

# Zusammenfassung: Optimale Steuerung

## Robots and Control

$$\begin{aligned}
 f - mg &= \underbrace{mx''}_{\substack{\text{Kräftebilanz} \\ \text{Newtons 2nd Law}}} = \frac{d}{dt} [mx'(t)] \\
 f - \frac{\partial}{\partial x} \underbrace{[mgx(t)]}_{\substack{\text{Potentielle Energie } V}} &= \frac{d}{dt} \frac{\partial}{\partial x'} \underbrace{\left[ m \frac{1}{2} x'(t)^2 \right]}_{\substack{\text{kinetische Energie } T}} \\
 f - \frac{\partial V}{\partial x} = \frac{d}{dt} \frac{\partial T}{\partial x'} &\Rightarrow f = \frac{d}{dt} \frac{\partial L}{\partial x'} - \frac{\partial L}{\partial x}
 \end{aligned}$$

Wobei  $L = T - V = \frac{1}{2}mx'(t)^2 - mgx(t)$  gilt.  
Für generalisierte Koordinaten  $\mathbf{q} = \theta \in \mathbb{R}^n$

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \theta'_i} \right) - \frac{\partial T}{\partial \theta_i} + \frac{\partial V}{\partial \theta_i} = \tau_i \quad i = 1, \dots, n$$

## Optimal Control Approach to Robust Control of Manipulators

Ziel: Finde Steuerung  $u$  mit  $x(t) \xrightarrow{t \rightarrow \infty} 0$  unter

$$x'(t) = A(x(t)) + B(x)u(t) + F(x)$$

Dazu definieren wir das Hilfsproblem:

$$\begin{aligned}
 \min_u \int_0^\infty Qx(t)^2 + Ru(t)^2 dt, \\
 x'(t) = A(x(t)) + B(x)u(t), x(0) = x_0
 \end{aligned}$$

Die HJB liefert

$$\begin{aligned}
 0 &= \min_u \left[ \underbrace{Qx^2 + Ru^2 + V'(x) \cdot (A(x) + B(x)u)}_{Qx^2 + R\mathbf{u}^{*2} + V'(x) \cdot (A(x) + B(x)\mathbf{u}^*)} \right] \\
 - \underbrace{(Qx^2 + R\mathbf{u}^{*2})}_{\geq 0} &= \underbrace{V'(x) \cdot (A(x) + B(x)\mathbf{u}^*)}_{\frac{d}{dt} V(x(t)) \leq 0}
 \end{aligned}$$

Einsetzen des Originalproblems in  $V$

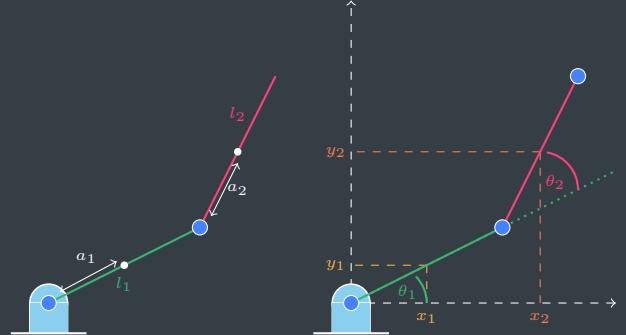
$$\begin{aligned}
 \frac{d}{dt} V(x) &= V'(x) \cdot \underbrace{(A(x) + B(x)u(t) + F(x))}_{x'(t)} \\
 &= -(Qx^2 + Ru^2) + V'(x)F(x) \leq 0
 \end{aligned}$$

Für  $|V'(x)F(x)| \leq \gamma(Qx^2 + Ru^2)$ ,  $\gamma < 1$ . Da  $x_0 \neq 0$  gilt  $\frac{d}{dt} V(x(t)) < 0$  und  $V(x(t)) \geq 0$

$$\lim_{t \rightarrow \infty} V(x(t)) = 0 \Leftrightarrow \lim_{t \rightarrow \infty} x(t) = 0$$

## Beispiel Two Link Revolute Manipulator

Für den Two-Link Revolute Manipulator definieren wir



$$x_1 = a_1 \cdot \cos(\theta_1) \quad y_1 = a_1 \cdot \sin(\theta_1)$$

Gelenk 2 befindet sich in  $(l_1 \cos(\theta_1), l_1 \sin(\theta_1))$ . Der Schwerpunkt von Link 2 ist zusätzlich noch  $a_2$  vom Gelenk 2 entfernt und besitzt eine absolute Winkelausrichtung  $\theta_1 + \theta_2$ . Daher:

$$\begin{aligned}
 x_2 &= l_1 \cdot \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \\
 y_2 &= l_1 \cdot \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2)
 \end{aligned}$$

Die Euler-Lagrange Gleichung des Systems

$$D(\theta)\theta'' + C(\theta, \theta')\theta' + G(\theta) = \tau$$

## Herleitung der kinetischen Energie

Für die translatorische Energie von Link 1 gilt:

$$\begin{aligned} T_{1,trans} &= \frac{m_1}{2} \cdot (x'_1(t)^2 + y'_1(t)^2) \\ &= \frac{m_1}{2} \cdot ((-a_1 \sin(\theta_1) \theta'_1)^2 + (a_1 \cos(\theta_1) \theta'_1)^2) \\ &= \frac{m_1}{2} \cdot a_1^2 \cdot \theta'_1(t)^2 \end{aligned}$$

Es gilt  $T_1 = T_{1,trans} + T_{1,rot}$  und somit

$$T_1 = \frac{m_1}{2} a_1^2 \theta'_1(t)^2 + \frac{I_1}{2} \theta'_1(t)^2 = \frac{m_1 a_1^2 + I_1}{2} \theta'_1(t)^2$$

Analog für die Energie an Link 2 gilt:

$$T_2 = \underbrace{\frac{m_2}{2} \cdot (x'_2(t)^2 + y'_2(t)^2)}_{\text{TRANSLATORISCH}} + \underbrace{\frac{I_2}{2} \cdot (\theta'_1(t) + \theta'_2(t))^2}_{\text{ROTATORISCH}}$$

$$\begin{aligned} T(\theta_1, \theta'_1, \theta_2, \theta'_2) &= \frac{1}{2} \begin{pmatrix} \theta'_1 & \theta'_2 \end{pmatrix} D(\theta_1, \theta_2) \begin{pmatrix} \theta'_1 \\ \theta'_2 \end{pmatrix} \\ &= \frac{1}{2} D_{11} \theta'_1(t)^2 + D_{12} \theta'_1 \theta'_2 + \frac{1}{2} D_{22} \theta'_2(t)^2 \end{aligned}$$

$$\begin{aligned} x'_2(t) &= -l_1 \sin(\theta_1) \theta'_1 - a_2 \sin(\theta_1 + \theta_2) (\theta'_1 + \theta'_2) \\ y'_2(t) &= l_1 \cos(\theta_1) \theta'_1 + a_2 \cos(\theta_1 + \theta_2) (\theta'_1 + \theta'_2) \end{aligned}$$

$$\begin{aligned} x'_2(t)^2 + y'_2(t)^2 &= \left[ -l_1 \sin(\theta_1) \theta'_1 - a_2 \sin(\theta_1 + \theta_2) (\theta'_1 + \theta'_2) \right]^2 + \left[ l_1 \cos(\theta_1) \theta'_1 + a_2 \cos(\theta_1 + \theta_2) (\theta'_1 + \theta'_2) \right]^2 \\ &= l_1^2 \sin(\theta_1)^2 \theta'^2_1 + 2l_1 a_2 \sin(\theta_1) \sin(\theta_1 + \theta_2) \theta'_1 (\theta'_1 + \theta'_2) + \underbrace{a_2^2 \sin(\theta_1 + \theta_2)^2 (\theta'_1 + \theta'_2)^2}_{a_2^2 \sin(\theta_1 + \theta_2)^2 \cdot (\theta'^2_1 + 2\theta'_1 \theta'_2 + \theta'^2_2)} \\ &\quad + l_1^2 \cos(\theta_1)^2 \theta'^2_1 + 2l_1 a_2 \cos(\theta_1) \cos(\theta_1 + \theta_2) \theta'_1 (\theta'_1 + \theta'_2) + \underbrace{a_2^2 \cos(\theta_1 + \theta_2)^2 (\theta'_1 + \theta'_2)^2}_{a_2^2 \cos(\theta_1 + \theta_2)^2 \cdot (\theta'^2_1 + 2\theta'_1 \theta'_2 + \theta'^2_2)} \\ &= \left[ \underbrace{l_1^2 \sin(\theta_1)^2 + l_1^2 \cos(\theta_1)^2}_{=l_1^2} + \underbrace{a_2^2 \sin(\theta_1 + \theta_2)^2 + a_2^2 \cos(\theta_1 + \theta_2)^2}_{=a_2^2} \right] \theta'^2_1 \\ &\quad + 2l_1 a_2 \theta'_1 \cdot \left[ \underbrace{\sin(\theta_1) \sin(\theta_1 + \theta_2) + \cos(\theta_1) \cos(\theta_1 + \theta_2)}_{\cos(\theta_2)} \right] (\theta'_1 + \theta'_2) \\ &\quad + 2 \cdot \left[ \underbrace{a_2^2 \sin(\theta_1 + \theta_2)^2 + a_2^2 \cos(\theta_1 + \theta_2)^2}_{=a_2^2} \right] \theta'_1 \theta'_2 + \left[ \underbrace{a_2^2 \sin(\theta_1 + \theta_2)^2 + a_2^2 \cos(\theta_1 + \theta_2)^2}_{=a_2^2} \right] \theta'^2_2 \\ &= \left[ l_1^2 + a_2^2 + 2l_1 a_2 \cos(\theta_2) \right] \theta'^2_1 + \left[ 2a_2^2 + 2l_1 a_2 \cos(\theta_2) \right] \theta'_1 \theta'_2 + a_2^2 \theta'^2_2 \end{aligned}$$

Für die kinetische Energie gilt somit

$$\begin{aligned}
T &= T_1 + T_2 \\
&= \frac{m_1 a_1^2 + I_1}{2} \theta_1'^2 + \frac{m_2}{2} \cdot (x_2'(t)^2 + y_2'(t)^2) + \frac{I_2}{2} \cdot \underbrace{(\theta_1' + \theta_2')^2}_{\theta_1'^2 + 2\theta_1'\theta_2' + \theta_2'^2} \\
&= \left[ \frac{m_1 a_1^2 + I_1 + m_2 \cdot (l_1^2 + a_2^2 + 2l_1 a_2 \cos(\theta_2)) + I_2}{2} \right] \theta_1'^2 + \left[ \frac{m_2 \cdot (2a_2^2 + 2l_1 a_2 \cos(\theta_2)) + 2I_2}{2} \right] \theta_1' \theta_2' \\
&\quad + \left[ \frac{m_2 a_2^2 + I_2}{2} \right] \theta_2'^2 \\
&= \frac{1}{2} (\theta_1' \quad \theta_2') \underbrace{\begin{pmatrix} m_1 a_1^2 + I_1 + m_2 \cdot (l_1^2 + a_2^2 + 2l_1 a_2 \cos(\theta_2)) + I_2 & m_2 \cdot (a_2^2 + l_1 a_2 \cos(\theta_2)) + I_2 \\ m_2 \cdot (a_2^2 + l_1 a_2 \cos(\theta_2)) + I_2 & m_2 a_2^2 + I_2 \end{pmatrix}}_{D(\theta_1, \theta_2) = D(\theta)} \begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix} \\
&= \frac{1}{2} D_{11} \theta_1'^2 + D_{12} \theta_1' \theta_2' + \frac{1}{2} D_{22} \theta_2'^2
\end{aligned}$$

## Aufstellen der Coriolis-Matrix

Es gilt

$$\frac{\partial T}{\partial \theta_1} = \frac{1}{2} \frac{\partial D_{11}}{\partial \theta_1} \theta_1'^2 + \frac{\partial D_{12}}{\partial \theta_1} \theta_1' \theta_2' + \frac{1}{2} \frac{\partial D_{22}}{\partial \theta_1} \theta_2'^2(t)^2$$

$$\frac{\partial T}{\partial \theta_2} = \frac{1}{2} \frac{\partial D_{11}}{\partial \theta_2} \theta_1'^2 + \frac{\partial D_{12}}{\partial \theta_2} \theta_1' \theta_2' + \frac{1}{2} \frac{\partial D_{22}}{\partial \theta_2} \theta_2'^2(t)^2$$

$$\frac{\partial T}{\partial \theta_1'} = D_{11} \theta_1' + D_{12} \theta_2'$$

$$\frac{\partial T}{\partial \theta_2'} = D_{12} \theta_1' + D_{22} \theta_2'$$

Für die zeitlichen Ableitungen der Größen gilt

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T}{\partial \theta_1'} &= \underbrace{\frac{d}{dt} D_{11}}_{\frac{\partial D_{11}}{\partial \theta_1} \theta_1' + \frac{\partial D_{11}}{\partial \theta_2} \theta_2'} \cdot \theta_1' + D_{11} \theta_1'' + \underbrace{\frac{d}{dt} D_{12}}_{\frac{\partial D_{12}}{\partial \theta_1} \theta_1' + \frac{\partial D_{12}}{\partial \theta_2} \theta_2'} \cdot \theta_2' + D_{12} \theta_2'' \\
&= \frac{\partial D_{11}}{\partial \theta_1} \theta_1'^2 + \frac{\partial D_{11}}{\partial \theta_2} \theta_1' \theta_2' + D_{11} \theta_1'' + \frac{\partial D_{12}}{\partial \theta_1} \theta_1' \theta_2' + \frac{\partial D_{12}}{\partial \theta_2} \theta_2'^2 + D_{12} \theta_2'' \\
&= \frac{\partial D_{11}}{\partial \theta_1} \theta_1'^2 + \left[ \frac{\partial D_{11}}{\partial \theta_2} + \frac{\partial D_{12}}{\partial \theta_1} \right] \theta_1' \theta_2' + \frac{\partial D_{12}}{\partial \theta_2} \theta_2'^2 + D_{11} \theta_1'' + D_{12} \theta_2''
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T}{\partial \theta_2'} &= \underbrace{\frac{d}{dt} D_{12}}_{\frac{\partial D_{12}}{\partial \theta_1} \theta_1' + \frac{\partial D_{12}}{\partial \theta_2} \theta_2'} \cdot \theta_1' + D_{12} \theta_1'' + \underbrace{\frac{d}{dt} D_{22}}_{\frac{\partial D_{22}}{\partial \theta_1} \theta_1' + \frac{\partial D_{22}}{\partial \theta_2} \theta_2'} \cdot \theta_2' + D_{22} \theta_2'' \\
&= \frac{\partial D_{12}}{\partial \theta_1} \theta_1'^2 + \frac{\partial D_{12}}{\partial \theta_2} \theta_1' \theta_2' + D_{12} \theta_1'' + \frac{\partial D_{22}}{\partial \theta_1} \theta_1' \theta_2' + \frac{\partial D_{22}}{\partial \theta_2} \theta_2'^2 + D_{22} \theta_2''
\end{aligned}$$

Es gilt

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T}{\partial \theta'_1} - \frac{\partial T}{\partial \theta_1} &= \left[ \frac{\partial D_{11}}{\partial \theta_1} - \frac{1}{2} \frac{\partial D_{11}}{\partial \theta'_1} \right] \theta'^2_1 + \left[ \frac{\partial D_{11}}{\partial \theta_2} + \frac{\partial D_{12}}{\partial \theta_1} - \frac{\partial D_{12}}{\partial \theta'_1} \right] \theta'_1 \theta'_2 + \left[ \frac{\partial D_{12}}{\partial \theta_2} - \frac{1}{2} \frac{\partial D_{22}}{\partial \theta_1} \right] \theta'^2_2 \\
&\quad + D_{11} \theta''_1 + D_{12} \theta''_2 \\
&= \frac{1}{2} \frac{\partial D_{11}}{\partial \theta_1} \theta'^2_1 + \frac{\partial D_{11}}{\partial \theta_2} \theta'_1 \theta'_2 + \left[ \frac{\partial D_{12}}{\partial \theta_2} - \frac{1}{2} \frac{\partial D_{22}}{\partial \theta_1} \right] \theta'^2_2 + D_{11} \theta''_1 + D_{12} \theta''_2 \\
&= \underbrace{\left[ \frac{1}{2} \frac{\partial D_{11}}{\partial \theta_1} \theta'_1 + \frac{\partial D_{11}}{\partial \theta_2} \theta'_2 \right]}_{C_{11}} \theta'_1 + \underbrace{\left[ \frac{\partial D_{12}}{\partial \theta_2} \theta'_2 - \frac{1}{2} \frac{\partial D_{22}}{\partial \theta_1} \theta'_2 \right]}_{C_{12}} \theta'_2 + D_{11} \theta''_1 + D_{12} \theta''_2
\end{aligned}$$
  

$$\begin{aligned}
\frac{d}{dt} \frac{\partial T}{\partial \theta'_2} - \frac{\partial T}{\partial \theta_2} &= \left[ \frac{\partial D_{12}}{\partial \theta_1} - \frac{1}{2} \frac{\partial D_{11}}{\partial \theta_2} \right] \theta'^2_1 + \left[ \frac{\partial D_{12}}{\partial \theta_2} + \frac{\partial D_{22}}{\partial \theta_1} - \frac{\partial D_{12}}{\partial \theta'_2} \right] \theta'_1 \theta'_2 + \left[ \frac{\partial D_{22}}{\partial \theta_2} - \frac{1}{2} \frac{\partial D_{22}}{\partial \theta_2} \right] \theta'^2_2 \\
&\quad + D_{12} \theta''_1 + D_{22} \theta''_2 \\
&= \underbrace{\left[ \frac{\partial D_{12}}{\partial \theta_1} \theta'_1 - \frac{1}{2} \frac{\partial D_{11}}{\partial \theta_2} \theta'_1 \right]}_{C_{21}} \theta'_1 + \underbrace{\left[ \frac{1}{2} \frac{\partial D_{22}}{\partial \theta_2} \theta'_2 + \frac{\partial D_{22}}{\partial \theta_1} \theta'_1 \right]}_{C_{22}} \theta'_2 + D_{12} \theta''_1 + D_{22} \theta''_2
\end{aligned}$$

Für die Einträge der Matrix  $C$  gilt

$$\begin{aligned}
C_{11} &= \frac{1}{2} \underbrace{\frac{\partial D_{11}}{\partial \theta_1}}_{=0} \theta'_1 + \underbrace{\frac{\partial D_{11}}{\partial \theta_2}}_{=-2m_2 l_1 a_2 \sin(\theta_2)} \theta'_2 = -2m_2 l_1 a_2 \sin(\theta_2) \theta'_2 \\
C_{12} &= \underbrace{\frac{\partial D_{12}}{\partial \theta_2}}_{=-m_2 l_1 a_2 \sin(\theta_2)} \theta'_2 - \frac{1}{2} \underbrace{\frac{\partial D_{22}}{\partial \theta_1}}_{=0} \theta'_2 = -m_2 l_1 a_2 \sin(\theta_2) \theta'_2 \\
C_{21} &= \underbrace{\frac{\partial D_{12}}{\partial \theta_1}}_{=0} \theta'_1 - \frac{1}{2} \underbrace{\frac{\partial D_{11}}{\partial \theta_2}}_{=-2m_2 l_1 a_2 \sin(\theta_2)} \theta'_1 = m_2 l_1 a_2 \sin(\theta_2) \theta'_1 \\
C_{22} &= \frac{1}{2} \underbrace{\frac{\partial D_{22}}{\partial \theta_2}}_{=0} \theta'_2 + \underbrace{\frac{\partial D_{22}}{\partial \theta_1}}_{=0} \theta'_1 = 0
\end{aligned}$$

Es gilt somit

$$C \cdot \theta'(t) = \begin{pmatrix} -2m_2 l_1 a_2 \sin(\theta_2) \theta'_2 & -m_2 l_1 a_2 \sin(\theta_2) \theta'_2 \\ m_2 l_1 a_2 \sin(\theta_2) \theta'_1 & 0 \end{pmatrix} \begin{pmatrix} \theta'_1 \\ \theta'_2 \end{pmatrix}$$

## Gravitationsterme

Wählen wir  $y = 0$  als Nullhöhe, so ist

$$\begin{aligned}
V &= m_1 \cdot g \cdot \textcolor{brown}{y}_1 + m_2 \cdot g \cdot \textcolor{brown}{y}_2 \\
&= m_1 g a_1 \sin(\theta_1) + m_2 g (l_1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2))
\end{aligned}$$

Für die partiellen Ableitungen erhalten wir

$$\begin{aligned}
\frac{\partial V}{\partial \theta_1} &= m_1 g a_1 \cos(\theta_1) + m_2 g (l_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2)) \\
\frac{\partial V}{\partial \theta_2} &= m_2 g (l_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2))
\end{aligned}$$

## Numerische Lösung - Beispiel Eulerverfahren