Exercise 12 – Bayesian inference and Data assimilation

Group : Duckies

1.1

The variance of the sample mean $\bar{g}_X$ can be calculated as follows:

Since X_i are iid samples from the distribution π, $g(X_i)$ are also iid. Let $\sigma^2 = Var(g(X))$. Then, by the properties of variance, we have:

$$Var(\bar{g}_X) = Var\left(\frac{1}{N}\sum_{i=1}^{N} g\left(X_i\right)\right) = \frac{1}{N^2}Var\left(\sum_{i=1}^{N} g\left(X_i\right)\right) = \frac{1}{N^2}\sum_{i=1}^{N} Var\left(g\left(X_i\right)\right) = \frac{N\sigma^2}{N^2} = \frac{\sigma^2}{N}$$

So, the variance of the sample mean $\bar{g}_X$ is $\frac{\sigma^2}{N}$, where σ^2 is the variance of $g(X)$.|

1.2

This is a result of the importance sampling technique. The idea is to estimate the expected value of $g(X)$ with respect to the distribution π by using samples from another distribution π'.

We can derive the result as follows:

$$\pi[g] = \mathbb{E}_\pi[g(X)] = \int g(x)\pi(x)dx = \int g(x)\frac{\pi(x)}{\pi'(x)}\pi'(x)dx = \mathbb{E}_{\pi'}\left[\frac{\pi(Y)}{\pi'(Y)}g(Y)\right]$$

So, we have shown that $\pi[g] = \mathbb{E}\left[\frac{\pi(Y)}{\pi'(Y)}g(Y)\right]$.

1.3

The variance of the new sample mean $\bar{g}_Y$ can be calculated as follows:

Let $Z_i = \frac{\pi(Y_i)}{\pi'(Y_i)}g(Y_i)$. Since Y_i are iid samples from the distribution π', Z_i are also iid. Let $\sigma_Z^2 = Var(Z_i)$. Then, by the properties of variance, we have:

$$Var(\bar{g}_Y) = Var\left(\frac{1}{N}\sum_{i=1}^{N} Z_i\right) = \frac{1}{N^2}Var\left(\sum_{i=1}^{N} Z_i\right) = \frac{1}{N^2}\sum_{i=1}^{N} Var\left(Z_i\right) = \frac{N\sigma_Z^2}{N^2} = \frac{\sigma_Z^2}{N}$$

So, the variance of the new sample mean $\bar{g}_Y$ is $\frac{\sigma_Z^2}{N}$, where σ_Z^2 is the variance of Z_i.

1.4

Let's denote $h(Y) = \frac{\pi(Y)}{\pi'(Y)} g(Y)$. Then, the variance of $\bar{g}_Y$ can be expressed as:

$$Var(\bar{g}_Y) = \frac{1}{N} Var_{\pi'}(h(Y)) = \frac{1}{N}\left(\mathbb{E}_{\pi'}[h(Y)^2] - \mathbb{E}_{\pi'}[h(Y)]^2\right)$$

Since $\mathbb{E}_{\pi'}[h(Y)] = \pi[g]$ is a constant, we can minimize the variance of $\bar{g}_Y$ by minimizing $\mathbb{E}_{\pi'}[h(Y)^2]$.

Let's expand the expression for $\mathbb{E}_{\pi'}[h(Y)^2]$:

$$\mathbb{E}_{\pi'}[h(Y)^2] = \int h(y)^2 \pi'(y) dy = \int \left(\frac{\pi(y)}{\pi'(y)} g(y)\right)^2 \pi'(y) dy = \int \frac{\pi(y)^2}{\pi'(y)} g(y)^2 dy$$

To minimize this expression, we can use the Cauchy-Schwarz inequality:

$$\int \frac{\pi(y)^2}{\pi'(y)} g(y)^2 dy \geq \left(\int \sqrt{\frac{\pi(y)^2}{\pi'(y)}} |g(y)| \sqrt{\pi'(y)} dy\right)^2 = \left(\int \pi(y)|g(y)|dy\right)^2$$

The equality holds when $\sqrt{\frac{\pi(y)^2}{\pi'(y)}} |g(y)| = k\sqrt{\pi'(y)}$ for some constant k. Solving for $\pi'(y)$, we get:

$$\begin{aligned} \sqrt{\frac{\pi(y)^2}{\pi'(y)}} |g(y)| &= k\sqrt{\pi'(y)} \\ \Leftrightarrow \frac{\pi(y)^2}{\pi'(y)} g(y)^2 &= k^2 \pi'(y) \\ \Leftrightarrow \pi'(y) &= \frac{\pi(y)^2 g(y)^2}{k^2} \end{aligned}$$

Since $\int \pi'(y) dy = 1$, we have:

$$k^2 = \int \frac{k^2 \pi(y)^2 g(y)^2}{k^2} dy = \int \pi(y)^2 g(y)^2 dy$$

So, the optimal choice of π' that minimizes the variance of $\bar{g}_Y$ is:

$$\boxed{\pi'(y) = \frac{\pi(y)^2 g(y)^2}{\int \pi(x)^2 g(x)^2 dx}}$$

In practice, however, it might not be possible to achieve this optimal choice of π', and one would need to propose a suitable distribution that can represent $\pi[g]$ well or is easy to sample from.

1.5

Here is a pseudo-code for the importance resampling algorithm to generate N samples from π:

```
1. Generate N samples Y_i from the proposal distribution pi'
2. Calculate the unnormalized weights tilde_w_i = C * pi(Y_i) / pi'(Y_i)
for i = 1,...,N
3. Normalize the weights w_i = tilde_w_i / sum(tilde_w_i) for i =
1,...,N
```

```
4. Resample X_i from Y_i with probability mass function w_i for i =
1,...,N
5. Return the resampled values X_i for i = 1,...,N
```

This algorithm generates N samples Y_i from the proposal distribution π', calculates the importance weights w_i, and then resamples X_i from Y_i with probability mass function w_i. The resulting samples X_i can be used to approximate the distribution π.

2.1

Here is a pseudo-code for implementing the particle filter algorithm:

```
1. Initialize N particles x_0^i from the prior distribution p_0 for i =
1,...,N
2. Set the initial weights w_0^i = 1/N for i = 1,...,N
3. for n = 1 to T do
4.     for i = 1 to N do
5.         Sample x_n^i from the transition kernel p_n(x_n | x_{n-1}^i)
6.         Update the weight w_n^i = w_{n-1}^i * q_n(y_n | x_n^i)
7.     end for
8.     Normalize the weights w_n^i = w_n^i / sum(w_n^i) for i = 1,...,N
9.     Resample (x_n^i, w_n^i) with probability mass function w_n^i for
i = 1,...,N
10.     Set the weights of the resampled particles to w_n^i = 1/N for i
= 1,...,N
11. end for
12. Return (x_T^i, w_T^i) for i = 1,...,N
```

This algorithm generates N particles x_0^i from the prior distribution p_0 and sets their initial weights to $w_0^i = 1/N$. Then, it iterates over time from $n = 1$ to T. At each time step, it samples new particles x_n^i from the transition kernel $p_n(x_n|x_{n-1}^i)$ and updates their weights using the observation likelihood $q_n(y_n|x_n^i)$. The weights are then normalized and the particles are resampled with replacement according to their weights. The weights of the resampled particles are reset to $w_n^i = 1/N$. The final output is the set of particles and their weights at time T, which approximates the posterior distribution $\pi_{X_T|Y_{1:T}}$.

2.2

Here is an updated version of the pseudo-code that includes a resampling step based on the effective sample size (ESS):

```
1. Initialize N particles x_0^i from the prior distribution p_0 for i =
1,...,N
```

```
2. Set the initial weights w_0^i = 1/N for i = 1,...,N
3. for n = 1 to T do
4.     for i = 1 to N do
5.         Sample x_n^i from the transition kernel p_n(x_n | x_{n-1}^i)
6.         Update the weight w_n^i = w_{n-1}^i * q_n(y_n | x_n^i)
7.     end for
8.     Normalize the weights w_n^i = w_n^i / sum(w_n^i) for i = 1,...,N
9.     Calculate the effective sample size ESS = 1 / sum(w_n^i)^2
10.     if ESS < gamma * N then
11.         Resample (x_n^i, w_n^i) with probability mass function w_n^i
for i = 1,...,N
12.         Set the weights of the resampled particles to w_n^i = 1/N
for i = 1,...,N
13.     end if
14. end for
15. Return (x_T^i, w_T^i) for i = 1,...,N
```

This updated version of the algorithm includes an additional input parameter $\gamma \in (0, 1)$ that determines the threshold for resampling based on the effective sample size (ESS). At each time step, after updating and normalizing the weights, the algorithm calculates the ESS as $ESS = 1/\sum(w_n^i)^2$. If the ESS is less than γN, then the particles are resampled with replacement according to their weights and their weights are reset to $w_n^i = 1/N$. This resampling step helps to prevent weight degeneracy and maintain a large effective sample size.

3.1

Here is a Python code that implements the importance resampling algorithm to generate 10,000 samples of X from the given mixture of Gaussian random variables using a Gaussian proposal distribution with $\sigma^2 = 1$ or $\sigma^2 = 4$:

In [2]:

```python
import numpy as np
from scipy.stats import norm

def importance_resampling(N, sigma2):
    # Set the proposal distribution pi' = N(0, sigma^2)
    pi_prime = norm(loc=0, scale=np.sqrt(sigma2))

    # Generate N samples Y_i from the proposal distribution pi'
    Y = pi_prime.rvs(size=N)

    # Define the target distribution pi(x)
    def pi(x):
```

```
        return 0.5 * norm.pdf(x, loc=-4, scale=1) + 0.5 * norm.pdf(x, loc=4,
scale=1)

    # Calculate the unnormalized weights tilde_w_i = pi(Y_i) / pi'(Y_i)
    tilde_w = pi(Y) / pi_prime.pdf(Y)

    # Normalize the weights w_i = tilde_w_i / sum(tilde_w_i)
    w = tilde_w / np.sum(tilde_w)

    # Resample X_i from Y_i with probability mass function w_i
    X = np.random.choice(Y, size=N, p=w)

    return X

# Generate 10,000 samples of X using a Gaussian proposal with sigma^2 = 1
X1 = importance_resampling(N=10000, sigma2=1)

# Generate 10,000 samples of X using a Gaussian proposal with sigma^2 = 4
X2 = importance_resampling(N=10000, sigma2=4)
```

This code defines a function `importance_resampling` that takes as input the number of samples `N` and the variance `sigma2` of the Gaussian proposal distribution. The function generates `N` samples `Y` from the Gaussian proposal distribution `pi_prime`, calculates the importance weights `w`, and then resamples `X` from `Y` with probability mass function `w`. The resulting samples `X` can be used to approximate the target distribution `pi(x)`. The code generates two sets of samples `X1` and `X2` using different proposal distributions with `sigma2 = 1` and `sigma2 = 4`, respectively.

3.2

Here is a Python code that plots the histograms of the samples generated using the importance resampling algorithm with different proposal distributions, and overlays them with the true density $\pi(x)$:

In [3]:

```
import matplotlib.pyplot as plt

# Define the target distribution pi(x)
def pi(x):
    return 0.5 * norm.pdf(x, loc=-4, scale=1) + 0.5 * norm.pdf(x, loc=4,
scale=1)

# Plot the histogram of the samples generated using a Gaussian proposal with
sigma^2 = 1
```

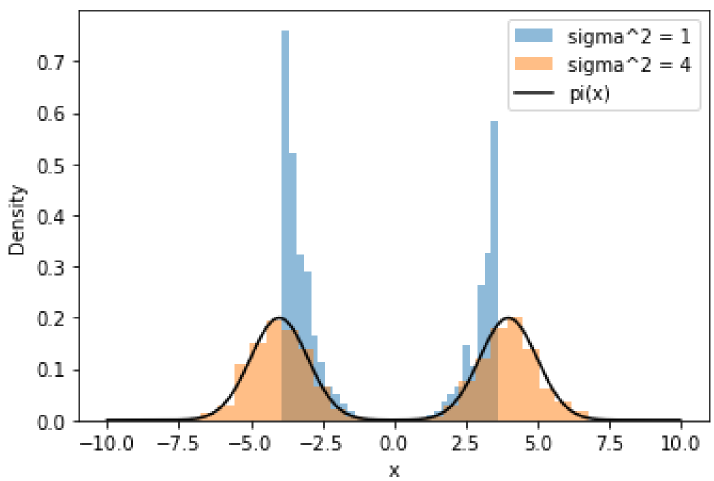
```
plt.hist(X1, bins=30, density=True, alpha=0.5, label='sigma^2 = 1')

# Plot the histogram of the samples generated using a Gaussian proposal with
sigma^2 = 4
plt.hist(X2, bins=30, density=True, alpha=0.5, label='sigma^2 = 4')

# Overlay the true density pi(x)
x = np.linspace(-10, 10, 1000)
plt.plot(x, pi(x), 'k', label='pi(x)')

# Add labels and legend
plt.xlabel('x')
plt.ylabel('Density')
plt.legend()

# Show the plot
plt.show()
```



This code uses the `matplotlib` library to plot the histograms of the samples `X1` and `X2` generated using different proposal distributions with `sigma2 = 1` and `sigma2 = 4`, respectively. It also plots the true density `pi(x)` as a black line on top of the histograms.

The choice of `sigma2` for the proposal distribution can affect the quality of the samples generated by the importance resampling algorithm. If `sigma2` is too small, then the proposal distribution might not cover the entire support of the target distribution `pi(x)`, resulting in samples that are concentrated in a small region of the space. On the other hand, if `sigma2` is too large, then the proposal distribution might generate many samples in regions where `pi(x)` is very small, resulting in a large variance for the importance weights and a high variance for the importance sampling estimator. A good choice of `sigma2` should balance these two considerations and generate samples that are representative of the target distribution `pi(x)`.