Dynamics of Robots

We use the principle of least action [3] with $L = T - V$ where T is the kinetic energy and V is the potential energy to get the Euler-Lagrange equations:

$$\delta \int L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\, \mathrm{d}t = 0 \Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{\boldsymbol{q}}} - \frac{\partial T}{\partial \boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}} = 0$$

For a rigid body system, we can express the kinetic energy T as: [4]

$$T = \frac{1}{2}\dot{\boldsymbol{q}}^T M(\boldsymbol{q})\dot{\boldsymbol{q}} \text{with } M(\boldsymbol{q})\text{: Inertia matrix}$$

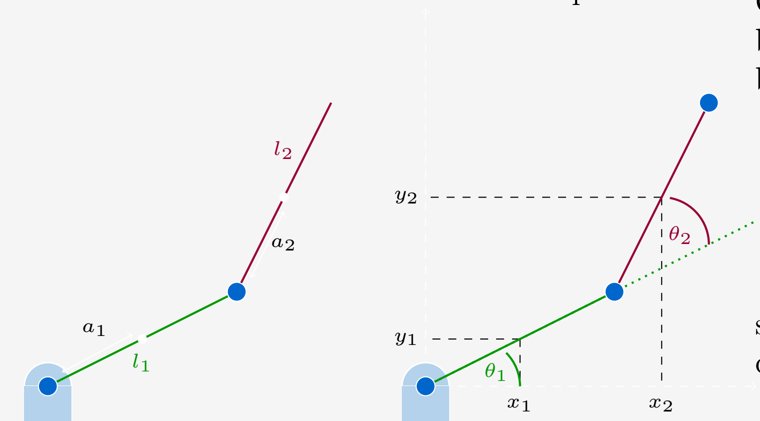
We can compute the Inertia matrix $M(\boldsymbol{q})$ using the kinetic energy T of the system:

$$M(\mathbf{q}) = \sum_{i=1}^{n} m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_i J_{\omega_i} \quad J_v = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \quad J_\omega = \frac{\partial \boldsymbol{\omega}}{\partial \dot{\mathbf{q}}}$$

where m_i is the mass of link i, I_i is the inertia tensor of link i, J_{v_i} is the linear velocity Jacobian of link i, and J_{ω_i} is the angular velocity Jacobian of link i.

Two Link Revolute

We use the generalized coordinates $\boldsymbol{q} = [\theta_1, \theta_2]$ for the two-link revolute manipulator.



By using the coordinates of the centers of mass,

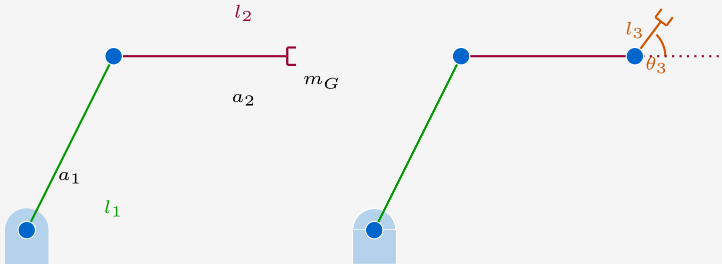
$$\boldsymbol{r}_1 = \begin{pmatrix} a_1 \cdot \cos(\theta_1) \\ a_1 \cdot \sin(\theta_1) \\ 0 \end{pmatrix} \quad \boldsymbol{r}_2 = \begin{pmatrix} l_1 \cdot \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \\ l_1 \cdot \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix}$$

we can derive the Jacobians J_{v_i} and J_{ω_i} and thus the inertia matrix $M(\boldsymbol{q})$. Plugging $M(\boldsymbol{q})$ into the Euler-Lagrange equations, we have the following equations of motion for a robotic manipulator:

$$\begin{aligned} M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q}) &= \boldsymbol{\tau} \\ C(\boldsymbol{q}, \dot{\boldsymbol{q}})&\text{: Coriolis and centrifugal matrix} \\ G(\boldsymbol{q})&\text{: Gravity vector} \\ \boldsymbol{u}&\text{: Joint torques} \end{aligned}$$

Gripper extension

We can model the gripper as an additional point mass m_G at the end of the second link or as an additional third link with length l_3 and mass m_3.



The second formulation requires the addition of a third generalized coordinate θ_3, which represents the angle of the gripper relative to the second link. The new generalized coordinates are $\boldsymbol{q} = [\theta_1, \theta_2, \theta_3]$. The inertia matrix $M(\boldsymbol{q})$ must be recalculated to account for the additional link.

Foundations of Control Theory

Our formulation of an optimal control problem is based on the procedure outlined in [1]. Let $[t_0, t_f]$ be a fixed time interval,

$$\begin{aligned} \Phi &: \mathbb{R}^{n_{\boldsymbol{x}}} \to \mathbb{R}, \\ L &: [t_0, t_f] \times \mathbb{R}^{n_{\boldsymbol{x}}} \to \mathbb{R}, \\ f &: [t_0, t_f] \times \mathbb{R}^{n_{\boldsymbol{x}}} \times \mathbb{R}^{n_{\boldsymbol{u}}} \to \mathbb{R}^{n_{\boldsymbol{x}}} \end{aligned}$$

sufficient smooth functions and $\mathcal{U} \subset \mathbb{R}^{n_{\boldsymbol{u}}}$ a closed convex non-empty set.

$$\min_{\boldsymbol{u}} \int_{t_0}^{t_f} L(t, \boldsymbol{x}(t), \boldsymbol{u}(t))\, \mathrm{d}t + \Phi(\boldsymbol{x}(t_f))$$

with respect to $\boldsymbol{x} \in W_{1,\infty}^{n_{\boldsymbol{x}}}([t_0, t_f]), \boldsymbol{u} \in L_{\infty}^{n_{\boldsymbol{u}}}([t_0, t_f])$ and subject to

$$\dot{\boldsymbol{x}}(t) = f(t, \boldsymbol{x}(t), \boldsymbol{u}(t)),\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0,\ \boldsymbol{u}(t) \in \mathcal{U}$$

The necessary conditions for optimality are given by minimimum principle (see [1]):

$$\dot{\boldsymbol{p}} = -H_{\boldsymbol{x}} \; \boldsymbol{p}(t_f) = \Phi_{\boldsymbol{x}}(\boldsymbol{x}(t_f))$$
$$H(t,\boldsymbol{x},\boldsymbol{u},\boldsymbol{p}) = L(t,\boldsymbol{x},\boldsymbol{u}) + \boldsymbol{p}^T f(t,\boldsymbol{x},\boldsymbol{u})$$
$$\boldsymbol{u}^* = \arg\min_{\boldsymbol{u}\in\mathcal{U}} H(t,\boldsymbol{x}^*,\boldsymbol{u},\boldsymbol{p}^*)$$

Linearisation Method

Let $\mathbf{x} = \begin{bmatrix}\mathbf{q}\\ \dot{\mathbf{q}}\end{bmatrix}$ we use the state space representation of the manipulator dynamics (compare [2]).

$$\underbrace{\begin{bmatrix}\dot{\mathbf{q}}\\ \ddot{\mathbf{q}}\end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix}\dot{\mathbf{q}}\\ M^{-1}(\mathbf{q})\left[\boldsymbol{u} - C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - G(\mathbf{q})\right]\end{bmatrix}}_{f(\mathbf{x},\boldsymbol{u})}$$

We linearize the system around an operating point:

$$\begin{aligned}\dot{\boldsymbol{x}} &= f(\boldsymbol{x},\boldsymbol{u})\\ &\approx \underbrace{f(\boldsymbol{x}^*,\boldsymbol{u}^*)}_{=0} + \underbrace{\frac{\partial f}{\partial x}(\boldsymbol{x}^*,\boldsymbol{u}^*)}_{A(\boldsymbol{x}^*,\boldsymbol{u}^*)}(\boldsymbol{x}-\boldsymbol{x}^*) + \underbrace{\frac{\partial f}{\partial u}(\boldsymbol{x}^*,\boldsymbol{u}^*)}_{B(\boldsymbol{x}^*,\boldsymbol{u}^*)}(\boldsymbol{u}-\boldsymbol{u}^*)\\ &= A(\boldsymbol{x}^*,\boldsymbol{u}^*)\bar{\boldsymbol{x}} + B(\boldsymbol{x}^*,\boldsymbol{u}^*)\bar{\boldsymbol{u}}\end{aligned}$$

We get the equivalent linear system:

$$\dot{\bar{\boldsymbol{x}}}(t) = A\bar{\boldsymbol{x}}(t) + B\bar{\boldsymbol{u}}(t) \; \bar{\boldsymbol{x}} = \mathbf{0} \Leftrightarrow \boldsymbol{x} = \boldsymbol{x}^*$$

By assuming $f(\boldsymbol{x}^*,\boldsymbol{u}^*) = \mathbf{0}$ we get $\ddot{\boldsymbol{q}}^* = \dot{\boldsymbol{q}}^* = \mathbf{0}$. Thus we have the following derivates for our specific two link manipulator:

$$A = \begin{bmatrix}\mathbf{0} & I\\ M^{-1}(\boldsymbol{q}^*)\left[-\frac{\partial G}{\partial \boldsymbol{q}}(\boldsymbol{q}^*)\right] & \mathbf{0}\end{bmatrix} \; B = \begin{bmatrix}\mathbf{0}\\ M^{-1}(\boldsymbol{q}^*)\end{bmatrix}$$

Solving using LQR

We can solve the nonlinar optimal control problem

$$\min_u \int_0^\infty (\boldsymbol{q}-\boldsymbol{q}_f)^T Q(\boldsymbol{q}-\boldsymbol{q}_f) + (\boldsymbol{u}-\boldsymbol{u}_f)^T R(\boldsymbol{u}-\boldsymbol{u}_f)\,\mathrm{d}t$$
$$\text{s.t. } M(\boldsymbol{q})\ddot{\boldsymbol{q}} + C(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + G(\boldsymbol{q}) = \boldsymbol{u}; \quad \boldsymbol{u}_f - G(\boldsymbol{q}_f) = \mathbf{0}$$

by solving the linearized problem using LQR:

$$\min_{\bar{\boldsymbol{u}}} \int_0^\infty \bar{\boldsymbol{x}}^T Q\bar{\boldsymbol{x}} + \bar{\boldsymbol{u}}^T R\bar{\boldsymbol{u}}\,\mathrm{d}t \quad \dot{\bar{\boldsymbol{x}}} = A\bar{\boldsymbol{x}} + B\bar{\boldsymbol{u}};$$

with $\bar{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{x}_f$ and $\bar{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{u}_f$. The solution K of the Riccati equation:

$$Q - KBR^{-1}B^TK + KA + A^TK = 0$$

gives us the optimal feedback law:

$$\boldsymbol{u}_i = G(\boldsymbol{q}_f) - R^{-1}B^TK(\boldsymbol{x}_i - \boldsymbol{x}_f)$$
$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + h\cdot f(\boldsymbol{x}_i,\boldsymbol{u}_i)$$

Gradient Methods

We can also solve the optimal control problem

$$\min_{\boldsymbol{u}} \int_{t_0}^{t_f} (\boldsymbol{u}-\boldsymbol{u}_f)^T\cdot R\cdot(\boldsymbol{u}-\boldsymbol{u}_f)\,\mathrm{d}t + \Phi(\boldsymbol{x}(t_f))$$
$$\Phi(\boldsymbol{x}(t_f)) = (\boldsymbol{q}(t_f)-\boldsymbol{q}_f)^T\cdot Q\cdot(\boldsymbol{q}(t_f)-\boldsymbol{q}_f)$$
$$\dot{\boldsymbol{x}} = f(\boldsymbol{x},\boldsymbol{u}) = \begin{bmatrix}\dot{\boldsymbol{q}}\\ M^{-1}(\boldsymbol{q})\cdot(-C(\boldsymbol{q},\dot{\boldsymbol{q}})\cdot\dot{\boldsymbol{q}} - G(\boldsymbol{q}) + \boldsymbol{u})\end{bmatrix}$$

The adjoint equation is given by:

$$\dot{\boldsymbol{p}} = -\frac{\partial f}{\partial \boldsymbol{x}}^T\cdot\boldsymbol{p}, \quad \boldsymbol{p}(t_f) = \begin{bmatrix}2Q(\boldsymbol{q}(t_f)-\boldsymbol{q}_f)\\ \mathbf{0}\end{bmatrix}$$

The gradient of the Hamiltonian is given by:

$$\nabla J(\boldsymbol{u}) = 2R\cdot(\boldsymbol{u}-\boldsymbol{u}_f) + \begin{bmatrix}\mathbf{0}_2\\ M^{-1}(\boldsymbol{q})\end{bmatrix}\cdot\begin{bmatrix}\boldsymbol{p}_{\boldsymbol{q}}\\ \boldsymbol{p}_{\dot{\boldsymbol{q}}}\end{bmatrix}$$

1) Forward integration of the state equation

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + h\cdot f(\boldsymbol{x}_i,\boldsymbol{u}_i) \Rightarrow \boldsymbol{X} = \{\boldsymbol{x}_0,\ldots,\boldsymbol{x}_N\}$$

2) Backward integration of the adjoint equation

$$\boldsymbol{p}_N = \begin{bmatrix}2Q(\boldsymbol{q}(t_f)-\boldsymbol{q}_f)\\ \mathbf{0}\end{bmatrix} \boldsymbol{p}_{i-1} = \boldsymbol{p}_i + h\cdot\begin{bmatrix}\mathbf{0}_2 & I_2\\ \frac{\partial\ddot{\boldsymbol{q}}}{\partial\boldsymbol{q}} & \frac{\partial\ddot{\boldsymbol{q}}}{\partial\dot{\boldsymbol{q}}}\end{bmatrix}\cdot\boldsymbol{p}_i$$

using finite differences to compute

$$\boldsymbol{x}_\pm = \boldsymbol{x} \pm \epsilon\cdot\boldsymbol{e}_j \; \frac{\partial\ddot{\boldsymbol{q}}}{\partial x_j} \approx \frac{\ddot{\boldsymbol{q}}(\boldsymbol{x}_+) - \ddot{\boldsymbol{q}}(\boldsymbol{x}_-)}{2\epsilon} \Rightarrow \boldsymbol{P} = \{\boldsymbol{p}_0,\ldots$$

3) Gradient descent for $i = 0,\ldots,N-1$:

$$\boldsymbol{u}_{i+1} = \boldsymbol{u}_i - \alpha_j\cdot\left[2R\cdot(\boldsymbol{u}_i-\boldsymbol{u}_f) + M^{-1}(\boldsymbol{q}_i)\cdot\boldsymbol{P}_{\dot{\boldsymbol{q}},i}\right]$$

by using the armijo rule to α_j.

References

[1] Matthias Gerdts. *Optimal Control of ODEs and DAEs*. Berlin: De Gruyter, 2012.

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[3] Bruno Siciliano et al. *Robotics: Modelling, Planning and Control*. London: Springer, 2010. ISBN: 978-1849964507.

[4] Mark W. Spong, Seth Hutchinson, and M. Vidyasagar. *Robot Dynamics and Control*. 2nd. Hoboken, NJ: Wiley, 2004. ISBN: 978-0471649908.