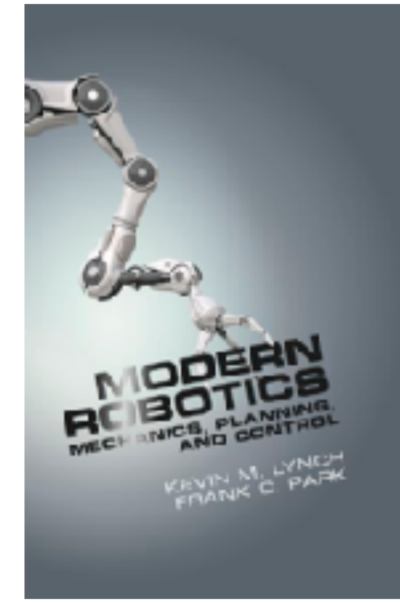
ROB-GY 6003
Foundations of robotics

Ludovic Righetti

Lecture III
Homogeneous transforms II - velocities

Organization

Course website(s)
All necessary material will be posted on Brightspace
Code will be posted on the Github site of the class

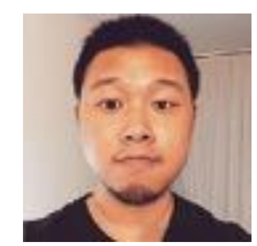


Discussions/Forum with Campuswire

Book
Modern Robotics by K. Lynch and F. Park
http://hades.mech.northwestern.edu/index.php/Modern_Robotics

Contact
ludovic.righetti@nyu.edu
Zoom office hours
Tuesdays 4 to 5pm

Course Assistant
Huaijiang Zhu
hzhu@nyu.edu



Zoom office hours
Wednesday 1:30 to 2:30pm

any other time by appointment

Planned class schedule (subject to changes)

Lecture 1 - 09/13	Introduction, sensors/actuators, configuration space		
Lecture 2 - 09/20	Rotations and homogeneous transforms	HW1	
Lecture 3 - 09/27	Homegenous transforms II / Velocities	HW2	
Lecture 4 - 10/04	Velocities II	HW3	
Lecture 5 - 10/11	Forward kinematics	HW4	
Lecture 6 - 10/12	Geometric Jacobian - Inverse Kinematics		
Lecture 7 - 10/18	Inverse Kinematics II	HW5	
Midterm - 10/25	Midterm		Project 1
Lecture 8 - 11/01	Control I (trajectory generation and resolved rate		
Lecture 9 - 11/08	Forces, duality kineto-statics	HW6	
Lecture 10 - 11/15	Dynamics		Project 2
Lecture 11 - 11/22	Control II (gravity compensation, impedance control)		
Lecture 12 - 11/29	Introduction to object manipulation		
Lecture 13 - 12/06	Introduction to legged robots		Project 3
Lecture 14 - 12/13	Going beyond the class, advanced topics		
Finals week	-		

Homework II

Go to https://prairielearn.engr.illinois.edu/pl/

Open now

Due in 1 week to get 100% (10/04)

10% bonus if you do it by Thursday (09/30)

One week late (10/11) you get 80%
Two weeks late (10/18) you get 50%
More than two weeks late you get 0%

Questions?

$$R = \begin{bmatrix} \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Today's goal

Homogeneous transform to describe rigid bodies + describe the angular velocity of a rigid body

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- How to compute the angular velocity of a rigid body in terms of its rotation matrix and its time derivative

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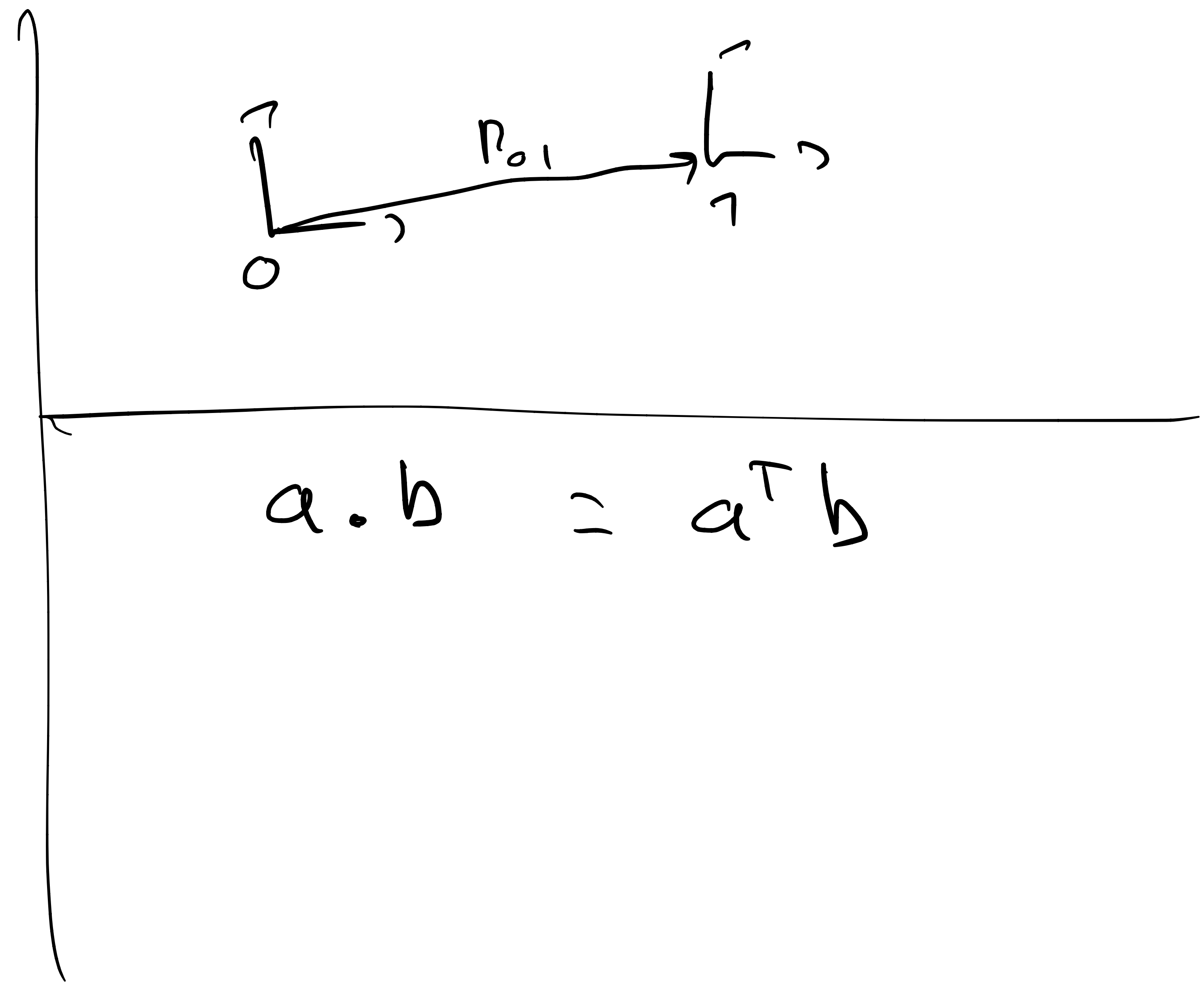
Recommended reading: Chapter 3 of Modern Robotics

Some notations

$\underline{p_{01}}$ $\begin{bmatrix} p^0_{01} \\ p^1_{01} \end{bmatrix}$

$p\{0\}$

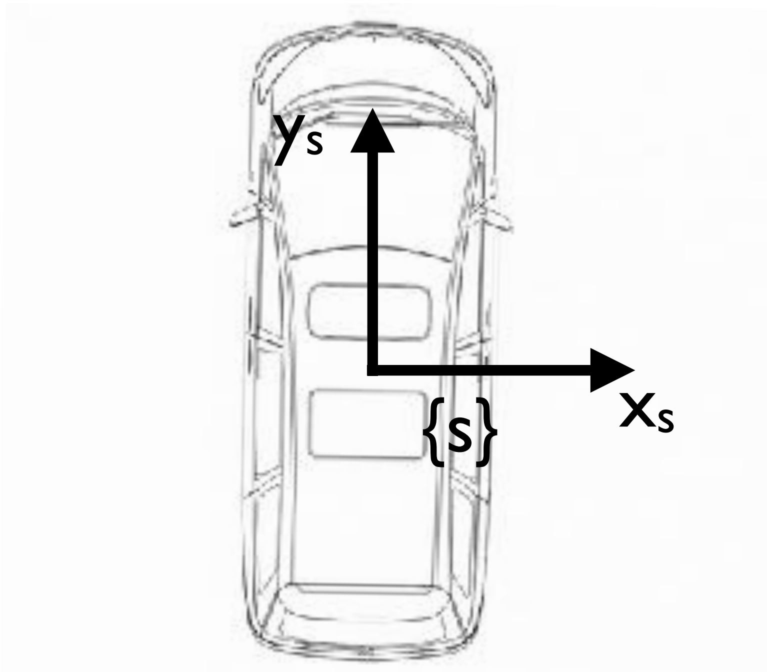
T_{23}

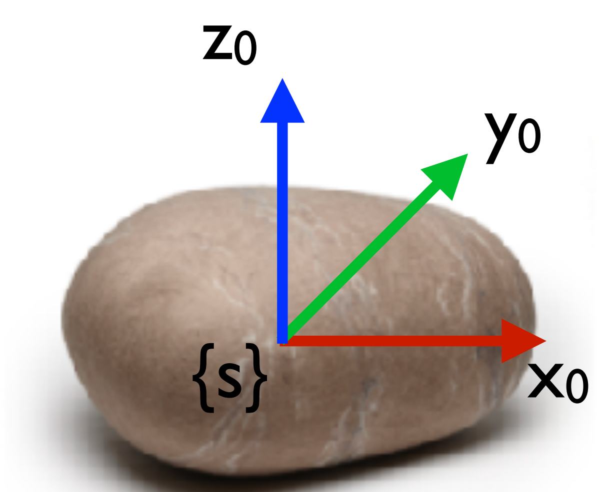


$a \cdot b = a^T b$

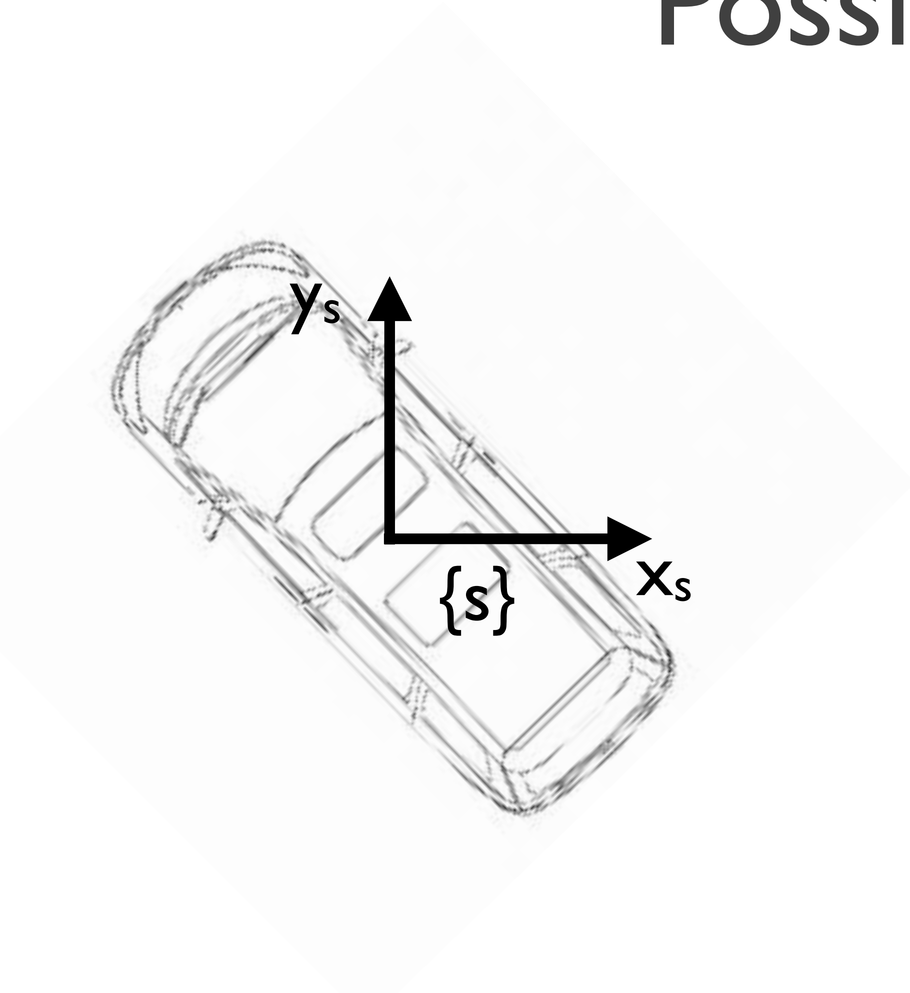
Recap from last week

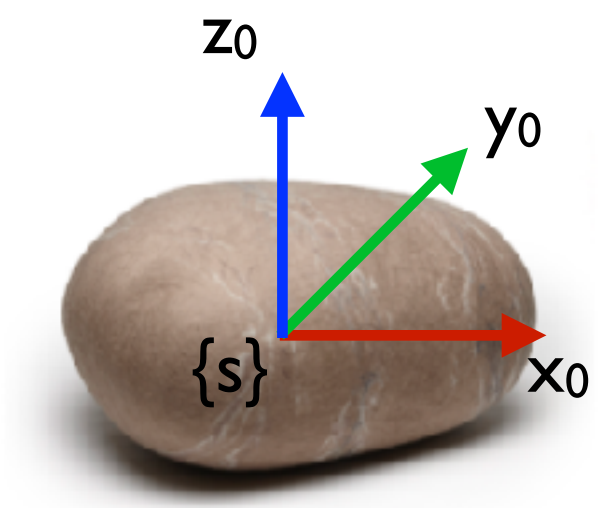
Possible rigid body motions



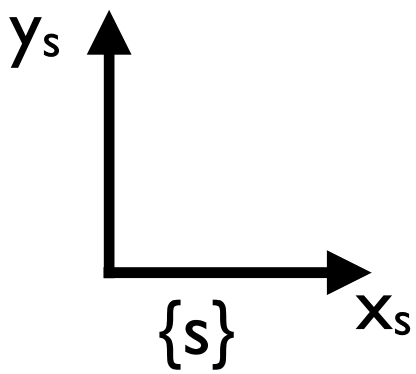


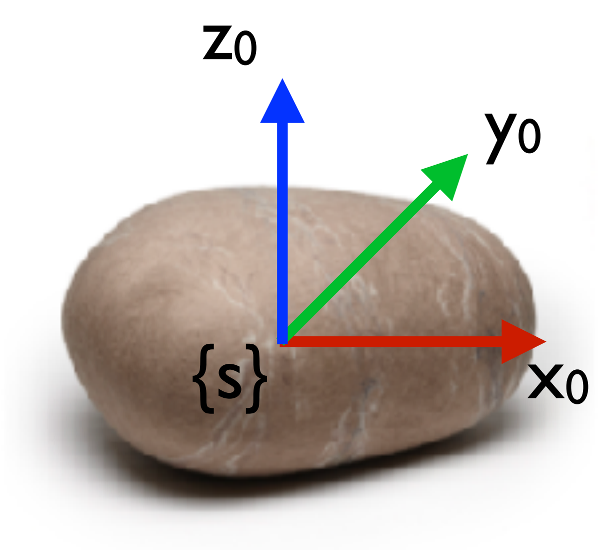
Possible rigid body motions



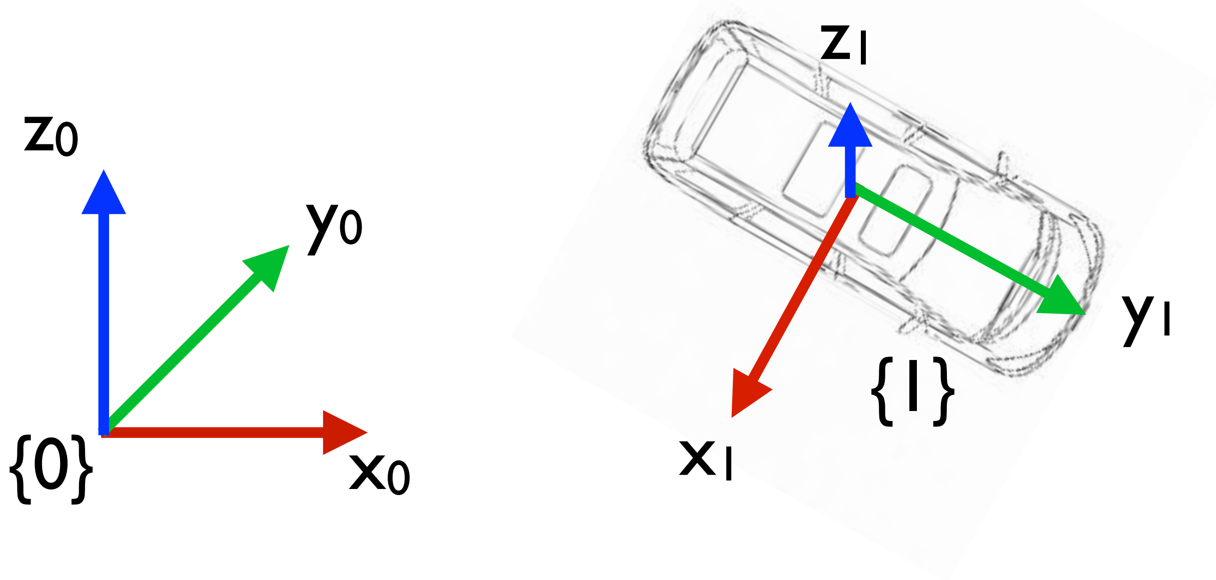


Possible rigid body motions





Rotations



The rotation matrix R_{01} which describes the orientation of frame 1 with respect to frame 0 has the form

$$R_{01} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}$$

The columns correspond to the coordinates of the basis vectors of frame 1 in the coordinate frame 0 (i.e. x_1 is a column vector containing the coordinates of the x axis of frame 1 expressed in the coordinates of frame 0)

Properties of rotation matrices

Properties of rotation matrices:

- A rotation matrix is orthogonal, it means that its columns (and rows) are orthogonal with each other
- $RR^T = R^T R = I$ ($R^T = R^{-1}$ is also a rotation matrix)
- $\det(R) = 1$
- the composition of two rotation matrices $R_1 R_2$ is also a rotation matrix
- the identity matrix I is also a rotation matrix
- rotations preserve distances

SO (2)

SO (3)

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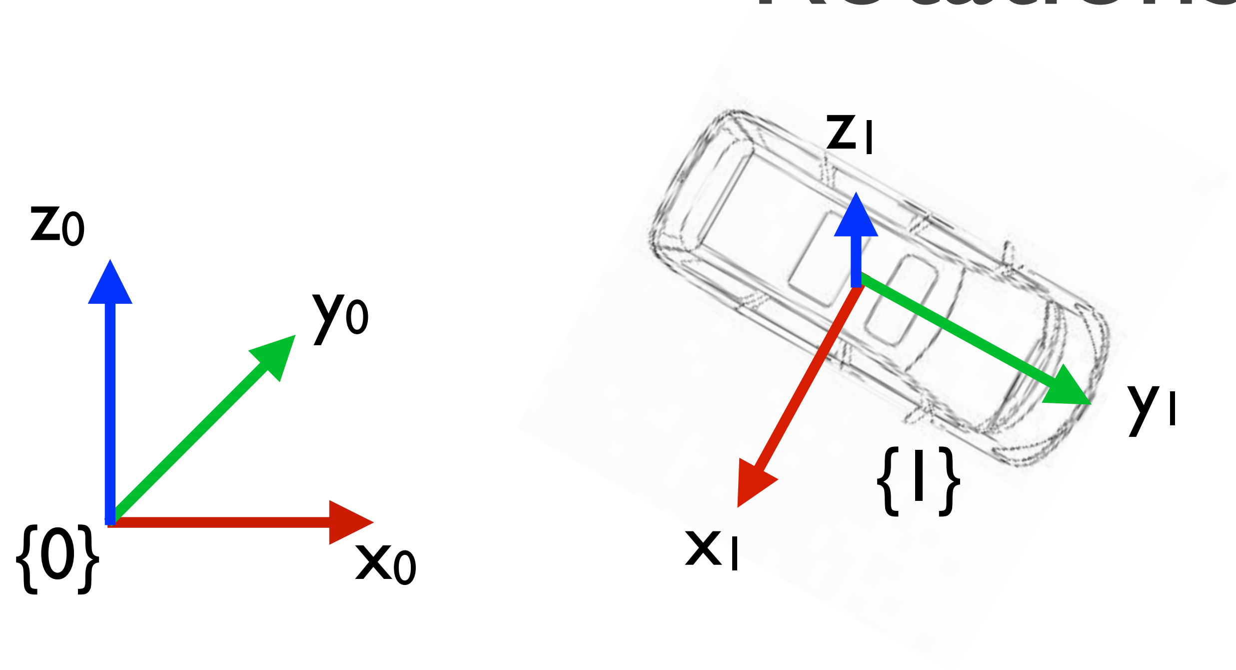
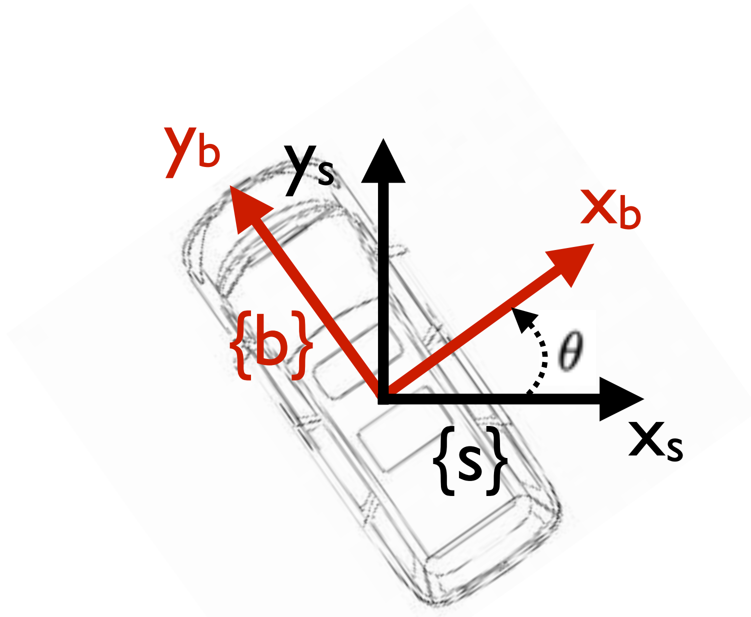
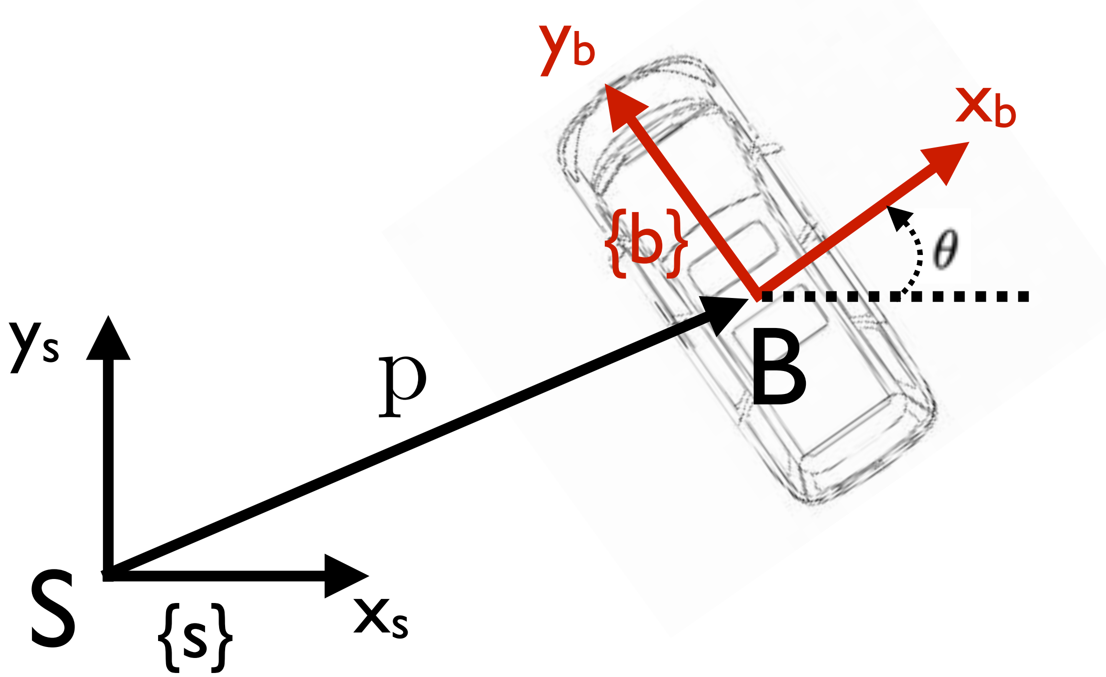
$$R \text{ is a rotation matrix} \Leftrightarrow \begin{array}{l} 1)\ RR^T = R^T R = I \\ 2)\ \det(R) = 1 \end{array}$$

Rotations



Rigid body transforms





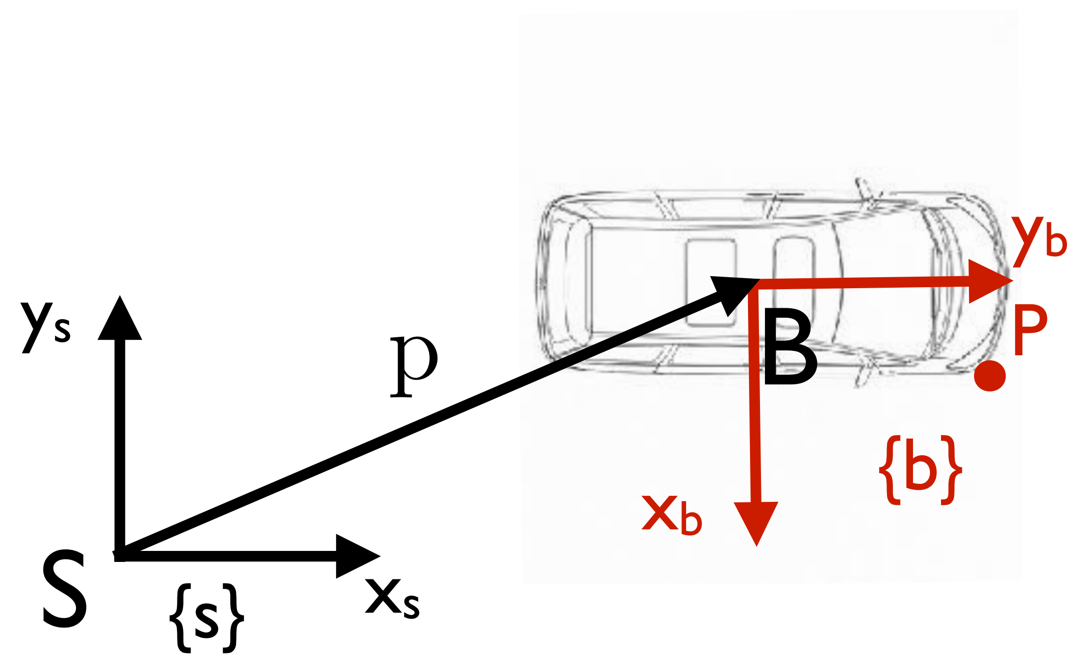
Any rigid body transformation can be described by a translation p and a rotation R. The position and orientation of frame B (the car) with respect to frame S (the spatial frame) is described by:

- p_{01} the position of the origin of B in the coordinates of S

- R_{01} the orientation of B with respect to S

As for pure rotations, we can
1) describe the position of an object
2) do a coordinate transform
3) move an object in space

Example



Homogeneous transforms

Rigid body transformations can be conveniently described by homogeneous matrices, which summarizes the position p_{01} and orientation R_{01}

$$T_{01} = \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix}$$

A point p with 3D (or 2D) coordinates p is then described with 4D (or 3D) coordinates as

$$\bar{p} = \begin{pmatrix} p \\ 1 \end{pmatrix}$$

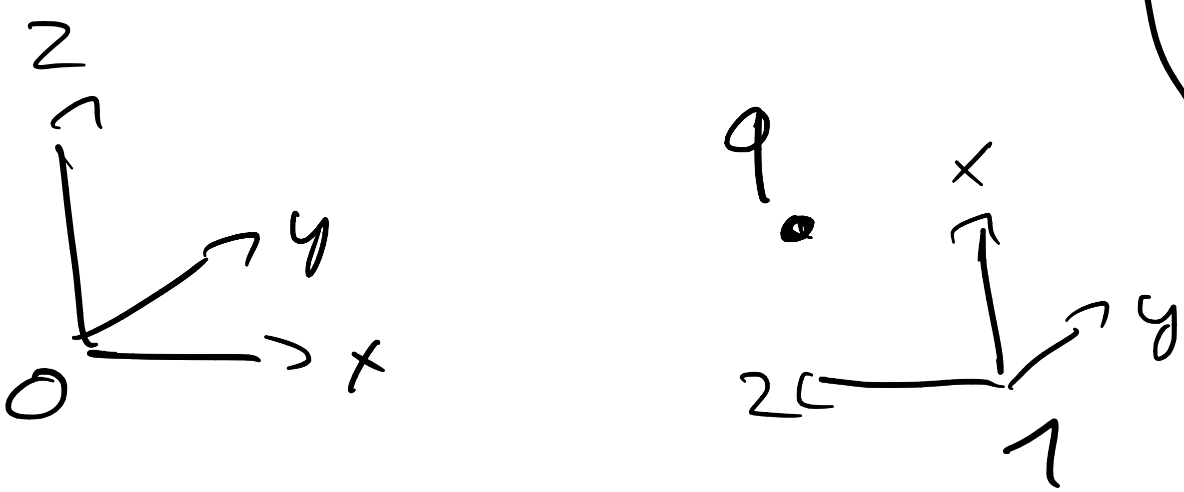
A vector v with 3D (or 2D) coordinates v is then described as a 4D (or 3D) coordinates as

$$\bar{v} = \begin{pmatrix} v \\ 0 \end{pmatrix}$$

Transformations of points and vectors

$$T_{01} = \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix} \qquad \bar{q}_1 = \begin{pmatrix} q_1 \\ 1 \end{pmatrix} \qquad q_0 \, ?$$

$$T_{01} \cdot \bar{q}_1 = \begin{pmatrix} R_{01} \, q_1 + p_{01} \\ 1 \end{pmatrix}$$



Transformations of points and vectors

$$T_{01} = \begin{bmatrix} R_{01} & p_{01} \\ 0 & 1 \end{bmatrix} \qquad \bar{v}_1 = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \qquad v_0 \,?$$

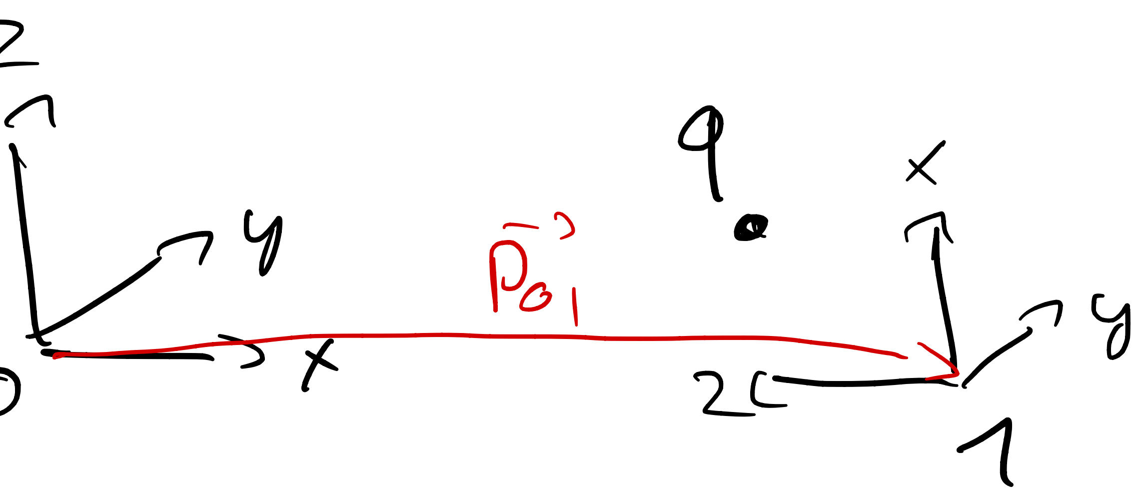
$$T_{01}\,\bar{v}_1 = \begin{pmatrix} R_{01} v_1 \\ 0 \end{pmatrix}$$

Inverse of an homogenous transform

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$T_{01} \qquad q_0 \qquad q_1 \, ?$$

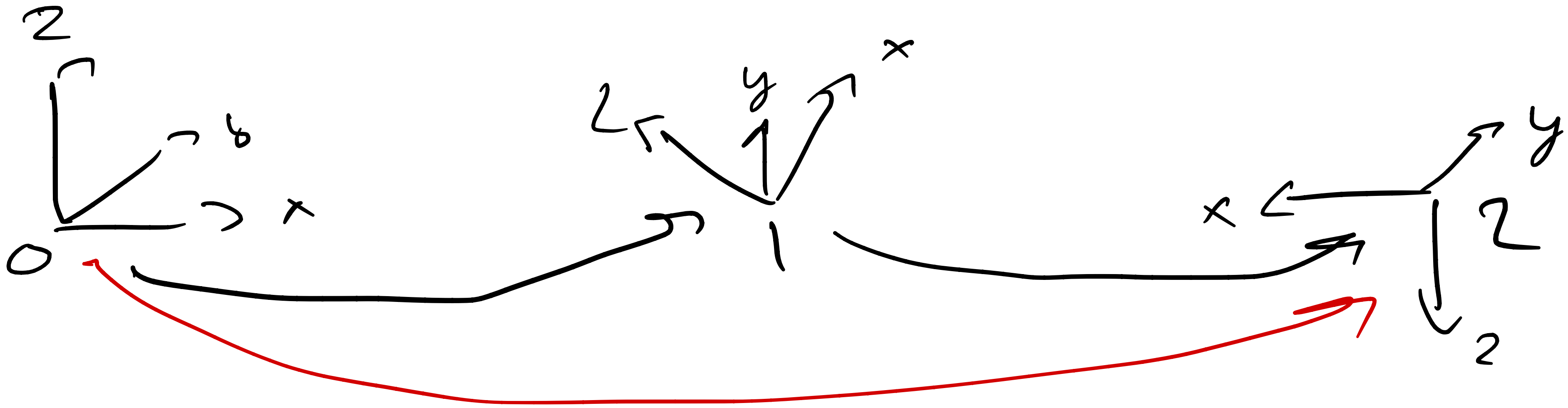
$$T_{01}^{-1} q_0 = T_{10} q_0 = \begin{bmatrix} R_{01}^T & -R_{01}^T p_{01} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} q_0 \\ 1 \end{pmatrix}$$



$$= \begin{pmatrix} R_{01}^T q_0 - R_{01}^T p_{01} \\ 1 \end{pmatrix}$$

$$v_0 \qquad v_1 \, ? \qquad T_{01}^{-1} \begin{pmatrix} v_0 \\ 0 \end{pmatrix} = \begin{pmatrix} R_{01}^T v_0 \\ 0 \end{pmatrix}$$

Composition of homogeneous transform



T_{01} $\quad T_{12}$ $\quad T_{02}$?

$\cancel{T_{02} = T_{12} T_{01}}$?

$T_{02} = T_{01} T_{12}$?

$T_{12} q_2 \to q_1$

$T_{01} q_1 \to q_0$

$T_{01} (T_{12} q_2) \to q_0$

Composition of homogeneous transform

$$T_1 = \begin{bmatrix} R_1 & P_1 \\ 0 & 1 \end{bmatrix} \qquad T_2 = \begin{bmatrix} R_2 & P_2 \\ 0 & 1 \end{bmatrix}$$

$$T_1 T_2 ? \qquad \begin{bmatrix} R_1 R_2 & R_1 P_2 + P_1 \\ 0 & 1 \end{bmatrix}$$

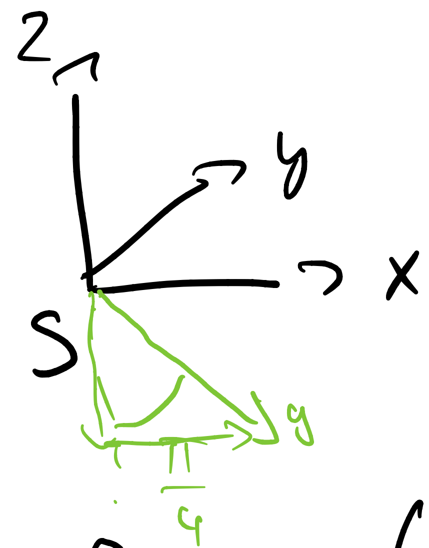
$$T_E \cdot T_1 = T_1 \cdot T_E = T_1 \qquad T_E = I$$

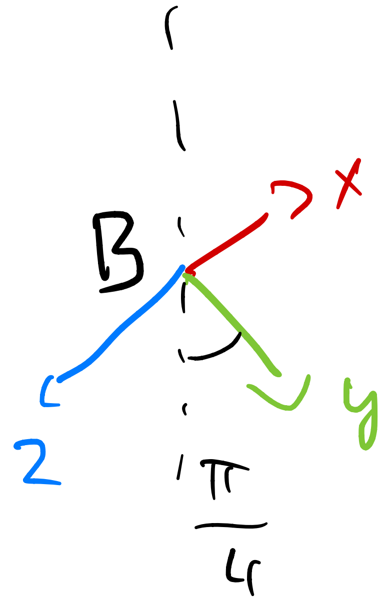
Set of <u>all</u> rigid body transforms

Special Euclidean Group

$SE(3)$ $\quad SE(2)$

Pose of an object





x points inside screen

y, z are in the plane

$$P_{SB} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$$

$$T_{SB} = \left[\begin{array}{ccc|c} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

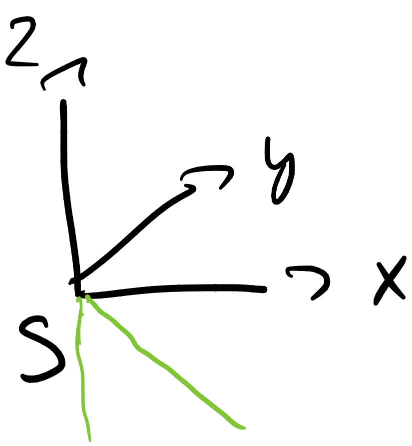
$$R_{z_B}\left(\frac{\pi}{2}\right) \cdot R_x\left(\frac{5\pi}{4}\right)$$

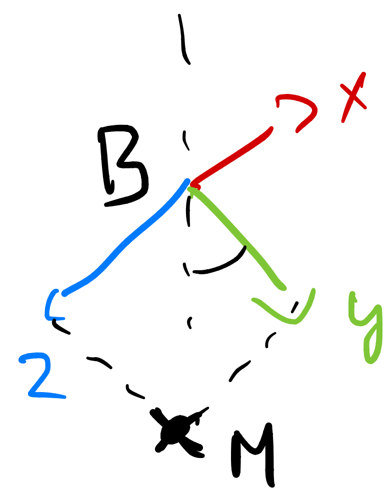
Coordinate transform

$$T_{SB} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_B = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

M_S ?



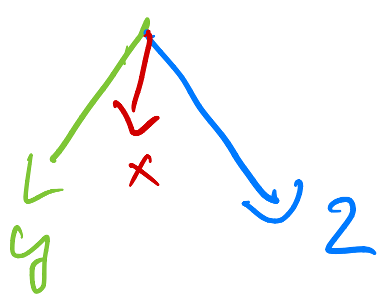


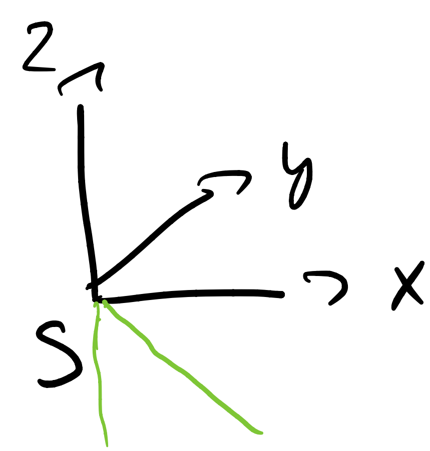
$$T_{SB} M_B = \begin{pmatrix} 5 \\ 0 \\ 1-\sqrt{2} \\ 1 \end{pmatrix}$$

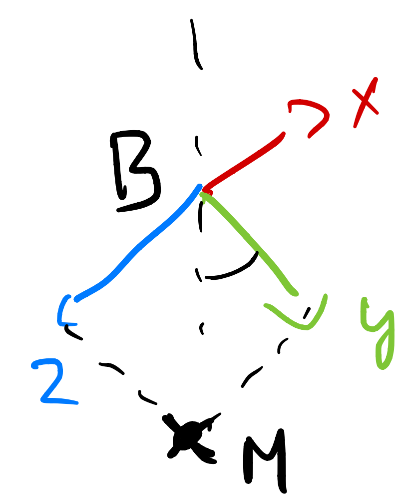
Move an object

$$T_{SB} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

rotate B along z_S π







x outside screen

$$\underline{T_{SB} M_B} \rightarrow M_S$$

$$T_{z_S} M_S \rightarrow M_S'$$

$$\boxed{T_{z_S} T_{SB}} M_B \rightarrow M_S'$$

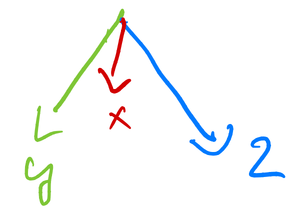
$$T_{z_S}(\pi) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

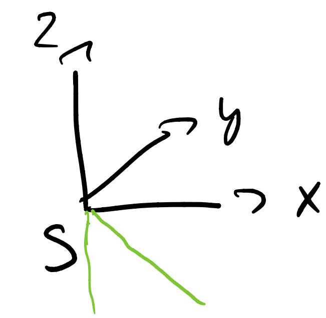
Move an object

$T_{2S} T_{SB}$?

$$= \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & \sqrt{2}/2 & -5 \\ -1 & 0 & 0 & 0 \\ 0 & -\sqrt{2}/2 & -\sqrt{2}/2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

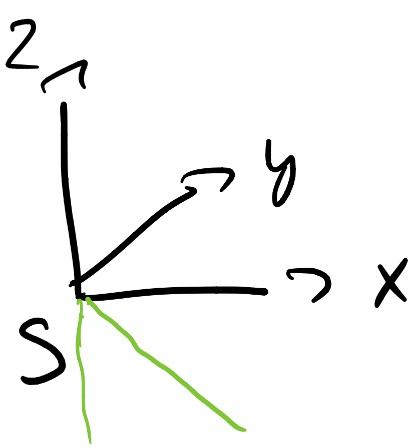
$$T_{SB} = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

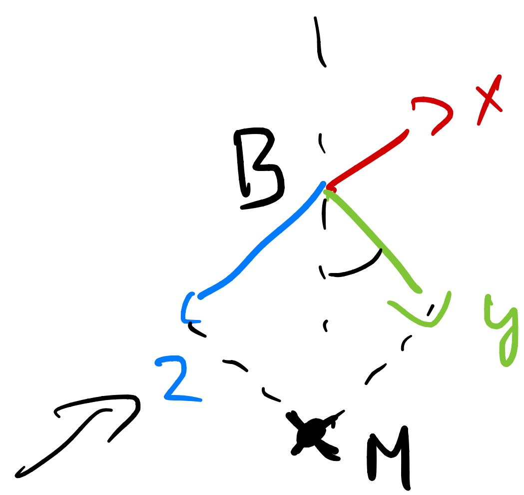




x outside screen

$$T_{2S}(\pi) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





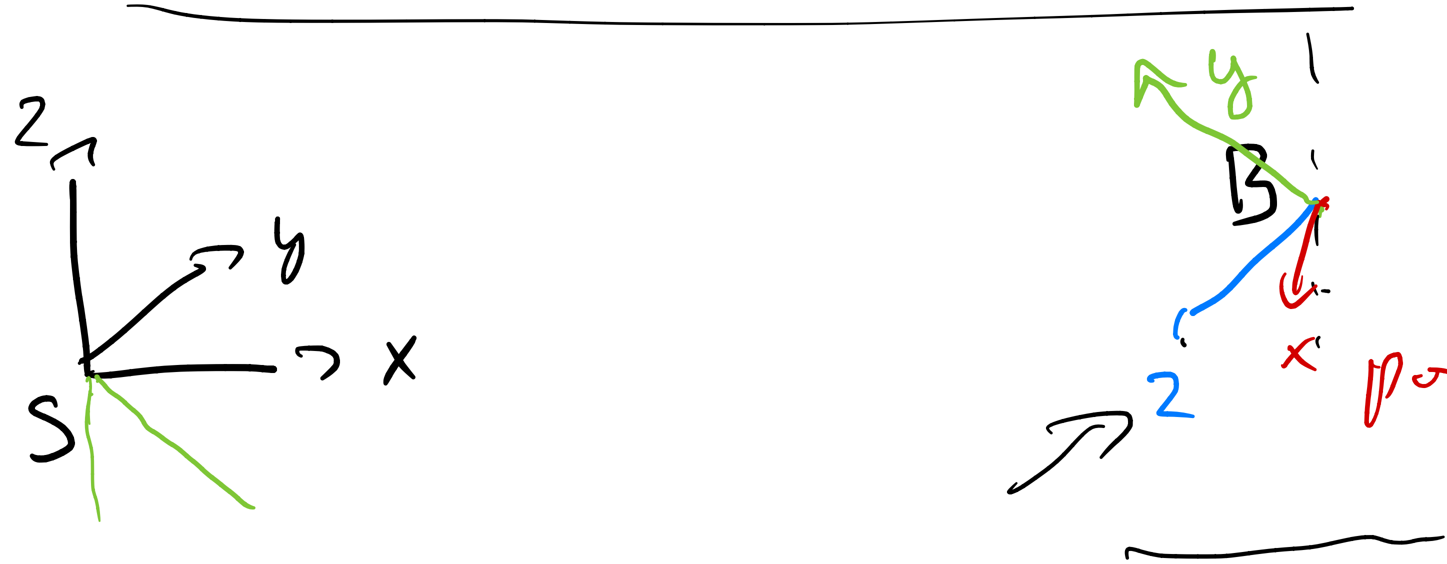
rotation π $\boxed{2B}$

↳ $T_{SB} T_{2B}$

Move an object

rotation π $\boxed{z_B}$

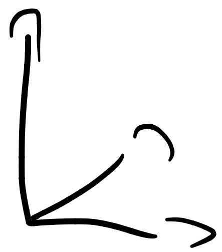
$\hookrightarrow T_{sB}\, T_{z_B}$



points outside

$$T_{sB}\, T_{z_B}(\pi) = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5 \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 5 \\ -1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Move an object



Spatial

Body

Move w.r.t. spatial

$$T_{move} \cdot T_{SB}$$

Move w.r.t. body

$$T_{SB} \cdot T_{move}$$

Why matrices to represent rotations?

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In 2D

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

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4 matrix entries but
only 1 free parameter

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4 matrix entries but
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In 3D

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Why matrices to represent rotations?

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9 matrix entries but only 3 free parameter

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9 matrix entries but
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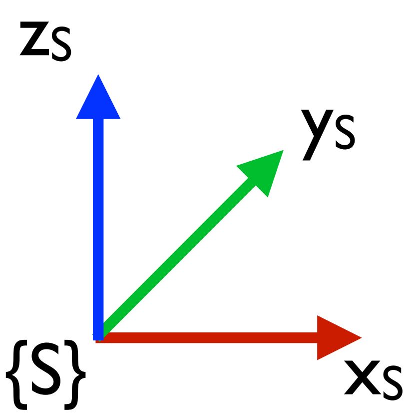
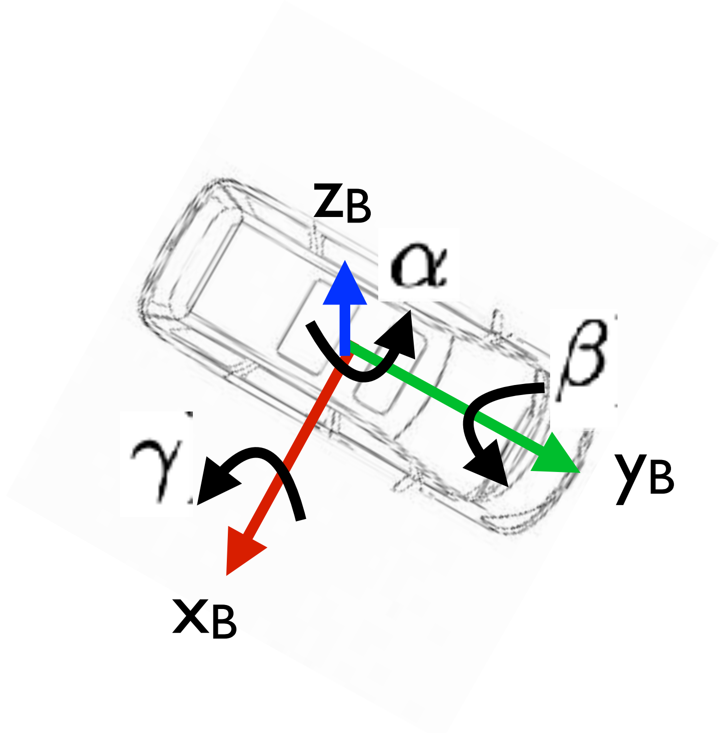
Why not use a 3 parameter representation in 3D?

ZYX Euler Angles

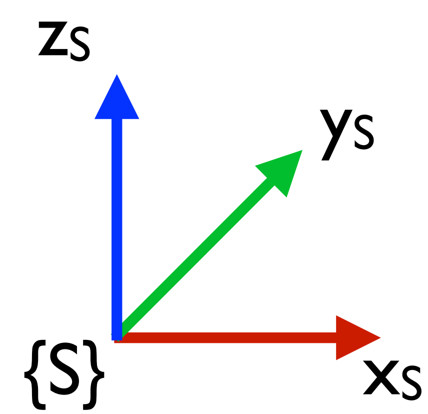


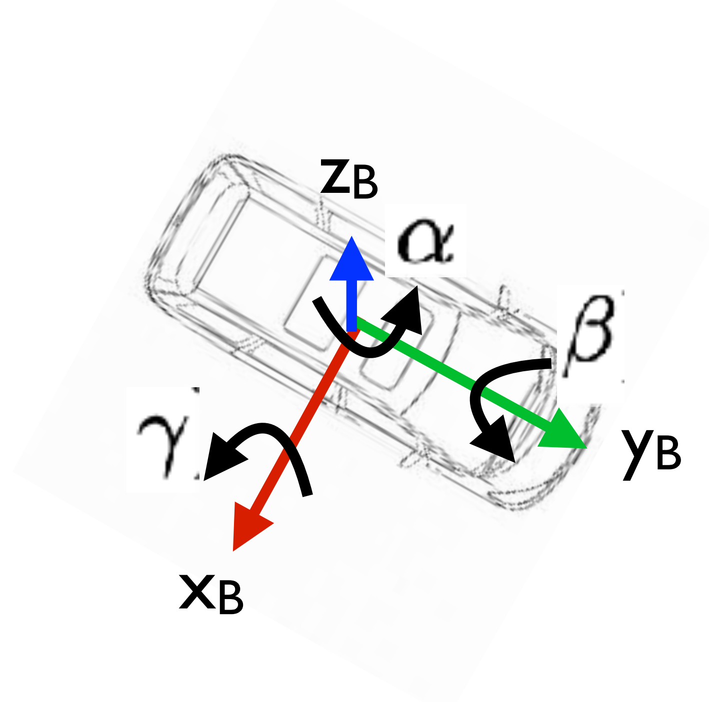


ZYX Euler Angles

Every rotation matrix R can be written as a composition of rotations along Z_B, Y_B and X_B

$$R = R_1(z_B, \alpha) R_2(y_B, \beta) R_3(x_B, \gamma)$$





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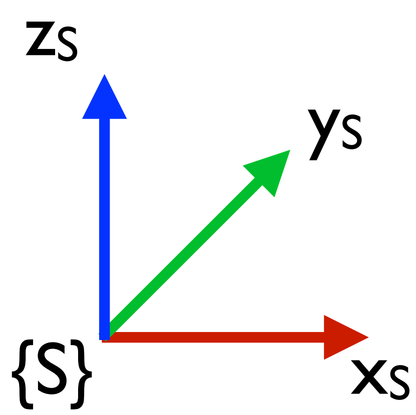
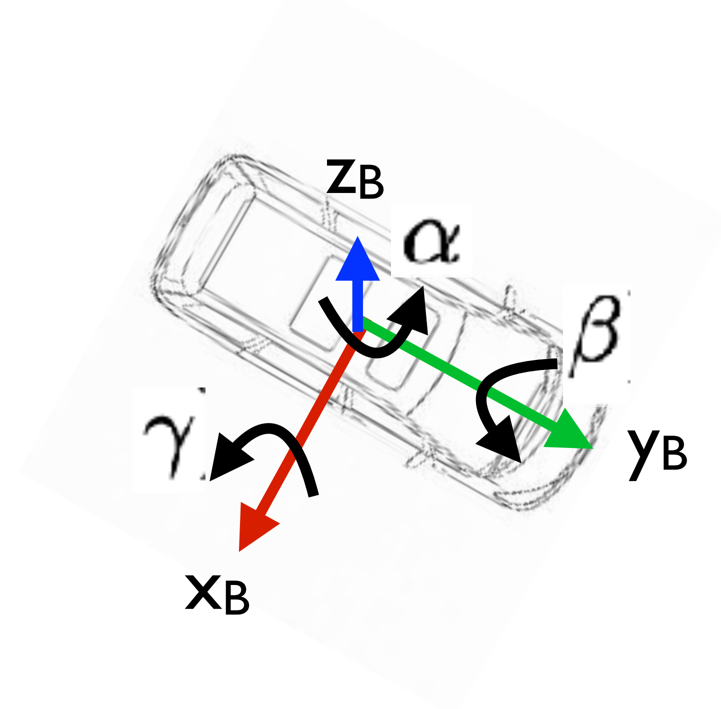




$$\mathrm{Rot}(\hat{z}, \alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}(\hat{y}, \beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \quad \mathrm{Rot}(\hat{x}, \gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}.$$

ZYX Euler Angles

$$R(\alpha,\beta,\gamma) = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

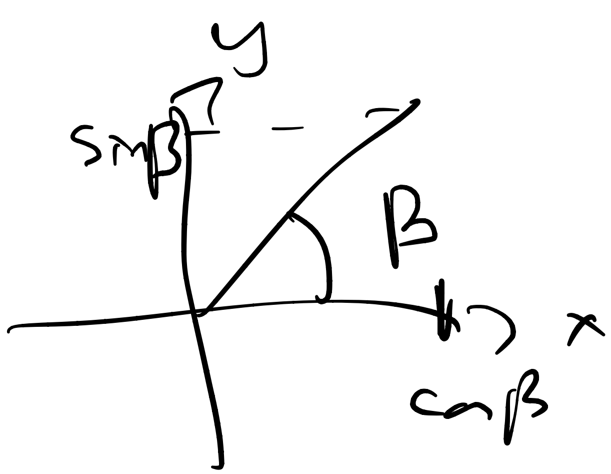
$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\cos^2\beta \sin^2\gamma + \cos^2\beta\cos^2\gamma$$

$$= \cos^2\beta$$

$$\sqrt{r_{32}^2 + r_{33}^2} = \cos\beta$$

$$\sin\beta = r_{31}$$



$$\beta = \operatorname{atan2}(\sin\beta, \cos\beta)$$

$$= \operatorname{atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right)$$

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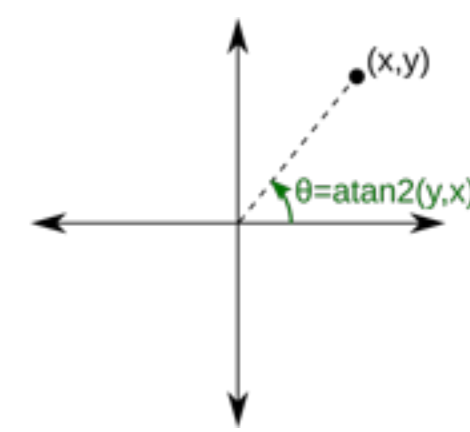
if $\beta \neq \pm\frac{\pi}{2}$

$$\beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \quad \text{with } -\frac{\pi}{2} < \beta < \frac{\pi}{2}$$

$$\begin{aligned} \alpha &= \operatorname{atan2}(r_{21}, r_{11}) \\ \gamma &= \operatorname{atan2}(r_{32}, r_{33}) \end{aligned}$$



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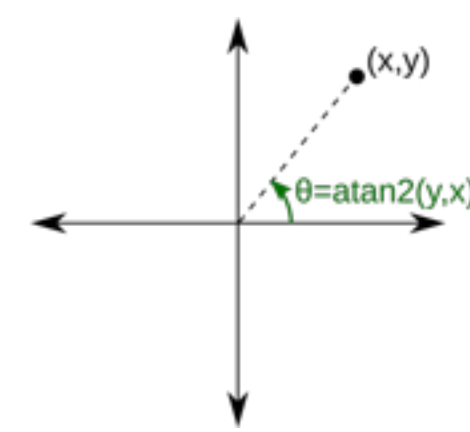

if $\beta = \pm\frac{\pi}{2}$ There is an infinite number of solutions for α and γ

$R(\alpha_1, \frac{\pi}{2}, \gamma_1) = R(\alpha_2, \frac{\pi}{2}, \gamma_2)$ whenever $\alpha_1 - \gamma_1 = \alpha_2 - \gamma_2$

ZYX Euler Angles

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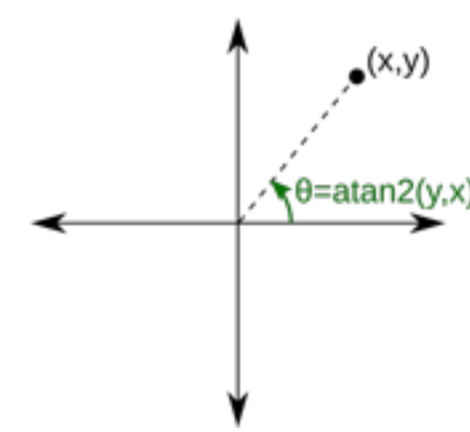


if $\beta = \pm\frac{\pi}{2}$ There is an infinite number of solutions for α and γ

$$R(\alpha_1, \tfrac{\pi}{2}, \gamma_1) = R(\alpha_2, \tfrac{\pi}{2}, \gamma_2) \text{ whenever } \alpha_1 - \gamma_1 = \alpha_2 - \gamma_2$$

This is called a "singularity" of the Euler angle representation, this is an issue when trying to compute the velocity of a rigid body as a function of the Euler angles

Other Euler angle representations

ZYZ Euler angles are define as $R(\alpha, \beta, \gamma) = Rot(\hat{z}, \alpha) Rot(\hat{y}, \beta) Rot(\hat{z}, \gamma)$

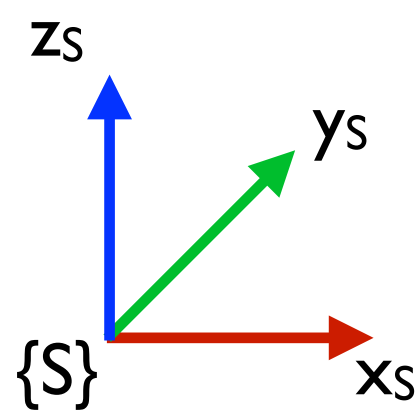
Any parametrization of this form will have similar singularity issues:

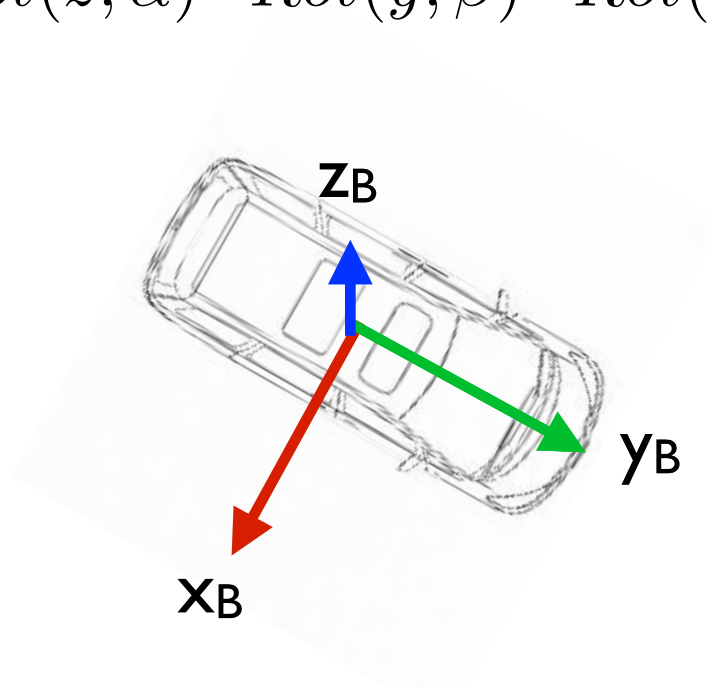
- Convenient to "visualize" a rotation
- Always a singularity => it is impossible to represent uniquely orientations with only 3 parameters!
- Problematic for computations (velocities or trajectories computations will be sensitive to singularity)

Roll-Pitch-Yaw

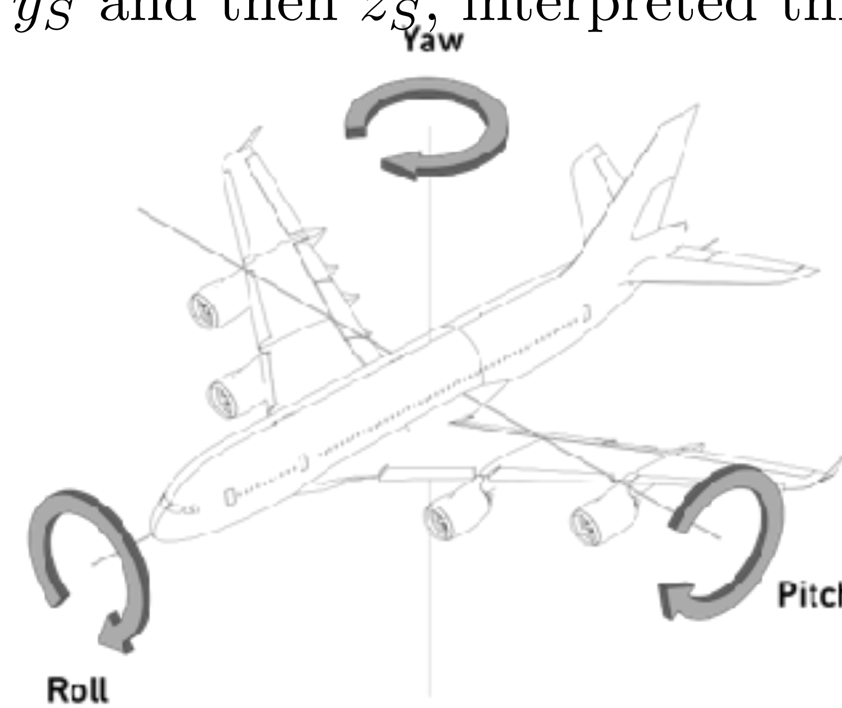
In ZYX Euler is thought as a body rotation around z_B, then y_B then x_B

$$R = Rot(\hat{z}, \alpha) \cdot Rot(\hat{y}, \beta) \cdot Rot(\hat{x}, \gamma)$$





This is the same as a spatial rotation around x_S, then y_S and then z_S, interpreted this way γ, β and α are called the roll-pitch-yaw angles.



Quaternions

Another popular representation is using unit quaternions

Quaternions

Another popular representation is using <u>unit</u> quaternions

A quaternion is a 4D generalization of complex numbers

$$q = q_0 + iq_1 + jq_2 + kq_3$$

with $i^2 = j^2 = k^2 = ijk = -1$ and
$ij = k,\ jk = i,\ ki = j$
$ji = -k,\ kj = -i,\ ik = -j$

$$||q|| = 1$$

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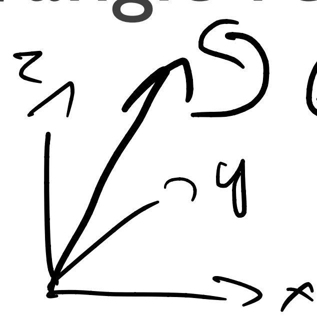
It can be thought as a 4D representation of the axis/angle representation

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ \hat{\omega}\sin(\theta/2) \end{bmatrix} \in \mathbb{R}^4$$



θ $\hat{\omega}$

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Quaternions do not have the issue of Euler angles

=> no singularity

Quaternions

From quaternions to rotation matrices and back

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Quaternions

From quaternions to rotation matrices and back

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$$\begin{aligned} q_0 &= \frac{1}{2}\sqrt{1 + r_{11} + r_{22} + r_{33}}, \\ \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} &= \frac{1}{4q_0} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - 2_{12} \end{bmatrix}. \end{aligned}$$

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$$R = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1 q_2 - q_0 q_3) & 2(q_0 q_2 + q_1 q_3) \\ 2(q_0 q_3 + q_1 q_2) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2 q_3 - q_0 q_1) \\ 2(q_1 q_3 - q_0 q_2) & 2(q_0 q_1 + q_2 q_3) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Quaternions

Multiplication of quaternions (non-trivial!)

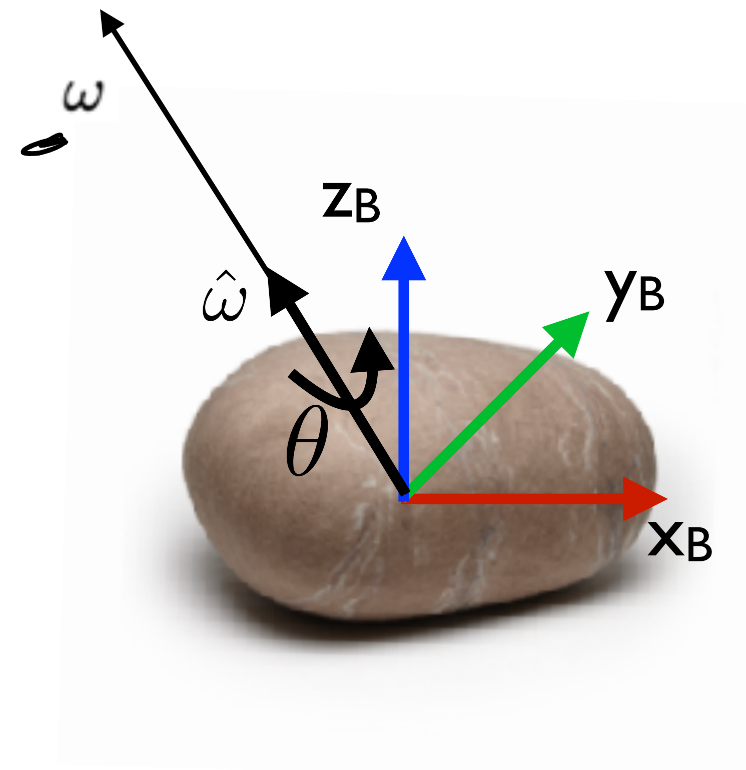
Quaternions

Multiplication of quaternions (non-trivial!)

$$q \cdot p = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \cdot \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3 \\ q_0p_1 + p_0q_1 + q_2p_3 - q_3p_2 \\ q_0p_2 + p_0q_2 - q_1p_3 + q_3p_1 \\ q_0p_3 + p_0q_3 + q_1p_2 - q_2p_1 \end{bmatrix}$$

Axis-angle

Every rotation can be defined by a (unitary) axis of rotation ω and an angle of rotation θ (instead of a composition of 3 rotations)



$\|\hat{\omega}\| = 1$

$\omega = \theta \cdot \hat{\omega}$

$\|\omega\| = \theta$

Axis-angle

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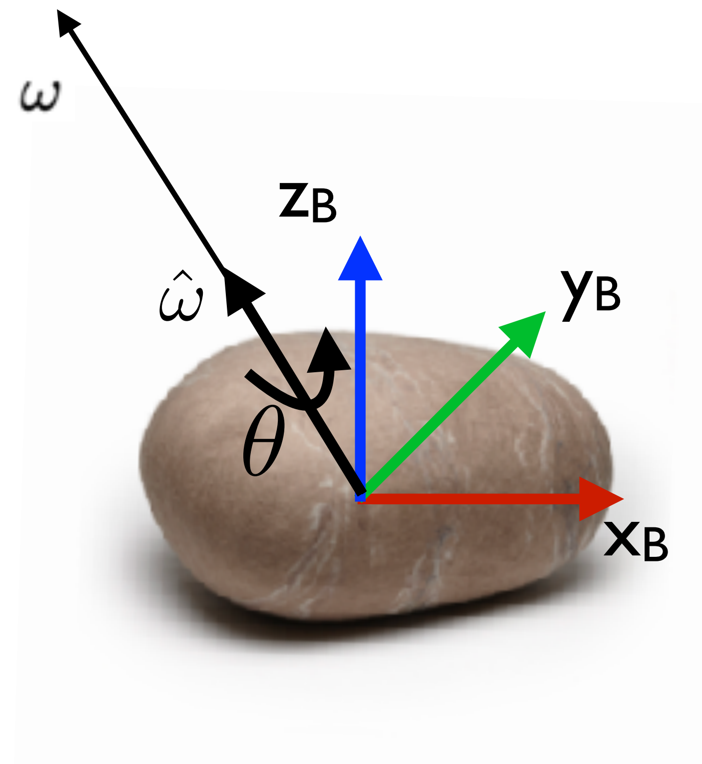
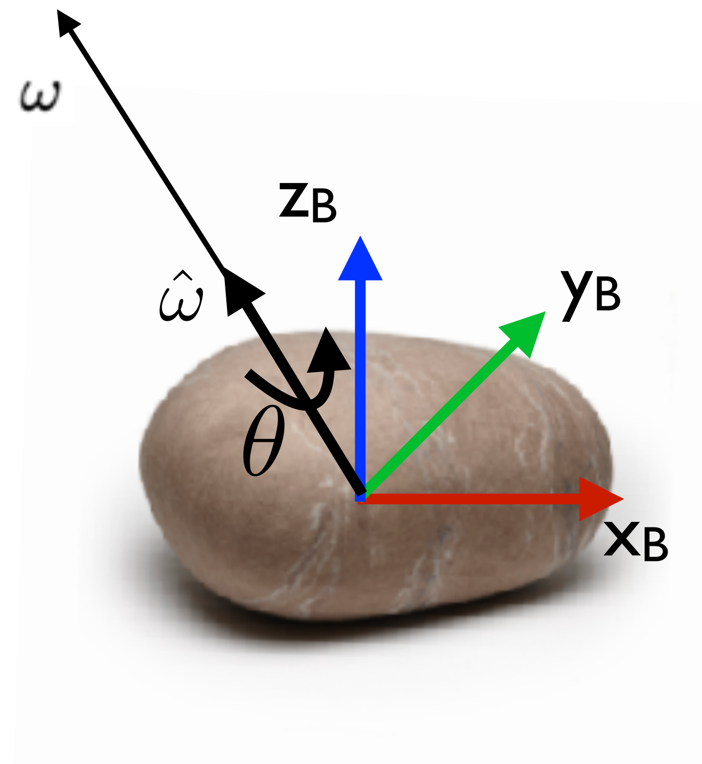


How do we find the associated rotation matrix?

Axis-angle

Every rotation can be defined by a (unitary) axis of rotation ω and an angle of rotation θ (instead of a composition of 3 rotations)

$\theta = 0$



How do we find the associated rotation matrix?

Singularity at 0 (for 0 angle, any axis of rotation can be chosen)
But velocities are well-defined (no loss of degrees of freedom)

Cross products

$$a \times b = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = \begin{bmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{bmatrix} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$a \times b = ||a|| \cdot ||b|| \cdot \hat{n} \cdot \sin\theta$$

Notation: $$[a] = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$a \times b = [a]b$$

Cross products: properties

$a \times a = 0$

$a \times b = -b \times a$

$a \times (b + c) = (a \times b) + (a \times c)$

$(ra) \times b = r(a \times b)$ where r is a scalar

$a \times (b \times c) + b \times (c \times a) + c \times (a \times b) = 0$ Jacobi identity