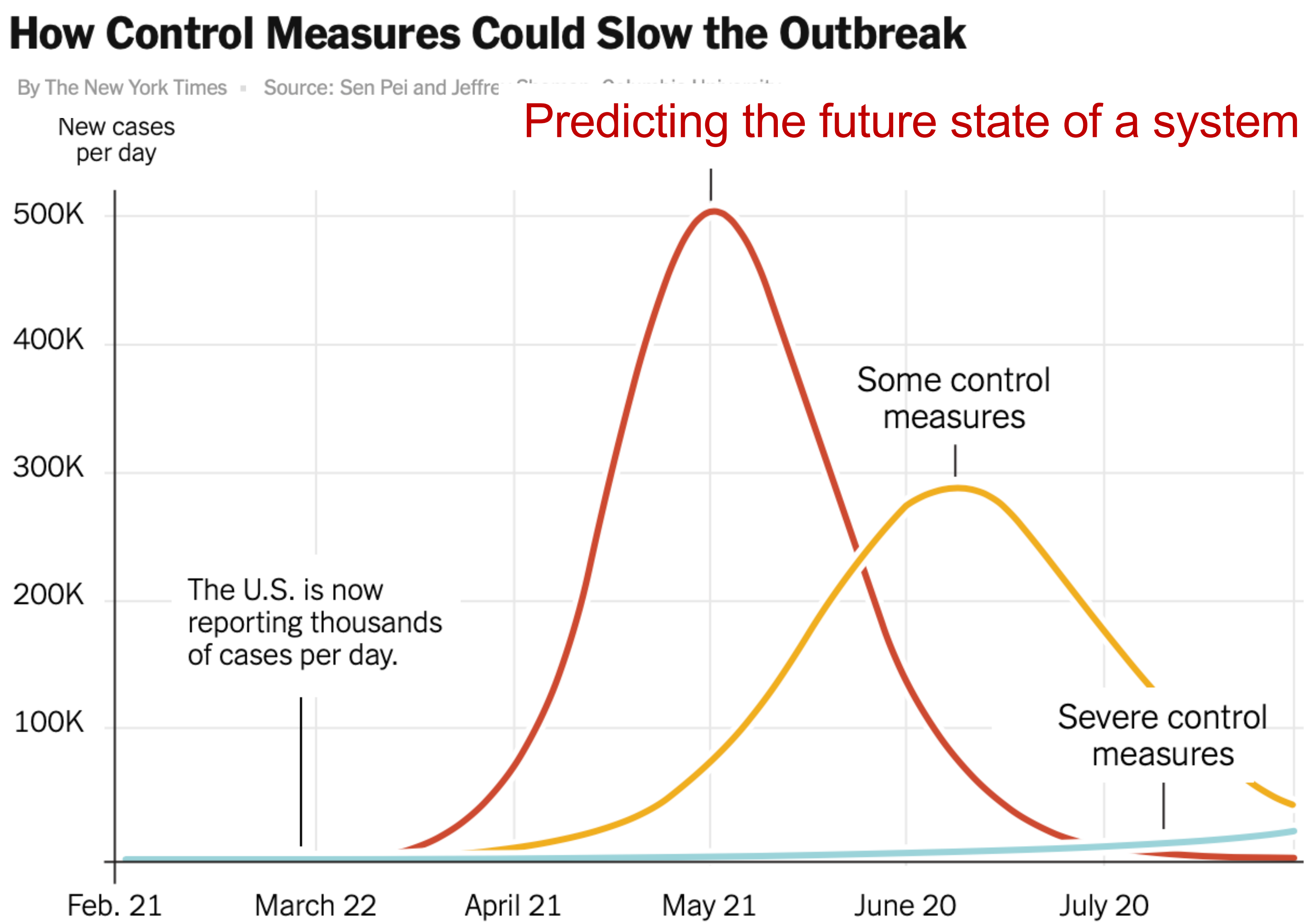
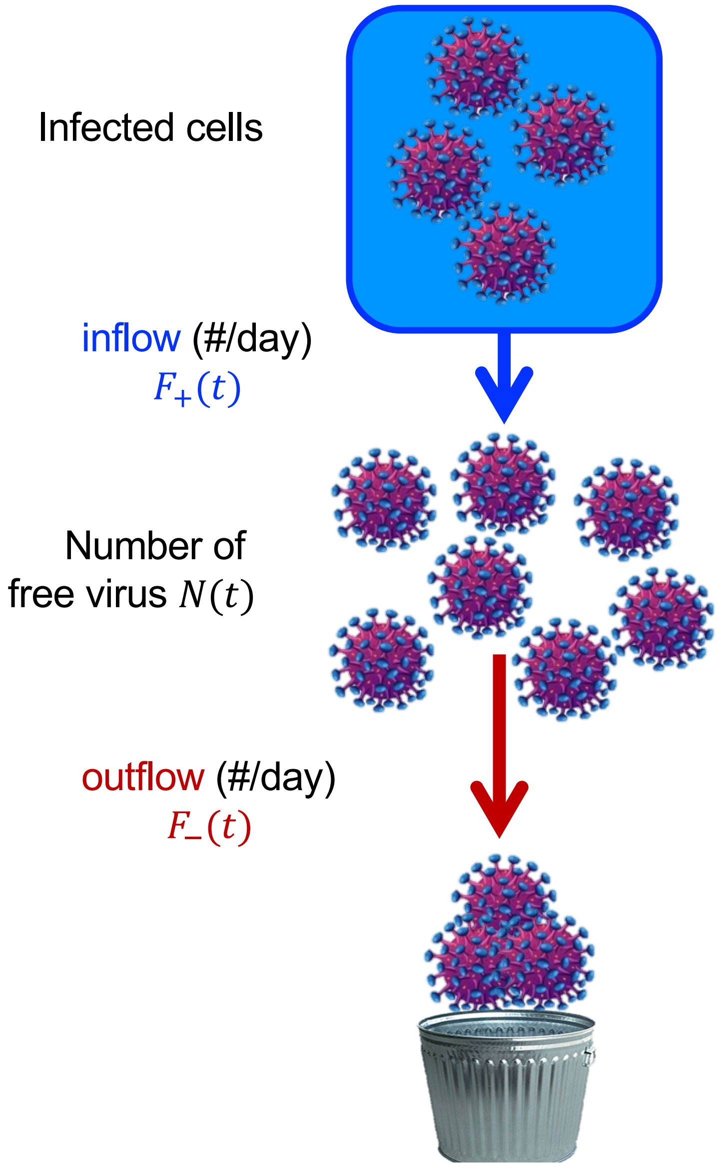
Predicting the future state of a system



Predicting the future state of a system



Rate of change of $N(t)$
will be given by

$$\frac{dN}{dt} = F_+(t) - F_-(t)$$

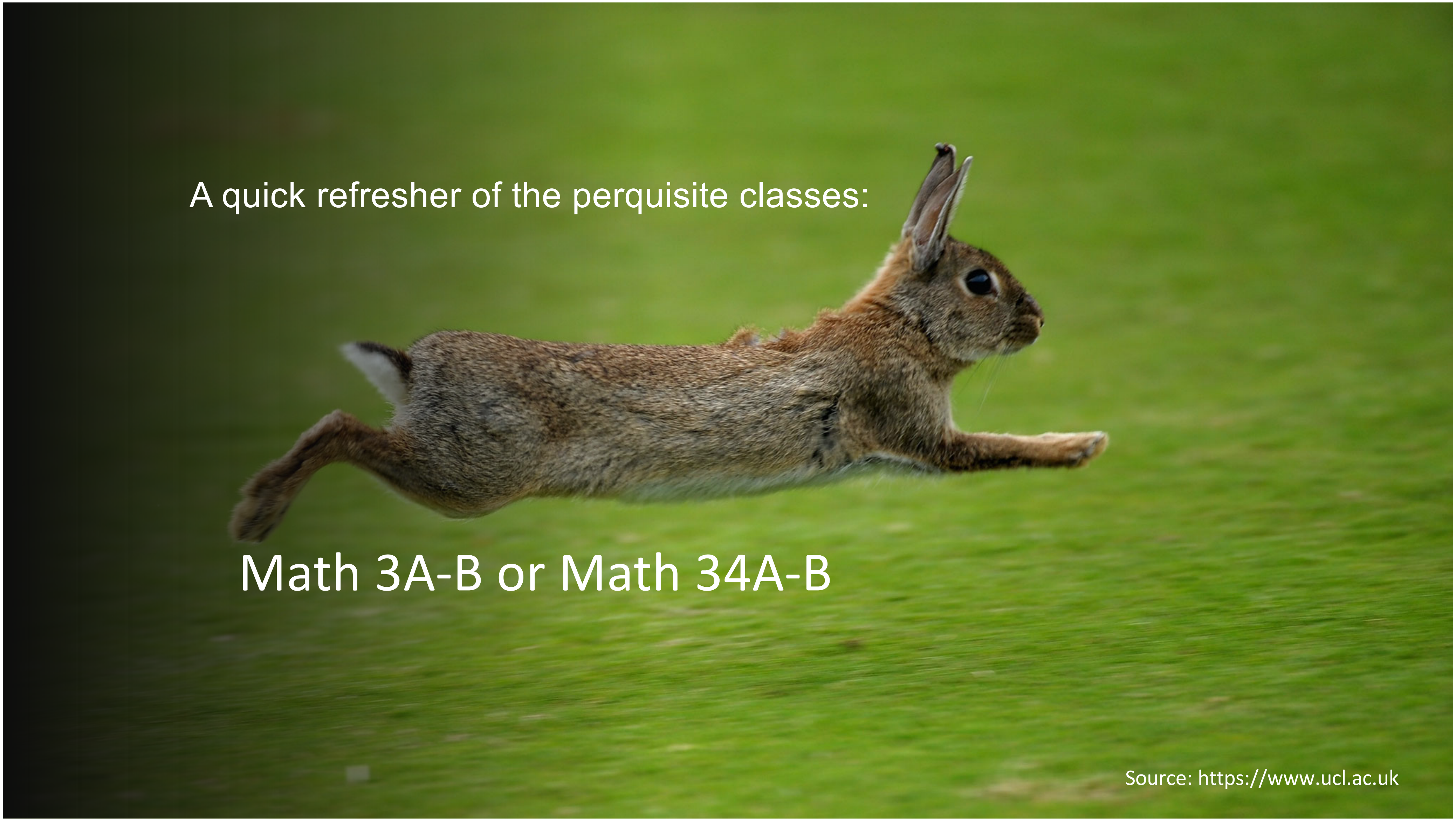
$$N(T)$$

Prediction of the number of virus
at a time T

$$\frac{\Delta x}{\Delta y}$$

$$\int_0^T f(s)\,ds$$

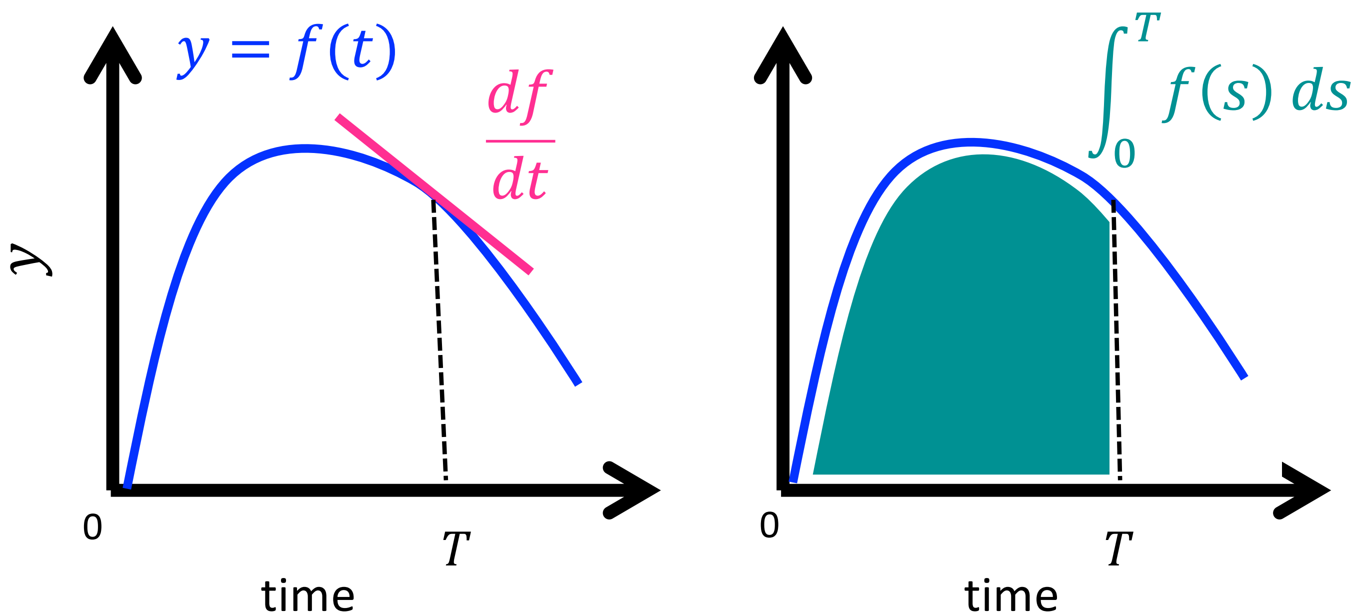
$$\frac{df}{dt}$$


A quick refresher of the perquisite classes:
Math 3A-B or Math 34A-B
Source: https://www.ucl.ac.uk

derivative $\frac{df}{dt} \Leftrightarrow$ integral $\int f(s)ds$ [!]

Derivative: "slope of tangent to the curve"

Integral: "area under the curve"



Intuitive understanding of $\frac{df}{dt}$ versus $\int ds$

<u>The coach's instructions:</u> "*Mickey, this is a 10-mile race, not a sprint race. Okay? Go steady. Set yourself a pace and maintain it until the end of the race.*"

Mickey, the rising star of the team

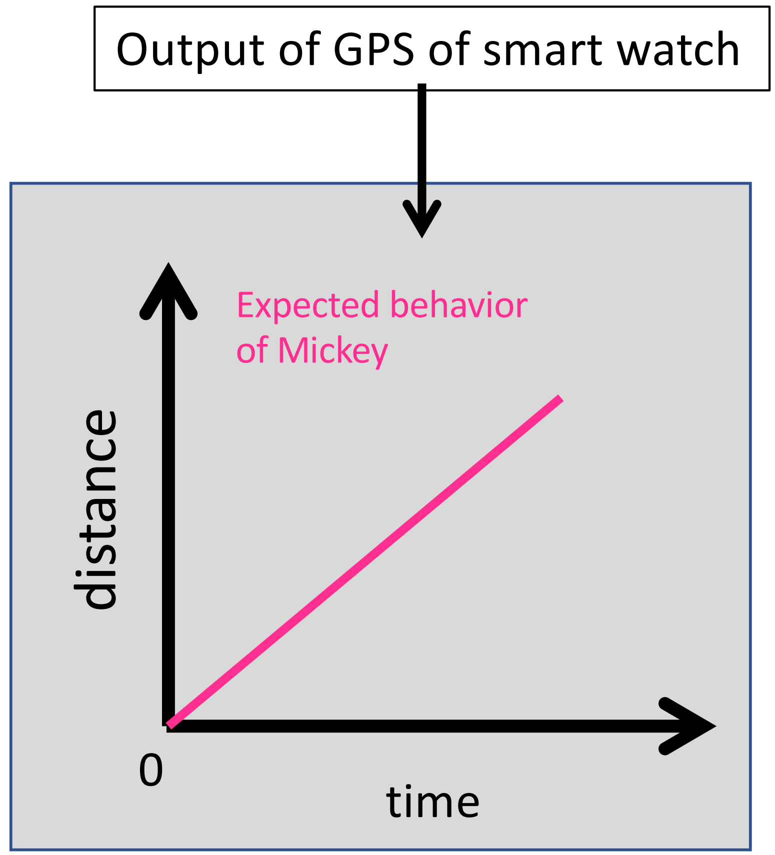
Results:

10 miles

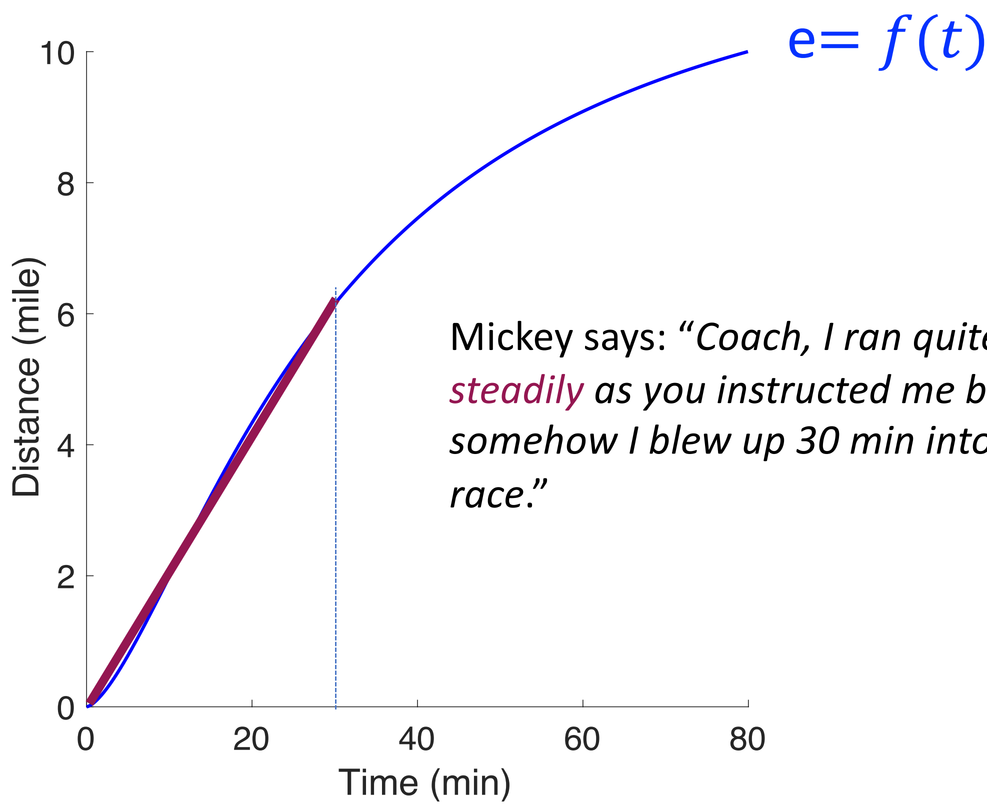
80 min

position: 213/215

The coach's analysis of the race



Graph: distance as a function of time



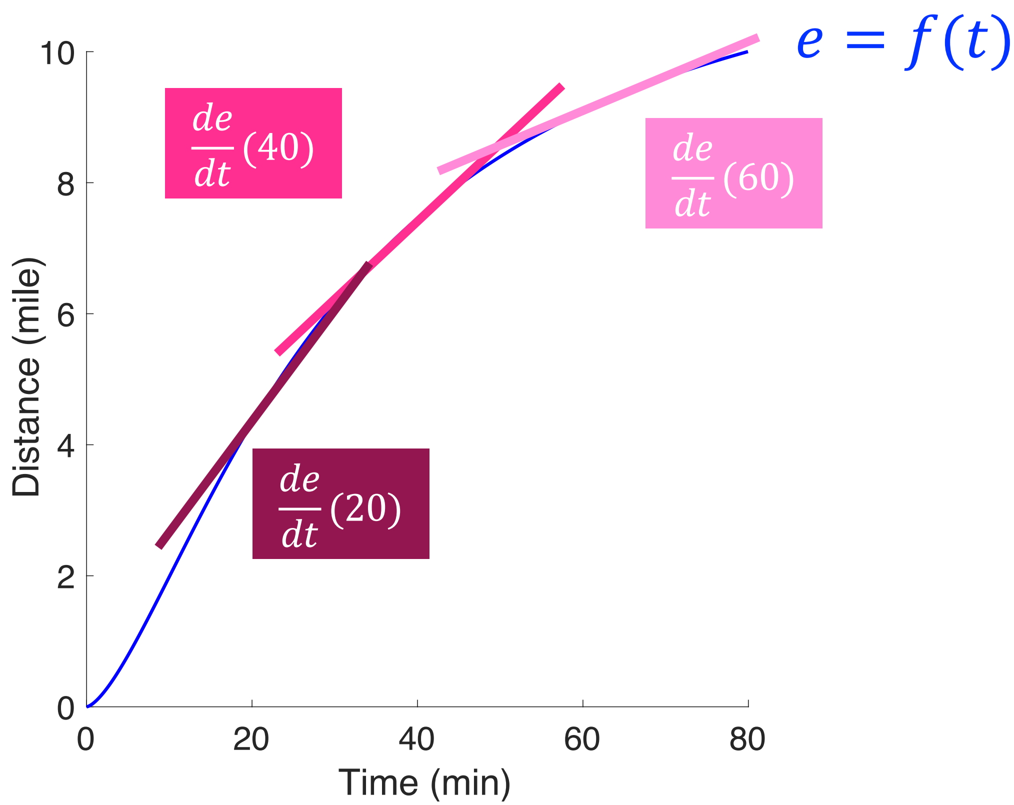
Mickey says: "*Coach, I ran quite steadily as you instructed me but somehow I blew up 30 min into the race.*"

Was the pace indeed constant?

speed/pace

[!] $$v(t) = \frac{de}{dt}$$

slope of the
tangent at
a given point



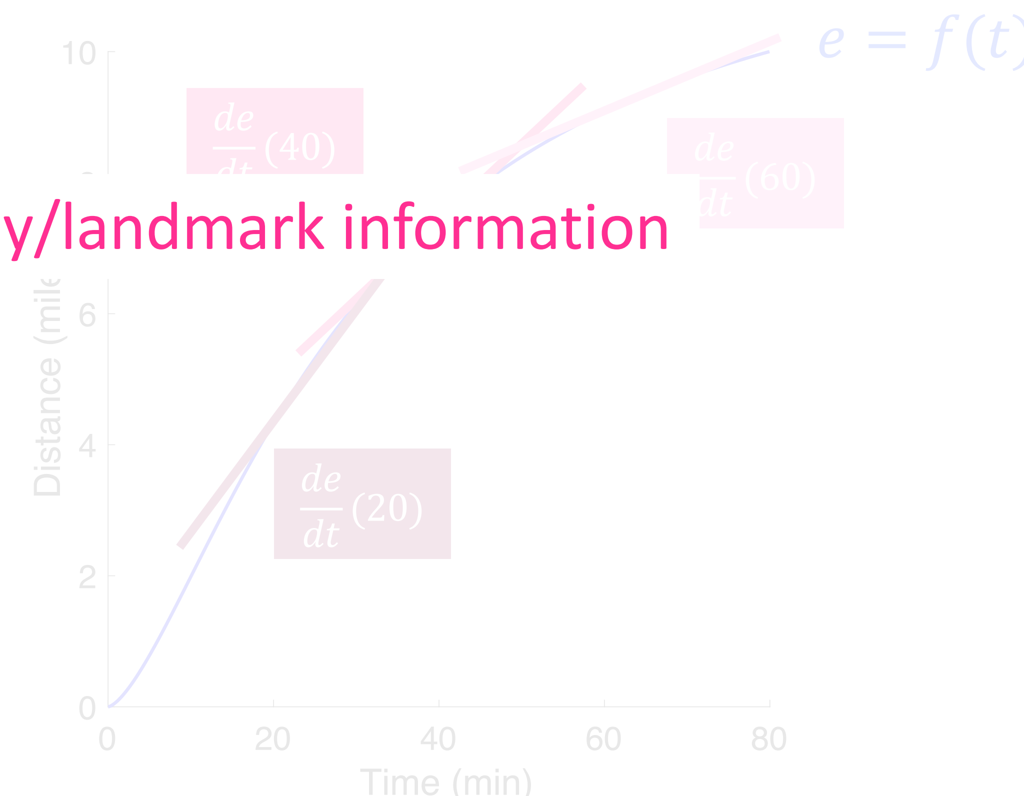
Was the pace indeed constant?

speed/pace

[!] $v(t) = \frac{de}{dt}$

Key/landmark information

slope of the
tangent at
a given point

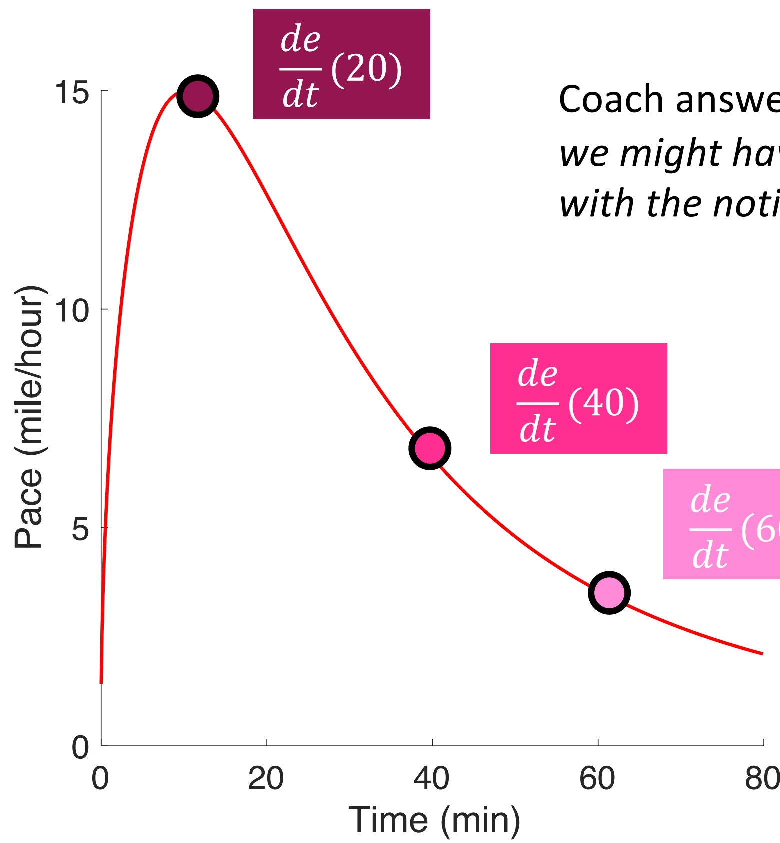


Graph: speed (derivative) as a function of time

Speed/pace

[!] $v(t) = \frac{de}{dt}$

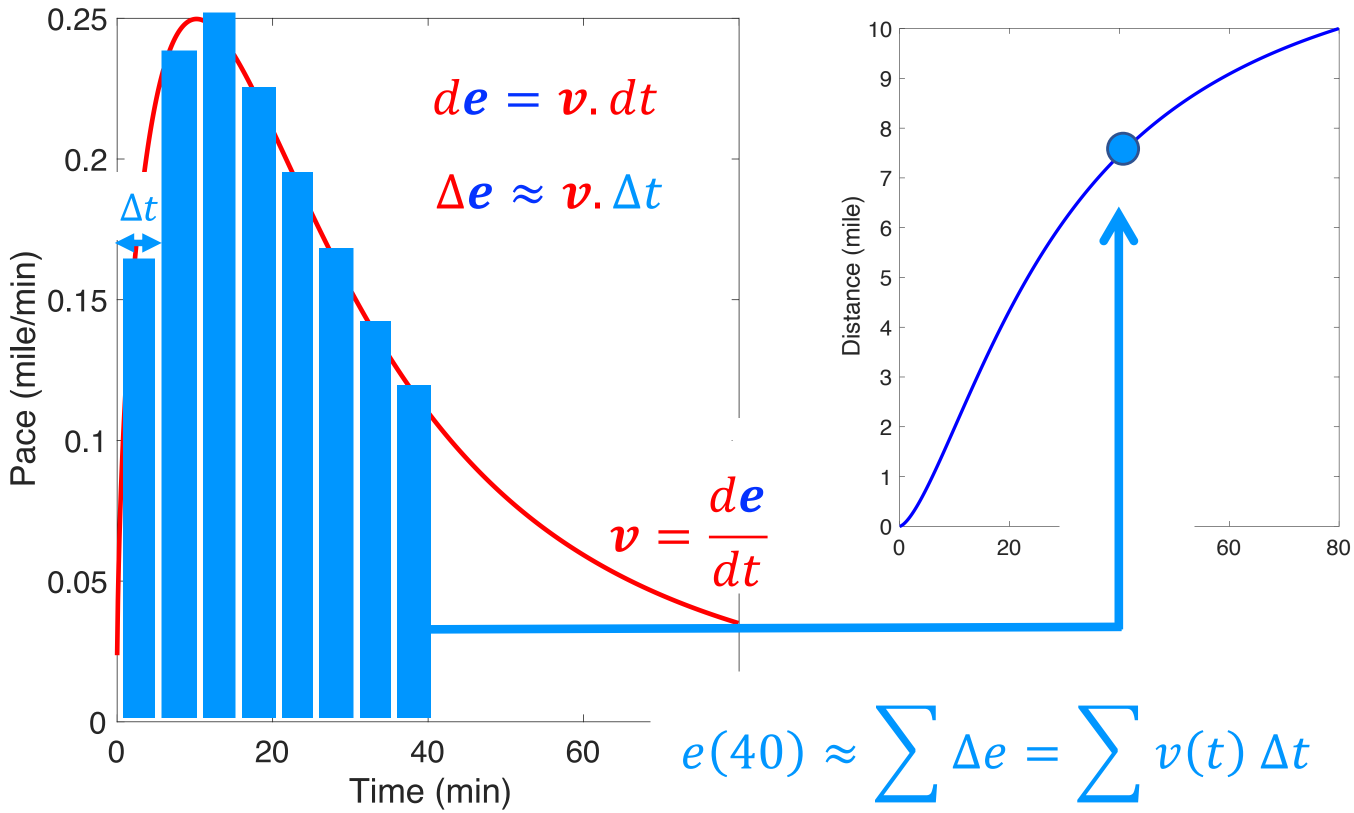
$e = f(t)$



Coach answers: "*Well, Mickey, we might have some work to do with the notion of steady pace.*"

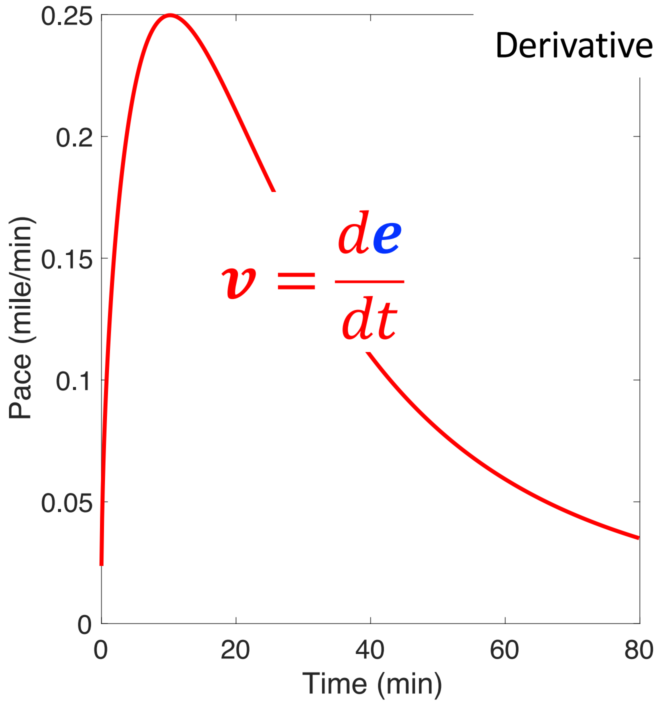
$$v = \frac{de}{dt} = \frac{df}{dt}$$

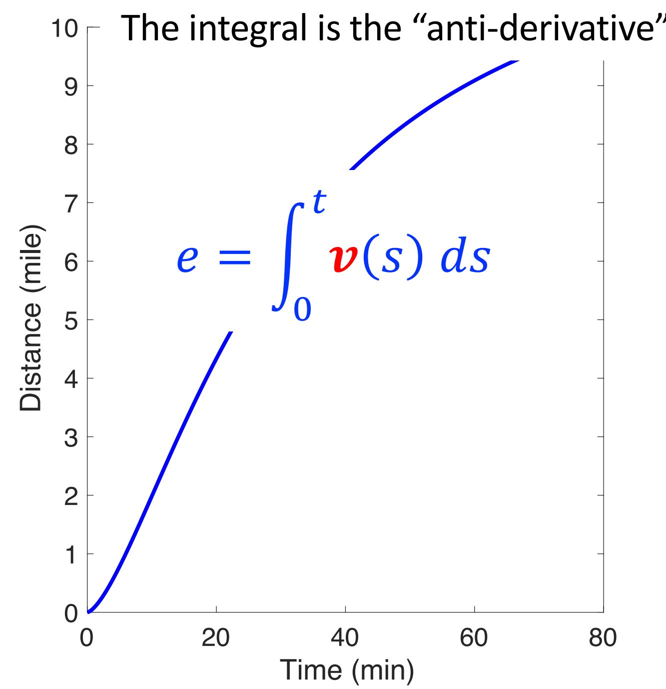
From a derivative $\frac{de}{dt}$ to its primitive $e(t)$



$$e(40) \approx \sum \Delta e = \sum v(t)\,\Delta t$$

derivative $\frac{df}{dt} \Leftrightarrow$ integral $\int f(s)ds$ [!]



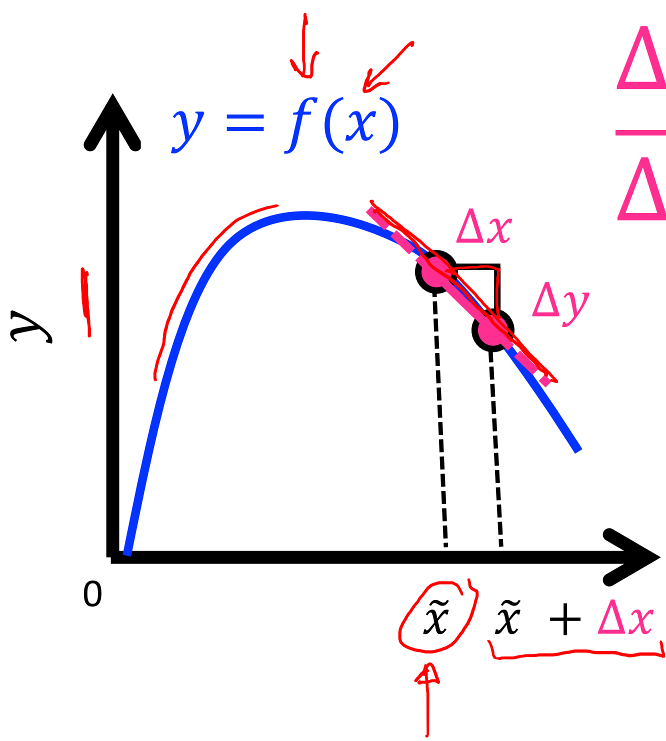


More about derivatives

Leonard Euler
(1707-1783)

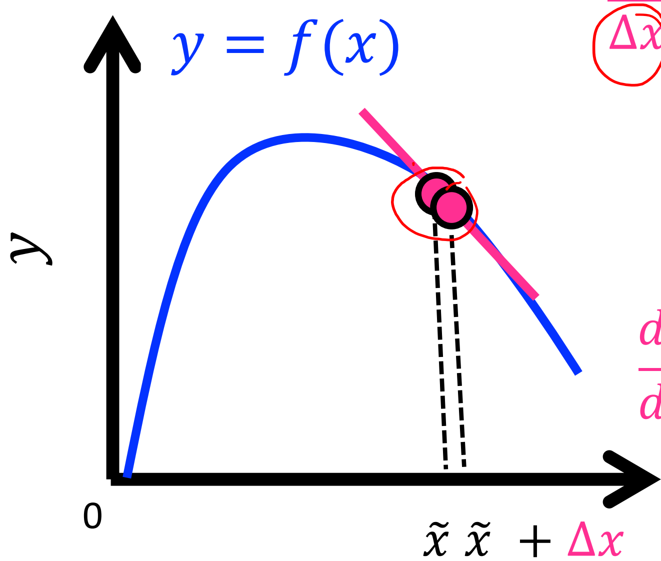
More about derivatives

The slope of the **secant** of curve $f(x)$ at point $\tilde{x}$ is:



$$\frac{\Delta y}{\Delta x} = \frac{f(\tilde{x}+\Delta x) - f(\tilde{x})}{\Delta x}$$

From a function to its derivative



The slope of the **secant** of curve $f(x)$ at point $\tilde{x}$ is:

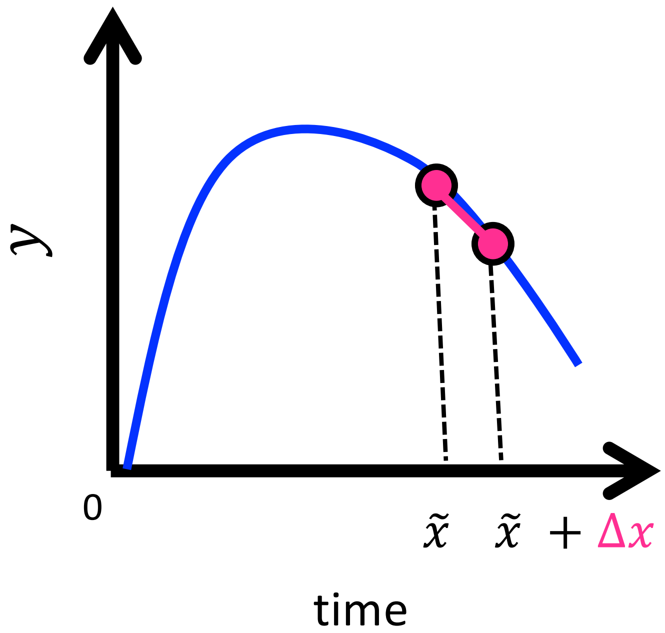
$$\frac{\Delta y}{\Delta x} = \frac{f(\tilde{x} + \Delta x) - f(\tilde{x})}{\Delta x}$$

The derivative of $f(x)$ at point $\tilde{x}$ is the slope of the tangent to the curve:

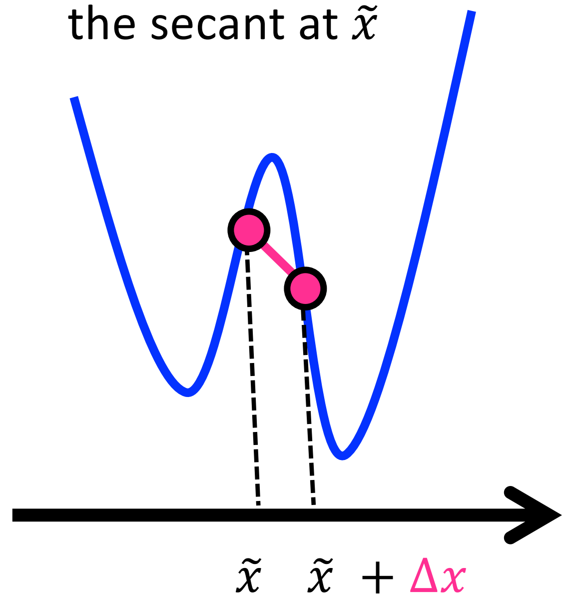
$$\frac{df}{dx}(\tilde{x}) = \lim_{\Delta x \to 0} \frac{f(\tilde{x} + \Delta x) - f(\tilde{x})}{\Delta x} \quad [!]$$

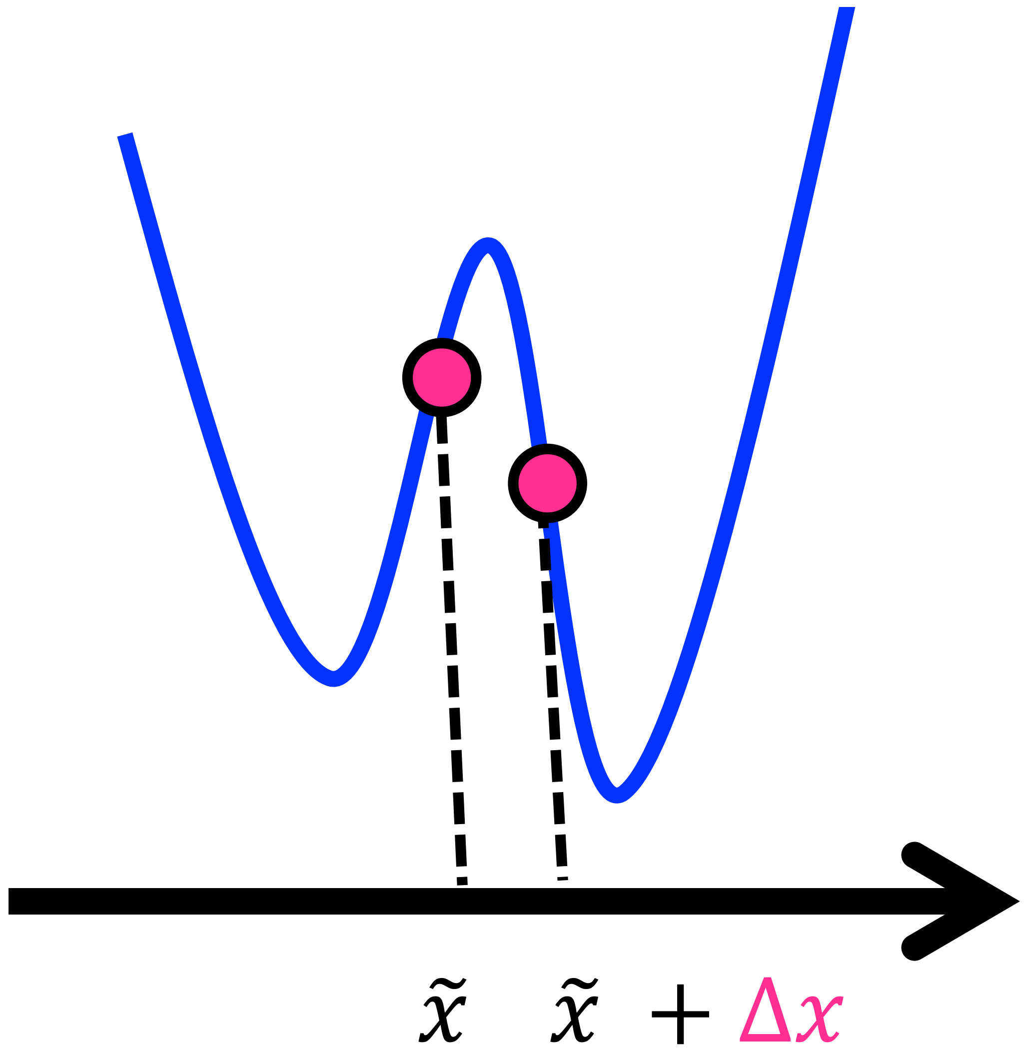
Limit $\Delta x \rightarrow 0$ is needed to avoid accidents

Here the slope is **well**-estimated by the secant at $\tilde{x}$



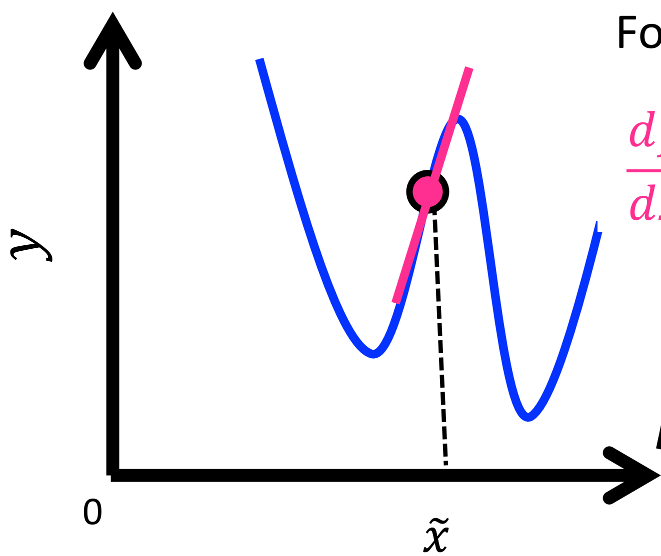
Here the slope is **badly**-estimated by the secant at $\tilde{x}$




$\tilde{x}$
$\tilde{x} + \Delta x$

From a derivative to its "primitive"

$$\frac{df}{dx}(\tilde{x}) = \lim_{\Delta x \to 0} \frac{f(\tilde{x} + \Delta x) - f(\tilde{x})}{\Delta x}$$



For Δx "small enough"…

$$\frac{df}{dx}(\tilde{x}) \approx \frac{f(\tilde{x} + \Delta x) - f(\tilde{x})}{\Delta x}$$

Isolate $f(\tilde{x} + \Delta x)$ as a function of the other terms

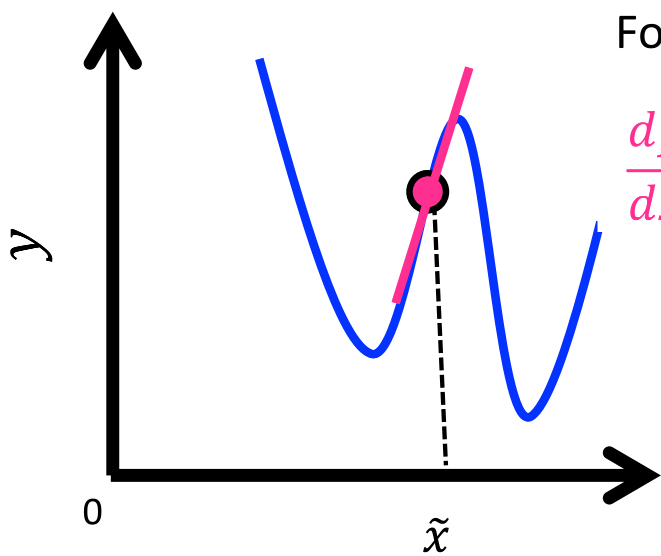
$$f(\tilde{x} + \Delta x) = \cdots$$

From a derivative to its “primitive”

$$\frac{df}{dx}(\tilde{x}) = \lim_{\Delta x \to 0} \frac{f(\tilde{x} + \Delta x) - f(\tilde{x})}{\Delta x}$$

For Δx “small enough”...

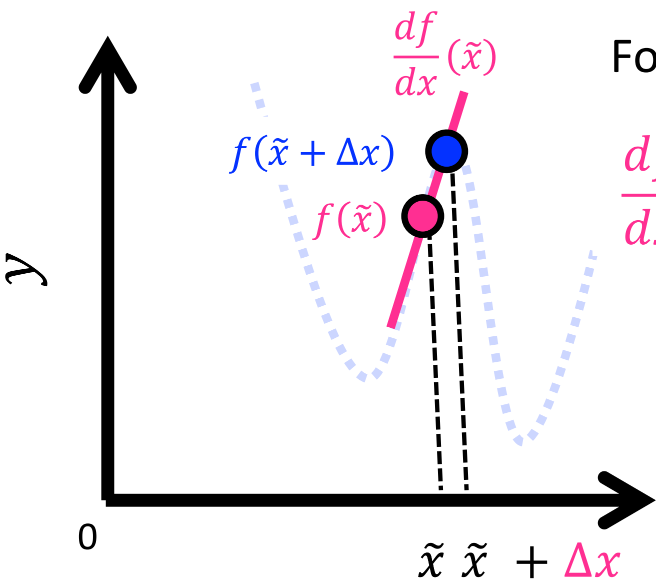
$$\frac{df}{dx}(\tilde{x}) \approx \frac{f(\tilde{x} + \Delta x) - f(\tilde{x})}{\Delta x}$$



$$f(\tilde{x} + \Delta x) = \cdots$$

From a derivative to its “primitive”

$$\frac{df}{dx}(\tilde{x}) = \lim_{\Delta x \to 0} \frac{f(\tilde{x}+\Delta x) - f(\tilde{x})}{\Delta x}$$



For Δx “small enough”…

$$\frac{df}{dx}(\tilde{x}) \approx \frac{f(\tilde{x}+\Delta x) - f(\tilde{x})}{\Delta x}$$

$$f(\tilde{x}+\Delta x) \approx f(\tilde{x}) + \frac{df}{dx}(\tilde{x}).\Delta x \quad [!]$$

prediction of future value of f

present value (known) of f

derivative of f

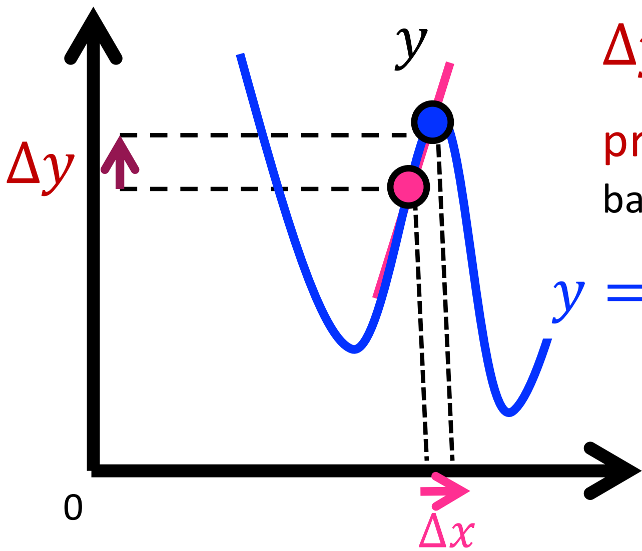
From a derivative to its "primitive"

$$\color{blue}{f(x+\Delta x)} \approx \underbrace{f(x)}_{y} + \underbrace{\frac{df}{dx}(x).\Delta x}_{\Delta y}$$

$$\Delta y = \frac{df}{dx}(x).\Delta x$$

predicted change

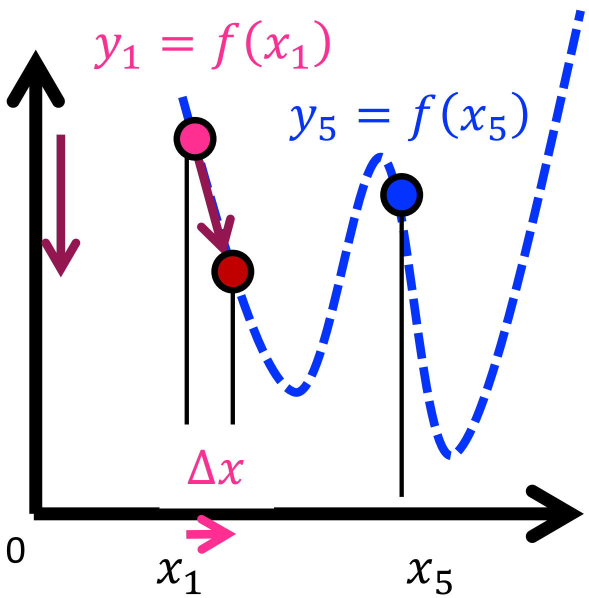
based on the secant (linear approximation)



$y = f(x)$

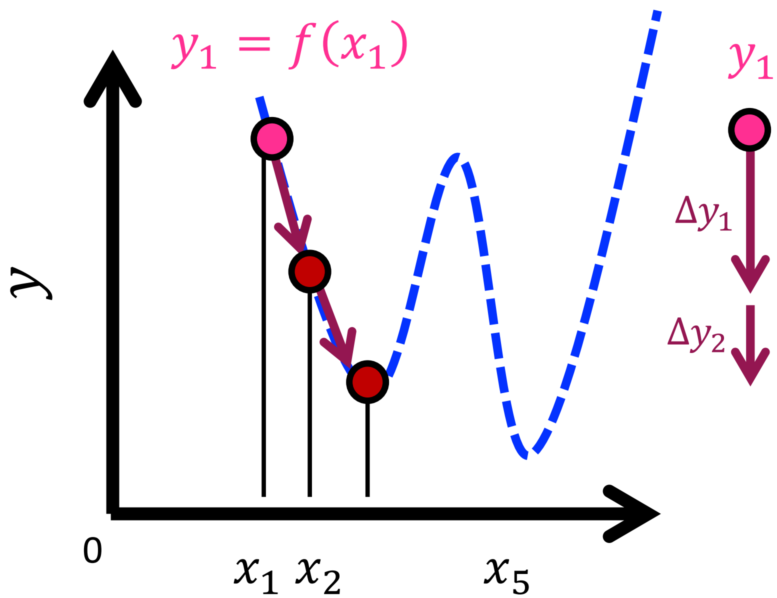
Combining linear approximations

$$f(x + \Delta x) \approx f(x) + \frac{df}{dx}(x).\Delta x = y + \Delta y$$



Combining linear approximations

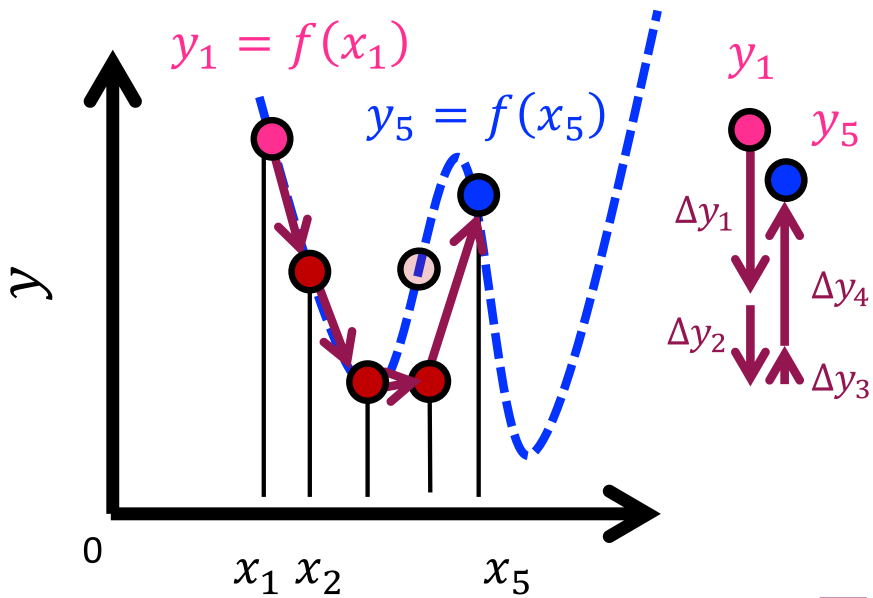
$$f(x + \Delta x) \approx f(x) + \frac{df}{dx}(x).\Delta x = y + \Delta y$$



$$f(x_1)$$
$$+\frac{df}{dx}(x_1).\Delta x$$
$$+\frac{df}{dx}(x_2).\Delta x$$

Combining linear approximations

$$f(x + \Delta x) \approx f(x) + \frac{df}{dx}(x).\Delta x = y + \Delta y$$



$$f(x_5) = f(x_1)$$

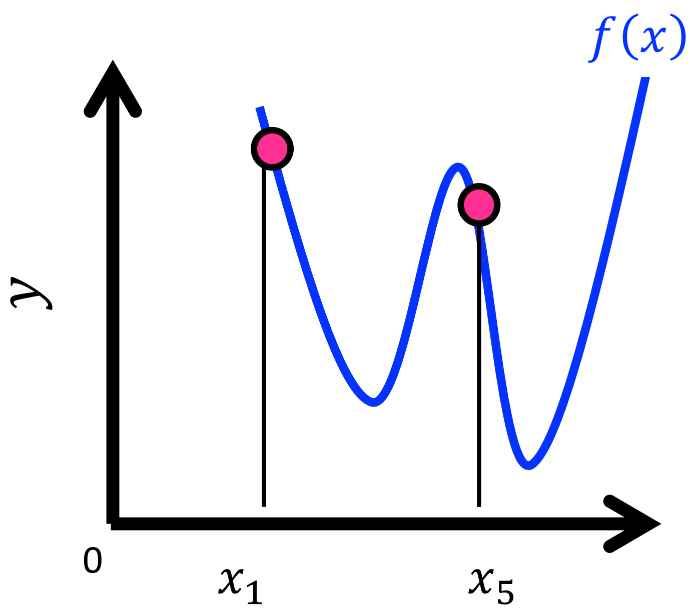
$$+ \frac{df}{dx}(x_1).\Delta x$$

$$+ \frac{df}{dx}(x_2).\Delta x$$

$$+ \frac{df}{dx}(x_3).\Delta x$$

$$\sum \frac{df}{dx}\Delta x \quad + \frac{df}{dx}(x_4).\Delta x$$

From a derivative to its "primitive"



$$f(x_5) \approx f(x_1) + \sum \frac{df}{dx}\Delta x$$

In the limit where $\Delta x \rightarrow 0$:

$$f(x_5) = f(x_1) + \int_{x_1}^{x_5} \frac{df}{dx} dx$$

$$f(x_5) - f(x_1) = \int_{x_1}^{x_5} \frac{df}{dx} dx$$

(fundamental theorem of calculus)

Think algorithmically

Numerical integration strategy

Goal: define $f(x)$ between x_i and x_f

Formula to be applied: $f(x+\Delta x) \approx f(x) + g(x).\Delta x$

Leonard Euler
(1707-1783)

I did this without having access to a computer!

Given:

$$\frac{df}{dx} = g(x)$$

lower boundary: x_i

higher boundary: x_f

initial condition: $f(x_i)$

Integration step: Δx

Number integration steps: N_{step}

[!] Numerical integration strategy

Think algorithmically

Goal: define $f(x)$ between x_i and x_f

Formula to be applied: $f(x+\Delta x) \approx f(x) + g(x).\Delta x$

1. Start from the initial condition:

$$x = x_i$$

2. Define the step size: $\Delta x = \dfrac{(x_f - x_i)}{N_{step}}$

3. Apply approximation & record prediction:

$$f(x+\Delta x) = f(x) + g(x).\Delta x$$

4. Update x : $x + \Delta x \rightarrow x$

5. Repeat steps 3 & 4 for N_{step} times

Given:

$$\frac{df}{dx} = g(x)$$

lower boundary: x_i

higher boundary: x_f

initial condition: $f(x_i)$

Integration step: Δx

Number integration steps: N_{step}