An Optimal Control Approach to Robust Control of Robot Manipulators

Feng Lin and Robert D. Brandt

***Abstract*—We present a new optimal control approach to robust control of robot manipulators in the framework of Lin *et al* [7]. Because of the unknown load placed on a manipulator and the other uncertainties in the manipulator dynamics, it is important to design a robust control law that will guarantee the performance of the manipulator under these uncertainties. To solve this robust control problem, we first translate the robust control problem into an optimal control problem, where the uncertainties are reflected in the performance index. We then use the optimal control approach to solve the robust control problem. We show that the solution to the optimal control problem is indeed a solution to the robust control problem. We illustrate this approach using a two-joint SCARA type robot, whose robust control is obtained by solving an algebraic Riccati equation.**

***Index Terms*— Manipulators, optimal control, robust control, uncertainties.**

I. Introduction

WE PROPOSE a new approach to robust control of robot manipulators. The approach translates a robust control problem into an optimal control problem in the framework of Lin *et al*. [7]–[14]. It solves the robust control problem by solving the optimal control problem, which is much easier to do in many cases.

The manipulator control problem to be solved can be described as follows. A robot manipulator is controlled to move an unknown object. To control the manipulator, the following uncertainties must be dealt with:

1) the weight of the object is unknown because the object itself is not known before-hand;
2) the friction and other parameters in the manipulator dynamics may be uncertain because it is difficult to model and measure them precisely.

Our goal is to design a robust control that can handle these uncertainties.

Such robust control problems have been studied extensively in the literature. A survey (up to 1991) on robust control of robots is given in [1]. In [1], various approaches are classified into five categories (see, e.g. [2], [5], [16], and [17]):

1) linear-multivariable approach;
2) passivity-based approach;
3) variable-structure controllers;
4) robust saturation approach;
5) robust adaptive approach.

More recently, parametric uncertainties are dealt with in [19] and the results are extended to include also nonparametric uncertainties [15]. Although our approach is fundamentally different from all the above approaches, it bears resemblance to the robust saturation approach in the sense that some kind of Lyapunov arguments is used.

We apply the optimal control approach first proposed by Lin *et al*. [9], [12]–[14] to find a robust control for manipulators. In [9], a robust control problem is introduced using the state space representation, where the uncertainty is a function of state and matching condition is assumed. Furthermore, there is no uncertainty in the control input matrix. This robust control problem is then translated into an optimal control problem, where the uncertainty is reflected in the performance index. It is shown in [9] that if the solution to the optimal control problem exists, then it is also a solution to the robust control problem. In [12], the matching condition is relaxed. If it is not totally matched, the uncertainty is decomposed into a matched component and an unmatched component. The robust control problem is similarly translated into an optimal control problem, except that an augmented control is introduced to compensate for the unmatched uncertainty. It is proved in [12] that under a condition on the magnitude of the unmatched uncertainty, the solution to the optimal control problem is a solution to the robust control problem. This optimal control approach has been successfully applied to hovering control of V/STOL aircraft [14] and active damping of vibration systems [13].

To apply this optimal control approach to the robust control problem for robot manipulators as described above, we must first extend the theoretical result in [12] by relaxing the assumption that no uncertainty is present in the input matrix. We show that this extension can be done as long as the matrix representing the uncertainty in the control input is positively semi-definite. This new result fits nicely with the robust control problem we want to solve for robot manipulators.

In Section II, we consider the differential equation describing the dynamics of robot manipulators and show how the uncertainties discussed above can be incorporated in the equation. We then formulate the robust control problem for robot manipulators.

In Section III, we extend a optimal control approach for solving the type of robust control problems proposed in Section II. We discuss three cases and show how the optimal

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control approach can be used. This section will establish theoretical backgrounds needed for the development in Section IV.

We then go back in Section IV to the robust control problem for robot manipulators proposed in Section II and derive a solution using the optimal control approach. Finally, in Section V we will illustrate our approach by applying it to a two-joint SCARA-type robot. We select this robot because its dynamics includes many of the coupling effects common to other types of robots.

Some of the advantages of our approach are as follows.

1) It is conceptually simple: Instead of solving a complicated robust control problem, we will solve a standard optimal control problem. The solution obtained is guaranteed to be robust.
2) It is practically useful: In many cases, solving the optimal control problem only involves solving an algebraic Riccati equation. This is much easier to do than to tackle the robust control problem directly.
3) It is flexible: While always guaranteeing the robustness, our approach also allows compromise between fast response time and small control input by adjusting the relative weights of states and control inputs in the cost function.

II. Manipulator Dynamics

The dynamics of a robot manipulator is well understood (see e.g. [6], [18]) and is given by

$$M(q)\ddot{q} + V(q, \dot{q}) + F(\dot{q}) + G(q) = \tau$$

where q consists of the joint variables, τ is the generalized forces, $M(q)$ the inertia matrix, $V(q, \dot{q})$ the Coriolis/centripetal vector, $G(q)$ the gravity vector, and $F(\dot{q})$ the friction vector. For simplicity, we denote

$$N(q, \dot{q}) = V(q, \dot{q}) + F(\dot{q}) + G(q).$$

There are uncertainties in $M(q)$ and $N(q, \dot{q})$ due to, say, unknown load on the manipulator and unmodeled frictions. We assume the following bounds on the uncertainties.

1) There exist positive definite matrices $M_o(q)$ and $M_{\min}(q)$ such that[1]

$$M_o(q) \geq M(q) \geq M_{\min}(q) > 0.$$

2) There exist $N_o(q, \dot{q})$ and a nonnegative function $n_{\max}(q, \dot{q})$ such that $n_{\max}(0, 0) = 0$[2] and

$$||N(q, \dot{q}) - N_o(q, \dot{q})|| \leq n_{\max}(q, \dot{q}).$$

[1] Here by the inequality $A \geq B$, we mean $A - B$ is semi-positive definite.

[2] This implies that the gravity vector is known at the equilibrium 0. This assumption is not difficult to satisfy because in most applications, one can select coordinates so that the gravity at $q = 0$ is canceled.

Our control objective is to move the robot manipulator from some initial position to $(q, \dot{q}) = (0, 0)$. A more general control objective can be studied without much difficulty if we introduce a desired trajectory.

Define the state variables to be

$$x_1 = q$$
$$x_2 = \dot{q}$$

and the control to be

$$u = M_o(q)^{-1}[\tau - N_o(q, \dot{q})].$$

Then

$$\begin{aligned}
\dot{x}_1 &= \dot{q} = x_2\\
\dot{x}_2 &= \ddot{q} = M(q)^{-1}[\tau - N(q, \dot{q})]\\
&= M(q)^{-1}[\tau - N_o(q, \dot{q})] + M(q)^{-1}[N_o(q, \dot{q}) - N(q, \dot{q})]\\
&= M(q)^{-1}M_o(q)u + M(q)^{-1}[N_o(q, \dot{q}) - N(q, \dot{q})]\\
&= M(x_1)^{-1}M_o(x_1)u + M(x_1)^{-1}[N_o(x_1, x_2)\\
&\quad - N(x_1, x_2)].
\end{aligned}$$

Let us define

$$h(x) = M(x_1)^{-1}M_o(x_1) - I$$

and

$$f(x) = M(x_1)^{-1}(N_o(x_1, x_2) - N(x_1, x_2)).$$

Then the state equation is given by

$$\dot{x} = Ax + B(u + h(x)u) + Bf(x)$$

where

$$x = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix};$$
$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix};$$
$$B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Our goal is to control the above system under the uncertainties $f(x)$ and $h(x)$. In the next section, we will present an optimal control approach to solve this type of robust control problems.

III. Optimal Control Approach

In this section we first review an optimal control approach to the robust control problem (details can be found in [7], [9], and [12]). We then extend it to handle uncertainty in the control input matrix. We start with the the following simple case.

A. Matched Uncertainty

Consider the following nonlinear system:

$$\dot{x} = A(x) + B(x)u + B(x)f(x)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. $B(x)f(x)$ models the uncertainty in the system dynamics. Since the uncertainty is in the range of $B(x)$, we say that the *matching condition* is satisfied. We will assume $f(0) = 0$ and $A(0) = 0$ so that $x = 0$ is an equilibrium. In this paper, stability is always with respect to $x = 0$. Our goal is to solve the following robust control problem.

1) Robust Control Problem: Find a feedback control law $u = u_0(x)$ such that the closed-loop system

$$\dot{x} = A(x) + B(x)u_0(x) + B(x)f(x)$$

is globally asymptotically stable for all uncertainties $f(x)$ satisfying the condition that there exists a nonnegative function $f_{\max}(x)$ such that[3]

$$||f(x)|| \le f_{\max}(x).$$

We would like to translate this robust control problem into the following optimal control problem.

2) Optimal Control Problem: For the following auxiliary system

$$\dot{x} = A(x) + B(x)u$$

find a feedback control law $u = u_0(x)$ that minimizes the following cost functional

$$\int_0^\infty [f_{\max}(x)^2 + ||x||^2 + \gamma^2 ||u||^2]\, dt$$

where $\gamma \neq 0$ is a design parameter.

The relation between the robust control problem and the optimal control problem is shown in the following theorem [9].

Theorem 1 If the solution to the optimal control problem exists, then it is a solution to the robust control problem.

By this theorem, we can translate the robust control problem into the optimal control problem. In many cases, the optimal control problem is much easier to solve. For example, if it happens that the system is linear

$$\dot{x} = Ax + Bu + Bf(x)$$

and the uncertainty $f(x)$ satisfies the following condition: $||f(x)||^2$ is quadratically bounded as follows:

$$f(x)^T f(x) \le x^T P x$$

for some positive definite matrix P. Then the optimal control problem reduces to the following linear-quadratic-regulator (LQR) problem: For the auxiliary system

$$\dot{x} = Ax + Bu$$

find a feedback control law $u = u_0(x)$ that minimizes the following cost function:

$$\int_0^\infty (x^T P x + x^T x + \gamma^2 u^T u)\, dt.$$

The solution to this LQR problem always exists and can be obtained by solving an algebraic Riccati equation.

It is noteworthy that our result holds for all possible $\gamma \neq 0$. γ can be used to adjust the relative weights of states and control inputs. For a large γ, the control (i.e., feedback gain) will be small (at the expense of response time). Nevertheless, the robustness is guaranteed for all $\gamma \neq 0$. Since the optimal selection of γ is beyond the scope of this paper, we shall take $\gamma = 1$ in the rest of the paper.

[3] By our assumption $f(0) = 0, f_{\max}(0) = 0$.

B. Unmatched Uncertainty

Now we assume that uncertainty is not in the range of $B(x)$. Consider the following nonlinear system:

$$\dot{x} = A(x) + B(x)u + C(x)f(x)$$

where $C(x)$ is a matrix of dimension $n \times p$ and $C(x) \neq B(x)$. Again, we will assume $f(0) = 0$ and $A(0) = 0$ so that $x = 0$ is an equilibrium. We need to solve the following robust control problem.

1) Robust Control Problem: Find a feedback control law $u = u_0(x)$ such that the closed-loop system

$$\dot{x} = A(x) + B(x)u_0(x) + C(x)f(x)$$

is globally asymptotically stable for all uncertainties $f(x)$ satisfying the following conditions.

1) There exists a nonnegative function $f_{\max}(x)$ such that

$$||B(x)^+ C(x) f(x)|| \le f_{\max}(x).$$

where $^+$ denotes the Moore-Penrose pseudo-inverse.

2) There exists a nonnegative function $g_{\max}(x)$ such that

$$||f(x)|| \le g_{\max}(x).$$

Obviously, the second assumption implies the first, but we spell out each of them so that we can use the weakest possible assumption in a particular situation.

We decompose the uncertainty $C(x)f(x)$ into the sum of a matched component and an unmatched component by projecting $C(x)f(x)$ onto the range of $B(x)$. Thus, the decomposition becomes

$$C(x)f(x) = B(x)B(x)^+C(x)f(x) + [I - B(x)B(x)^+] \cdot C(x)f(x).$$

This robust control problem is translated into the following optimal control problem.

2) Optimal Control Problem: For the following auxiliary system

$$\dot{x} = A(x) + B(x)u + [I - B(x)B(x)^+]C(x)v$$

find a feedback control law $(u_0(x), v_0(x))$ that minimizes the following cost functional:

$$\int_0^\infty [f_{\max}(x)^2 + \rho^2 g_{\max}(x)^2 + \beta^2 ||x||^2 + ||u||^2 + \rho^2 ||v||^2]\, dt$$

where ρ and β are some constants that serve as design parameters.

We can prove the following result [12].

Theorem 2 If one can choose ρ and β such that the solution to the optimal control problem denoted by $(u_0(x), v_0(x))$ exists and the following condition is satisfied

$$2\rho^2 ||v_0(x)||^2 \le \beta'^2 ||x||^2, \quad \forall x \in \mathbb{R}^n$$

for some $\beta' < \beta$, *then* $u_0(x)$, the u-component of the solution to the optimal control problem, is a solution to the robust control problem.

Comparing with the case of matched uncertainty, an additional condition is needed to ensure that the unmatched part

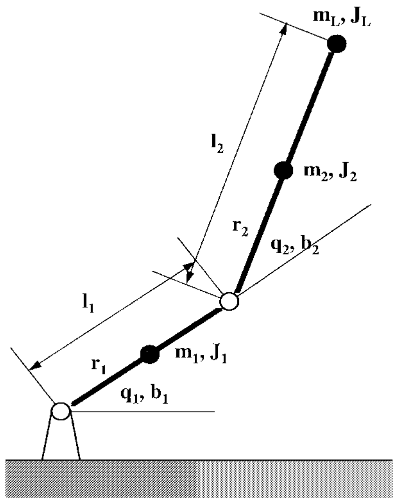

Fig. 1. SCARA robot.

of the uncertainty is not too large. As shown in [12], this condition is sufficient but not necessary. By proper choice of ρ and β, this sufficient condition can be satisfied. One example is given in [14].

C. Uncertainty in the Input Matrix

We now generalize our optimal control approach to robust control of nonlinear systems reviewed above to handle uncertainty in the control input matrix.

Consider the following nonlinear system

$$\dot{x} = A(x) + B(x)(u + h(x)u) + C(x)f(x)$$

where $B(x)h(x)$ models the uncertainty in the input matrix. The robust control problem is described as follows.

1) Robust Control Problem: Find a feedback control law $u = u_0(x)$ such that the closed-loop system

$$\dot{x} = A(x) + B(x)(u_0(x) + h(x)u_0(x)) + C(x)f(x)$$

is globally asymptotically stable for all uncertainties $f(x)$ and $h(x)$ satisfying the following conditions.

1) $h(x) \geq 0$.
2) There exists a nonnegative function $f_{\max}(x)$ such that

$$\|B(x)^+C(x)f(x)\| \leq f_{\max}(x).$$

3) There exists a nonnegative function $g_{\max}(x)$ such that

$$\|f(x)\| \leq g_{\max}(x).$$

We decompose the uncertainty $C(x)f(x)$ as before.

$$C(x)f(x) = B(x)B(x)^+C(x)f(x) + [I - B(x)B(x)^+] \cdot C(x)f(x).$$

The corresponding optimal control problem is as follows.

2) Optimal Control Problem: For the following auxiliary system

$$\dot{x} = A(x) + B(x)u + [I - B(x)B(x)^+]C(x)v$$

find a feedback control law $(u_0(x), v_0(x))$ that minimizes the following cost functional:

$$\int_0^\infty [f_{\max}(x)^2 + \rho^2 g_{\max}(x)^2 + \beta^2\|x\|^2 + \|u\|^2 + \rho^2\|v\|^2]\, dt$$

where ρ and β are design parameters.

The relation between the robust control problem and the optimal control problem is shown in the following theorem.

Theorem 3 If one can choose ρ and β such that the solution to the optimal control problem denoted by $(u_0(x), v_0(x))$ exists and the following condition is satisfied

$$2\rho^2\|v_0(x)\|^2 \leq \beta'^2\|x\|^2, \quad \forall x \in \mathbb{R}^n$$

for some $\beta' < \beta$, then $u_0(x)$, the u-component of the solution to the optimal control problem, is a solution to the robust control problem.

Proof: To simplify the notation, in this proof we will eliminate, when possible, the explicit reference to x when we denote a function of x. Define

$$V(x_0) = \min_{u\in\mathbb{R}^m, v\in\mathbb{R}^p} \int_0^\infty [f_{\max}(x)^2 + \rho^2 g_{\max}(x)^2 + \beta^2\|x\|^2 + \|u\|^2 + \rho^2\|v\|^2]\, dt$$

to be the minimum cost of the optimal control that brings the auxiliary system from the initial state x_0 to its equilibrium ($x = 0$). The Hamilton-Jacobi-Bellman equation gives us

$$\min_{u\in\mathbb{R}^m, v\in\mathbb{R}^p} [f_{\max}^2 + \rho^2 g_{\max}^2 + \beta^2\|x\|^2 + \|u\|^2 + \rho^2\|v\|^2 + V_x^T(A + Bu + (I - BB^+)Cv)] = 0$$

where $V_x(x) = \nabla V(x)$. Therefore, if $(u_0(x), v_0(x))$ is the solution to the optimal control problem, then

$$f_{\max}^2 + \rho^2 g_{\max}^2 + \beta^2\|x\|^2 + \|u_0\|^2 + \rho^2\|v_0\|^2 + V_x^T[A + Bu_0 + (I - BB^+)Cv_0] = 0$$

$$2u_0^T + V_x^T B = 0$$
$$2\rho^2 v_0^T + V_x^T[I - BB^+]C = 0.$$

We now show that u_0 is a solution to the robust control problem, i.e., the equilibrium $x = 0$ of

$$\dot{x} = A + B(u_0 + hu_0) + Cf$$

is globally asymptotically stable for all admissible uncertainties f and h. To do this, we show that $V(x)$ is a Lyapunov function. Clearly

$$V(x) > 0, x \neq 0;$$
$$V(x) = 0, x = 0.$$

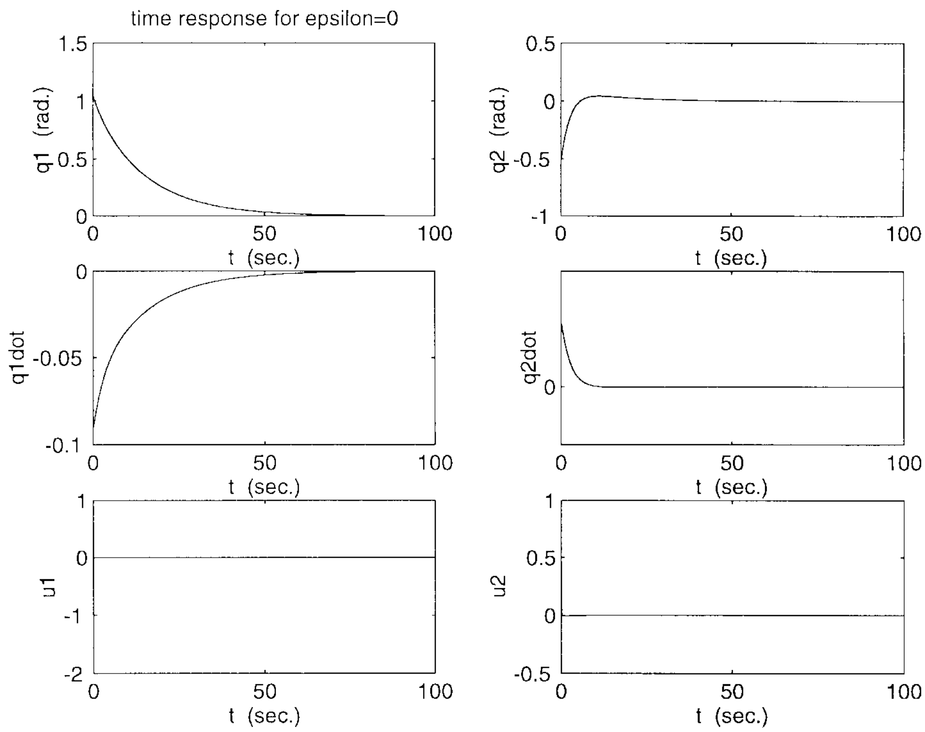

Fig. 2. Response of SCARA robot for $m_L = 0$ oz.

To show $\dot{V}(x) = dV(x)/dt < 0$ for $x \neq 0$, we have

$$\begin{aligned}\dot{V}(x) &\\ &= V_x^T[A + Bu_0 + Cf + Bhu_0] \\ &= V_x^T[A + Bu_0 + (I - BB^+)Cv_0 \\ &\quad + BB^+Cf + (I - BB^+)C(f - v_0) + Bhu_0] \\ &= V_x^T[A + Bu_0 + (I - BB^+)Cv_0] \\ &\quad + V_x^T BB^+Cf + V_x^T(I - BB^+)C(f - v_0) + V_x^T Bhu_0 \\ &= -f_{\max}^2 - \rho^2 g_{\max}^2 - \beta^2||x||^2 - ||u_0||^2 - \rho^2||v_0||^2 \\ &\quad - 2u_0^T B^+Cf - 2\rho^2 v_0^T(f - v_0) - 2u_0^T hu_0.\end{aligned}$$

Since

$$\begin{aligned}-||u_0||^2 - 2u_0^T B^+Cf &= -||u_0 + B^+Cf||^2 + ||B^+Cf||^2 \\ -2\rho^2 v_0^T f &\leq \rho^2(||v_0||^2 + ||f||^2) \\ -2u_0^T hu_0 &\leq 0\end{aligned}$$

we then have

$$\begin{aligned}\dot{V}(x) &\leq -[f_{\max}^2 - ||B^+Cf||^2] - \rho^2[g_{\max}^2 - ||f||^2] \\ &\quad - ||u_0 + B^+Cf||^2 + 2\rho^2||v_0||^2 - \beta^2||x||^2 \\ &\leq 2\rho^2||v_0||^2 - \beta^2||x||^2 \\ &= 2\rho^2||v_0||^2 - \beta'^2||x||^2 - (\beta^2||x||^2 - \beta'^2||x||^2) \\ &\leq -(\beta^2 - \beta'^2)||x||^2 < 0.\end{aligned}$$

Thus, the conditions of the Lyapunov local stability theorem are satisfied. Consequently, there exists a neighborhood $\mathcal{N} = \{x\colon ||x|| \leq c\}$ for some $c > 0$ such that if $x(t)$ enters $\mathcal{N}$, then

$$\lim_{t \to \infty} x(t) = 0.$$

But $x(t)$ cannot remain forever outside $\mathcal{N}$. Otherwise

$$||x(t)|| \geq c$$

for all $t \geq 0$. Then

$$\begin{aligned}V(x(t)) - V(x(0)) &= \int_0^t \dot{V}(x(\tau))\, d\tau \\ &\leq -(\beta^2 - \beta'^2) \int_0^t ||x(\tau)||^2\, d\tau \\ &\leq -(\beta^2 - \beta'^2)c^2 t.\end{aligned}$$

Letting $t \to \infty$, we have

$$V(x(t)) \leq V(x(0)) - (\beta^2 - \beta'^2)c^2 t \to -\infty$$

which contradicts the fact that $V(x(t)) \geq 0$ for all $x(t)$. Therefore

$$\lim_{t \to \infty} x(t) = 0$$

no matter where the trajectory begins. ■

Because of the above theorems, we can translate the robust control problem into the optimal control problem. So instead of solving the robust control problem, we can solve the optimal control problem. This is easier to do for robot manipulators. Of course, if the translated optimal control problem has no solution, then nothing can be said about the solution to the robust control problem.

IV. Robust Control of Manipulator

Let us now apply the above optimal control approach to robust control of robot manipulators. As shown in Section II, the manipulator dynamics can be formulated as

$$\dot{x} = Ax + B(u + h(x)u) + Bf(x)$$

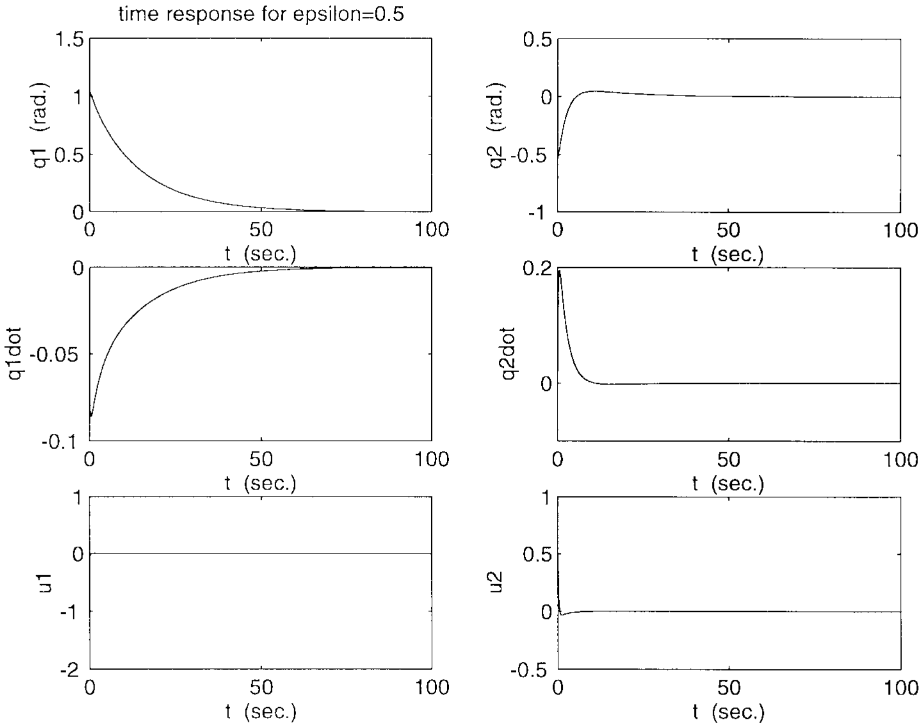

Fig. 3. Response of SCARA robot for $m_L = 5$ oz.

where

$$h(x) = M(x_1)^{-1}M_o(x_1) - I$$

and

$$f(x) = M(x_1)^{-1}(N_o(x_1, x_2) - N(x_1, x_2)).$$

The uncertainties $f(x)$ and $h(x)$ have the following bounds:

$$h(x) = M(x_1)^{-1}M_o(x_1) - I \geq 0$$

and

$$\begin{aligned} ||f(x)|| &= ||M(x_1)^{-1}(N_o(x_1, x_2) - N(x_1, x_2))|| \\ &\leq ||M(x_1)^{-1}|| \times ||(N_o(x_1, x_2) - N(x_1, x_2))|| \\ &\leq ||M_{\min}(x_1)||^{-1} n_{\max}(x_1, x_2). \end{aligned}$$

If we define $g_{\max}$ to be a bound for $||f(x)||$, for example, we can take $g_{\max} = ||M_{\min}(x_1)||^{-1} n_{\max}(x_1, x_2)$, then we have reformulated the robust control problem into the form studied in Section III. Note that the matching condition holds for robot manipulators, because $C(x) = B(x)$. Hence we can select $\rho = 0$ and $\beta = 1$ so that the sufficient condition

$$2\rho^2 ||v_0(x)||^2 \leq \beta'^2 ||x||^2 \quad \forall x \in \mathbb{R}^n$$

is always satisfied.

Although for the above $f(x)$, $||f(x)||^2$ may not be quadratically bounded, we can, in many cases, find the largest physically feasible region of x and determine a quadratic bound for $||f(x)||^2$. Assume such a quadratic bound is given by

$$f(x)^T f(x) \leq x^T P x$$

for some positive definite matrix P. Then, as shown in Section III, we only need to solve the following LQR problem: For the system

$$\dot{x} = Ax + Bu$$

find a feedback control law $u(x)$ that minimizes the cost function

$$\int_0^\infty (x^T P x + x^T x + u^T u)\, dt.$$

The solution can be obtained by solving the following algebraic Riccati equation:

$$A^T S + SA + P + I - SBB^T S = 0$$

and the optimal control is given by

$$u_0 = -B^T S x.$$

To solve the Riccati equation, we can take the advantage of the special structure of A and B

$$A = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}$$
$$S = \begin{bmatrix} S_1 & S_2 \\ S_2 & S_3 \end{bmatrix}.$$

Then the Riccati equation reduces to

$$\begin{aligned} P_1 + I - S_2^2 &= 0 \\ S_1 + P_2 - S_2 S_3 &= 0 \\ 2S_2 + P_3 + I - S_3^2 &= 0 \end{aligned}$$

and the optimal control is obtained as

$$u_0 = -(P_1 + I)^{1/2} x_1^T - (2(P_1 + I)^{1/2} + P_3 + I)^{1/2} x_2^T.$$

This is the robust control law for the robot manipulator.

V. Example

We now illustrate the optimal control approach by an example of a two-joint SCARA-type robot (see e.g., [3] and [4]). The configuration of the robot manipulator and its parameters are shown in Fig. 1.

The robot manipulator has two joint variables: two angles q_1 and q_2. The inertia matrix is

$$M(q) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where

$$\begin{aligned} M_{11} &= J_1 + m_1 r_1^2 + J_2 + m_2(l_1^2 + r_2^2 + 2l_1 r_2 \cos q_2) \\ &\quad + J_L + m_L(l_1^2 + l_2^2 + 2l_1 l_2 \cos q_2) \\ M_{12} &= M_{21} = J_2 + m_2(r_2^2 + l_1 r_2 \cos q_2) + J_L + m_L l_2^2 \\ M_{22} &= J_2 + m_2 r_2^2 + J_L + m_L l_2^2. \end{aligned}$$

The centripetal vector is

$$V(q, \dot{q}) = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

where

$$\begin{aligned} V_1 &= (m_2 l_1 r_2 + m_L l_1 l_2)(\dot{q}_2^2 + 2\dot{q}_1 \dot{q}_2) \sin q_2 \\ V_2 &= (m_2 l_1 r_2 + m_L l_1 l_2)\dot{q}_1^2 \sin q_2. \end{aligned}$$

The friction vector is

$$F(q, \dot{q}) = \begin{bmatrix} b_1 \dot{q}_1 \\ b_2 \dot{q}_2 \end{bmatrix}.$$

Without loss of generality, we assume that the gravity vector $G(q) = 0$, otherwise, it is easy to calculate the additional torque to balance the gravity. The values of the parameters are listed below

$$\begin{aligned} m_1 &= 13.86 \text{ oz} \\ m_2 &= 3.33 \text{ oz} \\ J_1 &= 62.39 \text{ oz-in/rad/s}^2 \\ J_2 &= 16.70 \text{ oz-in/rad/s}^2 \\ r_1 &= 6.12 \text{ in} \\ r_2 &= 3.22 \text{ in} \\ l_1 &= 8 \text{ in} \\ l_2 &= 6 \text{ in} \\ b_1 &= 0.2 \text{ oz-in/rad/s} \\ b_2 &= 0.5 \text{ oz-in/rad/s.} \end{aligned}$$

The load is unknown, so we assume

$$\begin{aligned} m_L &= 10\epsilon \text{ oz} \\ J_L &= 60\epsilon^2 \text{ oz-in/rad/s}^2 \end{aligned}$$

where $\epsilon \in [0, 1]$.

With these values, we can calculate $M(q)$ as shown at the bottom of the page and $N(q, \dot{q})$ as follows:

$$N(q, \dot{q}) = \begin{bmatrix} (86 + 480\epsilon)(\dot{q}_2^2 + 2\dot{q}_1 \dot{q}_2) \sin q_2 + 0.2\dot{q}_1 \\ (86 + 480\epsilon)\dot{q}_1^2 \sin q_2 + 0.5\dot{q}_2. \end{bmatrix}.$$

Therefore, we can select the following $M_o(q)$ and $N_o(q, \dot{q})$:

$$\begin{aligned} M_o(q) &= \begin{bmatrix} 1906 + 1132 \cos q_2 & 471 + 86 \cos q_2 \\ 471 + 86 \cos q_2 & 471 \end{bmatrix} \\ N_o(q, \dot{q}) &= \begin{bmatrix} 566(\dot{q}_2^2 + 2\dot{q}_1 \dot{q}_2) \sin q_2 + 0.2\dot{q}_1 \\ 566\dot{q}_1^2 \sin q_2 + 0.5\dot{q}_2 \end{bmatrix}. \end{aligned}$$

The state equation is

$$\dot{x} = Ax + B(u + h(x)u) + Bf(x)$$

where

$$\begin{aligned} [x_1 x_2 x_3 x_4]^T &= [q_1 \quad q_2 \quad \dot{q}_1 \quad \dot{q}_2]^T \\ h(x) &= M(q)^{-1} M_o(q) - I \geq 0 \end{aligned}$$

and $f(x)$ as shown at the bottom of the previous page.

Since x_1 and x_2 take values only in the interval $[-\pi, \pi]$ and x_3 and x_4 are bounded by the limit on the speed of the

$$M(q) = \begin{bmatrix} 846 + 60\epsilon^2 + 1000\epsilon + 172 \cos q_2 + 960\epsilon \cos q_2 & 51 + 60\epsilon^2 + 360\epsilon + 86 \cos q_2 \\ 51 + 60\epsilon^2 + 360\epsilon + 86 \cos q_2 & 51 + 60\epsilon^2 + 360\epsilon \end{bmatrix}$$

$$\begin{aligned} f(x) &= M(q)^{-1}(N_o(q, \dot{q}) - N(q, \dot{q})) \\ &= \begin{bmatrix} 846 + 60\epsilon^2 + 1000\epsilon + 172 \cos q_2 + 960\epsilon \cos q_2 & 51 + 60\epsilon^2 + 360\epsilon + 86 \cos q_2 \\ 51 + 60\epsilon^2 + 360\epsilon + 86 \cos q_2 & 51 + 60\epsilon^2 + 360\epsilon \end{bmatrix}^{-1} \\ &\quad \cdot \begin{bmatrix} 480(1 - \epsilon)(x_4^2 + 2x_3 x_4) \sin x_2 + 0.2x_1 \\ 480(1 - \epsilon)x_3^2 \sin x_2 + 0.5x_2 \end{bmatrix} \end{aligned}$$

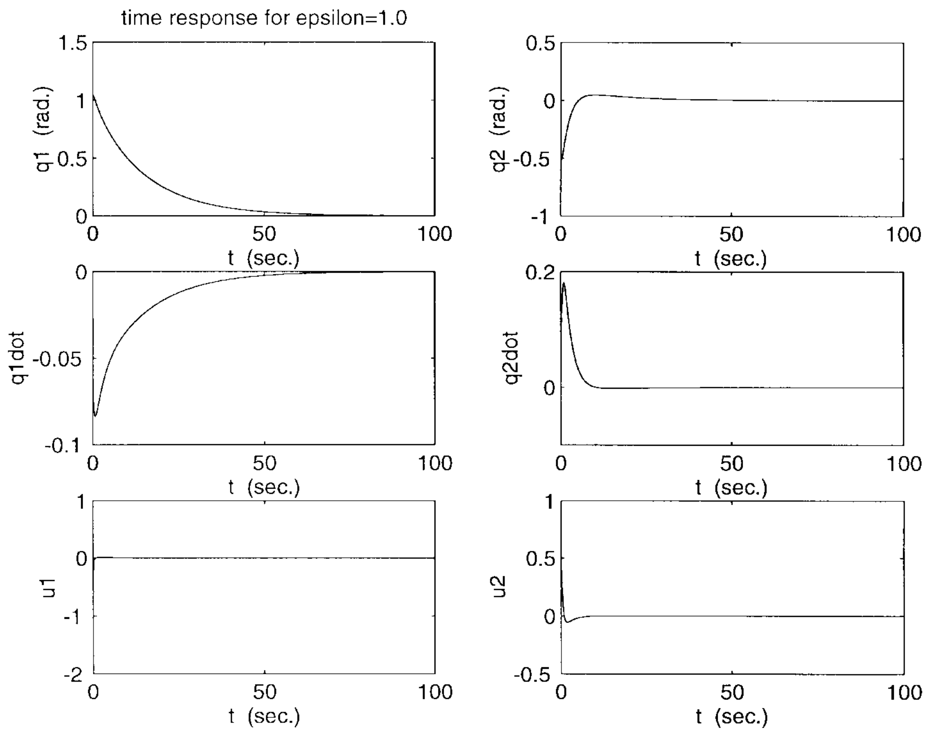

Fig. 4. Response of SCARA robot for $m_L = 10$ oz.

manipulator, we can find a quadratic bound for $||f(x)||^2$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 218 & 24 \\ 0 & 0 & 24 & 7 \end{bmatrix}.$$

The corresponding control law is

$$u_0 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 14.8038 & 1.3591 \\ 1.3591 & 2.8553 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}.$$

We have performed the following simulations for this robust control. We take the initial position of the manipulator to be $q_1 = 60^\circ = \pi/3$ and $q_2 = -30^\circ = -\pi/6$. The simulation results for $\epsilon = 0$, 0.5, 1 are shown in Figs. 2–4, respectively. From the figures, we can see that the control is indeed very robust with respect to the change in the load.

Note that the response (settling) times and the magnitudes of control inputs depend on the relative weights of states and control inputs in the cost function (i.e., γ discussed in the end of Section III-A, which we take to be 1). By using a small γ, we will achieve fast response times (at the expense of large control inputs and large overshoots). Such performance considerations are subjects of future research.

VI. Conclusion

We presented a new design method for robust control of manipulators. The method is based on an optimal control approach that we have developed recently [7]. This approach has been applied not only to manipulator control, but also to other difficult linear and nonlinear robust control problems [13], [14]. We believe that the approach holds great promise to a wide range of practical problems. Unlike some other approaches to robust control where the success is measured in terms of the complexity of mathematics, we measure the success of our approach in terms of its simplicity, and hence applicability. Indeed, it is hard to imagine that any other robust control design methods can be simpler and more effective than ours: To use our optimal control approach, only an algebraic Riccati equation needs to be solved, while our results are as good as, if not better, than those based on much more complicated approaches.

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