

# Dynamics of Robots

We use the principle of least action [3] with  $L = T - V$  where  $T$  is the kinetic energy and  $V$  is the potential energy to get the Euler-Lagrange equations:

$$\delta \int L(t, \mathbf{q}, \dot{\mathbf{q}}) dt = 0 \Rightarrow \frac{d}{dt} \frac{\partial T}{\partial \dot{\mathbf{q}}} - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = 0$$

For a rigid body system, we can express the kinetic energy  $T$  as: [4]

$$T = \frac{1}{2} \dot{\mathbf{q}}^T M(\mathbf{q}) \dot{\mathbf{q}}$$

with  $M(\mathbf{q})$ : Inertia matrix

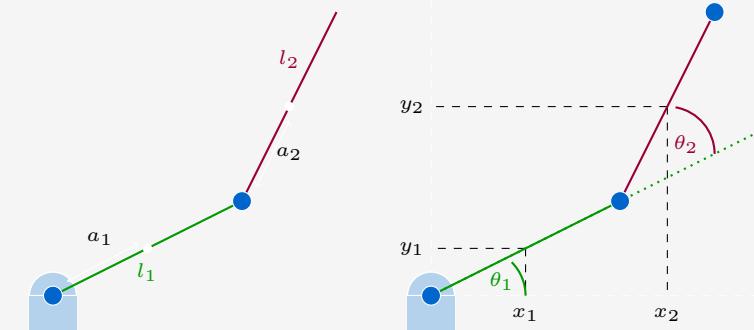
We can compute the Inertia matrix  $M(\mathbf{q})$  using the kinetic energy  $T$  of the system:

$$M(\mathbf{q}) = \sum_{i=1}^n m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_i J_{\omega_i} \quad J_v = \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \quad J_{\omega} = \frac{\partial \boldsymbol{\omega}}{\partial \dot{\mathbf{q}}}$$

where  $m_i$  is the mass of link  $i$ ,  $I_i$  is the inertia tensor of link  $i$ ,  $J_{v_i}$  is the linear velocity Jacobian of link  $i$ , and  $J_{\omega_i}$  is the angular velocity Jacobian of link  $i$ .

## Two Link Revolute

We use the generalized coordinates  $\mathbf{q} = [\theta_1, \theta_2]$  for the two-link revolute manipulator.



By using the coordinates of the centers of mass,

$$\mathbf{r}_1 = \begin{pmatrix} a_1 \cdot \cos(\theta_1) \\ a_1 \cdot \sin(\theta_1) \\ 0 \end{pmatrix} \quad \mathbf{r}_2 = \begin{pmatrix} l_1 \cdot \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2) \\ l_1 \cdot \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{pmatrix}$$

we can derive the Jacobians  $J_{v_i}$  and  $J_{\omega_i}$  and thus the inertia matrix  $M(\mathbf{q})$ . Plugging  $M(\mathbf{q})$  into the Euler-Lagrange equations, we have the following

equations of motion for a robotic manipulator:

$$M(\mathbf{q}) \ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} + G(\mathbf{q}) = \boldsymbol{\tau}$$

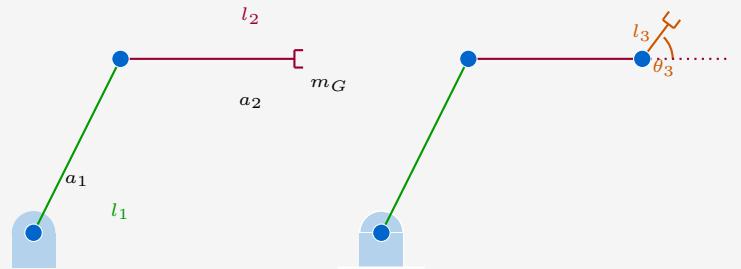
$C(\mathbf{q}, \dot{\mathbf{q}})$ : Coriolis and centrifugal matrix

$G(\mathbf{q})$ : Gravity vector

$\boldsymbol{\tau}$ : Joint torques

## Gripper extension

We can model the gripper as an additional point mass  $m_G$  at the end of the second link or as an additional third link with length  $l_3$  and mass  $m_3$ .



The second formulation requires the addition of a third generalized coordinate  $\theta_3$ , which represents the angle of the gripper relative to the second link. The new generalized coordinates are  $\mathbf{q} = [\theta_1, \theta_2, \theta_3]$ . The inertia matrix  $M(\mathbf{q})$  must be recalculated to account for the additional link.

## Foundations of Control Theory

Our formulation of an optimal control problem is based on the procedure outlined in [1]. Let  $[t_0, t_f]$  be a fixed time interval,

$$\begin{aligned} \Phi : \mathbb{R}^{n_x} &\rightarrow \mathbb{R}, \\ L : [t_0, t_f] \times \mathbb{R}^{n_x} &\rightarrow \mathbb{R}, \\ f : [t_0, t_f] \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} &\rightarrow \mathbb{R}^{n_x} \end{aligned}$$

sufficient smooth functions and  $\mathcal{U} \subset \mathbb{R}^{n_u}$  a closed convex non-empty set.

$$\min_{\mathbf{u}} \int_{t_0}^{t_f} L(t, \mathbf{x}(t), \mathbf{u}(t)) dt + \Phi(\mathbf{x}(t_f))$$

with respect to  $\mathbf{x} \in W_{1,\infty}^{n_x}([t_0, t_f])$ ,  $\mathbf{u} \in L_{\infty}^{n_u}([t_0, t_f])$  and subject to

$$\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{u}(t) \in \mathcal{U}$$

The necessary conditions for optimality are given

by minimum principle (see [1]):

$$\begin{aligned}\dot{\mathbf{p}} &= -H_{\mathbf{x}} \mathbf{p}(t_f) = \Phi_{\mathbf{x}}(\mathbf{x}(t_f)) \\ H(t, \mathbf{x}, \mathbf{u}, \mathbf{p}) &= L(t, \mathbf{x}, \mathbf{u}) + \mathbf{p}^T f(t, \mathbf{x}, \mathbf{u}) \\ \mathbf{u}^* &= \arg \min_{\mathbf{u} \in \mathcal{U}} H(t, \mathbf{x}^*, \mathbf{u}, \mathbf{p}^*)\end{aligned}$$

## Linearisation Method

Let  $\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix}$  we use the state space representation of the manipulator dynamics (compare [2]).

$$\underbrace{\begin{bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} \dot{\mathbf{q}} \\ M^{-1}(\mathbf{q}) [\mathbf{u} - C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} - G(\mathbf{q})] \end{bmatrix}}_{f(\mathbf{x}, \mathbf{u})}$$

We linearize the system around an operating point:

$$\begin{aligned}\dot{\mathbf{x}} &= f(\mathbf{x}, \mathbf{u}) \\ &\approx \underbrace{f(\mathbf{x}^*, \mathbf{u}^*)}_{=0} + \underbrace{\frac{\partial f}{\partial \mathbf{x}}(\mathbf{x}^*, \mathbf{u}^*)(\mathbf{x} - \mathbf{x}^*)}_{A(\mathbf{x}^*, \mathbf{u}^*)} + \underbrace{\frac{\partial f}{\partial \mathbf{u}}(\mathbf{x}^*, \mathbf{u}^*)(\mathbf{u} - \mathbf{u}^*)}_{B(\mathbf{x}^*, \mathbf{u}^*)} \\ &= A(\mathbf{x}^*, \mathbf{u}^*)\bar{\mathbf{x}} + B(\mathbf{x}^*, \mathbf{u}^*)\bar{\mathbf{u}}\end{aligned}$$

We get the equivalent linear system:

$$\bar{\mathbf{x}}(t) = A\bar{\mathbf{x}}(t) + B\bar{\mathbf{u}}(t) \quad \bar{\mathbf{x}} = \mathbf{0} \Leftrightarrow \mathbf{x} = \mathbf{x}^*$$

By assuming  $f(\mathbf{x}^*, \mathbf{u}^*) = \mathbf{0}$  we get  $\ddot{\mathbf{q}}^* = \dot{\mathbf{q}}^* = \mathbf{0}$ . Thus we have the following derivatives for our specific two link manipulator:

$$A = \begin{bmatrix} \mathbf{0} & I \\ M^{-1}(\mathbf{q}^*) \left[ -\frac{\partial G}{\partial \mathbf{q}}(\mathbf{q}^*) \right] & \mathbf{0} \end{bmatrix} \quad B = \begin{bmatrix} \mathbf{0} \\ M^{-1}(\mathbf{q}^*) \end{bmatrix}$$

## Solving using LQR

We can solve the nonlinear optimal control problem

$$\begin{aligned}\min_{\mathbf{u}} \int_0^\infty & (\mathbf{q} - \mathbf{q}_f)^T Q(\mathbf{q} - \mathbf{q}_f) + (\mathbf{u} - \mathbf{u}_f)^T R(\mathbf{u} - \mathbf{u}_f) dt \\ \text{s.t. } & M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + G(\mathbf{q}) = \mathbf{u}; \quad \mathbf{u}_f - G(\mathbf{q}_f) = \mathbf{0}\end{aligned}$$

by solving the linearized problem using LQR:

$$\min_{\bar{\mathbf{u}}} \int_0^\infty \bar{\mathbf{x}}^T Q \bar{\mathbf{x}} + \bar{\mathbf{u}}^T R \bar{\mathbf{u}} dt \quad \dot{\bar{\mathbf{x}}} = A\bar{\mathbf{x}} + B\bar{\mathbf{u}};$$

with  $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}_f$  and  $\bar{\mathbf{u}} = \mathbf{u} - \mathbf{u}_f$ . The solution  $K$  of the Riccati equation:

$$Q - KBR^{-1}B^T K + KA + A^T K = 0$$

gives us the optimal feedback law:

$$\begin{aligned}\mathbf{u}_i &= G(\mathbf{q}_f) - R^{-1}B^T K(\mathbf{x}_i - \mathbf{x}_f) \\ \mathbf{x}_{i+1} &= \mathbf{x}_i + h \cdot f(\mathbf{x}_i, \mathbf{u}_i)\end{aligned}$$

## Gradient Methods

We can also solve the optimal control problem

$$\begin{aligned}\min_{\mathbf{u}} \int_{t_0}^{t_f} & (\mathbf{u} - \mathbf{u}_f)^T \cdot R \cdot (\mathbf{u} - \mathbf{u}_f) dt + \Phi(\mathbf{x}(t_f)) \\ \Phi(\mathbf{x}(t_f)) &= (\mathbf{q}(t_f) - \mathbf{q}_f)^T \cdot Q \cdot (\mathbf{q}(t_f) - \mathbf{q}_f)\end{aligned}$$

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} \dot{\mathbf{q}} \\ M^{-1}(\mathbf{q}) \cdot (-C(\mathbf{q}, \dot{\mathbf{q}}) \cdot \dot{\mathbf{q}} - G(\mathbf{q}) + \mathbf{u}) \end{bmatrix}$$

The adjoint equation is given by:

$$\dot{\mathbf{p}} = -\frac{\partial f^T}{\partial \mathbf{x}} \cdot \mathbf{p}, \quad \mathbf{p}(t_f) = \begin{bmatrix} 2Q(\mathbf{q}(t_f) - \mathbf{q}_f) \\ \mathbf{0} \end{bmatrix}$$

The gradient of the Hamiltonian is given by:

$$\nabla J(\mathbf{u}) = 2R \cdot (\mathbf{u} - \mathbf{u}_f) + \begin{bmatrix} \mathbf{0}_2 \\ M^{-1}(\mathbf{q}) \end{bmatrix} \cdot \begin{bmatrix} \mathbf{p}_{\mathbf{q}} \\ \mathbf{p}_{\dot{\mathbf{q}}} \end{bmatrix}$$

1) Forward integration of the state equation

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h \cdot f(\mathbf{x}_i, \mathbf{u}_i) \Rightarrow \mathbf{X} = \{\mathbf{x}_0, \dots, \mathbf{x}_N\}$$

2) Backward integration of the adjoint equation

$$\mathbf{p}_N = \begin{bmatrix} 2Q(\mathbf{q}(t_f) - \mathbf{q}_f) \\ \mathbf{0} \end{bmatrix} \quad \mathbf{p}_{i-1} = \mathbf{p}_i + h \cdot \begin{bmatrix} \mathbf{0}_2 & I_2 \\ \frac{\partial \ddot{\mathbf{q}}}{\partial \mathbf{q}} & \frac{\partial \ddot{\mathbf{q}}}{\partial \dot{\mathbf{q}}} \end{bmatrix} \cdot \mathbf{p}_i$$

using finite differences to compute

$$\mathbf{x}_\pm = \mathbf{x} \pm \epsilon \cdot \mathbf{e}_j \frac{\partial \ddot{\mathbf{q}}}{\partial x_j} \approx \frac{\ddot{\mathbf{q}}(\mathbf{x}_+) - \ddot{\mathbf{q}}(\mathbf{x}_-)}{2\epsilon} \Rightarrow \mathbf{P} = \{\mathbf{p}_0, \dots\}$$

3) Gradient descent for  $i = 0, \dots, N-1$ :

$$\mathbf{u}_{i+1} = \mathbf{u}_i - \alpha_j \cdot [2R \cdot (\mathbf{u}_i - \mathbf{u}_f) + M^{-1}(\mathbf{q}_i) \cdot \mathbf{P}_{\dot{\mathbf{q}}, i}]$$

by using the armijo rule to  $\alpha_j$ .

## References

- [1] Matthias Gerdts. *Optimal Control of ODEs and DAEs*. Berlin: De Gruyter, 2012.
- [2] Amit Kumar, Shrey Kasera, and L. B. Prasad. “Optimal Control of 2-Link Underactuated Robot Manipulator”. In: *2017 International Conference on Innovations in Information Embedded and Communication Systems (ICI-IECS)*. Gorakhpur, India, 2017.
- [3] Bruno Siciliano et al. *Robotics: Modelling, Planning and Control*. London: Springer, 2010. ISBN: 978-1849964507.
- [4] Mark W. Spong, Seth Hutchinson, and M. Vidyasagar. *Robot Dynamics and Control*. 2nd. Hoboken, NJ: Wiley, 2004. ISBN: 978-0471649908.