Exercise 11 – *Bayesian inference and Data assimilation*

Due by: Tuesday, 4 July 2023, 23:59 (CEST)

Problem 1 In this exercise, we will do a very simple filtering and smoothing step by hand. The forward map is given as

$$X_{n+1} = \frac{1}{2}X_n + 1 + \Xi_n$$

where $\Xi_n \sim N(0, 1)$. The observation operator is given by

$$Y_n = X_n + \sqrt{2}\Sigma_n$$

with $\Sigma_n \sim N(0, 1)$. Assume $X_0 \sim N(-1, 2)$. All noise processes are independent.

1. Prediction step: What is the distribution of X_1?
2. Filtering step: What is the distribution of X_1 conditioned on $Y_1 = 2$?
3. Smoothing step: What is the distribution of X_0 conditioned on $Y_1 = 2$?
4. Now we want to implement this in pseudo code. Let the model be parameterized by:

$$X_{n+1} = \alpha X_n + \beta \Xi_n, \quad X_0 \sim N(m, 1)$$

with the same observation model and noise parameter. You also observe that $Y_1 = y$. Write a pseudo code, that given the inputs α, β, m and y will output the distribution of (i) X_1, (ii) X_1 conditioned on $Y_1 = y$, and (iii) X_0 conditioned on $Y_1 = y$.

Problem 2 In this exercise, we will implement a particle filter on the previous model.

1. Prediction step: Draw $N = 1000$ independent samples of X_0^i and Ξ_0^i for $i = 1, \ldots, N$. Compute X_1^i for each sample path.
2. Plot the histogram of particles and compare with the theoretical pdf obtained in P1.
3. Obtain (by hand) formula for the likelihood

$$\tilde{w}_i := \pi_{Y_1|X_1}[y \mid X_1^i]$$

Now implement it for the observation $Y_1 = 2$. You do not have to compute the common constant factor as we will normalize them anyways.

4. Filtering step: compute the normalized weight w_i for $i = 1, \ldots, N$ where

$$w_i := \frac{\tilde{w}_i}{\sum_{i=1}^N \tilde{w}_i}$$

5. Compute the weighted mean and the variance

$$\bar{m}_1 := \sum_{i=1}^N w_i X_1^i, \quad \bar{V}_1 := \sum_{i=1}^N w_i (X_1^i - \bar{m}_1)^2$$

6. Compute the "effective sample size" defined by

$$ESS := \frac{1}{\sum_{i=1}^N w_i^2}$$