Note Exercise-2

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The following variables are defined for each $L \in \{I, H, O\}$, where I stands for the input layer, H stands for hidden layer and O stands the output layer. For the index $k \in \mathbb{N}$ the following is assumed $0 \leq k \leq N_L$

N_L = the number of nurons in layer L

w_{ji}^{L} = the weight in layer L for neuron j with the incoming nuron i

$O_k^{L'}$ = the output of the k-th neuron in layer $L \in \{H, O\}$ which is produced by perception with the activation function $O_k^{L'} = sig(x_k^{L'})$

$x_k^{L'}$ = the input of the k-th neuron in layer $L' \in \{H, O\}$

$x_k^{L'} = \sum_{n=1}^{N_L} w_{nk}^{L} O_n^{(L-1)} + b_k$ with $(L-1)$ representing the previous layer

$sig(x)$ = the sigmoid function of x. It is the activation function for every neuron in the hidden and output layer.

t_k = the k^{th} target value

b_k^L = the bias of the k-th neuron in layer L

$$\frac{\delta E}{\delta \omega_{ji}^{H}} = \frac{\delta}{\delta \omega_{ji}^{H}} \frac{1}{2} \sum_{k \in N_o} (O_k - t_k)^2$$

$$\overset{(1)}{=} \sum_{k \in N_o} (O_k - t_k) * \frac{\delta}{\delta \omega_{ji}} O_k$$

$$= \sum_{k \in N_o} (O_k - t_k) * \frac{\delta}{\delta \omega_{ji}} sig(x_k) \quad \left[\because O_k = sig(x_k) \right]$$

$$= \sum_{k \in N_o} (O_k - t_k) * sig(x_k) * (1 - sig(x_k)) * \frac{\delta}{\delta \omega_{ji}} \sum_{l=1}^{N_H} \omega_{lk} O_l^H + b_k$$

$$\left[x_k = \sum_{l=1}^{N_o} \omega_{lk} \cdot O_l^H + b_k \text{ and } \right.$$

$$\left. \frac{\delta}{\delta \omega_{ji}} sig(x_k) = sig(x_k) (1 - sig(x_k)) \cdot \frac{\delta}{\delta \omega_{ji}} (x_k) \right]$$

$$\overset{(2)}{=} \sum_{k \in N_o} (O_k - t_k) * sig(x_k) * (1 - sig(x_k)) * \omega_{ik} \frac{\delta}{\delta \omega_{ji}} O_i^H$$

$$\overset{(3)}{=} \sum_{k \in N_o} (O_k - t_k) * sig(x_k) * (1 - sig(x_k)) * \omega_{ik} * sig(x_i^H) * (1 - sig(x_i^H)) * O_j^I$$

HEART'S

The following properties explain the equation –
Because we are looking at a weight w_{ji}^H of the hidden layer, every O_K is influenced by w_{ji}^H, therefore

$$\frac{\delta}{\delta w_{ji}} \sum_{K \in N_O} (O_K - t_K)^2 = 2(O_i - t_i) \frac{\delta}{\delta w_{ji}} (O_i - t_i)$$

$$= 2(O_i - t_i) \frac{\delta}{\delta w_{ji}} O_i$$

$$\frac{\delta}{\delta w_{ji}} \sum_{K \in N_O} (O_K - t_K) = 2 \sum_{l=1}^{N_O} (O_K - t_K) \frac{\delta}{\delta w_{ji}} (O_K) \quad (1)$$

only O_i^H is influenced by w_{ji}^H because

$$O_i^H = sig(x_i^H) \text{ with } x_i^H = \sum_{K \in N_H} w_{li} O_l^H + b_i \quad (2)$$

Equivalent to $\frac{\delta x_K}{\delta w_{ji}} = \frac{\delta}{\delta w_{ji}} \sum_{l=1}^{N_O} w_{lK} O_l^H + b_K = O_j^H$

the following holds

$$\frac{\delta O_i^H}{\delta w_{ji}^H} = Sig(x_i^H)(1 - sig(x_i^H)) \frac{\delta}{\delta w_{ji}^H} \sum_{j=1}^{N_H} (O_j^I w_{ji}^H + b_j)$$

$$= Sig(x_i^H)(1 - sig(x_i^H)) O_j^I \quad (3)$$