Ex1

$X \sim \text{Unif } \mathcal{X}_1$

$Y \sim \text{Unif } \mathcal{X}_2$

$\mathcal{X}_1 = \{a = 1, a_2 = 2, a_3 = 3\}.$

$\mathcal{X}_2 = \{b_1 = 1.5, b_2 = 2, b_3 = -1\}$

$P(X = a_i) = P(Y = b_i) = 1/3, \quad \forall i \in \{1, 2, 3\}$

$T \in \mathbb{R}^{3\times 3}$

$$J(T) = \sum_{i,j \in 1}^{3} t_{ij} |b_i - a_j|^2$$

we know that,

$$T^* = \arg \min_{T \in \mathbb{R}^{3\times 3}} J(T)$$

$t_{ij} = \mu_{(X,Y)}(i,j) \qquad \mu_{(X,Y)} \in \pi(\mu_X, \mu_Y)$

By coupling $|b_i - a_j|^2 \quad \forall i, j \in \{1, 2, 3\}$

we see that $|b_2 - a_2| = 0$ -

Since we want to minimize $J(T)$

We assign to t_{22} the maximum value $1/3$.

consequently, by calculating

we set, that, $t_{12} = t_{32} = t_{23} = t_{21} = 0$

$$T = \begin{pmatrix} t_{11} & 0 & t_{13} \\ 0 & 1/3 & 0 \\ t_{31} & 0 & t_{33} \end{pmatrix}$$

$$J(T) = t_{11}(b_1 - a_1)^2 + t_{13}(b_1 - a_3)^2 + t_{31}(b_3 - a_1)^2 + t_{33}(b_3 - a_3)^2$$

Now, by calculating,

$$t_{31} = 1/3 - t_{11}$$
$$t_{13} = 1/3 - t_{11} \quad \Longleftrightarrow \quad \begin{cases} t_{31} = t_{13} = 1/3 - t_{11} \\ t_{33} = t_{11} \end{cases}$$
$$t_{33} = 1/3 - t_{13} = t_{11}$$

$$\Rightarrow J(T) = 1/4 \, t_{11} + \left(\frac{9}{4} + 4\right)\left(\frac{1}{3} - t_{11}\right) + 16\, t_{11}$$
$$= \left(\frac{1}{4} + 16\right) t_{11} + \left(\frac{9+16}{4}\right) 1/3 - \frac{25}{4} t_{11}$$
$$= 10\, t_{11} + \frac{25}{12}$$

Since $t_{11} \in [0, 1/3]$ amd $J(T)$ is monotonous increasing,

$$t_{11}^* = \arg \min_{t_{11} \in [0, 1/3]} J(T) = 0$$

$$\Rightarrow t_{31}^* = t_{13}^* = 1/3 - t_{11}^* = 1/3$$

and $t_{33}^* = 0$

$$\Rightarrow T^* = \begin{pmatrix} 0 & 0 & 1/3 \\ 0 & 1/3 & 0 \\ 1/3 & 0 & 0 \end{pmatrix}$$

Exercise – 5

Problem : 2a (i)

Let $X \sim U[0,1]$ be a uniform random variable.

PDF π_X is,

$$\pi_X(x) = \begin{cases} 1, & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Since we are considering a quadrature rule of order $p = 2$, $f(x) \in \Pi_1(\mathbb{R})$, i.e

$$f(x) = a_0 + a_1 x$$

By definition 3.1 of numerical quadrature rule, we have

$$\bar{f} = \int_{\mathbb{R}} f(x)\, \pi_X(x)\, dx$$

$$= \int_0^1 (a_0 + a_1 x)\, dx$$

$$= \left[a_0 x + \frac{1}{2} a_1 x^2\right]_0^1$$

$$= a_0 + \frac{a_1}{2} \quad \dots \dots \text{(i)}$$

And,

$$\bar{f}_1 = \sum_{i=1}^{m} b_i f(c_i)$$

$$= \sum_{i=1}^{1} b_1 f(c_1) \quad [\because m = 1]$$

$$= b_1(a_0 + a_1 c_1) \quad \dots \dots \text{(ii)}$$

A quadrature rule is of order $p = 2$ if $\bar{f} = \bar{f}_1$ for all integrands $f(x) \in \Pi_1(\mathbb{R})$. By comparing two equation above,

$$b_1 = 1, \; a_1 = 0$$

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By using midpoint rule,

$c_1 = 1/2$ (midpoint of the interval [0,1])

$b_1 = 1$ (weight)

Now we get,

$$\int_0^1 f(x)dx \approx 1 f(1/2) \quad \left[\int_0^1 f(x)dx = b_1 * f(c_1)\right]$$

Problem : 2a (ii)

Classify all quadrature rules for $M=2$, that have order $p=3$.

Quadrature of order $p=3$ means that we have to consider $f(x) \in \Pi_2(\mathbb{R})$ i.e.

$$f(x) = a_0 + a_1 x + a_2 x^2$$

To find the quadrature rules for $M=2$ (two quadrature points) that have polynomials of degree 3, we need to find the weights and the nodes, which satisfies,

$$\int_0^1 f(x)\,dx \simeq b_1\, f(c_1) + b_2\, f(c_2) \, .$$

where, b_1, b_2 are the weights and c_1 and c_2 are the nodes (Points).

The points and weights achieved from the Gaussian-Legendre polynomials with $M=2$ and $P=3$ are,

$$c_1 = \frac{1 - 1/\sqrt{3}}{2}$$

$$c_2 = \frac{1 + 1/\sqrt{3}}{2}$$

$$b_1 = b_2 = 1/2 \, .$$

$$\therefore \int_0^1 f(x)\,dx \approx \frac{1}{2} f\left(\frac{1-1/\sqrt{3}}{2}\right) + \frac{1}{2} f\left(\frac{1+1/\sqrt{3}}{2}\right) .$$

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Exercise - 5

Problem 2b: Determine the ANOVA decomposition for,

$$f(x_1, x_2) = 12x_1 + 6x_2 - 6x_1x_2$$, under the uniform measure on $[0,1]^2$.

We seek the decomposition of $f(x_1, x_2)$ in the form:

$$f(x_1,x_2) = f_0 + f_1(x_1) + f_2(x_2) + f_{12}(x_1, x_2) \quad \text{--- (i)}$$

$$f_0 = \mathbb{E}[f(x)] = \int_0^1\int_0^1 f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^1\int_0^1 (12x_1 + 6x_2 - 6x_1x_2)\, dx_1\, dx_2$$

$$= \int_0^1 \left[6x_1^2 + 6x_1x_2 - 3x_1^2x_2\right]_0^1 dx_2$$

$$= \int_0^1 (6 + 6x_2 - 3x_2)\, dx_2$$

$$= \int_0^1 (6 + 3x_2)\, dx_2$$

$$= \left[6x_2 + \frac{3}{2}x_2^2\right]_0^1$$

$$= 6 + 3/2$$

$$= \frac{15}{2}$$

$$f_1(x_1) = \int_0^1 f(x_1, x_2)\cdot dx_2 - f_0$$

$$= \int_0^1 (12x_1 + 6x_2 - 6x_1x_2)\, dx_2 - f_0$$

$$= \left[12x_1x_2 + 3x_2^2 - 3x_1x_2^2\right]_0^1 - f_0$$

$$= 12x_1 + 3 - 3x_1 - \frac{15}{2}$$

$$= 9x_1 - \frac{9}{2}$$

$$f_2(x_2) = \int_0^1 (12x_1 + 6x_2 - 6x_1x_2)\,dx_1 - f_0$$

$$= \left[6x_1^2 + 6x_1x_2 - 3x_1^2x_2\right]_0^1 - f_0$$

$$= (6 + 6x_2 - 3x_2) - f_0$$

$$= 3x_2 + 6 - \frac{15}{2}$$

$$= 3x_2 - \frac{3}{2}$$

$$f_{12}(x_1, x_2) = f(x_1, x_2) - f_0 - f_1(x_1) - f_2(x_2)$$

$$= 12x_1 + 6x_2 - 6x_1x_2 - \frac{15}{2} - 9x_1 + \frac{9}{2} - 3x_2 + \frac{3}{2}$$

$$= 3x_1 + 3x_2 - 6x_1x_2 - \frac{3}{2}$$

for uniform distribution,

$$E[X] = \frac{a+b}{2} = \frac{1+0}{2} = \frac{1}{2}$$

$$Var(X) = \frac{1}{12}(b-a)^2 = \frac{1}{12}$$

$$\sigma_1^2 = Var(f_1(x_1))$$
$$= Var\left(9x_1 - \frac{9}{2}\right)$$
$$= Var(9x_1)$$
$$= 81\, Var(x_1)$$
$$= 81 \cdot \frac{1}{12}$$
$$= \frac{27}{4}$$

$$\sigma_2^2 = Var(f_2(x_2))$$
$$= Var(3x_2 - 3/2)$$
$$= 9\, Var(x_2)$$
$$= 9 \cdot \frac{1}{12}$$
$$= \frac{3}{4}$$

$$\sigma_{12}^2 = Var(f_{12}(x_1,x_2))$$

$$= Var(3x_1 + 3x_2 - 6x_1x_2 - 3/2)$$

$$= Var(3x_1 + 3x_2 - 6x_1x_2)$$

$$= Var(3x_1 + 3x_2) + Var(6x_1x_2) - 2\,cov(3x_1 + 3x_2, 6x_1x_2)$$

$$= Var(3x_1) + Var(3x_2) + 2\,cov(3x_1\, 3x_2) + Var(6x_1x_2) - 2\,Cov(3x_1, 6x_1x_2) - 2\,Cov(3x_2, 6x_1x_2)$$

$$= 9\,Var(x_1) + 9\,Var(x_2) + \ cov(\ x_1,x_2) + 36\,Var(x_1x_2) - 36\,Cov(x_1, x_1x_2) - 36\,(x_2, x_1x_2) \quad \text{— (ii)}$$

Since we know that,

$$Var(x_1x_2) = E[x_1^2 x_2^2] - (E[x_1x_2])^2$$

$$= E[x_1^2]\,E[x_2^2] - (E[x_1x_2])^2 \quad [\text{independent}]$$

$$= (Var(x_1) + E[x_1]^2)(Var(x_2) + E[x_2]^2) - (E[x_1x_2])^2$$

$$= \left(\frac{1}{12} + \frac{1}{4}\right)\left(\frac{1}{12} + \frac{1}{4}\right) - \left(\frac{1}{4}\right)^2$$

$$= 7/144$$

$$Cov(x_1, x_2) = 0 \quad [\text{independent}]$$

$$Cov(x_1\, x_1x_2) = Cov(x_2, x_1x_2) \quad [\text{i.i.d}]$$

$$cov(x_1, x_1x_2) = E[x_1x_1x_2] - E[x_1]\,E[x_1x_2]$$

$$= E[x_1^2]\,E[x_2] - E[x_1]^2\,E[x_2] \quad [\text{independent}]$$

$$= \left(\frac{1}{12} + \frac{1}{4}\right)\frac{1}{2} - \left(\frac{1}{2}\right)^2\frac{1}{2}$$

$$= \frac{1}{24}$$

Now, σ_{12}^2 can further be computed:

$$\sigma_{12}^2 = 9 \cdot \frac{1}{12} + 9 \cdot \frac{1}{12} + 6 \cdot 0 + 36 \cdot 7/144 - 36 \cdot \frac{1}{24} - 36 \cdot \frac{1}{24}$$

$$= \frac{1}{4}$$

$\therefore \ \sigma_1^2 = 27/4, \ \sigma_2^2 = 3/4, \ \sigma_{12}^2 = 1/4.$

Further, we can also say that,

σ_1^2 contributes most significantly to the total variance σ^2, since,

$$\sigma_1^2 > \sigma_2^2 > \sigma_{12}^2.$$