Sheet 5 Exercise 3

Probability of a randomly chosen node having degree k: p_k.

Probability of a randomly chosen edge having a node of degree k attached to it: $q_k = C \cdot k \cdot p_k$

Network: Power-law degree distribution ($2 < \gamma < 3$, k_min, k_max)

1. Find C

Solution
Start by noting that the q_k are probabilities and therefore sum up to 1 (i.e. a random edge has to connect to nodes of some degree between k_{min} and k_{max}):

$$\sum_{k=k_{min}}^{k_{max}} q_k = 1$$

By the definition of q_k, then:

$$\begin{aligned} &\sum_{k=k_{min}}^{k_{max}} Ckp_k = 1 \\ \Leftrightarrow &C \sum_{k=k_{min}}^{k_{max}} kp_k = 1 \\ \Leftrightarrow &C < k >= 1 \\ \Leftrightarrow &C = \frac{1}{< k >} \end{aligned}$$

(Recall that for such a network, we do indeed have the first moment $< k >= \sum_{k=k_{min}}^{k_{max}} kp_k$, as seen in statement (4.16) of the slides for $n = 1$.)
2. Determine the average degree of a random node's neighbors, bearing in mind that q_k is the probability of a neighbor having degree k.

Solution
The average degree of the neighbors, which we will call $< k_{nei} >$, is the expected value of the neighboring degrees, i.e.

$$< k_{nei} >= \sum_{k=k_{min}}^{k_{max}} P(\text{neighbor has degree } k) \cdot k = \sum_{k=k_{min}}^{k_{max}} q_k k = \sum_{k=k_{min}}^{k_{max}} Ck^2 p_k \overset{1.}{=} \frac{1}{< k >} \sum_{k=k_{min}}^{k_{max}} k^2 p_k = \frac{< k^2 >}{< k >}$$

This is perhaps the most expressive way of transforming $< k_{nei} >$, but for computations in the next part, it will prove helpful to further transform the above expression:

$$< k_{nei} >= \frac{1}{< k >} \sum_{k=k_{min}}^{k_{max}} k^2 \tilde{C} k^{-\gamma} = \frac{\tilde{C}}{< k >} \sum_{k=k_{min}}^{k_{max}} k^{2-\gamma}$$

where $\tilde{C} = \left(\sum_{k=k_{min}}^{k_{max}} k^{-\gamma} \right)^{-1}$

3. We did not get this right on the sheet, so here is a version using the continuum formalism which I now believe to be correct:

$$< k >= \int_{k_{min}}^{k_{max}} kp_k dk = \int_1^{1000} k \cdot k^{-2.3} dk = \int_1^{1000} k^{-1.3} dk = \left[\frac{-1}{0.3} k^{-0.3} \right]_1^{1000} = \frac{-1}{0.3} \cdot 1000^{-0.3} + \frac{1}{0.3} \approx 2,91$$

$$< k^2 >= \int_{k_{min}}^{k_{max}} k^2 p_k dk = \int_1^{1000} k^{-0.3} dk = \left[\frac{1}{0.7} k^{0.7} \right]_1^{1000} \approx 178,42$$

Thus

$$< k_n ei >= \frac{< k^2 >}{< k >} \approx 61,31$$

4. Try to explain the apparent paradox $< k > \quad < \quad < k_{nei} >$.

The average degree corresponds to checking a random node's degree. The chances of that node being a hub are not extremely high. By contrast, the chances of one of its k neighbors being a hub is much higher, so that their average degree, $< k_{nei} >$, will be superior.¡br¿ Considered differently, we could also say that the network is denser locally than it is globally: the global average is lower than the local average of a neighborhood, because the influence of a hub is felt more strongly locally than globally.