**Problem 2(i):**

To compute the probability mass function of X2 when X0 = 1, we need to calculate the probabilities of transitioning from state 1 to states 1, 2, 3, and 4 after two-time steps.

The transition probability matrix P is given by:

P =

⎡ 0 1/3 0 2/3 ⎤

⎢ 1/2 0 1/2 0 ⎥

⎢ 0 2/3 0 1/3 ⎥

⎣ 1/2 0 1/2 0 ⎦

To compute the probability mass function of X2, we can multiply the initial probability distribution X0 with the transition probability matrix P twice.

Let X0 = (1, 0, 0, 0) be the initial probability distribution.

After one time step:

X1 = X0 \* P

X1 = (1, 0, 0, 0) \*

⎡ 0 1/3 0 2/3 ⎤

⎢ 1/2 0 1/2 0 ⎥

⎢ 0 2/3 0 1/3 ⎥

⎣ 1/2 0 1/2 0 ⎦

X1 = (0, 1/3, 0, 2/3)

After two-time steps:

X2 = X1 \* P

X2 = (0, 1/3, 0, 2/3) \*

⎡ 0 1/3 0 2/3 ⎤

⎢ 1/2 0 1/2 0 ⎥

⎢ 0 2/3 0 1/3 ⎥

⎣ 1/2 0 1/2 0 ⎦

X2 = (0, 1/2, 0, 1/2)

Therefore, the probability mass function of X2, given X0 = 1, is:

P(X2 = 1) = 0

P(X2 = 2) = 1/2

P(X2 = 3) = 0

P(X2 = 4) = 1/2

**Problem 3(i):**

To find the value of w such that the event [X = x, Y = y] is equivalent to the event [X = x, W = w], we need to understand the relationship between X, Y, and W.

Given:

X ~ N(1, 1)

Y = X^2 + W, where W ~ N(0, 1)

Let's substitute the value of Y in terms of X and W:

Y = X^2 + W

Now, we want to find the value of W such that the event [X = x, Y = y] is equivalent to [X = x, W = w].

If [X = x, Y = y], then we have:

x = X

y = X^2 + W

Similarly, if [X = x, W = w], then we have:

x = X

w = W

To make [X = x, Y = y] equivalent to [X = x, W = w], we need to find the value of w such that y = x^2 + w.

Substituting the value of y:

y = x^2 + w

Rearranging the equation, we get:

w = y - x^2

Therefore, the value of w that makes the events [X = x, Y = y] and [X = x, W = w] equivalent is w = y - x^2.

**Problem 3(ii):**

To find the formula for πX|Y=y(x), the conditional density of X at x given Y = y, we can use Bayes' formula:

πX|Y=y(x) = πX(x) \* πY|X=x(y) / πY(y)

The prior density of X is given as X ∼ N(1, 1), which represents a normal distribution with mean 1 and variance 1. Therefore, πX(x) is the probability density function of a normal distribution with these parameters.

The conditional density of Y given X = x, πY|X=x(y), can be found by substituting the values into the equation Y = X^2 + W. In this case, W ∼ N(0, 1), which represents a normal distribution with mean 0 and variance 1. We can calculate the conditional density by using the properties of normal distributions.

The marginal density of Y, πY(y), can be found by integrating the joint density of X and Y.

The explicit formula for the normalization constant can be omitted.

**Problem 3(iii):**

import numpy as np

import matplotlib.pyplot as plt

from scipy.stats import norm

# Define the prior distribution parameters

prior\_mean = 1

prior\_variance = 1

# Define the observed value of Y

y = 2

# Define the range of x values

x\_values = np.linspace(-10, 10, 100)

# Calculate the prior density of X

prior\_density = norm.pdf(x\_values, prior\_mean, np.sqrt(prior\_variance))

# Calculate the conditional density of Y given X = x

conditional\_density = norm.pdf(y - x\_values\*\*2, 0, 1)

# Calculate the marginal density of Y

marginal\_density = np.trapz(prior\_density \* conditional\_density, x=x\_values)

# Calculate the conditional PDF of X given Y = y

conditional\_pdf = (prior\_density \* conditional\_density) / marginal\_density

# Plot the conditional PDF

plt.plot(x\_values, conditional\_pdf)

plt.xlabel('X')

plt.ylabel('Conditional PDF')

plt.title('Conditional PDF of X given Y = 2')

plt.show()

# Find the maximum a-posteriori (MAP) estimator of X given Y = y

map\_estimator = x\_values[np.argmax(conditional\_pdf)]

print("MAP Estimator of X given Y = 2:", map\_estimator)