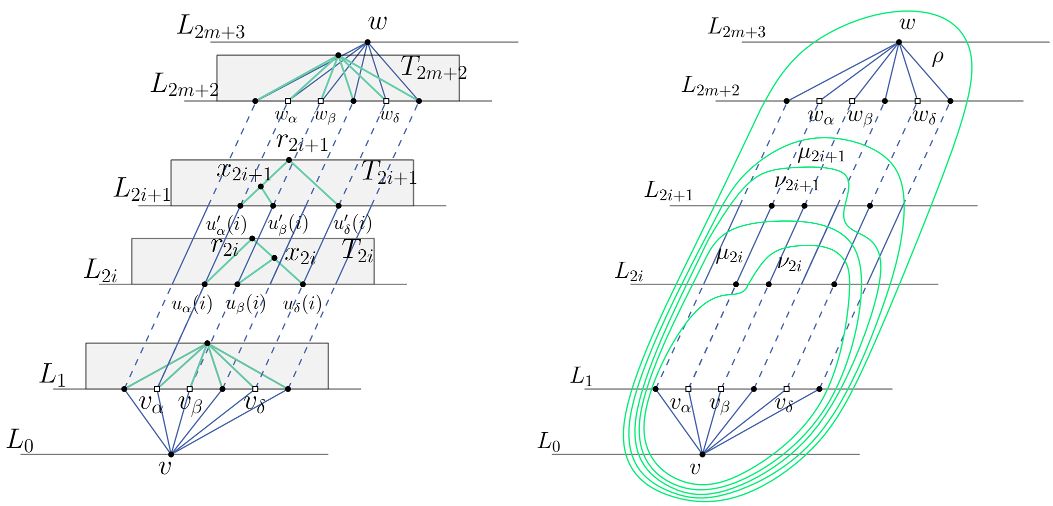


Fig. 1. Updates on the classification proposed by Schaefer in [12]. Dashed lines represent the boundaries between problems that were known to be polynomial-time solvable, problems that were known to be $\mathcal{NP}$-complete, and problems whose complexity was unknown before this paper. Solid lines represent the new boundaries according to the results of this paper. All the arcs representing reductions that can be transitively inferred are omitted. Results proved after [12] are equipped with references. Reductions and classes introduced in this paper are red. The prefix "proper" has been added to two classes in [12] to better clarify their nature.

T-Level Planarity problem was introduced by Wotzlaw *et al.* [13], who showed a quadratic-time algorithm for the case in which the number of vertices at each level is bounded by a constant.

The definition of *proper* naturally extends to cl-graphs and $\mathcal{T}$-level graphs. Note that, given any non-proper level graph G it is easy to construct a proper level graph G' that is level planar if and only if G is level planar. However, as mentioned above, there exists no trivial transformation from a non-proper cl-graph (a non-proper $\mathcal{T}$-level graph) to an equivalent proper cl-graph (resp., an equivalent proper $\mathcal{T}$-level graph).

In this paper we show that Clustered-Level Planarity and T-Level Planarity are $\mathcal{NP}$-complete for non-proper instances. Conversely, we show that both problems are polynomial-time solvable for proper instances. Our results have several consequences: (1) They narrow the gap between polynomiality and $\mathcal{NP}$-completeness in the classification of Schaefer [12] (see Fig. 1). The reduction of Schaefer between T-Level Planarity and SEFE-2 holds for proper instances [12]. (2) They allow to partially answer a question from [12] asking whether a reduction exists from CL-Planarity to SEFE-2. We show that such a reduction exists for proper instances and that a reduction from general instances would imply the $\mathcal{NP}$-hardness of SEFE-2. (3) They improve on [9] and [13] by extending the classes of instances which are decidable in polynomial-time for CL-Planarity and T-Level Planarity, respectively. (4) They provide the first, as far as we know, $\mathcal{NP}$-completeness for a problem that has all the constraints of clustered planarity problem (and some more).



(a) (b)

Fig. 2. Illustrations for the proof of (a) Theorem 1 and (b) Theorem 2.

The paper is organized as follows. The $\mathcal{NP}$-completeness proofs are in Section 2, while the algorithms are in Section 3. We conclude with open problems in Section 4.

2 NP-Hardness

In this section we prove that the T-LEVEL PLANARITY and the CL-PLANARITY problems are $\mathcal{NP}$-complete. In both cases, the $\mathcal{NP}$-hardness is proved by means of a polynomial-time reduction from the $\mathcal{NP}$-complete problem BETWEENNESS [11], that takes as input a finite set A of n objects and a set C of m ordered triples of distinct elements of A, and asks whether a linear ordering $\mathcal{O}$ of the elements of A exists such that for each triple $\langle\alpha, \beta, \delta\rangle$ of C, we have either $\mathcal{O} = <\dots, \alpha, \dots, \beta, \dots, \delta, \dots>$ or $\mathcal{O} = <\dots, \delta, \dots, \beta, \dots, \alpha, \dots>$.

Theorem 1. *T-LEVEL PLANARITY is $\mathcal{NP}$-complete.*

Proof: The problem trivially belongs to $\mathcal{NP}$. We prove the $\mathcal{NP}$-hardness. Given an instance $\langle A, C\rangle$ of BETWEENNESS, we construct an equivalent instance $(V, E, \gamma, \mathcal{T})$ of T-LEVEL PLANARITY as follows. Let $A = \{1, 2, \dots, n\}$ and let $m = |C|$. Graph (V, E) is composed of a set of paths connecting two vertices v and w. Refer to Fig. 2(a).

Initialize $V = \{v, w\}$ and $E = \emptyset$, with $\gamma(v) = 0$ and $\gamma(w) = 2m + 3$. Let $T_0 \in \mathcal{T}$ and $T_{2m+3} \in \mathcal{T}$ be trees with a single node v and w, respectively.

For each $j = 1, \dots, n$, add two vertices v_j and w_j to V, with $\gamma(v_j) = 1$ and $\gamma(w_j) = 2m + 2$. Add edges (v, v_j) and (w, w_j) to E. Also, let $T_1 \in \mathcal{T}$ and $T_{2m+2} \in \mathcal{T}$ be two stars whose leaves are all the vertices of levels V_1 and V_{2m+2}, respectively. Further, for each $j = 1, \dots, n$, we initialize variable $last(j) = v_j$.

Then, for each $i = 1, \dots, m$, consider the triple $t_i = \langle\alpha, \beta, \gamma\rangle$. Add six vertices $u_\alpha(i)$, $u'_\alpha(i)$, $u_\beta(i)$, $u'_\beta(i)$, $u_\delta(i)$, and $u'_\delta(i)$ to V with $\gamma(u_\alpha(i)) = \gamma(u_\beta(i)) = \gamma(u_\delta(i)) = 2i$ and $\gamma(u'_\alpha(i)) = \gamma(u'_\beta(i)) = \gamma(u'_\delta(i)) = 2i + 1$. Also, add edges $(last(\alpha), u_\alpha(i))$, $(last(\beta), u_\beta(i))$, $(last(\delta), u_\delta(i))$, $(u_\alpha(i), u'_\alpha(i))$, $(u_\beta(i), u'_\beta(i))$, and $(u_\gamma(i), u'_\gamma(i))$ to E. Further, set $last(\alpha) = u'_\alpha(i)$, $last(\beta) = u'_\beta(i)$, and $last(\delta) = u'_\delta(i)$. Let $T_{2i} \in \mathcal{T}$ be a binary tree with a root r_{2i}, an internal node x_{2i} and a leaf $u_\alpha(i)$ both adjacent to r_{2i}, and with leaves $u_\beta(i)$ and $u_\delta(i)$ both adjacent to x_{2i}. Moreover, let $T_{2i+1} \in \mathcal{T}$ be a binary tree with a root r_{2i+1}, an internal node x_{2i+1} and a leaf $u'_\delta(i)$ both adjacent to r_{2i+1}, and with leaves $u'_\alpha(i)$ and $u'_\beta(i)$ both adjacent to x_{2i+1}.

Finally, for each $j = 1, \dots, n$, add an edge $(last(j), w_j)$ to E.

The reduction is easily performed in $O(n + m)$ time. We prove that $(V, E, \gamma, \mathcal{T})$ is $\mathcal{T}$-level planar if and only if $\langle A, C\rangle$ is a positive instance of BETWEENNESS.

Suppose that $(V, E, \gamma, \mathcal{T})$ admits a $\mathcal{T}$-level planar drawing Γ. Consider the left-to-right order $\mathcal{O}_1$ in which the vertices of level V_1 appear along L_1. Construct an order $\mathcal{O}$ of the elements of A such that $\alpha \in A$ appears before $\beta \in A$ if and only if $v_\alpha \in V_1$ appears before $v_\beta \in V_1$ in $\mathcal{O}_1$. In order to prove that $\mathcal{O}$ is a positive solution for $\langle A, C\rangle$, it suffices to prove that, for each triple $t_i = \langle \alpha, \beta, \delta\rangle \in C$, vertices v_α, v_β, and v_δ appear either in this order or in the reverse order in $\mathcal{O}_1$. Note that tree T_{2i} enforces $u_\alpha(i)$ not to lie between $u_\beta(i)$ and $u_\delta(i)$ along L_{2i}; also, tree T_{2i+1} enforces $u'_\delta(i)$ not to lie between $u'_\alpha(i)$ and $u'_\beta(i)$ along L_{2i+1}. Since the three paths connecting v and w and passing through v_α, v_β, and v_δ do not cross each other in Γ and since they contain $u_\alpha(i)$ and $u'_\alpha(i)$, $u_\beta(i)$ and $u'_\beta(i)$, and $u_\delta(i)$ and $u'_\delta(i)$, respectively, we have that v_α, v_β, and v_δ appear either in this order or in the reverse order in $\mathcal{O}_1$.

Suppose that an ordering $\mathcal{O}$ of the elements of A exists that is a positive solution of BETWEENNESS for instance $\langle A, C\rangle$. In order to construct Γ, place the vertices of V_1 and V_{2m+2} along L_1 and L_{2m+2} in such a way that vertices $v_j \in V_1$ and $w_j \in V_{2m+2}$, for $j = 1, \dots, n$, are assigned x-coordinate equal to s if j is the s-th element of $\mathcal{O}$. Also, for $i = 1, \dots, m$, let $t_i = \langle \alpha, \beta, \delta\rangle \in C$. Place vertices $u_\lambda(i)$ and $u'_\lambda(i)$, with $\lambda \in \{\alpha, \beta, \delta\}$, on L_{2i} and L_{2i+1}, respectively, in such a way that $u_\lambda(i)$ and $u'_\lambda(i)$ are assigned x-coordinate equal to s if λ is the s-th element of $\mathcal{O}$. Finally, place v and w at any points on L_0 and L_{2m+3}, respectively, and draw the edges of E as straight-line segments. We prove that Γ is a $\mathcal{T}$-level planar drawing of $(V, E, \gamma, \mathcal{T})$. First note that, by construction, Γ is a level planar drawing of (V, E, γ). Further, for each $i = 1, \dots, m$, vertices $u_\alpha(i)$, $u_\beta(i)$, and $u_\delta(i)$ appear along L_{2i} either in this order or in the reverse order; in both cases, the order is compatible with tree T_{2i}. Analogously, vertices $u'_\alpha(i)$, $u'_\beta(i)$, and $u'_\delta(i)$ appear along L_{2i+1} either in this order or in the reverse order; in both cases, the order is compatible with tree T_{2i+1}. Finally, the order in which vertices of V_0, V_1, V_{2m+2}, and V_{2m+3} appear along L_0, L_1, L_{2m+2}, and L_{2m+3}, respectively, are trivially compatible with T_0, T_1, T_{2m+2}, and T_{2m+3}. □

Note that the reduction described in Theorem 1 can be modified in such a way that $\mathcal{T}$ contains only binary trees by removing levels V_1 and V_{2m+2}. Indeed, the presence of these two levels was only meant to simplify the description of the relationship between the order of the elements of A and the order of the paths between v and w.

Theorem 2. CLUSTERED-LEVEL PLANARITY *is $\mathcal{NP}$-complete.*

Proof: The problem trivially belongs to class $\mathcal{NP}$. We prove the $\mathcal{NP}$-hardness. Given an instance $\langle A, C\rangle$ of BETWEENNESS, we construct an instance $(V, E, \gamma, \mathcal{T})$ of T-LEVEL PLANARITY as in the proof of Theorem 1; then, starting from $(V, E, \gamma, \mathcal{T})$, we construct an instance (V, E, γ, T) of CL-PLANARITY that is cl-planar if and only if $(V, E, \gamma, \mathcal{T})$ is $\mathcal{T}$-level planar. This, together with the fact that $(V, E, \gamma, \mathcal{T})$ is $\mathcal{T}$-level planar if and only if $\langle A, C\rangle$ is a positive instance of BETWEENNESS, implies the $\mathcal{NP}$-hardness of CL-PLANARITY. Refer to Fig. 2(b).

Cluster hierarchy T is constructed as follows. Initialize T with a root ρ. Let $w \in V_{2m+3}$ and $w_j \in V_{2m+2}$, for $j = 1, \dots, n$, be leaves of T that are children of ρ; add an internal node μ_{2m+1} to T as a child of ρ. Next, for $i = m, \dots, 1$, let $u'_\delta(i)$ be a leaf of T that is child of μ_{2i+1}; add an internal node ν_{2i+1} to T as a child of μ_{2i+1}; then, let $u'_\alpha(i)$ and $u'_\beta(i)$ be leaves of T that are children of ν_{2i+1}; add an internal node μ_{2i} to T as a child of ν_{2i+1}. Further, let $u_\alpha(i)$ be a leaf of T that is a child of μ_{2i}; add an internal node ν_{2i} to T as a child of μ_{2i}; then, let $u_\beta(i)$ and $u_\delta(i)$ be leaves of T that are children of ν_{2i}; add an internal node μ_{2i-1} to T as a child of ν_{2i}. Finally, let vertices $v \in V_0$ and $v_j \in V_1$, for $j = 1, \dots, n$, be leaves of T that are children of μ_1.

We prove that (V, E, γ, T) is cl-planar if and only if $(V, E, \gamma, \mathcal{T})$ is $\mathcal{T}$-level planar.

Suppose that (V, E, γ, T) admits a cl-planar drawing Γ. Construct a $\mathcal{T}$-level planar drawing Γ^* of $(V, E, \gamma, \mathcal{T})$ by removing from Γ the clusters of T. First, observe that the drawing of (V, E, γ) in Γ^* is level-planar, since it is level-planar in Γ. Further, for each $i = 1, \dots, m$, vertex $u_\alpha(i)$ does not appear between $u_\beta(i)$ and $u_\gamma(i)$ along line L_{2i}, since $u_\beta(i), u_\gamma(i) \in \nu_{2i}$ and $u_\alpha(i) \notin \nu_{2i}$; analogously, vertex $u'_\delta(i)$ does not appear between $u'_\alpha(i)$ and $u'_\beta(i)$ along line L_{2i+1}, since $u'_\alpha(i), u'_\beta(i) \in \nu_{2i+1}$ and $u'_\delta(i) \notin \nu_{2i+1}$. Hence, the order of the vertices of V_{2i} and V_{2i+1} along L_{2i} and L_{2i+1}, respectively, are compatible with trees T_{2i} and T_{2i+1}. Finally, the order in which vertices of V_0, V_1, V_{2m+2}, and V_{2m+3} appear along lines L_0, L_1, L_{2m+2}, and L_{2m+3}, respectively, are trivially compatible with T_0, T_1, T_{2m+2}, and T_{2m+3}.

Suppose that $(V, E, \gamma, \mathcal{T})$ admits a $\mathcal{T}$-level planar drawing Γ^*; we describe how to construct a cl-planar drawing Γ of (V, E, γ, T). Assume that Γ^* is a straight-line drawing, which is not a loss of generality [8]. Initialize $\Gamma = \Gamma^*$. Draw each cluster α in T as a convex region $R(\alpha)$ in Γ slightly surrounding the border of the convex hull of its vertices and slightly surrounding the border of the regions representing the clusters that are its descendants in T. Let j be the largest index such that V_j contains a vertex of α. Then, $R(\alpha)$ contains all and only the vertices that are descendants of

α in T; moreover, any two clusters α and β in T are one contained into the other, hence $R(\alpha)$ and $R(\beta)$ do not cross; finally, we prove that no edge e in E crosses more than once the boundary of $R(\alpha)$ in Γ. First, if at least one end-vertex of e belongs to α, then e and the boundary of $R(\alpha)$ cross at most once, given that e is a straight-line segment and that $R(\alpha)$ is convex. All the vertices in $V_0 \cup \ldots \cup V_{j-1}$ and at least two vertices of V_j belong to α, hence their incident edges do not cross the boundary of $R(\alpha)$ more than once. Further, all the vertices in $V_{j+1} \cup \ldots \cup V_{2m+3}$ have y-coordinates larger than every point of $R(\alpha)$, hence edges between them do not cross $R(\alpha)$. It remains to consider the case in which e connects a vertex x_1 in V_j not in α (there is at most one such vertex) with a vertex x_2 in $V_{j+1} \cup \ldots \cup V_{2m+2}$; in this case e and $R(\alpha)$ do not cross given that x_1 is outside $R(\alpha)$, that x_2 has y-coordinate larger than every point of $R(\alpha)$, and that $R(\alpha)$ is arbitrarily close to the convex hull of its vertices. □

3 Polynomial-Time Algorithms

In this section we prove that problems T-LEVEL PLANARITY and CL-PLANARITY become polyomial-time solvable if restricted to proper instances.

3.1 T-LEVEL PLANARITY

We start by describing a polynomial-time algorithm for T-LEVEL PLANARITY. The algorithm is based on a reduction to the *Simultanoues Embedding with Fixed Edges* problem for two graphs (SEFE-2), that is defined as follows.

A *simultanoues embedding with fixed edges* (SEFE) of two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on the same set of vertices V consists of two planar drawings Γ_1 and Γ_2 of G_1 and G_2, respectively, such that each vertex $v \in V$ is mapped to the same point in both drawings and each edge of the *common graph* $G_\cap = (V, E_1 \cap E_2)$ is represented by the same simple curve in the two drawings. The SEFE-2 problem asks whether a given pair of graphs $\langle G_1, G_2 \rangle$ admits a SEFE [5]. The computational complexity of the SEFE-2 problem is unknown, but there exist polynomial-time algorithms for instances that respect some conditions [2,5,6,7,12]. We are going to use a result by Bläsius and Rütter [7], who proposed a quadratic-time algorithm for instances $\langle G_1, G_2 \rangle$ of SEFE-2 in which G_1 and G_2 are 2-connected, and the common graph $G_\cap$ is connected.

In the analysis of the complexity of the following algorithms we assume that the internal nodes of the trees in $\mathcal{T}$ in any instance $(V, E, \gamma, \mathcal{T})$ of T-LEVEL PLANARITY and of tree T in any instance (V, E, γ, T) of CL-PLANARITY have at least two children. It is easily proved that this is not a loss of generality; also, this allows us to describe the size of the instances in terms of the size of their sets of vertices.

Lemma 1. *Let $(V, E, \gamma, \mathcal{T})$ be a proper instance of T-LEVEL PLANARITY. There exists an equivalent instance $\langle G_1^*, G_2^* \rangle$ of SEFE-2 such that $G_1^* = (V^*, E_1^*)$ and $G_2^* = (V^*, E_2^*)$ are 2-connected, and the common graph $G_\cap = (V^*, E_1^* \cap E_2^*)$ is connected. Further, instance $\langle G_1^*, G_2^* \rangle$ can be constructed in linear time.*

Proof: We describe how to construct instance $\langle G_1^*, G_2^* \rangle$. Refer to Fig. 3.

Graph $G_\cap$ contains a cycle $\mathcal{C} = t_1, t_2, \ldots, t_k, q_k, p_k, q_{k-1}, p_{k-1}, \ldots, q_1, p_1$, where k is the number of levels of $(V, E, \gamma, \mathcal{T})$. For each $i = 1, \ldots, k$, graph $G_\cap$ contains a copy $\overline{T_i}$ of tree $T_i \in \mathcal{T}$, whose root is identified with vertex t_i, and contains two stars P_i and Q_i centered at vertices p_i and q_i, respectively, whose number of leaves is as follows. For each vertex $u \in V_i$ such that an edge $(u, v) \in E$ exists connecting u to a vertex $v \in V_{i-1}$, star P_i contains a leaf vertex $u(P_i)$; also, for each vertex $u \in V_i$ such that an edge $(u, v) \in E$ exists connecting u to a vertex $v \in V_{i+1}$, star Q_i contains a leaf vertex $u(Q_i)$. We also denote by $u(\overline{T_i})$ a leaf of $\overline{T_i}$ corresponding to vertex $u \in V_i$.

Graph G_1^* contains $G_\cap$ plus a set of edges defined as follows. For $i = 1, \ldots, k$, consider each vertex $u \in V_i$. Suppose that i is even. Then, G_1^* has an edge connecting the leaf $u(\overline{T_i})$ of $\overline{T_i}$ corresponding to u with either the leaf $u(Q_i)$ of Q_i corresponding to u, if it exists, or with the center q_i of Q_i, otherwise; also, for each edge in E connecting a vertex $u \in V_i$ with a vertex $v \in V_{i-1}$, graph G_1^* has an edge connecting the leaf $u(P_i)$ of P_i corresponding to u with the leaf $v(Q_{i-1})$ of Q_{i-1} corresponding to v (such leaves exist by construction). Suppose that i is odd. Then, graph G_1^* has an edge between $u(\overline{T_i})$ and either $u(P_i)$, if it exists, or the center p_i of P_i, otherwise.

Graph G_2^* contains $G_\cap$ plus a set of edges defined as follows. For $i = 1, \ldots, k$, consider each vertex $u \in V_i$. Suppose that i is odd. Then, G_2^* has an edge connecting $u(\overline{T_i})$ with either the leaf $u(Q_i)$ of Q_i corresponding to u,

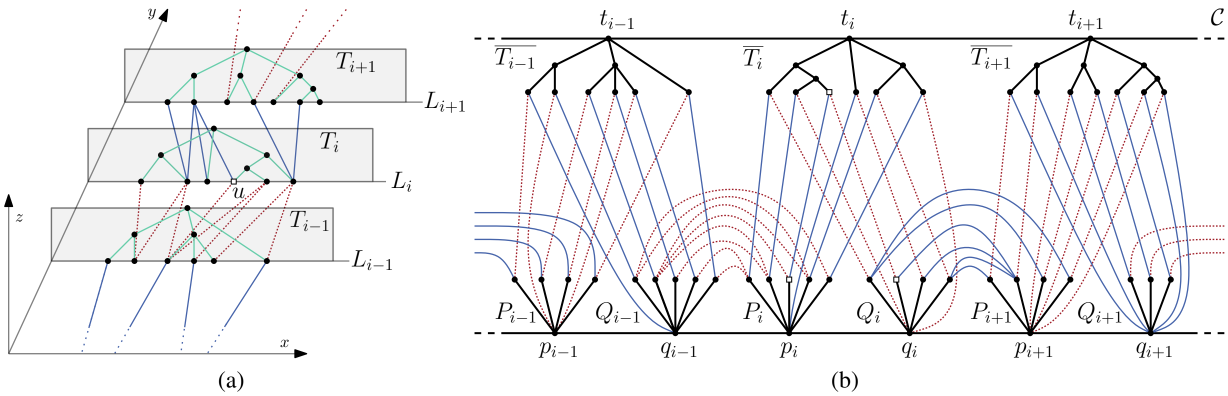


Fig. 3. Illustration for the proof of Lemma 1. Index i is assumed to be even. (a) A T-level planar drawing Γ of instance $(V, E, \gamma, \mathcal{T})$. (b) The SEFE $\langle \Gamma_1, \Gamma_2 \rangle$ of instance $\langle G_1^*, G_2^* \rangle$ of SEFE-2 corresponding to Γ. Correspondence between a vertex $u \in V_i$ and leaves $u(\overline{T_i}) \in \overline{T_i}$, $u(P_i) \in P_i$, and $u(Q_i) \in Q_i$ is highlighted by representing all such vertices as white boxes.

if it exists, or with the center q_i of Q_i, otherwise; also, for each edge in E connecting a vertex $u \in V_i$ with a vertex $v \in V_{i-1}$, graph G_2^* has an edge $(u(P_i), v(Q_{i-1}))$. Suppose that i is even. Then, graph G_2^* has an edge between $u(\overline{T_i})$ and either $u(P_i)$, if it exists, or p_i, otherwise.

It is easy to see that $G_\cap$ is connected and that $\langle G_1^*, G_2^* \rangle$ can be constructed in polynomial time. We prove that G_1^* and G_2^* are 2-connected, that is, removing any vertex v disconnects neither G_1^* nor G_2^*. If v is a leaf of either $\overline{T_i}$ or P_i or Q_i, with $1 \le i \le k$, then removing v disconnects neither G_1^* nor G_2^*, since $G_\cap$ remains connected. If v is an internal node (the root) of $\overline{T_i}$, or P_i, or Q_i, say of $\overline{T_i}$, with $1 \le i \le k$, then removing v disconnects $G_\cap$ into $m = \deg(v)$ (resp. $m = \deg(v) - 1$) components, namely one component $\overline{T_i}(v)$ containing all the vertices of $\mathcal{C}$ (resp. all the vertices of $\mathcal{C}$, except for v) and $m-1$ subtrees $\overline{T_i}^j$ of $\overline{T_i}$, with $j = 1, \dots, m-1$, rooted the children of v; however, by construction, each $\overline{T_i}^j$ is connected to $\overline{T_i}(v)$ via at least an edge $(u(\overline{T_i}), u(P_i)) \in E_1^*$ and an edge $(u(\overline{T_i}), u(Q_i)) \in E_2^*$, or vice versa, incident to one of its leaves $u(\overline{T_i})$.

Observe that, if $(V, E, \gamma, \mathcal{T})$ has $n_\mathcal{T}$ nodes in the trees of $\mathcal{T}$ (where $|V| < n_\mathcal{T}$), then $\langle G_1^*, G_2^* \rangle$ contains at most $3n_\mathcal{T}$ vertices. Also, the number of edges of $\langle G_1^*, G_2^* \rangle$ is at most $|E| + 2n_\mathcal{T}$. Hence, the size of $\langle G_1^*, G_2^* \rangle$ is linear in the size of $(V, E, \gamma, \mathcal{T})$ and it is easy to see that $\langle G_1^*, G_2^* \rangle$ can be constructed in linear time.

We prove that $\langle G_1^*, G_2^* \rangle$ admits a SEFE if and only if $(V, E, \gamma, \mathcal{T})$ is $\mathcal{T}$-level planar.

Suppose that $\langle G_1^*, G_2^* \rangle$ admits a SEFE $\langle \Gamma_1^*, \Gamma_2^* \rangle$. We show how to construct a drawing Γ of $(V, E, \gamma, \mathcal{T})$. For $1 \le i \le k$, let $\Theta(\overline{T_i})$ be the order in which the leaves of $\overline{T_i}$ appear in a pre-order traversal of $\overline{T_i}$ in $\langle \Gamma_1^*, \Gamma_2^* \rangle$; then, let the ordering $\mathcal{O}_i$ of the vertices of V_i along L_i be either $\Theta(\overline{T_i})$, if i is odd, or the reverse of $\Theta(\overline{T_i})$, if i is even.

We prove that Γ is $\mathcal{T}$-level planar. For each $i = 1, \dots, k$, $\mathcal{O}_i$ is compatible with $T_i \in \mathcal{T}$, since the drawing of $\overline{T_i}$, that belongs to $G_\cap$, is planar in $\langle \Gamma_1^*, \Gamma_2^* \rangle$. Suppose, for a contradiction, that two edges $(u, v), (w, z) \in E$ exist, with $u, w \in V_i$ and $v, z \in V_{i+1}$, that intersect in Γ. Hence, either u appears before w in $\mathcal{O}_i$ and v appears after z in $\mathcal{O}_{i+1}$, or vice versa. Since i and $i+1$ have different parity, either u appears before w in $\Theta(\overline{T_i})$ and v appears before z in $\Theta(\overline{T_{i+1}})$, or vice versa. We claim that, in both cases, this implies a crossing in $\langle \Gamma_1^*, \Gamma_2^* \rangle$ between paths $(q_i, u(Q_i), v(P_{i+1}), p_{i+1})$ and $(q_i, w(Q_i), z(P_{i+1}), p_{i+1})$ in $\langle G_1^*, G_2^* \rangle$. Since the edges of these two paths belong all to G_1^* or all to G_2^*, depending on whether i is even or odd, this yields a contradiction. We now prove the claim. The pre-order traversal $\Theta(Q_i)$ of Q_i (the pre-order traversal $\Theta(P_{i+1})$ of P_{i+1}) in $\langle \Gamma_1^*, \Gamma_2^* \rangle$ restricted to the leaves of Q_i (of P_{i+1}) is the reverse of $\Theta(\overline{T_i})$ (of $\Theta(\overline{T_{i+1}})$) restricted to the vertices of V_i (of V_{i+1}) corresponding to leaves of Q_i (of P_{i+1}). Namely, each leaf $x(Q_i)$ of Q_i ($y(P_{i+1})$ of P_{i+1}) is connected to leaf $x(\overline{T_i})$ of $\overline{T_i}$ ($y(\overline{T_{i+1}})$ of $\overline{T_{i+1}}$) in the same graph, either G_1^* or G_2^*, by construction. Hence, the fact that u appears before (after) w in $\Theta(\overline{T_i})$ and v appears before (after) z in $\Theta(\overline{T_{i+1}})$ implies that u appears after (before) w in $\Theta(Q_i)$ and v appears after (before) z in $\Theta(P_{i+1})$. In both cases, this implies a crossing in $\langle \Gamma_1^*, \Gamma_2^* \rangle$ between the two paths.

Suppose that $(V, E, \gamma, \mathcal{T})$ admits a $\mathcal{T}$-level planar drawing Γ. We show how to construct a SEFE $\langle \Gamma_1^*, \Gamma_2^* \rangle$ of $\langle G_1^*, G_2^* \rangle$. For $1 \leq i \leq k$, consider the order $\mathcal{O}_i$ of the vertices of level V_i along L_i in Γ. Since Γ is $\mathcal{T}$-level planar, there exists an embedding Γ_i of tree $T_i \in \mathcal{T}$ that is compatible with $\mathcal{O}_i$. If i is odd (even), then assign to each internal vertex of $\overline{T_i}$ the same (resp. the opposite) rotation scheme as its corresponding vertex in Γ_i. Also, if i is odd, then assign to p_i (to q_i) the rotation scheme in G_1^* (in G_2^*) such that the paths connecting p_i (q_i) to the leaves of $\overline{T_i}$ (either with an edge or passing through a leaf of the corresponding star of $G_\cap$) appear in the same clockwise order as the vertices of V_i appear in $\mathcal{O}_i$; if i is even, then assign to p_i (to q_i) the rotation scheme in G_2^* (in G_1^*) such that the paths connecting p_i (q_i) to the leaves of $\overline{T_i}$ appear in the same counterclockwise order as the vertices of V_i appear in $\mathcal{O}_i$. Finally, consider the embedding $\Gamma_{i,i+1}$ obtained by restricting Γ to the vertices and edges of the subgraph induced by the vertices of V_i and V_{i+1}. If i is odd (even), then assign to the leaves of Q_i and of P_{i+1} in G_1^* (in G_2^*) the same rotation scheme as their corresponding vertices have in $\Gamma_{i,i+1}$. This completes the construction of $\langle \Gamma_1^*, \Gamma_2^* \rangle$.

We prove that $\langle \Gamma_1^*, \Gamma_2^* \rangle$ is a SEFE of $\langle G_1^*, G_2^* \rangle$. Since the rotation scheme of the internal vertices of each $\overline{T_i}$ are constructed starting from an embedding of Γ_i of tree $T_i \in \mathcal{T}$ that is compatible with $\mathcal{O}_i$, the drawing of $\overline{T_i}$ is planar. Further, since the rotation schemes of p_i (of q_i) are also constructed starting from $\mathcal{O}_i$, there exists no crossing between two paths connecting t_i and p_i (t_i and q_i), one passing through a leaf $u(\overline{T_i})$ of $\overline{T_i}$ and, possibly, through a leaf $u(P_i)$ of P_i (through a leaf $u(Q_i)$ of Q_i), and the other passing through a leaf $v(\overline{T_i})$ of $\overline{T_i}$ and, possibly, through a leaf $v(P_i)$ of P_i (through a leaf $v(Q_i)$ of Q_i). Finally, since the rotation schemes of the leaves of Q_i and P_{i+1} are constructed from the embedding $\Gamma_{i,i+1}$ obtained by restricting Γ to the vertices and edges of the subgraph induced by the vertices of V_i and V_{i+1}, there exist no two crossing edges between leaves of Q_i and of P_{i+1}. □

We remark that a reduction from T-LEVEL PLANARITY to SEFE-2 was described by Schaefer in [12]; however, the instances of SEFE-2 obtained from that reduction do not satisfy any conditions that make SEFE-2 known to be solvable in polynomial-time.

Theorem 3. *Let $(V, E, \gamma, \mathcal{T})$ be a proper instance of T-LEVEL PLANARITY. There exists a quadratic-time algorithm that decides whether $(V, E, \gamma, \mathcal{T})$ is $\mathcal{T}$-level planar.*

Proof: By Lemma 1, an instance $\langle G_1, G_2 \rangle$ of SEFE-2 can be constructed in linear time such that G_1 and G_2 are 2-connected, the common graph $G_\cap$ is connected, and $\langle G_1, G_2 \rangle$ is a positive instance of SEFE-2 if and only if $(V, E, \gamma, \mathcal{T})$ is $\mathcal{T}$-level planar. The statement follows from the fact that there exists a quadratic-time algorithm [7] that decides whether $\langle G_1, G_2 \rangle$ is a positive instance of SEFE-2. □

3.2 CLUSTERED-LEVEL PLANARITY

In the following we prove that the polynomial-time algorithm to decide the existence of a $\mathcal{T}$-level planar drawing of a proper instance $(V, E, \gamma, \mathcal{T})$ of T-LEVEL PLANARITY can be also employed to decide in polynomial time the existence of a cl-planar drawing of a proper instance (V, E, γ, T) of CL-PLANARITY.

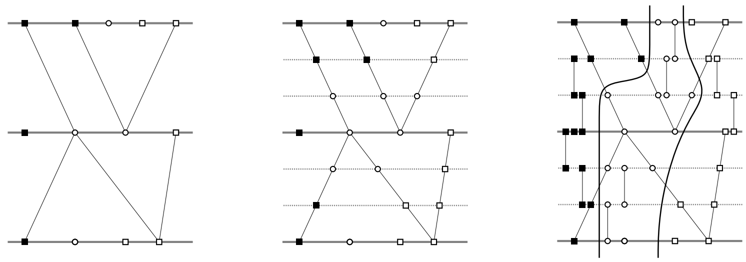
A proper cl-graph (V, E, γ, T) is *μ-connected between two levels* V_i and V_{i+1} if there exist two vertices $u \in V_\mu \cap V_i$ and $v \in V_\mu \cap V_{i+1}$ such that edge $(u, v) \in E$. For a cluster $\mu \in T$, let $\gamma_{\min}(\mu) = \min \{i | V_i \cap V_\mu \neq \emptyset\}$ and let $\gamma_{\max}(\mu) = \max \{i | V_i \cap V_\mu \neq \emptyset\}$. A proper cl-graph (V, E, γ, T) is *level-μ-connected* if it is μ-connected between levels V_i and V_{i+1} for each $i = \gamma_{\min}(\mu), \dots, \gamma_{\max}(\mu) - 1$. A proper cl-graph (V, E, γ, T) is *level-connected* if it is μ-level-connected for each cluster $\mu \in T$.

Our strategy consists of first transforming a proper instance of CL-PLANARITY into an equivalent level-connected instance, and then transforming such a level-connected instance into an equivalent proper instance of T-LEVEL PLANARITY.

Lemma 2. *Let (V, E, γ, T) be a proper instance of* CLUSTERED-LEVEL PLANARITY. *There exists an equivalent level-connected instance $(V^*, E^*, \gamma^*, T^*)$ of* CLUSTERED-LEVEL PLANARITY. *Further, the size of $(V^*, E^*, \gamma^*, T^*)$ is quadratic in the size of (V, E, γ, T) and $(V^*, E^*, \gamma^*, T^*)$ can be constructed in quadratic time.*

Proof: The construction of $(V^*, E^*, \gamma^*, T^*)$ works in two steps. See Fig. 4.

First, we transform (V, E, γ, T) into an equivalent instance (V', E', γ', T'). Initialize $V' = V$, $E' = E$, and $T' = T$. Also, for each $i = 1, \dots, k$ and for each vertex $u \in V_i$, set $\gamma'(u) = 3(i-1)+1$. Then, for each $i = 1, \dots, k-1$,



(a) (b) (c)

Fig. 4. Illustration for the proof of Lemma 2. (a) Instance (V, E, γ, T) with flat hierarchy containing clusters $\mu_{\blacksquare}$, $\mu_{\square}$, and $\mu_{\circ}$. (b) Insertion of dummy vertices in (V, E, γ, T) to obtain (V', E', γ', T'). (c) Level-connected instance $(V^*, E^*, \gamma^*, T^*)$ obtained from (V', E', γ', T').

consider each edge $(u, v) \in E$ such that $\gamma(u) = i$ and $\gamma(v) = i + 1$. Add two vertices d_u and d_v to V', and replace (u, v) in E' with three edges (u, d_u), (d_u, d_v), and (d_v, v). Set $\gamma'(d_u) = 3(i - 1) + 2$ and $\gamma'(d_v) = 3i$. Finally, add d_u (d_v) to T' as a child of the parent of u (of v) in T'.

Second, we transform (V', E', γ', T') into an equivalent level-connected instance $(V^*, E^*, \gamma^*, T^*)$. Initialize $V^* = V'$, $E^* = E'$, $\gamma^* = \gamma'$, and $T^* = T'$. Consider each cluster $\mu \in T'$ according to a bottom-up visit of T'. If there exists a level V'_i, with $\gamma'_{\min}(\mu) \leq i < \gamma'_{\max}(\mu)$, such that no edge in E' connects a vertex $u \in V'_i \cap V'_\mu$ with a vertex $v \in V'_{i+1} \cap V'_\mu$, then add two vertices u^* and v^* to V^*, add an edge (u^*, v^*) to E^*, set $\gamma^*(u^*) = i$ and $\gamma^*(v^*) = i + 1$, and add u^* and v^* to T^* as children of μ.

Observe that, for each cluster $\mu \in T$ and for each level $1 \leq i \leq 3k - 2$, at most two dummy vertices are added to $(V^*, E^*, \gamma^*, T^*)$. This implies that $|V^*| \in O(|V|^2)$. Also, the whole construction can be performed in $O(|V|^2)$ time.

Claim 1 *(V', E', γ', T') is equivalent to (V, E, γ, T).*

Proof: Suppose that (V, E, γ, T) admits a cl-planar drawing Γ; we show how to construct a cl-planar drawing Γ' of (V', E', γ', T'). Initialize $\Gamma' = \Gamma$. We scale Γ' up by a factor of 3 and we vertically translate it so that the vertices in V'_1 lie on line $y = 1$. After the two affine transformations have been applied (i) no crossing has been introduced in the drawing, (ii) every edge is still drawn as a y-monotone curve, (iii) for $i = 1, \dots, k$, the vertices of level $V_i = V'_{3(i-1)+1}$ are placed on line $y = 3(i-1)+1$, that we denote by $L'_{3(i-1)+1}$, and (iv) the order in which vertices of $V_i = V'_{3(i-1)+1}$ appear along $L'_{3(i-1)+1}$ is the same as the order in which they appeared along L_i. For each $i = 1, \dots, k - 1$, consider each edge $(u, v) \in E$ such that $\gamma(u) = i$ and $\gamma(v) = i + 1$. Place vertices d_u and d_v in Γ' on the two points of the curve representing (u, v) having y-coordinate $3(i - 1) + 2$ and $3i$, respectively. Then, the curves representing in Γ' any two edges in E' are part of the curves representing in Γ' any two edges in E. Hence Γ' is a cl-planar drawing of (V', E', γ', T').

Suppose that (V', E', γ', T') admits a cl-planar drawing Γ'; we show how to construct a cl-planar drawing Γ of (V, E, γ, T). Initialize $\Gamma = \Gamma'$. For each $i = 1, \dots, k - 1$, consider each path (u, d_u, d_v, v) such that $\gamma'(u) = 3(i - 1) + 1$, $\gamma'(d_u) = 3(i - 1) + 2$, $\gamma'(d_v) = 3i$, and $\gamma'(v) = 3i + 1$; remove vertices d_u and d_v, and their incident edges in E' from Γ; draw edge $(u, v) \in E$ as a curve obtained as a composition of the curves representing edges (u, d_u), (d_u, d_v), and (d_v, v) in Γ'. Scale Γ down by a factor of 3 and vertically translate it so that the vertices of V_1 lie on line $y = 1$. After the two affine transformations have been applied (i) no crossing has been introduced in the drawing, (ii) every edge is still drawn as a y-monotone curve, (iii) for $i = 1, \dots, k$, the vertices of level V_i are placed on line $y = i$, and (iv) the order in which vertices of $V_i = V'_{3(i-1)+1}$ appear along L_i is the same as the order in which they appeared along $L'_{3(i-1)+1}$. Since Γ' is cl-planar, this implies that Γ is cl-planar, as well. $\square$

Instance (V', E', γ', T') is such that, if there exists a vertex $u \in V'_j$, with $1 \leq j \leq 3(k - 1) + 1$, that is adjacent to two vertices $v, w \in V'_h$, with $h = j \pm 1$, then u, v, and w have the same parent node $\mu \in T'$; hence, (V', E', γ', T') is μ-connected between levels V'_j and V'_h.

Claim 2 *$(V^*, E^*, \gamma^*, T^*)$ is equivalent to (V', E', γ', T').*

Proof: Suppose that $(V^*, E^*, \gamma^*, T^*)$ admits a cl-planar drawing Γ^*; we show how to construct a cl-planar drawing Γ' of (V', E', γ', T'). Initialize $\Gamma' = \Gamma^*$ and remove from V', E', and Γ' all the vertices and edges added when constructing Γ^*. Since all the other vertices of V' and edges of E' have the same representation in Γ' and in Γ^*, and since Γ^* is cl-planar, drawing Γ' is cl-planar, as well.

Suppose that (V', E', γ', T') admits a cl-planar drawing Γ'; we show how to construct a cl-planar drawing Γ^* of $(V^*, E^*, \gamma^*, T^*)$. Initialize $\Gamma^* = \Gamma'$. Consider a level V'_i, with $1 \leq i \leq 3(k-1)$, such that vertices $u^*, v^* \in \mu$ with $\gamma'(u^*) = i$ and $\gamma'(v^*) = i+1$, for some cluster $\mu \in T$, have been added to $(V^*, E^*, \gamma^*, T^*)$. By construction, (V', E', γ', T') is not μ-connected between levels V'_i and V'_{i+1}. As observed before, this implies that no vertex $u \in V'_i \cap V'_\mu$ exists that is connected to two vertices $v, w \in V'_{i+1}$, and no vertex $u \in V'_{i+1} \cap V'_\mu$ exists that is connected to two vertices $v, w \in V'_i$. Hence, vertices u^* and v^*, and edge (u^*, v^*), can be drawn in Γ^* entirely inside the region representing μ in such a way that u^* and v^* lie along lines L'_i and L'_{i+1} and there exists no crossing between edge (u^*, v^*) and another edge. □

This concludes the proof of the lemma. □

Lemma 3. *Let (V, E, γ, T) be a level-connected instance of* Clustered-Level Planarity. *There exists an equivalent proper instance $(V, E, \gamma, \mathcal{T})$ of $\mathcal{T}$-*Level Planarity. *Further, the size of $(V, E, \gamma, \mathcal{T})$ is linear in the size of (V, E, γ, T) and $(V, E, \gamma, \mathcal{T})$ can be constructed in quadratic time.*

Proof: We construct $(V, E, \gamma, \mathcal{T})$ from (V, E, γ, T) as follows. Initialize $\mathcal{T} = \emptyset$. For $i = 1, \ldots, k$, add to $\mathcal{T}$ a tree T_i that is the subtree of the cluster hierarchy T whose leaves are all and only the vertices of level V_i. Note that the set of leaves of the trees in $\mathcal{T}$ corresponds to the vertex set V. Since each internal node of the trees in $\mathcal{T}$ has at least two children, we have that the size of $(V, E, \gamma, \mathcal{T})$ is linear in the size of (V, E, γ, T). Also, the construction of $(V, E, \gamma, \mathcal{T})$ can be easily performed in $O(|V|^2)$ time.

We prove that $(V, E, \gamma, \mathcal{T})$ is $\mathcal{T}$-level planar if and only if (V, E, γ, T) is cl-planar.

Suppose that $(V, E, \gamma, \mathcal{T})$ admits a $\mathcal{T}$-level planar drawing Γ^*; we show how to construct a cl-planar drawing Γ of (V, E, γ, T). Initialize $\Gamma = \Gamma^*$. Consider each level V_i, with $i = 1, \ldots, k$. By construction, for each cluster $\mu \in T$ such that there exists a vertex $v \in V_i \cap V_\mu$, there exists an internal node of tree $T_i \in \mathcal{T}$ whose leaves are all and only the vertices of $V_i \cap V_\mu$. Since Γ^* is $\mathcal{T}$-level planar, such vertices appear consecutively along L_i. Hence, in order to prove that Γ is a cl-planar drawing, it suffices to prove that there exist no four vertices u, v, w, z such that (i) $u, v \in V_i$ and $w, z \in V_j$, with $1 \leq i < j \leq k$; (ii) $u, w \in V_\mu$ and $v, z \in V_\nu$, with $\mu \neq \nu$; and (iii) u appears before v on L_i and w appears after z on L_j, or vice versa. Suppose, for a contradiction, that such four vertices exist. Note that, we can assume $j = i \pm 1$ without loss of generality, as (V, E, γ, T) is level-connected. Assume that u appears before v along L_i and w appears after z along L_j, the other case being symmetric. Since Γ^* is $\mathcal{T}$-level planar, all the vertices of V_μ appear before all the vertices of V_ν along L_i and all the vertices of V_μ appear after all the vertices of V_ν along L_j. Also, since (V, E, γ, T) is level-connected, there exists at least an edge (a, b) such that $a \in V_i \cap V_\mu$ and $b \in V_j \cap V_\mu$, and an edge (c, d) such that $c \in V_i \cap V_\nu$ and $d \in V_j \cap V_\nu$. However, under the above conditions, these two edges intersect in Γ and in Γ^*, hence contradicting the hypothesis that Γ^* is $\mathcal{T}$-level planar.

Suppose that (V, E, γ, T) admits a cl-planar drawing Γ; we show how to construct a $\mathcal{T}$-level planar drawing Γ^* of $(V, E, \gamma, \mathcal{T})$. Initialize $\Gamma^* = \Gamma$. Consider each level V_i, with $i = 1, \ldots, k$. By construction, for each internal node w of tree $T_i \in \mathcal{T}$, there exists a cluster $\mu \in T$ such that the vertices of $V_i \cap V_\mu$ are all and only the leaves of the subtree of T_i rooted at w. Since Γ is cl-planar, such vertices appear consecutively along L_i. Hence, Γ^* is $\mathcal{T}$-level planar. □

We get the following.

Theorem 4. *Let (V, E, γ, T) be a proper instance of* Clustered-Level Planarity. *There exists an $O(|V|^4)$-time algorithm that decides whether (V, E, γ, T) admits a cl-planar drawing.*

Proof: By Lemma 2, it is possible to construct in $O(|V|^2)$ time a level-connected instance (V', E', γ', T') of CL-Planarity that is cl-planar if and only if (V, E, γ, T) is cl-planar, with $|V'| = O(|V|^2)$. By Lemma 3, it is possible to construct in $O(|V'|^2)$ time a proper instance $(V', E', \gamma', \mathcal{T}')$ of T-Level Planarity that is $\mathcal{T}$-level planar if and only if (V', E', γ', T') is cl-planar. Finally, by Theorem 3, it is possible to test in $O(|V'|^2)$ time whether $(V', E', \gamma', \mathcal{T}')$ is $\mathcal{T}$-level planar. □

4 Open Problems

Several problems are opened by this research:

1. The algorithm in [10] for testing level planarity and the algorithm in [9] for testing CL-PLANARITY for level-connected instances in which the level graph is a proper hierarchy both have linear-time complexity. The algorithm in [13] for testing T-LEVEL PLANARITY for instances in which the number of vertices on each level is bounded by a constant has quadratic-time complexity. Although our polynomial-time algorithms solve more general problems than the ones cited above, they are less efficient. Hence, there is room for future research aiming at improving our complexity bounds.
2. Our $\mathcal{NP}$-completeness result on the complexity of CL-PLANARITY exploits a cluster hierarchy whose depth is linear in the number of vertices of the underlying graph. Does the $\mathcal{NP}$-hardness hold even when the hierarchy is flat or has a depth that is sublinear in the number of vertices?
3. The $\mathcal{NP}$-hardness of CL-PLANARITY presented in this paper is, to the best of our knowledge, the first hardness result for a variation of the clustered planarity problem in which none of the c-planarity constraints is dropped. Is it possible to use similar techniques to tackle the more intriguing problem of determining the complexity of CLUSTERED PLANARITY?

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