Statistical Data Analysis

Exercise 2

December 12, 2022

Here we can define the following variables for each $L \in \{I, H, O\}$, where I stands for the input layer, H stands for the hidden layer and O stands for the output layer. For the index $k \in \mathbb{N}$ the following is assumed $0 \leq k \leq N_L$.

N_L is the number of neurons in layer L.

w^L_{ji} is the weight in layer L for neuron j with the incoming neuron i.

$O^{L'}_k$ is the output of the k-th neuron in layer $L' \in \{H, O\}$ which is produced by the perceptron with the activation function:

$$O^{L'}_k = sig(x^{L'}_k)$$

$x^{L'}_k$ is the input of the k-th neuron in layer $L' \in \{H, O\}$. If $(L - 1)$ represents the previous layer, then:

$$x^{L'}_k = \sum_{n=1}^{N_L} w^L_{nk} O^{(L-1)}_n + b^O_k$$

$sig(x)$ is the sigmoid function of x. It is the activation function for every neuron in the hidden and output layer.

t_k is the k-th target value

b^L_k is the bias of the k-th neuron in layer L

Here . . .

$$O_k = sig(x_k) \tag{1}$$

Where . . .

$$x_k = \sum_{l=1}^{N_O} w^O_{lk} . O^H_l + b_k \tag{2}$$

And . . .

$$\frac{\delta}{\delta w^O_{ji}} sig(x_k) = sig(x_k)(1 - sig(x_k)) \frac{\delta}{\delta w^O_{ji}} (x_k) \tag{3}$$

Because of (1) only O_i is influenced by w_{ji}, so,

$$\frac{\delta}{\delta w^O_{ji}} \sum_{k=1}^{N_O} (O_k - t_k)^2 = 2(O_i - t_i) \frac{\delta}{\delta w^O_{ji}} (O_i - t_i) = 2(O_i - t_i) \frac{\delta}{\delta w^O_{ji}} O_i \tag{4}$$

Because we are looking at a weight w_{ji}^H of the hidden layer, every O_k is influenced by w_{ji}^H, therefore (also (4)):

$$\frac{\delta}{\delta w_{ji}^O}\sum_{k=1}^{N_O}(O_k - t_k)^2 = 2\sum_{l=1}^{N_O}(O_k - t_k)\frac{\delta}{\delta w_{ji}^O}(O_k) \tag{5}$$

Only O_i^H is influenced by w_{ji}^H because:

$$O_i^H = sig(x_i^H)\ with\ x_i^H = \sum_{k=1}^{N_H} w_{li}^O O_l^H + b_i^O \tag{6}$$

Also,the following holds:

$$\frac{\delta O_i^H}{\delta w_{ji}^H} = sig(x_i^H)(1 - sig(x_i^H))\frac{\delta}{\delta w_{ji}^H}\sum_{j=1}^{N_H}(O_j^I w_{ji}^H + b_j) = sig(x_i^H)(1 - sig(x_i^H))O_j^I \tag{7}$$

...... [Because $\forall l \neq j : \frac{\delta}{\delta w_{ji}^O} w_{lk}^O.O_l^H + b_k^O = 0$]

Now, we have to **determine** $\frac{\delta E}{\delta w_{ji}^H}$

$$\frac{\delta E}{\delta w_{ji}^H} = \frac{\delta}{\delta w_{ji}^H}\frac{1}{2}\sum_{k\in N_O}(O_k - t_k)^2$$

$$= \sum_{k\in N_O}(O_k - t_k) * \frac{\delta}{\delta w_{ji}} O_k \qquad \langle\text{from (5)}\rangle$$

$$= \sum_{k\in N_O}(O_k - t_k) * \frac{\delta}{\delta w_{ji}} sig(x_k) \qquad \langle\text{from (1)}\rangle$$

$$= \sum_{k\in N_O}(O_k - t_k) * sig(x_k) * (1 - sig(x_k)) * \frac{\delta}{\delta w_{ji}}\sum_{l=1}^{N_H} w_{lk}^O O_l^H + b_k^O \qquad \langle\text{from (2) and (3)}\rangle$$

$$= \sum_{k\in N_O}(O_k - t_k) * sig(x_k) * (1 - sig(x_k)) * w_{ik}^O \frac{\delta}{\delta w_{ji}} O_i^H \qquad \langle\text{from (6)}\rangle$$

$$= \sum_{k\in N_O}(O_k - t_k) * sig(x_k) * (1 - sig(x_k)) * w_{ik} * sig(x_i^H) * (1 - sig(x_i^H)) * O_j^I \qquad \langle\text{from (7)}\rangle$$