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Optimal Control of 2-Link Underactuated Robot Manipulator

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***Abstract*—Robot manipulators are nonlinear multi-link system, having higher disturbances and uncertainties. Thus, the controlling of robot manipulator is matter of concern. In this paper for deriving the dynamics of robot manipulator Euler Lagrange (EL) method is used. This paper proposes a nonlinear optimal control approach of 2-link robot manipulator. At first stage linearization of robot manipulator is performed at its operating point using Taylor series expansion. For optimal control, Linear Quadratic Regulator (LQR) is designed after linearization of nonlinear dynamics of 2-link robot. This uses the feed forward gain to control the robot manipulator and to achieve desired position and velocity. This proposed paper shows that Linear Quadratic Regulator (LQR) controlled robot manipulator is locally stable.**

***Index Terms*—Robot manipulator, Euler Lagrange (EL), Taylor series, LQR.**

I. Introduction

Robots are introduced in industries for increasing productivity, quality improvement and for replacement of human in hazardous task. Robot manipulators are multi-link nonlinear system [1]. Mostly robot manipulators are driven by actuators which may be electric, hydraulic or pneumatic. Robot manipulator dynamics describes how in the response of these actuator forces the robot moves.

In this paper 2-link under actuated planer robot is considered for discussion of optimal control. Recently much attention has been taken in the field of robotics to control of mechanical systems, where the number of control inputs are lesser than that of generalized co-ordinates, such type of system is known as under actuated system [2]. The under actuated system also used in mobile robot systems. For example, manipulator arm is attached to mobile platform, space platform or sea vehicles. In this paper the 2-link robot one joint is attached with dc motor for actuation and other link is un- actuated (no motor is connected at the joint), the second link is as a simple pendulum and the motion of second link is controlled by actuation of first link. Under actuated robot manipulator control is the most challenging task in robotics [3] because of presence of nonlinearity in dynamics of robot with unity controlling element. Due to occurrence of gravity part robot manipulator dynamics does not identify precise state feedback linearization [3]. Dynamic of 2-link robot is written in paper by using kinetic and potential energy of links using Euler Lagrange approach [20-23]. If robot manipulator dynamic model is controllable then for optimal control Linear Quadratic Regulator (LQR) can be designed with the help of feed forward gain. For Linear Quadratic Regulator (LQR) design feed forward gain can be obtained by solving Algebraic Recatti Equation (ARE) [13-19]. Because of the dynamic equations of robot manipulators are nonlinear, due to this reason in optimal control of robot manipulator for Linear Quadratic Regulator(LQR) design the dynamics of robot manipulator should be linearized.

In this presented paper section II represent the mathematical modeling of 2-Link robot manipulator and in section III linearization of nonlinear dynamics of robot manipulator is performed. In section IV for optimal control linear quadratic regulator is designed, and in section V simulation and results and in VI conclusions are discussed.

II. Mathematical Modeling

A. 2-link Robot manipulator

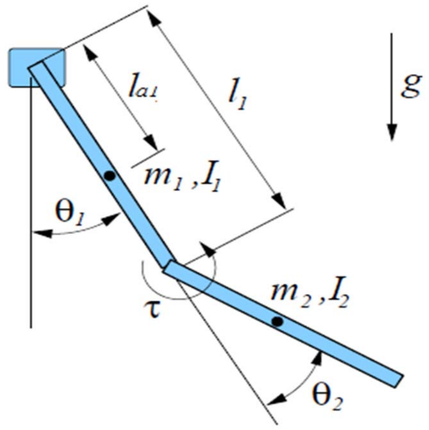

Fig.1. 2-Link Robot Manipulator

In this paper for optimal control of robot considering a 2-link robot arm which is acting in vertical plane against the gravitational force. Here the considered robot arm is in underactuated condition, an actuator is connected with elbow (at joint-2) but no actuator is connected at shoulder (at joint-1) as shown in Fig.1.The most common control task in robot is swing-up control task in which the elbow (joint-2) torque is used to balance and transfer the robot arm in vertical arrangement.

The link mass is concentrated at l_{a1} distance for link-1 and at l_{a2} distance from link-2 and joint variable vectors are $q = [\theta_1 \theta_2]^T$ and torque vector is $\tau = [M_1 \; M_2]^T$, as shown in figure θ_1 is shoulder joint angle and θ_2 is elbow joint angle. The moment of inertia I_1 and I_2 are considered about the pivots.

TABLE.1

Parameters of 2-Link Robot	
$\boldsymbol{m_1\,(Kg)}$	1
$\boldsymbol{m_2(\,Kg)}$	1
$\boldsymbol{I_1(\,Kg/m^2\,)}$	0.083
$\boldsymbol{I_2\,(Kg/m^2)}$	0.33
$\boldsymbol{l_{a1}(m)}$	0.5
$\boldsymbol{l_{a2}\,(m)}$	1
$\boldsymbol{l_1\,(m)}$	1
$\boldsymbol{l_2\,(m)}$	2
$\boldsymbol{g\,(m\,/s^2)}$	9.8

B. Equation of Motion

The kinematics are given by

$$X_1 = \begin{bmatrix} l_1 \sin\theta_1 \\ -l_1 \cos\theta_1 \end{bmatrix} \text{ And } X_2 = X_1 + \begin{bmatrix} l_2\,(\sin\theta_1 + \theta_2) \\ -l_1 \cos(\theta_1 + \theta_2) \end{bmatrix} \quad (1)$$

For link-1 kinetic energy is $T_1 = \frac{1}{2} I_1 \dot{\theta}_1^2$ (2)

And for link-2 kinetic energy is

$$T_2 = \frac{1}{2}(m_2\, {l_1}^2 + I_2 + 2m_2 l_1 l_{a2} \cos\theta_2)\dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + (I_2 + m_2 l_1 l_{a2} \cos\theta_2)\,\dot{\theta}_1 \dot{\theta}_2 \quad (3)$$

Now potential energy [4] for link-1 and link-2 is

$$V = -m_1 g l_{a1} \cos\theta_1 - m_2 g(l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \quad (4)$$

Lagrangian for entire arm is

$$L = (T_1 + T_2) - (V_1 - V_2) \quad (5)$$

Thus, we can write Lagrangian equation in the form of

$$L = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2}(m_2\, {l_1}^2 + I_2 + m_2 l_1 l_{a2} \cos\theta_2)\dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + (I_2 + m_2 l_1 l_{a2} \cos\theta_2)\dot{\theta}_1 \dot{\theta}_2 + m_1 g l_{a1} \cos\theta_1 m_2 g(l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)) \quad (6)$$

According to Euler-Lagrange

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = M_1 \quad (7)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = M_2 \quad (8)$$

Thus, torque equation can be written *as*

$$M_1 = m_1 l_{a1}^2 + m_2\big(\,{l_1}^2 + l_{a2}^2 + 2 l_1 l_{a2} \cos\theta_2\big) + I_1 + I_2)\ddot{\theta}_1 + (m_2(l_{a2}^2 + l_1 l_{a2} \cos\theta_2) + I_2)\ddot{\theta}_2 - m_2 l_1 l_{a2} \sin\theta_2\, \dot{\theta}_2^2 - 2m_2 l_1 l_{a2} \sin\theta_2\, \dot{\theta}_2 \dot{\theta}_1 + (m_1 l_{a1} + m_2 l_1) g \cos\theta_1 + m_2\, l_{a2} g \cos(\,\theta_1 + \theta_2) \quad (9)$$

$$M_2 = (m_2(l_{a2}^2 + l_1 l_{a2} \cos\theta_2) + I_2)\ddot{\theta}_1 + (m_2 l_{a2}^2 + I_2)\ddot{\theta}_2 + m_2 l_1 l_{a2} \sin\theta_2 \dot{\theta}_1^2 + m_2 l_{a2} g cos(\theta_1 \,+\, \theta_2) \quad (10)$$

2-link robot manipulator considered having dc motor is used to generate torque at elbow (joint 2) and no motor is used at joint one (shoulder) [2], thus $M_1 = 0$.

The equation M_1 and M_2 can be arranged in matrix vector form as follows:

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \quad (11)$$

Where

$M_{11} = m_1 l_{a1}^2 + m_2\big(\,{l_1}^2 + l_{a2}^2 + 2 l_1 l_{a2} \cos\theta_2\big) + I_1 + I_2)$

$M_{12} = m_2(l_{a2}^2 + l_1 l_{a2} \cos\theta_2) + I_2 = M_{21}$

$M_{22} \,= m_2 l_{a2}^2 + I_2$

$H_1 = -m_2 l_1 l_{a2} \sin\theta_2\, \dot{\theta}_2^2 - 2m_2 l_1 l_{a2} \sin\theta_2\, \dot{\theta}_2 \dot{\theta}_1$

$H_2 = m_2 l_{m1} l_2 \sin\theta_2 \dot{\theta}_1^2$

$G_1 = (m_1 l_{a1} + m_2 l_1) g \cos\theta_1 + m_2 l_{a2} g \cos(\,\theta_1 + \theta_2)$

$G_2 = m_2 l_{a2} g cos(\theta_1 + \theta_2)$

Dynamic equation of robot can be written as

$$M(\ddot{q}) + H(q, \dot{q}) + G(q) = B\tau \quad (12)$$

In this $M(\ddot{q})$ is inertia matrix, H $(q, \dot{q})$ is Coriolis matrix and centrifugal vector and G(q) belongs to gravity vector of the manipulator.

III. Linearization of Nonlinear Dynamics of Robot Manipulator

Linearization using Taylor series expansion:

Let $x_1 = \theta_1 \qquad \dot{\theta}_1 = \dot{x}_1 = x_3$

$x_2 = \theta_2 \qquad \dot{\theta}_2 = \dot{x}_2 = x_4$

Let us consider in above dynamic equations of robot manipulator

$$m_1 l_{a1}^2 = \emptyset_1 \quad (13)$$

$$m_2 l_{a2}^2 = \emptyset_2 \quad (14)$$

$$m_2 l_1^2 = \emptyset_3 \quad (15)$$

$$m_2 l_1 l_{a2} = \emptyset_4 \quad (16)$$

$$m_2 l_{a2}^2 + I_2 = \emptyset_5 \quad (17)$$

$$m_2 l_{a2} g = \emptyset_6 \quad (18)$$

$$(m_1 l_{a1} + m_2 l_1) g = \emptyset_7 \quad (19)$$

Then torque equations can be written as

$$M_1 = \big((\emptyset_1 + \emptyset_2 + \emptyset_3 + 2\emptyset_4 cos x_2) + I_1 + I_2\big)\ddot{\theta}_1 + (\emptyset_5 + \emptyset_4 cos x_2)\ddot{\theta}_2 - \emptyset_4 sin x_2 x_4^2 - 2\emptyset_4 \sin x_2\, x_3 x_4 + \emptyset_7 cos x_1 + \emptyset_6 \cos(x_1 + x_2) \quad (20)$$

$$M_2 = (\emptyset_2 + \emptyset_4 cos x_2 + I_2)\ddot{\theta}_1 + \emptyset_5 \ddot{\theta}_2 + \emptyset_4 sin x_2\, x_3^2 + \emptyset_6 \cos(x_1 + x_2) \quad (21)$$

Because of dc motor is applied at joint -2, so applied torque at joint -1is $M_1 = 0$

Considering initially at equilibrium $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = 0$ then $\dot{\theta}_1 = 0$ and $\dot{\theta}_2 = 0$

Then final state space equation for 2-link robot may be written as:

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\frac{d}{dt}\begin{bmatrix}\theta_1\\\theta_2\\\dot{\theta}_1\\\dot{\theta}_2\end{bmatrix}=\begin{bmatrix}f_1\\f_2\\f_3\\f_4\end{bmatrix} \tag{22}$$

$$f_1 = x_3 \tag{23}$$

$$f_2 = x_4 \tag{24}$$

$$f_3 = \frac{M_2}{\emptyset_5+\emptyset_4 cosx_2} - \frac{\emptyset_4 sinx_2 x_3^2}{\emptyset_5+\emptyset_4 cosx_2} - \frac{\emptyset_6 \cos(x_1+x_2)}{\emptyset_5+\emptyset_4 cosx_2} - \frac{\emptyset_5 \ddot{\theta_2}}{\emptyset_5+\emptyset_4 cosx_2} \tag{25}$$

$$f_4 = \frac{\emptyset_4 sinx_2 x_4^2+2\emptyset_4 sinx_2 x_4 x_3}{\emptyset_5+\emptyset_4 cosx_2} - \frac{\emptyset_7 cosx_1}{\emptyset_5+\emptyset_4 cosx_2} - \frac{\emptyset_6 \cos(x_1+x_2)}{\emptyset_5+\emptyset_4 cosx_2} - \frac{((\emptyset_1+\emptyset_2+\emptyset_3+2\emptyset_4 cosx_2)+I_1+I_2)\ddot{\theta}_1}{\emptyset_5+\emptyset_4 cosx_2} \tag{26}$$

Linearization about a fixed point $(\frac{\pi}{2}, 0)$ with the help of Taylor series expansion is performed using,

$$\dot{x} = f(x,u) \approx f(x^*,u^*) + \left[\frac{\partial f}{\partial x}\right](x-x^*) + \left[\frac{\partial f}{\partial u}\right](u-u^*) \tag{27}$$

After partial differentiation put x=x^* and u=u^*

Now assume that, in linearization of robot manipulator dynamic equation, about the fixed point $f(x^*,u^*)$ is zero. Then the linearized dynamic equation in state space form is

$$\dot{x}=\begin{bmatrix}\dot{q}\\ M^{-1}(q)[Bu-H(q,\dot{q})\dot{q}\ -G(q)]\end{bmatrix} \tag{28}$$

$$\dot{x} \approx \mathrm{A}\,(x-x^*) + \mathrm{B}\,(u-u^*) \tag{29}$$

Where

$x - x^* = \tilde{x}$ and $u - u^* = \tilde{u}$

$$\dot{x}=\dot{A}\tilde{x} + B\tilde{u} \text{ and } y = C\tilde{x} + D\tilde{u} \tag{30}$$

After applying Taylor expansion about equilibrium point $x^* and\ u^*$,the equation obtained in form of simple state space as

$$\mathrm{A}=\begin{bmatrix}0 & I\\ -M^{-1}\frac{\partial G}{\partial q} & -M^{-1}C\end{bmatrix} \tag{31}$$

And

$$\mathrm{B}=\begin{bmatrix}0\\ -M^{-1}B\end{bmatrix} \tag{32}$$

Robot manipulator is in under actuated condition and no motor is provided at joint-1 thus applied torque at joint one is zero [4]. Using Jacobian, the matrix A and B also can be written as

$$A=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}\end{pmatrix} \mid x^*, u^*$$

$$B=\begin{pmatrix}\frac{\partial f_1}{\partial M_1} & \frac{\partial f_1}{\partial M_2} & \cdots & \frac{\partial f_1}{\partial M_m}\\ \frac{\partial f_2}{\partial M_1} & \frac{\partial f_2}{\partial M_2} & \cdots & \frac{\partial f_2}{\partial M_m}\\ \cdots & \cdots & \cdots & \cdots\\ \frac{\partial f_n}{\partial M_1} & \frac{\partial f_n}{\partial M_2} & \cdots & \frac{\partial f_n}{\partial M_m}\end{pmatrix} \mid x^*, u^*$$

The linearized system matrixes we obtained are:

$$\mathrm{A}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 12.49 & -12.54 & 0 & 0\\ -14.49 & 29.36 & 0 & 0\end{bmatrix} \mathrm{B}=\begin{bmatrix}0\\0\\-2.98\\5.98\end{bmatrix}$$

$$\mathrm{C}=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \mathrm{D} = [0]$$

IV. LINEAR QUADRATIC REGULATOR (LQR) DESIGN

The one well known method of optimal control theory of pole placement method is Linear Quadratic Regulator (LQR) [7]. The optimal pole location can be evaluated by LQR algorithm, which is based on the performance index. The optimal performance index will be defined firstly in order to find the optimal feedforward gain and after that Algebraic Recati Equation(ARE) will be solved [6].

To achieve finest performance in according to provided amount of performance, LQR is found to be a best control method. To minimize the objective function J, LQR controller is designed with the help of a state feedback K. [9-12]. To realize certain negotiation among the application of control determination, magnitude and the rate of response that will confirms a stable system, the feedback gain matrix is used, as this will minimize the objective function.

For continuous time, linear system is described by:

$$\dot{x} = Ax + Bu \text{ And } y = Cx + Du \tag{33}$$

Cost functional defined as:

$$J = \int (x^T Q x + u^T R u)dt \tag{34}$$

Where Q and R are the weighting matrixes, Q is positive semi definite matrix and R is Positive definite symmetric matrix.

An energy function can be taken as, performance index (J). so, that for minimizing the performance index, the total energy of the closed-loop system should be keeps small. In performance index (*J*) the control input (u) and the state x are biased such that to keep performance index (*J*) is minimum, control input u and state x can't be too large. Performance index (*J*) is surely finite if it is minimized and since it is an infinite integral of x, this is implying that as time t tends to infinity the x tends to zero. It turns that the closed loop system will be guaranteed stable.

The positive semi definite matrices Q (of nxn) and positive definite matrix R (of m×m) are chosen by the design engineer. Depending on how these design parameters are selected, the closed-loop system will present different response.

To keep performance index *J* minimum, if design engineer select Q large enough then the state x is minimum minimum and the control input u must be small when the R is selected large. This mean that, the larger values of Q result are closed-loop system matrix $(A - BK)$ poles are lie in left of the s-plane, so that the state decays to zero [19].

Scalar quantity $x^T Qx$ is always positive or zero for all functions x at each time t, if selected Q is positive semi-definite. If R is positive definite then the scalar quantity $u^T Ru$ is always positive for all values of control u at each time t. This guarantees that in term of eigenvalues, the eigenvalues of R should be positive and eigenvalues of Q should be non-negative. The matrices Q and matrix R should be selected diagonally and all elements of matrix R and matrix Q must be chosen positive, may be probability of few zeros on its diagonal and it is note that the matrix R is must be invertible [19].

LQR Theorem 1: The closed loop system (A-BK) must be asymptotically stable if the system (A, B) is reachable and if matrix Q is a positive semi definite matrix and matrix R is positive definite matrix. [19].

LQR Theorem 2: The closed loop system (A-BK) must be asymptotically stable if the system (A, B) is stabilizable and (A, Q) is observable and if R is positive definite matrix and Q is positive semi definite matrix [19].

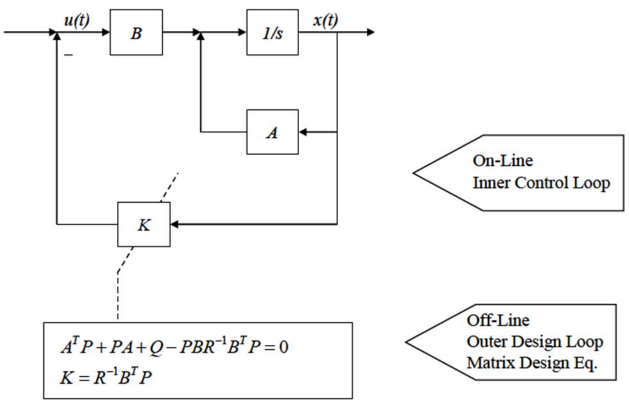

Fig.2. Linear Quadratic Regulator

Feedback control law for state is

$u = -Kx$ (35)

Tends to $\dot{x} = (A - BK)x$ (36)

Here K is derived from minimization of cost function.

LQR gain vector K can be obtaining by

$K = R^{-1}B^T P$ (37)

Where P is positive definite symmetric matrix.

P can be obtaining by Algebraic Recatti Equation (ARE):

$A^T P + PA - PBR^{-1}B^T P + Q = 0$ (38)

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \\ P_{41} & P_{42} & P_{43} & P_{44} \end{bmatrix}$$

where in Positive definite symmetric matrix P:

$$P_{12} = P_{21}, P_{31} = P_{13}, P_{32} = P_{22}, P_{42} = P_{23}, P_{34} = P_{43}$$

$$P = 1.0e + 04 * \begin{bmatrix} 1.2513 & 0.5589 & 0.5469 & 0.2685 \\ 0.5589 & 0.2511 & 0.2445 & 0.1202 \\ 0.5469 & 0.2445 & 0.2391 & 0.1174 \\ 0.2685 & 0.1202 & 0.1174 & 0.0577 \end{bmatrix}$$

Here considering Q= $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and R = [1]

Using equation $K = R^{-1}B^T P$ optimal control gain K can be obtained as:

$$k = [-242.52 \quad -96.33 \quad -104.59 \quad -49.05]$$

and by using MAT Lab command

[P, s, K] = care (A, B, Q, R) the poles of the system can be obtained as

$$s = -10.2427, -3.6577. -2.2855, -2.1966$$

Here all poles of system are lie in left half of s-plane, thus it can be say that system is stable.

V. Simulation and Results

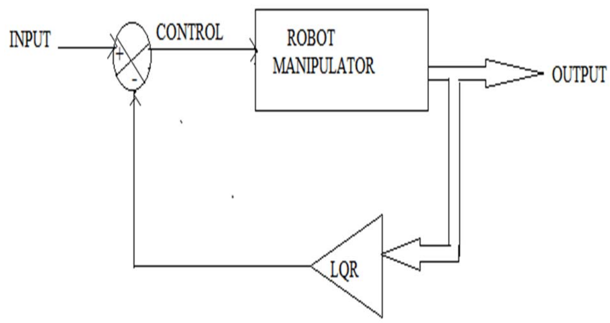

Fig.3. LQR control Robot Manipulator

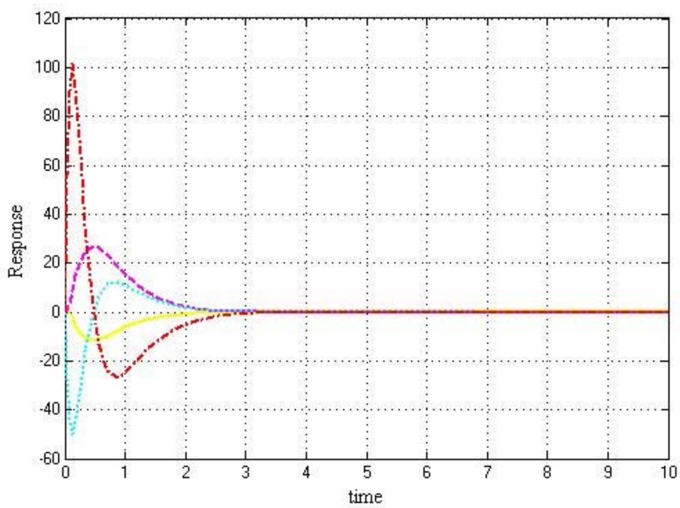

Fig.4. State response of system using LQR with initial condition

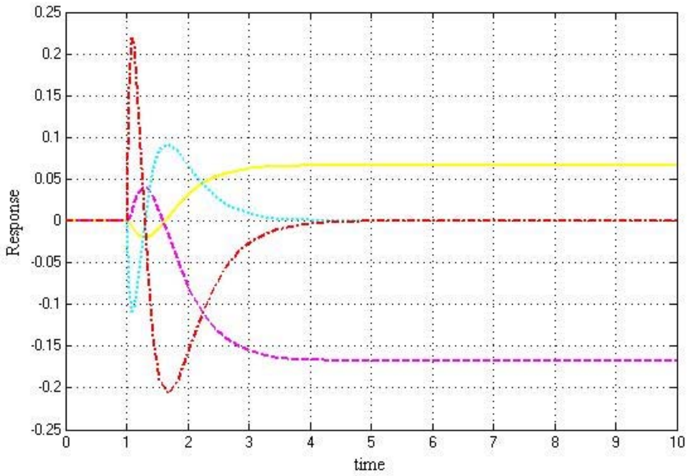

Fig.5. State response of system using LQR with zero initial condition

The closed loop response analysis with system using Linear Quadratic Regulator (LQR) in feedback is performed in MAT LAB environment.

Figure (4) shows the responses of 2-link robot manipulator together with their initial conditions (π/2,0,0,0). And Figure (5) shows response of robot manipulator when initial condition is zero. Both state response of robot manipulator shows that the response of system depends on initial conditions, thus system is locally stable.

VI. Conclusion

Using feed forward gain Linear Quadratic Regulator (LQR) is designed, and Linear Quadratic Regulator (LQR) controlled system on its different initial conditions shows different state responses which shows that system responses depends on initial conditions. Because of system response depends on initial conditions thus it can be say that LQR controlled system is locally stable. Linear Quadratic Regulator (LQR) is totally a feedback controller to establish feedback the information of error on the variables is used, which bound the utilizing potential of this controller. By using LQR control the desired angular position and velocity control is possible with better performance. Depending on the value of Q the closed loop system matrix poles are lie in left half of s-plane and state response decay faster to zero. For future scope this project simple LQR control is can be replaced by LQR and PID control. And in the same way we can replace this optimal controller with time optimal controller still for better performance. This research work can be elongated in various directions, to examine and balancing multi- links robot manipulators, adding states for more complicated and complex problems. And the same process as stated in this paper can be implement for control of multi-link robot manipulators when actuating motors are provided at the all joints.

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