Algebraic Geometry

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1. Sheaf Theory

1.1. From Presheaves to Sheaves

Sheaf theory, now ubiquitous in geometry and beyond, owes its origin to the categorical idea that spaces are best understood through functions on them; sheaves are, accordingly, structures designed to accommodate functional data associated to spaces, which often appear "bundled" with respect to open sets. The datum underlying a sheaf is nothing but a functor.

Definition 1.1.1. Fix a topological space X, a category C. A functor $\mathscr{F}\colon \mathsf{Op}(X)^{\mathrm{op}} \to \mathsf{C}$ is a **presheaf of objects in C on X**, C being the **base category**. When C is concrete, we adopt the following terminology to describe the datum of $\mathscr{F}$:

1. For each open $U \subseteq X$, elements of $\mathscr{F}(U)$ are **sections of $\mathscr{F}$ over U**; **global sections** refer to sections of $\mathscr{F}$ over X.
2. The pushforward $r_{V,U}\colon \mathscr{F}(V) \to \mathscr{F}(U)$ (also written i^*) of each inclusion $i\colon U \to V$ is a **restriction map**. For $f \in \mathscr{F}(V)$, we write $f|_U$ for $r_{V,U}(f)$.

Remark. The nomenclature of "section" is historical: the classical approach to sheaves was via "étale spaces" of topological spaces, to which one associates a genuine sheaf of sections. We will see shortly that every sheaf indeed arises in this way. Nevertheless, it is essential to view sheaves from the outset as being comprised of functions regardless of appearance; only through constant analogy to the behavior of functions will the sheaf-theoretic definitions we are to make seem perfectly natural.

Example 1.1.2. Fix topological spaces X, Y. Then the **(pre)sheaf of continuous maps to Y on X** sends each open $U \subseteq X$ to $\mathscr{C}(U) = \mathrm{Hom}_{\mathsf{Top}}(U, Y)$ and each inclusion $i\colon U \to V$ to precomposition with i: $r_{V,U}\colon \mathscr{C}(V) \to \mathscr{C}(U)$ sending $\varphi\colon V \to Y$ to $U \xrightarrow{i} V \xrightarrow{\varphi} Y$. Given a composition $U \xrightarrow{i} V \xrightarrow{j} W$, clearly $(ji)^* = i^* j^*$, i.e. the following diagram commutes:

$$\begin{tikzcd} & \mathscr{C}(V) \arrow[dr, "i^*"] & \\ \mathscr{C}(W) \arrow[ur, "j^*"] \arrow[rr, "(ji)^*"] & & \mathscr{C}(U) \end{tikzcd}$$

It is equally evident that the pushforward $r_{U,U}$ of $\mathrm{id}\colon U \to U$ is the identity on $\mathscr{C}(U)$.

$\mathscr{C}$, however, possesses a characteristic property beyond functoriality that qualifies it as a *sheaf*: a continuous map on any open $U \subseteq X$ is uniquely determined by its restrictions to

U_i, where $\{ U_i \}$ is any open cover of U, from which synthetic data it may be reconstructed if the restrictions agree on intersections. We are not describing any exotic phenomenon—this is merely the pasting lemma from topology, stated in a slightly different manner.

This sort of behavior is all that we will ask of sheaves, which are to be understood as the natural gadgets for describing local data that can be lifted to global datum without any obstruction. Now by the above, we intuit that local-global inconsistencies cannot possibly originate from within $\mathsf{Op}(X)$, so rather than expecting the presheaf to enable gluing from thin air, we merely stipulate its *preserving* the state of things in $\mathsf{Op}(X)$. We will stop speaking in riddles and make clear what we mean categorically.

Lemma 1.1.3. *For X a topological space, $\mathsf{Op}(X)$ is bicomplete.*

Proof. Equalizers in $\mathsf{Op}(X)$ are identity arrows: there is at most one map between any two objects U, V, so $e\colon E \to U$ is an isomorphism; but there are no non-trivial isomorphisms in $\mathsf{Op}(X)$. Similarly for coequalizers. A set of open sets $\{ U_\alpha \}$ has as its product the interior of their intersection, taking as projection maps the natural inclusions, and as its coproduct their union, again with inclusions as canonical maps. $\square$

Proposition 1.1.4 (Categorical Pasting Lemma)**.** *Fix an open set $U \in \mathsf{Op}(X)$ and an open cover $\{ U_\alpha \}$ thereof. Denote by U the full completion of $\{ U_\alpha \}_{\alpha \in I}$ in $\mathsf{Op}(X)$. Then U is the colimit of the inclusion diagrams $\mathsf{U} \to \mathsf{Op}(X)$ and, with all open sets seen as subspaces of X, $\mathsf{U} \to \mathsf{Top}$.*

Proof. The first part is trivialized by the preceeding lemma. For the latter, let Y be a space admitting compatible morphisms $f_J\colon (\bigcap_{\beta \in J} U_\beta)^{\mathrm{o}} \to Y$ for all $J \subseteq I$. By the pasting lemma, the datum of morphisms $f_{\alpha\beta}$ from pairwise intersections already determines a unique morphism $f\colon U \to Y$ compatible with $f_{\alpha\beta}$. (This is the germ of Thm X.) That it is compatible with the rest follows from the compatibility within $\{ f_J \}$. $\square$

It is clear from the proof that all there is to gluing is encapsulated by U fulfilling the universal property of the prescribed colimit; functions are but morphisms $U \to Y$. A presheaf *preserving* this colimit then surely deserves the title of sheaf[1]. This might even be apparently too stringent a condition, and to address this concern we will promptly show that it agrees with the classical, self-explanatory definition of a sheaf.

Definition 1.1.5. A presheaf $\mathscr{F}\colon \mathsf{Op}(X) \to \mathsf{C}$ is **sheaf** if for any full complete subcategory U, $\mathscr{F}(\varinjlim \mathsf{U}) = \varprojlim \mathscr{F} \circ \mathsf{U}$.

[1] or "chief"... get it?

Deducing the classical axioms hinges on making judicious use of morphisms $\{*\} \to U_i$, which essentially *pick out* compatible local sections.

Theorem 1.1.6 (Sheaf Axioms)**.** *A presheaf $\mathscr{F}\colon \mathsf{Op}(X) \to \mathsf{C}$ with concrete base category admitting a forgetful functor preserving and reflecting small limits is a sheaf iff it satisfies the following axioms:*

Locality *Fix any open $U \subseteq X$ and open cover $\{U_i\}$ thereof. Given a pair $f_1, f_2 \in \mathscr{F}(U)$, if $f_1|_{U_i} = f_2|_{U_i}$ for all i, then $f_1 = f_2$. A presheaf satisfying locality is **separated**.*

Glubility *Given $f_i \in \mathscr{F}(U_i)$ for all i, if $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i, j, then there exists some $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$ for each i.*

Proof. It will suffice to take $\mathsf{C} := \mathsf{Set}$: for U the forgetful functor, since the implications are all concerning equality of images of morphisms under the forgetful functor, $U(f_1) = U(f_2)$ is a sheaf iff $U \circ \mathscr{F}$ is a sheaf. Recall that an element of $\mathscr{F}(U)$ is merely a function $\{*\} \to \mathscr{F}(U)$. We first show locality. It suffices to show that $f\colon \{*\} \to \mathscr{F}(U)$ is uniquely determined by the subset $\{ f_i\colon \{*\} \to \mathscr{F}(U_i) \} \subseteq \left\{ f_J\colon \{*\} \to F(\mathscr{F}(\bigcap_{\alpha \in J} U_\alpha{}^{\circ})) \right\}$, where the latter a fortiori uniquely determines f by the universal property. In other words, we must show that $\{\varphi_i\}$ determines $\{\varphi_J\}$. But this immediate from compatibility with restrictions.
Again we note that provided arrows $\{*\} \to \mathscr{F}(U_i \cap U_j)$ for all i, j compatible under restrictions, the universal property furnishes a morphism $\{*\} \to \mathscr{F}(U)$, i.e. an element of $\mathscr{F}(U)$. Now observe that arrows $\{\varphi_i\colon \{*\} \to \mathscr{F}(U_i)\}$ extends to arrows $\{\varphi_{ij}\colon \{*\} \to \mathscr{F}(U_i \cap U_j)\}$ compatible under restrictions iff $r_{U_i, U_i \cap U_j} \circ \varphi_i = \varphi_{ij} = r_{U_j, U_i \cap U_j} \iff f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, hence gluability.
For the converse, fix $X \in \mathsf{C}$, and let $\varphi_{ij}\colon X \to \mathscr{F}(U_i \cap U_j)$ be maps. We must exhibit a unique $\varphi\colon X \to \mathscr{F}(U)$. First recall that we require any such map to commute with the restrictions, i.e. for all i, j, $r_{U_i \cap U_j} \circ \varphi = \varphi_{ij}$. Locality then guarantees that φ is uniquely determined. On the other hand, the value of φ at each $f \in \mathscr{F}(U)$ may be determined, due to gluability, by those of φ_{ii} at $f|_{U_i}$. $\square$

When the open cover consists of disjoint sets, its image under a sheaf is very easy to compute.

Proposition 1.1.7. *For disjoint $\{U_i\}$, $\mathscr{F}(\bigcup U_i) = \prod \mathscr{F}(U_i)$.*

Proof. Disjointness amounts to that the indexing category is discrete, with underlying set $\{U_i\}$. Now recall that an n-fold product is precisely the limit over a diagram indexed in a discrete category of cardinality n. $\square$

However, with regards to computability, our current definition is arguably the worst one: the indexing category consists of a large number of vertices and arrows, and the limit over such an arbitrary diagram might be unclear. Our saving grace is that certain simplifications of the indexing category can yield the same limit, up to canonical isomorphism, and these will make our lives much easier. Even better, it turns out that the criterion concerns solely a functor between the indexing categories, and thus works regardless of what sheaf is in question.

Definition 1.1.8. Fix (possibly small) categories I, J. A functor $T\colon I \to J$ is final if for all functors $F\colon J \to \mathsf{C}$, the natural morphism $\varinjlim F \circ T \to \varinjlim F$ is an isomorphism.

Theorem 1.1.9. *$T\colon I \to J$ is final iff for each $x \in J$, the comma category x/T is non-empty and connected.*

Proof. Objects are pairs (a, f) where $f\colon x \to T(a)$ is a morphism, and morphisms are maps $a \to b$ making the obvious diagram commute. □

Corollary 1.1.10. *A presheaf $\mathscr{F}$ on X is a sheaf iff for any open $U \subseteq X$ and a cover $\{ U_i \}_{i\in I}$ thereof, $\mathscr{F}$ preserves limits of the following shape:*

1. *the full subcategory of $\mathsf{Op}(X)$ with objects $\{ U_i \cap U_j \}$ for all pairs $i, j \in I$, not necessarily distinct,*
2. *the subcategory $\coprod_{i,j} U_i \cap U_j \overset{\coprod i_{U_i\cap U_j, U_i}}{\underset{\coprod i_{U_i\cap U_j, U_j}}{\rightrightarrows}} \coprod_k U_k$ of Top.*

Proposition 1.1.11. *A C-presheaf $\mathscr{F}$ is a sheaf iff for every open cover $\{ U_i \}$ of an open $U \subseteq X$, $\mathscr{F}(\bigcup U_i)$ equalizes the diagram $\prod_k \mathscr{F}(U_k) \overset{r_{U_i, U_i\cap U_j}}{\underset{r_{U_j, U_i\cap U_j}}{\rightrightarrows}} \prod_{i,j} \mathscr{F}(U_i \cap U_j)$.*

Proof. We need only show that the diagram over which the limit is taken equates that which is given in the definition above, namely $\{ U_i \cap U_j \}$. Indeed, since there are no inclusions $U_i \to U_j$ for any $i \neq j$, the latter may be realized as a collection of disjoint cospans

$$\begin{tikzcd} \mathscr{F}(U_i) \arrow[d, "r_{U_i, U_i\cap U_j}"] & \\ \mathscr{F}(U_i \cap U_j) & \mathscr{F}(U_j). \arrow[l, "r_{U_j, U_i\cap U_j}"'] \end{tikzcd}$$

It then suffices to show that the collection of arrows $\{ r_{U_i, U_i\cap U_j} \}$ uniquely determines an arrow $\prod_k \mathscr{F}(U_k) \to \prod_{i,j} \mathscr{F}(U_i \cap U_j)$; the case of $\{ r_{U_j, U_i\cap U_j} \}$ is handled identically. Indeed, fixing i, we have for all j the morphism $r_{U_i,U_i\cap U_j}\colon \mathscr{F}(U_i) \to \mathscr{F}(U_i \cap U_j)$, which uniquely determines a morphism $\mathscr{F}(U_i) \to \prod_j \mathscr{F}(U_i \cap U_j)$. The desired morphism is then obtained by multiplying the induced maps associated to each i. $\square$

Corollary 1.1.12. *Let A be an abelian category. Then an A-presheaf $\mathcal{F}$ is a sheaf iff the following sequence is exact:*

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \prod_k \mathscr{F}(U_k) \xrightarrow{r_{U_i,U_i\cap U_j} - r_{U_j,U_i\cap U_j}} \prod_{i,j} \mathscr{F}(U_i \cap U_j).$$

Proof. We may rewrite the diagram which $\mathscr{F}(U)$ equalizes as

$$\begin{tikzcd} \prod_k \mathscr{F}(U_k) \arrow[r, shift left, "r_{U_i,U_i\cap U_j} - r_{U_j,U_i\cap U_j}"] \arrow[r, shift right, "0"'] & \prod_{i,j} \mathscr{F}(U_i \cap U_j). \end{tikzcd}$$

In other words, the map $\mathscr{F}(U) \to \prod_k \mathscr{F}(U_k)$ is the kernel of $r_{U_i,U_i\cap U_j} - r_{U_j,U_i\cap U_j}$, i.e. the sequence is exact at $\prod_k \mathscr{F}(U_k)$. Exactness at $\mathscr{F}(U)$ follows from that kernels are a fortiori mono. $\square$

Remark. Let us reexamine Corollary 1.1.12 with the above in mind. Once one expands the definitions, it is easy to see that exactness at $\mathscr{F}(U)$ is precisely locality, that at $\prod_k \mathscr{F}(U_k)$ gluability. Even when the category is not abelian, being an equalizer alone ensures that $\mathscr{F}(U) \to \prod_k \mathscr{F}(U_k)$ is mono, which property again corresponds to locality.

Sheaves thus defined may seem unwieldy in the sense that the axioms must be checked for an *arbitrary* open cover; it is precisely due to the "localizable" nature of sheaves that we have the economic notion of a *sheaf on a base*, i.e. we may obtain a sheaf from only its values on a base of the topological space. This trick will prove to be invaluable when we define affine schemes.

Definition 1.1.13. Fix a topological space X, a base $\mathcal{B}$. The category of **C-presheaves on $\mathcal{B}$** is the functor category C_{B}, where by B we understand the complete full subcategory of $\mathsf{Op}(X)$ satisfying $\mathrm{Ob}(\mathsf{B}) = \mathcal{B}$. The **associated C-presheaf** $\hat{\mathscr{F}}$ on X sends any open $U \subseteq X$ to $\varprojlim_{B_\alpha \subseteq U} \mathscr{F}(B_\alpha)$, where $B_\alpha \in \mathcal{B}$, and inclusion $U \to V$ to the morphism $\mathscr{F}(V) \to \mathscr{F}(U)$ induced from the morphisms $\mathscr{F}(V) \to \mathscr{F}(B_\alpha)$. This assignment is functorial, sending a morphism $\mathscr{F} \to \mathscr{G}$ to the morphism $\hat{\mathscr{F}} \to \hat{\mathscr{G}}$ whose components $\hat{\mathscr{F}}(U) \to \hat{\mathscr{G}}(U)$ are limits of the morphisms $\mathscr{F}(B_\alpha) \to \mathscr{G}(B_\alpha)$.

Theorem 1.1.14. *$\hat{\mathscr{F}}$ is a sheaf iff for all $B \in \mathcal{B}$, $\mathscr{F}(B) = \varprojlim_{B_\alpha \subseteq B} \mathscr{F}(B_\alpha)$, where $B_\alpha \in \mathcal{B}$, in which case $\mathscr{F}$ is said to be a* ***sheaf on*** $\mathcal{B}$*. Such presheaves on $\mathcal{B}$ constitute a subcategory $\mathsf{C}_\mathsf{B}^{sh} \subseteq \mathsf{C}_\mathsf{B}$. Furthermore, $\hat{-}\colon \mathsf{C}_\mathsf{B}^{sh} \to \mathsf{C}_X$ is an equivalence of categories.*

Proof. The forward direction is immediate from the uniqueness (up to unique isomorphism) of limits. Conversely, let $\{ B_\alpha \}$ be a basic cover for an open $U \subseteq X$. It will suffice to show that U is cofinal in the directed set of *all* basic subsets of U, once we identify $\mathscr{F}(B)$ with $\hat{\mathscr{F}}(B)$ for every basic B. Indeed, every $B_\alpha \cap B_\beta$ can be covered by basic open sets; □

Remark. In practice we will check this condition by verifying separately *base locality* and *base gluability*, which are obtained from Proposition 1.1.6 by requiring every open set therein to be basic. This substitution also allows the proof of equivalence to be carried over verbatim.

Now that we have the axioms down, we will provide some examples of presheaves and sheaves to build intuition. First, to any object in the base category we may associate a presheaf which "adds no redundant information:"

Example 1.1.15. Fix a category C, $A \in \mathsf{C}$. The **constant presheaf** $\underline{A}_{\text{pre}}$ on X sends each open set U to X and each inclusion to the identity. This is tautologically a presheaf, as there are no restrictions other than the identity.

In general, the constant presheaf is not a sheaf. Indeed, if A is not the final object, namely a singleton, $\underline{A}_{\text{pre}}(\varnothing) = A$ fails to be a final object. More directly, if we endow $X = \{ x, y \}$ with the discrete topology, gluability fails: pick the cover of singletons, which, encompassing no non-trivial intersections, renders the compatibility assumption in gluability vacuous. Now we may just choose distinct $\underline{A}_{\text{pre}}(x) \ni a \neq b \in \underline{A}_{\text{pre}}(y)$, as a global section in $\underline{A}_{\text{pre}}(U) = A$ cannot possibly restrict to distinct values.

It turns out that one may "sheafify" (c.f. §1.3) $\underline{A}_{\text{pre}}$ to obtain

Example 1.1.16. The **constant sheaf** $\underline{A}$ is defined as the sheaf of continuous maps to A, endowed with the discrete topology. By considering the fiber of $f(p)$, one easily sees that this is equivalent to the set of *locally constant* maps $U \to Y$, that is, for any $p \in U$, there is a neighborhood on which the map is constant.

We now present some variations to the theme of sheaves of continuous maps.

Example 1.1.17. The **presheaf of bounded functions** assigns to each open $U \subseteq X$ the set of bounded functions from U to $\mathbb{R}$. Note that this is *not* a sheaf: take the cover

$\{ B_{1+\epsilon}(p) \colon p \in \mathbb{Z} \}$ for U the entire space. Notice, the function $f \colon \mathbb{R} \to \mathbb{R}$ sending $x \rightsquigarrow \lfloor x \rfloor$ is not bounded, but each $f|_{B_{1+\epsilon}(p)}$ is. Clearly these cannot be glued into a bounded map, so $\mathscr{F}$ fails the gluability axioms.

Example 1.1.18. Let $\mu \colon Y \to X$ be a continuous map. Then the **sheaf of sections associated to** μ sends each open $U \subseteq X$ to the set of continuous sections to μ, i.e. $s \colon U \to Y$ such that $\mu \circ s = \mathrm{id}$, and inclusions to restrictions as usual, since evidently restricting a section yields another section. Functoriality is then immediate. There is also nothing to be said regarding locality, with sections being functions. For gluability we need only show that the resultant function is a section. To this end, we check on elements: any $x \in U$ must lie in U_i for some i, so $\mu \circ s(x) = \mu \circ s|_{U_1}(x) = x$.

Sheaves of continuous maps also play well with enrichment. For instance,

Example 1.1.19. If Y is a topological group, then the continuous maps to Y form a sheaf of groups. Observe first that to each $\{ f \colon X \to Y \}$ possesses a group structure, with operation $(f, g) \rightsquigarrow f + g := m \circ (f \times g) \circ \Delta_X$, whose continuity follows from that of m, and the constant map to 0 as the identity. The evident inverse $\iota \circ f$ remains continuous, as the inverting operator $\iota \colon Y \to Y$ is a continuous anti-automorphism on Y. Functoriality is immediate once we show $r_{V,U} \colon \mathscr{F}(V) \to \mathscr{F}(U)$ is a group homomorphism. Indeed, $(f + g)|_U = f|_U + g|_U$ is routine from definition. Lastly, the additional structure does not jeopardize locality and gluability, with the underlying sets unaltered.

1.2. Functor Categories

Any deeper study of sheaf theory hinges on a notion of morphisms of sheaves. Since presheaves are just functors, there is a *natural* definition.

Definition 1.2.1. A **morphism of presheaves** $\mathscr{F}, \mathscr{G} \colon \mathsf{Op}(X) \to \mathsf{C}$ is a natural transformation $\mathscr{F} \to \mathscr{G}$. That is, it is a collection of natural maps $\mu_U \colon \mathscr{F}(U) \to \mathscr{G}(U)$ making the following diagram commutes for every inclusion $i \colon V \to U$:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "i^*"] \arrow[d, "\mu_U"] & \mathscr{F}(V) \arrow[d, "\mu_V"] \\ \mathscr{G}(U) \arrow[r, "i^*"] & \mathscr{G}(V) \end{tikzcd}$$

The **category of C-presheaves on a space** X is denoted $\mathsf{C}_X^{\text{pre}}$, and the full subcategory of C-sheaves C_X.

Remark. It is often the case that a functor between sheaf categories can be immediately extended to one between the analogous presheaf categories, for functoriality rarely follows from the sheaf axioms. As such, henceforth we opt to define functors between sheaf categories whenever possible, leaving it to the reader to infer the analogous functor between presheaf categories.

Continuing immediately from our example above, we have two flavors of *forgetful* morphisms:

Example 1.2.2. The **forgetful morphism** from the presheaf of bounded functions to the sheaf of continuous maps does nothing but "forget" the boundedness of the functions. It is routine to verify naturality. On the other hand, the **forgetful functor** $\mathsf{Top}^X \to \mathsf{Set}^X$ merely forgets the group structure of each $\mathscr{F}(U)$, under which the sheaf of continuous maps to a topological group gets sent to the usual sheaf of continuous maps.

Definition 1.2.3. Fix a subcategory U of $\mathsf{Op}(X)$. The **restriction functor** $\mathsf{C}_X \to \mathsf{C}_\mathsf{U}$ sends each sheaf $\mathscr{F}$ on X to the restricted sheaf $\mathscr{F}|_\mathsf{U}$ defined on each open $V \in \mathsf{U}$ as $\mathscr{F}(V)$ and each morphism $\mu \colon \mathscr{F} \to \mathscr{G}$ to the restricted morphism $\mu|_\mathsf{U}$ sending $\mathscr{F}|_\mathsf{U}(V) = \mathscr{F}(V) \rightsquigarrow \mu(\mathscr{F}(V))$, which indeed is a subset of $\mathscr{G}(V) = \mathscr{G}|_\mathsf{U}(V)$. Functoriality is immediate.

We now undertake the task of understanding adjunctions and (co)limits in presheaf categories. Fortunately, just as functions often inherit properties of their codomains, functor categories possess all adjunctions and (co)limits of their base category. This should not come as a surpise, for the Yoneda embedding is fully faithful.

Lemma 1.2.4. *Given an adjunction* $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{D} \arrow[l, bend left, "G"] \end{tikzcd}$, *, there is for any small category I an adjunction* $\begin{tikzcd} \mathsf{C}^I \arrow[r, bend left, "F_*"] \arrow[r, phantom, "\perp"] & \mathsf{D}^I \arrow[l, bend left, "G_*"] \end{tikzcd}$, *where F_* (resp. G_*) denotes post-composition with F (resp. G).*

Proof. Let $\eta\colon 1_{\mathsf{C}} \to GF$, $\varepsilon\colon FG \to 1_{\mathsf{D}}$ denote the unit and counit of $F \dashv G$, respectively. We claim that the new unit $\eta_*\colon 1_{\mathsf{C}^I} \to G_*F_*$ and counit $\eta_*\colon F_*G_* \to 1_{\mathsf{D}^I}$ have as components

$$\eta_*(H \in \mathsf{C}^I) = \eta H\colon H \to GFH, \quad \varepsilon_*(H \in \mathsf{D}^I) = \varepsilon H\colon FGH \to H.$$

We will only show naturality of η_* and one triangle identity; the other cases are dual. Indeed, the naturality square

$$\begin{tikzcd} H \arrow[r, "\eta_* H"] \arrow[d, "\varphi"'] & GFH \arrow[d, "GF\varphi"] \\ K \arrow[r, "\eta_* K"] & GFK \end{tikzcd} \qquad \begin{tikzcd} H(x) \arrow[r, "\eta_{*H(x)}"] \arrow[d, "\varphi_x"'] & GFH(x) \arrow[d, "GF(\varphi_x)"] \\ K(x) \arrow[r, "\eta_{*K(x)}"] & GFK(x), \end{tikzcd}$$

where $x \in I$, commute immediately by the naturality of η. Similarly, the commutativity of the triangle identity

$$\begin{tikzcd} F_* \arrow[r, "F_*\eta_*"] \arrow[dr, "1_{F_*}"'] & F_*G_*F_* \arrow[d, "\varepsilon_* F_*"] \\ & F_*, \end{tikzcd} \qquad \begin{tikzcd} FH \arrow[r, "F\eta H"] \arrow[dr, "1_{FH}"'] & FGFH \arrow[d, "\varepsilon FH"] \\ & FH, \end{tikzcd}$$

for $H \in \mathsf{C}^I$, follows from that of η and ε and the functoriality of whiskering. $\square$

Lemma 1.2.5. *Fix a category C. For all diagrams $D\colon J \to \mathsf{C}^I$, $(\varprojlim_J D)(A) = \varprojlim_{j\in J} D_j(A)$, with $\pi_\alpha\colon \varprojlim_J D \to D(J_\alpha)$ defined on each $A \in I$ as $\pi_{\alpha_A}\colon \varprojlim_{j\in J} D_j(A) \to D_\alpha(A)$, provided the relevant limits exist in C. The analogous statement for colimits follows by duality.*

Proof. Recall that $\varprojlim\colon \mathsf{C}^J \to \mathsf{C}$ is right adjoint to the constant diagram functor $\Delta\colon \mathsf{C} \to \mathsf{C}^I$. It thus suffices to prove that $\varprojlim_*\colon (\mathsf{C}^J)^I \to \mathsf{C}^I$ is as prescribed and that $\Delta_*\colon \mathsf{C}^I \to (\mathsf{C}^J)^I$

is the constant diagram functor, both after precomposition with the canonical isomorphism $(\mathsf{C}^I)^J \to (\mathsf{C}^J)^I$, which sends a diagram $D\colon J \to \mathsf{C}^I$ to the diagram $D'\colon I \to \mathsf{C}^J$ sending an object $i \in I$ to the functor D_i sending $j \rightsquigarrow D_j(i)$, $\varphi\colon j \to j'$ to $D(\varphi)(i)$ and a morphism $\psi\colon i \to i'$ to the natural transformation $D_i \to D_{i'}$ whose components are $D_j(\psi)\colon D_i(j) \to D_{i'}(j)$. As such, the image of D under $\varprojlim_*$ sends $A \in I$ to $A \in I$ to $\varprojlim_J D'(A)$. Is there an elegant way to do this? $\square$

Theorem 1.2.6. *If* C *is an abelian category, so is* C_X^{pre}.

Proof. Recall that a category is abelian if it has a zero object, binary biproducts, (co)kernels, and all monos and epis are normal. With the former three properties all concerning the existence of (co)limits, they are inherited by C_X from C by virtue of the above lemma. But a moment of thought reveals that the last property is easily carried over as well, with normality being a statement about a morphism that is the (co)limit of a diagram. $\square$

With a functor category containing more data than its base category, we would expect it to have more structure and desirable properties. Indeed, witness the small miracle that functor categories over monoidal categories are always *closed*.

Lemma 1.2.7. *Every presheaf* $F\colon \mathsf{C} \to \mathsf{Set}$ *is a colimit of representables.*

Proposition 1.2.8. *A morphism* $\varphi\colon \mathscr{F} \to \mathscr{G}$ *is an isomorphism iff* $\varphi(U)$ *is injective for all* U *and* $\operatorname{im}\varphi = \mathscr{G}$.

As such, if our base category has products, so does the corresponding category of presheaves. It is thus natural to look for an internal Hom functor:

Definition 1.2.9. Let $\mathscr{F}, \mathscr{G} \in \mathsf{C}_X^{\text{pre}}$. Then the **sheaf Hom** $\mathscr{H}(\mathscr{F}, \mathscr{G})$ is defined by the data $\mathscr{H}(\mathscr{F}, \mathscr{G})(U) = \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$. Show that this is a sheaf of sets on X

Proposition 1.2.10. *If* C *has a binary product,* $\mathcal{H}\wr\Updownarrow(\mathscr{F}, -)$ *is right adjoint to the functor* $- \times \mathscr{G}$*, i.e. we have for all* $\mathscr{F}, \mathscr{H} \in \mathsf{C}_X$ *natural isomorphisms*

$$\operatorname{Hom}(\mathscr{F} \times \mathscr{G}, \mathscr{H}) \cong \operatorname{Hom}(\mathscr{F}, \mathcal{H}\wr\Updownarrow(\mathscr{G}, \mathscr{H})).$$

Proof. Fix a natural transformation $\mu\colon \mathscr{F} \times \mathscr{G} \to \mathscr{H}$, which has components $\mu_U\colon \mathscr{F}(U) \times \mathscr{G}(U) \to \mathscr{H}(U)$. We propose the map sending $\mathscr{F}(U) \rightsquigarrow (\mathscr{F}(G))$ must produce unique corresponding components $\mathscr{F}(U) \to \operatorname{Hom}(\mathscr{G}|_U, \mathscr{H}|_U)$. $\square$

Example 1.2.11. 2.3.J Let $\mathscr{O}_{\mathbb{C}}$ be the sheaf of holomorphic functions, $\mathscr{F}$ be the presheaf of functions admitting a holomorphic logarithm. $\mathscr{F}$ is not a sheaf... don't know complex analysis

Definition 1.2.12. Fix an open cover $\{ U_i \}$ for a topological space X. The presheaf on X **glued from** presheaves $\mathscr{F}_i$ on U_i is the limit of the diagram with vertices (i, j) for $\mathscr{F}_i|_{U_i \cap U_j}$, edges restriction maps $(i, i) \to (i, j)$ and transition isomorphisms $\phi_{ij} \colon (i, j) \to (j, i)$.

1.3. Sites and Topoi

Definition 1.3.1. A **Grothendieck topology** J on a category C is the assignment to each object U in C of a collection of sets of arrows $\{ U_i \to U \}$, called **coverings** of U, satisfying the following axioms:

1. If $V \to U$ is an isomorphism, then the $\{ V \to U \}$ is a covering.
2. If $\{ U_i \to U \}$ is a covering, then for any $V \to U$, the fibered products $U_i \times_U V$ exist, and the collection of projections $\{ U_i \times_U V \to V \}$ is a covering.
3. Given a $\{ U_i \to U \}$ a covering and a covering $\{ V_{ij} \to U_i \}$ for each i, the collection of composites $\{ V_{ij} \to U_i \to U \}$ is a covering.

Any such pair (C, J) is said to be a **site**.

1.4. The Direct Image and its Adjoints

From our previous discussion, one should acquire the impression that (pre)sheaves are amalgamations of topologically local data which aspire to be glued into global datum. Heading in the opposite direction, we ask how datum is concentrated on a even smaller locality, say a point. This leads us to the notion of a stalk.

Definition 1.4.1. Fix a presheaf $\mathscr{F} \colon \mathsf{Op}(X) \to \mathsf{C}$, $p \in X$. The **stalk** $\mathscr{F}_p$ **of** $\mathscr{F}$ **at** p is the colimit $\varinjlim_{\mathsf{Op}(X)_p} \mathscr{F}$ where $\mathsf{Op}(X)_p$ denotes the full subcategory of $\mathsf{Op}(X)$ of open sets containing p. By **stalkification at** p we understand the composition

$$\mathsf{C}_X^{\text{pre}} \xrightarrow{\ r\ } \mathsf{C}_{\mathsf{Op}(X)_p}^{\text{pre}} \xrightarrow{\text{colim}} \mathsf{C},$$

where r denotes the restriction functor. The **germ of** $f \in \mathscr{F}(U)$ **at** p is the image of f in $\mathscr{F}_p$.

Proposition 1.4.2. *The stalk of $\mathscr{F}_p$ at p is the set of germs at p modulo the equivalence relation $(f, U) \sim (g, V) \iff$ there exists a $W \subseteq U \cap V$ such that $f|_W = g|_W$.*

Proof. Write $\mathscr{S}$ for the construction in question. For each U we define $\alpha_U \colon \mathscr{F}(U) \to \mathscr{S}$ as sending $f \rightsquigarrow [f, U]$. This is compatible with restrictions since $\alpha_U(f)$ and $\alpha_V(g)$ should agree precisely when there is a mutual subspace on which f and g agree. Now let X be a set such that for each U there is a map $x_U \colon \mathscr{F}(U) \to X$ compatible with restrictions. Then commutativity forces the map $x \colon \mathscr{S} \to X$ to send $[f, U]$ to $x_U(f)$. This is well-defined, as for $(g, V) \sim (f, U)$, $x_U(f) = x_W \circ r_{U,W}(f) = x_W \circ r_{V,W}(g) = x_V(g)$, where $W \subseteq U \cap V$ is the neighborhood on which f, g agree. $\square$

Definition 1.4.3. Let X be a topological space, $p \in X$, A an object of a category C, $i_p \colon \{\, p \,\} \to X$ the inclusion. Then the **skyscraper sheaf** $i_{p,*}\underline{A}$ sends $U \rightsquigarrow \begin{cases} S & \text{if } p \in U, \\ T & \text{otherwise,} \end{cases}$ where T denotes the terminal object in C.

Theorem 1.4.4. *The stalkification functor is left-adjoint to the skyscraper sheaf functor.*

Proof. Fix $S \in \mathsf{C}$, $p \in X$, a presheaf $\mathscr{F}$ on X. We must exhibit a natural bijection

$$\operatorname{Hom}(\mathscr{F}_p, S) \cong \operatorname{Hom}(\mathscr{F}, i_{p,*}\underline{S}).$$

Since for any U not containing p, $i_{p,*}\underline{S}$ is the terminal object, to construct a natural transformation it suffices to define the components on U containing p. For each morphism $\varphi\colon \mathscr{F}_p \to S$, consider the natural transformation whose component on U containing p is the composition $\mathscr{F}(U) \xrightarrow{\iota} \mathscr{F}_p \xrightarrow{\varphi} S$, where ι is the canonical injection into the colimit. We take the inverse to be the map induced by the components $\mathscr{F}(U) \to S$ for all U containing p; that they are inverse to each other is an easy application of the universal property. □

Proposition 1.4.5. *Fix a continuous map $\pi\colon X \to Y$, a C-(pre)sheaf $\mathscr{F}\colon \mathsf{Op}(X) \to \mathsf{C}$. Then for all $p \in X$, there is natural morphism of stalks $(\pi_*\mathscr{F})_{\pi(p)} \to \mathscr{F}_p$.*

Proof. Write $q := \pi(p)$. Since $(\pi_*\mathscr{F})_q = \varinjlim \pi_*\mathscr{F}(U)$, it suffices to exhibit morphisms from $\pi_*\mathscr{F}(V) \to \mathscr{F}_p$ for all V containing q. But $\pi_*\mathscr{F}(V) = \mathscr{F}(\pi^{-1}(V))$, and since $\pi^{-1}(V) \ni p$, we may simply take the canonical injections. □

is the pushforward of the constant sheaf $\underline{S}$ under i_p

We now propose some general theory that encompasses these constructions.

Definition 1.4.6. Fix a functor $\Pi\colon \mathsf{Op}(X) \to \mathsf{Op}(Y)$. Then the **pushforward** or **direct image under** Π is simply the pre-composition functor $\Pi^*\colon \mathsf{C}_X^{\text{pre}} \to \mathsf{C}_Y^{\text{pre}}$.

The astute reader will quickly realize that this encompasses the restriction functor defined in the last section, by taking Π to be the inclusion. On the other hand, in applications we often take Π to be the induced functor of a continuous map $\pi\colon Y \to X$, mapping an open $U \subseteq X$ to $\pi^{-1}(U)$.

It turns out the adjoints to this functor are instances of the categorical notion of a *Kan extension*, which we will not use extensively. Nevertheless, we find that its introduction will streamline the notational quagmire to ensue otherwise, so we proceed accordingly.

Definition 1.4.7.

Definition 1.4.8. The **inverse image functor under** π is the composition

$$\mathsf{C}_Y^{\text{pre}} \mathsf{C}_{\mathsf{Op}(Y)_{\pi(U)}}$$

$$\begin{array}{cccc} \pi^{-1}\colon \mathsf{C}_Y^{\text{pre}} & \longrightarrow & \mathsf{C}_X^{\text{pre}} & \\ \mathscr{F} & \mapsto & \pi^{-1}\mathscr{F} & , \\ \big\downarrow{\scriptstyle\mu} & \mapsto & \big\downarrow{\scriptstyle\pi^{-1}\mu} & \\ \mathscr{G} & \mapsto & \pi^{*}\mathscr{G} & \end{array}$$

$$\pi^{-1}\colon \mathsf{C}_Y^{\text{pre}} \to$$

Definition 1.4.9. Fix functors $F, G\colon$ **Kan extension**

TO-DO: inverse images commute with restrictions, and left-exactness maybe

1.5. Sheafification

With sheaves constituting a full subcategory of the category of presheaves, it is natural to ask for a way to freely generate sheaves from presheaves. This process is fittingly named *sheafification*. Other than practical benefits, seeking a construction for this will prove to be a pedagogical device for us to grasp the essence of the sheaf axioms.

Definition 1.5.1. The **sheafification** functor $L\colon \mathsf{C}_X^{\text{pre}} \to \mathsf{C}_X$ is uniquely characterized by the property of being left adjoint to the inclusion functor $i\colon \mathsf{C}_X \to \mathsf{C}_X^{\text{pre}}$, i.e. there are for $\mathscr{F} \in \mathsf{C}_X^{\text{pre}}$, $\mathscr{G} \in \mathsf{C}_X$ natural isomorphisms

$$\operatorname{Hom}(L(\mathscr{F}), \mathscr{G}) \cong \operatorname{Hom}(\mathscr{F}, U(\mathscr{G}))$$

As such, $L(\mathscr{F})$ is characterized by the universal property that every morphism $\mathscr{F} \to \mathscr{G}$, where $\mathscr{G} \in \mathsf{C}_X$, factors uniquely through $S(\mathscr{F})$:

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d] & U(L(\mathscr{F})) \arrow[dl, dashed] \\ U(\mathscr{G}) & \end{tikzcd} \qquad \begin{tikzcd} L(\mathscr{F}) \arrow[d, dashed] \\ \mathscr{G} \end{tikzcd}$$

Intuitively, the failure of gluability is the *shortage* of global information to which local information can be assembled; one would thus expect sheafification to infuse the presheaf with requisite formal global sections to accommodate all collections of local sections which do not yet form global sections. Locality, on the other hand, is a far more subtle property: it turns out, somewhat surprisingly, that the inability to distinguish between distinct global sections using only local information stems from *extra local fluff* which must be shredded for locality to be satisfied, not the lack of local imprints of the global differences. As evidence we present the following construction, which does not conflate existing local information only if the locality axiom is already satisfied.

Definition 1.5.2. Let $\mathscr{F} \in \mathsf{C}_X^{\text{pre}}$. Its **sheaf of stalks** $S_{\mathscr{F}}\colon \mathsf{Op}(X) \to \mathsf{C}$ sends each open $U \subseteq X$ to $\prod_{p \in U} \mathscr{F}_p$ and each inclusion $i\colon U \to V$ to the induced morphism $\prod_{p \in V} \mathscr{F}_p \to \prod_{p \in U} \mathscr{F}_p$.

Verification. This is immediate from that $\{ \mathscr{F}_p \}$ is cofinal in $\left\{ \prod_{p \in U_i \cap U_j} \mathscr{F}_p \right\}$, with $\{ U_i \}$ an open cover of an open $U \subseteq X$. $\square$

Remark. Recall that on a very heuristic level, stalks encode local information of presheaves. It is thus not entirely surprising that bundling together all stalks yields a sheaf.

Proposition 1.5.3. *Let $\mathscr{F}$ be a separated presheaf. Then there is a natural transformation $\sigma\colon \mathscr{F} \to S_{\mathscr{F}}$ whose components $\sigma_U\colon \mathscr{F}(U) \to \prod_{p\in U} \mathscr{F}_p$ sending $f \rightsquigarrow \prod_{p\in U}[f]_p$ are injective.*

Proof. For open $V \subseteq U$, consider the naturality diagram:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d] & \prod_{p\in U} \mathscr{F}_p \arrow[d] \\ \mathscr{F}(V) \arrow[r] & \prod_{p\in V} \mathscr{F}_p. \end{tikzcd}$$

Its commutativity amounts to that $\prod_{p\in V}[f|_V]_p = \prod_{p\in V}[f]_p$. Indeed, $[f|_V]_p = [f]_p$: the maps agree on V.
Now take $f, f' \in \mathscr{F}(U)$ agreeing under σ_U. Then for all $p \in U$, there exist a neighborhood $V_x \subseteq U$ with $f|_{V_x} = f|_{V'_x}$. This furnishes an open cover of U on every set of which f and f' agree, so the equality of the sections themselves follows by locality. $\square$

This does not immediately suffice for the intent of sheafification, as we may have added excess global sections not needed for merely accommodating all local sections. In particular, the sheaf of stalks of a sheaf $\mathscr{F}$ is not necessarily $\mathscr{F}$. Consider an S-valued constant sheaf, where each $\mathscr{F}(U)$ consists of locally constant functions $U \to S$. The stalk at each point p can be identified with S (it is the subset of the set of all locally constant functions whose domain contains p where functions whose value on the neighborhood of p on which they are constant agree are identified), hence $\mathscr{S}_{\underline{S}}(U) \cong \mathrm{Hom}_{\mathsf{Set}}(U, S)$, which is strictly larger than $\underline{S}(U)$.

On the other hand, with $\mathscr{S}_{\mathscr{F}}$ being a sheaf for any presheaf $\mathscr{F}$, we can rest assured that all necessary global information for ensure gluability have been added, all fatal excess compromising locality removed, so it remains to remove enough information to secure us the most efficient way by which to obtain sheaves from presheaves. To this end, it turns out that it is sufficient for the functor to preserve sheaves.

Definition 1.5.4. An element $\prod_{p\in U}[f_p] \in \prod_{p\in U} \mathscr{F}_p$ consists of **compatible germs** if there is an open cover $\{ U_i \}$ of U such that for all i, there exists a section $\tilde{f}_i \in \mathscr{F}(U_i)$ whose germ at all $p \in U_i$ is $[f_p]$.

Proposition 1.5.5. *Let $\mathscr{F}$ be a sheaf. Then for any open $U \subseteq X$, the image of $\mathscr{F}(U)$ is precisely the set of choices of compatible germs in $\mathscr{S}_{\mathscr{F}}(U)$.*

Proof. The image of a section manifestly gives rise to a choice of compatible germs: take the singleton cover $\{U\}$. Now let $\prod_{p\in U}[f_p]$ be a choice of compatible germs under the open cover $\{U_i\}$. A section f over U with $\sigma_U(f) = \prod_{p\in U}[f_p]$ will be given by gluability if we can show that the sections $\tilde{f}_i$ over U_i given by compatibility agree on intersections. Indeed, $\tilde{f}_i$ and $\tilde{f}_j$ have the same germ (namely $[f_p]$) at all $p \in U_i \cap U_j$, i.e. there is a neighborhood V_p of p on which they agree. Another application of gluability proves they agree on the entirety of $U_i \cap U_j$. □

Before we prove that taking the choices of compatible germs at each open $U \subseteq X$ gives the sheafification, we show some preliminary results on morphisms of sheaves.

Proposition 1.5.6. *Given presheaves $\mathscr{F}, \mathscr{G}$, if a natural transformation $\mu, \lambda\colon \mathscr{F} \to \mathscr{G}$ is mono (resp. epi), then it is mono (resp. epi) on stalks.*

Proof. Notice, stalkification is a filtered colimit, so monos are preserved in every category where filtered colimits preserve finite limits. The statement for epi follows from that stalkification is a left-adjoint functor. □

Proposition 1.5.7. *Given sheaves $\mathscr{F}, \mathscr{G}$, if $\mu, \lambda\colon \mathscr{F} \to \mathscr{G}$ are morphisms inducing the same maps on each stalk, then $\mu = \lambda$.*

Proof. Consider the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\mu"] \arrow[d, "\sigma_U"] & \mathscr{G}(U) \arrow[d, hook, "\sigma_U"] \\
\prod_{p\in U} \mathscr{F}_p \arrow[r, "\tilde{\mu}"] & \prod_{p\in U} \mathscr{G}_p
\end{tikzcd}$$

To show that μ_U is uniquely determined by $\tilde{\mu}$ it suffices to show that $\operatorname{im}\tilde{\mu}$ lies in $\operatorname{im}\sigma_U^{\mathscr{G}}$, with $\sigma_U^{\mathscr{G}}$ being injective. Indeed, $\prod_{p\in U}[\mu(f)]_p = \sigma_U^{\mathscr{G}}(\mu(f))$. □

Proposition 1.5.8. *Given sheaves $\mathscr{F}, \mathscr{G}$, $\mu\colon \mathscr{F} \to \mathscr{G}$ is epi iff it is epi on stalks.*

Proof.

$\Rightarrow$) This is immediate from that

$\Leftarrow$) □

Proposition 1.5.9. *Given sheaves $\mathscr{F}, \mathscr{G}$, $\mu\colon \mathscr{F} \to \mathscr{G}$ is an isomorphism iff it induces an isomorphism on all stalks.*

Proof. The forward direction is obvious: the inverse of the bottom map sends $\prod_{p\in U}[g]_p \rightsquigarrow \prod_{p\in U}[\mu_U^{-1}(g)]_p$. For the converse, again consider the diagram above. Since we have replaced $\prod_{p\in U}\mathscr{G}_p$ with $\operatorname{im}\sigma_U^{\mathscr{G}}$ and inverted $\sigma_U^{\mathscr{G}}$, injectivity of μ follows immediately from that of $\tilde{\mu}$. Surjectivity essentially follows from that elements of $\operatorname{im}\sigma_U^{\mathscr{G}}$ consists of compatible germs: for every $g \in \mathscr{G}(U)$ and $p \in U$, there exists $[f_p] \in \mathscr{F}_p$ such that $[\mu(f_p)] = [g]_p$, i.e. there is open cover $\{ V_p \}$ such that $\mu(f_p|_{V_p}) = \mu(f_p)|_{V_p} = g|_{V_p}$. By naturality, $\mu(f_p|_{V_p\cap V_q}) = g|_{V_p\cap V_q} = \mu(f_q|_{V_p\cap V_q})$, so injectivity of μ proves that f_p and f_q agree on $V_p \cap V_q$, and the result follows from gluability. □

Remark. The above propositions are not true for presheaves in general. Let $X = \{ x, y \}$, $\mathscr{F} \in \mathsf{Set}_X^{\text{pre}}$ be the presheaf sending every open set to itself, $\mathscr{G}$ the one sending every proper open set to itself and $\{ x, y \} \rightsquigarrow \{ x, y, z \}$. Then the natural transformation of obvious inclusions induces isomorphisms on stalks (both of which are singletons), but is not itself an isomorphism as $\{ x, y \} \to \{ x, y, z \}$ is not a bijection. Of course, this also says that the morphism of stalks does not determine a morphism of presheaves.

Proposition 1.5.10. *Let $\mathscr{F} \in \mathsf{C}_X^{pre}$. Then for all open $U \subseteq X$, $L_{\mathscr{F}}(U)$ is isomorphic to the set of choices of compatible germs in $\prod_{p\in U}\mathscr{F}_p$.*

Proof. We first show that this indeed defines a sheaf. Fix an open cover $\{ U_i \}$ of U. Since $S_{\mathscr{F}}$ is a sheaf, we have an isomorphism $\prod_{p\in U}\mathscr{F}_p \to \varprojlim \prod_{p\in U_i}\mathscr{F}_p$. The same maps may be used as in the sheaf of stalks; we need only show the induced map is well-defined. We have the obvious projections $L_{\mathscr{F}}(U) \to L_{\mathscr{F}}(U_i)$. Given an object X with arrows to $L_{\mathscr{F}}(U_i)$, the induced map Then $L_{\mathscr{F}}(U_i) = \varprojlim L_{\mathscr{F}}(U_i)$ □

Definition 1.5.11. Fix a presheaf $\mathscr{F} \colon \mathsf{Op}(X) \to \mathsf{C}$. The **étale space** $\text{Ét}(\mathscr{F})$ (or space of sections) over $\mathscr{F}$ is the topological space on $\coprod_{p\in X}\mathscr{F}_p$ endowed with the colimit topology with respect to the section $\tilde{s} \colon U \to \text{Ét}(\mathscr{F})$ sending $p \rightsquigarrow (p, [s]_p)$ associated to every $s \in \mathscr{F}(U)$ for open $U \subseteq X$. The canonical map $\pi \colon F \to X$ sends $(p, [f]) \rightsquigarrow p$.

Proposition 1.5.12. *The sets $\{ (x, [s]_x) \colon x \in U, s \in \mathscr{F}(U) \}$ for open $U \subseteq X$ form a base of F.*

Proposition 1.5.13. *π is a local homeomorphism.*

Proposition 1.5.14. *The sheaf of sections associated to π is the sheafification of $\mathscr{F}$.*

Proof. □

1.6. Gluing Sheaves

We assume for the rest of this section that the base category has a closed monoidal structure $(\mathsf{C}, \otimes)$, so that it is enriched over itself. We will also write $U_{ijk\ldots}$ for $U_i \cap U_j \cap U_k \cap \ldots$.

Theorem 1.6.1. *There is a closed monoidal structure on C_X^{pre} given pointwise by $\mathscr{F} \otimes \mathscr{G}(U) = \mathscr{F}(U) \otimes \mathscr{G}(U)$ and $\mathscr{H}om(\mathscr{F}, \mathscr{G})(U) = \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$.*

Proof. We first verify that these bifunctors are well-defined. Indeed, the monoidal product sends an inclusion $U \rightsquigarrow V$ to the restriction $r^{\mathscr{F}}_{V,U} \otimes r^{\mathscr{G}}_{V,U} \colon \mathscr{F}(V) \otimes \mathscr{G}(V) \to \mathscr{F}(U) \otimes \mathscr{G}(U)$. On the other hand, the internal Hom is well-defined on objects as $\mathsf{C}_X^{\mathrm{pre}}$ is canonically C-enriched (once we replace $\mathsf{Op}(X)$ with free V-enriched category it generates), and on morphisms by sending $U \rightsquigarrow V$ to the morphism $\mathrm{Hom}(\mathscr{F}|_V, \mathscr{G}|_V) \to \mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ defined on each $\varphi \colon \mathscr{F}|_V \to \mathscr{G}|_V$ as the morphism on U defined on each open subset $U' \subseteq U$ as $\varphi(U')$ (why is this a morphism? we only know it's a set map). Verifying the monoidal coherence conditions is left as an exercise for the reader.
It remains to demonstrate the natural bijection $\mathrm{Hom}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}(\mathscr{F}, \mathscr{H}om(\mathscr{G}, \mathscr{H}))$ for the adjunction. Indeed, natural transformations are computed pointwise, and due to the closed monoidal structure on C we have $\mathrm{Hom}(\mathscr{F}(U), \mathscr{H}(U)) \cong \mathrm{Hom}(\mathscr{F}(U), \mathrm{Hom}(\mathscr{G}(U), \mathscr{H}(U)) = \mathrm{Hom}(\mathscr{F}(U), \mathscr{H}om(\mathscr{G}, \mathscr{H})(U))$ $\square$

Theorem 1.6.2. *If $\mathscr{G}$ is a sheaf, then $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ is a sheaf.*

Proof. We deduce from the left-adjointness of the sheaf monoidal product the following chain of isomorphisms: $\varprojlim \mathscr{H}om(\mathscr{F}, \mathscr{G})(U_i) \cong \varprojlim \mathrm{Hom}(\mathscr{F} \otimes h_{U_i}, \mathscr{G}) \cong \mathrm{Hom}(\varinjlim \mathscr{F} \otimes h_{U_i}, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F} \otimes \varinjlim h_{U_i}, \mathscr{G})$. It remains to show that $\mathrm{Hom}(\mathscr{F} \otimes \varinjlim h_{U_i}, \mathscr{G}) \cong \mathrm{Hom}(\mathscr{F} \otimes h_U, \mathscr{G})$, with the latter being $\mathscr{H}om(\mathscr{F}, \mathscr{G})(U)$. Manifestly $\mathscr{H} := \varinjlim h_{U_i}$ sends W to the terminal object 1 if $W \subseteq U_i$ for some U_i, the collection of which open sets is denoted $\mathcal{V}$, and the initial object 0 otherwise. This leaves one canonical way for it to send morphisms, as $\mathscr{H}(V) = 1 \implies \mathscr{H}(U) = 1$ if $U \subseteq V$. Since there are compatible morphisms $h_{U_i} \to h_U$ whose component at V is the unique map $\mathrm{Hom}(V, U_i) \to \mathrm{Hom}(V, U)$, there is an induced morphism $u \colon \mathscr{H} \to h_U$. We claim that precomposition with $\mathrm{id}_{\mathscr{F}} \otimes u$ gives the desired isomorphism $\mathrm{Hom}(\mathscr{F} \otimes h_U, \mathscr{G}) \to \mathrm{Hom}(\mathscr{F} \otimes \mathscr{H}, \mathscr{G})$. Fix a morphism, $\varphi \colon \mathscr{F} \otimes \mathscr{H} \to \mathscr{G}$. We define the component at W of a morphism $\tilde{\varphi} \colon \mathscr{F} \otimes h_U \to \mathscr{G}$ to send $\{x, *\} \in \mathscr{F}(W) \otimes \{*\}$ to the unique section glued from $\varphi_V(x, \{*\})$ for V in the cover $\mathcal{V} \cap \mathsf{Op}(W)$ of W. $\square$

Corollary 1.6.3 (Morphisms glue)**.** *For any open $U \subseteq X$ and an open cover $\{ U_i \}$, a set of morphisms of sheaves $\varphi_i \colon \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ agreeing on intersections ($\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$) gives rise to a unique morphism $\varphi \colon \mathscr{F}|_U \to \mathscr{G}|_U$ such that $\varphi|_{U_i} = \varphi_i$.*

It turns out that sheaves glue as well; our object is then to determine an equivalence of categories between C_X and a category of gluing data of sheaves. Gluing sheaves, surprisingly, is very similar to gluing sections. First, both are with respect to a fixed open cover $\{ U_i \}$ of X. Within a sheaf, $\mathscr{F}(U)$ is the limit of the diagram $\mathscr{F}(U_i) \longrightarrow \mathscr{F}(U_i \cap U_j) \longleftarrow \mathscr{F}(U_j)$. Given a collection of sheaves $\{ \mathscr{F}_i \}$ on U_i, we can regard each as a sheaf on X using the direct image. If we naively use the same diagram as above, namely $\mathscr{F}_i \longrightarrow \mathscr{F}|_{U_i \cap U_j} \longleftarrow \mathscr{F}_j$, we realize there is some ambiguity: both $\mathscr{F}_i$ and $\mathscr{F}_j$ can be restricted to $U_i \cap U_j$! But it is unwarranted to make a choice here, as if we choose the restriction to be from $\mathscr{F}_i$, $\mathscr{F}_j$ no longer admits a map to it. To avoid making any unwarranted choice, we thus incorporate both $\mathscr{F}_i|_{U_{ij}}$ and $\mathscr{F}_j|_{U_{ij}}$ into the gluing data of sheaves.

Definition 1.6.4. Fix a topological space X. A **gluing datum** with respect to a cover $\{ U_i \}$ of X is a pair $(\mathscr{F}_i, \varphi_{ij})$ in which each $\mathscr{F}_i$ is a sheaf on U_i, each $\varphi_{ij} \colon \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ a transition isomorphism. A morphism of gluing data is a collection of natural transformations $(\Phi_i \colon \mathscr{F}_i \to \mathscr{G}_i)$ such that $\Phi_j \circ \varphi_{ij}^F = \varphi_{ij}^G \circ \Phi_i$. We thus obtain a category GD_{U_i} of gluing data w.r.t. $\{ U_i \}$.
The **gluing functor** $\mathrm{Gl} \colon \mathsf{GD}_{U_i} \to \mathsf{C}_X$ is the limit functor on the diagram D with vertices $(i,j) := \iota_*(\mathscr{F}_i|_{U_i \cap U_j})$ and morphisms $\{ r_{ij} \colon (i,i) \to (i,j) \} \cup \{ \varphi_{ij} \}$ precomposed with the inclusion $\mathsf{GD}_{U_i} \to (\mathsf{C}_X)^D$.

Verification. We demonstrate that the inclusion is indeed functorial. Given a morphism in GD_{U_i}, consider the morphism whose component at (i,j) is $\Phi_i|_{(i,j)}$. Naturality w.r.t. φ_{ij} follows from the compatibility condition in GD_{U_i}, whereas that w.r.t. r_{ij} is tautological. □

Proposition 1.6.5. Gl *is split essentially surjective.*

Proof. We exhibit a right quasi-inverse: consider the functor $\mathrm{Sp} \colon \mathsf{C}_X \to \mathsf{GD}_{U_i}$ sending a sheaf $\mathscr{F}$ on X to the gluing datum $(\mathscr{F}|_{U_i}, \mathrm{id}_{\mathscr{F}|_{U_i \cap U_j}})$, a morphism $\Phi \colon \mathscr{F} \to \mathscr{G}$ to $(\Phi|_{U_i} \colon \mathscr{F}_i \to \mathscr{G}_i)$. We must show that $\mathscr{F}$ is the limit of the image of $(\mathscr{F}_i, \varphi_{ij})$ under the inclusion into $(\mathsf{C}_X)^D$. Indeed, since limits of sheaves are computed pointwise, this amounts to that for all U, $\mathscr{F}(U)$ is the limit of the diagram made up of $\mathscr{F}(U \cap U_i) \to \mathscr{F}(U \cap U_i \cap U_j) \leftarrow \mathscr{F}(U \cap U_j)$, which is immediate from definition, taking $\{ U \cap U_i \}$ to be the open cover. □

Remark. We could have defined Gl for *presheaves* instead, but it would then not be essentially surjective. As can be seen, the sheaf axioms are necessary for proving that $\mathscr{F}$ is the limit; the idea is that not all presheaves on X arise from gluing.

Proposition 1.6.6. *The restriction of* Gl *to the full subcategory of* GD_{U_i} *whose objects satisfy the* ***cocycle condition*** $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ *is an equivalence of categories.*

$$\begin{tikzcd}
\mathscr{F}_i|_{U_i \cap U_j \cap U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathscr{F}_k|_{U_i \cap U_j \cap U_k} \\
& \mathscr{F}_j|_{U_i \cap U_j \cap U_k} \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

Proof. Sp remains a right quasi-inverse, for clearly the gluing datum obtained from a sheaf $\mathscr{F}$ satisfies the cocycle condition. We now show fully faithfulness. That is, for any pair of gluing datum $(\mathscr{F}_i, \varphi_{ij})$, $(\mathscr{G}_i, \psi_{ij})$ satisfying the cocycle condition, any morphism (Φ_i) of gluing data arises from a unique morphism $\Phi\colon \mathscr{F} \to \mathscr{G}$, where $\mathscr{F} := \mathrm{Gl}(\mathscr{F}_i)$, $\mathscr{G} := \mathrm{Gl}(\mathscr{G}_i)$, for which $\Phi|_{U_i} = \Phi_i$ (up to some representative). It then suffices to show that the canonical maps $\varphi_i\colon \mathscr{F}|_{U_i} \to \mathscr{F}_i$ are isomorphisms, and that $\varphi_i|_{U_i \cap U_j} = \varphi_{ij}$; the result then follows from Corollary 1.6.3, as compatibility on intersections follows from the naturality of gluing data morphisms.

Since the restriction functor is right adjoint, this amounts to that $\mathscr{F}_i$ is the limit of the diagram involved in Gl, restricted to U_i. By the cocycle condition, we can take the projections to be as follows:

$$\begin{tikzcd}
& \mathscr{F}_j|_{U_{ij}} \arrow[r] & \mathscr{F}_j|_{U_{ijk}} \arrow[d, shift left, "\varphi_{jk}"] \\
\mathscr{F}_i \arrow[ur, "\pi_j"] \arrow[dr, "\pi_k"'] & & \\
& \mathscr{F}_k|_{U_{ik}} \arrow[r] & \mathscr{F}_k|_{U_{ijk}}, \arrow[u, shift left, "\varphi_{kj}"]
\end{tikzcd}$$

where π_j is given by $\varphi_{ij} \circ r_{U_j}$. Since $\mathscr{F}_i$ is itself in the diagram and $\pi_i = \mathrm{id}_{\mathscr{F}_i}$, evidently it is the desired limit. $\square$

1.7. Ringed Spaces and $\mathscr{O}_X$-Modules

Our work in this section will vindicate our opening claim, which may be what has kept the reader intrigued thus far, that sheaves provide a unifying language for all of geometry; by the end we will have described various versions of manifolds (and their morphisms) in the language of sheaf theory, the same language that will enable us to extract the geometry out of algebraic contexts in Chapter 2.

Definition 1.7.1. A **ringed space** is a pair $(X, \mathscr{O}_X)$, with X a topological space, $\mathscr{O}_X$ a sheaf of rings on X termed the **structure sheaf** of the ringed space. Sections of $\mathscr{O}_X$ over an open set U are **functions on** U, and functions on X **global functions**.

Remark. It is *not* an abuse of notation to refer to arbitrary elements of the sheaf as "functions," for we will see that they genuinely behave like functions, and it is beneficial to think of them as such.

One might reasonably question the need to define ringed spaces separately: clearly the space is part of the datum of the structure sheaf. It is the morphisms between them that distinguish ringed spaces from mere sheaves of rings: to account for the underlying topological spaces, a continuous map must be part of the datum of such a morphism. There then arises the challenge of mapping between sheaves defined over different spaces: the direct image construction comes to the rescue.

Definition 1.7.2. A **morphism of ringed spaces** $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a continuous map $\pi \colon X \to Y$ together with a **pullback morphism** of sheaves $\pi^\# \colon \mathscr{O}_Y \to \pi_* \mathscr{O}_X$. We define the composition of $(f, f^\#) \colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $(g, g^\#) \colon (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ to be $(g \circ f, g_*(f^\#) \circ g^\#)$. The **category of ringed spaces** is denoted RS.
We denote the morphism $\pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ corresponding to $\pi^\#$ by $\pi^\flat$. These, similar to the above, compose as $f^\flat \circ f^{-1}(g^\flat)$.

Of course, the pullback morphism is entirely contingent on the behavior of the underlying continuous map: its component on some $U \subseteq Y$ has as codomain $\mathscr{O}_X(\pi^{-1}(U))$. Formally, the contravariance of the pullback map derives from the very definition of continuity, as a priori the only function between $\mathsf{Op}(Y)$ and $\mathsf{Op}(X)$ is given by taking preimages, which necessarily runs opposite to direction of π. This matches what happens in practice: viewing the global sections of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$) as functions $X \to Z$ (resp. $Y \to Z$) for some other topological space Z, it is clear that the effect of π can only be via pre-composition, which is of course contravariant. For instance,

Definition 1.7.3. The **sheaf $\mathscr{C}^k_X$ of C^k-maps on** a C^k-manifold X assigns to each open $U \subseteq X$ the $\mathbb{K}$**-algebra of C^k-maps** on U, $\mathrm{Hom}_{C^k}(U, \mathbb{K})$, and to each inclusion $i \colon U \to V$ the restriction $r_{V,U} \colon \mathscr{C}^k(V) \to \mathscr{C}^k(U)$ resulting from precomposition with i. By $\mathbb{K}$ we understand *either* $\mathbb{R}$ and $\mathbb{C}$; homolorphic functions are said to be C^ω.

Proposition 1.7.4. *A continuous map of C^k-manifolds $\pi \colon X \to Y$ is C^k iff pre-composition with π produces a morphism $\mathscr{C}^k_Y \to \pi_* \mathscr{C}^k_X$.*

Proof. The forward direction is immediate: for any open $V \subseteq Y$ and C^k-map $f \colon V \to \mathbb{K}$, the composition $f \circ \pi$ is C^k; naturality $f \circ i_{U,V} \circ \pi = i_{\pi^{-1}(U),\pi^{-1}(V)} \circ f \circ \pi$ amounts to the definition of inclusion. To see the converse, let $\varphi \colon U \to \mathbb{K}^m$ be a chart in X, $\psi \colon V \to \mathbb{K}^n$ a chart in Y. We must show that $\psi \circ \pi \circ \varphi^{-1}$ is C^k. Writing $\psi := (\psi_1, \ldots, \psi_n)$, it will suffice to show that $\psi_i \circ \pi \circ \varphi^{-1}$ is C^k for each i. Indeed, by supposition we have that $\psi_i \circ \pi$ is C^k, which is preserved by composition with φ^{-1}, a local diffeomorphism. □

Remark. Unfortunately, the above may not be recast in a categorical light, say as a certain bijection of homsets, at least not in its current form. There is a key attribute of manifolds amiss if we merely consider them as ringed spaces and, more importantly, smooth maps as ringed space morphisms. This desired property is the set-theoretic triviality that given a smooth map $\pi \colon X \to Y$ sending $p \rightsquigarrow q$, a function on $U \subseteq Y$ vanishing at q should vanish on p after pre-composition: $f \circ \pi(p) = f(q) = 0$. The language of ringed spaces is not tailored to such statements, as in general it does not make sense to speak of the value of a section at a point.

As a conflation of a topological space and a sheaf, it is clear that ringed spaces should inherit constructions natural to both. For starters, the notion of "open subspace" behaves as expected.

Definition 1.7.5. Fix a ringed space $(X, \mathscr{O}_X)$. For an open $U \subseteq X$, write $\mathscr{O}_U := \mathscr{O}_X|_U$. The ringed space $(U, \mathscr{O}_U)$ is said to be an **open subspace** of $(X, \mathscr{O}_X)$, with an **inclusion morphism** $j \colon (U, \mathscr{O}_U) \to (X, \mathscr{O}_X)$ having as its underlying continuous map the set-theoretic inclusion and and for its component on $V \subseteq X$ the canonical restriction $\mathscr{O}_X(V) \rightsquigarrow \mathscr{O}_X(j^{-1}(V))$. Manifestly $\mathscr{O}_U = j^{-1}\mathscr{O}_X$.

Definition 1.7.6. Let $(Y, \mathscr{O}_Y)$, $(X, \mathscr{O}_X)$ be ringed spaces. A morphism $(Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ is an **open embedding** (or **open immersion**) if it factors as

$$(Y, \mathscr{O}_Y) \xrightarrow{\sim} (U, \mathscr{O}_U) \xrightarrow{j} (X, \mathscr{O}_X).$$

Remark. There is also a notion of *closed* embeddings of ringed spaces. But as can be seen, closed subsets are not an essential part of the datum of sheaves, so the definition is much harder to motivate and thus relegated to Chapter 2.

Proposition 1.7.7. *A morphism $\pi\colon Y \to X$ is an open embedding iff π is a homeomorphism onto an open subset of X and the map $\pi^{-1}\mathscr{O}_X \to \mathscr{O}_Y$ corresponding to the pullback is an isomorphism.*

Proof. The forwards direction is clear. For the converse, write π' for the homomeomorphism given by π, V for $\pi(Y)$. We clearly have the factorization $\pi = j \circ \pi'$ where both sides are regarded as continuous maps. It follows that $\pi^{-1}\mathscr{O}_X = \pi'^{-1}j^{-1}\mathscr{O}_X = \pi'^{-1}\mathscr{O}_V$, so the isomorphism induced from the pullback is precisely the induced map $\pi'^{-1}\mathscr{O}_V \to \mathscr{O}_Y$. Since the other pullback $j^{-1}\mathscr{O}_Y \to \mathscr{O}_V$ is the identity, we are done. $\square$

Proposition 1.7.8 (Universal Property of Open Embeddings)**.** *Let $\pi\colon X \to Y$ be a morphism of ringed spaces. For open embeddings $i\colon U \hookrightarrow X$, $j\colon V \hookrightarrow Y$ such that $\pi(i(U)) \subseteq j(V)$, there is a unique morphism $\pi|_U\colon U \to V$ making the following diagram commute:*

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[d, dashed, "\pi|_U"'] & X \arrow[d, "\pi"] \\ V \arrow[r, "j"] & Y. \end{tikzcd}$$

Proof. Let us first prove this for when V is a subspace. The existence and uniqueness of this factorization of π as a continuous map is evident. Since j is an open subspace, $j^\flat$ is the identity, and $\pi|_U^{-1}\mathscr{O}_V = \pi|_U^{-1}j^{-1}\mathscr{O}_Y = i^{-1}\pi^{-1}\mathscr{O}_Y$. It follows that the $\pi|_U$ is uniquely determined by the composition $i^\flat \circ i^{-1}(\pi^\flat)$.

The full statement follows if we consider the following diagram:

$$\begin{tikzcd} & U \arrow[r, "i"] \arrow[dl, dashed, "f"] \arrow[d, dashed, "g"] & X \arrow[d, "\pi"] \\ V \arrow[r, "\sim", "\varphi"'] & V' \arrow[r, "j'"] & Y, \end{tikzcd}$$

wherein V' denotes the open subspace of Y through which the open embedding $V \hookrightarrow Y$ factors. Notice, commutativity forces $j' \circ g = \pi \circ i = j \circ f = j' \circ \varphi \circ f$. Since we have already shown the universal property for open subspaces, the proof of the corollary below shows that j' is a monomorphism. As such, f is uniquely determined as $\varphi^{-1} \circ g$. $\square$

In particular, any morphism factors uniquely through its set-theoretic image, and if π is an open immersion, we know that the isomorphism through which it factors is unique. It will thus be harmless to identify an open subset $U \subseteq X$ with the ringed subspace $(U, \mathscr{O}_U)$.

Corollary 1.7.9. *Open immersions are monomorphisms in* RS.

Proof. Let $f\colon Y \to X$ be an open immersion, Z a ringed space, $g_1, g_2\colon Z \to Y$ morphisms such that $f \circ g_1 = f \circ g_2$. That $g_1 = g_2$ is immediate from the following diagram:

$$\begin{tikzcd} & & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] \arrow[urr, dashed, "g"] & Y \arrow[r, "f"'] & X. \end{tikzcd}$$

□

Remark. It turns out that open embeddings are precisely the *open sets* in the etále topology imposed on the *category* of locally ringed spaces. This probably went right over your head, and is completely irrevelant for the time being. But it will be a key ingredient when we define *stacks* in the far future. (Make precise once the section on sites is done)

Like topological spaces and sheaves, ringed spaces glue as well. This also occurs on a categorical level, i.e. it be formulated as an equivalence of categories between RS and the category of *gluing data*. As before, we first glue morphisms.

Theorem 1.7.10. *Given ringed space* $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$, *an open cover* $X = \bigcup U_i$, *morphisms of ringed spaces* $\pi_i\colon U_i \to Y$ *agreeing on the overlaps, i.e.* $\pi_i|_{U_i \cap U_j} = \pi_j|_{U_i \cap U_j}$, *there exists a unique morphism of ringed spaces* $\pi\colon X \to Y$ *such that* $\pi|_{U_i} = \pi_i$.

Proof. That π is well-defined as a continuous map on X follows from that continuous maps from open subsets of X to Y forms a sheaf. Since $\pi_i^\flat|_{U_i \cap U_j} = (\pi_i|_{U_i \cap U_j})^\flat = (\pi_j|_{U_i \cap U_j})^\flat = \pi_j^\flat|_{U_i \cap U_j}$ and $\pi^{-1}\mathscr{O}_Y|_{U_i} = (\pi|_{U_i})^{-1}\mathscr{O}_Y = \pi_i^{-1}\mathscr{O}_Y$, Corollary 1.6.3 furnishes a unique map $\pi^\flat\colon \pi^{-1}\mathscr{O}_Y \to \mathscr{O}_X$ such that $\pi^\flat|_{U_i} = \pi_i^\flat$. Uniqueness is evident. □

Corollary 1.7.11. *Given ringed spaces* $(X, \mathscr{O}_X), (Y, \mathscr{O}_Y)$, *there is a sheaf of sets on* X *which sends each open* $U \subseteq X$ *as* $\operatorname{Hom}(X|_U, Y)$ *and each inclusion* $U \to V$ *to the obvious restriction.*

Proof. Applying the above to $(U, \mathscr{O}_X|_U), (Y, \mathscr{O}_Y)$ yields the result. □

Remark. Note that this *almost* forms a ringed space: while there is an obvious way to add and multiply the pullback morphisms, that the obstruction comes from the lack of an operation on continuous maps between spaces with no extra structure. This is already an indicator that RS is not as well-behaved a category as categories of sheaves, much like how R-Mod is "nicer" than Ring. This is one of the motivations for defining "modules over ringed spaces."

In defining a gluing datum, we are led by the principle that whatever is given here should apply equally if the sheaf if forgotten, that is, if taken entirely in Top. Let us then consider how the gluing datum of sheaves can be modified to fit the situation at hand. Fundamentally, in order to glue, we always need a set of spaces and transition maps ensuring they agree on intersections. But while the restricted sheaves needed not be specified, the intersection subspaces need to be here, as a priori there is no common space from which they can be restricted: that is precisely what we are trying to obtain via gluing! It is even clear from this that gluing of the underlying topological spaces must occur "prior" to that of the sheaves, which must use that glued topological space as its atlas.

Definition 1.7.12. Let $\mathsf{C} = \mathsf{Top}, \mathsf{RS}$. A **gluing datum of C-objects** w.r.t. an indexing set I is a triple of sets $(\{ X_i \}_{i \in I}, \{ U_{ij} \}, \{ \varphi_{ij} \})$, where each X_i is a space in C, U_{ij} an open subspace thereof (we assume $U_{ii} = X_i$), and $\varphi_{ij} \colon U_{ij} \to U_{ji}$ a transition morphism, such that the following conditions hold:

1. $\varphi_{ij} \circ \varphi_{ji} = \mathrm{id}$;
2. $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$;
3. $\varphi_{ik}|_{U_{ij} \cap U_{ik}} = \varphi_{jk}|_{U_{ji} \cap U_{jk}} \circ \varphi_{ij}|_{U_{ij} \cap U_{ik}}$.

A shorthand for this gluing datum is (X_i). A morphism of gluing data is a collection of C-morphisms $(\Phi_i \colon X_i \to Y_i)$ such that $\Phi_j \circ \varphi_{ij}^X = \varphi_{ij}^Y \circ \Phi_i$. The resultant *category of gluing data* is denoted $\mathsf{C}\text{-}\mathsf{GD}_I$.
The **gluing functor** $\mathrm{Gl} \colon \mathsf{C}\text{-}\mathsf{GD}_I \to \mathsf{C}$ is the colimit functor on the diagram D comprised of

$$\begin{tikzcd} U_{ij} \arrow[r] \arrow[d, shift left, "\varphi_{ij}"] & X_i \\ U_{ji}, \arrow[r] \arrow[u, shift left, "\varphi_{ji}"] & X_j \end{tikzcd}$$

for $i, j \in I$, precomposed with the inclusion $\mathsf{C}\text{-}\mathsf{GD}_I \to \mathsf{C}^D$.

Remark. Notice how unlike with sheaves, we have imposed the cocycle condition from the get-go. This is not just because the condition has already been motivated; rather, proving that the gluing functor on a category with more structure is well-defined will hinge on the same result for the "underlying" categories, together with a full suite of implications regarding the canonical maps, which can be deduced only using the cocycle condition.

Since Top is cocomplete, the gluing functor for topological spaces is well-defined. It turns out that so is RS, but of course we have not proven this yet. Thus we must construct the colimit directly, to which end, we require a few auxiliary results on the topological gluing that follow from the cocycle condition.

Lemma 1.7.13. *Let (X_i) be a gluing datum of topological spaces, $X := \mathrm{Gl}(X_i)$. Then the canonical maps $\varphi_i \colon X_i \to X$ are open embeddings with images U_i, and the following holds:*

1. $X = \bigcup U_i$,
2. $\varphi_i(U_{ij}) = U_i \cap U_j$,
3. $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$.

Proof. By the cocycle condition, each X_i is the colimit of D restricted to U_i. To show that φ_i is an open embedding, □

Proposition 1.7.14. *The gluing functor* $\mathrm{Gl}\colon \mathsf{RSGD}_I \to \mathsf{RS}$ *is an equivalence of categories. Furthermore, the canonical maps $\varphi_i \colon X_i \to \mathrm{Gl}_{\mathsf{RS}}(X_i)$ agree with the topological ones, and are ringed space open embeddings.*

Proof. Let (X_i) be a gluing datum. Put $X := \mathrm{Gl}_{\mathsf{Top}}(X_i)$, and by the above, we may write $\phi_i \colon X_i \to U_i$ for the unique homeomorphisms through which the canonical maps factor. We thus have sheaves, $\mathscr{F}_i := \phi_{i*}\mathscr{O}_{X_i}$ on U_i. Applying ϕ_i to the transition isomorphisms $\varphi_{ij}^{\#} \colon \mathscr{O}_{X_i}|_{U_{ij}} \to {\varphi_{ij}}_*\mathscr{O}_{X_j}|_{U_{ji}}$, we obtain isomorphisms $\phi_{ij} \colon \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$. The cocycle condition is manifestly satisfied, and we thus obtain a unique (up to isomorphism) sheaf $\mathscr{F}$ on X such that $\mathscr{F}|_{U_i} = \mathscr{F}_i$. □

The category of ringed spaces, however, is not fully suited to geometric spaces, for reasons explained in the remark after Proposition 1.7.4. That is, we need a notion of evaluation that must be safely transported by morphisms. It turns out, however, that we can emulate evaluation using stalks:

Proposition 1.7.15. *For all $p \in X$, $\mathscr{C}_p^k$ forms a local ring.*

Proof. We remark that the ring structure on the stalk is given by $[f, U] + [g, V] = [f + g, U \cap V]$, $[f, U][g, V] = [fg, U \cap V]$, leaving the verification to the reader. To see that it is a local ring, we first show that the ideal $I(p) := \left\{ [f] \in \mathscr{C}_p^k \colon f(p) = 0 \right\}$ of germs vanishing at p is maximal. (We will meet this ideal again, using them as surrogates for classical points in the algebro-geometric context. This point of view will be put on solid grounding when we inroduce schemes in Chapter 2.) Indeed, $\mathscr{C}_p^k / I(p)$ is a field, as the sequence

$$0 \longrightarrow I(p) \longrightarrow \mathscr{C}_p^k \xrightarrow{[f] \rightsquigarrow f(p)} \mathbb{K} \longrightarrow 0$$

is exact. (See below for why this is well-defined.)

The uniqueness of $I(p)$ as a maximal ideal of $\mathscr{C}_p^k$ would follow from that every element of $[f, U] \in \mathscr{C}_p^k \setminus I(p)$ is invertible, as then any ideal involving them would be the entirety of $\mathscr{C}_p^k$. This in turn amounts to finding a sufficiently small neighborhood V on which f is uniformly non-zero, since evidently $(f, U) \sim (f, V)$. Let U be a neighborhood of $f(p)$ which does not contact zero, and simply take $f^{-1}(U)$ to be the desired neighborhood. $\square$

Remark. A small tangent: this is the first reason local rings are "local." Stalks are local (in the intuitive sense) entities, and their type as rings should certainly enjoy the property of being "local!" We will give the second reason when discussing geometric interpretations of localizations.

The punchline is that $f(p) = \pi([f])$, where π denotes the canonical projection $\mathscr{C}_p^k \to \mathscr{C}_p^k / I(p)$, which is only well-defined because we *do* have an evaluation that is well-behaved in the sense that all representatives of a germ agree at the point. We may thus employ the "old French trick" of turning a theorem into a definition: we will transform the result that stalks form local rings, which had to be deduced from the existence of evaluation, into the *criterion* for a ringed space to have evaluation.

Definition 1.7.16. A ringed space $(X, \mathscr{O}_X)$ is a **locally ringed space** if its stalks are local rings. The unique maximal ideal of $\mathscr{O}_{X,p}$ is denoted $\mathfrak{m}_p$, the **residue field** $\kappa(p) := \mathscr{O}_{X,p}/\mathfrak{m}_p$. The **value at** p of any function f on an open subset $U \subseteq X$ containing p is $f(p) := \pi_p(f)$, where π_p denotes the composition $\mathscr{O}(U) \longrightarrow \mathscr{O}_{X,p} \longrightarrow \kappa(p)$.

Here is some topological evidence this is a sensible definition:

Proposition 1.7.17. *The set of points on which a function f on a locally ringed space $(X, \mathscr{O}_X)$ vanishes, $V(f) := \{ x \in X \colon [f] \in \mathfrak{m}_x \}$, is closed. Equivalently, the locus $D(f)$ where f is nonzero is open. Furthermore, f vanishes nowhere iff it is invertible.*

Proof. Notice, f is nonzero at a point p iff its germ there is invertible: indeed, an element of $\mathfrak{m}_p$ cannot be invertible, lest $\mathfrak{m}_p$ the the entire stalk, and f being non-invertible would imply

that (f) is proper, hence contained in a maximal ideal, namely $\mathfrak{m}_p$. Now let $p \in X$ be a point on which $[f]_p$ is invertible. Denote the function whose germ is its inverse by g. Then $[f]_p[g]_p = [1]_p$, furnishing a neighborhood U of p on which fg is uniformly 1. Given $q \in U$, clearly $[f]_q[g]_q = [fg]_q = [1]_q$, so $[f]_q$ is indeed invertible.
Now if f vanishes nowhere, then the locus where $[f]_p$ is invertible is X. We construct an inverse to f. Let g_p be as above. We restrict each g_p to the neighborhood U_p on which $ff'_p = 1$. It remains to show that functions in $\{ g_p \}$ agree on intersections $U_p \cap U_q$; the global inverse will then be yielded by the sheaf axioms. Indeed, for every $x \in U_p \cap U_q$, $[g_p]_x = [f]_x^{-1} = [g_q]_x$, so there is a neighborhood of x on which they agree. The converse is obvious: the germ of the global inverse is inverse to that of f at every point. $\square$

Remark. The distinction between f vanishing at p and $[f]_p$ being 0 is a subtlety that might be elusive at first. The latter, which may be thought of as having an open neighborhood U of p where $f(U) = 0$, is strictly stronger than the former, which literally translates to $f(p) = 0$, as a nonzero $[f]_p$ can still lie in $\mathfrak{m}_p$. It is then not too surprising, as we recall, that the support of a section is closed: it is the (tounge-in-cheek) preimage of a closed set U^c, whereas the nonvanishing locus is the preimage of $\{ 0 \}^c$, which is open. This distinction is an important one: stalks determine sections, but as we will see in the next section, there are locally ringed spaces in which functions are not determine by their values on points.

It will turn out to be important that V and D are topological invariants, in the following sense:

Proposition 1.7.18. *Let X, Y be locally ringed spaces. If there is an isomorphism of ringed spaces $\pi\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$, then for all $f \in \Gamma(Y, \mathscr{O}_Y)$, $V_Y(f) \cong V_X(\pi^\# f)$.*

Corollary 1.7.19. *The values of a function on a dense subset of X determines its values at all points of X.*

Proof. Assume that f vanishes on a dense subset K of X. Then $X = \overline{K} \subseteq V(f)$. $\square$

Crucially, it is possible to transmit values of functions across morphisms.

Proposition 1.7.20. *Fix a morphism of ringed spaces $\pi\colon X \to Y$. For every $p \in X$, there is an induced morphism of stalks $\pi^\#\colon \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$.*

Proof. It suffices to exhibit compatible morphisms $\mathscr{O}_Y(U) \to \mathscr{O}_{X,p}$ for every open $U \subseteq Y$

containing $\pi(p)$. Indeed, we have the diagram

$$\begin{tikzcd} \Gamma(V,\mathscr{O}_Y) \arrow[r] & \Gamma(\pi^{-1}(V),\mathscr{O}_X) \arrow[dr] & \\ & & \mathscr{O}_{X,x}. \\ \Gamma(U,\mathscr{O}_Y) \arrow[r] \arrow[uu] & \Gamma(\pi^{-1}(U),\mathscr{O}_X) \arrow[uu] \arrow[ur] & \end{tikzcd}$$

$\square$

Corollary 1.7.21. *Fix a morphism of ringed spaces $\pi\colon X \to Y$. Then the following diagram commutes*

$$\begin{tikzcd} \Gamma(Y,\mathscr{O}_Y) \arrow[r, "\pi^\#"] \arrow[d] & \Gamma(X,\mathscr{O}_X) \arrow[d] \\ \mathscr{O}_{Y,\pi(x)} \arrow[r, "\pi_x^\#"] & \mathscr{O}_{X,x}. \end{tikzcd}$$

The condition that $q = \pi(p) \wedge f(q) = 0 \implies f \circ \pi(p) = 0$ then translates into

Definition 1.7.22. A **morphism of locally ringed spaces** $\pi\colon X \to Y$ is a morphism of ringed spaces such that for all $p \in X$, the induced morphism $\pi_p^\#\colon \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$ is a local homomorphism. That is, $\pi_p^\#(\mathfrak{m}_{\pi(p)}) \subseteq \mathfrak{m}_p$, or equivalently $(\pi_p^\#)^{-1}(\mathfrak{m}_p) = \mathfrak{m}_{\pi(p)}$. The resultant *subcategory of locally ringed spaces* is denoted $\mathsf{LRS} \subset \mathsf{RS}$.

That vanishing functions pull back immediately ensures that *nonzero* functions also do:

Proposition 1.7.23. *Morphisms of locally ringed spaces $\pi\colon X \to Y$ induces inclusions $\kappa(\pi(p)) \to \kappa(p)$. Hence invertible functions pull back.*

Proof. It will suffice to show that the composition $\mathscr{O}_{Y,\pi(p)} \xrightarrow{\pi^\#} \mathscr{O}_{X,p} \longrightarrow \mathscr{O}_{X,p}/\mathfrak{m}_p$ has a kernel containing $\mathfrak{m}_{\pi(p)}$. But this is immediate from that $\pi^\#$ is local. $\square$

Proposition 1.7.24. *Let $\pi\colon X \to Y$ be an LRS morphism. Then for all $f \in \Gamma(Y,\mathscr{O}_Y)$, $\pi^{-1}(D(f)) = D(\pi^\#(f))$.*

Proof. Evidently the equality amounts to that $[f] \in \mathfrak{m}_{\pi(x)}$ iff $[\pi^\#(f)] \in \mathfrak{m}_x$. But this is immediate from Corollary 1.7.21 and the locality of $\pi_x^\#$. $\square$

LRS is reflective subcategory.

The requirements on pulling back do not get in the way of the gluing properties, and Theorem 1.7.10 admits the following specialization.

Theorem 1.7.25. *Morphisms of locally ringed spaces glue.*

Proof. One sees that the induced morphism $\pi^\#\colon \mathscr{O}_{Y,\pi(p)} \to \mathscr{O}_{X,p}$ is equal to $\pi_i^\#\colon \mathscr{O}_{Y,\pi_i(p)} \to \mathscr{O}_{U_i,p}$ composed with the unique isomorphism $\mathscr{O}_{U_i,p} \to \mathscr{O}_{X,p}$ for any U_i containing p by simple cofinality arguments. $\square$

Theorem 1.7.26. *Locally ringed spaces glue.*

It is time to close off the circle of ideas involving manifolds and locally ringed spaces.

Theorem 1.7.27. *There is a fully faithful functor from the category of C^k-manifolds to* LRS.

Proof. We define the functor F as sending X to $(X, \mathscr{C}_X^k)$ and each C^k-map to the morphism of Proposition 1.7.4. That said morphism is local is obvious from the preceding discussion: indeed, if $f \in \mathfrak{m}_{Y,p} \cong I_Y(\pi(p))$, then $f \circ \pi \in I_X(p)$. We claim that any morphism $(\pi, \psi)\colon (X, \mathscr{C}_X^k) \to (Y, \mathscr{C}_Y^k)$ is pre-composition with π, from which the result would follow, as then the assignment $(\pi, \psi) \to \pi$ would be an explicit inverse to F, with π being C^k by Proposition 1.7.4.
Consider the diagram

$$\begin{tikzcd}
f \arrow[d, squiggly] & \mathscr{C}^k(V) \arrow[r, "\psi"] \arrow[d] \arrow[dr] & \pi_*\mathscr{C}^k(V) \arrow[d] & \psi_f \arrow[d, squiggly] \\
{[f]_{\pi(p)}} \arrow[d, squiggly] & I_Y(\pi(p)) \arrow[r, dashed, "\psi^\#"] \arrow[d, "\mathrm{ev}_{\pi(p)}"] \arrow[dr] & I_X(p) \arrow[d, "\mathrm{ev}_p"] & {[\psi_f]_p} \arrow[d, squiggly] \\
f(\pi(p)) & \mathbb{K} \cong \kappa_Y(\pi(p)) \arrow[r, dashed] & \mathbb{K} \cong \kappa_X(p) & \psi_f(p),
\end{tikzcd}$$

where $V \subseteq Y$ is open and $p \in \pi^{-1}(V)$. Since $\psi^\#$ is induced, the top square commutes. Taking the bottommost map to be that in Proposition 1.7.23, the bottom square also commutes. Now the only ring homomorphism $\mathbb{K} \to \mathbb{K}$ is the identity. $\square$

We now move to the second theme of this section: modules over ringed spaces.

Definition 1.7.28. A **module over a ringed space** $\mathscr{O}_X$ is a morphism of sheaves $\mathscr{O}_X \to \mathscr{E}(\mathscr{F}_X)$, where $\mathscr{F}_X$ is a sheaf of abelian groups.

By the Tensor-Hom adjunction proven earlier, we may reinterpret this data as

Proposition 1.7.29. *An $\mathscr{O}_X$-module $\mathscr{F}$ is a morphism $\mathscr{O}_X \otimes \mathscr{F}_X \to \mathscr{F}_X$. Explicitly, it is a set of morphisms $\mathscr{O}_X(U) \otimes \mathscr{F}_X(U) \to \mathscr{F}_X(U)$ compatible under restriction, i.e. the following diagram commutes:*

$$\begin{tikzcd} \mathscr{O}_X(U) \otimes \mathscr{F}_X(U) \arrow[r] \arrow[d, "r^{\mathscr{O}}_{U,V} \otimes r^{\mathscr{F}_X}_{U,V}"'] & \mathscr{F}_X(U) \arrow[d, "r_{U,V}"] \\ \mathscr{O}_X(V) \otimes \mathscr{F}_X(V) \arrow[r] & \mathscr{F}_X(V). \end{tikzcd}$$

Morphisms of $\mathscr{O}_X$-modules are then precisely those natural transformations that preserve the module structure:

Definition 1.7.30. A **morphism of $\mathscr{O}_X$ modules** $\mathscr{F}$, $\mathscr{G}$ is a natural transformation $\mu\colon \mathscr{F} \to \mathscr{G}$ such that for all U, μ_U is linear:

$$\begin{tikzcd} \mathscr{O}_X(U) \otimes \mathscr{F}(U) \arrow[r, "\mathrm{id}_U \otimes \mu_U"] \arrow[d] & \mathscr{O}_X(U) \otimes \mathscr{G}(U) \arrow[d] \\ \mathscr{F}(U) \arrow[r, "\mu_U"] & \mathscr{G}(U). \end{tikzcd}$$

The resultant subcategory of Ab_X is denoted $\mathscr{O}_X$-Mod.

Proposition 1.7.31. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. For each $p \in X$, $\mathscr{F}_p$ is an $\mathscr{O}_{X,p}$-module.*

Proof. To exhibit a morphism $\mathscr{O}_{X,p} \to \mathrm{End}(\mathscr{F}_p)$, it suffices to find compatible morphisms $\mathscr{O}_X(U) \to \mathrm{End}(\mathscr{F}_p)$ for all open $U \subseteq X$ containing p. We are given morphisms $\mathscr{O}_X(U) \to \mathrm{End}(\mathscr{F}(U))$, or equivalently, $\mathscr{O}_X(U) \otimes \mathscr{F}(U) \to \mathscr{F}(U)$, each of which we compose with the inclusion $\mathscr{F}(U) \to \mathscr{F}_p$. Now for a fixed U, we precompose each $\mathscr{O}_X(V) \otimes \mathscr{F}(V) \to \mathscr{F}_p$ with $r_{U,V} \otimes \mathrm{id}$, where $V \subseteq U$, to obtain a cofinal set of morphisms $\phi_V\colon \mathscr{O}_X(U) \otimes \mathscr{F}(V) \to \mathscr{F}_p$. Note that these are indeed compatible: for $W \subseteq V$, $\phi_W \circ (\mathrm{id} \otimes r_{V,W}) = \phi_V$ by the $\mathscr{O}_X$-module structure. Taking the colimit, we obtain out desired morphism $\mathscr{O}_X(U) \to \mathrm{End}(\mathscr{F}_p)$, with tensor products commuting with colimits as a left adjoint functor. Compatibility is again due to the $\mathscr{O}_X$-module structure. $\square$

Example 1.7.32. Let $(X, \mathscr{C}^k_X)$ be a C^k-manifold, $\pi\colon V \to X$ a vector bundle. Then the sheaf of C^k-sections $\sigma\colon X \to V$ is a $\mathscr{C}^k_X$-module. Indeed, we may multiply sections $s\colon U \to V$ and functions $f\colon U \to \mathbb{R}$ pointwise using the canonical $\mathbb{R}$-module structure on V. This is the quintessential $\mathscr{O}_X$-module; we will see algebro-geometric analogs very soon.

4.3.8

2. Schemes

2.1. Why Spec?

Underlying all of algebraic geometry is the idea that intrinsic to algebra is geometry. On a basic level there is nothing mysterious, and indeed we have been accustomed to this duality since grade school, perhaps without realizing at the time the profundity of this insight. Consider a polynomial $y - x^2 \in \mathbb{C}[x, y]$. The algebra yields little intuition point-blank, and the natural move is to appeal to geometry and identify it with the complex-valued function $f(x, y) := y - x^2$ on the complex manifold $\mathbb{C}^2$. The roots of f in $\mathbb{C}[x, y]$ gives rise to a closed subspace $V(f)$ of $\mathbb{C}^2$, called the **vanishing set** or **graph** of f, which in the present case is a parabola centered at the origin. Armed with knowledge of all vanishing sets, we can recover f itself, for the locus at which f takes the value $a \in \mathbb{C}$ is precisely $V(f(x, y) - a)$. In sum, we have distilled the natural process by which a "coordinate ring" ($\mathbb{C}[x, y]$) gives rise to a topological space ($\mathbb{C}^2$), in such a way that elements of the ring correspond to closed subspaces.

The first goal of modern algebraic geometry is to make precise this correspondence in the general case, manifesting the geometry inherent in *any ring*. We demand furthermore that this correspondence be reversible and well-behaved categorically. That is, we seek an *equivalence of category* whose domain is Ring. This functor will be known as Spec, its target "the category of affine schemes."

It is perhaps prudent to begin by looking at the problem in reverse: can we describe well-known geometry in terms of algebra? To this end, take a compact Hausdorff topological space M. Our work with ringed spaces has taught us that we should never study a space in isolation from its structure sheaf, which in the case of manifolds captures the datum of coordinate charts in a coordinate-free way. It turns out that more is true: the topological space itself can be recovered from the ring of global sections by assigning to each point $p \in M$ its *vanishing ideal* $I(p) := \{ f \in \mathscr{C}(M) \colon f(p) = 0 \}$ (an old friend to the attentive reader)!

Theorem 2.1.1. *Maximal ideals of $\mathscr{C}(M)$ are precisely the vanishing ideals, and $I(p) = I(q) \implies p = q$. Hence $I(-) \colon p \rightsquigarrow I(p)$ defines a correspondence*

$$\{ p \in M \} \rightleftarrows \operatorname{Spec}_m \mathscr{C}(M) := \{ \textit{maximal ideals of } \mathscr{C}(M) \}.$$

Proof. That $I(p)$ is maximal is proven using the same exact sequence as in Chapter 1, and that $I(p) \neq I(q)$ for $p \neq q$ follows from Urysohn's lemma: as $\{p\}$, $\{q\}$ are closed there is a continuous function which is 0 at p and 1 at q. It remains to show that any ideal I not contained in some $I(p)$ is the whole of $\mathscr{C}(M)$. In such an ideal, for all $p \in P$ there exists some $f_p \in I$ with $f(p) \neq 0$. Take an open cover of M by $\{D(f_p)\}$ where $D(f) := \{p \in M : f(p) \neq 0\}$. Compactness furnishes a finite set of functions $\{f_i\}$ such that for every $p \in M$, there exists some f_i with $f_i(p) \neq 0$. Now manifestly $\sum f_i^2$ is uniformly non-zero, hence invertible, and it follows that $I = \mathscr{C}(M)$. □

Judicious choice of a topology on $\mathscr{C}(M)$ upgrades this bijection to a homeomorphism:

Definition 2.1.2. The **Zariski topology** on $\{I(p)\}$ is generated by the base of **distinguished open sets**

$$\tilde{D}(f) := \{I(p) : f \notin I(p)\}$$

for each $f \in \mathscr{C}(M)$.

Proposition 2.1.3. *$I(-): M \to \mathrm{Spec}_m \mathscr{C}(M)$ is a homeomorphism.*

Proof. It will suffice to show that $I(-)$ sends $D(f) := \{p \in M : f(p) \neq 0\}$ to $\tilde{D}(f)$, as $\{D(f) : f \in \mathscr{C}(M)\}$ forms a base for M. But it is immediate from definition that for $f \in \mathscr{C}(M)$, $f(p) \neq 0$ iff $f \notin I(p)$. □

Notice that $I(-)$ and $D(-)$ involves nothing specific to M in their definitions and may be carried over without change to arbitrary rings. It turns out that they are *almost* the correct gadgets to consider in the general setting: we may view ring elements as *functions*, ideals *points*. The gap, however, is that taking the set of maximal ideals is not *functorial*. That is, one seeks for each morphism of rings $\varphi : A \to B$ a map $\varphi^* : \{\text{maximal ideals of } B\} \to \{\text{maximal ideals of } A\}$. There is no good candidate for this map. When taking the set of *all* ideals, the canonical choice is to merely take inverse images; but the inverse image of a maximal ideal needs no longer to be maximal (a non-maximal prime ideal is the inverse image of the unique maximal ideal in its localization). The contravariance is a feature, not a bug: in general homomorphisms do not send ideals to ideals.

Prime ideals are precisely what we need to fix this defect (c.f. Theorem 2.1.6). In what follows we will keep mentioning maximal ideal analogs of constructions, which we prematurely note should be thought of as encoding the "classical" counterpart to scheme-theoretic geometry, where points (but not necessarily functions) "behave normally." The reader will see very soon how "abnormal" points can be for us.

Definition 2.1.4. Fix a ring A. The **spectrum** $\operatorname{Spec} A$ (resp. **maximal spectrum** $\operatorname{Spec}_m A$) of A is the set of prime (resp. maximal) ideals of A. A point p is denoted $\mathfrak{p}$ when viewed as a prime ideal, and conversely a prime ideal $\mathfrak{p}$ is written as $[\mathfrak{p}]$ or p when viewed as a point. Elements f of A are **functions** on $\operatorname{Spec} A$ (resp. $\operatorname{Spec}_m A$), with **value** $f(p) = \pi(f)$ at x, where $\pi\colon A \to A/\mathfrak{p}$ is the projection. Since π is a homomorphism, functions may be added and multiplied pointwise.

Definition 2.1.5. Let A be a ring, $f \in A$. The **distinguished open set**

$$D(f) := \{\, p \in \operatorname{Spec} A\colon f \notin \mathfrak{p} \,\}$$

consists of the loci on $\operatorname{Spec} A$ where f doesn't vanish. Put $D_m(f) := D(f) \cap \operatorname{Spec}_m A$. The **Zariski topology** on $\operatorname{Spec} A$ is the one generated by the base of distinguished open sets $\{\, D(f)\colon f \in A \,\}$.

Verification. We must show that the distinguished open sets indeed form a synthetic base for $\operatorname{Spec} A$. To this end, we show that they cover $\operatorname{Spec} A$ and that $D(f) \cap D(g) = D(fg)$. Indeed, if $p \in \operatorname{Spec} A$ is contained in no $D(f)$, then all f are contained in $\mathfrak{p}$, i.e. $\mathfrak{p} = \operatorname{Spec} A$. On the other hand, notice that by primality $f \notin \mathfrak{p} \wedge g \notin \mathfrak{p} \implies fg \notin \mathfrak{p}$, proving the forward inclusion. The converse is the defining property of ideals. $\square$

Remark. We will use "spectrum" and "affine scheme" interchangeably; but beware that we have not yet endowed it with all the structure it deserves.

The definition of evaluation warrants some explanation. First, it may be seen as a generalization of the fact from elementary ring theory that the isomorphism $\varphi_a\colon k[x]/(x - a) \cong k$ is induced from the evaluation map $\mathrm{ev}_a\colon k[x] \to k$ sending $f \rightsquigarrow f(a)$; in this case, π is precisely $\varphi_a^{-1} \circ \mathrm{ev}_a$. Second, we note its formal similarity to LRS evaluation, which is also implemented as quotienting by prime ideals. We should thus be able to rely on the intuition developed in that more general setting.

As promised, Spec in this crude form is already functorial.

Theorem 2.1.6. Spec *defines a contravariant functor* $\mathsf{Ring} \to \mathsf{Top}$ *sending a ring homomorphism* $\varphi\colon A \to B$ *to a map* $\varphi^*\colon \operatorname{Spec} B \to \operatorname{Spec} A$ *sending* $\mathfrak{p} \rightsquigarrow \varphi^{-1}(\mathfrak{p})$.

Proof. The key is to show that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal: functoriality is immediate, as $\varphi^{-1} \circ \psi^{-1} = (\psi \circ \varphi)^{-1}$. Indeed, given $x, y \in A$ with $\varphi(x), \varphi(y) \in \mathfrak{p}$, $\varphi(x + y) = \varphi(x) + \varphi(y) \in \mathfrak{p}$, and furthermore if y is an arbitrary element of A, $\varphi(xy) = \varphi(x)\varphi(y) \in \mathfrak{p}$, with $\mathfrak{p}$ being an

ideal. To see primality, suppose xy satisfies $\varphi(xy) = \varphi(x)\varphi(y) \in \mathfrak{p}$. Then by primality of $\mathfrak{p}$ at least one of $\varphi(x)$ and $\varphi(y)$ lies in $\mathfrak{p}$, i.e. one of x and y lies in $\varphi^{-1}(\mathfrak{p})$.
That φ^* is continuous will follows from Proposition 2.2.2 (4) and Proposition 2.2.3, which together show that it pulls closed sets back to closed sets. $\square$

Beware that maps between affine schemes as topological spaces do not always arise from ring homomorphisms (c.f. Example 2.2.25). But proceeding teleologically, we should expel all maps insolent to the algebra of the underlying rings from civilized discourse, as once reasonable maps of the form $p \rightsquigarrow \varphi^{-1}(p)$ are included, the desired equivalence of categories should force there to be no more. That is, one should be immediately alerted by the presence of any continuous map not arising from a homomorphism and ponder what precipitated such blasphemy. Systematically ruling out such pathologies is among the myriad reasons why the structure sheaf of "algebraic functions" is the most important aspect of an affine scheme, alongside the presence of properties, such as that of "reducedness," idiosyncratic to schemes can only be picked up by working with sections on open subsets. Accordingly, $\mathscr{O}(U)$ should accommodate all *rational functions* that may be undefined outside of U, and function evaluation on affine schemes as locally ringed spaces should coincide with the one given above. We defer formally defining the structure sheaf to §2.3, but exhort the reader to think function-theoretically from the outset.

2.2. Zariski Topology

Our primary task in this section is to understand features of the Zariski topology distinguishing it from the classical picture, introducing key, recurrent examples on the way. We will begin with a more conventional definition of the Zariski topology, in terms of *closed sets*:

Definition 2.2.1. Let A be a ring, S a subset thereof. The **vanishing set** of S is the closed subspace

$$V(S) := \{ [\mathfrak{p}] \in \operatorname{Spec} A \colon S \subseteq \mathfrak{p} \}.$$

Put $V_m(S) := V(S) \cap \operatorname{Spec}_m$. The **vanishing ideal** of a subset $T \subseteq \operatorname{Spec} A$ is $I(T) := \bigcap_{[\mathfrak{p}] \in T} \mathfrak{p}$, which consists of functions vanishing on all of T. $I_m(T)$ is defined analogously for $T \subseteq \operatorname{Spec}_m A$.

Verification. Indeed, $V(S) = \bigcap_{f \in S} V(f) = \bigcap_{f \in S} D(f)^c = (\bigcup_{f \in S} D(f))^c$. □

Appealing to the discussion in §2.1, we note that $V(f)$ should be viewed as the *graph* of f, now realized through the spectrum. More generally, that of a set of functions is the intersection of the respective graphs, which in some occasions is the region cut out by various hypersurfaces.

Vanishing ideals are dual to vanishing sets in a sense that will be made precise by the end of this section. Geometrically, $I(T)$ consists of the algebraic functions whose graphs *pass through* the points of T. In the special case when T is a singleton, this amounts to the important philosophy that *points are but the functions which vanish on them*, with every point being tautologically its own vanishing ideal. This is of course an echo of Theorem 2.1.1, whose classical utility was witnessed in Proposition 1.7.15. It turns out the converse statement that "functions are but the points they vanish on" is, however ludicrous it might sound, not true. As Vakil mentions is noted by Mumford, "it is this aspect of schemes which was most scandalous when Grothendieck defined them." But as with all things scheme theory, this is feature, not a bug: this topologically undetectable legroom possessed by functions, among other more sophisticated reasons, allows us to "count with multiplicity" in degenerate situation, something only dreamed of by classical algebraic geometers. On the flip side, it is one more reason for the structure sheaf to take the utmost precedence.

Proposition 2.2.2. *Fix a ring A.*

1. *Let $S \subseteq A$. Then $V(S) = V((S))$. Dually, for $T \subseteq \operatorname{Spec} A$, $I(T) = I(\overline{T})$.*
2. *Given a set of ideals $\{ I_\alpha \}$, $\cap_\alpha V(I_\alpha) = V(\sum_\alpha I_\alpha)$.*

3. *Given ideals I, J, $V(I) \cup V(J) = V(IJ) = V(I \cap J)$. Dually, for subsets $S, T \subseteq \operatorname{Spec} A$, $I(S \cup T) = I(S) \cap I(T)$.*
4. *For $\varphi\colon A \to B$ a a ring homomorphism, $S \subseteq A$, $\varphi^{*-1}(V(S)) = V(\varphi(S))$, and thus for any $f \in A$, $\varphi^{*-1}(D(f)) = D(\varphi(f))$.*

Proof. For (1), the backwards inclusion is obvious, and the forwards follows from that $S \subseteq \mathfrak{p} \implies (S) \subseteq \mathfrak{p}$, an elementary property of ideals. Since $T \subseteq \overline{T}$, the dual statement would follow from that $I(T) \subseteq I(\overline{T})$. Now $T \subseteq V(I(T))$, so $\overline{T} \subseteq V(I(T))$, and it suffices to show that $I(T) \subseteq I(V(I(T)))$. Indeed, given $f \in I(T)$, we have that for any $[\mathfrak{q}] \in V(I(T)) \implies \mathfrak{q} \supseteq I(T)$, $f \in \mathfrak{q}$. For (2), suppose that $\mathfrak{p}$ lies in all $V(I_\alpha)$, namely that $I_\alpha \subseteq \mathfrak{p}$ for all α. Then $\sum I_\alpha \subseteq \mathfrak{p}$ follows from that $\mathfrak{p}$ is a subgroup. The converse is trivial, as individual elements of I_α are 1-fold finite sums. For (3), $\mathfrak{p} \in V(I) \cup V(J) \iff I \subseteq \mathfrak{p} \vee J \subseteq \mathfrak{p}$. That $IJ \subseteq \mathfrak{p}$ again follows from closure. The converse is primality: if there are $a \in I$, $b \in J$ both not in $\mathfrak{p}$, then $ab \notin \mathfrak{p}$. The second equality follows from a chain of inclusion: $V(I \cap J) \supseteq V(I) \cup V(J)$, as $I \cap J \subseteq I_k \implies V(I_k) \subseteq V(I \cap J)$, and $V(I \cap J) \subseteq V(IJ)$, with $IJ \subseteq I \cap J$. The dual statement is immediate from the commutativity of intersections. For (4), note that $\varphi^{*-1}(V(S)) = \{\, \mathfrak{p} \in \operatorname{Spec} B \colon \varphi^{-1}(\mathfrak{p}) \in V(S) \,\} = \{\, \mathfrak{p} \colon S \subseteq \varphi^{-1}(\mathfrak{p}) \,\} = \{\, \mathfrak{p} \colon \varphi(S) \subseteq \mathfrak{p} \,\} = V(\varphi(S))$. □

Proposition 2.2.3. *The Zariski topology on* $\operatorname{Spec} A$ *has as closed sets vanishing sets of all subsets of A.*

Proof. Fix a closed $V \subseteq \operatorname{Spec} A$. Then $V^c = \bigcup_\alpha D(f_\alpha) \implies V = \bigcap_\alpha D(f_\alpha)^c = \bigcap_\alpha V(f_\alpha) = \bigcap_\alpha V((f_\alpha)) = V(\sum_\alpha (f_\alpha))$. □

In other words, we have decreed that the graphs of functions on A shall be the *only* closed sets in $\operatorname{Spec} A$; as such, most open sets in the Zariski topology are *huge*, being the empty space left out by some hypersurface.

Remark. If it is not clear why vanishing sets need to be closed, it might be worthwhile to rewrite $V(f)$ evocatively as $f^{-1}(0)$, where the algebraic function f is reasonably assumed to be continuous and 0 is a closed point in some sensible space, say $\mathbb{R}^n$.

We now introduce the gadget that literally underlies all classical algebraic geometry, the stage on which polynomials display their geometry.

Definition 2.2.4. For k a ring, the **affine n-space over k** is $\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$, where $x_1, \ldots, x_n$ are called **coordinate functions**.

Remark. It is of little harm to use polynomial rings as a mental model for arbitrary rings and affine spaces arbitrary affine schemes; polynomials then serve as avatars for algebraic functions. This blanket assumption does has its pitfalls, as affine spaces are indeed among the nicest affine schemes, enjoying properties that evade general affine schemes. Over time we will gradually refine this model to accommodate more exotic schemes, hopefully without compromising the intuition it provides.

Example 2.2.5. The intuition that vanishing sets are graphs can be immediately put to test. Thinking of $V(xy, yz) \subseteq \mathbb{A}^3_{\mathbb{C}}$ as the union of the xz-plane $y = 0$ and the y-axis $x = z = 0$, one would expect that it contains the x-axis $y = z = 0$. Indeed, $V(y, z) \subseteq V(xy, yz)$, as $(xy, yz) \subseteq (y, z)$.
The work style of most algebraic geometers can be snappily described as *thinking in geometry, proving in algebra*. Above is our first victory due to this mindset, but it will not be the last. It is as much an elegant philosophy as it is a practical guideline.

We will determine the structure of affine spaces by order of dimension, which corresponds positively to complexity. As a prelude, note that $\mathbb{A}^0_k = \operatorname{Spec} k$ is trivial: the only prime ideal is 0. The **affine line** $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ is non-trivial, but still fairly simple, as $k[x]$ is a PID:

Proposition 2.2.6. *For A a PID,* $\operatorname{Spec} A = \{\, 0 \,\} \cup \{\, (x) \colon x \text{ irreducible} \,\}$.

Proof. Recall that in general prime principal ideals must be generated by an irreducible element, while the converse holds for UFDs, hence PIDs. □

As such, the affine line consists of 0 and the principal ideals generated by all irreducible polynomials. Similarly, $\operatorname{Spec} \mathbb{Z}$ consists precisely of the ideals corresponding to prime numbers. We reiterate that in either case, the ring elements (polynomials and integers) ought to be viewed as functions taking values on various loci: for instance, taking $k = \mathbb{C}$, the non-trivial prime ideals are $(x - a)$ for each $a \in \mathbb{C}$, and for $f \in k[x]$, $f([(x - a)]) = f(a)$; 5 takes the value 1 at $[(2)]$ and 2 at $[(3)]$.

Classically, the base field k is always assumed to be algebraically closed, and for good reason: if k is a finite field, for instance, there is no sensible topology on k^n other than the discrete topology. Thus surfaces the first advantage of considering all prime ideals: regardless of the base field, the affine line always boasts an abundance of points.

Proposition 2.2.7. *For k a field, $\mathbb{A}^1_k$ is infinite.*

Proof. It suffices to show that there are infinitely many irreducible polynomials in $k[x]$. Assume there are only finitely many $\{\, f_i \,\}$. But then consider $\prod f_i + 1$. □

There is a *stratification* of points in the affine line. To begin with, 0 is somehow distinct from all other points, and

Proposition 2.2.8. *The closed points in* Spec A *are precisely the maximal ideals.*

Proof. Maximality of an ideal $\mathfrak{m}$ means precisely that $V(\mathfrak{m}) = \{ [\mathfrak{m}] \}$. Conversely, if $V(S) = \{ \mathfrak{p} \}$ for some prime ideal $\mathfrak{p}$, then $\{ \mathfrak{p} \} \subseteq V(\mathfrak{p}) \subseteq V(S) = \{ \mathfrak{p} \}$, proving maximality. $\square$

Since $k[x]$ is a PID, all non-trivial prime ideals are maximal. Thus this is indeed the whole picture, albeit quite a peculiar one: 0 is never closed, so $\mathbb{A}^1_k$ is never Hausdorff. As such, it *never* agrees with the classical topology on k, if there is any to begin with. It is tempting, for instance, to think of $\mathbb{A}^1_{\mathbb{C}}$ as the complex plane, or at least something resembling it. *It is not.* Even if we only look at the subspace of maximal ideals, the Zariski topology is much coarser: only finite subspaces of $\mathbb{C}$ are closed in $\mathbb{A}^1_{\mathbb{C}}$, since polynomials only admit finitely many solutions!

Remark. Warning: this is not to be confused with the motivating example involving compact Hausdorff spaces in §2.1; there are far more function declared algebraic there. This again suggests that *functions* are higher class inhabitants of the land of schemes than the underlying topological space, and that eventually we will have to make do without paying too much mind to the space.

To put this stratification on solid ground, we introduce the notion of *generic points*: closed subspaces can admit generic points which stand in or represent the entire subspace, despite being only a point. On a related note, we can make precise the "size" of points and the "containment" of one in another, which sound absurd out of context. But here, the closed points indeed situate themselves "within" 0, i.e. *specializes* 0.

Definition 2.2.9. Fix a topological space X. A point $p \in X$ is a **generic point** of a closed subspace K if $\overline{\{ x \}} = K$. A point $x \in X$ is a **specialization** of some $y \in X$, and y a **generization** of x, if $x \in \overline{\{ y \}}$.

Proposition 2.2.10. *Let X be a topological space, U a non-empty open subspace. Then a generic point of X remains generic in U, and the converse hold if U is dense.*

Proof. Let x be a generic point of X. Then by denseness, $x \in Y$, and indeed $\overline{x}_U = \overline{X} \cap U = X \cap U = U$. The converse follows from the transitivity of denseness. $\square$

Proposition 2.2.11. *For any ring A, 0 is a generic point of in* Spec A*, hence generizes all points in* Spec A*.*

Proof. Notice, 0 lies in $D(f)$ for all $f \neq 0$, and $D(0) = \varnothing$. $\square$

From this we can extrapolate an immensely useful slogan: *the smaller the ideal, the bigger the point*! This is conveyed formally by the following two results.

Lemma 2.2.12. *For $S \subseteq \operatorname{Spec} A$, $V(I(S)) = \overline{S}$. Hence $V(\mathfrak{p}) = \overline{\{\, \mathfrak{p} \,\}}$.*

Proof. Let $\mathfrak{q} \in V(I(S))$, so that $\bigcap_{\mathfrak{p} \in S} \mathfrak{p} \subseteq \mathfrak{q}$. We must show that if $\mathfrak{q} \notin S$, then it is a limit point of S. Let U be a punctured neighborhood of $\mathfrak{q}$. If it were to contain no point in S, then $S \subseteq U^c \setminus \{\, \mathfrak{q} \,\}$, the latter being the vanishing set of some $T \subseteq A$. In particular, $T \subseteq \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$. But this would imply $\mathfrak{q} \in U^c \setminus \mathfrak{q}$, a contradiction. For the second claim, note that $I([\mathfrak{p}]) = [\mathfrak{p}]$. $\square$

Remark. This fact is entirely transparent geometrically: the subspace cut out by graphs covering the points in S should be the graph traced out by the points themselves! This will stand in stark contrast to $I(V(f))$, which will turn out to not always recover (f) and cement once again in our minds how functions are higher entities than points.

Theorem 2.2.13. *Fix A a ring. Then $[\mathfrak{q}]$ is a specialization of $[\mathfrak{p}]$ in $\operatorname{Spec} A$ iff $\mathfrak{p} \subseteq \mathfrak{q}$.*

Proof. Indeed, $[\mathfrak{q}] \in \overline{\{\, [\mathfrak{p}] \,\}} = V([\mathfrak{p}]) \iff \mathfrak{p} \subseteq \mathfrak{q}$. $\square$

Our study of $\mathbb{A}^1$ is summarized in the following figure:

$\boxed{\text{Vakil Figure 3.1}}$

Namely, the affine line is a mesh of 0-dimensional "traditional" points corresponding to the maximal ideals, with no particular ordering, lying on a 1-(complex) dimensional line embodied, or loomed over by the 1-dimensional generic point 0. Note that we are using the term "dimension" heuristically, whose meaning as a topological property will be given in the future.

It is time to bump the dimension up one notch.

Proposition 2.2.14. *For k a field, the **affine plane** $\mathbb{A}^2_k$ consists of 0, the principal ideals generated by irreducibles in $k[x, y]$, and*

$$\{\, (f, g) \colon f \text{ irreducible in } k[x], g \text{ irreducible in } k(\alpha)[y] \text{ where } f = \operatorname{minP}_k(\alpha) \,\}.$$

Proof. As before, principal prime ideals are necessarily generated by irreducible elements (which in turn always generate prime ideals, with $k[x,y]$ a UFD), so it suffices to consider when $\mathfrak{p} \neq 0$ is a non-principal prime ideal. Since $k[x,y]$ is furthermore Noetherian, we can write $\mathfrak{p} = (f_i)$ for irreducible $f_i \in k[x,y]$, within which we take distinct f, g. Note that they are also coprime in $k(x)[y]$, since writing $k[x,y] = k[x][y]$ we see by Gauss' lemma that f and g are primitive in $k(x)[y]$ ($f = lg \implies l \in k[x]^\times = k$). By the Euclidean algorithm on $k(x)[y]$, the GCD satisfies $\gcd(f,g) = fc + gd \in (f,g)_{k(x)[y]}$. Clearing denominators yields an $h \in k[x] \cap (f,g)$. By primality, one of the irreducible pieces a of h must lie in $\mathfrak{p}$, i.e. $(a) \subseteq \mathfrak{p}$. Now prime ideals in $k[x][y]$ containing (a) correspond to those in $(k[x]/(a))[y] = k(\alpha)[y]$ for some algebraic α with $\mathrm{minP}_k(\alpha) = a$. The latter is a PID, so $\mathfrak{p} = (b) + (a)$ for some irreducible $b \in k(\alpha)[y]$, as desired. We end by noting that (a,b) is maximal, hence prime, with $k[x,y]/(a,b) \cong k(\alpha,\beta)$. $\square$

The increase in algebraic dimension is reflected in the topological dimension: we now have points $[(f(x,y))]$ that represent whole 1-dimensional curves, and the generic point 0 has been promoted to the status of a 2-dimensional entity. The maximal ideals (f,g), of course, remain closed. The 1-dimensional points are not maximal, so cannot be closed; similar to 0, their size can be gauged with their closure. The extra generic points provide new grounds for us to test Theorem 2.2.13:

Example 2.2.15. That the point $(2,4) \in \mathbb{C}^2$ lies on the parabola $y = x^2$ translates to that $[(x-2, y-4)]$ is a specialization of $[(y-x^2)]$ in $\mathbb{A}^2_{\mathbb{C}}$. Indeed, $y - x^2 = -(x-2)^2 - 4(x-2) + (y-4)$!

$\boxed{\text{Vakil Figure 3.3, 3.8}}$

Unfortunately, higher affine spaces are much more complicated. It is already impossible to give in our current language a complete taxonomy of points in $\mathbb{A}^3_{\mathbb{C}}$. The $0, 2, 3$-dimensional parts are simple enough: in reversed order, these are 0, (f) for $f \in \mathbb{C}[x,y,z]$ irreducible, and $(x-a, y-b, z-c)$, respectively. While it is not difficult to produce *some* 1-dimensional prime ideals, such as (x,y) (for $k[x,y,z]/(x,y) \cong k(\alpha,\beta)[z]$ is a domain), to enumerate them all is a combinatorial nightmare.

That *all* maximal ideals are generated by three linear polynomials in different variables is a consequence of the *Nullstellensatz*, which will be proved in later sections using elimination theory. Its precise statement is irrelevant, or even misleading to us: the form taken by closed points does not matter insofar no algebraic property of the underlying ideal is reflected in the topology other than maximality. Of course, when an explicit form is given, we may still readily determine which points it specializes, etc.

It is clear that a better approach is warranted. The move is to wave our algebro-geometric wand and appeal to the geometry, setting up a bijection between points and certain subspaces of the affine scheme we can easily visualize that will be applicable to more than just affine spaces—an entry in the grand algebra-geometry dictionary we are assembling. Before taking on this monumental task, we must acquire a larger stock of examples, so that we can fully appreciate its significance when the moment arrives. First, let us tie up loose ends by considering how affine spaces over different fields relate to each other.

Example 2.2.16. In $\mathbb{A}^1_{\mathbb{R}}$, there are prime ideals generated by irreducible *quadratic* polynomials, and at these points functions evaluate to linear polynomials $ax+b$, where x should be tacitly thought of as i. The effect is then the gluing of complex conjugate pairs. To illustrate, whereas x used to take the distinct values i and $-i$ at $(x-i)$ and $(x+i)$, respectively, in $\mathbb{A}^1_{\mathbb{C}}$, the two points are now merged or "glued" in $\mathbb{A}^1_{\mathbb{R}}$, encompassed by the ideal (x^2+1). This generalizes perfectly to arbitrary fields and higher dimensions: we simply glue together all *Galois conjugates*.

Theorem 2.2.17. *For a field k, $\mathbb{A}^n_k \cong \mathbb{A}^n_{\overline{k}}/\operatorname{Gal}(\overline{k}/k)$.*

Proof. 3.2.I Talk about Galois descent $\square$

There are ways other than increasing the dimension by which we can acquire new schemes from old. This is not as mysterious as it sounds: any reasonable operation extending a scheme will have as its root an operation on the ring of global sections, with none other than the Spec functor serving as the interpreter. Now, our primary tools in commutative algebra are localizing multiplicative submonoids quotienting by an ideal. Somewhat surprisingly, the canonical maps for both correspond to inclusions of subschemes (which for now means nothing more than a subspace). Even more so, these will turn out to be the basis for the notions of open and closed subschemes, respectively, and *are* just these notions in the affine case.

Proposition 2.2.18. *For a surjective homomorphism $\varphi\colon A \to B$, $\operatorname{Spec}\varphi$ is a homeomorphism onto $V(\ker\varphi)$. Hence for any ideal I of A, $\operatorname{Spec} A/I$ is naturally a closed subspace of $\operatorname{Spec} A$.*

Proof. Note that surjective ring homomorphisms always reflect containment of ideals, i.e., that $\varphi^{-1}(I) \subseteq \varphi^{-1}(J) \implies I \subseteq J$, since in this case $\varphi(\varphi^{-1}(I)) = I$. By functoriality, $\operatorname{Spec}\varphi$ is injective as a set map, so we need only show it is furthermore a closed map and compute its image. For any $S \subseteq B$, $\varphi^*(V(S)) = \{ \varphi^{-1}(\mathfrak{p}) \colon S \subseteq \mathfrak{p} \} = \{ \varphi^{-1}(\mathfrak{p}) \colon \varphi^{-1}(S) \subseteq \varphi^{-1}(\mathfrak{p}) \}$. It will suffice to show this is equal to $V(\varphi^{-1}(S))$, i.e. that all $\mathfrak{q} \in \operatorname{Spec} A$ containing $\varphi^{-1}(S)$ are of the form $\varphi^{-1}(\mathfrak{p})$ for some $\mathfrak{p} \in \operatorname{Spec} B$. Since $V(S) = V((S))$, we may assume that S is an ideal, and in particular that $0 \in S$, so $\ker \varphi \subseteq \mathfrak{q}$. Indeed, recall that with φ surjective, there is a bijection between prime ideals of A containing $\ker \varphi$ and prime ideals of B given by φ^{-1}. Finally, note that $\varphi^*(\operatorname{Spec} B) = \varphi^*(V(0)) = V(\ker \varphi)$. □

This leads us to the notion of affine varieties, a fancy name for hyperplanes in affine spaces. Unsurprisingly, these objects are the backbone of classical algebraic geometry, and as we start to understand them better, one should shift their default image of an affine scheme to that of an affine variety.

Definition 2.2.19. For k algebraically closed, $\operatorname{Spec} k[x_1, \ldots, x_n]/I$ is said to be an **affine n-prevariety** over k, with $k[x_1, \ldots, x_n]/I$ being its **coordinate ring**.

Remark. There are technical reasons forcing us to exercise caution and refer to these affine schemes as prevarieties. The most tangible one is "irreducibility:" we want varieties to be atomic, in the sense that if a prevariety can be decomposed into two proper sub-prevarieties, there is no reason not to just directly study these smaller objects instead. These conditions will be very relevant to the bijection we are shooting for. Indeed, the conditions are imposed precisely to ensure that the bijection can be applied to varieties. In sum, we have the following chain of class inclusions:

$$\{ \text{affine prevarities} \} \supseteq \{ \text{affine varieties} \} \supseteq \{ \text{affine spaces} \}.$$

A similar chain will hold for "projective" and "algebraic" varieties.

Proposition 2.2.20. *Let S be a multiplicative submonoid of a ring A. For $\varphi \colon A \to S^{-1}A$ the canonical map, φ^* is a homeomorphism onto* $\operatorname{im} \varphi^* = \{ \mathfrak{p} \in \operatorname{Spec} A \colon \mathfrak{p} \cap S = \varnothing \}$.

Proof. We must exhibit a continuous inverse of φ^*. Consider the restriction of the map sending an ideal $\mathfrak{p}$ of A to its *extension* $\mathfrak{p}^e = \{ (a, s) \colon a \in \mathfrak{p}, s \in S \}$, which is manifestly an ideal of $S^{-1}A$, to $\operatorname{Spec} A$. We must show $\mathfrak{p} \in \operatorname{Spec} A$ implies that $\mathfrak{p}^e$ is also prime. Suppose that $(a_1a_2, s_1s_2) \in \mathfrak{q}_{\mathfrak{p}}$. Then $(a_1a_2, s_1s_2) = (b, t)$, i.e. there exists $u \in S$ such that $us_1s_2b = uta_1a_2$. Now $uta_1a_2 \in \mathfrak{p}$, so with $u, t \notin \mathfrak{p}$, by primality $\mathfrak{p}$ we may assume without loss of generality that $a_1 \in \mathfrak{p}$, which implies that $(a_1, s_1) \in \mathfrak{q}_{\mathfrak{p}}$.

To prove the image is as prescribed, we must show that for $\mathfrak{p} \in \operatorname{Spec} A$, $\mathfrak{p} \cap S = \varnothing \iff$ there exists $\mathfrak{q} \in \operatorname{Spec} S^{-1}A$ such that $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Indeed, if $\mathfrak{p} \in \operatorname{Spec} A$ does not intersect

S, then consider $\mathfrak{q}_{\mathfrak{p}} := \{ (a,s) \colon a \in \mathfrak{p}, s \in S \}$. This is evidently an ideal ; we must show it is also prime. It is clear that $\varphi^{-1}(\mathfrak{q}_{\mathfrak{p}}) = \mathfrak{p}$. For the converse, note that if there exists $s \in S$ belonging to $\varphi^{-1}(\mathfrak{q})$, i.e. such that $(s,1) = \varphi(s) \in \mathfrak{q}$, then $\mathfrak{q}$ would contain an invertible element, hence not be prime.
Since φ^* is injective as a map from the set of all ideals of $S^{-1}A$, it certainly remains injective when restricted to $\operatorname{Spec} S^{-1}A$. Thus it remains to show it is an open map. Indeed, for prime ideals if $\mathfrak{q}_{\mathfrak{p}} = \mathfrak{q}_{\mathfrak{p}}$ □

A small aside: localizing and quotienting commute. The extension construction introduced in the proof above allows us to state this succinctly.

Proposition 2.2.21. *Let I be an ideal of a ring A, S a multiplicative subset thereof. Then $S^{-1}(A/I) \cong S^{-1}A/I^e$.*

Proof. Recall that localization is a left adjoint functor from the category of pairs (R, M) to that of pairs (S, N), where R, S are rings, M, N submonoids of the multiplicative monoid of R and $S^\times$, respectively. Now A/I may be By left adjoints preserve colimits, it will suffice to show that $S^{-1}A/I^e$ □

Corollary 2.2.22. $\operatorname{Spec} A_f \cong D_{\operatorname{Spec} A}(f)$*. Hence* $\operatorname{Spec} A_f$ *is naturally an open subspace of* $\operatorname{Spec} A$*.*

Proof. By primality, $\mathfrak{p} \cap \{ f^n \} = \varnothing \iff f \notin \mathfrak{p}$. □

Example 2.2.23. It is remarkable that this holds even when the canonical map is not injective. Let A be the variety $\mathbb{C}[x,y]/(xy)$. Then $D_{\operatorname{Spec} A}(x)$, being the complement of the y-axis, should intuitively be the x-axis with the origin removed. That is, the geometry tells us to expect $\operatorname{Spec} A_x \cong \operatorname{Spec} \mathbb{C}[x]_x$, even though x is a zero-divisor in A. Indeed, the natural map $A_x \to \mathbb{C}[x]_x$ induced from the composition $\mathbb{C}[x,y]/(xy) \overset{y \rightsquigarrow 0}{\longrightarrow} \mathbb{C}[x] \longrightarrow \mathbb{C}[x]_x$, has its inverse is induced from the composition $\mathbb{C}[x] \longrightarrow \mathbb{C}[x,y] \longrightarrow \mathbb{C}[x,y]/(xy) \longrightarrow (\mathbb{C}[x,y]/(xy))_x$.

Corollary 2.2.24. *For $\mathfrak{p} \in \operatorname{Spec} A$, $\operatorname{Spec} A_{\mathfrak{p}} \cong \{ \mathfrak{q} \in \operatorname{Spec} A \colon \mathfrak{q} \subseteq \mathfrak{p} \}$.*

Proof. The isomorphism amounts to that $\mathfrak{q} \cap (A \setminus \mathfrak{p}) = \varnothing \iff \mathfrak{q} \subseteq \mathfrak{p}$. Assume that $\mathfrak{q} \not\subseteq \mathfrak{p}$, i.e. that there exists $f \in \mathfrak{q} \setminus \mathfrak{p}$. Then clearly $f \in A \setminus \mathfrak{p}$. The backwards direction is immediate. □

While we have considered sets of prime ideals *containing* a certain prime ideal, i.e. vanishing sets, the set above is entirely new to us, to be only very loosely interpreted here as the space of all generizations of a point. For instance, $\operatorname{Spec} k[x]_{(x)} = \{ (x), 0 \}$, and $\operatorname{Spec} \mathbb{C}[x,y]_{(x,y)}$ consists of all points vanishing at $(0,0)$: the 0-dimensional point (x,y), 1-dimensional "irreducible curves through the origin" $(f(x,y))$ with $f(0,0) = 0$, such as $(y - x^2)$, and the 2-dimensional generic point 0. Once we define the structure sheaf, this will be understood to be the correct notion of "zooming in", as opposed to say $V(\mathfrak{p})$, which ostensibly (on the level of closed points) achieves the same effect of casting away points other than $\mathfrak{p}$. As viscerally put by Vakil, $\operatorname{Spec} A_p$ is a "shred" of the scheme at p; we must not entirely lose sight of the rest of the scheme. (Figure 3.5)

In general, $\operatorname{Spec} S^{-1}A$ is identified with a subspace of $\operatorname{Spec} A$ that is neither open nor closed. For instance, $\operatorname{Spec} \mathbb{C}[x]_{(0)}$ corresponds to subspace of $\operatorname{Spec} \mathbb{C}[x]$ consisting of just the generic point 0, which is tautologically non-closed and non-open as its complement is infinite. We will see this pathological behavior mirrored in how general open subsets of an affine scheme need not be affine (cf example). On the contrary, this should be a signal that the one case in which it *is* naturally open, namely $D(f)$, is all the more precious to us. Indeed, as we venture further into scheme theory, the affine base $\{ D(f) \}$ will prove to be absolutely indispensable, with many problems trivialized when reduced to the affine case.

Since we have already opened this can of worms, we might as well see it through to the end and attempt to understand continuous maps between schemes in general. For starters, let us present an instance of a "blasphemous morphism" as described in §2.1.

Example 2.2.25. Consider the map π on $\mathbb{A}^1_{\mathbb{C}}$ that swaps (x) and $(x-1)$ and leaves all other points fixed: with $\mathbb{C}[x]$ a PID, any closed set is of the form $V(f)$ for some $f \in \mathbb{C}[x]$. $V(f) = \pi^{-1}(V(f))$ if both or neither point lies in it, so we need only consider when exactly one is contained. Assume without loss of generality that it is (x). Then f must be of the form $x^m g(x)$ where $m \leq \deg(f)$ and $\{ 0, 1 \} \subseteq D(g)$. As such, $\pi^{-1}(V(f)) = \pi^{-1}(V(g) \cup V(x^m)) = V(g) \cup \{ (x-1) \}$ is closed, and π continuous. π cannot possibly arise from a homomorphism by way of Theorem 2.1.6: if there were such an endomorphism $\varphi \in \operatorname{End}(\mathbb{C}[x])$ that $\pi(p) = \varphi^{-1}(\mathfrak{p})$, then $\varphi^{-1}((x-a)) = (x-b) \implies \varphi(x-a) \in (x-b) \implies \varphi_{x-a}(b) = 0 \implies \varphi_x(b) = a$. That is, $\varphi_x \in \mathbb{C}[x]$ is defined at a as $\begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } a = 1, \\ a & \text{otherwise.} \end{cases}$ But having its only root at 1, φ_x must be of the form $(x-1)^n$, which evaluates to 1 at 2, a contradiction.

We now supply an explicit form for maps between affine prevarieties:

Proposition 2.2.26. *For k a field, put $A := k[x_1, \dots, x_m], B := k[y_1, \dots, y_n]$. Fixing $f_1, \dots, f_n \in B$, let $\varphi\colon B \to A$ be the k-algebra morphism sending $y_i \rightsquigarrow f_i$. Then for all ideals $I \subseteq B$, $J \subseteq A$ such that $\varphi(I) \subseteq J$, the induced map* $\operatorname{Spec} A/J \to \operatorname{Spec} B/I$ *sends each closed point $(a_1, \dots, a_m)$ to $(f_1(a_1, \dots, a_m), \dots, f_n(a_1, \dots, a_m))$.*

Proof. The map is obtained by applying Spec to the following induced morphism: compose φ with $B \to B/I$. By the universal property of quotients, there is an induced map $B/I \to A/J$, since $\varphi(J) \subseteq I \implies I \subseteq \ker \tilde{\varphi}$.

The result amounts to that $\varphi^{-1}(x_1 - a_1, \dots, x_m - a_m) = (y_1 - f_1(a_1, \dots, a_m), y_n - f_n(a_1, \dots, a_m))$. $\varphi(y_i - f_i(a_1, \dots, a_m)) =$ One direction is clear: for each $\varphi(y_i - f_i(a_1, \dots, a_m)) = f_i - f_i(a_1, \dots, a_m) \in (x_1 -)$ We must show that $\varphi^{-1}((a_1, \dots, a_m)) = (f_1(a_1, \dots, a_m), \dots,)$ $\square$

Consider a morphism $\operatorname{Spec} \mathbb{C}[x, y]/(y - x^2) \to \operatorname{Spec} \mathbb{C}[x, y, z]/(y - x^2, z - y^2)$ sending $(x - a, y - b) \rightsquigarrow (x - a, y - b, z - b^2)$. What is the corresponding homomorphism? The claim is that is should send $x \rightsquigarrow x, y \rightsquigarrow y, z \rightsquigarrow y^2$. $\varphi^{-1}(x - a, y - b) = (x - a, y - b, z - b^2)$. $y^2 - b^2$

We are ready to undertake the task of finding the bijection. It should not come as a surprise that it will result from certain restrictions of $V(-)$ and $I(-)$. Not only have we already seen a hint of this in Lemma 2.2.12, this is indeed what we have been alluding to the entire time with such statements as "curves are closed" and "points represent curves." If not, we exhort the reader to pause and reread the whole section until this becomes second-nature.

It is salient to find an interpretation for $I \circ V$. For any $f \in A$, $I(V(f))$ is the intersection of the prime ideals containing it. Viewing this as the totality of functions whose graphs cover $V(f)$, we see that it measures the extent to which f defects from its graph, that is, it contains functions distinct from f which nevertheless have a graph covering that of f. Ditto for $I_m(V_m(f))$: it consists of functions vanishing at the *closed* points in $V(f)$, and hence always contains $I(V(f))$.

In what follows, this construction will be named, and these geometric interpretations formalized.

Definition 2.2.27. For A a ring, the **radical** of a subset S is $\sqrt{S} := I(V(S))$. The **Jacobson radical** thereof is $J(S) := I_m(V_m(S))$. An ideal is **(J-)radical** if it equals its own (Jacobson) radical. The **(J-)nilradical** of A is $\mathfrak{N}(A) := \sqrt{0}$ (resp. $\mathfrak{J}(A) := J(0)$), where the ring may be omitted when it is clear from context. Elements of $\mathfrak{N}$ are **nilpotent**. A ring is **reduced** (resp. **J-semisimple**) if 0 is radical (resp. J-radical).

Proposition 2.2.28. *For functions $f, g \in A$, $V(f) \subseteq V(g) \iff g \in \sqrt{f}$. Hence $D(f) = \varnothing \iff f \in \mathfrak{N}$. Likewise for D_m, V_m, $J(f)$.*

Proof. Indeed, $g \in \sqrt{f}$ iff $g \in \mathfrak{p}$ for all $\mathfrak{p} \in V(f)$ iff $f \in \mathfrak{p}$ implies $g \in \mathfrak{p}$ iff $V(f) \subseteq V(g)$. □

Theorem 2.2.29. *Two functions in a ring A agreeing on a subset $S \subseteq \operatorname{Spec} A$ (resp. $\operatorname{Spec}_m A$) differ by an element of $I(\overline{S})$ (resp. $I_m(\overline{S})$). In particular, for closed S it suffices to check agreement on a dense subset.*

Proof. If $f, g \in A$ agree on S, then $f - g$ vanishes on S, i.e. $f - g \in \bigcap_{[\mathfrak{p}] \in S} \mathfrak{p} = I(S) = I(\overline{S})$. The final claim follows by taking $S = V(f)$. □

Corollary 2.2.30. *A function $f \in A$ is determined up to multiples by $V(f)$ (resp. $V_m(f)$) iff (f) is radical (resp. Jacobson radical). Functions are uniquely determined by their values on points (resp. closed point) iff the ring is reduced (resp. J-semisimple).*

Proof. Apply the above, using $V(f)$ and $V(0) = \operatorname{Spec} A$ for S, respectively. □

Corollary 2.2.31. *Let A be a ring, K a subset of $\operatorname{Spec} A$. Write $V_K(S) := V(S) \cap K$, $I_K(T) = I(T \cap K)$ for $S \subseteq A$, $T \subseteq \operatorname{Spec} A$. Then $\mathfrak{N} = I_K(V_K(0))$ iff K is dense in $\operatorname{Spec} A$.*

Proof. Let $f \in I_K(V_K(0))$. Then $K \subseteq V(f)$, so $f \in I(\overline{K})$. Clearly $f \in \mathfrak{N}$ iff $I(\overline{K}) \subseteq I(\operatorname{Spec} A)$ iff $\operatorname{Spec} A \subseteq \overline{K}$, applying V to both sides for the forward implication. □

Corollary 2.2.32. *For A a ring, $\mathfrak{N} = \mathfrak{J}$ iff Spec_m is dense in $\operatorname{Spec} A$.*

Proof. Let $f \in \mathfrak{J}$. Then $\operatorname{Spec}_m A \subseteq V(f)$, so $f \in I(\overline{\operatorname{Spec}_m A})$. Clearly $f \in \mathfrak{N}(A)$ iff $I(\overline{\operatorname{Spec}_m A}) \subseteq I(\operatorname{Spec}_m)$ iff $\operatorname{Spec}_m \subseteq \overline{\operatorname{Spec}_m A}$, applying V to both sides for the forward implication. □

Remark. The geometric intuition here is crystal clear: $\mathfrak{N} = \mathfrak{J}$ means precisely that functions vanishing on the closed points vanish on all points, i.e. that maximal ideals suffice for measuring how much functions are determined by their values, a task only dense subsets are capable of performing.

As we alluded to earlier, the notion of reducedness will be one of the two ingredients in the definition of affine varieties, as they should be "classical" in the sense of lacking nilpotence. We will thus pause to study some of its properties.

Proposition 2.2.33. *Integral domains are reduced.*

Proof. 0 is prime in any integral domain R. □

Remark. Hence functions on affine spaces, and more generally spectra of polynomial rings over integral domains are determined by their values. It will turn out that they are in fact determined by values on closed points; but this has been reduced to a topological issue by Corollary 2.2.32, so had better be relegated to §2.4, where topological properties of general schemes are discussed.

By virtue of the following, shreds of an affine scheme over a reduced ring still have functions determined by their values.

Proposition 2.2.34. *Localizations of a reduced ring are reduced.*

Proof. Fix a reduced ring A, a multiplicative submonoid S, $a/b \in S^{-1}A$. Suppose that $(a/b)^n = 0$ for some n, so that $ta^n = 0$ for some $t \in S$, $n \geq 0$. Then of course $(ta)^n = 0$, so ta is nilpotent, hence 0. It follows that $a/b = 0$. $\square$

To make this result geometric, we note that invertibility amounts to vanishing at all points by Proposition 1.7.17. Of course, we do not yet have the resources to prove that it applies here, but this is only for intuition, and we *will* show that affine spaces are LRS's in the next section. Thus our interpretation is the following: shrinking the space to make certain functions vanish nowhere does not introduce functions vanishing everywhere. We will see a generalization in Corollary 2.9.4. The same cannot be said for quotients, but a stronger result holds in the sense that a quotient of ring A may be reduced even when A itself is not.

Proposition 2.2.35. *For I an ideal of a ring A, A/I is reduced iff I is radical.*

Proof. Reducedness amounts to that $\mathfrak{N}(A) = \bigcap_{\mathfrak{p} \in A} \mathfrak{p} = 0$. It follows from Proposition 2.2.18 that by taking π^{-1} on both sides, this is indeed equivalent to that $I = \ker \pi = \bigcap_{\mathfrak{p} \in A} \pi^{-1}(\mathfrak{p}) = \sqrt{I}$. $\square$

Remark. We will see that affine varieties in fact all arise from quotienting affine spaces by radical ideals: the other desired property falls out if we mandate primality. This, of course, results from the ordering of our desiderata: we want affine varieties to play well with the bijection given by $I(-)$ and $V(-)$, and radicality is in turn a properpty in terms of these functions.

Example 2.2.36. The ring of **dual numbers** $k[\epsilon]/(\epsilon^2)$, for k a field, offers the quintessential example of a ring in which functions are not determined by their values on points. Indeed, $\operatorname{Spec} k[\epsilon]/(\epsilon^2) = \{ (\epsilon) \}$, as (ϵ^2) is not itself prime in $k[\epsilon]$ and $f \mid \epsilon^2 \implies f = \epsilon$ (f cannot be a unit due by primality). Is this proof valid? Then 0-valued functions need only be divisible by ϵ (consider ϵ itself!), and functions of the same value differ by multiples of ϵ. We should note that $k[\epsilon]$ is reduced.

That nilpotents cannot be detected by the topology is reiterated in the following result.

Proposition 2.2.37. *If $I \subseteq \mathfrak{N}(A)$ is an ideal of nilpotents, then $\operatorname{Spec}\pi$ is a homeomorphism.*

It is time we developed a calculus of radicals.

Proposition 2.2.38. *An ideal is radical iff it is a vanishing ideal.*

Proof. If J a radical ideal, then $I = \sqrt{I} = I(V(J))$, a vanishing ideal. Conversely, $I(S) = I(\overline{S}) = I(V(I(S))) = \sqrt{I(S)}$. □

Proposition 2.2.39. *Let I, J be ideals. Then we have the following:*

1. *If I is prime, then it is radical.*
2. $V(\sqrt{I}) = V(I)$.
3. $\sqrt{\sqrt{I}} = \sqrt{I}$.
4. $\sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$.
5. *For $\varphi\colon A \to B$ a ring homomorphism, $I \subseteq B$ an ideal, $\overline{\varphi^{*-1}(V(I))} = V(\varphi^{-1}(I))$.*

Proof. For (1), note that any prime ideal $\mathfrak{p}$ is equal to $I(\{\mathfrak{p}\})$. For (2), we need only show that $V(I) \subseteq V(\sqrt{I})$, as $I \subseteq \sqrt{I}$. Indeed, for a prime ideal $[\mathfrak{p}]$, if $I \subseteq \mathfrak{p}$, then $\sqrt{I} \subseteq \mathfrak{p}$, with $\mathfrak{p}$ being among the prime ideals that contain I. (3) follows: $\sqrt{\sqrt{I}} = I(V(\sqrt{I})) = I(V(I)) = \sqrt{I}$. For (4), notice that $\sqrt{I \cap J} = I(V(I \cap J)) = I(V(I) \cup V(J)) = I(V(I)) \cap I(V(J)) = \sqrt{I} \cap \sqrt{J}$. (5) is EGA I.1.2.2 (ii). Add Corollary (1.2.7). □

We may now give an explicit form for nilpotent elements, and more generally elements of radicals.

Theorem 2.2.40. *For $S \subseteq A$, $r \in A$, the following are equivalent:*

1. $r \in \sqrt{S}$.
2. $(A/(S))_r = 0$.
3. *There exists some $n \in \mathbb{Z}^+$ such that $r^n \in (S)$.*

Proof. We first prove the equivalence for the nilradical, namely when $S = 0$, in the order (1) $\implies$ (2) $\implies$ (3) $\implies$ (1). First, as $A/0 \cong A$, A_r is not the zero ring, so must contain a maximal ideal. But as to be shown later in the section, prime ideals of A_r correspond to those of A not containing r (the reader is invited to prove this manually). Hence $r \notin \sqrt{0}$, with all prime ideals containing 0. Second, if $A_r = 0$, i.e. so that $(0,1) = (1,1)$, then $(r^n, r^n) = (0,1)$ for any $n \in \mathbb{Z}^+$, furnishing an $r^m \in \left\{ r^k : k \in \mathbb{N} \right\}$ such that $r^m(r^n) = r^{m+n} = 0$. Lastly, if there exists some $n \in \mathbb{Z}^+$ for which $r^n = 0$, then for any prime ideal $\mathfrak{p}$, $r^n \in \mathfrak{p} \implies r \in \mathfrak{p}$. Now the general statement follows from that prime ideals containing (S) correspond to those in $A/(S)$, i.e. $r \in \sqrt{S} \iff [r] \in \mathfrak{N}(A/(S))$, and applying the above. Indeed, $(A/(S))/0_{[[r]]} = A/(S)_{[r]}$, and $[r]^n = [0]$ iff $r^n \in (S)$. $\square$

Remark. How to interpret 2 geometrically?

We are ready to prove the bijection.

Theorem 2.2.41. *$I(-)$ and $V(-)$ give an inclusion-reversing bijection*

$$\{ \textit{closed subsets of } \operatorname{Spec} A \} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \{ \textit{radical ideals of } A \}.$$

Proof. Indeed, for a closed subset K of $\operatorname{Spec} A$, $V(I(K)) = \overline{K} = K$, and for a radical ideal J of A, $I(V(J)) = \sqrt{J} = J$. The correspondence is obviously inclusion-reversing. $\square$

Recall that our initial motivation for proving this monumental result was the humble task of counting points. This will be answered by restricting $V(-)$ to $\operatorname{Spec} A$. As promised, the image consists precisely of those closed subsets which cannot be decomposed into meaningful pieces:

Definition 2.2.42. A topological space X is **reducible** if it is the union of two proper closed subsets, or equivalently, if it contains two disjoint non-empty open sets. A non-empty space that is not reducible is **irreducible**.

Remark. Irreducibility is not of much use outside of algebraic geometry: certainly disconnected spaces are reducible, and so are Hausdorff spaces! Of course, most spaces we care about in classical geometry are Hausdorff.

Proposition 2.2.43. *Under the bijection of Theorem 2.2.41, irreducible closed subsets of* $\operatorname{Spec} A$ *correspond to prime ideals of A. Hence spectra of integral domains are irreducible.*

Proof. We will show that the irreducible closed subsets surject via $I(-)$ onto the prime ideals. To this end, we show a double inclusion. Let K be a closed subset such that $I(K)$ is not prime. Then there exists $x, y \notin I(K)$ such that $xy \in I(K)$. Put $J_1 := (I(K), x), J_2 := (I(K), y)$; clearly $J_1 J_2 \subseteq I(K) \subsetneq J_i$. Then $V(J_i) \subsetneq K = V(I(K)) \subseteq V(J_1 J_2) = V(J_1) \cup V(J_2) \subseteq K$. That is, K is reducible. Now let $\mathfrak{p}$ be a radical ideal, and suppose that $V(\mathfrak{p}) = K_1 \cup K_2$ for proper closed subsets K_1, K_2. We may write $K_1 = V(I_1), K_2 = V(I_2)$ for distinct radical ideals I_1, I_2, so that $I(V(\mathfrak{p})) = \sqrt{\mathfrak{p}} = \mathfrak{p} = I(V(I_1) \cup V(I_2)) = I(V(I_1 I_2)) = I_1 I_2$. In particular, $\mathfrak{p} \subsetneq I_i$. Hence we may pick $x_i \in I_i \setminus \mathfrak{p}$ whose product inevitably lies in $\mathfrak{p}$, as desired. $\square$

Corollary 2.2.44. *Each irreducible closed subset of* $\operatorname{Spec} A$ *has precisely one generic point.*

Proof. Notice, $[I(K)]$ is a generic point for any irreducible closed $K \subseteq \operatorname{Spec} A$, as $\overline{\{\, [I(K)] \,\}} = V(I(\{\, [I(K)] \,\})) = V(I(K)) = \overline{K} = K$. Furthermore, a prime $\mathfrak{p} \neq I(K)$ can be written as $I(K')$ for some irreducible closed $K' \neq K$, so the same argument proves uniqueness. $\square$

Remark. The only element of surprise is that we have imposed no conditions on the nilradical: the above holds even when functions are not determined by their values! Certainly the generic points are quite capacious, in the sense that functions vanishing on it *vanish up to nilpotents*. The point cannot *see* the functions it represents; it is merely the graph, the imprint or "projection" of functions, which are rightful inhabitants of some phantom "higher-dimensional" space, looming atop the relatively "flat" affine space. In the ring of dual numbers, the affine line cannot "see" the parabola $y = x^2$; its sensibility is limited to the intersection points, unable to distinguish the curve from such derelicts as $x = 0$ which nonetheless affect it in the same way. By the same token, affine spaces over non-algebraically closed fields cannot distinguish between Galois conjugates. This is why the *structure sheaf*, as we will soon define, is the heart of the affine scheme, for it overcomes the underlying space's short-sightedness by daring to come into direct contact with the functions themselves.

We end this section by defining affine varieties.

Definition 2.2.45. For k a field, an affine prevariety $\operatorname{Spec} k[x_1, \ldots, x_n]/I$ is an **affine variety** if I is prime.

Proposition 2.2.46. *Affine varieties are irreducible and reduced.*

Proof. Reducedness follows from Proposition 2.2.35, and irreducibility from that $V(I) \cong \operatorname{Spec} k[x_1, \ldots, x_n]/I$. $\square$

Remark. Note that our convention is not universal: some sources only require I to be radical, in which case irreducibility of the prevariety is not guaranteed.

2.3. The Category AffSch

As we have alluded to at various points, the one remaining, and most essential structure of an affine scheme is its *structure sheaf*. To define it, we need only specify sections on basic open sets. But this has already been done for us: since the homeomorphisms $\operatorname{Spec} A_f \cong D(f)$ are induced from ring homomorphisms, they had better be isomorphisms of ringed spaces. But then $A_f = \Gamma(\operatorname{Spec} A_f, \mathscr{O}_{\operatorname{Spec} A_f}) \cong \mathscr{O}_{\operatorname{Spec} A}(D(f))$! For our desired equivalence of categories to hold, this is forcibly the case.

Notice how this aligns with our expectation prior to our excursion into intricacies of the Zariski topology: since functions on $D(f)$ are allowed to be undefined on $V(f)$, the formal inverses of f^n, albeit not global sections, are bona fide elements of $\mathscr{O}_{\operatorname{Spec} A}(D(f))$. On the other hand, starting only with knowledge of the ring A, there are no other functions that can be reasonably adjoined: everything must be "algebraic."

Definition 2.3.1. The **structure sheaf** $\mathscr{O}_{\operatorname{Spec} A}$ of $\operatorname{Spec} A$ sends each $D(f)$ to A_f and each inclusion $D(f) \to D(g)$ to the natural map $A_g \to A_f$.

Proof $\mathscr{O}_{\operatorname{Spec} A}$ is well-defined. We first show that that A_f can be expressed in terms of $D(f)$ by exhibiting an isomorphism $A_f \to S^{-1}A$, where $S := \{\, g \in A \colon D(f) \subseteq D(g) \,\}$. We take it to be induced from the inclusion $A \to S^{-1}A$, as $f \in S$. It suffices to show that S is localized once we invert f, as then the induced map $S^{-1}A \to A_f$ is automatically the inverse. Let $g \in A$ be such that $D(f) \subseteq D(g) \iff V(g) \subseteq V(f) \iff f \in \sqrt{g}$, furnishing some $n \in \mathbb{Z}^+$ with $f^n \in (g)$. Hence the image of (g) under $A \to A_f$ is the entire ring, i.e. g must itself be a unit.
From this definition, we see that the restrictions are well-defined: if $D(f) \subseteq D(g)$, then $S_g \subseteq S_f$, i.e. $D(g) \subseteq D(h) \implies D(f) \subseteq D(h)$. Functoriality is evident. □

Remark. That functions containing an invertible element in their radical are themselves invertible has a geometric interpretation, again appealing to Proposition 1.7.17. It is simply that if $V(f) = \varnothing$, $V(f) \supseteq V(g) \implies V(g) = \varnothing$!

The reader may protest that we lied in our definition: by our proof, $\mathscr{O}_{\operatorname{Spec} A}(D(f))$ is only defined up to canonical isomorphism! This can be easily ameliorated by setting it to $S^{-1}A$ instead, but only with hindsight: the construction would not have been clear a priori.

The proof that $\tilde{M}$ is indeed an $\mathscr{O}_{\operatorname{Spec} A}$-module, and hence $\mathscr{O}_{\operatorname{Spec} A} = \tilde{A}$ is a sheaf (of abelian groups, but of course the natural ring structure on each $\mathscr{O}_{\operatorname{Spec} A}(D(f))$ module structure), will make use of a crucial topological property of affine schemes that may come as a surprise:

Definition 2.3.2. A topological space is **quasicompact** if every open cover thereof has a finite subcover.

Remark. This property may be known as simply *compactness* to topologists, and we trust it to be an entirely familiar notion to the reader. The prefix "quasi" is unfortunate: it is merely to disambiguate our definition from the one that additionally imposes Hausdorffness. Albeit cumbersome, the emphasis on non-Hausdorfness is not misplaced: most spaces in algebraic geometry, affine schemes included, are not Hausdorff.

Theorem 2.3.3. *For A a ring,* $\operatorname{Spec} A$ *is quasicompact.*

Proof. Since $\{ D(f) \colon f \in A \}$ forms a base for A, it suffices to show that any cover consisting of distinguished open sets has a finite subcover. This amounts to proving the equivalence

$$\{ D(f_\alpha) \} \text{ covers } \operatorname{Spec} A \iff (\{ f_\alpha \}) = A \iff \exists \text{ finitely-supported } \{ a_\alpha \} \colon \sum a_\alpha f_\alpha = 1.$$

For the distinguished open sets of the finitely many f_i whose corresponding a_i is non-zero would then cover $\operatorname{Spec} A$: if there were some $\mathfrak{p} \in \operatorname{Spec} A$ such that $\mathfrak{p} \notin D(f_i) \iff f_i \in \mathfrak{p}$ for all i, then $\sum a_\alpha f_\alpha$ would vanish at $\mathfrak{p}$.
The first equivalence is straightforward: $\exists\, D(f_\alpha) \colon \mathfrak{p} \in D(f_\alpha)$ iff $\exists\, f_\alpha \colon f_\alpha \notin \mathfrak{p}$, so that $\{ D(f_\alpha) \}$ covers amounts to that no prime ideal contains $(\{ f_\alpha \})$. But any proper ideal is contained in a maximal ideal, and A is not prime. Now observe that the third condition is equivalent to that $1 \in (\{ f_\alpha \})$, and an ideal contains a unit iff it is the entire ring. Note that this is just like any other partitions-of-unity argument, except that genuine functions to $[0, 1]$ are replaced with ring elements. $\square$

One should not be guided by classical topological intuition to conceive of this sort of quasicompactness. As can be seen from the proof, the source of the finite covers is algebraic: there are no infinite linear combinations in ideals. With $D(f)$, an open subspace, being compact, no version of Heine-Borel could hold here. Instead of owing their compactness to being "the next best thing to finitude", as we classically conceive of them to be, quasicompact sets here attain their status not by their own virtue, but by the corpulence of *every* open subset. It is as if the "small" open subsets were intentionally removed from the Zariski topology.

The situation in $\mathbb{A}^1$ is particularly clear: the complement of any $D(f)$ is a finite set (indeed, due to the algebraic fact that polynomials have finitely many roots), so clearly finite subcovers are always attainable. In general affine spaces, while vanishing sets of nonzero polynomials are no longer finite in general, quasicompactness captures how they are still sufficiently "low-dimensional" for distinguished opens to easily cover. This theme will be more thoroughly explored when we deal with dimension theory proper; for now let us continue discussing the structure sheaf.

Proof $\mathscr{O}_{\operatorname{Spec} A}$ is a sheaf. We first show base locality. Let $D_f = \bigcup_\alpha D_{f_\alpha}$, $s \in A$ be such that $s|_{D_{f_\alpha}} = 0$ for all α. That is, there exists d_α such that $f_\alpha^{d_\alpha} s = 0$. Now $D(f_\alpha^{d_\alpha}) = D(f_\alpha)$, so compactness of $D_f \cong \operatorname{Spec} A_f$ furnishes a finitely-supported set $\{ r_\alpha \} \subseteq A$ with $\sum r_\alpha f_\alpha^{d_\alpha} = 1$. Hence $0 = \sum r_\alpha (f_\alpha^{d_\alpha} s) = (\sum r_\alpha f_\alpha^{d_\alpha}) s = s$.
To show base gluability, we fix a cover $D_f = \bigcup_\alpha D_{f_\alpha}$ and functions $s_\alpha / f_\alpha^{d_\alpha}$ (where $s_\alpha \in A$) on D_{f_α} which agree on overlaps. Writing $g_\alpha := f_\alpha^d$, from $D(f_i) = D(g_i)$ we have that $(g_\alpha g_\beta)^{m_{\alpha\beta}} (s_\alpha g_\beta - s_\beta g_\alpha) = 0$ for all α, β. By compactness we may take a finite subcover $\{ D_{f_i} \}$ and put $m := \max \{ m_{ij} \}$. Then writing $h_i := g_i^{m+1}$, $k_i := g_i^m s_i$, certainly $k_i h_j = h_i k_j$. It follows from $D(h_i) = D(g_i)$ that $\sum r_i h_i = 1$, as above. We claim that $r = \sum r_i k_i$ is the desired function on A. Indeed, it restricts to $s_j / f_j^{d_j} = s_j / g_j = k_j / h_j$ on $D(h_j)$: $r h_j = \sum r_i k_i h_j = \sum r_i h_i k_j = k_j$. We conclude that r restricts to s_α for all α by constructing r's from covers $\{ D_{f_i} \} \cup \{ D_{f_\alpha} \}$ as above and noting that by locality, they must agree with the r above. □

With this, we have established that affine schemes carry the structure of a ringed space, so morphisms of affine schemes should at least be morphisms of ringed spaces. Spec can thus be interpreted as a functor to RS.

Proposition 2.3.4. Spec *defines a contravariant functor* $\mathsf{Ring}^{op} \to \mathsf{RS}$.

Proof. We must determine the pullback $\varphi^\# \colon \mathscr{O}_{\operatorname{Spec} A} \to \pi_* \mathscr{O}_{\operatorname{Spec} B}$ associated to each ring homomorphism $\varphi \colon A \to B$, where $\pi \colon \operatorname{Spec} B \to \operatorname{Spec} A$ is induced by φ. It will suffice to define its components on distinguished open sets. By Proposition 2.2.2 (4), $\pi^{-1} D(f) = D(\varphi(f))$. Thus we may define the component on $D(f) \subseteq \operatorname{Spec} A$ as the natural map $\varphi_f \colon A_f \to B_{\varphi(f)}$ composed with the canonical isomorphisms to $S^{-1}A$ and $S^{-1}B$. To see that this is independent of the choice of f, it will suffice to show that $S_{\varphi(f)} = S_{\varphi(g)}$ for any $g \in A$ with $D(f) = D(g)$; for then we see by the diagram below that both maps are induced from $A \xrightarrow{\varphi} B \longrightarrow S^{-1}B$:

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d] \arrow[dr] & B \arrow[dr] \arrow[drr] & & \\
S^{-1}A \arrow[r] & A_f \arrow[r, "\varphi_f"] & B_{\varphi(f)} \arrow[r] & S^{-1}B.
\end{tikzcd}$$

The equality in turn amounts to that $D(\varphi(f)) = D(\varphi(g))$, which is, of course, immediate from Proposition 2.2.2 (4). It remains to show naturality. That is, for $D(g) \subseteq D(f)$, the following diagram must commute:

$$\begin{tikzcd}
A_f \arrow[r, "\varphi_f"] \arrow[d, "r_1"] & B_{\varphi(f)} \arrow[d, "r_2"] \\
A_g \arrow[r, "\varphi_g"] & B_{\varphi(g)}.
\end{tikzcd}$$

Indeed, since by naturality of localization $\varphi_f \circ r_{A,A_f} = r_{B,B_{\varphi(f)}} \circ \varphi$ for all f, both paths are induced from $r_{B,B_{\varphi(g)}} \circ \varphi \colon A \to B_{\varphi(g)}$: $\varphi_g \circ r_1 \circ r_{A,A_f} = \varphi_g \circ r_{A,A_g} = r_{B,B_{\varphi(g)}} \circ \varphi = r_2 \circ r_{B,B_{\varphi(f)}} \circ \varphi = r_2 \circ \varphi_f \circ r_{A,A_f}$. □

But recall that general ringed space do not have a good notion of evaluation, whereas affine schemes do. It is then plausible to conjecture that they are in fact *locally* ringed spaces, with morphisms inducing local homomorphisms on stalks. The proof that affine schemes have local stalks will be almost identical to that for the following mild generalization of $\mathscr{O}_X$, which will be fundamental to our future study of *quasicoherent sheaves*.

Definition 2.3.5. Let M be an A-module. The **associated $\mathscr{O}_{\operatorname{Spec} A}$-module** of M sends each $D(f)$ to $S_f^{-1}M$ and each inclusion $D(f) \to D(g)$ to $\iota \otimes \mathrm{id}_M$, where ι is the natural map $S_g^{-1}A \to S_f^{-1}A$.

Verification. The proof that $\tilde{M}$ is a sheaf of abelian groups is entirely analogous to the proof that $\mathscr{O}_{\operatorname{Spec} A}$ is a sheaf, as we have only used properties of s as an element of an A-module in the proof. On the other hand, for every f, $\tilde{M}(D(f)) = S^{-1}M \cong S^{-1}A \otimes M$ is tautologically a natural $S^{-1}A = \mathscr{O}_{\operatorname{Spec} A}(U)$-module. □

Remark. We will see in §2.5 that associated modules may be axiomatized as precisely those $\mathscr{O}_{\operatorname{Spec} A}$-modules which are locally presented; such sheaves are said to be *quasicoherent*.

Theorem 2.3.6. *Let $\mathfrak{p}$ be a prime ideal of A. Then there is a natural isomorphism $\tilde{M}_{\mathfrak{p}} \to M_{\mathfrak{p}}$. Hence affine schemes are LRS's.*

Proof. Here we consider $\tilde{M}$ as a sheaf on the distinguished base. It suffices to show that $M_\mathfrak{p}$ satisfies the universal property of the stalk. Indeed, for all $f \in A$ such that $\mathfrak{p} \in D(f) \iff f \notin \mathfrak{p}$, we have a canonical inclusion $\tilde{M}(D(f)) \cong M_f \to M_\mathfrak{p}$. That compatible A-module morphisms $\pi_f \colon M_f \to N$ determine a unique morphism $\pi \colon M_\mathfrak{p} \to N$ is a standard result on localizations.
The consequence does not follow formally, since $\tilde{A}$ and $\mathscr{O}_{\operatorname{Spec} A}$, despite having the same underlying abelian group, belong to different categories. Instead, we will just note that the same proof applies, since all scalar multiplications involved when taking $M = A$ can be viewed as multiplications in a ring. $\square$

With this result, we are finally able to define affine schemes in their full generality.

Definition 2.3.7. An **affine scheme** is a locally ringed space $(X, \mathscr{O})$ that is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some ring A. The full subcategory thereof is denoted AffSch.

Having absorbed affine schemes into the general framework of locally ringed spaces, we may reconcile our two notions of evaluation. The residue field of any point $p \in \operatorname{Spec} A$ is $\kappa(p) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \cong K(A/\mathfrak{p})$, the fraction field of our naive avatar for evaluation, $A/\mathfrak{p}$. The key difference is that while we were only able to evaluate global sections with the latter, we can evaluate sections on *arbitrary* open containing $\mathfrak{p}$ with the former. Nevertheless, the two means of evaluation do agree on $\operatorname{Spec} A$:

Proposition 2.3.8. *Let A be a ring, $f \in A$. Then $V(f) = \underline{V}(f)$, where by $\underline{V}$ we understand the vanishing set defined in Proposition 1.7.17.*

Proof. The equality amounts to that $f \in \mathfrak{p} \iff [f] \in \mathfrak{m}_\mathfrak{p}$. But this is evident, as $\mathscr{O}_{X,p} \cong A_\mathfrak{p}$, by which $[f]$ gets sent to $(f, 1)$; clearly $f \in \mathfrak{p} \iff (f, 1) \in \mathfrak{p}A_\mathfrak{p}$. $\square$

Remark. Evaluation is also the correct lens from which to conceive of $\operatorname{Spec} A_\mathfrak{p}$. Introducing formal inverses to all functions in A which do not vanish at $\mathfrak{p}$, $A_\mathfrak{p}$ allows its functions to be undefined outside the vicinity of $\mathfrak{p}$, as this does not impact evaluability at p. As such, the mysterious point constitution of $\operatorname{Spec} A_\mathfrak{p}$ can be justified as to accommodate all the points which can be evaluated concomitant of the evaluability at $\mathfrak{p}$. That is, if $\mathfrak{q}$ generizes $\mathfrak{p}$, then $\mathfrak{q} \subseteq \mathfrak{p}$, so if a function f/g is defined at $\mathfrak{p}$, i.e., that $\mathfrak{p} \in D(g) \implies g \notin \mathfrak{p}$, then it is also defined at $\mathfrak{q}$, as $g \notin \mathfrak{q} \implies \mathfrak{q} \in D(g)$. As an example, let us take $A := \mathbb{C}[x, y]$, $p := (x, y)$. Then if a function f/g cannot be evaluated at (x), g must be divisible by x. But this clearly makes it undefined at (x, y) as well.

This interpretation solicits a renewed understanding of evaluation at a generic point $\mathfrak{p}$. A function defined on $\mathfrak{p}$ need *not* be defined on every point of $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$! As shown, evaluability at $\mathfrak{p}$ is implied by that at any of its specializations. The converse, as stated, is trivially true: $\mathfrak{p}$ is its own specialization. It no longer holds if we require the specialization $\mathfrak{q}$ to be proper, i.e. for $\mathfrak{q}$ to strictly contain $\mathfrak{p}$. Such a $\mathfrak{q}$ might not exist, and even when it does, $g \notin \mathfrak{p}$ does not imply that $g \notin \mathfrak{p}$.

The final act in our little trilogy on affine schemes, of course, is to prove the equivalence of categories we have been using as motivation throughout. To this end we will prove a slightly stronger result that should further enshrine LRS as the "correct setting" to do scheme theory. Before all this, however, we will set the stage by showing an analogous, but weaker result for $\mathscr{O}_{\operatorname{Spec} A}$-modules, whose proof is quite trivial.

Theorem 2.3.9. *The assignment $M \rightsquigarrow \tilde{M}$ gives a fully faithful and exact functor $A\text{-}\mathsf{Mod} \to \mathscr{O}_{\operatorname{Spec} A}\text{-}\mathsf{Mod}$.*

Proof. We first exhibit a bijection $\operatorname{Hom}_{A\text{-}\mathsf{Mod}}(M, N) \cong \operatorname{Hom}_{\mathscr{O}_{\operatorname{Spec} A}\text{-}\mathsf{Mod}}(\tilde{M}, \tilde{N})$ for any $M, N \in A\text{-}\mathsf{Mod}$. We set the forwards map to send a morphism $\varphi \colon M \to N$ to the morphism $\tilde{\varphi} \colon \tilde{M} \to \tilde{N}$ whose component on $D(f)$ is induced from the composition $M \xrightarrow{\varphi} N \longrightarrow N_f$, the backwards map to send a morphism $\Phi \colon \tilde{M} \to \tilde{N}$ to its component on $D(1)$. It is routine to verify that these are mutually inverse.
For exactness, recall that localization is an exact functor, and that exactness of a sequence of sheaves may be checked on stalks. The result then follows from Theorem 2.3.6. $\square$

Theorem 2.3.10. *The global sections functor is left-adjoint to* Spec*:*
$$\begin{tikzcd} \mathsf{LRS} \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathsf{Ring}^{op} \arrow[l, bend left, "\operatorname{Spec}"] \end{tikzcd}.$$

Proof. We must exhibit a natural bijection $\operatorname{Hom}(A, \mathscr{O}_X(X)) \cong \operatorname{Hom}(X, \operatorname{Spec} A)$ for $A \in \mathsf{Ring}$, $X \in \mathsf{LRS}$. To each LRS morphism $\Phi \colon X \to \operatorname{Spec} A$ we associate the component on $D(1)$ of its pullback, i.e. $\Phi^{\#}_{\operatorname{Spec} A} \colon A \to \mathscr{O}(X)$. The LRS morphism associated to each ring homomorphism $\varphi \colon A \to \mathscr{O}_X(X)$ will be constructed in stages, first as a set map, then (verified to be) a continuous map, and finally a full-fledged LRS morphism. For motivation, let us consider the following result:

Lemma 2.3.11. *For any LRS morphism $\pi \colon X \to \operatorname{Spec} A$, $\pi(x)$ is the inverse image of $\mathfrak{m}_x$ under the composition $\alpha_x \colon A \xrightarrow{\pi^{\#}} \mathscr{O}_X(X) \xrightarrow{i} \mathscr{O}_{X,x}$.*

Proof. Recall from Corollary 1.7.21 that we have a diagram

$$\begin{array}{ccc} A & \xrightarrow{\pi^\#} & \mathscr{O}_X(X) \\ \downarrow & & \downarrow \\ A_{\mathfrak{m}_{\pi(x)}} & \xrightarrow{\pi_x^\#} & \mathscr{O}_{X,x}. \end{array}$$

By definition, $\mathfrak{m}_x$ pulls back along $\pi^\#$ to $\mathfrak{m}_{\pi(x)}$. Now clearly $\mathfrak{m}_{\pi(x)}$ pulls back along the left localization map to the prime ideal corresponding to $\pi(x)$. $\square$

Proof of Theorem 2.3.10, continued. For a ring homomorphism $\varphi\colon A \to \mathscr{O}_X(X)$, we define the corresponding morphism of locally ringed spaces $\Phi\colon X \to \operatorname{Spec} A$ on points by the preceding lemma, sending $x \rightsquigarrow [\alpha_x^{-1}(\mathfrak{m}_x)]$, where $\alpha_x := i_{\mathscr{O}_{X,x}} \circ \varphi$. Continuity is readily checked on the base of distinguished open sets: for any $f \in A$, $\Phi^{-1}(D(f)) = \{\, x \in X\colon f \notin \alpha_x^{-1}(\mathfrak{m}_x) \,\} = \{\, x \in X\colon f \notin \varphi^{-1} i^{-1}(\mathfrak{m}_x) \,\} = \{\, x \in X\colon [\varphi(f)] \notin \mathfrak{m}_x \,\} = D(\Phi^\#(f))$, which is open by Proposition 1.7.17. We now construct the pullback on the base: for any open $D(f) \subseteq \operatorname{Spec} A$, we take it to be the composition $\Gamma(\mathscr{O}_{\operatorname{Spec} A}, D(f)) \xrightarrow{\sim} A_f \longrightarrow \Gamma(\mathscr{O}_X, D(\varphi(f)))$, where the second map is induced from $r_{D(\varphi(f))} \circ \varphi$, with f being invertible in the image, again by Proposition 1.7.17. The argument at the end of the proof of Proposition 2.3.4 can be adapted to show naturality. To see that the maps on stalks are local, consider the diagram above, in which the upper path is seen to be α_x. Then indeed $\alpha_x^{-1}(\mathfrak{m}_x) = i^{-1}{\Phi_x^\#}^{-1}(\mathfrak{m}_x) \implies \mathfrak{m}_{\Phi(x)} \supseteq i(\Phi(x)) = i(\alpha_x^{-1}(\mathfrak{m}_x)) \supseteq {\Phi_x^\#}^{-1}(\mathfrak{m}_x)$.
It remains to show that we have indeed defined inverses. Clearly $\varphi \to \Phi \to \varphi$. For the other direction, we have to show that Φ is the unique LRS moprhism $X \to \operatorname{Spec} A$ having $\Phi^\#_{\operatorname{Spec} A}$ as its global component. Indeed, by the lemma, the morphism is determined on points by the $\Phi^\#_{\operatorname{Spec} A}$, and by naturality its component on any $D(f)$ is determined by the global one. We omit the proof of naturality. $\square$

Corollary 2.3.12. $\operatorname{Spec}\colon \mathsf{Ring}^{op} \to \mathsf{AffSch}$ *is an anti-equivalence of categories. Hence* AffSch *is bicomplete.*

Proof. Simply take the X above to be affine. Note that it is precisely Spec that generates a morphism of affine schemes from a global pullback map! $\square$

Corollary 2.3.13. $\operatorname{Spec}\mathbb{Z}$ *is final in* LRS.

Proof. Recall that $\mathbb{Z}$ is initial in Ring. Hence for any LRS X, we have $\operatorname{Hom}(X, \operatorname{Spec}\mathbb{Z}) \cong \operatorname{Hom}(\mathbb{Z}, \mathscr{O}_X(X)) \cong \{\, * \,\}$. $\square$

Beware, however, that in general (co)limits in AffSch and LRS do not agree. See, for instance, Example 2.4.30.

Corollary 2.3.14. *For any f in a ring A, the open subspace $(D(f), \mathscr{O}_{\operatorname{Spec} A}|_{D(f)})$ is affine.*

Proof. Notice, $\Gamma(\mathscr{O}_{\operatorname{Spec} A}, D(f)) = S_f^{-1}A \cong A_f = \Gamma(\mathscr{O}_{\operatorname{Spec} A_f}, \operatorname{Spec} A_f)$. □

We end by noting that Theorem 2.3.9 likewise extends to an adjunction. The constructions are by no menas identical, however, as here both functors are *covariant* rather than contravariant. Furthermore, there is no need to deal in stalks, as the morphisms involved are of mere sheaves.

Theorem 2.3.15. *Fix a base ring A. The assignment $M \rightsquigarrow \tilde{M}$ is left-adjoint to the global sections functor:*
$$\begin{tikzcd} A\text{-}\mathsf{Mod} \arrow[rr, bend left, "\sim"] \arrow[rr, phantom, "\perp"] & & \mathscr{O}_{\operatorname{Spec} A}\text{-}\mathsf{Mod}. \arrow[ll, bend left, "\Gamma"] \end{tikzcd}$$

Proof. We must exhibit a natural bijection $\operatorname{Hom}_A(\tilde{M}, \mathscr{F}) \cong \operatorname{Hom}_{\mathscr{O}_{\operatorname{Spec} A}}(M, \Gamma(\operatorname{Spec} A, \mathscr{F}))$. Like before, the forwards map can be taken to be simply the global component; what is non-trivial is constructing a moprhism $\tilde{M} \to \mathscr{F}$ from a ring homomorphism $\varphi\colon M \to \Gamma(\operatorname{Spec} A, \mathscr{F})$. Let us take the component on $D(f)$ to be the composition $\tilde{M}(D(f)) \to M_f \to \mathscr{F}(D(f))$, wherein $M_f \to \mathscr{F}(D(f))$ is induced from the composition $r_{\mathscr{F}(\operatorname{Spec} A), \mathscr{F}(D(f))} \circ \varphi$, with $\mathscr{F}(D(f))$ an A_f-module. We must show that these constitute a morphism of $\mathscr{O}_{\operatorname{Spec} A}$-modules. Indeed, each component is evidently linear, and naturality is plain from the following diagram:

$$\begin{tikzcd} & M \arrow[d] \arrow[dr] & \\ \tilde{M}(D(f)) \arrow[r] \arrow[d] & M_f \arrow[r, dashed] \arrow[d] & \mathscr{F}(D(f)) \arrow[d] \\ \tilde{M}(D(g)) \arrow[r] & M_g \arrow[r, dashed] & \mathscr{F}(D(g)). \end{tikzcd}$$

It is immediate from construction that the two maps thus defined are mutual inverses. For brevity, we will only show naturality in the first argument. Indeed, given a morphism $\varphi\colon M \to N$, $\mathscr{O}_{\operatorname{Spec} A}$-module morphism $\lambda\colon \tilde{M} \to \mathscr{F}$, clearly $\lambda_{\operatorname{Spec} A} \circ \varphi = (\lambda \circ \tilde{\varphi})_{\operatorname{Spec} A}$: we need only look at the global component. □

2.4. Schemes

There are various reasons why working with affine schemes alone is not satisfactory. For one, the equivalence $\mathsf{AffSch} \simeq \mathsf{Ring}^{\mathrm{op}}$, while conveniently allowing us to convert virtually all of affine geometry into commutative algebra, restricts our scope to those spaces with datum not surpassing the fairly limited expressiveness of a ring. Many spaces of classical interest, projective spaces (c.f. §2.6) included, find no incarnation in AffSch. In a similar vein, despite bicomplete, (co)limits in AffSch differ vastly from those in LRS. Most sub-LRS's, for instance, are not affine.

We have already explored the category LRS as a general setting for *all* of geometry. It is due to this generality, however, that it will not suit our study of algebraic geometry, to which end we must make salient use of *affine charts*. The motion from affine schemes to schemes will thus be entirely analogous to the that from $\mathbb{R}^n$ to topological manifolds:

Definition 2.4.1. A **scheme** is a ringed space $(X, \mathscr{O}_X)$ such that every point x admits an **affine neighborhood**, i.e. a neighborhood $(U, \mathscr{O}_X|_U)$ isomorphic to some affine scheme. We denote by Sch the full subcategory of LRS whose objects are schemes.

Verification. Let $x \in X$ with an affine neighborhood U. Then since U is an LRS, $\mathscr{O}_{X,p} = \mathscr{O}_{U,p}$ is a local ring. □

In this introductory section, we will develop some basic machinery of schemes that will be applied persistently. We will also introduce some classes to which properties of schemes and morphisms should belong. To begin, we recall the notion of an open subspace from LRS, though in this case there is the additional structure of *affines*.

Definition 2.4.2. An LRS-theoretic open subspace of a scheme X is an **open subscheme**, and similarly **open embeddings** of schemes are defined exactly as they are for LRS. An open subscheme $(U, \mathscr{O}_U)$ is said to be an **affine open** if U is an affine scheme.

Remark. Putting a subscheme structure on *closed* subsets is more delicate, as the sections on such a scheme are not an intrinsic datum of the sheaf. As seen in §2.2, we do have a good definition in the affine case, namely any affine scheme of the form $\operatorname{Spec} A/I$, for I an ideal of A. Generalizing this to schemes will hinge on the notion of *quasicoherent ideal sheaves* and thus relegated to §2.6.

Proposition 2.4.3. *Affine opens form a base for the topology on a scheme X. Hence open subschemes are schemes.*

Proof. Let $U \subseteq X$ be open, $x \in U$. Then $\operatorname{Spec} A \cap U$ is an open subspace of its affine neighborhood $\operatorname{Spec} A$, and thus admits a cover $\{ \operatorname{Spec} A_f \}$ of distinguished open sets. Since $x \in \operatorname{Spec} A \cap U$, there exists an $f \in A$ for which $x \in \operatorname{Spec} A_f \subseteq U$, as desired. We conclude by recalling that open subspaces of an open subscheme are also open subspaces of X. $\square$

Manifestly the universal property is still satisfied, with Sch a full subcategory of LRS. The same proof then applies to show that open embeddings are monomorphisms in Sch.

Nevertheless, not all open subschemes are affine (this much is clear)—even when the scheme is itself affine.

Example 2.4.4. Put $X := \mathbb{A}_k^2$, $A := \Gamma(X, \mathscr{O}_X)$. The open subscheme $(U, \mathscr{O}_U)$, where $U := X \setminus \{ [(x, y)] \}$, called the "punctured affine plane", has A as its ring of global sections. Hence it is not affine.

Proof. Clearly U admits the affine cover $\{ D(x), D(y) \}$. Recall that $\mathscr{O}(D(f)) = A_f$ and $D(x) \cap D(y) = D(xy)$, so $\mathscr{O}(U)$ may be computed as the pullback of $A_y \longrightarrow A_{xy} \longleftarrow A_x$. Since both arrows are injections, x, y not being zero-divisors, $\mathscr{O}(U)$ is simply $A_x \cap A_y = A$. If U were affine, then necessarily $(U, \mathscr{O}_X|_U) \cong (\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) \cong X$. But $V(x, y)$, which is an irreducible closed subset in $\mathbb{A}_k^2$, is in fact empty in U, contracting Proposition 1.7.18 (which will hitherto be used without reference). $\square$

Remark. The reason that we get no new functions after removing a point is that for any function f, $V(f)$, if non-empty, is of higher "dimension" than a point, which is 0-dimensional, i.e. there is no function which *only* vanishes at $(0, 0)$. A remarkable consequence is that any function on $\mathbb{A}_k^2 \setminus \{ (0, 0) \}$ extends to the entirety of $\mathbb{A}_k^2$. This will be vastly generalized by the *algebraic Hartog Lemma*: one can always extend functions defined on the complement of a set of codimension less than or equal to two to the entire scheme.

More can be said about the affine base. The intersection of two affines admits not just any affine, but a *distinguished* one in the following sense.

Lemma 2.4.5. *The intersection of two affine opens U, V of a scheme X admits an open cover by simultaneously distinguished affine opens of U and V.*

Proof. We first demonstrate transitivity of distinction in affines. That is, given an affine scheme $\operatorname{Spec} A$, for any $f \in A$, $g \in \mathscr{O}_{\operatorname{Spec} A}(D(f))$, $D_{D(f)}(g)$ is distinguished in $\operatorname{Spec} A$, i.e. there exists some $h \in A$ for which $D_{D(f)}(g) = D(h)$. Under the identification $\mathscr{O}_{\operatorname{Spec} A}(D(f)) \cong A_f$, we can write g as h/f^n for some $h \in A$. Then $D_{D(f)}(g) = D_{D(f)}(h) = D(h) \cap D(f) =$

$D(hf)$, as needed.

Let $p \in U \cap V$. Since the intersection is an open subspace of U, there exists some $f \in \mathscr{O}_X(U)$ for which $p \in D_U(f) \subseteq U \cap V$. Viewing $D_U(f)$ as an open subspace of V, we may similarly concoct some $g \in \mathscr{O}_X(V)$ for which $p \in D_V(g) \subseteq D_U(f)$. We claim that $D_V(g) = D_{D_U(f)}(i_V^\#(g))$, for $i\colon D_U(f) \to V$ the inclusion morphism; it then follows that $D_V(g)$ is distinguished in $D_U(f)$, hence U by the above. Indeed, i, a morphism of affine schemes, is induced from $i_V^\#$, so we may conclude from Proposition 2.2.2 (4) that the latter is equal to $i^{-1}(D_V(g)) = D_V(g)$. □

This result is far more useful than it may seem, stemming from the fact that as opposed to distinguished inclusions, arbitrary inclusions $\operatorname{Spec} A \to \operatorname{Spec} B$ are hardly understood. In other words, we may distill from the category defined by the affine base the following "nice" subcategory.

Definition 2.4.6. The **distinguished base** is the subcategory of the affine base consisted of only inclusions of the form $\operatorname{Spec} A_f \to \operatorname{Spec} A$.

Proposition 2.4.7. *For any scheme X, the distinguished base is a final subcategory of the affine base. Hence there is an equivalence of categories between sheaves on X and sheaves on its distinguished base.*

Proof. Let us denote the affine base C, the distinguished base by D. We must show that the inclusion functor $i\colon \mathsf{D} \to \mathsf{C}$ is final. First, note that for any $x \in \mathsf{C}$, the object $j \in \mathsf{D}$ to whose image under i x admits an arrow can be taken to be simply x itself, as an object of D. Now let there be arrows $U \longleftarrow x \longrightarrow V$. By the lemma, there indeed exists a W contained and distinguished in both U and V. Hence x/T is non-empty and connected, as desired. □

Here is another consequence, which in a nutshell tells us that a property holding for distinguished opens informs us about whether it holds for *all* affine opens.

Definition 2.4.8. A property P of affine opens is said to be **affine local** if the existence of an affine open cover $\{ U_i \}$ of X for which $P(U_i)$ holds for all i implies that $P(U)$ holds for *every* affine open U of X.

Remark. Note that when we speak of a property of either affine opens or opens, we really understand a property evaluable on the Zariski site on Sch or on the subsite comprised of all distinguished bases. That is, we are rarely interested in a property that evaluates only subschemes of a single fixed scheme; rather, properties are understood to be functors from $\mathsf{Sch} \to \{ \top, \bot \}$, much like topological properties are in some sense functors $\mathsf{Top} \to \{ \top, \bot \}$.

Proposition 2.4.9. *Affine locality is stable under conjunction: given a set of affine local properties $\{ P_i \}$, $\bigwedge_i P_i$ is affine local.*

Proof. Assume that $\forall\, j(\bigwedge_i P_i(U_j))$, for $\{ U_j \}$ an affine cover of X. Then for each i, $\forall\, j(P_i(U_j))$ implies that $P_i(U)$ for every affine open U. That is, $\bigwedge P_i(U)$ holds. □

Lemma 2.4.10 (Affine Communication)**.** *A property P of affine opens is affine local iff for any affine open U, the following conditions hold:*

Heredity $P(U) \implies P(D_U(f))$ *for all* $f \in \mathscr{O}_X(U)$*;*
Gluability *If* $U = \bigcup_{i=1}^n D_U(f_i)$ *and* $P(D_U(f_i))$ *for all* i*, then* $P(U)$*.*

Proof. Let $\{ U_i \}$ be an affine open cover of a fixed scheme X in which each U_i enjoys P. For any affine open U of X, $\{ U \cap U_i \}$ covers U_i, which by the lemma can be refined into a distinguished affine cover $\{ D_\alpha \}$ in which each D_α is distinguished in some U_i, hence enjoys P by heredity. Applying gluability to the finite subcover obtained via compactness, we conclude that $P(U)$ holds.
The converse is trivial if we apply the affine locality of P to U as a scheme in itself, taking $\{ U \}$ and $\{ D_U(f_i) \}$ as affine covers, respectively. □

Remark. The conditions of the affine communication lemma can be seen as describing the presheaf of sets $\mathscr{P}$ on the distinguished base sending U to $\{ * \}$ if $P(U)$, $\varnothing$ otherwise. Heredity is precisely that $\mathscr{P}$ is well-defined, while gluability asserts that it is a sheaf. Note since $\mathscr{P}(U_i \cap U_j)$ has precisely one element, agreement on intersections is trivial.
A **local** property P is then naively defined to be a property of *all* open subschemes of a fixed scheme X making $\mathscr{P}$ a sheaf on $\mathsf{Op}(X)$; one then obtains the traditional definition by dropping all appearances of "affine" in that of affine-locality. That restricting $\mathscr{P}$ to the distinguished base still yields a sheaf is tantamount to that locality implies affine locality. More strikingly, an affine-local property is uniquely extended to a local property of all open subschemes, by virtue of Proposition 2.4.7.
However, locality is not too useful a notion in practice, as arbitrary open subschemes are badly behaved, unqualified heredity (**strict heredity**) is in fact very stringent a condition. What we can do instead is to begin with affine heredity and let $\{ U_i \}$ be an affine cover on which P holds. Glueability for this restricted case would imply that $P(X)$ holds, provided that $P(U_i \cap U_j)$ holds for all i, j—this is precisely agreement on intersections, which in this case is non-trivial! Let us extrapolate this into a definition.

Definition 2.4.11. An affine-local property P of open subschemes of a fixed scheme X is **quasi-local** if the existence of an affine cover $\{ U_i \}$ such that $P(U_i \cap U_j)$ holds for all i, j, not necessarily distinct, implies that $P(X)$ holds.

We now move to discuss some conditions on *classes* to which morphisms belong. Classes are equivalently properties: given a property P, we have a class of morphisms satisfying P, and given a class C, we have the property of membership in C. It is thus acceptable to use the two notions interchangeably. The conditions we introduce below (together with *stability under base change*, c.f. §2.6) are very lax coherence conditions that should hold for every class. Failing these coherence conditions is a sign that the property of hand is not idiomatic to scheme theory, and thus should be enhanced to satisfy them.

Definition 2.4.12. A class C of morphisms of schemes is **categorical** if it includes all isomorphisms (this may be relaxed in practice) and is *closed under composition*: if $f\colon X \to Y$, $g\colon Y \to Z$ lie in C, then $g \circ f \in C$. It is **(affine) local on the target** if for any morphism $f\colon X \to Y$, the following properties hold:

Restriction If $f \in C$, then for any (affine) open $U \subseteq Y$, $f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ belongs to C.

Gluing Given an (affine) open cover $\{ U_i \}$ of U, $\forall\, i(f|_{f^{-1}(U_i)} \in C)$ implies that $f|_{f^{-1}(U)} \in C$.

Manifestly being local on the target is a stronger condition than that of being affine local on the target. But locality on the target often follows from affine-locality in the following way:

Proposition 2.4.13. *A* ***restrictable*** *property P of morphisms, i.e. a property satisfying the restriction condition of locality on the target, is local on the target iff it is affine-local on the target.*

Proof. The forwards direction is trivial. For the converse, let $\{ U_i \}$ be an open cover of an open $U \subseteq Y$ such that $P(\pi|_{\pi^{-1}(U_i)})$ for all i. We refine $\{ U_i \}$ into an affine cover $\{ V_j \}$, so that each V_j is contained in some U_i. By restrictability, $P(\pi|_{\pi^{-1}(U_i)}|_{\pi^{-1}(V_j)}) = P(\pi|_{\pi^{-1}(V_j)})$ holds, so by affine gluing $P(\pi)$ holds, and P is local on the target. □

Remark. The name "restrictable" is non-standard; it will be seen to be equivalent to "stability under base change by open embeddings" once we discuss fibered products.

Here is a closely related qualifier which nevertheless need not be satisfied by even reasonable classes.

Definition 2.4.14. A class C of morphisms of schemes is **(affine) local on the source** if given a morphism $f\colon X \to Y$, $f \in C$ implies that for any (affine) open $U \subseteq X$, $f \circ i_U \in C$, and given an (affine) open cover $\{ U_i \}$ of U, $\forall\, i\colon f \circ i_{U_i} \in C$ implies that $f \circ i_U \in C$.

Proposition 2.4.15. *The class of isomorphisms of schemes is categorical, local on the target, but not even affine local on the source.*

Proof. The class of isomorphisms is tautologically categorical. To see that it is local on the target, note that the inverse of $f|_{f^{-1}(U)}$ is given by $f^{-1}|_U$, and that the local inverses $(f|_{f^{-1}(U_i)})^{-1}$ glue to the unique morphism $g\colon U \to f^{-1}(U)$ such that for all U_i, $f \circ g|_{U_i} = \mathrm{id}_{U_i}$ and $g \circ f|_{f^{-1}(U_i)} = \mathrm{id}_{f^{-1}(U_i)}$. By the locality of morphisms, $f \circ g = \mathrm{id}_U$, $g \circ f = \mathrm{id}_{f^{-1}(U)}$. That isomorphisms are not affine local on the source is evident set-theoretically: take the identity map on $\mathbb{A}^1$, and consider the restriction to $D(x)$. $\square$

Proposition 2.4.16. *The class of open embeddings is categorical, local on the target, but not even affine local on the source.*

Proof. Manifestly isomorphisms are open embeddings. For closure under composition, consider the following diagram:

$$(X, \mathscr{O}_X) \xrightarrow{\sim} (U, \mathscr{O}_Y|_U) \xrightarrow{i} (Y, \mathscr{O}_Y) \xrightarrow[\varphi]{\sim} (V, \mathscr{O}_Z|_V) \longrightarrow (Z, \mathscr{O}_Z).$$

Indeed, $\varphi \circ i$ is, being a restriction of φ, an isomorphism.

For locality on the target, let $f\colon X \to Y$ be a morphism. Restriction is composition with an open embedding, so the first condition holds by stability under composition. Now let $\{ U_i \}$ be a cover of U such that for all i, $f^{-1}(U_i) \to U_i \to X$ is an open embedding. Now since isomorphisms are local on the target, the isomorphisms to U_i glue to an isomorphism $f^{-1}(U) \to U$, and of course U is an open subscheme of X.

For the counterexample, take k a field, $X := \operatorname{Spec} k^2$, $Y := \operatorname{Spec} k$, and consider the morphism $\pi\colon X \to Y$ induced from the diagonal map $\Delta\colon k \to k^2$. Writing e_1 for the function $(0,1)$ on k^2, e_2 for $(1,0)$, evidently $p_1 := 0 \times k = D(e_1)$ and $p_2 := k \times 0 = D(e_2)$ are the only points of $\operatorname{Spec} k^2$, and indeed $\Delta^{-1}(p_1) = 0 = \Delta^{-1}(p_2)$. That is, both points, being open, are sent to the unique point of $\operatorname{Spec} k$, so the restriction of π to either is a homeomorphism. Furthermore, $(\pi \circ i_{p_1})^{-1}\mathscr{O}_Y \to \mathscr{O}_{p_1}$ is an isomorphism: only the global component $\mathscr{O}_Y(0) \to \mathscr{O}_{p_1}(p_1)$ is non-trivial, and it is simply the composition $k \to k^2 \to k^2_{e_1}$, which is evidently an isomorphism. Hence $\pi \circ i_{p_1}$ is an open embedding. Similarly for $\pi \circ i_{p_2}$. But π is not itself an open embedding. $\square$

This is really the extent of meaningful classes of morphisms we have so far come across, and as we encounter more classes, it will become customary for us to immediately test them against these conditions. But there is a natural way (or two!) for us to procure properties of morphisms out of properties of schemes, and a surprising amount of theory can still be developed in this very abstract setting. The subsequent discourse will also bridge the notions of affine locality and affine locality on the target.

Definition 2.4.17. A property P of schemes is **strongly affine local** if for all morphisms $\pi\colon X \to Y$, the property of an affine open $U \subseteq Y$ that $P(\pi^{-1}(U))$ is affine-local, or, equivalently, $\pi_*\mathscr{P}$ is a sheaf.

Definition 2.4.18. Let P be a property of schemes. The **associated (affine) property of morphisms**, denoted $\hat{P}$ (resp. $\tilde{P}$), holds for a morphism $\pi\colon X \to Y$ if for every $U \subseteq Y$ that makes $P(U)$ hold (resp. is affine), we have $P(\pi^{-1}(U))$. P is said to be **affine-respecting** if for any morphism π, $\hat{P}(\pi) \iff \tilde{P}(\pi)$.

Proposition 2.4.19. *Let P be a property of schemes. Then $\hat{P}$ and $\tilde{P}$ are restrictable.*

Proof. Let $\pi\colon X \to Y$ be a morphism of schemes. If $\hat{P}(\pi)$ (resp. $\tilde{P}(\pi)$), then manifestly for any open $U \subseteq Y$, the inverse image of any open $V \subseteq U$ that enjoys P (resp. is affine), under $\pi|_{\pi^{-1}(U)}$ is $\pi^{-1}(V) \cap \pi^{-1}(U) = \pi^{-1}(V)$, for which P holds. Hence $\hat{P}(\pi|_{\pi^{-1}(U)})$ (resp. $\tilde{P}(\pi|_{\pi^{-1}(U)})$) holds. $\square$

Lemma 2.4.20. *Let P be a property of schemes. Then $\tilde{P}$ is local on the target if P is strongly affine local.*

Proof. Let $\pi\colon X \to Y$ be a morphism. By Proposition 2.4.13 and the above, we need only show that $\tilde{P}$ satisfies the affine gluing condition. Let $U \subseteq Y$ be an affine open, $\{ U_i \}$ an affine cover thereof for which $\hat{P}(\pi|_{\pi^{-1}(U_i)})$ holds for all i. Then in particular $P(\pi^{-1}(U_i))$ for all i, so by affine locality we have that $P(\pi^{-1}(U))$. Since U is arbitrary, it follows that $\hat{P}(\pi)$, and thus $\hat{P}(\pi|_{\pi^{-1}(U)})$ holds. $\square$

Additionally, strongly affine local properties can always be recovered from their associated affine property of morphisms.

Lemma 2.4.21. $\operatorname{Spec}\mathbb{Z}$ *is final in the category of schemes.*

Proof. This is immediate from Corollary 2.3.13, as Sch is a full subcategory of LRS. $\square$

Proposition 2.4.22. *Let P be a strongly affine local property of schemes. Then P holds for a scheme X iff $\tilde{P}$ holds for the canonical map $\pi\colon X \to \operatorname{Spec}\mathbb{Z}$. If P holds for affines, then the conditions are furthermore equivalent to $\hat{P}$ holding for the canonical map.*

Proof. Let $U \subseteq \operatorname{Spec}\mathbb{Z}$ be an affine open. By supposition, $P(\pi^{-1}(\operatorname{Spec}\mathbb{Z}))$ holds. But evidently $\{\operatorname{Spec}\mathbb{Z}\}$ covers $\operatorname{Spec}\mathbb{Z}$, so by affine locality $P(\pi^{-1}(U))$ holds for all affines $U \subseteq \operatorname{Spec}\mathbb{Z}$, that is, $\tilde{P}$ holds. The converse is obvious: simply take $\pi^{-1}(\operatorname{Spec}\mathbb{Z})$.
The second statement follows from that all open subschemes of $\operatorname{Spec}\mathbb{Z}$ are affine, with $\mathbb{Z}$ a PID. $\square$

Proposition 2.4.23. *For any property of schemes P, $\hat{P}$ is categorical.*

Proof. Let $f\colon X \to Y$, $g\colon Y \to Z$ enjoy $\hat{P}$. Then given $U \subseteq Z$ enjoying P, $(g \circ f)^{-1}(U) = g^{-1}(f^{-1}(U))$ evidently enjoys P. Hence $\hat{P}(g \circ f)$ holds. $\square$

Categoricality is precisely what we need to bridge $\hat{P}$ and $\tilde{P}$; this is not surprising, as a priori $\tilde{P}$ needs not be categorical.

Proposition 2.4.24. *Let P be a strongly affine local property of schemes that holds for affines. Then it is affine-respecting iff $\tilde{P}$ is categorical.*

Proof. The forwards direction is immediate, as $\hat{P}$ is categorical. For the converse, fix a morphism $\pi\colon X \to Y$. That $\hat{P}(\pi) \implies \tilde{P}(\pi)$ is immediate. For the other direction, let $U \subseteq Y$ be an open subscheme enjoying P, which is tantamount by Proposition 2.4.22 to that $\tilde{P}(i)$, for $i\colon U \to \operatorname{Spec}\mathbb{Z}$. Since $\tilde{P}$ is restrictable, $\tilde{P}(\pi|_{\pi^{-1}(U)})$ also holds. But $\tilde{P}$ is categorical, so the composition, or equivalently the canonical map $\pi^{-1}(U) \to \operatorname{Spec}\mathbb{Z}$ enjoys $\tilde{P}$. Hence $P(\pi^{-1}(U))$ holds, from which $\hat{P}(\pi)$ immediately follows. $\square$

Remark. This is not really a criterion; we have just exchanged the burden of proving affine-respect to the more intractable one of proving categoricality. More practical sufficient conditions will be developed in the next section.

We conclude this section with a discussion of gluing schemes, which is indeed identical to the procedure for gluing locally ringed spaces, as Sch is a full subcategory of LRS, and affines survive gluing.

Proposition 2.4.25. *For any gluing datum $\{X_i\}$ of schemes, $\mathrm{Gl}_{\mathsf{LRS}}(X_i)$ is the colimit over the usual diagram, restricted to Sch. Furthermore, $\mathrm{Gl}_{\mathsf{LRS}}$ restricts to an equivalence of categories $\mathsf{Sch}\text{-}\mathsf{GD}_I \to \mathsf{Sch}$.*

Proof. Since Sch is a full subcategory of LRS, it suffices to show that schemes glue to a scheme. But this is immediate from Proposition 1.7.13: $\mathrm{Gl}_{\mathsf{LRS}}(X_i)$ admits an open cover by schemes, and hence affine schemes. □

Corollary 2.4.26. *Given schemes X, Y, a morphism of ringed spaces $\pi\colon X \to Y$ is a morphism of schemes iff for any affine opens $\operatorname{Spec} A \subseteq X$, $\operatorname{Spec} B \subseteq Y$ with $\pi(\operatorname{Spec} A) \subseteq \operatorname{Spec} B$, the restriction $\pi|_{\operatorname{Spec} A}\colon \operatorname{Spec} A \to \operatorname{Spec} B$ is a morphism of schemes.*

Proof. The forwards direction is immediate, and the backwards direction follows from that morphisms of schemes glue. □

The coproduct, for instance, may be viewed as a trivial gluing construction.

Lemma 2.4.27. *The empty scheme $\operatorname{Spec} 0$ is the initial object in Sch.*

Proof. Notice, $\operatorname{Spec} 0$ has no points. Thus its structure sheaf consists only of global sections, and $\mathscr{O}_{\operatorname{Spec} 0}(\varnothing) = 0$. Since $\varnothing$ is initial in Top, for any scheme X, $\operatorname{Hom}(\operatorname{Spec} 0, X)$ is determined by $\operatorname{Hom}(\mathscr{O}_X, i^*(\mathscr{O}_{\operatorname{Spec} 0}))$, where i is the unique continuous map $\varnothing \to X$. For any open $U \subseteq X$, $i^*(\mathscr{O}_{\operatorname{Spec} 0})(U) = \mathscr{O}_{\operatorname{Spec} 0}(i^{-1}(U)) = \mathscr{O}_{\operatorname{Spec} 0}(\varnothing) = 0$, so each component of the pullback map is uniquely determined. Compatibility is tautological, as all $\mathscr{O}_X(U)$ are mapped onto 0. □

Proposition 2.4.28. *The coproduct in Sch of a set of schemes $\{ X_i \}$ is glued from the datum $(\{ X_i \}, \{ \varnothing \}, \{ \mathrm{id} \})$.*

Proof. It is clear that for all i, $X_i|_\varnothing = \operatorname{Spec} 0$. Hence the usual gluing diagram degenerates into a diagram of disconnected points $\{ X_i \}$, whose colimit is just the coproduct. □

Proposition 2.4.29. *Let $\{ X_i \}$ be an infinite set of non-empty schemes. Then $\coprod X_i$ is not quasicompact.*

Proof. $\{ X_i \}$ is itself an open cover of $\coprod X_i$ admitting no finite subcover. In fact, it admits no subcover whatsoever, as the X_i are disjoint. □

Example 2.4.30. The infinite coproduct $A := \coprod_{i=1}^\infty \operatorname{Spec} A_i$ of a non-empty affine scheme in LRS is not affine, as it is not quasicompact by the above. The infinite coproduct in AffSch, on the other hand, exists as $\operatorname{Spec}(\prod_{i=1}^\infty A_i)$, and is thus quasicompact.
Notice, however, that the two do have the *same* ring of global sections: taking $\{ \operatorname{Spec} A_i \}$ to be the open cover of A, we have by Proposition 1.1.7 that $\mathscr{O}_A(A) = \prod_{i=1}^\infty A_i$, witnessing how general schemes are *not* determined by their global sections.
There was, in fact, no need to compute the ring by hand. Since $\Gamma\colon \mathsf{Sch} \to \mathsf{Ring}^{\mathrm{op}} \cong \mathsf{AffSch}$ is left-adjoint, the gluing colimit gets sent to the same colimit with vertices $\operatorname{Spec} \Gamma(U_{ij})$.

Gluing can also produce fairly pathological spaces.

Definition 2.4.31. Let X_1, X_2 be copies of $\mathbb{A}^1$, from which we form a gluing datum using subspaces $U_{ij} := D_{X_i}(x)$ and transition isomorphisms φ_{ij} induced from $k[x_j]_{x_i} \to k[x_i]_{x_i}$ sending $x_j \rightsquigarrow x_i$. The resultant scheme is the **affine line with doubled origin**.

Intuitively, the affine lines are glued uniformly together everywhere except at the origin, thus leaving a dangling copy thereof. This scheme will thus be our first example of a "non-separated" scheme, with separatedness taken to be the algebro-geometric analog of Hausdorffness.

Proposition 2.4.32. *The affine line with doubled origin is not affine.*

Proof. Suppose for contradiction that the scheme in question, denoted X, is affine. Then since $\Gamma\colon \mathsf{Sch} \to \mathsf{Ring}^{op}$ is left-adjoint, $\Gamma(X, \mathscr{O}_X)$ is the limit of $k[x] \longrightarrow k[x]_x \cong k[y]_y \longleftarrow k[y]$. It is easy to see that $k[t]$ satisfies the universal property, and thus that X is the affine line. It follows that $V(t)$ must be a singleton. But this cannot be the case: by the gluing construction, t vanishes on the images of both $[(x)]$ and $[(y)]$, which are nevertheless not identified in X. $\square$

2.5. Topological Properties

Let us immediately put the minimal machinery developed in the previous section to use by considering some topological properties of schemes and morphisms, which are arguably as simple as properties get.

2.5.1. Quasicompactness, Quasiseparatedness

Definition 2.5.1. A property P of schemes is **finitely additive** if given any scheme X and a finite set $\{ U_i \}$ of open subschemes thereof enjoying P, $P(\bigcup U_i)$ holds.

Proposition 2.5.2. *A scheme is quasicompact iff it can be written as a finite union of affine open subschemes. Hence quasicompactness is finitely additive.*

Proof. The desired finite cover is obtained by applying quasicompactness to the cover of affine opens, and conversely a finite union of quasicompact spaces is quasicompact. □

Proposition 2.5.3. *Every nonempty quasicompact scheme X has a closed point.*

Proof. Let $\{ U_i \}$ be a finite affine cover of X, which may be assumed to be irredundant by finiteness. Then $(\bigcup_{i=2}^n U_i)^c$ is a non-empty closed subset of X, hence U_1. Recall that every closed subset of an affine scheme corresponds to a radical ideal which must be contained in a maximal ideal; this amounts to that $(\bigcup_{i=2}^n U_i)^c$ contains a point x closed in U_1. Now $\overline{\{ x \}} \subseteq (\bigcup_{i=2}^n U_i)^c \subseteq U_1$, so indeed x is closed in X as well. □

Remark. This does *not*, however, imply that the set of closed points is dense in a quasicompact scheme. The white-lie proof goes as follows: the closed points are dense iff every non-empty open subspace U contains a closed point; indeed, U necessarily contains an affine scheme, which is quasicompact, hence a closed point. The problem is that a point is closed in the affine open is not necessarily closed in X, which is an obvious, yet subtle point.

Proposition 2.5.4. *Quasicompactness is affine-respecting. Thus it makes sense to speak of quasicompactness of morphisms.*

Proof. One direction is trivial. Let $V \subseteq Y$ be quasicompact, so that it admits a finite affine cover $\{ U_i \}$. Then evidently $\pi^{-1}(V) = \pi^{-1}(\bigcup U_i) = \bigcup \pi^{-1}(U_i)$ is quasicompact. □

Lemma 2.5.5. *Quasicompactness is strongly affine local.*

Proof. Fix a morphism $\pi\colon X \to Y$; we must show that the property P of an affine open $U \subseteq Y$ that $\pi^{-1}(U)$ is quasicompact is affine-local. We first show locality. Assume that $\pi^{-1}(U)$ is quasicompact, so that it admits a finite affine cover $\{ V_i \}$. Now $\pi|_{V_i}\colon V_i \to U$ is a morphism of affine schemes, so by Proposition 2.2.2 (4) $\pi|_{V_i}^{-1}(D(f))$ is affine. It follows that $\pi^{-1}(D(f)) = \bigcup \pi^{-1}(D(f)) \cap V_i = \bigcup \pi|_{V_i}^{-1}(D(f))$, as desired. Glauability is immediate from finite additivity. $\square$

Corollary 2.5.6. *Quasicompactness is categorical and local on the target.*

Definition 2.5.7. A topological space is **quasiseparated** if the intersection of any two quasicompact open subsets is quasicompact.

Quasiseparatedness belongs to the large swarth of algebro-geometric properties that will replace and extend classical ones, itself being a weak analogue of Hausdorffness. We say weak because even the affine line with doubled origin is quasiseparated (the proof of which will be shortly delayed), whilst its analog in topology is by no means Hausdorff. The correct analog is "separatedness," which will be seen in §2.6 to be a property of the diagonal map $\Delta\colon X \to X \times X$. It turns out that even quasiseparatedness can be stated in terms of Δ, which definition is much more susceptible to categorical methods.

Proposition 2.5.8. *Affine schemes are quasiseparated.*

Proof. Let U, V be quasicompact open subsets of an affine scheme $\operatorname{Spec} A$, so that $U = \bigcup_{i=1}^n D(f_i)$, $V = \bigcup_{i=1}^m D(g_i)$. Now $U \cap V$ is simply $\bigcup_{i,j} D(f_i) \cap D(g_j) = \bigcup_{i,j} D(f_i g_j)$, hence quasicompact. $\square$

The practical utility of quasiseparatedness should be apparent from the outset: often we know that a property of schemes holds for all affines, and that it is quasi-local. Showing that the property holds globally then amounts simply to that $P(U \cap V)$ holds for affine opens U, V. In the frequent case that finite additivity holds, this would follow immediately from quasiseparatedness. Without a finite affine cover, however, this could become entirely intractable: hardly any property is infinitely additive. Additionally, Lemma 2.4.5 is easily strengthened for quasiseparated schemes to furnish *finite* distinguished affine covers.

The correct definition of quasiseparatedness is very cumbersome, as intersections of arbitrary quasicompact opens are difficult to characterize. It turns out that it suffices to check on a single affine cover:

Proposition 2.5.9. *For a scheme X, the following conditions are equivalent:*

1. *X is quasiseparated;*
2. *the intersection of any two affine opens is quasicompact;*
3. *X admits an affine cover in which every pairwise intersection is quasicompact.*

Proof. (1) $\implies$ (2) is immediate from the quasicompactness of affine schemes, and the converse is easily seen if we write both quasicompact opens as a finite union of affines. (2) $\implies$ (3) is obvious, so it remains to show that (3) $\implies$ (2).
Let $\{ U_i \}$ be the cover furnished by (3). Let P be the property of an affine open V that for all U_i, $V \cap U_i$ is quasicompact. The affine-locality of P would imply that it actually holds for all affines opens V, as it does hold on $\{ U_i \}$. Similarly, let Q be the property of an affine open U that for every affine open V of X, $U \cap V$ is quasicompact. P holding for all affine opens amounts to Q holding for $\{ U_i \}$, so (2) would in turn follow from the affine-locality of Q.
P and Q are both properties of an affine open U of the following form: given a set of affine opens $\{ V_i \}$, $U \cap V_i$ is quasicompact for all i. But notice that $U \cap V_i = \pi_i^{-1}(U)$ for $\pi_i \colon U \cap V_i \to U$ the inclusion, so it can be further rephrased as: given a set of morphisms $\{ \pi_i \}$ to U, $\pi_i^{-1}(U)$ is quasicompact for all i. Now, by Lemma 2.5.5, the property of $\pi_i^{-1}(U)$ being quasicompact for a single π_i is affine-local; the result follows from Proposition 2.4.9. $\square$

We adopt the acronym **qcqs** for schemes that are both quasicompact and quasiseparated. What makes these schemes worthwhile of us making a fuss is the following simple characterization:

Corollary 2.5.10. *A scheme X is qcqs iff X admits a finite affine cover in which every pairwise intersection admits a finite affine cover.*

Proof. This is immediate from Proposition 2.5.9 and the definition of quasicompactness. $\square$

(1) $\iff$ (2) in the proposition allows us to apply the affine communication lemma to show that quasiseparatednesss is strongly affine local, using the same trick as in Lemma 2.5.5.

Proposition 2.5.11. *Quasiseparatedness is strictly hereditary and strongly affine local.*

Proof. Let X be a quasiseparated scheme, U an open subscheme thereof. Any two quasicompact open subschemes of U remain quasicompact in X, so by quasiseparatedness their intersection is quasicompact. Hence U is quasiseparated.
Thus to show that it is strongly affine local, it will suffice to show gluability. Namely, given a morphism $\pi \colon X \to Y$, an affine open $U = \bigcup_{i=1}^n D(f_i)$ of Y for which $\pi^{-1}(D(f_i))$

is quasiseparated for all i, we must show that $\pi^{-1}(U)$ is quasiseparated. To this end, let V_1, V_2 be affine open subsets of $\pi^{-1}(U)$. Then $V_1 \cap V_2 = \bigcup_{i=1}^n V_1 \cap V_2 \cap \pi^{-1}(D(f_i)) = \bigcup_{i=1}^n (V_1 \cap \pi^{-1}(D(f_i))) \cap (V_2 \cap \pi^{-1}(D(f_i))) = \bigcup_{i=1}^n \pi|_{V_1}^{-1}(D(f_i)) \cap \pi|_{V_2}^{-1}(D(f_i))$. Evidently it will suffice to show that each summand is compact. Indeed, with $\pi|_{V_i} \colon V_i \to U$ being a morphism of affine schemes, $\pi|_{V_i}^{-1}(D(f_i))$ is affine; now view both as affine opens of $\pi^{-1}(D(f_i))$, which is quasiseparated. $\square$

Motivation for weak-locality: we have to extend properties of affines to a property beyond them.

Definition 2.5.12. A property P of schemes is **weakly local** if for any scheme X, $P(X)$ holds iff there exists an affine cover $\{ U_i \}$ wherein $P(U_i \cup U_j)$ for all i, j.

Proposition 2.5.13. *Quasiseparatedness is weakly local.*

Proof. Let X be a scheme. If X is quasiseparated, then we have by condition (3) of Proposition 2.5.9 an affine cover $\{ U_i \}$ in which every $U_i \cap U_j$ is quasicompact. It follows that for all i, j, $U_i \cup U_j$ is quasiseparated, with $\{ U_i, U_j \}$ as the needed cover. Conversely, let $\{ U_i \}$ be an affine cover of X wherein $U_i \cup U_j$ is quasiseparated for all i, j. Then since U_i, U_j are affine opens of $U_i \cup U_j$, $U_i \cap U_j$ is quasicompact. It follows that X is quasiseparated. $\square$

Proposition 2.5.14. *Fix a morphism $\pi \colon X \to Y$. If for all affine opens $U \subseteq Y$, $\pi^{-1}(U)$ is quasiseparated, then for all qcqs open $V \subseteq Y$, $\pi^{-1}(V)$ is quasiseparated.=*

Proof. Fix a qcqs open $U = \bigcup_{i=1}^n U_i$ of Y for which $\pi^{-1}(U_i)$ is quasiseparated for all i. We must show that $\pi^{-1}(U)$ is quasiseparated. To this end, let V_1, V_2 be affine open subsets of $\pi^{-1}(U)$. Then $V_1 \cap V_2 = \bigcup_{i=1}^n V_1 \cap V_2 \cap \pi^{-1}(U_i) = \bigcup_{i=1}^n (V_1 \cap \pi^{-1}(U_i)) \cap (V_2 \cap \pi^{-1}(U_i))$. Evidently it will suffice to show that each summand is compact; in fact, with each $\pi^{-1}(U_i)$ being quasiseparated, this amounts to showing each $W \cap \pi^{-1}(U_i)$ is quasicompact, denoting by W either V_1 or V_2. Since W is quasicompact, it admits a finite affine cover $\{ W_j \}$ where each W_j is contained in some $\pi^{-1}(U_{\alpha(j)})$. Then with $\{ W_j \cap \pi^{-1}(U_i) \}$ a finite cover of $W \cap \pi^{-1}(U_i)$, it in turn suffices to show that each $W_j \cap \pi^{-1}(U_i)$ is quasicompact. Notice, $W_j \cap \pi^{-1}(U_i) = \pi|_{W_j}^{-1}(U_i)$, where $\pi|_{W_j} \colon W_j \to U_i \cup U_j$. We claim that this is a quasiseparated morphism. Indeed, any affine open $O \subseteq U_i \cup U_j$ is an affine open of Y, so $\pi^{-1}(O)$ is quasiseparated. Now $\pi|_{W_j}^{-1}(O) = \pi^{-1}(O) \cap W_j$, which must be quasiseparated by strict heredity. But $U_i \cup U_j$ is quasiseparated, so the result follows from supposition. (bad) $\square$

Proposition 2.5.15. *Let P be a property on open subschemes of a fixed scheme X. If P is weakly local, strictly hereditary, and holds for affines, then P is affine-respecting.*

Proof. Fix a morphism $\pi\colon X \to Y$. That $\hat{P}(\pi) \implies \tilde{P}(\pi)$ is immediate, as P holds on affines. For the converse, we first show that $\tilde{P}(\pi)$ implies that for all quasicompact open $V \subseteq Y$ such that $P(V)$, $P(\pi^{-1}(V))$ holds. TODO, see above.
Now let $V \subseteq Y$ be any open subscheme enjoying P, $\{ W_i \}$ an affine cover thereof wherein $P(W_i \cup W_j)$ for all i, j not necessarily distinct. It follows by the above that $P(\pi^{-1}(W_i) \cup \pi^{-1}(W_j))$ holds for all i, j. For each i, let $\mathcal{U}_i := \{ U_{ij} \}$ be an affine cover of $P(\pi^{-1}(W_i))$. Then $\bigcup_i \mathcal{U}_i$ is an affine cover of $\pi^{-1}(V)$, wherein $U_{ij} \cup U_{kl} \subseteq \pi^{-1}(W_i) \cup \pi^{-1}(W_k)$ enjoys P for all i, j, k, l by strict heredity. Hence $P(\pi^{-1}(V))$ holds by weak locality. ☐

Corollary 2.5.16. *Quasiseparatedness is affine-respecting. Thus it makes sense to speak of quasiseparatedness of morphisms.*

Proposition 2.5.17. *Any morphism from a quasiseparated scheme is quasiseparated.*

Proposition 2.5.18. *The affine line with doubled origin is quasiseparated.*

Proof. Let us take the standard cover consisting of the images of the two affine lines. Their intersection is the complement of the two origins. ☐

We can, however, already conjure a space that is definitively not quasiseparated.

Example 2.5.19. Affine ∞-space with doubled origin.

2.5.2. Irreducible Spaces

Irreducible subspaces have proven to be a natural and essential notion in the affine case, corresponding precisely to affine varieties. It turns out to be just as applicable in the case of general schemes (c.f. Theorem 2.5.27). Nevertheless, irreducibility is ultimately a *topological* notion, and can be made sense of in arbitrary topological spaces (though Sch is where it really shines). As such, we enter here into a discussion of some implications immediate from its definition.

Proposition 2.5.20. *Any non-empty open subspace of an irreducible topological space is dense and itself irreducible.*

Proof. Let X be irreducible. The complement of a non-empty non-dense open set U is a non-empty proper closed set. Now $U \cup U^c = X \implies \overline{U} \cup U^c = X$; but $\overline{U}$ is also a proper closed set. Now if U were reducible, then there would be disjoint non-empty opens $U_1, U_2 \subseteq U$. But with U being itself open, U_1, U_2 are disjoint non-empty open subsets of X, contradicting irreducibility. ☐

Recall that an irreducible closed subspace can be thought of as a "piece" of a space; it is then reasonable to ask for the maximal *components* by which the space is made up of.

Definition 2.5.21. Fix a topological space X. An **irreducible component** of X is a maximal irreducible subset.

Remark. Now this may be juxtaposed with another familiar notion concerning *division* of a space: *connected components.* The key difference lies in that irreducible component, unlike connected ones, need not be disjoint. Furthermore, the decomposition into irreducibles is always finer:

Proposition 2.5.22. *Irreducible topological spaces are connected.*

Proof. This is tautological, if X is disconnected, then it is the disjoint union of non-empty opens U, V. It then certainly reducible. $\square$

Remark. The converse is not true. A simple counterexample is provided by the affine prevariety $\operatorname{Spec}\mathbb{C}[x,y]/(xy)$, which is manifestly reducible: $V(xy) := V(x) \cup V(y)$. It is, however connected.

Proposition 2.5.23. *A subspace is irreducible iff its closure is. Hence irreducible components are closed.*

Proof. Let $V \subseteq X$ be a subspace. If V is reducible, then $V = V_1 \cup V_2$ for V_1, V_2 proper and closed in V. Then there exists K_1, K_2 closed in X such that $V_i = K_i \cap V$. Put $W_i := K_i \cap \overline{V}$. Manifestly, $W_1 \cup W_2$ cover $\overline{V}$. They are also proper, as $V_i \setminus K_i \subseteq \overline{V_i} \setminus K_i$. For the converse, let $\overline{V}$ be reducible, so that $\overline{V} = W_1 \cup W_2$ for W_1, W_2 proper and closed in $\overline{V}$, hence X. Then $V_i := W_i \cap V$ is closed, and manifestly $V = V_1 \cup V_2$. Furthermore, if V_i were not proper, then it would be a closed set containing V which is properly contained in the closure. $\square$

The following proposition shows that *irreducible components* indeed deserve their name:

Proposition 2.5.24. *Every topological space X is the union of its irreducible components.*

Proof. We must show that every point $p \in X$ is contained in an irreducible component. To this end, consider the poset I of all irreducible subsets containing p, ordered by inclusion. Since manifestly $\{ p \}$ is contained in I, it will suffice to show that every chain in I has an upper bound, so that Zorn's lemma completes the proof. We claim that an upper bound of any chain S is given by $\bigcup S$. Since containment is evident, this amounts to showing that

$\bigcup S$ is irreducible. Assume to the contrary that it contains non-empty disjoint open subsets U_1, U_2. Then there are elements S_1, S_2 of S such that $S_i \cap U_i \neq \varnothing$. Now S is a chain, so we may assume without loss of generality that S_1 contains S_2, which in turn implies that $S_1 \cap V \neq \varnothing$ as well. But then $S_1 \cap U$, $S_1 \cap U$ are disjoint non-empty open subsets of S_1, contradicting its irreducibility. □

In the finite case, the irreducible decomposition is unique:

Proposition 2.5.25. *Let X be a topological space. If $X = \bigcup_{i=1}^n X_i$ for each X_i a irreducible closed subset, then $\{$ irreducible components of $X \} \subseteq \{ X_i \}$. The opposite inclusion holds if $\{ X_i \}$ is irredundant.*

Proof. Let Y be an irreducible component of X. Then it is covered by $\{ Y \cap X_i \}$; in fact, it must be equal to one of them by irreducibility (for convenience, we can first refine $\{ X_i \}$ into an irredundant cover). But $Y \subseteq X_i \implies Y = X_i$ by maximality. Now assume that $\{ X_i \}$ is irredundant. Then each X_i is contained in an irreducible component Y, which has been shown to be X_j for some j. But by irredundance, it must be that $X_i = X_j = Y$. □

It is now easy to see that the irreducible components of an open subspace are intimately connected to those of the ambient space which intersect the subspace.

Theorem 2.5.26. *Let X be a topological space, U an open subspace. Then there is an inclusion-preserving bijection*

$$\{ \textit{irreducible closed subsets of } X \textit{ meeting } U \} \underset{g}{\overset{f}{\rightleftarrows}} \{ \textit{irreducible closed subsets of } U \},$$

where f sends $Z \rightsquigarrow Z \cap U$, g sends $K \rightsquigarrow \overline{K}$. In particular, this restricts a bijection between irreducible components of X meeting U and irreducible components of U.

Proof. That f is well-defined follows from Proposition 2.5.20, viewing $Z \cap U$ as an open subspace of Z. That of g is immediate from Proposition 2.5.23. Now $g \circ f(Z) = \overline{Z \cap U}_X = \overline{Z \cap U}_Z = Z$, since $Z \cap U$ is a non-empty open subspace of Z, hence dense in Z. On the other hand, $f \circ g(K) = \overline{K}_X \cap U = \overline{K}_U = K$. Both maps are evidently inclusion-preserving, and thus restricting the bijection to maximal elements, namely irreducible components, is warranted. □

Remark. It is somewhat surprising that Theorem 2.5.26 coupled with Proposition 2.2.43 does not yield new insights on points. Let us consider, for instance, an affine scheme Spec A

and the open subspace U corresponding to $\operatorname{Spec} A_{\mathfrak{p}}$ for some prime ideal $\mathfrak{p}$. The theorems together tell us that the points of U correspond to points of X whose closures meet U. Namely, for any such point $\mathfrak{x}$, there exists some $\mathfrak{q} \in V(\mathfrak{x})$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. But then $\mathfrak{x} \subseteq \mathfrak{q}$, and thus this condition amounts to merely that $\mathfrak{x} \subseteq \mathfrak{p}$: the original description of $\operatorname{Spec} A_{\mathfrak{p}}$ from Corollary 2.2.24.

We now show the analog of Proposition 2.2.43 for general schemes.

Theorem 2.5.27. *There is a bijection between irreducible closed subsets and points of any scheme X.*

Proof. Consider the map V sending each point in X to its closure, which is irreducible by Proposition 2.5.23. We claim that the inverse I is given by the map sending an irreducible closed subset K to the generic point x of $K \cap U$, which is unique by Corollary 2.2.44, for any affine open U intersecting K non-trivially, which can be taken to be the affine neighborhood of any point in K.

We must show that the choice of affine open is immaterial. Since $K \cap U$ is a non-empty open, hence dense subset of K, we see by Proposition 2.2.10 that x is a generic point of K. The same proposition asserts that x is the unique generic point of $K \cap V$ for any other affine open V meeting K. Thus I is well-defined.

We have shown in passing that $V \circ I = \mathrm{id}_{\mathrm{irred}}$, i.e. that $\overline{\{\, x \,\}}_X = K$ for $x = I(K)$. On the other hand, $I \circ V = \mathrm{id}_{\mathrm{pt}}$ amounts to that for any $x \in X$, the generic point of $\overline{\{\, x \,\}}_X \cap U$ is x, which, of course, follows from that x is the generic point of $\overline{\{\, x \,\}}_X$. $\square$

Remark. One might think that compared to the situation in AffSch, some is lost: whereas the affine bijection between prime ideals and irreducible closed subsets was derived from the larger bijection between radical ideals and arbitrary closed subsets, the one here is deduced purely topologically, unable to impart any algebraic information. This tragic state of things is due not to the generality of arbitrary schemes, but our present lack of technology. In §2.8, once quasicoherent ideal sheaves and closed embeddings have been defined, we will be able to state a bijection involving all closed subsets.

It is clear that a space can be qualified by conditions on its irreducible subspaces. For schemes, the requirement that the set of irreducible components be finite, surprisingly, is intimately connected to the fundamental property of *Noetherian*ness of a ring. On the other hand, it turns out that lengths of chains of irreducible subspaces indicate the *dimension* of a space, in the algebraic sense that we have used in §2.2. These will be the two topics we explore in the remainder of this section.

2.5.3. Noetherian Conditions

Definition 2.5.28. A topological space X is **Noetherian** if it satisfies the *descending chain condition* that any chain $Z_1 \supseteq Z_2 \supseteq \dots$ of closed subsets eventually stabilizes, i.e. there exists an $n \in \mathbb{Z}^+$ such that $Z_n = Z_{n+i}$ for all $i \in \mathbb{N}$. Equivalently, the open subsets satisfy the evident **ascending chain condition**.

Proposition 2.5.29. *Let X be a topological space. The following are equivalent:*

1. *X is Noetherian.*
2. *Every non-empty set of open subsets of X, partially ordered by inclusion, has a maximal element.*
3. *Every open subspace of X is quasicompact.*
4. *Every subspace of X is quasicompact.*

Proof. (1) $\implies$ (2) is immediate from Zorn's lemma and the ascending chain condition. For (2) $\implies$ (3), let U be an open subspace, $\{ U_\alpha \}$ an open cover thereof. Let S be the set of of all finite unions of elements of the cover. We claim that the maximal element $\bigcup_{i \in I} U_i$ contains U, for I a finite indexing set. Indeed, let $x \in U \setminus \bigcup U_i$, so that $x \in U_\alpha$ for some $\alpha \notin I$. In particular, $x \in \bigcup U_i \cup U_\alpha$. But this is a finite union, so it cannot be that $\bigcup U_i \subsetneq \bigcup U_i \cup U_\alpha$, a contradiction. For (3) $\implies$ (4), let Y be a subspace. Let $\{ U_\alpha \} \subseteq \mathsf{Op}(Y)$ be an open cover. For each i, there exists $V_\alpha \subseteq \mathsf{Op}(X)$ such that $U_\alpha = V_\alpha \cap Y$. In particular $Y = (\bigcup V_\alpha) \cap Y$. Now $\bigcup V_\alpha$ is open, and, being quasicompact, admits a finite subcover $\{ V_i \}$. It follows that $Y = (\bigcup V_i) \cap Y = \bigcup (V_i \cap Y) = \bigcup U_i$. That is, Y is quasicompact, as needed. It remains to show that (4) $\implies$ (1). Let $U_1 \subseteq U_2 \subseteq \dots$ be an ascending chain of open subsets. Then $\bigcup_{i=1}^{\infty} U_i$ is quasicompact, and thus admits a finite subcover $\{ U_i \}_{i \in I}$. Take $m := \max(I)$. Evidently, $\bigcup_{i \in I} U_i = U_m$, as $U_i \subseteq U_j$ for all $j \geq i$. It follows that the chain terminates at m: for all $n \geq m$, $U_n \subseteq \bigcup_{i=1}^{\infty} U_i = U_m$. □

Corollary 2.5.30. *A finite union of Noetherian spaces is Noetherian.*

Proof. Let $X = \bigcup X_i$, U be an open subspace. Then $U = \bigcup U \cap X_i$ is a finite union of quasicompact spaces, hence itself quasicompact. □

It is then evident that knowing a scheme is Noetherian will be immensely useful in pratice, without which condition we would have to work very hard to exhibit quasicompact subsets of even affine opens (hence distinguished bases): in general, quasicompactness is not hereditary. On the other hand, condition (2) in the above yields a form of transfinite *induction* applicable to Noetherian spaces, which makes working with them even more pleasant.

Definition 2.5.31. A partially ordered set $(S, <)$ is **well-founded** if any non-empty subset T has a minimal element with respect to $<$.

Proposition 2.5.32. *A topological X is Noetherian iff the set of closed subsets of X is well-founded.*

Proof. This is immediate from (2) in Proposition 2.5.29, as every non-empty set of closed subsets arises from taking the complement of each element of a non-empty set of open subsets. $\square$

Lemma 2.5.33 (Principle of Noetherian Induction)**.** *Let $(X, <)$ be a well-founded set, P a property of elements of X. Then if for all $y \in X$, $P(z)$ for all $z < y$ implies $P(y)$, then $P(x)$ for all $x \in X$.*

Proof. Let S be the set of elements of X for which $P(x)$ does not hold. Since X is well-founded, S admits a minimal element m. But then it is vacuously true that $P(z)$ for all $z < y$, as such z do not exist, so $P(m)$ holds, a contradiction. Hence $S = \varnothing$, as desired. $\square$

The connection of the Noetherian condition to irreducible components now becomes clear.

Proposition 2.5.34. *Let X be a Noetherian topological space. Then every non-empty closed subset Y admits a unique irredundant cover $\{ Y_i \}$ by irreducible closed subsets. Hence Noetherian topological spaces have finitely many irreducible components.*

Proof. We proceed by Noetherian induction. Let $Y \subseteq X$ be a non-empty closed subset such that every closed $Z \subsetneq Y$ can be written as an irredundant finite union of irreducible closed subsets. Note that Y may be assumed to be reducible, as otherwise it is a one-fold union. Thus Y can be written as the union of two proper non-empty closed subsets. But with both being an irredundant finite union of irreducible closed subsets, so is Y. Uniqueness follows from Proposition 2.5.25. $\square$

We now reveal the promised connection to Noetherian rings, which by now should come as a surprise, given the bijection between the closed subsets of an affine scheme $\operatorname{Spec} A$ and the radical ideals of A.

Proposition 2.5.35. *If a ring R is Noetherian, then* $\operatorname{Spec} R$ *is a Noetherian topological space.*

Proof. Let $Z_1 \supseteq Z_2 \supseteq \dots$ be a descending chain of closed subsets in $\operatorname{Spec} R$. By Theorem 2.2.41, this corresponds to an ascending chain $I(Z_1) \subseteq I(Z_2) \subseteq \dots$ of radical ideals in R, which must terminate at say n. Applying $V(-)$ yields back the original descending chain, which we now know terminates. □

Remark. It is intuitively clear that the converse cannot be true, as an ascending chain of arbitrary ideals does not give rise to a descending chain of closed subsets. Indeed, consider $k[x_1, x_2, \dots]/(x_1, x_2^2, \dots)$. This is a non-Noetherian ring whose spectrum is nevertheless the one point space.

Proposition 2.5.36. *The minimal prime ideals of a ring A are in bijection with the irreducible components of* $\operatorname{Spec} A$. *Hence* $\operatorname{Spec} A$ *is irreducible iff A has only one minimal prime ideal.*

Proof. This is immediate from Proposition 2.2.43. □

Corollary 2.5.37. *Noetherian rings have finitely many minimal prime ideals.*

Let us end with a discussion of *Noetherian schemes*, which are unexpectedly not defined as merely having Noetherian underlying spaces.

Definition 2.5.38. A scheme is **locally Noetherian** if it admits an affine cover $\{ U_i \}$ in which each U_i is isomorphic to the spectrum of a Noetherian ring. A locally Noetherian and quasicompact scheme is **Noetherian**.

Proposition 2.5.39. *The underlying topological space of a Noetherian scheme is Noetherian. Hence it has finitely many irreducible components.*

Proof. This is immediate from Corollary 2.5.30. □

Proposition 2.5.40. *Locally Noetherian schemes are quasiseparated.*

Proof. Let X be a locally Noetherian schemes, so that it admits a cover $\{ U_i \}$ by affine schemes isomorphic to spectra $\operatorname{Spec} A_i$ of Noetherian rings. For any i, j, $U_i \cap U_j$ is an open subspace of U_i, hence quasicompact. □

When working with affine schemes, affine-local properties depend *only* on the underlying rings, witnessing its versatility in that it can be employed even to purely algebraic problems.

Theorem 2.5.41. *A ring $A = (f_1, \ldots, f_n)$ is Noetherian iff for all i, A_{f_i} is Noetherian. Hence the property P of affine schemes of being the spectrum of a Noetherian ring is affine-local.*

Proof. The forward direction follows from the stability of Noetherianness under localization. For the converse, suppose for contradiction that there is a strictly increasing chain of ideals $J_1 \subsetneq J_2 \subsetneq \ldots$ in A. For each i, consider $J_1 \otimes_A A_{f_i} \subsetneq J_2 \otimes_A A_{f_i} \subsetneq \ldots$. TODO The implication follows from the affine communication lemma. □

Corollary 2.5.42. *An affine scheme is the spectrum of a Noetherian ring iff it is a locally Noetherian scheme.*

Proof. One direction is immediate. An affine scheme that is locally Noetherian is certainly Noetherian, so it admits a finite affine cover $\{ U_i \}$ where each U_i is isomorphic to the sepctrum of a Noetherian. Since distinction is transitive and Noetherianness is stable under localization, we may assume that each U_i is a distinguished open $D(f_i)$. The result then follows from the above. □

Proposition 2.5.43. *Any morphism from a Noetherian scheme is quasicompact.*

2.5.4. Krull Dimension

The topological notion of dimension to be defined momentarily does not align with classical expectations. For instance, it deems $\mathbb{R}^n$ as 0-dimensional for any $n \in \mathbb{N}$. Instead, it aims to capture the stratification mentioned in §2.2, or more precisely, the maximum number of nested strata that can be fit in a space.

Definition 2.5.44. The **(Krull) dimension** of a topological space X is the integer

$$\dim X := \sup\{ \ell(C_0 \subsetneq C_1 \subsetneq \ldots) \colon C_i \text{ is closed and irreducible} \}.$$

The **(Krull) dimension** of a ring A is defined analogously as

$$\dim A := \sup\{ \ell(\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \ldots) \colon \mathfrak{p}_i \text{ is a prime ideal} \}.$$

Remark. This definition is not as foreign as it appears. While not necessarily presented as such in a first course on linear algebra, the dimension of a vector space can be defined as the supremum of the length of ascending chains of subspaces. This may seem like an unnecessary extravagance in the linear context: the deciding chain will always arise from a choice of basis, whose cardinality determines the dimension. But it is nevertheless reassuring to know that the Krull dimension has an umbral presence from the very start.

Proposition 2.5.45. *For A a ring,* $\dim A = \dim \operatorname{Spec} A$.

Proof. This is immediate from Proposition 2.2.43. □

Dimension brings about the fruit of our classifications of prime ideals:

Example 2.5.46. A prime ideal of a PID A (say $\mathbb{Z}$ or $k[t]$) is either 0 or (f) for some irreducible $f \in A$, so $\dim A = 1$. $k[x,y]$ possesses one further type of prime ideal, so is 2-dimensional. Meanwhile, k and $k[\epsilon]/(\epsilon^2)$, haing 1-point underlying spaces, are 0-dimensional.

Proposition 2.5.47. *Let us determine the affine n-space over $\mathbb{Z}$. Evidently even $\mathbb{A}^1_{\mathbb{Z}}$ will not be a PID, as $\mathbb{Z}$ is not a field. It is, however, integral. Consider the morphism induced by the inclusion $i\colon \mathbb{Z} \hookrightarrow \mathbb{Z}[x_1, \dots, x_n]$. For each prime $\mathfrak{p} = (p) \in \mathbb{Z}$, we have $i^{-1}(\mathfrak{p}) = i^{-1}(V(\mathfrak{p})) = V(i(\mathfrak{p})) \cong \operatorname{Spec} \mathbb{Z}[x_1, \dots, x_n]/(p) \cong \mathbb{A}^n_{\mathbb{F}_p}$. Now 0 is not a closed point, so this computation doesn't work for it. $\pi^{-1}(0) = \{\, \mathfrak{q} \colon \mathfrak{q} \cap \mathbb{Z} = 0 \,\}$. We claim that this is homeomorphic to $\operatorname{Spec} \mathbb{Q}[x_1, \dots, x_n]$ via the map $\mathfrak{q} \rightsquigarrow \mathfrak{q}\mathbb{Q}[x_1, \dots, x_n]$.*
The outcome of this consideration is a taxonomy of the prime ideals of $\mathbb{Z}[x_1, \dots, x_n]$ with respect to prime integers; on top of each $\mathbb{F}_p[x_1, \dots, x_n]$, there is the point (0), so the dimension of $\mathbb{A}^n_{\mathbb{Z}}$ is $\dim k[x_1, \dots, x_n] + 1$.

The dimension of a reducible space will be the supremum of that of its irreducible components. A space is **equidimensional**, or **pure dimension** n if its irreducible components have the same dimension.

Proposition 2.5.48. *A scheme has dimension n iff it admits an open cover by affine open subsets with dimensions having n as their supremum.*

Proof. Let $C_1 \subsetneq \cdots \subsetneq C_n$ be a chain of irreducible closed subsets. Theorem 2.5.26. Then any $x \in C_1$ lies in all C_i and admits an affine open neighborhood U. The chain thus gives rise to a length n chain of closed irreducibles in U. Evidently any affine open V cannot have a dimension exceeding n, as then taking the closure of the defining chain would yield a chain of closed irreducible in X with length exceeding n. The converse is easy: immediately we have that the dimension is at least n. By the same argument as above, a chain of length exceeding n can be restricted to one affine. □

Proposition 2.5.49. *A Noetherian scheme of dimension* 0 *has a finite number of points.*

Proof. Noetherian schemes have finitely many irreducible components, each of which must be a point as otherwise we have a chain of length at least 1. $\square$

Proposition 2.5.50. *A surjection of* $\varphi\colon A \to B$ *of rings, where* A, B *are of the same finite dimension and* B *is an integral domain, an isomorphism.*

Proof. It will suffice to show injectivity, i.e. that $\ker\varphi = 0$. Now $B \cong A/\ker\varphi$, so $n := \dim A = \dim B = \dim A/\ker\varphi$. Fix a chain of primes in $A/\ker\varphi$ of length n, which corresponds to a length-n chain of primes in A containing $\ker\varphi$. But since B is integral, $\ker\varphi$ is prime, and if it is non-zero, we may add 0 to the chain to make it length $n+1$. $\square$

Definition 2.5.51. The **codimension** of an *irreducible* subspace $Y \subseteq X$ is the integer

$$\operatorname{codim}_X Y := \sup\big\{\, \ell(C_0 = \overline{Y} \subsetneq C_1 \subsetneq \dots) \colon C_i \text{ is closed and irreducible} \,\big\}.$$

The **codimension** or **height** of a prime ideal $\mathfrak{p} \subseteq A$ is defined analogously as

$$\operatorname{codim}_A \mathfrak{p} := \sup\{\, \ell(\mathfrak{p}_0 = \mathfrak{p} \supsetneq \mathfrak{p}_1 \supseteq \dots) \colon \mathfrak{p}_i \text{ is prime} \,\}.$$

Remark. Naively, one would think to define codimension as $\dim X - \dim Y$. This violates the *locality* of this notion, which should remain invariant under suitable restrictions of the "ambient" space. For instance, let us consider the disjoint union of a point and a line. The naive codimension of the point is 1, but intuitively it should be 0, as the dimensionality of the line does not affect the vicinity of the point. There is thus risk of ambiguity in defininig codimension for non-irreducible subspaces.
Note that the supremum definition again applies in linear algebra.

Proposition 2.5.52. $\operatorname{codim}_{\operatorname{Spec} A}[\mathfrak{p}] = \operatorname{codim}_A \mathfrak{p} = \dim A_{\mathfrak{p}}$.

Proof. Recall that prime ideals in $\operatorname{Spec} A_{\mathfrak{p}}$ are in inclusion-preserving bijection with those of A contained in $\mathfrak{p}$. $\square$

This readily extends to schemes:

Proposition 2.5.53. *For* Y *an irreducible subspace of a scheme* X *with a generic point* η*,* $\operatorname{codim}_X Y = \dim \mathscr{O}_{X,\eta}$.

Proof. Note that Y may be assumed closed, as $\operatorname{codim}_X Y = \operatorname{codim}_X \overline{Y}$. Fix an affine neighborhood U of η. We claim that $\operatorname{codim}_U U \cap Y = \operatorname{codim}_X Y$; the result then follows from the above, as $U \cap Y = \overline{\{\eta\}}$. Indeed, since the bijection of Theorem 2.5.26 is inclusion-preserving, irreducible closed subspaces in U containing $U \subseteq Y$ correspond to those in X containing Y. □

Proposition 2.5.54. *If Y is an irreducible closed subset of a scheme X, then* $\operatorname{codim}_X Y + \dim Y \leq \dim X$.

Proof. A chain in Y together with a chain in X starting from $\overline{Y}$ gives rise to a chain in X, so the sum of their lengths would be bounded above by $\dim X$. □

2.6. The Relative Point of View

A key insight of Grothendieck that is the cornerstone of modern algebraic geometry is that *morphisms should take precedence over objects*. One already takes up the relative point of view, albeit unknowingly, when working with field extensions: instead of studying a fixed field in isolation, we study morphisms out of a base field and work in the *coslice category* Fld/k. Since Spec is contravariant, field extensions evidently correspond to morphisms $\operatorname{Spec} F \to \operatorname{Spec} k$. A category of *relative schemes* is thus a *slice category* over a fixed *base scheme*.

Definition 2.6.1. Fix a scheme S. The **category of S-schemes** is the slice category Sch/S whose objects are **structural morphisms** $X \to S$. In this case, S is said to be the **base scheme**. Morphisms between S-schemes X, Y are scheme morphisms $X \to Y$ making the obvious triangle commute.

Remark. The morphism underlying an S-scheme X is "structural" in the sense that composing the global pullback map $\Gamma(S, \mathscr{O}_S) \to \Gamma(X, \mathscr{O}_X)$ with the suitable restriction gives a $\Gamma(S, \mathscr{O}_S)$-algebra structure on every $\mathscr{O}_X(U)$. Note that if $S \cong \operatorname{Spec} A$, a structure map $X \to S$ is determined by the ring homomorphism $A \to \Gamma(X, \mathscr{O}_X)$, so it will be of no harm in this case to refer to S-schemes as A-schemes.

Proposition 2.6.2. $\mathsf{Sch} \cong \mathsf{Sch}/\mathbb{Z}$. *More generally,* $\operatorname{Spec} A$ *is final in the category of* A*-schemes.*

Proof. This is immediate from Lemma 2.4.21 and that in any slice category C/x, x is final. □

Thanks to this proposition, results about S-schemes will always be applicable to traditional schemes, so it is at the very least harmless to adopt the relative point of view. Another proponent of its legitimacy is the Yoneda lemma: to determine an object X up to canonical isomorphism, we need only understand the representable functor h_X, whose data consists of all morphisms to X. Namely, all there is to a scheme X is captured in Sch/X.

Definition 2.6.3. The **functor of points** of a scheme X is its image $h_X := \operatorname{Hom}(-, X)$ under the contravariant Yoneda embedding $h \colon \mathsf{Sch} \to \mathsf{Set}^{\mathsf{Sch}}$. In the sequel we will use X as a shorthand for h_X, and call the elements of $X(Z)$ **Z-valued points** of X.

Let us recall the Yoneda Lemma in order to introduce some idiosyncratic notation:

Lemma 2.6.4 (Yoneda Lemma)**.** *Given a scheme X, a contravariant functor $F\colon \mathsf{Sch}^{op} \to \mathsf{Set}$, we have a natural bijection $\mathrm{Hom}(X, F) \cong F(X)$ sending $\Phi\colon X \to F$ to $\Phi_X(\mathrm{id}_X)$ and $\zeta \in F(X)$ to the natural transformation whose component on T sends a morphism $f\colon T \to X$ to $F(f)\zeta$, which we write as $f^*\zeta$ or $\zeta|_T$.*

The Yoneda lemma in particular yields a criterion for detecting representable functors, which in fact underlies all universal properties. Note that the criterion is stated entirely within Set, allowing us to invoke set-theoretic constructions rather than construct everything categorically by hand.

Proposition 2.6.5. *Let X be a scheme, F a functor $\mathsf{Sch}^{op} \to Set$, $\Phi\colon X \to F$ a morphism, $\zeta \in F(X)$ the element associated to Φ. Then Φ is an isomorphism iff ζ is the **universal family** of F, in the sense that for every scheme T, $x \in F(T)$, there exists a unique morphism $f\colon T \to X$ for which $x = F(f)\zeta$.*

Proof. Recall that $\operatorname{im}\Phi_T$ consists entirely of elements of the form $F(g)\zeta$, where $g\colon T \to X$ is a morphism. That the desired f exists and uniquely determined is exactly the injectivity and surjectivity of Φ. □

Example 2.6.6. Consider the global sections functor $\Gamma\colon \mathsf{Sch}^{\mathrm{op}} \to Set$ sending $X \rightsquigarrow \Gamma(X, \mathscr{O}_X)$, where the latter is regarded as a set, and $f\colon X \to Y$ to the pullback $f^{\#}\colon \Gamma(Y, \mathscr{O}_Y) \to \Gamma(X, \mathscr{O}_X)$. This functor is represented by $\operatorname{Spec}\mathbb{Z}[x]$: for all schemes X, we have $\Gamma(X, \mathscr{O}_X) \cong \mathrm{Hom}(\mathbb{Z}[x], \Gamma(X, \mathscr{O}_X)) \cong \mathrm{Hom}(X, \operatorname{Spec}\mathbb{Z}[x])$. The universal family is then $\Phi(\mathrm{id}_{\operatorname{Spec}\mathbb{Z}[x]})$, which corresponds to the identity on $\mathbb{Z}[x]$, and hence x.

The lemma can be strengthened when working over A-schemes: it suffices to consider the restriction of the functor of points to the category of affine schemes:

Theorem 2.6.7. *For A a ring, the composition*

$$\mathsf{Sch}/A \xrightarrow{h} \mathsf{Set}^{pre}_{\mathsf{Sch}/A} \xrightarrow{r} \mathsf{Set}^{pre}_{\mathsf{AffSch}/A} \xrightarrow{\simeq} \mathsf{Set}^{A\text{-}\mathsf{Alg}}$$

is fully faithful.

Proof. We need only show that r is fully faithful on the essential image of h. That is, for any A-schemes h_X, h_Y, the map $\mathrm{Hom}(h_X, h_Y) \to \mathrm{Hom}(h^a_X, h^a_Y)$ induced by r, where, for instance, by h^a_X we understand $r(h_X)$, is a bijection. But this is merely that morphisms glue: given a morphism $\varphi\colon h^a_X \to h^a_Y$, we define $\psi\colon h_X \to h_Y$ on any R-scheme U as the map sending a morphism $\pi\colon U \to X$ to the unique morphism $U \to Y$ glued from $\varphi_{U_i}(\pi|_{U_i})$, for $\{ U_i \}$ an affine cover of U. $\square$

In particular, schemes are equivalently functors $\mathsf{Ring} \to \mathsf{Set}$, subject to some conditions. To illustrate how the functorial point of view is genuinely useful in practical, we note that affine schemes now acquire a particularly terse description as a functor:

Corollary 2.6.8. *An affine R-scheme* $\operatorname{Spec} A$ *corresponds to the covariant* $h^A\colon R\text{-}\mathsf{Alg} \to \mathsf{Set}$ *sending* $B \rightsquigarrow \operatorname{Hom}(A, B)$.

Proof. Recall that the final equivalence in the above composition is brought about by the global sections functor. As such, $h_{\operatorname{Spec} A}$ sending $\operatorname{Spec} B$ to $\operatorname{Hom}(\operatorname{Spec} B, \operatorname{Spec} A)$ becomes the prescribed functor. $\square$

Notice how we were able to drop the entire technical baggage comprised of prime ideals, open sets, and the structure sheaf. Yet another advantage of working with morphisms then surfaces: they require less data to define. This is not particular to Sch: recall that to define functors, we must specify the target of **both** objects and morphisms, whereas for natural transformation we only need one morphism for each object of the mutual domain category, the rest of the data relegated to the *property* of naturality. In many future situations, whereas defining a scheme explicitly might be borderline impossible, specifying its functor of points can be much easier.

For instance, in Sch/k, the affine space $\mathbb{A}^n_k$ corresponds to the functor sending $B \rightsquigarrow B^n$, as the polynomial ring is then the free k-algebra on n generators. For arbitrary varieties, the functor can be interpreted in a way just as down-to-earth.

Proposition 2.6.9. *In* Sch/k*,* $\operatorname{Spec} k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$ *corresponds to the functor sending A to the set of A-valued solutions to the homogeneous equations* $f_1(x_1, \ldots, x_n) = \cdots = f_r(x_1, \ldots, x_n) = 0$.

Proof. We are merely computing $\operatorname{Hom}(k[x_1, \ldots, x_n]/(f_1, \ldots, f_r), A)$. By the universal property of the quotient, to induce such a morphism from a morphism $\mathbb{Z}[x_1, \ldots, x_n] \to A$, which assigns a value to each x_i, it is necessary and sufficient for all f_i to vanish. $\square$

TODO: Group schemes

Crucially, we need not work over a fixed base. A morphism of schemes $\pi\colon S \to T$ gives rise to a natural functor $\pi_*\colon \mathsf{Sch}/S \to \mathsf{Sch}/T$ acting on an S-scheme $X \to S$ by post-composition. But this alone is not satisfactory. For instance, given a scheme X, we ask for the universal way to regard it as an S-scheme, for S an arbitrary scheme, with respect to the base change morphism $S \to \mathbb{Z}$, which in this case is uniquely determined. In other words, we seek a left-adjoint functor to π_*. This turns out to be precisely the fibered product. Of course, we do not yet know if it exists, and constructing it is surprisingly strenuous and will fall out from some very general machinery at the end of section. But for the time being we will assume that it does.

Proposition 2.6.10. *Let $\pi\colon S \to T$ be a morphism. The functor $\mathsf{Sch}/T \to \mathsf{Sch}/S$ sending a T-scheme $X \to T$ to $X \times_T S \to S$ is right adjoint to π_*. This adjoint pair is collectively said to be **base change**.*

Proof. We must show that for all S-schemes X, T-schemes Y, there is a natural bijection $\mathrm{Hom}_T(\pi \circ X, Y) \cong \mathrm{Hom}_S(X, Y \times_T S)$. Consider the diagram

$$\begin{tikzcd}
X \arrow[dr, dashed, "f"] \arrow[drr, bend left, "X"] \arrow[ddr, bend right, "g"'] & & \\
& Y \times_T S \arrow[r] \arrow[d] & S \arrow[d, "\pi"] \\
& Y \arrow[r, "Y"] & T.
\end{tikzcd}$$

Indeed, the condition that $f\colon X \to Y \times_T S$ be an S-morphism forces the induced map $X \to S$ to be the structural morphism, so by the universal property such morphisms are in bijection with morphisms $g\colon X \to Y$ such that $Y \circ g = \pi \circ X$. But this is precisely the definition of a T-morphism $\pi \circ X \to Y$. Naturality is evident. $\square$

To build facility with fibered products, we develop a yoga for concocting new pullback squares out of old, which immediately pays dividends in the form of a renewed (and improved) definition of quasiseparated morphisms. Note that most of the proofs here will work in any category, but we work tentatively in Sch for convenience.

Proposition 2.6.11. *Fix morphisms $X \to S, Y \to S$, $Z \to Y$. Then $Z \times_S X \cong Z \times_Y (Y \times_S X)$. Diagrammatically, this amounts to that the outer rectangle is Cartesian:*

$$\begin{tikzcd}
Z \times_Y (Y \times_S X) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y \times_S X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "a"] \\
Z \arrow[r, "c"] & Y \arrow[r, "b"] & S
\end{tikzcd}$$

Proof. Fix a scheme T, morphisms $f\colon T \to X$, $g\colon T \to Z$. Since the right square is Cartesian, f and $c \circ g$ determine a unique morphism $u\colon T \to Y \times_S X$ subject to the conditions $f = p_2 \circ u$, $c \circ g = p_1 \circ u$. Now the left square is also Cartesian, so u and g determine a unique moprhism $v\colon T \to Z \times_Y (Y \times_S X)$ such that $u = \pi_2 \circ v$, $g = \pi_1 \circ v$. It follows from the condition on u that $p_2 \circ u = p_2 \circ \pi_2 \circ v \implies f = p_2 \circ \pi_2 \circ v$. Hence $Z \times_Y (Y \times_S X)$ satisfies the universal property of the desired fibered product. □

Corollary 2.6.12. *For any morphism of S-schemes $X \to Y$, S-scheme Z, the square below is Cartesian, and the triangle commutes:*

$$\begin{tikzcd}
Z \times_S X \arrow[r] \arrow[d] & Z \times_S Y \arrow[d] \\
X \arrow[r] & Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
Z \times_S X \arrow[r] \arrow[dr] & Z \times_S Y \arrow[d] \\
 & Z.
\end{tikzcd}$$

Proposition 2.6.13. *Fix S-scheme morphisms $f\colon X \to Y$, $g\colon X' \to Y'$. Then the canonical map $f \times_S g\colon X \times_S X' \to Y \times_S Y'$ factors as follows:*

$$\begin{tikzcd}
 & X \times_S X' \arrow[r] \arrow[dl] & X \times_S Y' \arrow[r] \arrow[dl, "\pi_2"] \arrow[dr] & Y \times_S Y' \arrow[dr] & \\
X' \arrow[r, "g"'] & Y' & & X \arrow[r, "f"'] & Y.
\end{tikzcd}$$

Proof. Recall that the canonical map is constructed as follows:

$$\begin{tikzcd}
 & X \arrow[r] \arrow[dr] & Y \arrow[d] \\
X \times_S X' \arrow[ur] \arrow[r, dashed, "u"] \arrow[dr] & Y \times_S Y' \arrow[ur] \arrow[dr] & S \\
 & X' \arrow[r] \arrow[ur] & Y'. \arrow[u]
\end{tikzcd}$$

We must show that the prescribed map fits into this diagram, i.e. the squares consisting of

u and f, g, respectively commute. To this end, consider the following diagram:

$$\begin{tikzcd}
X \times_S X' \arrow[r] \arrow[d] & X \times_S Y' \arrow[r] \arrow[dl] & Y \times Y' \arrow[d] \\
X \arrow[rr] & & Y.
\end{tikzcd}$$

Here, the quadrilateral commutes by construction, and the triangle commutes by Corollary 2.6.12. The proof for the square involving g is dual. □

In the case where two Y-schemes are naturally Z-schemes, there is yet another Cartesian square that can be made. This strange-looking square involves the *diagonal morphism*:

Definition 2.6.14. The **diagonal morphism** of any morphism $\pi\colon X \to Y$ is the morphism $\Delta_\pi\colon X \to X \times_Y X$ induced from id_X and id_X. Given a class of morphisms P, we denote by $P\Delta$, read P**-diagonal**, the class of morphisms π for which $\Delta_\pi \in P$.

Proposition 2.6.15 (Magic Square)**.** *Given morphisms $f\colon X_1 \to Y$, $g\colon X_2 \to Y$, $h\colon Y \to Z$, the following square is Cartesian:*

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X_1 \times_Z X_2 \arrow[d, "f \times_Z g"] \\
Y \arrow[r, "\Delta"] & Y \times_Z Y.
\end{tikzcd}$$

Proof. Throughout we make free use of Corollary 2.6.12. Note that the map π_1 is constructed as in the following diagram:

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[rr, "\pi_{X_1}"] \arrow[dr, dashed, "\pi_1"] \arrow[dd, "\pi_{X_2}"] & & X_1 \arrow[dd] \arrow[dddr] & \\
& X_1 \times_Z X_2 \arrow[ur, "p_2"] \arrow[dl, "p_1"] & & \\
X_2 \arrow[rr] \arrow[drrr] & & Y \arrow[dr] & \\
& & & Z.
\end{tikzcd}$$

We first show that the diagram commutes. By the universal property of $Y \times_Z Y$, this amounts to showing that $\pi_{Y,1} \circ (f \times_Z g) \circ \pi_1 = \pi_2 = \pi_{Y,2} \circ (f \times_Z g) \circ \pi_1$. By Proposition 2.6.13, $f \times_Z g$

factors as

$$\begin{tikzcd}
& X_1 \times_Z X_2 \arrow[rr, "X_1 \times_Z g"] \arrow[dl, "p_1"] & & X_1 \times_Z Y \arrow[rr, "f \times_Z Y"] \arrow[dl, "p'"] \arrow[dr, "q"] & & Y \times_Z Y \arrow[dr] & \\
X_2 \arrow[rr, "g"'] & & Y & & X_1 \arrow[rr, "f"'] & & Y.
\end{tikzcd}$$

Then indeed $\pi_{Y,1} \circ (f \times_Z g) \circ \pi_1 = f \circ q \circ (X_1 \times_Z g) \circ \pi_1 = f \circ p_2 \circ \pi_1 = f \circ \pi_{X_1} = \pi_2$, and likewise $\pi_{Y,2} \circ (f \times_Z g) \circ \pi_1 = p' \circ (X_1 \times_Z g) \circ \pi_1 = g \circ p_1 \circ \pi_1 = g \circ \pi_{X_2} = \pi_2$.
Now fix morphisms $\alpha\colon T \to X_1 \times_Z X_2$, $\beta\colon T \to Y$ such that $(f \times_Z g) \circ \alpha = \Delta \circ \beta \implies \pi_{Y_1} \circ (f \times_Z g) \circ \alpha = \pi_{Y_2} \circ (f \times_Z g) \circ \alpha$. By the diagram above, α gives rise to morphisms $p_1 \circ \alpha\colon T \to X_2$, $q \circ (X_1 \times_Z g) \circ \alpha\colon T \to X_1$. If we can show that $g \circ p_1 \circ \alpha = f \circ q \circ (X_1 \times_Z g) \circ \alpha$, then there would be a unique morphism $u\colon T \to X_1 \times_Y X_2$ for which $\pi_{X_1} \circ u = q \circ (X_1 \times_Z g) \circ \alpha = p_2 \circ \alpha$, $\pi_{X_2} \circ u = p_1 \circ \alpha$. To this end, we claim that $g \circ p_1 = \pi_{Y_2} \circ (f \times_Z g)$, $f \circ q \circ (X_1 \times_Z g) = \pi_{Y,1} \circ (f \times_Z g)$. Since $g \circ p_1 = p' \circ (X_1 \times_Z g)$, the first equation amounts to that $p' = \pi_{Y,2} \circ (f \times_Z Y)$. The second equation is immediate from that $f \circ q = \pi_{Y,1} \circ (f \times_Z Y)$. It remains to show that the conditions on u amount to that (1) $\pi_1 \circ u = \alpha$ and (2) $\pi_2 \circ u = \beta$. That (1) and (2) imply the earlier conditions is by construction: $p_2 \circ \pi_1 = \pi_{X_1}$, and $p_1 \circ \pi_1 = \pi_{X_2}$. Now by the universal property of $X_1 \times_Z X_2$, $p_1 \circ \pi_1 \circ u = p_1 \circ \alpha$ and $p_2 \circ \pi_1 \circ u = p_2 \circ \alpha$, which by construction are equal to the earlier conditions, together implies (1). (2) follows from (1): $\Delta \circ \beta = (f \times_Z g) \circ \alpha = (f \times_Z g) \circ \pi_1 \circ u = \Delta \circ \pi_2 \circ u$, and composing both sides with $\pi_{Y,1}$ yields the result. □

Remark. The magic square may be interpreted as *pulling back $f \times_Z g$ along* Δ: very heuristically, going against Δ removes all information about Z, but retains everything about Y, with the two copies thereof $Y \times_Z Y$ not really being distinct.

Definition 2.6.16. A class C of morphisms of schemes is **stable under base change** if given $f\colon X \to Z$ in C, for all $g\colon Y \to Z$, the projection $X \times_Z Y \to Y$ lies in C.

Stability under base change is yet another property that should be possessed by all reasonable morphisms. Indeed, we will now reserve the term **reasonable** for categorical classes of morphisms stable under base change and local on the target.

Proposition 2.6.17. *Isomorphisms and open embeddings are stable under base change.*

Proof. It will suffice to show that Z satisfies the universal property of the fibered product, with the second projection being the identity: the two induced maps would then be mutually inverse. This in turn follows from the second pullback square.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "\mathrm{id}"] \arrow[dr, dashed, shift left] \arrow[ddr, bend right, "\alpha^{-1}\circ\beta"'] & & \\
& X \times_Y Z \arrow[r] \arrow[d] \arrow[ul, dashed, shift left] & Z \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"] & Y,
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[drr, bend left, "f"] \arrow[dr, dashed, "f"] \arrow[ddr, bend right, "g"'] & & \\
& Z \arrow[r, "\mathrm{id}"] \arrow[d, "\alpha^{-1}\circ\beta"] & Z \arrow[d, "\beta"] \\
& X \arrow[r, "\alpha"] & Y.
\end{tikzcd}$$

Now let $i\colon U \xrightarrow[\varphi]{\sim} V \longrightarrow Z$ be an open embedding, and consider the following diagram:

$$\begin{tikzcd}
T \arrow[drr, bend left, "f"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "g"'] & & \\
& \rho^{-1}(V) \arrow[r, "j"] \arrow[d] & Y \arrow[d, "\rho"] \\
& U \arrow[r, "i"] & Z,
\end{tikzcd}$$

where the projection $\rho^{-1}(V) \to U$ is given $\varphi^{-1} \circ \rho|_{\rho^{-1}(V)}$, and j is the inclusion of an open subscheme. We must show that f factors through $\rho^{-1}(V)$. Indeed, $\rho \circ f = i \circ g$ factors through V, so set-theoretically, $f(T) \subseteq \rho^{-1}(V)$. By the universal property of open immersions, it admits a unique factorization h through $\rho^{-1}(V)$. It remains to show commutativity, namely that $g = \varphi^{-1} \circ \rho|_{\rho^{-1}(V)} \circ u$. Indeed, $\rho|_{\rho^{-1}(V)} = \rho \circ j$, so the composite is $\varphi^{-1} \circ \rho \circ j \circ u = \varphi^{-1} \circ \rho \circ f = \varphi^{-1} \circ i \circ g = g$. $\square$

Proposition 2.6.18. *Let P be a categorical property stable under base change. Then given morphisms $f\colon X \to Y$, $g\colon X' \to Y'$ of S-schemes satisfying P, $f \times_S g\colon X \times_S X' \to Y \times_S Y'$ also satisfies P. That is, categorical classes of morphisms stable under base change are stable under products.*

Proof. This is immediate from the factorization given in Proposition 2.6.13. $\square$

Proposition 2.6.19 (Cancellation Theorem)**.** *Let P be a categorical property stable under base change. Then given a diagram*

$$\begin{tikzcd}
X \arrow[rr, "\pi"] \arrow[dr, "\tau"'] & & Y \arrow[dl, "\rho"] \\
& Z, &
\end{tikzcd}$$

if $\rho \in P\Delta$ and $\tau \in P$, then $\pi \in P$.

Proof. This is immediate from the diagram

$$\begin{tikzcd}
& X = X \times_Y Y \arrow[dl] \arrow[rr] & & X \times_Z Y \arrow[rr, "\tau \times_Z Y"] \arrow[dl] \arrow[dr] & & Z \times_Z Y = Y \arrow[dr] & \\
Y \arrow[rr, "\Delta_\rho"] & & Y \times_Z Y & & X \arrow[rr, "\tau"] & & Z
\end{tikzcd}$$

and the stability under base change of P. $\square$

Of course, Grothendieck did not champion the relative point of view just for its formal elegance: it allows us to convert the onus of concocting structure to that of proving a property. The quintessential example of such a move is the equivalence of the existence of a limit to the *representability* of its functor of points:

Proposition 2.6.20. *In a category C, the limit of $D\colon J \to \mathsf{C}$ exists iff $\varprojlim_J h \circ D$ is representable.*

In what follows we will work towards a criterion, due to Grothendieck, for the representability of a "Zariski sheaf." Our ambient functor category will be $\mathsf{Set}_{\mathsf{Sch}/S}$, discarding the comfort of A-schemes and working with functors from arbitrary S-schemes instead, which, even when not representable, can be interpreted as geometric spaces in a way we do not yet have the technology to explicate.

Definition 2.6.21. A functor $F\colon \mathsf{Sch}/S \to \mathsf{Set}$ is a **Zariski sheaf** if it is a sheaf, suitably restricted, on every S-scheme X.

Remark. We will see that such are precisely the sheaves on the *big Zariski site* on Sch/S. We cannot yet state it as such, however, since we have not demonstrated the existence of fibered products. This does inform us, however, to consider coverings.

Our first step towards Grothendieck's criterion is to situate representable functors within a subclass of the class of Zariski sheaves.

Proposition 2.6.22. *Any representable functor $h_X\colon \mathsf{Sch}/S \to \mathsf{Set}$ is a Zariski sheaf.*

Proof. Fix an S-scheme Y. We must show that for each open $U \subseteq Y$, an open cover $\{U_i\}$ thereof, and $f_i \in \operatorname{Hom}(U_i, X)$ compatible on intersections, there exists a unique $f \in \operatorname{Hom}(U, X)$ restricting to each f_i on U_i. But this is merely that morphisms glue. $\square$

What extra conditions must we impose on Zariski sheaves to make them representable? To make use of the sheaf conditions, it had better involve some *open covering by representables.* If we had the Zariski topology on Sch/S, we can take the canonical topology on $\mathsf{Set}_{\mathsf{Sch}/S}$ and the natural notion of covering morphisms falls right out. Doing this properly would take us too far afield, so we have to make some definitions that may seem a bit contrived at first. To begin with, we shall extend the notion of representability to morphisms.

Definition 2.6.23. A morphism $F \to G$ of presheaves $\mathsf{Sch}/S^{\mathrm{op}} \to \mathsf{Set}$ is **representable** if for all schemes X, morphisms $X \to G$, the fibered product $F \times_G X$ is representable.

Heuristically, this stands for that F *as a "G-scheme" is a scheme over all "scheme bases."* If we assume the existence of fibered products, then clearly any morphism of schemes $X \to Y$ is representable. However, a priori F and G need not be themselves representable for the morphism to be, and on the contrary, if F alone is representable, then the morphisms $F \to G$ can still be overly unruly (they are not in bijection with scheme morphisms), precluding the projection $F \times_G T \to T$ from being a scheme morphism.

Definition 2.6.24. Let P be a categorical property of morphisms of schemes. Then P **extends** to a representable morphism $F \to G$ if for all schemes X, morphisms $X \to G$, the second projection $F \times_G X \to X$ enjoys P. If P assigns morphisms of schemes to a class C, we may colloquially say that it assigns a representable morphism to which it extends to C as well.

Remark. P must be categorical for this to be well-defined: for implicit to the statement "$F \times_G X \to X$ enjoys P" is a choice of representation; and it is well-known that representations are only defined up to canonical isomorphisms. At the cost of a slightly weakened definition, we can afford to be reckless with regard to choosing representations: showing P holds for *any* choice of representation will suffice.

Of particular interest to us at the moment is the property of being an open embedding. These will serve as the "open sets" in our "cover," to be glued using the Zariski sheaf condition.

Proposition 2.6.25. *A morphism $U \to X$ is an open embedding iff the corresponding $h_U \to h_X$ is.*

Proof. The forwards direction is just that open embeddings are stable under base change. For the converse, we may simply take the the second scheme to be X to obtain the Cartesian square

$$\begin{array}{ccc} h_U \times_{h_X} h_X & \longrightarrow & h_X \\ \downarrow & & \downarrow \mathrm{id} \\ h_U & \xrightarrow{i} & h_X. \end{array}$$

Now clearly (h_U, id, i) fills in the diagram, so the underlying map of i is an open embedding, as required. □

Open embeddings of functors also admit a sort of universal property; the following may be viewed as evidence our definitions above were sensible.

Lemma 2.6.26. *A subfunctor $\iota\colon F \to G$ is an open embedding (F is then an **open subfunctor** of G) iff for all pairs (X, ζ), where X is a scheme, $\zeta \in G(X)$, there exists an open subscheme $U_\zeta \subseteq X$ such that a morphism $f\colon Y \to X$ factors through U_ζ iff $G(f)\zeta \in F(Y)$. Such factorizations, if existent, are unique.*
In this case, given a morphism $\Phi\colon X \to G$ with associated element $\zeta \in G(X)$, U_ζ represents $F \times_G X$, and under this identification the scheme morphism corresponding to $F \times_G X \to X$ is precisely the inclusion $U_\zeta \to X$.

Proof. Let $\iota\colon F \to G$ be an open embedding, (X, ζ) be as prescribed. Denote by Φ the morphism $X \to G$ corresponding to ζ, and let U_ζ be the open subscheme of X representing $F \times_G X$. Write $\pi\colon U_\zeta \to F$, $i\colon U_\zeta \to X$ for the projections. Let $\alpha\colon Y \to X$ be a morphism. If $f\colon Y \to X$ factors through some $g\colon Y \to U_\zeta$, then indeed $f^*\zeta = \Phi(f) = (\Phi \circ i)(g) = (\iota \circ \pi)(g) \in F(Y)$. On the other hand, let us assume that $f^*\zeta \in F(Y)$. We must exhibit a unique morphism $u\colon Y \to U_\zeta$ such that $f = i \circ u$. By the universal property of fibered products, it suffices to instead exhibit a unique morphism $g\colon Y \to F$ such that $\iota \circ g = \Phi \circ f$. Given a scheme Z, a morphism $h\colon Z \to Y$, $f(h) = f \circ h$, which in turn gets sent to $(f \circ h)^*\zeta = G(h)(f^*\zeta)$. Now since $G(h)$ is an inclusion, and $G(h)(f^*\zeta) \in F(Z)$, so we may declare it to be the image of h under g. Clearly β is uniquely determined by commutativity. Conversely, let X be a scheme, $\Phi\colon X \to G$ a morphism, ζ the corresponding element of $G(X)$. We claim that the functor represented by the open subscheme U_ζ satisfies the universal property of $F \times_G X$. We first construct the projections. The morphism $i\colon U_\zeta \to X$ is taken as associated to the subscheme inclusion; the component on Y of the morphism $U_\zeta \to F$ sends a morphism $f\colon Y \to U_\zeta$ to $(i \circ f)^*\zeta$, which has codomain $F(Y)$ as $i \circ f$ tautologically factors through U_ζ. Naturality follows from that of the subfunctor inclusion: given a scheme Z, a morphism $g\colon Y \to Z$, morphism $f\colon Z \to U_\zeta$, we indeed have $F(g)(G(i \circ f)\zeta) =$

$G(g)(G(i \circ f)\zeta) = G(i \circ f \circ g)\zeta$. Commutativity is obvious.
Now let H be a functor, $\alpha\colon H \to X$, $\beta\colon H \to F$ be morphisms. We must exhibit a unique induced morphism $f\colon H \to U_\zeta$. To this end, fix a scheme Y. Then each morphism $f\colon Y \to X$ in the image of α_Y factors uniquely through U_ζ, since $f^*\zeta \in F(Y)$ by commutativity. We may thus construct a map $u\colon H \to U_\zeta$ sending $x \in H(Y)$ to the morphism obtained from restricting the codomain of $\alpha(x)$ to U_ζ. Furthermore, u is uniquely determined, since by the definition of $U_f \to X$, $u(x)$ must factor through $\alpha(x)$.

$$\begin{tikzcd}
H \arrow[drr, bend left, "\alpha"] \arrow[dr, dashed, "u"] \arrow[ddr, bend right, "\beta"'] & & \\
& U_\zeta \arrow[r, "i"] \arrow[d] & X \arrow[d, "\Phi"] \\
& F \arrow[r, "\iota"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
f \arrow[r, squiggly, "i"] \arrow[d, squiggly] & i \circ f \arrow[d, squiggly, "\Phi_Y"] \\
(i \circ f)^*\zeta \arrow[r, squiggly, "\iota"] & (i \circ f)^*\zeta
\end{tikzcd}$$

□

This lemma will be immensely useful for proving properties about open subfunctors: in the original definition no determinate representation has been chosen, so we must work painstakingly with the universal property. But now we may fix a conveninent representation of $F \times_G X$, whose universal property can be shown set-theoretically.

Definition 2.6.27. A **Zariski covering** of a functor $F\colon \mathsf{Sch}/S^{\mathrm{op}} \to \mathsf{Set}$ is a family $(f_i\colon F_i \to F)$ of open subfunctors such that for any S-scheme morphism $X \to F$, the images of the open embeddings $U_i \to X$ corresponding to $F_i \times_F X \to X$ cover X.

Lemma 2.6.28. *(f_i) is a Zariski covering iff for every scheme X, $\zeta \in F(X)$, there exists an open covering $\{\, U_i \,\}$ of X such that $\zeta|_{U_i} \in F_i(U_i)$.*

Proof. Let $\zeta \in F(X)$, Φ be its associated morphism $X \to F$. Then for all i, the scheme U_i representing $F_i \times_F X$ and characterized by the property that a morphism $f\colon T \to X$ factors through U_i iff $F(f)\zeta \in F_i(T)$ is precisely the image of the open embedding i_{U_i} associated to $F_i \times_F X \to X$. Now i_{U_i} factors trivially through U_i, so indeed $F(i_{U_i})\zeta \in F_i(U_i)$.
For the converse, for each i, let $V_i \to X$ be an open embedding corresponding to $F_i \times_F X \to X$. It will suffice to show that $i_{U_i} = i_{V_i} \circ g_i$ for some morphism $U_i \to V_i$: for then i_{U_i} also factors through the image of V_i in X, i.e. said image contains U_i. Now since $\zeta|_{U_i} \in F_i(U_i)$, we have that $(\zeta|_{U_i}, i_{U_i}) \in F_i \times_F X(U_i)$: $f_i(\zeta|_{U_i}) = F(i_{U_i})\zeta = \Phi(i_{U_i})$. This then corresponds to a unique morphism $g_i\colon U_i \to F_i \times_F X \to V_i$. To conclude the proof, we show that the

Yoneda isomorphism $\mathrm{Hom}(U_i, V_i) \cong (F_i \times_F X)(U_i)$ sends g_i to $(-, i_{V_i} \circ g_i)$, which, agreeing with $(\zeta|_{U_i}, i_{U_i})$, yields the desired equality. Indeed, the universal family of the isomorphism $\Psi\colon V_i \to F_i \times_F X$ is $\Psi(\mathrm{id}_{V_i}) = \xi \in F_i \times_F X(V_i) = F_i(V_i) \times_{F(V_i)} \mathrm{Hom}(V_i, X)$. Then the natuaral transformation obtained from precomposing with g_i sends a morphism $h\colon T \to U_i$ to $F_i \times_F X(g_i(h))\xi$. To obtain the corresponding element of $(F_i \times_F X)(U_i)$, we take $h := \mathrm{id}_{U_i}$. $\square$

Theorem 2.6.29 (Grothendieck's Representability Criterion)**.** *A functor* $F\colon \mathsf{Sch}/S^{op} \to \mathsf{Set}$ *is representable iff it is a Zariski sheaf and admits a Zariski covering by representable subfunctors.*

Proof. Only the converse is non-trivial. Let F be a Zariski sheaf, $(f_i\colon F_i \to F)$ a Zariski covering thereof by representables. We will glue the schemes X_i associated to each F_i via a universal family $\zeta_i \in F_i(X_i)$ to a scheme X. For all i, we apply Lemma 2.6.26 to the open subfunctor $F_j \to F$ to obtain an open subscheme $i_{ij}\colon U_{ij} \to X_i$ such that for every scheme T, a morphism $f\colon T \to X_i$ factors through U_{ij} iff $F(f)\zeta_i \in F_j(T)$. Now i_{ij} is trivially factorizable, so we have that $F(i_{ij})\zeta_i \in F_j(U_{ij})$. Proposition 2.6.5, applied to ζ_j, U_{ij}, and this element, yields a morphism $\varphi'_{ij}\colon U_{ij} \to X_j$ for which $F_j(\varphi'_{ij})\zeta_j = F(i_{ij})\zeta_i$. Since $F(i_{ij})\zeta_i \in F_i(U_{ij})$, φ'_{ij} factors through U_{ji} by the above, and we set the resultant map $U_{ij} \to U_{ji}$ to be φ_{ij}.
We first show that $\varphi_{ji} \circ \varphi_{ij} = \mathrm{id}_{U_{ij}}$. Indeed, with (X_j, ζ_j) being universal, it follows from $F(i_{ij})\zeta_i = F(\varphi'_{ij})\zeta_j = F(i_{ji} \circ \varphi_{ij})\zeta_j = F(\varphi_{ij})F(i_{ji})\zeta_j = F(\varphi_{ij})F(\varphi'_{ji})\zeta_i = F(\varphi_{ij})F(\varphi_{ji})F(i_{ij})\zeta_i$ that $i_{ij} = i_{ij} \circ \varphi_{ji} \circ \varphi_{ij}$. The desired equality follows from that i_{ij} is mono.
We now show that $\varphi_{ij}(U_{ijk}) = U_{jik}$, where $U_{ijk} := U_{ij} \cap U_{ik}$, $U_{jik} := U_{ji} \cap U_{jk}$. For the forward inclusion, it suffices by the universal property of open embeddings of schemes to show that $\phi := \varphi'_{ij}|_{U_{ijk}}$ factors through both U_{ji} and U_{jk}. This in turn amounts to that $F(\phi)\zeta_j \in F_j(U_{ijk}) \cap F_k(U_{ijk})$. Indeed, $F(\varphi'_{ij} \circ i_{ijk,ij})\zeta_j = F(i_{ijk,ij})F(\varphi'_{ij})\zeta_j = F(i_{ijk,ij})F(i_{ij})\zeta_i = F(i_{ijk})\zeta_i = F(i_{ijk,ik})F(i_{ik})\zeta_i$. For the opposite inclusion, it will suffice to show that i_{jik} can be written as $\phi \circ \psi$ for some $\psi\colon U_{jik} \to U_{ijk}$; for then $\mathrm{im}(i_{jik}) = U_{jik} \subseteq \mathrm{im}(\phi) = \varphi_{ij}(U_{ijk})$:

$$\begin{tikzcd}
U_{jik} \arrow[r, "\psi"] \arrow[rrr, bend right, "i_{jik}"'] & U_{ijk} \arrow[r] \arrow[rr, bend left, "\phi"] & \varphi_{ij}(U_{ijk}) \arrow[r] & X_j.
\end{tikzcd}$$

We take $\psi'\colon U_{jik} \to X_i$ to be the morphism obtained from Proposition 2.6.5 using ζ_i, U_{jik}, and $F(i_{jik})\zeta_j = F(i_{jik,ji})F(i_{ji})\zeta_j$ (which by naturality lies in $F_i(U_{jik})$), satisfying

$F(\psi')\zeta_i = F(i_{jik})\zeta_j$. Now since $\zeta_j \in F_j(X_j)$, by naturality $F(\psi')\zeta_i \in F_j(U_{jik})$, while at once $F(i_{jik})\zeta_j = F(i_{jik,ji})F(i_{jk})\zeta_j \in F_k(U_{jik})$. Thus ψ' factors through both U_{ij} and U_{ik}, and hence U_{ijk}. We claim that the unique map $U_{jik} \to U_{ijk}$ through which ψ' factors is our desired ψ. Indeed, $F(\phi \circ \psi)\zeta_j = F(\psi)F(i_{ijk,ij})F(\varphi'_{ij})\zeta_j = F(\psi)F(i_{ij} \circ i_{ijk,ij})\zeta_i = F(i_{ijk} \circ \psi)\zeta_i = F(\psi')\zeta_i = F(i_{ijk})\zeta_j$. It remains for the sake of gluing schemes to verify the cocycle condition. That is, we must show that $\varphi_{jk}|_{U_{jik}} \circ \varphi_{ij}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}}$. This would follow from that $\varphi'_{jk}|_{U_{jik}} \circ \varphi_{ij}|_{U_{ijk}} = \varphi'_{ik}|_{U_{ijk}}$, with i_{kij} being a monomorphism. Indeed, $F(\varphi'_{ik} \circ i_{ijk,ik})\zeta_k = F(i_{ijk,ik})F(\varphi'_{ik})\zeta_k = F(i_{ijk,ik})F(i_{ik})\zeta_i = F(i_{ijk})\zeta_i = F(\varphi'_{ij} \circ i_{ijk,ij})\zeta_j = F(\varphi_{ij} \circ i_{ijk,ij})F(i_{jik,jk})F(i_{jk})\zeta_j = F(\varphi_{ij} \circ i_{ijk,ij})F(\varphi'_{jk} \circ i_{jik,jk})\zeta_k$.

A scheme X admitting an open cover $\{ U_i \}$ and isomorphisms $\varphi_i \colon X_i \to U_i$ satisfying $\varphi_i(U_{ij}) = U_i \cap U_j$ and $\varphi'_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_i \cap U_j}$ has thus been procured. Clearly $\xi_i := F_i(\varphi_i^{-1})\zeta_i$ is the universal family of U_i: for every scheme T, $x \in F_i(T)$, the unique morphism $f \colon T \to X_i$ for which $x = F_i(f)\zeta_i$ gives rise to the composite $\varphi_i \circ f$, for which indeed $x = F_i(\varphi_i \circ f)F_i(\varphi_i^{-1})\zeta_i$. It is necessarily unique, as U_i also represents F_i. Notice, $F(\varphi_i^{-1}|_{U_i \cap U_j})\zeta_i = F(\varphi_i^{-1}|_{U_i \cap U_j})F(\varphi'_{ij})\zeta_j = F(\varphi_j^{-1}|_{U_i \cap U_j})\zeta_j$, and it follows that $\xi_i|_{U_i \cap U_j} = \xi_j|_{U_i \cap U_j}$. That is, $\{ \xi_i \}$ are compatible on intersections, so the Zariski sheaf condition furnishes a unique element $\zeta \in F(X)$ for which $\zeta|_{U_i} = \xi_i$. We claim that (X, ζ) represents F. To this end, it will suffice to show that ζ is a universal family. Let T be a scheme, $x \in F(T)$. Since (f_i) is a Zariski covering, we have an open cover $\{ V_i \}$ of T for which $x|_{V_i} \in F_i(V_i)$ for all i. Now for each i, with ξ_i being the universal family of $U_i \to F_i$, there exists a unique morphism $f_i \colon V_i \to U_i$ for which $x|_{V_i} = F_i(f_i)\xi_i$. Notice, for all i, j, $F(f_i|_{V_i \cap V_j})\xi_i = F(i_{V_i \cap V_j})x|_{V_i} = x|_{V_i \cap V_j} = F(i_{V_i \cap V_j})x|_{V_j} = f_i|_{V_i \cap V_j}\xi_j$, so since morphisms of schemes glue, we obtain a unique morphism $f \colon T \to X$ restricting to on each V_i to $i_{U_i,X} \circ f_i$. That $x = F(f)\zeta$ is precisely that $x|_{V_i} = F_i(f_i)\xi_i = F(f_i)F(i_{U_i,X})\zeta = F(f \circ i_{V_i})\zeta = F(f)\zeta|_{V_i}$, with F a Zariski sheaf. As can be seen, this condition thus uniquely determines f. □

Remark. The proof may look intimidating, but in fact the overall structure is very clear and there is a natural choice at every step. Before or instead of reading our proof, we invite the reader to do it themselves as an especially enlightening exercise.

Corollary 2.6.30. *Let $\alpha \colon X \to Z$, $\beta \colon Y \to Z$ be morphisms of schemes. Then the fibered product $X \times_Z Y$ exists.*

Proof. We prove as a lemma that fibered products of *affine* schemes exists. Let $C \to A$, $C \to B$ be ring homomorphisms. We claim that $\operatorname{Spec}(A \otimes_C B) \cong \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B$, which would follows from that for any scheme X, $\operatorname{Hom}(X, \operatorname{Spec} A) \times_{\operatorname{Hom}(X, \operatorname{Spec} C)}$

$\mathrm{Hom}(X, \operatorname{Spec} B) \cong \mathrm{Hom}(X, \operatorname{Spec}(A \otimes_C B))$. By the Γ-Spec adjunction, this is tantamount to that $\mathrm{Hom}(A, \Gamma(X)) \times_{\mathrm{Hom}(C,\Gamma(X))} \mathrm{Hom}(B, \Gamma(X)) \cong \mathrm{Hom}(A \otimes_C B, \Gamma(X))$, which holds as $\mathrm{Hom}(-, \Gamma(X))$ is cocontinuous.

For arbitrary scheme morphisms $\alpha\colon X \to Z$, $\beta\colon Y \to Z$, the existence of $X \times_Z Y$ amounts to the representability $h_X \times_{h_Z} h_Y$. That this is a Zariski sheaf follows immediately from that limits commute. By Grothendieck's criterion and the above, it remains to exhibit an affine Zariski covering of $X \times_Z Y$. To this end, let $\{ Z_i \}$ be an affine cover of Z, $\{ X_{ij} \}$ an affine cover of $\alpha^{-1}(Z_i)$, $\{ Y_{ik} \}$ an affine cover of $\beta^{-1}(Z_i)$. We first show that every $X_{ij} \times_{Z_i} Y_{ik}$ is an open subfunctor of $X \times_Z Y$. To this end, let (T, ζ) be a pair. Now ζ corresponds to a morphism $T \to X \times_Z Y$, which gives rise to unique morphisms $\zeta_1\colon T \to X$ and $\zeta_2\colon T \to Y$ such that $\alpha \circ \zeta_1 = \beta \circ \zeta_2$. We claim that $U_\zeta := \zeta_1^{-1}(X_{ij}) \cap \zeta_2^{-1}(Y_{ij})$ is our desired open subscheme. Indeed, let $f\colon T' \to T$ be any morphism of schemes. By the universal property of open embeddings, f factors through U_ζ iff $f(T') \subseteq \zeta_1^{-1}(X_{ij}) \cap \zeta_2^{-1}(Y_{ik})$ iff $\zeta_1(f(T')) \subseteq X_{ij}$ and $\zeta_2(f(T')) \subseteq Y_{ik}$. By construction, $X \times_Z Y(f)\zeta = (\zeta_1 \circ f, \zeta_2 \circ f)$, so this condition is indeed equivalent to that $X \times_Z Y(f)\zeta \in X_{ij} \times_{Z_i} Y_{ik}(T')$. $\square$

Proposition 2.6.31. *A morphism $\pi\colon X \to Y$ is quasiseprated iff Δ_π is quasicompact.*

Proof. For any affine open $U \subseteq Y$, affine opens V, W mapping to U, $V \cap W$ is quasicompact. $V \cap W = V \times_U W$ Inverse of affine is quasiseparated iff $\square$

2.7. Quasicoherent Sheaves

As opposed to arbitrary $\mathscr{O}_X$-modules over ringed spaces, the class of sheaves we describe in this section cater directly to the affine cover of their base scheme, the reduction to which constitutes the bulk of most arguments, as evident in the previous sections. The correct generalization to Sch of the module-ring relation is thus that of *quasicoherent sheaves* to schemes. On the other hand, quasicoherent sheaves are to arbitrary $\mathscr{O}_X$-modules as schemes are to locally ringed spaces.

Definition 2.7.1. Fix X a scheme. An $\mathscr{O}_X$-module $\mathscr{F}$ is **quasicoherent** if there exists an affine open cover $\{ U_i \}$ of X such that for all i, $\mathscr{F}|_{U_i} \cong \tilde{M}$ for some A-module M. Such $\mathscr{O}_X$-modules constitute a full subcategory of $\mathscr{O}_X$-Mod denoted QCoh_X.

Let us begin by reconciling this with the more general notion of quasicoherence used in other parts of geometry, where X is not necessarily a scheme.

Lemma 2.7.2. *For any affine scheme* $\operatorname{Spec} A$, $\mathscr{O}_{\operatorname{Spec} A}^{\oplus I} \cong \widetilde{A^{\oplus I}}$.

Proof. We will show that the sheaves are isomorphic on the distinguished base. For any $f \in A$, $\widetilde{A^{\oplus I}}(D(f)) = S_f^{-1} A^{\oplus I} \cong (S_f^{-1} A)^{\oplus I}$. Since limits of sheaves are computed pointwise, we also have $\mathscr{O}_{\operatorname{Spec} A}^{\oplus I}(D(f)) \cong (\mathscr{O}_{\operatorname{Spec} A}(D(f)))^{\oplus I} = (S_f^{-1} A)^{\oplus I}$. Hence we have exhibited an isomorphism. To show naturality, let $g \in A$ be such that $D(g) \subseteq D(f)$. Naturality is evident. $\square$

Proposition 2.7.3. *An $\mathscr{O}_X$-module $\mathscr{F}$ is quasicoherent iff it is* ***locally presented****, i.e. every point in X admits a neighborhood U on which there exists an exact sequence*

$$\mathscr{O}_U^{\oplus I} \longrightarrow \mathscr{O}_U^{\oplus J} \longrightarrow \mathscr{F}|_U \longrightarrow 0.$$

Proof. Let $p \in X$. Then there exists an affine open U in the prescribed cover containing p such that $\mathscr{F}|_U \cong \tilde{M}$ for some A-module M. Let $A^{\oplus I} \xrightarrow{i} A^{\oplus J} \xrightarrow{j} M \longrightarrow 0$ be a presentation of M. Now taking the associated $\mathscr{O}_{\operatorname{Spec} A}$-module is right exact, so the preceeding lemma furnishes the desired exact sequence of $\mathscr{O}_{\operatorname{Spec} A}$-modules.
Now for the converse. Let us take the affine cover comprised of the open neighborhoods U, which may be assumed affine, with restriction being right exact, associated by quasicoherence to each point of X. Then U admits a presentation $\mathscr{O}_U^{\oplus I} \to \mathscr{O}_U^{\oplus J} \to \mathscr{F}|_U \to 0$. We claim that $\mathscr{F}|_U$ is associated to the module M presented as $A^{\oplus I} \longrightarrow A^{\oplus J} \longrightarrow M \longrightarrow 0$. Again by the right exactness of $M \rightsquigarrow \tilde{M}$, we conclude that $\mathscr{F}|_U \cong \tilde{M}$. $\square$

Proposition 2.7.4. *If $\mathscr{F}$ is quasicoherent, then for all affine opens U, $\mathscr{F}|_U \cong \tilde{M}$ for some $\Gamma(U, \mathscr{O}_X)$-module M.*

Proof. We must show that the property of affine opens $U \subseteq X$ that $\mathscr{F}|_U \cong \tilde{M}$ for some $\Gamma(U, \mathscr{O}_X)$-module M is affine local. Locality is immediate: for $U \subseteq X$ an affine open, if $\mathscr{F}|_U \cong \tilde{M}$, then for any $D(f) \subseteq U$, we claim that $\widetilde{M_f} \cong \tilde{M}|_{D(f)} \cong (\mathscr{F}|_U)|_{D(f)} = \mathscr{F}|_{D(f)}$. Indeed, $\tilde{M}(D(f)) \cong M_f$, and the equality follows from Theorem 2.3.15.
For gluability, let $\{ D(f_i) \}$ be a finite affine cover of U such that for each i there is a $\Gamma(D(f_i), \mathscr{O}_X)$-module M_i admitting an isomorphism $\phi_i \colon \tilde{M}_i \to \mathscr{F}|_{D(f_i)}$. We claim that there is an isomorphism $\phi \colon \widetilde{\Gamma(U, \mathscr{F})} \to \mathscr{F}|_U$. Note that if $\mathscr{F}|_U$ were to be isomorphic to any associated $\mathscr{O}_U$-module, it must be that of $\Gamma(U, \mathscr{F})$ due to the functoriality of Γ, and that Theorem 2.3.15 does *not* help here a priori: only by restricting the codomain to sheaves of the form $\tilde{M}$ do we get a conservative functor (in fact an equivalence of categories), and to apply this we must prove the result at hand! Instead, we show that both $\mathscr{O}_U$-modules arise from a common gluing datum satisfying the cocycle condition. This will be $(\tilde{M}_i \phi_{ij})$, where $\phi_{ij} \colon \tilde{M}_i|_{D(f_i f_j)} \to \tilde{M}_j|_{D(f_i f_j)}$ is the composition $\phi_j|_{D(f_i f_j)}^{-1} \circ \phi_i|_{D(f_i f_j)}$ and gives rise to an isomorphism $\varphi_{ij} \colon M_{ij} \to M_{ji}$, for $M_{ij} := (M_i)_{f_j}$.
Writing $M := \Gamma(U, \mathscr{F})$, it will suffice to exhibit isomorphisms $\tilde{\psi}_i \colon \tilde{M}|_{D(f_i)} \cong \tilde{M}_i$ making the following triangle commute: Since all $\mathscr{O}_{D(f_i f_g)}$-modules involved are associated, we need only show the result for the global components. Recall Corollary 1.1.12, M is the kernel of the map $\prod_i M_i \to \prod_{i \neq j} M_{ij}$ sending $m \in M_i$ to M □

Corollary 2.7.5. *All quasicoherent $\mathscr{O}_{\operatorname{Spec} A}$-modules are of the form $\tilde{M}$ for some A-module M. Hence $\mathsf{QCoh}_{\operatorname{Spec} A} \cong A\text{-}\mathsf{Mod}$.*

Proof. $\tilde{M}$ is certainly quasicoherent, taking $\{ \operatorname{Spec} A \}$ as the affine open cover. But if $\mathscr{F}$ is quasicoherent, then by the above $\mathscr{F} = \mathscr{F}|_{\operatorname{Spec} A} \cong \tilde{M}$. □

Proposition 2.7.6. *An $\mathscr{O}_X$-module $\mathscr{F}$ is quasicoherent iff for every distinguished inclusion* $\operatorname{Spec} A_f \to \operatorname{Spec} A$, *the map* $\alpha_f \colon \Gamma(\operatorname{Spec} A, \mathscr{F})_f \to \Gamma(\operatorname{Spec} A_f, \mathscr{F})$ *induced from the canonical map* $\Gamma(\operatorname{Spec} A, \mathscr{F}) \to \Gamma(\operatorname{Spec} A_f, \mathscr{F})$ *is an isomorphism.*

Proof. For $\mathscr{F}$ quasicoherent, if $\mathscr{F}|_{\operatorname{Spec} A} \cong \tilde{M}$, then for any distinguished inclusion $\operatorname{Spec} A_f \to$

$\operatorname{Spec} A$, we have the diagrams

$$
\begin{tikzcd}
\Gamma(\operatorname{Spec} A, \mathscr{F}) \arrow[r, "\sim", "\varphi_{\operatorname{Spec} A}"'] \arrow[d] & M \arrow[d] \\
\Gamma(\operatorname{Spec} A_f, \mathscr{F}) \arrow[r, "\sim", "\varphi_{\operatorname{Spec} A_f}"'] & M_f,
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma(\operatorname{Spec} A, \mathscr{F})_f \arrow[drr, dashed, "\beta_f"'] \arrow[drrr, dashed, "\alpha_f"] & & & \\
\Gamma(\operatorname{Spec} A, \mathscr{F}) \arrow[u] \arrow[r, "\sim", "\varphi_{\operatorname{Spec} A}"'] & M \arrow[r] & M_f \arrow[r, "\sim", "\varphi^{-1}_{\operatorname{Spec} A}"'] & \Gamma(\operatorname{Spec} A_f, \mathscr{F}),
\end{tikzcd}
$$

where the square is from naturality and the monstrosity on the right contains two maps induced from localization. The indicated map is indeed α, since by the naturality square the three right-facing maps on the second row compose to $\Gamma(\operatorname{Spec} A, \mathscr{F}) \to \Gamma(\operatorname{Spec} A_f, \mathscr{F})$. To show that α_f is an isomorphism, it will suffice to show that β_f is, as $\varphi^{-1}_{\operatorname{Spec} A}$ is an isomorphism, and $\varphi^{-1}_{\operatorname{Spec} A} \circ \beta_f = \alpha_f$ by the universal property of α_f. But this is immediate from the functoriality of localization, with β_f being the image of $\varphi_{\operatorname{Spec} A}$.

For the converse, we claim that if $\Gamma(\operatorname{Spec} A, \mathscr{F}) \cong M$, then $\mathscr{F}|_{\operatorname{Spec} A} \cong \tilde{M}$. To this end, we exhibit a natural isomorphism $\tilde{M} \to \mathscr{F}|_{\operatorname{Spec} A}$ when restricted to the distinguished affine base of $\operatorname{Spec} A$. Consider the following diagram:

$$
\begin{tikzcd}
\Gamma(\operatorname{Spec} A, \mathscr{F}) \arrow[rr, "\sim", "\varphi"'] \arrow[d] \arrow[dr] & & M \arrow[d] \\
\Gamma(\operatorname{Spec} A_f, \mathscr{F}) & \Gamma(\operatorname{Spec} A, \mathscr{F})_f \arrow[l, dashed, "\sim", "\alpha_f"'] \arrow[r, dashed, "\sim", "\varphi_f"'] & M_f.
\end{tikzcd}
$$

Taking the component on $D(f)$ to be $\alpha_f \circ \varphi_f^{-1}$, naturality is evident. $\square$

Lemma 2.7.7 (Qcqs Lemma)**.** *For X a qcqs scheme, $\mathscr{F}$ a quasicoherent sheaf on X, if $f \in \Gamma(X, \mathscr{O}_X)$, then there is an isomorphism $\Gamma(X, \mathscr{F})_f \to \Gamma(D(f), \mathscr{F})$ making the following diagram commute:*

$$
\begin{tikzcd}
\Gamma(X, \mathscr{F}) \arrow[rr] \arrow[dr] & & \Gamma(D(f), \mathscr{F}) \\
& \Gamma(X, \mathscr{F})_f \arrow[ur, "\sim"] &
\end{tikzcd}
$$

Proof. Denote by $\{ U_i := \operatorname{Spec} A_i \}$ the finite affine cover of X wherein every pairwise intersection $U_{ij} := \operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ admits a finite cover $\{ U_{ijk} := \operatorname{Spec} A_{ijk} \}$. Since for every affine open $\operatorname{Spec} A$, $\operatorname{Spec} A \cap D(f) = \operatorname{Spec} A_f$, $\{ V_i := \operatorname{Spec}(A_i)_f \}$ is likewise a finite affine cover of $D(f)$ wherein every pairwise intersection $V_{ij} := \operatorname{Spec} A_i \cap \operatorname{Spec} A_j \cap D(f)$ is covered by $\{ \operatorname{Spec} A_{ijk} \cap D(f) \} = \{ V_{ijk} := \operatorname{Spec}(A_{ijk})_f \}$. By locality, for each (i, j), the natural map $\mathscr{F}(V_{ij}) \to \bigoplus_k \mathscr{F}(V_{ijk})$ is injective, so Corollary X furnishes an exact sequence

$$0 \longrightarrow \Gamma(D(f), \mathscr{F}) \xrightarrow{e} \textstyle\bigoplus_i (A_i)_f \xrightarrow{\varphi} \bigoplus_{ij,k} (A_{ijk})_f,$$

where φ is the map $\oplus_{ij,k} r_{V_{ij},V_{ijk}} \circ (\oplus_{i,j} r_{V_i,V_{ij}} - \oplus_{j,i} r_{V_j,V_{ij}})$. It will suffice to exhibit a map $e' \colon \Gamma(X, \mathscr{F})_f \to (\bigoplus A_i)_f$ which leaves the sequence exact after replacing e, for then we have the diagram

$$\begin{tikzcd} \Gamma(X, \mathscr{F}) \arrow[r, "r"] \arrow[dr, "l"'] & \Gamma(D(f), \mathscr{F}) \arrow[r, "e"] & (\bigoplus_i A_i)_f \\ & \Gamma(X, \mathscr{F})_f. \arrow[u, "\sim"', "u"] \arrow[ur, "e'"'] & \end{tikzcd}$$

Viewing the LES's as equalizer diagrams, the triangle on the right manifestly commutes. Then given $f \in \Gamma(X, \mathscr{F})$, we have $er(f) = e'l(f) = eul(f)$. But e is injective, so $r(f) = ul(f)$, as desired.

Consider the exact sequence associated to $\Gamma(X, \mathscr{F})$ like above:

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \textstyle\bigoplus_i A_i \xrightarrow{\psi} \bigoplus_{ij,k} A_{ijk},$$

where ψ denotes the map $\oplus_{ij,k} r_{U_{ij},U_{ijk}} \circ (\oplus_{i,j} r_{U_i,U_{ij}} - \oplus_{j,i} r_{U_j,U_{ij}})$. Since the localization functor is exact and thus commutes with finite biproducts, applying it to the above yields

$$0 \longrightarrow \Gamma(X, \mathscr{F})_f \longrightarrow \textstyle\bigoplus_i (A_i)_f \longrightarrow \bigoplus_{ij,k} (A_{ijk})_f.$$

It remains to show that the rightmost map is φ. Indeed, r_{U_i,V_i} and $r_{U_{ijk},V_{ijk}}$ are themselves pulled back from distinguished affine inclusions, so we have the diagram

$$\begin{tikzcd} A_i \arrow[r] \arrow[d] & A_{ijk} \arrow[d] \\ (A_i)_f \arrow[r] & (A_{ijk})_f. \end{tikzcd}$$

Hence localizing $r_{U_i,U_{ijk}}$ yields $r_{V_i,V_{ijk}}$. □

Example 2.7.8. The above needs not hold when X is not quasicompact. Consider an infinite coproduct $X := \coprod_{i=1}^{\infty} \operatorname{Spec} k[x]$. The ring of global sections is $\prod_{i=1}^{\infty} k[x]$, and it is immediate from the sheaf axioms that $\Gamma(D(f), \tilde{A}) \cong \prod_{i=1}^{\infty} k[x]_x$, where f is the function $(x, \dots)$ restricting to x on each $\operatorname{Spec} k[x]$. The element $(1/x, 1/x^2, \dots)$ in the latter is certainly not an element of $\Gamma(X, \mathscr{O}_X)_f$, for the coordinates of its elements all have denominators bounded above by some x^n.

A crucial reason quasicoherent sheaves are defined the way they are is so that they form an *abelian subcategory* of $\mathscr{O}_X$-Mod.

Theorem 2.7.9. *QCoh_X is an abelian subcategory of $\mathscr{O}_X$-Mod.*

Proof. It will suffices to show that QCoh_X is closed under finite biproducts, kernels, and cokernels. To this end, recall that (co)limits are computed pointwise, up to sheafification. $\square$

2.8. Finiteness Conditions

Our outlining in this section of the somewhat pedantic finiteness conditions is driven by the very practical need to axiomaitize the classical notion of an algebraic variety. As abstract scheme theory is nevertheless most commonly applied to classical situations, many morphisms will turn out to enjoy these properties. It is thus salient to extract as much about their behavior as possible while working in the generality of properties of morphisms.

Let us begin by relativizing the notion of an affine scheme, now viewed as a property. It is trivially affine-respecting, so the associated property of morphisms is unequivocally well-defined, and the theory developed in §2.4 applies nicely. More precisely,

Definition 2.8.1. A morphism of schemes $\pi\colon X \to Y$ is **affine** if for every affine open $U \subseteq Y$, $\pi^{-1}(U)$ is affine.

Proposition 2.8.2. *Affine morphisms are qcqs.*

Proof. This is immediate from that affine schemes are qcqs. □

Proposition 2.8.3. *Affinity is strongly affine local.*

Proof. Fix a morphism $\pi\colon X \to Y$. We first show heredity: for an affine open $U \subseteq Y$, if $\pi^{-1}(U)$ is affine, then $\pi^{-1}(D_U(f)) = \pi|_{\pi^{-1}(U)}(D_U(f))$ is affine by Proposition 2.2.2 (4), with $\pi|_{\pi^{-1}(U)}\colon \pi^{-1}(U) \to U$ a morphism of affine schemes. For gluability, writing $U = \bigcup_{i=1}^n D(f_i)$, assume that $\pi^{-1}(D(f_i))$ is affine for all i. Writing $A := \Gamma(\pi^{-1}(U), \mathscr{O}_X)$, we have by the Γ-Spec adjunction the diagram

$$\begin{tikzcd} \operatorname{Spec} A \arrow[dr, dashed, "\beta"] & \\ \pi^{-1}(U) \arrow[u, "\alpha"] \arrow[r, "\pi"] & U. \end{tikzcd}$$

We claim that α is an isomorphism. Since isomorphisms are local on the target and $\{\, \beta^{-1}(D(f_i)) \,\}$ covers $\operatorname{Spec} A$, it will suffice to show for each i that $\alpha|_{\alpha^{-1}(\beta^{-1}(D(f_i)))} = \alpha|_{\pi^{-1}D(f_i)}$ is an isomorphism. Now with α a morphism of schemes, we have by Corollary 2.4.26 that $\alpha|_{\pi^{-1}D(f_i)}$ is induced from $\Gamma(\alpha|_{\pi^{-1}D(f_i)})\colon \Gamma(D(\beta^{\#} f_i), \mathscr{O}_{\operatorname{Spec} A}) \to \Gamma(D(\pi^{\#} f_i), \mathscr{O}_X)$. Consider the following diagram:

$$\begin{tikzcd}
 & \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) = \Gamma(\pi^{-1}(U), \mathscr{O}_X) \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\
 & \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})_{\beta^\# f_i} = \Gamma(\pi^{-1}(U), \mathscr{O}_X)_{\pi^\# f_i} \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
\Gamma(D(\beta^\# f_i), \mathscr{O}_{\operatorname{Spec} A}) \arrow[rr, "\Gamma(\alpha|_{\pi^{-1}D(f_i)})"'] & & \Gamma(D(\pi^\# f_i), \mathscr{O}_X).
\end{tikzcd}$$

The largest triangle commutes by naturality, whilst the interpolating triangles on the left and right are furnished by the qcqs lemma, with $\pi^{-1}(U)$ qcqs by locality on the target, $\operatorname{Spec} A$ trivially qcqs, and the structure sheaves trivially quasicoherent. Hence the lowest triangle commutes, and $\Gamma(\alpha|_{\pi^{-1}D(f_i)})$ is an isomorphism. $\square$

Corollary 2.8.4. *Let Z be a closed subset of $\operatorname{Spec} A$ for which there exists an open cover $\{ U_i \}$ of $\operatorname{Spec} A$ where for all i, $Z \cap U_i = V(f_i) \cap U_i$ for some $f_i \in A$. Then Z^c is affine.*

Often the need arises for us to consider properties of morphisms induced from ring homomorphisms rather than schemes, for which *globalization* process affine morphisms provide the scaffold. Among globalized properties are the finiteness conditions of *integrality* and *finiteness* and the important notion of being *closed embedding*, defined in §2.9.

Definition 2.8.5. Fix a property P of ring homomorphisms. An affine morphism of schemes $\pi \colon X \to Y$ enjoys $\tilde{P}$, the **globalization** of P, if for all affine opens $U \subseteq Y$, $\pi_U^\#$ enjoys P.

Proposition 2.8.6. *$\tilde{P}$ is restrictable, and affine-gluable if P is when viewed as a property of affine scheme morphisms.*

Proof. Let $\pi \colon X \to Y$ be an affine morphism of schemes, $U \subseteq Y$ be open. Then $\pi|_{\pi^{-1}(U)}$ satisfies $\tilde{P}$, as for any affine $V \subseteq U$, $(\pi|_{\pi^{-1}(U)})_V^\# = \pi_V^\#$ enjoys P. Now let $\{ U_i \}$ be an affine cover of U, such that for all i, $\pi|_{\pi^{-1}(U_i)}$ enjoys $\tilde{P}$. Let V be an affine open of U. Then it admits an affine cover $\{ V_i \}$, where each V_i lies in some U_i, so that Namely, for all affines $V \subseteq U_i$, $(\pi|_{\pi^{-1}(U_i)})_{V_i}^\# = \pi_{V_i}^\#$ enjoys P. Since P is affine local on the target, it follows that $\pi_V^\#$ enjoys P. $\square$

We now move on to finiteness conditions that are no longer induced from properties of schemes but genuine properties of morphisms mirroring conditions on field and ring extensions from algebra. Affine moprhisms can be viewed as the first stage of this analogy:

Proposition 2.8.7. *If L/k is a field extension, then* $\operatorname{Spec} L \to \operatorname{Spec} k$ *is affine.*

Proof. Since k is a field, $\operatorname{Spec} k$ is a one-point space. Hence the only affine open of $\operatorname{Spec} k$ is itself, whose inverse image is $\operatorname{Spec} L$. □

Our next notion will correspond to algebraic extensions. To define it, we recall some ideas from algebra:

Definition 2.8.8. Fix A a ring, B a subring. An element $a \in A$ is **integral over** B if it is a root of some monic polynomial in $B[x]$. A homomorphism $\varphi\colon B \to A$ is **integral** if every element of A is integral over $\varphi(B)$. If φ is an inclusion of rings, it is furthermore said to be an **integral extension**.

Proposition 2.8.9. *Integrality is preserved by quotients and localizations.*

Proof. It now follows from gluability that the restriction axiom may be checked on a distinguished affine. That is, we must show that if φ is integral, then for all $b \in B$, $\varphi_b\colon B_b \to A_{\varphi(b)}$ is integral. To this end, let $a/\varphi(b)^n \in A_{\varphi(b)}$. Then a is the root of some monic polynomial $f(t) \in \varphi(B)[t]$; but then $a/\varphi(b)^n$ evidently solves $f(\varphi(b)^n t) \in \varphi(B)[t] \subseteq \varphi_b(B_b)[t]$. □

As a property of ring homomorphisms, integrality is at once a property of morphisms of affine schemes.

Proposition 2.8.10. *Integrality is affine local on the target.*

Proof. Let $\varphi\colon B \to A$ be a ring homomorphism. Restrictability follows from the above, so we need only show gluability. Assume that $B = (b_1, \ldots, b_n)$, and that for each i, $\varphi_{b_i}\colon B_{b_i} \to A_{\varphi(b_i)}$ is integral. Let $a \in A$. Denote by $I \subseteq A[x]$ the ideal of polynomials f for which $f(a) = 0$, by $J \subseteq A$ the ideal of leading coefficients of elements of I. It suffices to show that $J = (1)$. By supposition, there exists for each i an $m_i \in \mathbb{Z}^+$ such that $a^{m_i} \in B_{b_i} + aB_{b_i} + \cdots + a^{m_i-1}B_{b_i}$. There then is a least positive integer t_i for which $a^{m_i} b_i^{t_i} \in B + aB + \cdots + a^{m_i-1}B$, obtained via clearing denominators. That is, $b_i^{t_i} \in J$. Now $D(b_i^{t_i}) = D(b_i)$, so we are done. □

Integral morphisms Finite morphisms

Finite type Finitely presented

2.9. Reduced Schemes and Rational Maps

We continue inching towards the definition of a variety by introducing another necessary condition, that of *reducedness*, which is to be the globalization of the eponymous notion in the affine case. To ensure that the property is local (P holding on an open cover implies P holds for the whole scheme), we define it stalk-locally.

Definition 2.9.1. A ringed space whose stalks are all reduced is itself **reduced**.

Evidently open subspaces of reduced spaces are reduced. It comes as a pleasant surprise that it is equivalent to define reducedness on open subspaces.

Proposition 2.9.2. *A ringed space X is reduced iff $\mathscr{O}_X(U)$ is reduced for all open $U \subseteq X$.*

Proof. If X is reduced, then for any open $U \subseteq X$, $f \in \mathscr{O}_X(U)$, $f^n = 0$ implies that its germ in every $\mathscr{O}_{X,p}$, for $p \in U$, is zero, being nilpotent in a reduced ring. That is, f lies in the kernel of the injective canonical map $\mathscr{O}_X(U) \to \prod_{p \in U} \mathscr{O}_{X,p}$, so is itself 0. Conversely, let $[f] \in \mathscr{O}_{X,p}$ be nilpotent, i.e. so that $[f^n] = 0$. Then there exists some open neighborhood U of p on which $f^n = 0$ for a representative f. But $\mathscr{O}_X(U)$ is reduced, so $f = 0 \implies [f] = 0$. □

Proposition 2.9.3. *A ring A is reduced iff* $\operatorname{Spec} A$ *is reduced.*

Proof. The forward direction follows from Proposition 2.2.34, the backwards the above and that $A = \mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A)$. □

Corollary 2.9.4. *A ring A is reduced iff $A_{\mathfrak{p}}$ is reduced for all $[\mathfrak{p}] \in \operatorname{Spec} A$.*

Corollary 2.9.5. *A scheme X is reduced iff there exists an affine cover $\{U_i\}$ such that $\mathscr{O}_X(U_i)$ is reduced for all i.*

Proof. The forward direction is immediate; the backwards direction follows from the locality of reducedness and Proposition 2.9.3. □

At this point one would reasonably expect functions on a reduced scheme to be determined by their values; as in the affine case, we develop a more general theory that relate functions to their values on any scheme. To this end, we seek a globalization of $I(-)$ which should yield for any subset of X the sections vanishing on it. Due to the global nature of schemes, a plain ideal will not suffice; the correct gadget is going to be, unsurprisingly, that of a quasicoherent sheaf:

Definition 2.9.6. Fix a scheme X. A sub-$\mathscr{O}_X$-module of $\mathscr{O}_X$ is $\mathscr{F}$ said to be an **ideal sheaf** on X. An ideal sheaf is **radical** if $\mathscr{F}(U)$ is a radical ideal for all open $U \subseteq X$.

Remark. Note that for every open $U \subseteq X$, sub-$\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F}(U)$ is a submodule, or equivalently ideal of $\mathscr{O}_X(U)$, hence the name ideal sheaf.

Definition 2.9.7. Fix X a scheme, Z a subset. The **vanishing ideal sheaf** on Z, denoted $\mathscr{I}_Z$, sends any open $U \subseteq X$ to the subset of $\mathscr{O}_X(U)$ consisting of functions vanishing on all points of $U \cap Z$. The vanishing ideal sheaf of X is said to be the **sheaf of nilpotents**, denoted $\mathscr{N}$.

Proposition 2.9.8. *For any $Z \subseteq X$, $\mathscr{I}_Z = \mathscr{I}_{\overline{Z}}$ is a quasicoherent radical ideal sheaf.*

Proof. That $\mathscr{I}_Z$ is an ideal sheaf is evident: vanishing is a local condition. For quasicoherence, let U be an affine open of X. Given $\mathfrak{p} \in U$, $f(\mathfrak{p}) = 0 \iff f \in \mathfrak{p}$, so $\mathscr{I}_Z(U)$ is precisely $I_U(Z \cap U) = I_U(\overline{Z \cap U}_U) = I_U(\overline{Z} \cap U)$, which corresponds to a radical ideal $J \subseteq \Gamma(U, \mathscr{O}_X)$. We claim that $\mathscr{I}_Z|_U = \tilde{I}$. Indeed, for any $D(g) \subseteq U$, $\mathscr{I}_Z(D(g)) = I_{D(g)}(\overline{Z} \cap D(g))$ is equal to $S_g^{-1}J$. $\square$

Proposition 2.9.9. *A scheme X is reduced iff $\mathscr{N}(X) = 0$.*

Proof. Since $\mathscr{N}$ is a sheaf, $\mathscr{N}(X) = 0$ iff there exists an affine cover $\{ U_i \}$ of X such that $0 = \mathscr{N}(U_i) = \mathfrak{N}(U_i)$ for all i. But $\mathfrak{N}(U_i) = 0$ iff $\mathscr{O}_X(U_i)$ is reduced iff U_i is reduced. The result then follows from Corollary 2.9.5. $\square$

A bit more can be said about $\mathscr{N}(X)$ when X is quasicompact.

Proposition 2.9.10. *If X is a quasicompact scheme, then $\mathscr{N}(X)$ is a nil ideal. That is, a global section f vanishing at all points is nilpotent.*

Proof. The restriction $f|_{U_i}$ of f to each affine open in a finite affine cover $\{ U_i \}$ is nilpotent in $\mathscr{O}_X(U_i)$, i.e. there exists some n_i for which $f|_{U_i}^{n_i} = 0$. Then clearly $f^{\operatorname{lcm}(\{ n_i \})}$ restricts to 0 in every $\operatorname{Spec} A_i$, so f is itself 0. $\square$

Remark. The above may fail if X is not quasicompact. Consider $\coprod_{n \in \mathbb{Z}^+} \operatorname{Spec} k[x]/(x^n)$. The global section restricting to x in each $\operatorname{Spec} k[x]/(x^n)$ vanishes at all points, but cannot itself be nilpotent, for for every m, x^m is non-zero in $k[x]/(x^{m+1})$.

We now present the desired global analog of Theorem 2.2.29, whose proof will be general sheaf nonsense.

Proposition 2.9.11. *Two functions on an open $U \subseteq X$ agreeing on a subset $S \subseteq U$ differ by an element of $\mathscr{I}_{\overline{S}}(U)$. In particular, for closed S it suffices to check agreement on a dense subset.*

Proof. Let f be a section on U vanishing on S. Then for all affine opens $V \subseteq U$, $f|_V = 0 \iff f|_V \in I_V(\overline{S}) = \mathscr{I}_{\overline{S}}(V)$. Since $\mathscr{I}_{\overline{S}}$ is a sheaf, there is a unique section in $\mathscr{I}_{\overline{S}}(U)$ restricting to each. This is precisely f, as $\mathscr{I}_{\overline{S}}$ is an ideal sheaf of $\mathscr{O}_X$. □

Corollary 2.9.12. *Functions on an open $U \subseteq X$ are determined by their values (on a dense subset) iff $\mathscr{N}(U) = 0$. In particular, a scheme is reduced iff all sections are determined by their values (on a dense subset).*

Remark. To show a scheme is reduced it does *not* suffice to check the reduceness of stalks at points in a dense subset K. This might be somewhat surprising, as reducedness is apparently detectable from a dense subset in the function-theoretic definition, and a priori this should see a reflection in any equivalent definition. There is subtlety here: although we are evaluating the sections only on the point in K, we are nevertheless still tapping into global information, namely the global sections themselves.
Consider $\operatorname{Spec} A$ for $A := k[x,y]/(x^2, xy)$, a k-algebra of finite type, k an algebraically closed field. Evidently this affine scheme is not reduced (consider x). We claim that the stalk $A_{(x)}$ at the generic point (x) is reduced. Indeed, $A_{(x)} \cong k(y)[x]/(x^2, xy)_{(x)} = k(y)[x]/(x)_{(x)} \cong k(y)_{(0)} = k(y)$ is a field, hence reduced. Reducedness of a scheme also does not follow from that of the ring of global sections, unlike in the affine case. TODO: projective scheme example, 5.2.3

There is one case where it does suffice to look at the closed points, however.

Lemma 2.9.13. *Let X be a scheme with a generic point x. Then for all $p \in X$, $\mathscr{O}_{X,x}$ is a localization of $\mathscr{O}_{X,p}$.*

Proof. Let $\operatorname{Spec} A$ be an affine open containing p. Since x is dense in X, $x \in \operatorname{Spec} A$. It follows that $\mathscr{O}_{X,p} = A_{\mathfrak{p}}$, $\mathscr{O}_{X,x} = A_{\mathfrak{x}}$. Now $x \in V(\mathfrak{p}) \cap \operatorname{Spec} A \implies x \in V_{\operatorname{Spec} A}(\mathfrak{p})$. Hence $I(V(\mathfrak{p})) = \mathfrak{p} \subseteq I(x) = \mathfrak{x}$. □

Remark. You may find in this lemma an echo of the discussion about $A_{\mathfrak{p}}$ in §2.3. That is, even for schemes it is "easier" to evaluate at generic points. That localizations contains more functions translates to that all functions evaluable at specializations, plus possibly more, are evaluable at the generic point.

Proposition 2.9.14. *A quasicompact scheme X is reduced iff its stalk at each closed point is reduced.*

Proof. Let $p \in X$. Then $\overline{\{p\}}$ is non-empty, quasicompact, and closed, hence contains a closed point x by Proposition 2.5.3. By supposition, $\mathscr{O}_{X,x}$ is reduced. But then so is $\mathscr{O}_{X,x}$ by the lemma above and Proposition 2.2.34. □

As evident from the discussion on affine varieties, reducedness often goes hand in hand with irreducibility. Miraculously, there is an algebraic condition that perfectly encapsulates their conjunction.

Definition 2.9.15. An ringed space X is **integral** if it is nonempty and $\mathscr{O}_X(U)$ is an integral domain for all nonempty open $U \subseteq X$.

Proposition 2.9.16. *An ringed space X is integral iff it is irreducible and reduced.*

Proof.

- $\Rightarrow$) By Proposition 2.9.2, it is manifest that X is reduced. Now if X is reducible, then there exist disjoint nonempty open sets U, V. But then $\mathscr{O}_X(U \cup V)$ is not integral, as the section restricting to 0 in $\mathscr{O}_X(U)$, 1 in $\mathscr{O}_X(V)$ multiplies to 0 with the section restricting to 1 in $\mathscr{O}_X(U)$, 0 in $\mathscr{O}_X(V)$.
- $\Leftarrow$) Since the property of being irreducible and reduced is clearly hereditary on opens, it suffices to show that the ring of global sections is integral. Indeed, given $f, g \in \mathscr{O}_X(X)$ such that $fg = 0$, we have $X = V(fg) = V(f) \cup V(g)$. Since both summands are closed by Proposition 1.7.17, it follows from irreducibility that one has to be the whole of X. □

Proposition 2.9.17. Spec A *is integral iff A is an integral domain.*

Proof. The forward direction is immediate. The converse follows from Proposition 2.2.43 and Proposition 2.9.3. □

Definition 2.9.18. The **function field** $K(X)$ of an integral scheme X is the stalk $\mathscr{O}_{X,\eta}$ of X at its generic point η. Elements of $K(X)$ are **rational functions** on X.

Proposition 2.9.19. *$K(X)$ is naturally identified with $K(A)$ for any nonempty affine open.*

Proof. Let U be an affine open. Since $\eta \in U$, $\mathscr{O}_{X,\eta} = \mathscr{O}_{U,\eta}$. But $\Gamma(A, \mathscr{O}_X)$ is integral, so η correspond to its unique generic point, 0. That is, $\mathscr{O}_{U,\eta} \cong A_{(0)} = K(A)$. □

Proposition 2.9.20. *Let X be an integral scheme. The restrictions $r_{U,V}\colon \mathscr{O}_X(U) \to \mathscr{O}_X(V)$ are injections unless $V \neq \varnothing$. Hence for any non-empty open subset U, the natural map $\mathscr{O}_X(U) \to K(X)$ is an inclusion.*

Proof. We must show that for each $s \in \mathscr{O}_X(U)$, $s|_V = 0 \implies s = 0$. To this end, it will suffice to show that s vanishes at all points in U, with U being reduced. By supposition, s vanishes at all points in V. But V, being a non-empty open subset, is dense in U, an irreducible subspace, so by Corollary 1.7.19 s in fact vanishes at all points, as desired.
For the implication, we identify $K(X)$ with $K(A)$ for some affine open $\operatorname{Spec} A \subseteq U$. Then $\mathscr{O}_X(U) \to \mathscr{O}_X(\operatorname{Spec} A)$ is an inclusion, and so is $A \to K(A)$, with A being integral. $\square$

Definition 2.9.21. Given an A-module M, $m \in M$ is a **torsion element** if there exists a non-zero divisor $a \in A$ such that $am = 0$. The **torsion submodule** M_{tors} consists of all torsion elements. M is **torsion-free** if $M_{\text{tors}} = 0$, or **torsion** if $M = M_{\text{tors}}$.
For X a scheme, **torsion-freeness** is defined stalkwise for $\mathscr{O}_X$-modules.

Proposition 2.9.22. *M is torsion iff $M \otimes_A K(A) = 0$.*

Proof. Let $m \in M$ such that $am = 0$ for non-zero divisor $a \in A$. Then for any $r \in K(A)$, we have $rm = r(a/a)m = (r/a)(am) = 0$. The converse is obvious. $\square$

2.10. Closed Embeddings

Recall that our model for closed subschemes of an affine scheme $\operatorname{Spec} A$ is provided by vanishing sets thereof, which are tantamount to spectra of the form $\operatorname{Spec} A/I$. The notion of an affine closed embedding is thus encoded by the projection map $A \to A/I$ inducing the inclusion $\operatorname{Spec} A/I \to \operatorname{Spec} A$. Noting that projections are precisely the surjective maps in Ring, we proceed as before and globalize the notion of surjectivity to obtain the scheme-theoretic definition of closed embeddings:

Definition 2.10.1. A morphism of schemes $\pi\colon X \to Y$ is a **closed embedding** (or **closed immersion**) if it is affine and for every affine open $U \subseteq Y$, the pullback map $\pi_U^\#$ is surjective. If $X \subseteq Y$ and the inclusion $X \hookrightarrow Y$ is a closed embedding, X is said to be a **closed subscheme** of Y.

Lemma 2.10.2. *A closed embedding maps the domain homeomorphically onto a closed subspace of the codomain.*

Proof. Let $\pi\colon X \to Y$ be a closed embedding. To show that that $\pi(X)$ is closed in Y, it will suffice to show that given an affine cover $\{ U_i \}$ of Y, $\pi(X) \cap U_i$ is closed in U_i for all i. Since $\pi_{U_i}^\#$ is surjective, we have by Proposition 2.2.18 and Corollary 2.4.26 that $\pi|_{\pi^{-1}(U_i)}\colon \pi^{-1}(U_i) \to U_i$ is a homeomorphism onto a closed subset of U_i, namely its image $\pi(X) \cap U_i$. $\square$

This leads us to an equivalent definition of closed embeddings applicable to general ringed spaces.

Proposition 2.10.3. *A morphism of scheme $\pi\colon X \to Y$ is a closed embedding iff it maps X homeomorphically onto a closed subspace of Y, and the pullback $\pi^\#\colon \mathscr{O}_Y \to \pi_*\mathscr{O}_X$ is surjective.*

Proof. The forward direction is immediate: that X is mapped homeomorphically onto a closed subspace is immediate from the lemma above, and that the pullback is epi follows from that the affine opens of Y form a base, and that the pullback on each affine open is surjective (make sure to cite the lemma). We now show the converse. Let $U \subseteq Y$ be an affine open with ring of global sections A. Then $\pi(X) \cap U$ is a closed subset of U, hence of the form $V(S) \cong \operatorname{Spec} A/I$ for some radical ideal $I \subseteq A$. It will suffice to show that $\pi^{-1}(U) \cong \operatorname{Spec} A/I$, in such a way that $\pi_U^\#$ is the canonical projection $A \to A/I$. Since the pullback morphism is surjective, for every $f \in \mathscr{O}_X(\pi^{-1}(U))$, there is an open cover $\{ U_i \}$ of U, which may be assumed distinguished affine by locality, with $f_i \in A_{g_i} := \mathscr{O}_Y(U)$ such that $\pi_{U_i}^\#(f_i) = f|_{\pi^{-1}(U_i)}$. $\square$

Proposition 2.10.4. *Closed embeddings are finite morphisms.*

Proof. Trivial. □

Proposition 2.10.5. *The class of closed embeddings is reasonable.*

Proof. □

Other than being a central part of the theory in its own right, the notion of a closed subscheme allows us to set up an I-V correspondence for schemes, generalizing simultaneously Theorem 2.2.41 and Theorem 2.5.27. Whilst naive ideals no longer suffice to describe closed subschemes, a sheaf-theoretic anlog will do. Indeed, to any closed embedding $\pi\colon X \to Y$ we can associate an exact sequence

$$0 \longrightarrow \mathscr{I}_{X/Y} \longrightarrow \mathscr{O}_Y \longrightarrow \pi^*\mathscr{O}_X,$$

where $\mathscr{I}_{X/Y} := \ker \pi^\#$ is the kernel of $\pi^\#$ in $\mathscr{O}_Y$-Mod. In fact, by Lemma (qcqs)

Theorem 2.10.6. *There is a bijection between radical ideal sheaf and vanishing ideal sheaves of closed subsets.*

Proof. Let $\mathscr{F}$ be a radical ideal sheaf on X. We must exhibit a V for which $\mathscr{F} = \mathscr{I}_V$. How? □

Every radical sheaf is the vanishing ideal sheaf of a closed subset.

Proposition 2.10.7. *Given a closed embedding $\pi\colon X \hookrightarrow Y$, $\pi^*\mathscr{O}_X$ is a quasicoherent sheaf on Y.*

Theorem 2.10.8.

Proposition 2.10.9. *A morphism of schemes $\pi\colon X \to Y$ is a closed embedding iff it factors as $X \xrightarrow{\sim} V \xhookrightarrow{i} Y$, for i the inclusion of a closed subscheme.*

Compare with universal prop of open embeddings.

Definition 2.10.10.

discuss how naive image is not sheaf this explains left-handedness: in presheaf category, quotienting messes up sheaf structure.

2.11. Separatedness and Varieties

Separated Morphisms Varieties Various Properties Applicable to finite-x things

2.12. Projective Schemes

Projective geometry is inherently counterintuitive, being one of the few non-metrical geometric theories that are not pathologies. As such we begin with a more classical outlook on the subject, constructing projective spaces in a way that should be familiar to topologists. Fix an algebraically closed field k.

Definition 2.12.1. The **projective n-space over k** is the quotient $\mathbb{P}^n_k := (\mathbb{A}^{n+1}_k \setminus \{ 0 \})/\sim$, where $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$. A set of **homogeneous coordinates** for a point $p \in \mathbb{P}^n_k$ is simply a representative thereof in $\mathbb{A}^{n+1}_k \setminus \{ 0 \}$, written as $[x_0 : \cdots : x_n]$.

The above presentation lends itself to the interpretation of $\mathbb{P}^n_k$ as the space of lines through the origin, i.e. 1-dimensional subspaces of $\mathbb{A}^{n+1}_k$; in topology, $\mathbb{RP}^n$ can be equivalently viewed as S^n with antipodal points identified. This may not be entirely transparent, since each projective point only appears to stand for an affine line, with no prospect of crossing the origin. To avoid this confusion, we will impose a taxonomy on the projective points, so that it makes sense to connect each with the origin, and so that the lines naturally split into disjoint regions which together cover the projective space.

As a set $\mathbb{P}^n_k$ consists of $n+1$ *affine patches* $U_i := \{ [x_0 : \cdots : x_n] \in \mathbb{P}^n_k \colon x_i \neq 0 \}$, each of which is isomorphic to $\mathbb{A}^n_k$ via $[x_0 : \cdots : x_n] \rightsquigarrow (x_0/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i)$. Henceforth we will write $x_{j/i}$ for x_j/x_i. Intrinsically, this amounts to picking the set of projective coordinates with $x_i = 1$ for all points in U_i; the division results from the necessary scaling of the other coordinates. Separating one affine patch yields our desired decomposition:

Proposition 2.12.2. $\mathbb{P}^n \cong \mathbb{A}^n \coprod \mathbb{P}^{n-1}$.

Proof. The complement of U_i in $\mathbb{P}^n$ is manifestly the set of points where $x_i = 0$. Shifting all coordinates after i forward, we see that this is simply $\mathbb{P}^{n-1}$. □

Remark. Note that this decomposition is only valid on the topological level; it does not respect the structure sheaf to be put on $\mathbb{P}^n$.

It follows from induction that $\mathbb{P}^n \cong \coprod_{i=0}^n \mathbb{A}^i$ (again, only as topological spaces). Since $\mathbb{P}^n$ for $n \geq 3$ cannot be visualized, we will specialize to the case of $\mathbb{P}^2$. Let us assume without loss of generality that the copy of $\mathbb{A}^2$ is taken to be U_0, so that any point therein is obtained from $(1, x_{1/0}, x_{2/0})$ in $\mathbb{A}^3$. Since these are genuine points, it makes sense to connect them with the origin, generating 1-dimensional subspaces that cover all of $\mathbb{A}^3$ except for the plane $x_0 = 0$, corresponding to the copy of $\mathbb{P}^1$ that we have not yet touched. Notice, $V(x_0) \cong \mathbb{P}^1$, so assuming the conclusion that $\mathbb{P}^1$ consists of all 1-dimensional subspaces of $\mathbb{A}^2$, we would have obtained all lines through the origin in $\mathbb{A}^3$.

But the way we arrive at this will be identical to the above: decomposing $\mathbb{P}^1 \cong V(x_0)$ as $(U_1 \setminus U_0) \coprod \mathbb{P}^0$, each $[0 : x_1 : x_2]$ is seen to be either $(0 : 1 : x_2/x_1)$ or $[0 : 0 : x_2]$. The segments connecting the former with the origin span all of $V_{\mathbb{A}}(x_0) \setminus V_{\mathbb{A}}(x_1)$, where by $V_{\mathbb{A}}$ we understand the "affine cone" of the polynomial, its vanishing set when viewed as a function on $\mathbb{A}^2$; the latter corresponds to $V(x_0) \cap V(x_1)$, which is itself the only 1-dimensional subspace in $(U_2 \setminus U_1) \setminus U_0$.

We now endow $\mathbb{P}^n$ with a topology and a structure sheaf, mimicking our work with affine schemes: we want the closed sets to be precisely the vanishing sets of polynomials, such that their complements form an affine base on which rings of sections are obtained via localization. The caveat is that not all polynomials $f \in k[x_0, \ldots, x_n]$ have a well-defined vanishing set in $\mathbb{P}^n$: it could very much be that f vanishes at one homogeneous coordinate but not at another (consider $\mathbb{P}^1$, $x^2 - y$, and $[1 : 1]$). Vanishing sets of *homogeneous* polynomials h, however, are well-defined: $h(\lambda x_0, \ldots, \lambda x_n) = \lambda^{\deg(h)} h(x_0, \ldots, x_n)$. To put this on a formal footing, we will briefly foray into the theory of graded rings.

Definition 2.12.3. A ring S is **$\mathbb{Z}$-graded** if it admits a decomposition into a direct sum of abelian groups $\bigoplus_{n \in \mathbb{Z}} S_n$, such that multiplication restricts to a bilinear map $S_m \times S_n \to S_{m+n}$. It is immediate that S_0 is a subring, S an S_0-algebra, S_n an S_0-module for $n \in \mathbb{Z}$. Thus S_0 is said to be the **base ring**, S a $\mathbb{Z}$-graded ring **over** S_0. A non-zero element is **homogeneous** of **degree** n if it lies in S_n; the n-summand of a non-zero heterogeneous element is its **degree** n **piece**. A non-homogeneous element is **heterogeneous**. A **homogeneous ideal** I is an ideal generated by homogeneous elements, i.e. I is equal to its **homogenization** $I^h = \bigoplus_{n \in \mathbb{Z}} I \cap S_n$. A subset or multiplicative submonoid is **homogeneous** if it consists entirely of homogeneous elements.

Proposition 2.12.4. *Fix a $\mathbb{Z}$-graded ring S.*

1. *An ideal is homogeneous iff it contains the homogeneous pieces of each of its elements.*
2. *The set of homogeneous ideals in S is closed under sums, products, intersections, and radicals.*
3. *A homogeneous ideal is prime iff it is proper and for any homogeneous $a, b \in S$, $ab \in I \implies a \in I \vee b \in I$. Furthermore, the homogenization of a prime ideal is prime.*
4. *Let I be a homogeneous ideal, T a homogeneous multiplicative submonoid. Then $T^{-1}S$ admits the gradation $T^{-1}S_n = \{\, a/b \colon \deg(a) - \deg(b) = n \,\}$, S/I the gradation*

$(S/I)_n = (S_n + I)/I$.

Proof. 1. Every element a of S, and thus of a homogeneous ideal I, admits a unique decomposition into graded pieces $\sum \pi_d(a)$, one which would be furnished by the generators, and conversely the set of all homogeneous pieces manifestly generates the ideal.

2. Closure is obvious for sums, products, and intersections. For radicals, simply observe that given $r = \sum_{d=m}^{l} r_d \in \sqrt{I}$, the highest degree piece of r^n is r_l^n, so $r_l \in I$. Applying this argument to $r - \sum_{d=l-k}^{l} r_l$ for every $0 \leq k \leq l - m$ yields the result.

3. The forward direction is trivial. For the converse, let $a = \sum a_d, b = \sum b_e \in S \setminus I$. Denote by m, n be the highest degrees for which there exist $a_m, b_n \notin I$. It will suffice to show that the $m + n$ piece of ab, that is, $\sum_{d+e=m+n} a_d b_e$ does not lie in I. For each $(d, e) \neq (m, n)$, either $d > m$ or $e > n$, both of which implies that one term, and hence $a_d b_e \in I$. Thus there exists some $x \in I$ for which $\sum_{d+e=m+n} a_d b_e = a_m b_n + x$. If this lies in I, then $a_m b_n \in I$, a contradiction.
   To show that $\mathfrak{p}^h$ is prime, we use the above and assume that $ab \in \mathfrak{p}^h$ for homogeneous $a, b \in S$. Then ab is homogeneous, and thus lies in $\mathfrak{p}$. Hence $a, b \in \mathfrak{p}$. It follows again from homogeneity that $a, b \in \mathfrak{p}^h$, as desired.

4. Since colimits commute, evidently $S/I = (\bigoplus S_n)/(\bigoplus I \cap S_n) \cong \bigoplus S_n/(I \cap S_n) \cong \bigoplus (S_n + I)/I$, where the last isomorphism is due to the second isomorphism theorem. Multiplication is indeed compatible with this grading, as elements $(S_n + I)/I$ are equivalence classes $[a]$ for $a \in S_n$, and given $[b] \in (S_m + I)/I$, $[a][b] = [ab] \in (S_{m+n} + I)/I$. We now show that $T^{-1}S \cong \bigoplus T^{-1}S_n$ and that multiplication respects this grading. Indeed, any $a \in S$ can be decomposed as $\sum a_d$, and $(\sum a_d, b) = \sum (a_d, b)$. Now given $(a_1, b_1) \in S_n$, $(a_2, b_2) \in S_m$, indeed $\deg(a_1 a_2) - \deg(b_1 b_2) = \deg(a_1) + \deg(a_2) - \deg(b_1) - \deg(b_2) = n + m$. $\square$

Definition 2.12.5. A $\mathbb{Z}$-graded ring S is $\mathbb{N}$**-graded** (or **graded** for short) if $S_n = 0$ for all $n < 0$. S is **finitely generated over** S_0 if it is of finite type as an S_0-algebra, i.e., if the **irrelevant ideal** $S_+ := \bigoplus_{n>0} S_n$ is finitely generated, and **generated in degree** 1 if it is generated by S_1 as an S_0-algebra.

Proposition 2.12.6. *Fix a graded ring S over A.*

1. *If S is finitely generated over A, then it is generated by finitely many homogeneous elements of positive degree.*
2. *S is Noetherian iff A is Noetherian and S is finitely generated over A.*

Proof. For (1), we denote by $\{ x_i \}$ the finite generating set of S_+. If any x_i is heterogeneous, then, being a homogeneous ideal, S_+ contains all homogeneous pieces of x_i. But there can only be finitely many, so traversing through all heterogeneous generators we would have obtained a finite, homogeneous generating set. For (2), notice that if S is Noetherian, then so is $A \cong S/S_+$, and as an ideal S_+ must be finitely generated. The converse is Hilbert's basis theorem. □

We return to projective geometry. While we can give the classical definition for projective vanishing sets of homogeneous polynomials right away, we prefer to first recast the underlying set of $\mathbb{P}^n$ using our new machinery so that we may swiftly introduce the Proj construction, which produces a scheme out of *any* graded ring and subsumes $\mathbb{P}^n$ as a very special case.

Theorem 2.12.7. *There is a a bijection*

$$\mathbb{P}^n \rightleftarrows \{ \textit{homogeneous prime ideals of } S \textit{ not containing } S_+ \}.$$

Proof. We first define the backwards map. Notice, $S_+ = (x_0, \dots, x_n)$, so any homogeneous prime ideal not containing S_+ is an element of $\mathbb{A}^{n+1} \setminus \{ 0 \}$. To define the other map, we must associate every class of prime ideals up to scalars with a homogeneous prime ideal. Let $\mathfrak{p} = (f_\alpha)$ be a prime ideal. For each heterogeneous f_α,

Talk about this in meeting.

It then seems reasonable to take as the ring of global sections all homogeneous polynomials in $k[x_0, \dots x_n]$. Is there any issue with this? This also allows us to reinterpret every point as a homogeneous prime ideal of $\Gamma(\mathscr{O}_{\mathbb{P}^n}, \mathbb{P}^n)$. □

Brief discussion; how S_+ stands for the origin in general, and containing it is to exclude strange cases where 0 is not closed; considering only homogeneous ideals takes care of scaling.

remark that Proj is not functorial, so it is not as natural as Spec.

Definition 2.12.8. For any graded ring S, $\operatorname{Proj} S$ is the set of all homogeneous prime ideals of S not containing S_+, topologized by the base of **projective distinguished open sets** $\{ D_+(f) \colon \exists\, n > 0 (f \in S_n) \}$, where

$$D_+(f) := \{ \mathfrak{p} \in \operatorname{Proj} S \colon f \notin \mathfrak{p} \}.$$

Verification. The proof is exactly that found under Definition 2.1.5, except that we conclude $\mathfrak{p} = S_+$ rather than $\operatorname{Spec} A$ when arguing that they cover. □

Definition 2.12.9. The **projective vanishing set** of any homogeneous subset $T \subseteq S$ is the closed subspace

$$V_+(T) := \{ \mathfrak{p} \in \operatorname{Proj} S \colon T \subseteq \mathfrak{p} \}.$$

The **projective vanishing ideal** of a subset $Z \subseteq \operatorname{Proj} S$ is the homogeneous ideal $I(Z) := \bigcap_{p \in Z} \mathfrak{p}$, which manifestly equates its affine counterpart when regarding Z as a subset of $\operatorname{Spec} S$.

Proposition 2.12.10. *Fix a graded ring S.*

1. *Let $T \subseteq S$ be a subset of homogeneous elements. Then $V_+(T) = V_+((T))$.*
2. *Given a set of ideals $\{ I_\alpha \}$, $\bigcap_\alpha V_+(I_\alpha) = V_+(\sum_\alpha I_\alpha)$.*
3. *Given homogeneous ideals I, J, $V_+(I) \cup V_+(J) = V_+(IJ) = V_+(I \cap J)$.*
4. *Any closed subset of $\operatorname{Proj} S$ is of the form $V_+(Z)$.*

Proof. The proofs of Proposition 2.2.2 and 2.2.3 apply if we use Proposition 2.12.4 (3) in deducing the first equality in (3). □

We may now make precise the notion of an *affine cone*, which is to $\operatorname{Proj} S$ as $\mathbb{A}^{n+1}$ is to $\mathbb{P}^n$. Geometrically, it can be thought of the set of rays emitting from a *cone point* at infinity to each point in $\operatorname{Proj} S$.

Definition 2.12.11. Given a graded ring S, $\operatorname{Spec} S$ is said to be the **affine cone** of $\operatorname{Proj} S$, which may be regarded as a subspace thereof.

Verification. Clearly, $D_+(f) = D(f) \cap \operatorname{Proj} S$. □

Lemma 2.12.12. *Let $Z \subseteq \operatorname{Proj} S$, $J \subseteq S$ a homogeneous ideal. Then $V_+(I(Z)) = \overline{Z}$, and if $V_+(J) \neq \varnothing$, $I(V_+(J)) = \sqrt{J}$.*

Proof. By the above, $\overline{Z} = V(I(Z)) \cap \operatorname{Proj} S$, which immediate implies the first statement. Since $V_+(J) \subseteq V(J)$, the backwards inclusion is clear. Now assume that $f \in I(V_+(J))$, that is, $f \in \mathfrak{p}$ for all $\mathfrak{p} \in V_+(J)$. Let $\mathfrak{q}$ be an arbitrary prime ideal containing J. Then $\mathfrak{q}^h$ is homogeneous, prime, and contains J, as all generators thereof, being homogeneous, lie in some $\mathfrak{q} \cap S_n$. Hence $f \in \mathfrak{q}^h \subseteq \mathfrak{q}$. □

Proposition 2.12.13. *For a homogeneous ideal $I \subseteq S_+$, $V_+(I) = \varnothing$ iff $\sqrt{I} \supseteq S_+ \iff \sqrt{I} = S_+ \vee \sqrt{I} = S$.*

Proof. Let $\mathfrak{p}$ be a prime ideal containing I. If $\mathfrak{p}$ does not contain S_+, then $\mathfrak{p}^h$ lies in $\operatorname{Proj} S$ and still contains I. The converse is immediate from that no point in $\operatorname{Proj} S$ can contain S_+. Finally, □

Remark. This is the formal justification for the term *irrelevant ideal.* explains the terminology Hence why irrelevant. This is also the thing that breaks functoriality.

Actually, the real reason, I believe, is that we want to have an affine base, so that $\operatorname{Proj} S$ is a scheme. If we do not exclude S_+, then although $\{ D_+(f) \}$ where f is *any* homogeneous polynomial still covers $\operatorname{Proj} S$, the DOS for when f is not of positive degree may not be affine. Nullstellensatz is then a corollary of LRS Nullstellensatz.

Theorem 2.12.14. *$I(-)$ and $V_+(-)$ give an inclusion-reversing bijection*

$$\{ \textit{closed subsets of } \operatorname{Proj} S \} \underset{V(-)}{\overset{I(-)}{\rightleftarrows}} \{ \textit{homogeneous radical ideals of } S_+ \}.$$

Proof. □

Theorem 2.12.15. *For any $f \in S_r$ where $r > 0$, $D_+(f) \cong \operatorname{Spec}(S_f)_0$.*

Proof. We will show that the restriction of the canonical map $i^*\colon D(f) \cong \operatorname{Spec} S_f \to \operatorname{Spec}(S_f)_0$ induced from the inclusion $i\colon (S_f)_0 \to S_f$ to $D_+(f) = D(f) \cap \operatorname{Proj} S$ is the desired homeomorphism. To this end, we first compute the image of $D_+(f)$ in $\operatorname{Spec} S_f$. Recall that the homeomorphism $D(f) \to \operatorname{Spec} S_f$ simply sends a prime ideal $\mathfrak{p}$ to its extension $\mathfrak{p}S_f$. Of course, $\{ \mathfrak{p}S_f \colon \mathfrak{p} \in \operatorname{Proj} S \}$ is precisely the set of homogeneous prime ideals of S_f. (Every element thereof is clearly homogeneous, but what about conversely? Surely we can deduce this using the grading of S_f. Let $\mathfrak{q}$ be a homogeneous ideal. Then each of its generators are of the form a/f^d where a is homogeneous. $(a_1, \ldots, a_n)$ is homogeneous. Is it prime? Grading argument. What if it's S_+? Then f is inverted)
We first show surjectivity. Let $\mathfrak{p} \in \operatorname{Spec}(S_f)_0$. We claim that the homogeneous ideal $\sqrt{\mathfrak{p}S_f}$ is prime in S_f. Indeed, given $a, b \in S_f$ such that $ab \in \sqrt{\mathfrak{p}S_f} \iff (ab)^d \in \mathfrak{p}S_f$, we have $(a^{dr}/f^{\deg(a^d)})(b^{dr}/f^{\deg(b^d)}) \in \mathfrak{p}$, so without loss of generality $a^{dr}/f^{\deg(a^d)} \in \mathfrak{p} \implies a^{dr} \in \mathfrak{p}S_f$. Now since $a \in \mathfrak{p} \iff i(a)^d = a^d \in \mathfrak{p}S_f$, $i^{-1}(\sqrt{\mathfrak{p}S_f}) = \mathfrak{p}$, as desired.
For injectivity, let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} S_f$ be homogeneous. Image $i^{-1}(\mathfrak{p}) = (\mathfrak{p}S_f) \cap (S_f)_0$? □

We need to also that in $\mathbb{P}^n$ these correspond to the affine charts.

Define I.

Theorem 2.12.16. *Projective Nullstellensatz*

Given any homogeneous polynomial, its vanishing set viewed as a polynomial in $\mathbb{A}^{n+1}$ is the **affine cone**. Affine cone disjoint union V is the **projective cone**.

It is then clear that graded rings is the correct context in which to do projective geometry.

The downside of the classical construction is that it obfuscates the sheaf-theoretic aspect of projective spaces and that it is not very susceptible to a scheme-theoretic description. The intrinsic reason is that quotients are not well-behaved in the category of schemes.

We take a similar approach here to associating a scheme to a graded ring, using Proj, as we did with usual rings using Spec. Prima facie our sole inspiration for determining the points will be $\mathbb{P}^n$ from above; that the conditions we extrapolate are sufficient will be made clear afterwards. With the ideal corresponding to 0 in $k[x_0, \dots x_n]$ being precisely the irrelevant ideal, our first step will be to require out points to not contain A_+. Of course this is vacuous in $\mathbb{A}^n$ and amounts to just removing A_+, a maximal ideal, but this is not a given in arbitrary graded rings. Indeed, it is not prime if S_0 has zero divisors, hence certainly not maximal. More concretely, $(2, x_0, \dots, x_n)$ is a maximal ideal in $\mathbb{Z}[x_0, \dots, x_n]$ containing the irrelevant ideal.

On the other hand, the identification $v \sim \lambda v$ amounts to having only *homogeneous* prime ideals: . Analogously, it is only with homogeneous ideals that we get to define vanishing sets.

It should be clear that the points should be prime ideals, and homogeneous, in order to define vanishing sets. $\operatorname{Proj} A$ should in any case be a subspace of $\operatorname{Spec} A$ The exclusion of 0 is expressed by requiring the points to not contain the irrelevant $x \setminus \{0\}$

Several properties we would like $\operatorname{Proj} A$ to have. Be a quotient

Definition 2.12.17. Fix a graded ring S. We define $\operatorname{Proj} S$ to be scheme with underlying set homogeneous prime ideals of S not containing the S_+

Projective schemes intuition, functoriality

Example 2.12.18. Let us practice translating between homogeneous coordinates on different affine charts, working in $\mathbb{P}^2$. The coordinate $(x_{0/1}, x_{2/1})$ in U_1 stands for $[x_0/x_1, 1, x_2/x_1]$, whereas $(x_{1/0}, x_{2/0})$ stands for $[1, x_1/x_0, x_2/x_0]$. Clearly these are the same classes, so scalar by λ should result in the same representative. We see from the first component that $x_1/x_0 = \lambda$. Then in $\mathbb{A}^3$. We want to malize the first coordinate, so we get $(1, x_1/x_0,)$

All polynomials that can be dehomogenized to $x_{1/0}^2 + x_{2/0}^5 = 1$. $x_1^2 x_0^3 + x_2^5 = x_0^5$. We can freely multiply both sides by any x_0^n. Meets $x_0 = 0$ at $[0, \lambda, 0]$.

The classical construction (cone mod Gm) is not isomorphic to Proj. However, in a precise sense the functor it defines sheafify to Proj. We will make this precise in the far future.

2.13. Normality and Factoriality

Definition 2.13.1. Fix R a ring, S a subring. An element $r \in R$ is **integral over** S if it is a root of some monic polynomial in S. Such elements constitute the **integral closure of** S **in** R. An integral A domain whose integral closure in $K(A)$ is itself is **integrally closed**. A scheme is **normal** if all its stalks are integrally closed domains. Accordingly, a ring whose spectrum is normal is itself **normal**.

Normality indicates that the scheme is "almost smooth." I suppose this would be made precise in the chapter on dimension theory.

Proposition 2.13.2. *Normal schemes are reduced.*

Proof. Integral domains are reduced. □

Proposition 2.13.3. *For A an integrally closed domain, S a multiplicative submonoid thereof not containing 0, $S^{-1}A$ is an integrally closed domain.*

Proof. Let $\alpha \in K(S^{-1}A) = K(A)$ be a root of a monic polynomial $x^n + \frac{a_{n-1}}{s_{n-1}}x^{n-1} + \cdots + \frac{a_0}{s_0}$ in $S^{-1}A$. Put $s := \prod_{i=0}^{n-1} s_i$. Then clearly $s\alpha$ solves the monic polynomial $x^n + \frac{sa_{n-1}}{s_{n-1}}x^{n-1} + \cdots + \frac{s^n a_0}{s_0}$ in A, and hence belongs to A. It follows that $\alpha \in S^{-1}A$. □

Lemma 2.13.4. *For A an integral domain, then $A = \bigcap A_{\mathfrak{m}}$.*

Proof. We must show that if $\alpha \notin A$, then there exists some maximal ideal $\mathfrak{m}$ for which $\alpha \notin A_{\mathfrak{m}}$. Consider the maximal ideal containing the **ideal of denominators** $I := \{ r \in A \colon \alpha r \in A \}$. Indeed, if $\alpha \in A_{\mathfrak{m}}$, then $\alpha = a/b$ for some $a \in A$, $b \in A \setminus \mathfrak{m} \subseteq A \setminus I$. But then $\alpha b \in A \implies b \in I$. □

Proposition 2.13.5. *For A an integral domain, the following are equivalent:*

1. *A is integrally closed.*
2. *A is normal.*
3. *$A_{\mathfrak{m}}$ is integrally closed for all $\mathfrak{m} \in \operatorname{Spec}_m A$.*

Proof. (1) $\implies$ (2) follows from the above, and (2) $\implies$ (3) is trivial. For (3) $\implies$ (1), let f be a monic polynomial in A. Then it is a monic polynomial in every $A_{\mathfrak{m}}$, so its roots in $K(A)$ lie in $\bigcap A_{\mathfrak{m}} = A$. □

Corollary 2.13.6. *If a scheme X is quasicompact or locally finite type over a field, then normality can be checked at closed points.*

Proof. 5.1.E, 5.3.F □

Proposition 2.13.7. *Coproducts of normal schemes are normal. Hence normal schemes need not be integral.*

Definition 2.13.8. A scheme is **factorial** if all its stalks are UFDs. Accordingly, A is factorial if $\operatorname{Spec} A$ is.

Proposition 2.13.9. *Any non-zero localization of a UFD is a UFD. Hence UFDs are factorial.*

Proof. Let A be a UFD, S a multiplicative submonoid thereof not containing 0. Let $\alpha \in S^{-1}A$, so that $\alpha = a/r$ for $a \in A$, $r \in S$. Write $\prod a_i$ for the unique factorization of a into irreducibles. We claim that $\prod a_i r^{-1}$ is the unique factorization of a/r. □

Proposition 2.13.10. *UFDs are integrally closed. Hence factorial schemes are normal.*

Proof. Let α be an element of the integral closure of A in $K(A)$, so that it is the root of a monic polynomial $f = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ in A. Then $\alpha = r/s$ for $r, s \in A$, which may be assumed to be coprime since A is in particular a GCD domain. Trivially, s divides $0 = f(r/s)s^n = r^n + a_{n-1}sr^n + \cdots + a_0 s^n$. But all terms other than r^n are themselves divisible by s, so $s \mid r^n$ as well. Now if s is not itself a unit, then since A is a UFD, there is a prime element p dividing s, and thus r, contradicting that $\gcd(r, s) \in A^\times$. □

Proposition 2.13.11. $\operatorname{Spec} \mathbb{Z}$*,* $\mathbb{A}^n$*, and* $\mathbb{P}^n$ *are factorial.*

Proof. It is plain that $\mathbb{Z}$ and $k[x_1, \ldots, x_n]$ are UFDs. Since factoriality is stalk-local, it will suffice to show that each distinguished affine open $\operatorname{Spec}(k[x_0, \ldots, x_n]_f)_0$ of $\mathbb{P}^n$ is factorial. This will follows from that the base ring of a graded UFD is itself a UFD, $k[x_0, \ldots, x_n]_f$ being a UFD. Indeed, any factorization of a 0-form in the graded ring, consisting only of other 0-forms, remains valid and unique in the base ring. □

Do examples later