AN INTRODUCTION TO C^∞-RINGS AND SYNTHETIC DIFFERENTIAL GEOMETRY

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Abstract. We present a self-contained exposition on C^∞-rings, objects generalizing smooth manifolds to allow for a better-behaved category that at the same time accommodates infinitesimal and singular spaces. We begin with an investigation of various categorical deficiencies Man^∞, from which point of view we introduce the basic definitions surrounding C^∞-rings. Following a survey of elementary categorical properties of C^∞-Ring, we systematically develop the theory of affine C^∞-schemes, proving a C^∞-version of the Nullstellensatz. The paper culminates in an exploration of C^∞-rings as a foundation for synthetic differential geometry (SDG).
Our main contribution comes in the form of an algebro-geometric reorganization of existing literature, in addition to proofs of several folklore theorems. The reader is assumed to have a working knowledge of smooth manifold theory, category theory, and scheme-theoretic algebraic geometry.

Contents

1. Introduction

The category of smooth manifolds Man^∞ is notoriously poorly-behaved. Its inadequacy as a category in which to do geometry boils down to two primary causes: formal categorical deficiency and a restrictive class of objects. Both sources manifestly point to the embedding of Man^∞ into a suitably large ambient category, the conception of which will therefore be our main objective.

We begin by making the first cause precise. Recall that classically, there is a chain of forgetful functors $\mathsf{Man}^\infty \longrightarrow \mathsf{Top} \longrightarrow \mathsf{Set}$, where both Top and Set are bicomplete, and Set is furthermore *Cartesian closed*, i.e. the Hom functor and the Cartesian product form an adjoint pair $(\times \dashv \mathrm{Hom})$, known as the *product-exponential* adjunction or, somewhat colloquially, *currying.* While it is common knowledge that Man^∞ is neither bicomplete nor Cartesian closed, a characterization of the *extent* to which it fails is in order. This will come in the form of a taxonomy of the (co)limits that do exist in Man^∞ and the homsets that do possess natural smooth structures.

The most rudimentary of all (co)limits are products and coproducts. Since manifolds are locally *finite-dimensional* Euclidean spaces, only *finite direct product* of manifolds, equipped with the product topology, are products in Man^∞. The second-countability axiom, on the other hand, implies that only *countable disjoint unions* qualify as coproducts in Man^∞. By themselves, the failure of these naive constructions does not preclude the existence of all products and coproducts in Man^∞. An abstract argument, however, is possible:

Proposition 1.1 (Non-existence of Infinite Products and Uncountable Coproducts)**.** *Any infinite collection $\{ M_\alpha \}$ of manifolds that are not singletons, viewed as a discrete diagram, admits no limit. Furthermore, a collection $\{ N_\alpha \}$ of manifolds admits no colimit if $|\{ N_\alpha \}| > \aleph_0$.*

Proof. The arguments are inspired by [7] and [8]. Since $U\colon \mathsf{Man}^\infty \to \mathsf{Top}$ is continuous (see below), the product necessarily has $\prod M_\alpha$ as its underlying set, as well as the usual canonical projections. Assume towards a contradiction that there is indeed a smooth structure on $\prod M_\alpha$ making it a product in Man^∞. Put $I := \{ M_\alpha \}$. It can be seen set-theoretically that for any finite subset $J \subseteq I$, the map $\prod_{\alpha\in J} M_\alpha \to \prod_{\alpha\in I} M_\alpha$, induced from the projections $\prod_{\alpha\in J} M_\alpha \to M_\alpha$ and constant maps onto the other components, is an injection. But then with I being infinite, $\dim \prod_{\alpha\in I} M_\alpha$ would also be infinite as an upper bound for $\{ \sum_{\alpha\in J} \dim M_\alpha \colon J \subseteq I \text{ finite} \}$.
Now assume that $I' := \{ N_\alpha \}$ does admit a colimit. For each α, let $\chi_\alpha\colon \coprod N_\alpha \to \{ 0, 1 \}$ be the indicator function of the image of N_α onto the Sierpínski space. Then for $\alpha \neq \alpha'$, necessarily N_α and $N_{\alpha'}$ will lie in different path components, for otherwise their image will be sent to the same point under χ_α. But second-countability implies that $\coprod N_\alpha$ has countably many connected components, whereby I' must be countable as well. □

By virtue of this result, we shall restrict our attention to *finite* (co)limits. While to show finite (co)completeness it suffices to exhibit all pullbacks (resp. pushouts), this does *not* apply to Man^∞, as the following example shows. As a classification of the (co)limits in Man^∞ is far from the goal of this paper, in its stead we will undertake the much more manageable task of simply enumerating certain “good” classes of diagrams which do admit (co)limits.

Our example of choice to show the non-existence of all pushouts will be the union of the x and y-axes in the plane, a ubiquitous example across all flavors of geometry.

Example 1.2. The span $\mathbb{R} \longleftarrow \{ 0 \} \longrightarrow \mathbb{R}$ admits no colimit in Man^∞.

Proof. The argument is derived from [6]. Assume for contradiction that this diagram does admit a pushout, denoted (X, i_1, i_2), with $O := i_1(0) = i_2(0)$. Consider the maps $f, g\colon X \to \mathbb{R}$ induced from $(\mathrm{id}_1, 0)$, $(0, \mathrm{id}_2)$, respectively, so that $f \circ i_1(t) = t = g \circ i_2(t)$, $f \circ i_2(t) = 0 = g \circ i_1(t)$. Write v_j for the tangent vector induced by i_j. Then $df_O(v_1) = (f \circ i_1)'(0) = \frac{d}{dt}|_{t=0} t = 1$, and likewise we have $df_O(v_2) = 0$, $dg_O(v_1) = 0$, $dg_O(v_2) = 1$. It follows that the map $F\colon (X, O) \to (\mathbb{R}^2, 0)$ induced by f and g has surjective differential at O (the image of dF_O contains two linearly independent vectors). [12, Prop 4.1] then furnishes a neighborhood U of O in which F is a submersion, hence an open map, as seen by passing to the normal form of [12, Thm 4.12]. But $F(X)$, the union of the x and y-axes, manifestly contains no open set. □

Remark. This non-example already hints at the exclusion of “singular spaces” from Man^∞: even this most tame form of self-intersection, of two intersecting lines, cannot be studied directly using smooth manifold theory.

On the other hand, the forgetful functor $U\colon \mathsf{Man}^\infty \to \mathsf{Set}$, being represented by the one-point space $\{ * \}$, is continuous, whereby limits which do exist cannot be as exotic: at least their underlying sets can be computed explicitly, and canonical maps agree with the set-theoretic ones. Furthermore, a map into the pullback $X \times_Z Y$ is smooth precisely when its composition with the inclusion into $X \times Y$ is. Thus the determination of limits in Man^∞ amounts entirely to finding the correct topology and smooth structure to impose on a given set. Beware that since $\mathsf{Man}^\infty \to \mathsf{Top}$ is *not* continuous, this smooth topology needs not align with the usual initial topology.

This already shows, for example, that completing the above span into a square with maps $\mathbb{R} \to \mathbb{R}^2$ yields a *pullback* square. An amusing twist on the situation, again using the union of axes, happens to show that in general pullbacks do *not* exist in Man^∞:

Example 1.3. The cospan $\mathbb{R}^2 \xrightarrow{m} \mathbb{R} \longleftarrow \{ 0 \}$, where m sends $(x, y) \rightsquigarrow xy$, admits no limit in Man^∞.

Proof. This construction is due to [4]. As noted, if the pullback existed, its underlying set is again the union of the x and y-axes in $\mathbb{R}^2$, $X := \{ (x, y) \in \mathbb{R}^2 : xy = 0 \}$. Assume for contradiction that there is a smooth structure on X making it a pullback, so that for all smooth manifolds M, a map $f: M \to X$ is smooth iff $\iota \circ f: M \to \mathbb{R}^2$ is smooth, for $\iota: X \to \mathbb{R}^2$ the canonical map. Then the maps $i_1, i_2: \mathbb{R} \to X$ sending t to $(0, t)$, $(t, 0)$, respectively, are smooth. By direct computation we see that $d(\iota \circ i_1)_0 = (1, 0)$, $d(\iota \circ i_1)_0 = (0, 1)$, whereby the image of $d\iota_{(0,0)}$ contains two linearly independent vectors. That is, ι has surjective differential at $(0, 0)$, and a contradiction is obtained as a porism to the previous example. □

We now outline the main classes of (co)limits that are known to exist in Man^∞. Note that most of these are simply categorical reformulations of results proven in [12]. We begin with colimits.

Proposition 1.4. *For X a smooth manifold and $R \subseteq X \times X$ an equivalence relation, X/R admits a unique smooth structure making $\pi: X \to X/R$ a smooth submersion iff R is a submanifold of $X \times X$ such that either $\pi_i: R \twoheadrightarrow X$ is a submersion. In this case, it is the coequalizer*

$$R \longrightarrow X \rightrightarrows X/R.$$

Proof. The forward direction is [12, Thm 4.30], while the reverse is given in [1, Thm 5.9.5]. □

Corollary 1.5. *There is a unique smooth structure on the quotient of a smooth manifold M by a smooth, proper, and free Lie group G action making the projection $\pi: M \to M/G$ a smooth submersion.*

Proof. This is [12, Thm 21.10]. □

On the other hand, the most important class of limits which exist consists of pullbacks over the following class of maps:

Definition 1.6. Two smooth maps $f: X \to Z$, $g: Y \to Z$ are **transversal** if for all $(x, y) \in X \times Y$ with $f(x) = z = g(y)$, $\operatorname{im} df + \operatorname{im} dg = T_z Z$.

Proposition 1.7. *Given transversal maps $f: X \to Z$, $g: Y \to Z$, the pullback $X \times_Z Y$ exists and is preserved by the tangent bundle functor.*

Proof. See [18]. □

Remark. Quotients by equivalence relations and fibered products, as demonstrated to exist conditionally in Man^∞ by Proposition 1.4 and 1.7, are the (co)limits of greatest interest in practice; it is therefore necessary for any sensible enlarging of Man^∞ to preserve them. But this is not all that we shall ask for: quotients by general equivalence relations need not be expressible as a colimit, whereby mere cocompleteness is insufficient, and in some precise sense naive pullbacks are not the right objects to consider. As the two transcend the nature of ordinary (co)limits in disparate ways, different techniques are called for: the former is handled by C^∞-stacks and orbifolds [10], the latter by derived manifolds [19]. In any case, C^∞-rings will provide a base model for either generality.

We now consider the failure of Cartesian closedness, i.e. for which smooth manifolds M, N is $\mathsf{Man}^\infty(M, N)$ itself a smooth manifold. For now we just record [17] and [20] as key references and note that in Man^∞ itself such smooth mapping spaces are fairly sparse. It appears that the classical solution is to endow the mapping spaces with the *Frechet topology* to render them as *infinite dimensional* smooth manifolds. There is, however, the question of why Cartesian closedness is even to be desired. For one, path spaces, i.e. $\mathsf{Man}^\infty(I, X)$, should ideally be objects inside the category, so that we may freely switch between the two forms of homotopy, i.e. $X \times I \to Y$ and $X \to Y^I$.

We now move on to address the second cause, namely the exclusion of key geometric spaces from Man^∞. Before setting out to provide instances of such spaces, let us first discuss a hidden, third defect of Man^∞: smooth manifolds are by nature coordinate-dependent, in the sense that one needs to pick an atlas for the specification of a structure, on which choice the smoothness of maps is contingent. While it is the natural state of physicists to be working with coordinates, we counter with the remark made by Herman Weyl in [23, p. 90] that "the introduction of numbers as coordinates is an act of violence." Since it is the duty of a student of mathematics to heed the advice of a mathematical giant, we proceed by suggesting an alternative framework that evades coordinates.

Instead of considering the manifold X itself, we may describe instead the highly non-Noetherian ring $C^\infty(X)$, from which to recover the smooth structure. A scaffold for recovering the isomorphism class of X from $C^\infty(X)$ can be found in [12, Prob 2.10]: starting with a topological manifold X and choosing an $\mathbb{R}$-subalgebra of $C^0(X)$, if there exists a smooth structure on X whose $C^\infty(X)$ coincides with this subalgebra, then the Hom functor $\mathsf{Man}^\infty(X, -)$, whence the smooth structure on X is fully determined. In particular, any other choice of atlas compatible with the present one will give rise to the same $C^\infty(X)$. In fact, from $C^0(X)$, the points and topology on X can also be recovered:

Theorem 1.8. *For M a compact Hausdorff topological space, maximal ideals of $C^0(M)$ are precisely the vanishing ideals $\{\, I(p) \colon p \in M \,\}$, where $I(p) := \left\{\, f \in C^0(M) \colon f(p) = 0 \,\right\}$, and $I(p) = I(q) \implies p = q$. Hence $I(-) \colon p \rightsquigarrow I(p)$ defines a correspondence*

$$\{\, p \in M \,\} \rightleftarrows \operatorname{Spec}_m C^0(M) := \left\{\, \textit{maximal ideals of } C^0(M) \,\right\}.$$

Furthermore, endowing $\{\, I(p) \,\}$ with the ***Zariski topology*** *generated by the base of* ***distinguished open sets*** $\left\{\, \tilde{D}(f) \colon f \in C^0(M) \,\right\}$ *where*

$$\tilde{D}(f) := \{\, I(p) \colon f \notin I(p) \,\},$$

$I(-) \colon M \to \operatorname{Spec}_m C^0(M)$ is a homeomorphism.

Proof. The maximality of $I(p)$ follows from the following short exact sequence:

$$0 \longrightarrow I(p) \longrightarrow C^0(M) \xrightarrow{[f] \rightsquigarrow f(p)} \mathbb{R} \longrightarrow 0.$$

That $I(p) \neq I(q)$ for $p \neq q$ follows from Urysohn's lemma: as $\{\, p \,\}$, $\{\, q \,\}$ are closed there is a continuous function which is 0 at p and 1 at q. It remains to show that an ideal I not contained in any $I(p)$ is the whole of $C^0(M)$. In such an ideal, for all $p \in P$ there exists some $f_p \in I$ with $f(p) \neq 0$. Take an open cover of M by $\{\, D(f_p) \,\}$ where $D(f) := \{\, p \in M \colon f(p) \neq 0 \,\}$. Compactness furnishes a finite set of functions $\{\, f_i \,\}$ such that for every $p \in M$, there exists some f_i with $f_i(p) \neq 0$. Now manifestly $\sum f_i^2$ is uniformly non-zero, hence invertible, and it follows that $I = C^0(M)$.
It remains to show that $I(-)$ sends $D(f) := \{\, p \in M \colon f(p) \neq 0 \,\}$ to $\tilde{D}(f)$, as $\left\{\, D(f) \colon f \in C^0(M) \,\right\}$ forms a base for M. But it is immediate from definition that for $f \in C^0(M)$, $f(p) \neq 0$ iff $f \notin I(p)$. □

Remark. The proof works for C^0 replaced with C^∞ if M admits a smooth structure. This is by virtue, of course, of the smooth analogue of Urysohn's lemma. We may then conclude that $\operatorname{Spec}_m C^\infty(M) \cong \operatorname{Spec}_m C^0(M)$.

One consequence of this approach is that if one does not axiomatize the admissible subalgebras (i.e. come up with an analytic condition for "arising from a smooth structure"), we potentially open the door to a significantly larger class of spaces that are not manifolds, including spaces with wild singularities. As stated at the outset, this is in part a desirable feature, though certainly some degree of restriction is needed in order for techniques reminiscent of manifold theory to still apply. Note also that compactness is a crucial hypothesis in the proof of the above; alleviating this condition will be one of the goals of the theory of C^∞-rings.

We end this introduction with a discussion of prototypical "infinitesimal spaces," the inclusion of which as geometric spaces may seem like some sort of "freshman's dream" to the educated mathematician, to whom it would appear utterly uncivilized to not treat them with analytic techniques. But this mindset in which so many are ingrained proves to be contrary to the *synthetic* mindset with which the theory was historically developed by its progenitors. Namely, while in thought we constantly engage with the idea of zooming into "infinitesimal neighborhoods" of a point, and for instance tangent vectors are but "infinitesimal curves," these ideas can hardly be taken literally and in practice requires much mechanical effort to articulate.

We illustrate the situation with the tangent bundle of a manifold. Despite the variety in the forms in which it may be presented, the most fundamental one appears to be as the representing object of the *derivation functor* $\mathsf{Man}^\infty \to \mathsf{Set}$ [5] sending each manifold X to the set of pairs (f, δ) where f is a smooth map $X \to Y$ and δ is a derivation $C^\infty(Y) \to C^\infty(X)$, and each smooth map to the evident set map induced by precomposition. In practice, however, one rarely works at this level of abstraction and treats individual tangent vectors as either tangent curves or maps $C^\infty(M) \to \mathbb{R}$, making the geometric nature of the tangent bundle fall short. One is then forced to work with tangent spaces *module-theoretically* and *pointwise*.

An alternative to this narrative is already suggested by algebraic geometry, where for an affine k-scheme B the *ring of dual numbers* $\operatorname{Spec} B[\epsilon]/(\epsilon^2)$ serves as a model for an infinitesimal locus, i.e. $\mathrm{Der}_k(A, B) \cong K\text{-}\mathsf{Alg}(A, B[\epsilon]/(\epsilon^2))$ [21]. As such, a tangent vector is simply a morphism $A \to B[\epsilon]/(\epsilon^2)$! Still, the module-theoretic side of the story is not lost, as by moving to the category k-Mod we still have $\mathrm{Der}_k(A, M) \cong K\text{-}\mathsf{Mod}(\Omega_{A/R}, M)$, for $\Omega_{A/R}$ the module of *relative Kähler differentials*. This corresponds to the cotangent bundle on the side of smooth manifolds.

The need to obtain a classifying space D of tangent vectors on a manifold M *internal* to the category, in such a way that M^D also belongs to the category, becomes all the more tantalizing. As such, this will be made into one of the axioms that a model of SDG must be satisfy, in addition to several others, the most powerful of which even allows one to perform synthetic integration.

Now tangent vectors only exemplify one type of infinitesimal objects, those of *nilpotent* nature. The other end of the spectrum consists of *invertible* nilpotent, e.g. infinitesimal numbers of the hyperreals which can be inverted to obtain infinite numbers. The discipline in charge of formalizing these is traditionally *non-standard analysis*, the methods of SDG lend another means of doing so, though there are fundamental differences between the two approaches that are too intricate to explicate in this introduction.

The program of SDG is much grander in scope than these few facts, and as learners ourselves we are woefully unqualified to be introducing the subject and can hardly do justice to its underlying philosophy; for a more elaborate and convincing summary, see the Preface to [9].

2. The Category of C^∞-Rings

A naive, or generic attempt at ameliorating the categorical defects of Man^∞ that fails to take into account the idiosyncrasies of its objects, smooth manifolds, will be to take its *free cocompletion*, i.e. using the covariant Yoneda embedding, which sends $X \rightsquigarrow h_X := \mathrm{Hom}(-, X)$, to view Man^∞ as a full subcategory of $\widehat{\mathsf{Man}^\infty} := \mathrm{Fun}((\mathsf{Man}^\infty)^{\mathrm{op}}, \mathsf{Set})$. What is unnatural about this approach is that in practice, one rarely studies maps *into* a manifold; the whole yoga of local coordinate computations revolves around maps *out of* the manifold, i.e. $\mathrm{Hom}(X, \mathbb{R}^n)$. One might justly protest that since local charts are *diffeomorphisms*, our remarks on directionality are somewhat superficial. We are rescued by the fact that the ability to break down these charts into *components*, i.e. functionals in $C^\infty(X) := \mathrm{Hom}(X, \mathbb{R})$, is peculiar to maps out of X, essentially by virtue of the universal property of products.

From this discussion it becomes evident that the *co*-Yoneda embedding will be more amenable to our task. As an aside, this choice of variance does have philosophical implications, as in taking $C^\infty(X)$ we are indeed moving from algebra to geometry, whereby a categorical dualizing is warranted. The first evidence for the effectiveness of this approach springs from the classical result that a smooth manifold can be functorially recovered from the $\mathbb{R}$-algebra $C^\infty(X)$, a fact that is apparently stronger than that which we are claiming here, that X may be recovered from the *functor* h^X. Continued investigations along this vein lead one to realize $C^\infty(X)$ as the ring of global sections of a sheaf $\mathscr{O}_X$, whence $(X, \mathscr{O}_X)$ forms a *locally ringed space*, and manifolds are thrust into the same framework as that of schemes. This line of thought, however, will not be pursued further in this paper; for a reference see [22].

The discrepancy between $C^\infty(X)$ and h^X, to be sure, is very much expected, as while the former result makes use of non-trivial geometry (the points of X, for instance, correspond to the maximal ideals of $C^\infty(X)$), the latter is conceived of purely formally. In spite of our primary objective being the axiomatization and study of the latter, the interplay between (generalizations of) these two objects will be maintained, and in Corollary X we will deduce the aforementioned classical result from the formal statement. To this end, we observe that the category $\mathrm{Fun}(\mathsf{Man}^\infty, \mathsf{Set})$, unfiltered, contains too much excess information for this desideratum to be met. More precisely, an admissible functor $A\colon \mathsf{Man}^\infty \to \mathsf{Set}$ should allow for $A(\mathbb{R})$ to carry a natural ring structure. The natural move is to leverage the ring structure on $\mathbb{R}$: we may pass the addition and multiplication maps on $\mathbb{R}$ (which are of course smooth) through A to obtain $A(a), A(m)\colon A(\mathbb{R}^2) \to A(\mathbb{R})$. The structure maps on $A(\mathbb{R})$, however, have as their domain $A(\mathbb{R})^2$, whereby A is required to *preserve finite products*. One readily verifies that the ring axioms are satisfied from this one condition alone.

We may also simplify the domain category, taking inspiration from functorial algebraic geometry. Recall that for A a ring, the composition

$$\mathsf{Sch}/A \xrightarrow{h} \widehat{\mathsf{Sch}/A} \xrightarrow{r} \widehat{\mathsf{AffSch}/A} \xrightarrow{\simeq} \mathsf{Set}^{A\text{-}\mathsf{Alg}}$$

is fully faithful. Since schemes are glued from affine schemes just as manifolds are built out of Euclidean spaces, we define CartSp to be full subcategory of Man^∞ of Euclidean spaces and note the following:

Proposition 2.1. *The composition* $\mathsf{Man}^\infty \xrightarrow{よ} \mathrm{Fun}((\mathsf{Man}^\infty)^{\mathrm{op}}, \mathsf{Set}) \xrightarrow{r} \mathrm{Fun}(\mathsf{CartSp}^{\mathrm{op}}, \mathsf{Set})$ *is fully faithful.*

Proof. We need only show that r is fully faithful on the essential image of よ. That is, for any smooth manifolds X, Y, the map $\mathrm{Hom}(h_X, h_Y) \to \mathrm{Hom}(h^a_X, h^a_Y)$ induced by r, where, for instance, by h^a_X we understand $r(h_X)$, is a bijection. But this is merely that morphisms glue: given a morphism $\varphi\colon h^a_X \to h^a_Y$, we define $\psi\colon h_X \to h_Y$ on any smooth manifold U as the map sending a morphism $\pi\colon U \to X$ to the unique morphism $U \to Y$ glued from $\varphi_{U_i}(\pi|_{U_i})$, for $\{ U_i \}$ an atlas of U. $\square$

Remark. This result serves only as heuristic evidence for the possibility of reducing the codomain of the copresheaf category to CartSp. Notably, functor categories are *not* self-dual: for instance, $\mathrm{Fun}(\mathsf{CartSp}, \mathsf{Set})^{\mathrm{op}} \ncong \mathrm{Fun}(\mathsf{CartSp}^{\mathrm{op}}, \mathsf{Set})$, whereby the above argument does not immediately dualize to the co-Yoneda embedding restricting to $\mathrm{Fun}(\mathsf{CartSp}, \mathsf{Set})$. We will, however, show that Man^∞ embed into $\mathrm{Fun}(\mathsf{CartSp}, \mathsf{Set})^{\mathrm{op}}$, but only after the basic machinery of C^∞-rings has been developed. That the presheaf category accommodates Man^∞ whilst the copresheaf category accommodates $(\mathsf{Man}^\infty)^{\mathrm{op}}$ is no coincidence, and in the broadest context the interplay between the presheaf and coprehseaf categories constitutes what is called *Isbell duality*.

We are now in a position to formally introduce C^∞-rings.

Definition 2.2. A C^∞**-ring** is a copresheaf $\mathfrak{A}\colon \mathsf{CartSp} \to \mathsf{Set}$ that preserves finite products. The **base ring** of $\mathfrak{A}$ is the set $A := \mathfrak{A}(\mathbb{R})$ whose $\mathbb{R}$-algebra structure is endowed by $\mathfrak{A}(a)$, $\mathfrak{A}(m)$, for $a, m\colon \mathbb{R}^2 \to \mathbb{R}$ the addition and multiplication maps on $\mathbb{R}$, respectively, and the map $\mathbb{R} \times \mathfrak{A}(\mathbb{R}) \to \mathfrak{A}(\mathbb{R})$ whose component on t is given by $\mathfrak{A}(\lambda_t)$, for λ_t multiplication by t on $\mathbb{R}$.
Taking a **morphism of** C^∞**-rings** to be simply a natural transformation $\mathfrak{A} \Rightarrow \mathfrak{B}$, we obtain a full subcategory $C^\infty\text{-}\mathsf{Ring}$ of $\mathrm{Fun}(\mathsf{CartSp}, \mathsf{Set})$. There is then a forgetful functor $U\colon C^\infty\text{-}\mathsf{Ring} \to \mathbb{R}\text{-}\mathsf{Alg}$ sending each C^∞-ring to its base ring and each morphism $\mathfrak{A} \to \mathfrak{B}$ to its leg at $\mathbb{R}$.

Remark. The base ring mostly determines the action of $\mathfrak{A}$: by definition $\mathfrak{A}(\mathbb{R}^n) \cong A^n$, and specifying the action of $\mathfrak{A}$ on all maps $\mathbb{R}^n \to \mathbb{R}$ suffices to determine all maps $\mathfrak{A}(f) = \mathfrak{A}(\mathbb{R}^n) \to \mathfrak{A}(\mathbb{R}^m)$, by the universal property of products. Furthermore, if there is any relation $f = \sum_i \prod_j g_{ij}$, then we have

$\mathfrak{A}(f) = \sum_i \prod_j \mathfrak{A}(g_{ij})$, as the ring operations on A are precisely $\mathfrak{A}(a)$ and $\mathfrak{A}(m)$, and $\mathfrak{A}$ preserves compositions.
The situation is even better for morphisms: a morphism $\Phi\colon \mathfrak{A} \to \mathfrak{B}$ is determined entirely by the homomorphism $\phi\colon A \to B$, as by naturality

$$\begin{tikzcd}
A^n \arrow[r] \arrow[dr, "\pi_i"'] & \mathfrak{A}(\mathbb{R}^n) \arrow[r, "\phi'"] \arrow[d, "\mathfrak{A}(\pi_i)"] & \mathfrak{B}(\mathbb{R}^n) \arrow[r] \arrow[d, "\mathfrak{B}(\pi_i)"'] & B^n \arrow[dl, "\pi_i"] \\
& A \arrow[r, "\phi"] & B,
\end{tikzcd}$$

the homomorphisms $\mathfrak{A}(\mathbb{R}^n) \to \mathfrak{B}(\mathbb{R}^n)$ must be obtained from $\phi^n\colon A^n \to B^n$ via compositions with the canonical isomorphisms. Furthermore, naturality may be checked only with respect to smooth maps $\mathbb{R}^n \to \mathbb{R}$, again by the universal property of products.

The primordial example of a C^∞-ring is the ring of smooth functions from a smooth manifold M to $\mathbb{R}$; this is evidently the image of M under the co-Yoneda embedding, and in the case when $M = \mathbb{R}^n$ is a *representable functor.* That it preserves products then follows from Hom functors being continuous in the second argument and the inclusion $C^\infty\text{-}\mathsf{Ring} \to \mathrm{Fun}(\mathsf{Man}^\infty, \mathsf{Set})$ is fully faithful, hence reflects limits. As in [15], we will define two sets associated to elements of $C^\infty(M)$:

Definition 2.3. Fix M a topological space. For any $f\colon M \to \mathbb{R}$, the **zeroset** of f is the closed subset $Z(f) := f^{-1}(0)$, whose complement we denote by U_f.

It turns out, however, that category of C^∞-ring is not at all confined to smooth manifolds; the following variation on the $C^\infty(M)$ construction already furnishes a plethora of "non-smooth" spaces.

Example 2.4. We define the category $\widetilde{\mathsf{CartSp}}$ to consist of objects all subsets of $\mathbb{R}^n$ and of morphisms all continuous map $\varphi\colon X \to Y$ for which exists a smooth map $U \to \mathbb{R}^m$ restricting to φ, for $U \supseteq X$ an open subset of $\mathbb{R}^n$. We extend $C^\infty\colon \mathsf{CartSp}^{\mathrm{op}} \to C^\infty\text{-}\mathsf{Ring}$ to a functor $\widetilde{\mathsf{CartSp}}^{\mathrm{op}} \to C^\infty\text{-}\mathsf{Ring}$ sending every $X \subseteq \mathbb{R}^n$ to the C^∞-ring

$$C^\infty(X) := \left\{ f\colon X \to \mathbb{R}\colon \exists \text{ open } U \supseteq X, \tilde{f} \in C^\infty(U) \text{ with } \tilde{f}|_X = f \right\}$$

whose action is given by post-composition (identifying the n-tuples in $C^\infty(X)^n$ with single functions $X \to \mathbb{R}^n$), and every $\varphi\colon X \to Y$ to φ^*, pre-composition with φ.
From this construction we can already see how $C^\infty\text{-}\mathsf{Ring}$ is much larger than Man^∞: even subsets of Euclidean space carrying no natural smooth structure can be made into C^∞-rings. As a somewhat drastic example, the Cantor set, by this construction is a C^∞-ring. This is fundamentally distinct from the imposition of smooth structures on the graph of non-differential functions by so-called "transport of structure:" in so doing the underlying topology of the graph is irrevocably altered.

The representable functor $C^\infty(\mathbb{R}^n)$ happens to provide a local left adjoint to the forgetful functor U, and will thus be entitled **free C^∞-ring on n generators.**

Lemma 2.5. *The map $\Phi\colon \mathbb{R}\text{-}\mathsf{Alg}(\mathbb{R}[x_1, \ldots, x_n], A) \to C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}^n), \mathfrak{A})$ sending*

$$(f\colon \mathbb{R}[x_1, \ldots, x_n] \to A) \rightsquigarrow (\psi\colon \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}) \to A \text{ such that } \pi_i \mapsto f(x_i))$$

is an isomorphism natural in the second component.

Proof. We first show that Φ is well-defined, i.e. that ψ is indeed defined by its values on π_i. To this end, consider the diagram

$$\begin{tikzcd}
\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^n) \arrow[r, "\psi^n"] \arrow[d, "f_*"'] & A^n \cong \mathfrak{A}(\mathbb{R}^n) \arrow[d, "\mathfrak{A}(f)"] \\
\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}) \arrow[r, "\psi"] & A.
\end{tikzcd}$$

The claim follows readily if one chases $\mathrm{id}_{\mathbb{R}^n} = (\pi_1, \dots, \pi_n)$. Since the map sending ψ to the $\mathbb{R}$-algebra homomorphism f with $f(x_i) = \psi(\pi_i)$ is evidently inverse to Φ, it remains to show naturality. Indeed, for any morphism $\mu\colon A \to B$, we have $\mu_* \circ \Phi_A(f) = (\pi_i \mapsto \mu_{\mathbb{R}}(f(x_i))) = \Phi_B \circ (\mu_{\mathbb{R}})_*(f)$. $\square$

Corollary 2.6. *For $\mathfrak{A}$ a C^∞-ring, each $a \in A$ corresponds uniquely to a C^∞-morphism $\zeta_a\colon C^\infty(\mathbb{R}) \to \mathfrak{A}$.*

Corollary 2.7. *$C^\infty(\mathbb{R}^0) = \mathbb{R}$ is the initial object in C^∞-Ring.*

In order to promote this to a full adjunction, we must investigate the limits and colimits in C^∞-Ring.

Proposition 2.8. *C^∞-Ring is complete and filtered-cocomplete. Furthermore, these limits and colimits are computed pointwise.*

Proof. Since $\mathrm{Fun}(\mathsf{CartSp}, \mathsf{Set})$ is complete, every diagram $D\colon J \to \mathrm{Fun}(\mathsf{CartSp}, \mathsf{Set})$ of finite product-preserving functors admits a pointwise computed limit, which continues to preserve finite products as

$$(\varprojlim D)(\prod \mathfrak{A}_i) = \varprojlim D_j(\prod \mathfrak{A}_i) \cong \varprojlim \prod D_j(\mathfrak{A}_i) \cong \prod \varprojlim D_j(\mathfrak{A}_i) = \prod (\varprojlim D)(\mathfrak{A}_i),$$

whence the completeness of C^∞-Ring follows. Now recall that a colimit over $D\colon J \to C^\infty$-Ring is *filtered* if J is a *filter category*, i.e. if every finite diagram therein admits a cocone, and that filtered colimits commute with finite limits. The argument above then applies mutatis mutandis. $\square$

Remark. This mirrors the theory of smooth manifolds, in which as we recall limits are also considerably easier to compute than colimits.

It turns out that C^∞-Ring is in fact cocomplete, albeit general colimits are not computed pointwise. To see this, we first consider coequalizers and quotients:

Proposition 2.9. *Fix a C^∞-ring $\mathfrak{A}$. For any ideal I of A, A/I is the base ring of a C^∞-ring $\mathfrak{A}/I$, unique up to unique isomorphism, that is the coequalizer*

$$\mathfrak{R} \underset{p_2 \circ \iota}{\overset{p_1 \circ \iota}{\rightrightarrows}} \mathfrak{A} \xrightarrow{\pi} \mathfrak{A}/I,$$

where $\mathfrak{R}$ is given as the equalizer $\mathfrak{R} \xrightarrow{\iota} \mathfrak{A} \times \mathfrak{A} \underset{\pi \circ p_2}{\overset{\pi \circ p_1}{\rightrightarrows}} \mathfrak{A}/I$, for p_1, p_2 the projections out of $\mathfrak{A} \times \mathfrak{A}$.

Proof. We first show that each map $\mathfrak{A}(f)\colon A^n \to A$ arising from $f\colon \mathbb{R}^n \to \mathbb{R}$ descends to a map $(A/I)^n \to A/I$. This map will be defined as sending $(a_1 + I, \dots, a_n + I) \rightsquigarrow \mathfrak{A}(f)(a_1, \dots, a_n) + I$. To show that it is well-defined, it suffices to show that given alternative representatives $b_1, \dots, b_n$, $\mathfrak{A}(f)(a_1, \dots, a_n) - \mathfrak{A}(f)(b_1, \dots, b_n) \in I$ (note that $(A/I)^n$ is *not* a ring, whereby a kernel-type argument does not work). By Taylor's theorem [12, Thm C.15] in the 0th-order (also known as Hadamard's lemma), we have the identity

$$f(x_1, \dots, x_n) - f(y_1, \dots, y_n) = \sum (x_i - y_i) g_i(x_1, \dots, x_n, y_1, \dots, y_n),$$

for g_i smooth maps $\mathbb{R}^{2n} \to \mathbb{R}$, which by the earlier remark corresponds to the identity

$$\mathfrak{A}(f)(a_1, \dots, a_n) - \mathfrak{A}(f)(b_1, \dots, b_n) = \sum (a_i - b_i) \mathfrak{A}(g_i)(a_1, \dots, a_n, b_1, \dots, b_n).$$

Now since $a_i - b_i \in I$ for all i, we conclude that $\mathfrak{A}(f)(a_1, \dots, a_n) - \mathfrak{A}(f)(b_1, \dots, b_n) \in I$, as desired. To see that the projection is a C^∞-morphism, write $\widetilde{\mathfrak{A}} := \mathfrak{A}/I$, and note that $\pi(\mathfrak{A}(f)(a_1, \dots, a_n)) = \mathfrak{A}(f)(a_1, \dots, a_n) + I = \widetilde{\mathfrak{A}}(f)(a_1 + I, \dots, a_n + I) = \widetilde{\mathfrak{A}}(f)(\pi^n(a_1, \dots, a_n))$.
By the construction of $\mathfrak{R}$, we have $\pi \circ p_1 \circ \iota = \pi \circ p_2 \circ \iota$, so it remains to exhibit for each compatible $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ a unique C^∞-morphism $\tilde{\pi}\colon \mathfrak{A}/I \to \mathfrak{B}$. Working in Ring, we see that the underlying homomorphism of $\tilde{\pi}$ exists and is uniquely determined. That it is indeed a C^∞-morphism follows from that π is one. $\square$

Proposition 2.10. *For any $f, g\colon \mathfrak{A} \to \mathfrak{B}$, we have a coequalizer diagram*

$$\mathfrak{A} \underset{g}{\overset{f}{\rightrightarrows}} \mathfrak{B} \xrightarrow{\pi} \mathfrak{B}/(f(a) - g(a)).$$

Proof. For $\mathfrak{C}$ an arbitrary C^∞-ring, C^∞-homomorphism φ, there is a unique ring homomorphism $\tilde{\varphi}\colon B/(f(a) - g(a)) \to C$ making the obvious diagram commute, as $B/(f(a) - g(a))$ is known to be the coequalizer in Ring. Since any C^∞-morphism $\mathfrak{B}/(f(a) - g(a)) \to \mathfrak{C}$ must descend to such a homomorphism, uniqueness is clear. That $\tilde{\varphi}$ indeed respects smooth maps follows from the same property of φ. $\square$

At this point we have already laid the requisite groundwork for extending $\mathbb{R}[x_1, \dots, x_n] \rightsquigarrow C^\infty(\mathbb{R}^n)$ to a global left adjoint. Having this result (or more precisely, the cocontinuity of the free functor) at the ready will aid us in constructing the remaining colimits by way of furnishing concrete descriptions of their underlying rings.

Theorem 2.11. *The functor $C^\infty\colon \mathbb{R}\text{-}\mathsf{Alg} \to C^\infty\text{-}\mathsf{Ring}$ sending*

$$\mathbb{R}[X] \rightsquigarrow \varinjlim_{S \subseteq X \text{ finite}} C^\infty(\mathbb{R}^S), \quad \mathbb{R}[X]/I \rightsquigarrow C^\infty(\mathbb{R}^X)/(\iota(I)),$$

where $\iota\colon \mathbb{R}[X] \to C^\infty(\mathbb{R}^X)$ is canonical inclusion, and every morphism to itself is left adjoint to the forgetful functor $U\colon C^\infty\text{-}\mathsf{Ring} \to \mathbb{R}\text{-}\mathsf{Alg}$.

Proof. Let us first make precise how ι is defined. Recall that $\mathbb{R}[X]$ is itself $\varinjlim_{S \subseteq X \text{ finite}} \mathbb{R}[S]$, whereby ι may be induced from the inclusions $\mathbb{R}[S] \to C^\infty(\mathbb{R}^S)$ identifying polynomials with *polynomial functions*. With this in mind, we show the adjunction isomorphism for when X is finite. For $\varphi\colon \mathbb{R}[S]/I \to A$ a ring homomorphism, put $\tilde{\varphi} := \varphi \circ \pi$, which corresponds under the isomorphism of Lemma 2.5 to a C^∞-morphism $\tilde{\psi}$. Now $\tilde{\psi}$ factors (uniquely) through $C^\infty(\mathbb{R}^S)/\iota(I)$ iff $\iota(I) \subseteq \ker \tilde{\psi}$. The condition for producing φ from ψ is deduced analogously. Hence it would follow from $\iota(I) \subseteq \ker \tilde{\psi} \iff I \subseteq \ker \tilde{\varphi}$ that $\varphi \rightsquigarrow \psi$ defines an isomorphism $\mathbb{R}\text{-}\mathsf{Alg}(\mathbb{R}[S]/I, A) \to C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}^S)/\iota(I), \mathfrak{A})$. This clearly holds, as every element of $\iota(I)$ is of the $\sum a_i \pi_i$ and $\tilde{\psi}$ sends $\pi_i \rightsquigarrow \tilde{\varphi}(x_i)$. The following formal argument lifts this isomorphism to the general case:

$$\begin{aligned}
C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}^X)/\iota(I), \mathfrak{A}) &\cong C^\infty\text{-}\mathsf{Ring}(\varinjlim C^\infty(\mathbb{R}^S)/\iota(I_S), \mathfrak{A}) \\
&\cong \varprojlim C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}^S)/\iota(I_S), \mathfrak{A}) \\
&\cong \varprojlim \mathbb{R}\text{-}\mathsf{Alg}(\mathbb{R}[S]/I_S, A) \\
&\cong \mathbb{R}\text{-}\mathsf{Alg}(\varinjlim \mathbb{R}[S]/I_S, A) \\
&\cong \mathbb{R}\text{-}\mathsf{Alg}(\mathbb{R}[X]/I, A),
\end{aligned}$$

where $I_S := i_S^{-1}(I)$. Naturality in either argument follows from straightforward diagram chases. $\square$

Remark. We will take $C^\infty(\mathbb{R}^X)$ as a *definition* for the set of smooth functions on the space $\mathbb{R}^X := \prod_{x \in X} \mathbb{R}$, i.e. as continuous maps $f\colon \mathbb{R}^X \to \mathbb{R}$ for which there exist some finite $S \subseteq X$, $\hat{f}\colon \mathbb{R}^S \to \mathbb{R}$ such that $f = \hat{f} \circ \pi_S$, a condition we will henceforth refer to as "smoothly depending on finitely many coordinates."

It remains to demonstrate the existence of binary coproducts. Other than implying the cocompleteness of $C^\infty\text{-}\mathsf{Ring}$, this would also provide a natural construction for C^∞-polynomial rings.

Definition 2.12. For X a set, $\mathfrak{A}$ a C^∞-ring, the **C^∞-polynomial ring in X over $\mathfrak{A}$**, denoted $\mathfrak{A}[X]$, is the coproduct $\mathfrak{A} \otimes_\infty C^\infty(\mathbb{R}^X)$.

Remark. Like how polynomials give rise to polynomial functions, elements of $\mathfrak{A}[t]$ may be evaluated at every point of $\mathfrak{A}$. Indeed, given $a \in \mathfrak{A}$, we define a C^∞-morphism $\mathrm{ev}_a\colon \mathfrak{A}[t] \to \mathfrak{A}$, entitled **evaluation at** a, sending each $f \in \mathfrak{A}[t]$ to the element of $\mathfrak{A}$, denoted $f(a)$, corresponding to

$$C^\infty(\mathbb{R}) \xrightarrow{\zeta_f} \mathfrak{A}[t] \cong \mathfrak{A} \otimes C^\infty(\mathbb{R}) \xrightarrow{\mathrm{id}_{\mathfrak{A}} \times \zeta_a} \mathfrak{A}.$$

Running through the various evaluation morphisms, we obtain a map $\mathfrak{A}[t] \to \mathrm{End}_{C^\infty\text{-}\mathsf{Ring}}(\mathfrak{A})$ sending each $f(t)$ to its *polynomial function* $\mathfrak{A} \to \mathfrak{A}$ sending $a \rightsquigarrow \mathrm{ev}_a(f)$.

Our strategy for constructing binary coproducts will hinge on the universal property of $C^\infty(\mathbb{R}^n)$. First we introduce some terminology:

Definition 2.13. A C^∞-ring is **finitely generated** if it is of the form $C^\infty(\mathbb{R}^n)/I$, and furthermore **finitely presented** if I is finitely generated.
The coproduct of two C^∞-rings $\mathfrak{A}, \mathfrak{B}$, if existent, will be denoted $\mathfrak{A} \otimes_\infty \mathfrak{B}$. More generally, given C^∞-morphisms $\mathfrak{A} \to \mathfrak{C}$, $\mathfrak{B} \to \mathfrak{C}$, the pushout will be denoted $\mathfrak{A} \otimes_{\mathfrak{C}} \mathfrak{B}$. When taking the pushout in Ring, we refer explicitly to the base rings and write $A \otimes_C B$.

As an aside, we note that morphisms of finitely generated C^∞-rings admit an explicit description:

Lemma 2.14. *For finitely generated C^∞-rings $\mathfrak{A} := C^\infty(\mathbb{R}^m)/I$, $\mathfrak{B} := C^\infty(\mathbb{R}^n)/J$, we have an isomorphism*

$$C^\infty\text{-}\mathsf{Ring}(\mathfrak{A}, \mathfrak{B}) \cong \{\, \varphi \in \mathsf{CartSp}(\mathbb{R}^n, \mathbb{R}^m) \colon \varphi^*(I) \subseteq J \,\}/\sim,$$

where $\varphi \sim \varphi'$ if $\pi_i \circ \varphi - \pi_i \circ \varphi' \in J$ for all i.

Proof. We define a map Φ sending each smooth $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ satisfying $\varphi^*(I) \subseteq J$ to the C^∞-morphism induced from $\pi_B \circ \varphi^*$. Since morphisms of free C^∞-algebras are determined by the image of π_i, we have

$$\varphi \sim \varphi' \iff (\forall\, f \in C^\infty(\mathbb{R}^m) : f \circ \varphi - f \circ \varphi' \in J) \iff \Phi(\varphi) = \Phi(\varphi'),$$

whereby Φ factors through the quotient by $\sim$ to yield an injective map. To see surjectivity, note that each $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ is the image of the smooth map $\mathbb{R}^n \to \mathbb{R}^m$ induced from lifts of the $\varphi(\pi_i)$. $\square$

Proposition 2.15. *Binary coproducts in C^∞-Ring exist.*

Proof. Since colimits commute, we have $\mathfrak{A}/I \otimes_\infty \mathfrak{B}/J \cong \mathfrak{A} \otimes_\infty \mathfrak{B}/(I, J)$. On the other hand, since $C^\infty(\mathbb{R}^n)$ is a free C^∞-ring, we have

$$\begin{aligned}
&C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}^n) \otimes_\infty C^\infty(\mathbb{R}^m), \mathfrak{A}) \\
&\cong C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}^n), \mathfrak{A}) \times C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}^m), \mathfrak{A}) \\
&\cong \mathbb{R}\text{-}\mathsf{Alg}(\mathbb{R}[x_1, \dots, x_n], A) \times \mathbb{R}\text{-}\mathsf{Alg}(\mathbb{R}[x_1, \dots, x_m], A) \\
&\cong \mathbb{R}\text{-}\mathsf{Alg}(\mathbb{R}[x_1, \dots, x_n] \otimes_{\mathbb{R}} \mathbb{R}[x_1, \dots, x_m], A) \\
&\cong \mathbb{R}\text{-}\mathsf{Alg}(\mathbb{R}[x_1, \dots, x_{n+m}], A) \cong C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}^n \times \mathbb{R}^m), \mathfrak{A}).
\end{aligned}$$

It therefore suffices to be able to write an arbitrary C^∞-ring as a filtered colimit of finitely-generated C^∞-rings. This follows from the following even stronger proposition. $\square$

Proposition 2.16. *Every C^∞-ring is a directed colimit of finitely presented C^∞-rings.*

Proof. Composing the C^∞ functor with the forgetful functor to Set yields a functor C^∞-$\mathsf{Ring} \to \mathsf{Set}$ sending each C^∞-ring to its underlying set, and whose left adjoint is given by the obvious composite. As the A-leg of the counit is a C^∞-epimorphism $\varepsilon\colon C^\infty(\mathbb{R}^A) \to \mathfrak{A}$, we may write $\mathfrak{A} \cong C^\infty(\mathbb{R}^A)/I$, where $I := \ker \varepsilon$. For S a finite subset of $\mathfrak{A}$, write i_S for the inclusion $C^\infty(\mathbb{R}^S) \to C^\infty(\mathbb{R}^A)$, and I_S for $i_S^{-1}(I)$. Then $\mathfrak{A} \cong \varinjlim_{S \subseteq \mathfrak{A} \text{ finite}} C^\infty(\mathbb{R}^S)/I_S$. It remains to show that each $C^\infty(\mathbb{R}^S)/I_S$ is in turn a directed colimit of finitely presented C^∞-rings. Indeed, $C^\infty(\mathbb{R}^S)/I_S \cong \varinjlim_{J \subseteq I_S \text{ fingen}} C^\infty(\mathbb{R}^S)/J$, as in Ring every ideal is the union of its finitely generated sub-ideals. $\square$

We have thus shown that

Theorem 2.17. C^∞-Ring *is cocomplete.*

Proof. Recall that a category is cocomplete iff it admits all binary coproducts and coequalizers. $\square$

As can be seen, C^∞-Ring is the product of truncating $\mathrm{Fun}(C^\infty, \mathsf{Set})$ by just the right amount so as to retain categorical completeness and limit its purview to only objects of classical interest, i.e. those of the form $C^\infty(\mathbb{R}^n)/I$. Indeed, we will see that for M a smooth manifold, $C^\infty(M)$ is always of this form, from which it can be proven that $C^\infty(-)$ yields a fully faithful functor $(\mathsf{Man}^\infty)^{\mathrm{op}} \to C^\infty$-$\mathsf{Ring}$, as promised.

By virtue of the Whitney embedding theorem, any smooth n-manifold M admits a proper smooth embedding into $\mathbb{R}^{2n}$, whence it suffices to consider closed subsets of $\mathbb{R}^n$. In the Zariski topology, every closed subscheme V of an affine scheme $\operatorname{Spec} A$ is of the form $V(I) \cong \operatorname{Spec} A/I$ for I the *vanishing ideal* of Z. This phenomenon persists for C^∞-rings:

Lemma 2.18. *Fix V a subset of $\mathbb{R}^n$. Then V is closed iff all continuous (resp. smooth) map $V \to \mathbb{R}$ extends to a (smooth) function on $\mathbb{R}^n$.*

Proof. Let $x \in \overline{V} \setminus V$, (x_n) a sequence in V converging to x. Then a continuous map f on V giving rise to a monotone sequence $(f(x_n)) \to \infty$, if existent, cannot be extended to the whole of $\mathbb{R}^n$, as then $f(x) = \lim_{n\to\infty} f(x_n) = \infty$. To construct f, note that $\{ x_n \}$ is a discrete, hence closed subset of V, as its unique limit point in $\mathbb{R}^n$, the metric completion of V, does not lie in V. The function $g \colon \{ x_n \} \to \mathbb{R}$ sending $x_n \rightsquigarrow n$, which is trivially continuous, thus extends via the Tietze extension theorem to the desired continuous map on V, which is normal as a metric space. The reverse implication follows immediately from the (smooth) Tietze extension theorem [16, Thm 35.1] [12, Lemma 2.26]. $\square$

Proposition 2.19. *For V a closed subset of $\mathbb{R}^n$, $C^\infty(V) \cong C^\infty(\mathbb{R}^n)/I(V)$, where $I(V)$ is the ideal of $C^\infty(\mathbb{R}^n)$ of functions vanishing at V.*

Proof. Consider the map $r^* \colon C^\infty(\mathbb{R}^n) \to C^\infty(V)$ given by restriction. The preceding lemma guarantees the surjectivity of this map. That $\ker r^* = I(V)$ follows essentially by definition. $\square$

Theorem 2.20. *The restriction of C^∞ to the full subcategory of $\widetilde{\mathsf{CartSp}}$ of closed subsets is fully faithful. In particular, $(\mathsf{Man}^\infty)^{\mathrm{op}}$ is equivalent to a full subcategory of C^∞-Ring.*

Proof. We must show that for all closed subsets $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$, there is an isomorphism $\widetilde{\mathsf{CartSp}}(X, Y) \cong C^\infty$-$\mathsf{Ring}(C^\infty(Y), C^\infty(X))$. To this end, we simply apply Lemma 2.14 and observe that $\varphi^*(I(Y)) \subseteq I(X)$ amounts precisely to that $\varphi(X) \subseteq Y$, and $\sim$ identifies those φ and φ' which agree in X. $\square$

Since ι is contravariant, Man^∞ can only ever be a full subcategory of C^∞-$\mathsf{Ring}^{\mathrm{op}}$ and not C^∞-Ring itself. This is once more a ramification of the interpretation of op as interpolating between algebra and geometry: Man^∞ consists of geometric objects, whereby only a direct connection to C^∞-$\mathsf{Ring}^{\mathrm{op}}$, the dual of an algebraic category, is attainable. For this reason, the aim of the subsequent section will be to provide a concrete manifestation of C^∞-$\mathsf{Ring}^{\mathrm{op}}$, henceforth called the *category of smooth loci* or *C^∞-affine schemes*, and the development in this setting of a theory of ideal-variety correspondence, analogous to the celebrated Nullstellensatz of algebraic geometry over commutative rings.

3. C^∞-Schemes: the Geometry of C^∞-Rings

In this section, we will detail a concrete geometric construction of the category C^∞-$\mathsf{Ring}^{\mathrm{op}}$, following [14]. While we lay no claim to originality, many of our proofs offer an algebraic alternative to the order-theoretic proofs found in the literature, and serve as justification for several results originally taken as folklore. Furthermore, our approach places special emphasis on fostering analogies with the algebro-geometric picture and develops the C^∞-spectrum construction in tandem with the classical *spectrum of a ring*, in hopes of allowing the reader to (partially) import their intuition from scheme theory. Just as Grothendieck's theory turns commutative algebra into affine algebraic geometry, we hope that the study of $C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}$ may rightfully gain the title of "affine C^∞-geometry."

For reasons of space we will forego motivating constructions that bear obvious resemblance to ones in algebraic geometry, for which excellent exposition abound in the literature. We will furthermore assume henceforth that the reader has a working knowledge of sheaf and ringed space theory.

Remark. Note that affine scheme theory is in fact *directly applicable* to C^∞-rings via the forgetful functor $U\colon C^\infty\text{-}\mathsf{Ring} \to \mathsf{Ring}$, and it is likely that *algebraic properties* can already be extracted at this level. Here by "algebraic properties" we understand the behavior of "rigid" functions in the C^∞-ring, exemplified in the case when it is of the form $C^\infty(M)$, for M a smooth manifold, by functions on M which in each coordinate patch is a polynomial function. This fails, however, to capture the differential-geometric information (say bump functions) crucial to the C^∞-ring, whereby a more specialized approach is warranted.

We begin by recalling that the functor $\mathrm{Spec}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{LRS}$ is characterized by its being right adjoint to the global sections functor. A similar adjunction should thus also pedigree the correctness of our construction. To this end, we evidently require the notion of a local C^∞-ringed space, which in turn necessitates the definition of local C^∞-rings and C^∞-ringed spaces.

Definition 3.1. A C^∞-ring $\mathfrak{A}$ is **local** if it has a unique maximal ideal $\mathfrak{m}$, in which case $\mathfrak{A}/\mathfrak{m}$ is said to be its **residue field** and denoted $\kappa(\mathfrak{A})$. A local C^∞-ring with residue field $\mathbb{R}$ is said to be **archimedean**. Given local C^∞-rings $\mathfrak{A}, \mathfrak{B}$, a C^∞-morphism $\mathfrak{A} \to \mathfrak{B}$ is **local** if it is a local homomorphism.

The primordial example of a local C^∞-rings is the ring of germs at a point of a smooth manifold. As formally the locality condition on C^∞-ring is identical to that on rings, this example foreshadows the result that rings of germs are stalks of affine C^∞-schemes.

Example 3.2. Fix M a smooth manifold. Recall that $C_p^\infty(M) = C^\infty(M)/I_g(p)$, for $I_g(p)$ the ideal of smooth functions vanishing on a neighborhood of p. We claim that $C_p^\infty(M)$ is an archimedean local C^∞-ring. Consider the ideal $I(p)$ of functions vanishing at p, which, containing $I_g(p)$, corresponds to an ideal $\widetilde{I(p)}$ of $I(p)$. Then $C_p^\infty(M)/\widetilde{I(p)} \cong \mathbb{R}$ follows from the short exact sequence in §1.

One surefire way of obtaining new local C^∞-rings from old is by taking quotients: this follows immediately from that quotients of ordinary local rings are local. This is perhaps exemplified by rings of formal power series:

Example 3.3. By Borel's theorem, the Taylor series expansion at 0 furnishes a surjective homomorphism $C_0^\infty(\mathbb{R}^n) \twoheadrightarrow \mathbb{R}[[x_1, \ldots, x_n]]$ with kernel $I_\infty(0)$, the ideal of functions *flat* at 0, i.e. $\frac{\partial^n f}{\partial^n x_i}(0) = 0$ for all i, n. As an aside, we note that the quintessential example of a flat function is $e^{-\frac{1}{x^2}}$. By our observation above, $C_0^\infty(\mathbb{R}^n)/I_\infty(0) \cong \mathbb{R}[[x_1, \ldots, x_n]]$ is a local C^∞-ring.

The pattern will also yield later the result that *every C^∞-domain is local.* This is clearly false in the case of ordinary rings, and indeed highlights one exotic aspect of C^∞-rings that can be properly interpreted only after we develop much machinery.

Remark. It also holds that every local C^∞-ring is Henselian (simple roots in residue field can be lifted to genuine roots in the local C^∞-ring), whereby there is no need to take completions of stalks as one does in arithmetic situations. For a proof see [15]. This reflects the smooth, or rather complete nature of the spaces C^∞-rings represent.

Definition 3.4. A C^∞**-ringed space** is a pair $(X, \mathscr{O}_X)$, for X a topological space, $\mathscr{O}_X$ a sheaf of C^∞-rings on X. A morphism $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a continuous map $\pi\colon X \to Y$ together with a morphism $\mathscr{O}_Y \to \pi_*\mathscr{O}_X$. The resultant category is denoted $C^\infty\text{-}\mathsf{RS}$. The subcategory of $C^\infty\text{-}\mathsf{RS}$

consisting of **local C^∞-ringed spaces**, i.e. $(X, \mathscr{O}_X)$ in which each stalk is a local C^∞-ring, and morphisms which induce local homomorphisms on stalks, is denoted $\mathsf{L}C^\infty\text{-}\mathsf{RS}$.
The **global sections functor** $\Gamma^\infty \colon C^\infty\text{-}\mathsf{RS} \to C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}$ sends each $(X, \mathscr{O}_X)$ to $\mathscr{O}_X(X)$ and each morphism $\pi \colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ to the Y-leg of $\pi^\sharp \colon \mathscr{O}_Y \to \pi_* \mathscr{O}_X$.

Remark. There does not appear not appear to be extensive work on the categories $C^\infty\text{-}\mathsf{RS}$ and $\mathsf{L}C^\infty\text{-}\mathsf{RS}$ in the literature, but it would be reasonable to expect both to be bicomplete, the former by fibered category nonsense, and the latter by techniques analogous to those described in [2].

The construction of the right adjoint, i.e. of affine C^∞-schemes, is however not as easily imported as the formalism of locally ringed spaces. To set the stage, recall that given a ring A, $\operatorname{Spec} A$ is a locally ringed space whose underlying space has as points all prime ideals of A and the Zariski topology, and whose structure sheaf is defined on each distinguished open $D(f)$ as the localization A_f. As pointed out in an earlier remark, since $\operatorname{Spec} \mathfrak{A}$ does not suffice for recovering the smooth structure, we can no longer take any prime ideal unconditionally as a point of our affine C^∞-scheme. Instead of taking a blind guess as to the precise condition on prime ideals needed, let us instead first define localizations of C^∞-rings (which will differ from ring-theoretic localizations, as there is no promise $S^{-1}A$ admits a natural C^∞-structure) and from there determine what is needed for $\mathfrak{A}_\mathfrak{p}$ to be a local C^∞-ring, an indubitable necessary condition. Of course, we shall abide by the tenet of always working in terms of universal properties.

Definition 3.5. Fix a C^∞-ringed space $\mathfrak{A}$, $f \in A$. Then the **C^∞-localization of $\mathfrak{A}$ with respect to** s is the C^∞-ring $\mathfrak{A}_s$, together with a canonical C^∞-morphism $\eta \colon \mathfrak{A} \to \mathfrak{A}_s$, such that for all C^∞-rings $\mathfrak{B}$, C^∞-morphisms $\varphi \colon \mathfrak{A} \to \mathfrak{B}$ with $\varphi(f)$ a unit, there is a unique C^∞-morphism $\mathfrak{A}_s \to \mathfrak{B}$ factoring through φ.

Recall that in Ring, $A_s \cong A[t]/(ts-1)$. The analogous statement holds in $C^\infty\text{-}\mathsf{Ring}$, thereby proving the existence of $\mathfrak{A}_s$:

Proposition 3.6. *For $\mathfrak{A}$, f as above, $\mathfrak{A}_f \cong \mathfrak{A}[t]/(tf-1)$.*

Proof. We readily verify the universal property: for $\mathfrak{B}$ a C^∞-ring, we have

$$\begin{aligned} \operatorname{Hom}(\mathfrak{A}[t]/(ts-1), \mathfrak{B}) &\cong \{\, \psi \in \operatorname{Hom}(\mathfrak{A}[t], \mathfrak{B}) \colon \psi(ts-1) = 0 \,\} \\ &\cong \{\, (\varphi, b) \in \operatorname{Hom}(\mathfrak{A}, \mathfrak{B}) \times B \colon b\varphi(s) = 1 \,\} \\ &\cong \{\, \varphi \in \operatorname{Hom}(\mathfrak{A}, \mathfrak{B}) \colon \varphi(s) \text{ is a unit} \,\}. \end{aligned}$$ ∎

Crucially, C^∞-localization is not to be confused with localization of rings: the elements of $\mathfrak{A}_s$ are *not* fractions f/s^n. It is therefore imperative to maintain consistent notation throughout that distinguishes the two. In fact, for finitely generated free C^∞-rings we may deduce a concrete form for $\mathfrak{A}_s$:

Proposition 3.7. *For all $f \in C^\infty(\mathbb{R}^n)$, $C^\infty(\mathbb{R}^n)_f \cong C^\infty(U_f)$.*

Proof. By virtue of the above proposition, we know that $C^\infty(\mathbb{R}^n)_f \cong C^\infty(\mathbb{R}^n)[t]/(tf-1) \cong C^\infty(\mathbb{R}^{n+1})/(yf-1)$, where by y we understand the last coordinate. It therefore suffices to construct an isomorphism from this quotient C^∞-ring to $C^\infty(U_f)$. Consider the map $\gamma \colon U_f \to \mathbb{R}^{n+1}$ sending $x \rightsquigarrow (x, \frac{1}{f(x)})$, whose image is $\Gamma := \{\, (x,y) \in \mathbb{R}^n \colon f(x)y = 1 \,\}$, the zeroset of the smooth map $F \colon \mathbb{R}^{n+1} \to \mathbb{R}$ sending $(x,y) \rightsquigarrow f(x)y - 1$. Since $\frac{\partial F}{\partial y} = f(x)$ is non-zero, hence surjective on Γ, $F|_\Gamma$ is a submersion. It then follows from [12, Cor 5.14] that Γ is a properly embedded, hence closed submanifold of $\mathbb{R}^{n+1}$. Evidently γ furnishes a diffeomorphism from U_f onto Γ, with inverse given by projection onto the first n coordinates. That is, $C^\infty(U_f) \cong C^\infty(\Gamma)$. Applying Proposition 2.19, we may in turn write $C^\infty(\Gamma) \cong C^\infty(\mathbb{R}^{n+1})/I(\Gamma)$, whence it remains to show that $I(\Gamma) = (yf-1) = (F)$. Since the zeroset of F is manifestly Γ, this amounts to that for each $H \in C^\infty(\mathbb{R}^{n+1})$ with $H|_\Gamma = 0$, there exists some $G \in C^\infty(\mathbb{R}^{n+1})$ with $H = FG$. This follows from the non-singular, i.e. $k = 1$ case

of the *Mather division theorem* [13, Thm 2.1], but let us instead present an elementary argument. If we can exhibit for each $p \in \Gamma$ a neighborhood U_p and a $G_p \in C^\infty(U_p)$ such that $H|_{U_p} = F|_{U_p} G_p$, we would be done, as picking a partition of unity $\{ \psi_i \}$ subordinate to the open cover $\{ U_i \} := \{ U_p \} \cup \{ \mathbb{R}^{n+1} \setminus \Gamma \}$ of $\mathbb{R}^{n+1}$ and setting $G := \sum \psi_i G_i$, we would have $H(p) = \sum \psi_i(p) H(p) = \sum F(p) \psi_i G_i(p) = FG(p)$ for each $p \in \mathbb{R}^{n+1}$. Here the G_i associated to $\mathbb{R}^{n+1} \setminus \Gamma$ is taken to be $\frac{H|_{\mathbb{R}^{n+1} \setminus \Gamma}}{F|_{\mathbb{R}^{n+1} \setminus \Gamma}}$, with F being non-zero outside of Γ.
As such, let $p \in \Gamma$. Since dF_p is surjective, [12, Prop 4.1] furnishes a neighborhood U of p in which F is a submersion, whence [12, Thm 4.12] in turn yields coordinates $(x_1, \dots, x_n, t)$ in which $F(x,t) = t$. That is, $\Gamma \cap U = \{ (x,t) \colon t = 0 \}$, so that $H(x, 0) = 0$. Our desired function $G \in C^\infty(U)$ may then be defined as

$$G(x,t) := \int_0^1 \frac{\partial H}{\partial t}(x, st)\, ds,$$

for then $H(x,t) = \int_0^t \frac{\partial H}{\partial u}(x,u)\, du = tG(x,t) = FG(x,t)$, where $u(s) = st$. □

Since colimits commute with C^∞-localization (as shown in the following proposition), the above normal form result can be extended to arbitrary C^∞-rings by taking presentations:

Proposition 3.8. *For $\mathfrak{A}$ a C^∞-ring, I an ideal, one has for each $f \in A$ the isomorphism $(\mathfrak{A}/I)_{[f]} \cong \mathfrak{A}_f/(\eta(I))$. Furthermore, given J a filtered category, $D \colon J \to C^\infty\text{-}\mathsf{Ring}$ a diagram, and $f_j \in D(j)$ for each $j \in J$, one has $(\varinjlim_J D)_f \cong \varinjlim_{j \in J} D(j)_{f_j}$, where f is represented by (f_j).*

Proof. For quotients, we have

$$\mathfrak{A}_f/(\eta(I)) \cong \mathfrak{A}[t]/(tf - 1, I) \cong (\mathfrak{A}[t]/(I))/(t[f] - 1) \cong (\mathfrak{A}/I)[t]/(t[f] - 1) \cong (\mathfrak{A}/I)_{[f]}.$$

Likewise, for directed colimits we have

$$(\varinjlim_J D)_f \cong (\varinjlim_J D)[t]/(tf - 1) \cong \varinjlim_{j \in J} D(j)[t]/(tf_j - 1) \cong \varinjlim_{j \in J} D(j)_{f_j}. \qquad \square$$

Corollary 3.9. *Given a C^∞-ring $\mathfrak{A}$ with presentation $C^\infty(\mathbb{R}^X)/I$, for all $f \in \mathfrak{A}$, one has $\mathfrak{A}_f \cong C^\infty(U_f)/(I|_{U_f})$, where $C^\infty(U_f)$ is the C^∞-ring of functions on $U_f \subseteq \mathbb{R}^X$ smoothly depending on finitely many coordinates, and $(I|_{U_f})$ is the ideal generated by restrictions of elements of I to U_f.*

Proof. Writing $C^\infty(\mathbb{R}^X)/I$ as $\varinjlim C^\infty(\mathbb{R}^S)/i_S^{-1}(I)$, the result follows immediately from the above proposition once we observe that the canonical map is given by restriction. □

Despite their vastly different constructions, ring and C^∞-localizations still share some fundamental attributes. For instance, one can still "clear denominators" in C^∞-localizations, despite the units in $\mathfrak{A}_f$ being no longer as simple as $\{ a^n \}$ together with the ones in $\mathfrak{A}$:

Proposition 3.10. *For all $r \in \mathfrak{A}_f$, there exist $a, b \in A$ such that $r\eta(b) = \eta(a)$ and $\eta(b)$ is a unit in $\mathfrak{A}_f$.*

Proof. We first show the result for when $\mathfrak{A} = C^\infty(\mathbb{R}^n)$. To this end, it will suffice to show the analytic statement that for open $U \subseteq \mathbb{R}^n$, $g \in C^\infty(U)$, there exist $h, k \in C^\infty(\mathbb{R}^n)$ with $U_k = U$ and $gk|_U = h|_U$. For by Proposition 3.7, $\mathfrak{A}_f$ is simply $C^\infty(U_f)$, and applying the claim with $U := U_f$ and $g := r$, our desired a, b are h, k, respectively, since $U_k = U$ indicates that k does not vanish in U, hence has invertible restriction to U. Since the proof of the claim requires familiarity with Frechet topologies and detracts from the premise of the paper, we simply refer the reader to [15, Thm 1.3].
The generalization to finitely generated C^∞-rings and arbitrary C^∞-rings hinges on Proposition 3.8. Indeed, for $\mathfrak{A} := C^\infty(\mathbb{R}^n)/I$, $[f] \in \mathfrak{A}$, $\mathfrak{A}_{[f]} \cong C^\infty(\mathbb{R}^n)_f/(\eta(I))$, whereby one may lift $[r] \in \mathfrak{A}_{[f]}$ to an element of $C^\infty(\mathbb{R}^n)_f$ and apply the above to obtain $a, b \in C^\infty(\mathbb{R}^n)$ with the desired properties, which are evidently preserved when projected to $\mathfrak{A}$. Now for $\mathfrak{B} := \varinjlim \mathfrak{B}_i$, $\mathfrak{B}_f \cong \varinjlim (\mathfrak{B}_i)_{f_i}$, for $(f_i) \in \coprod \mathfrak{B}_i$ a representative of f, which exists as directed colimits of C^∞-rings are computed pointwise. Here

again the desired a, b are obtained from the a_i, b_i found in each $(\mathfrak{B}_i)_{f_i}$ by virtue of the previous case. $\square$

Remark. If we omit η, we obtain the catchphrase that every element of $\mathfrak{A}_f$ can be realized as a fraction a/b for $a, b \in A$ and b invertible in $\mathfrak{A}_f$. Distinct representations are related by a unit of $\mathfrak{A}_f$.

The property that zeros in A_a arise from zero divisors $b \in A$ for which $ba^n = 0$ also admits a C^∞-analogue:

Proposition 3.11. *If $a \in A$ and $\eta(a) = 0$, then there exists $b \in A$ such that $ab = 0$ and $\eta(b)$ is a unit in $\mathfrak{A}_f$.*

Proof. As in the proof of Proposition 3.10, it suffices to show the result for finitely generated free C^∞-rings and appeal to general properties of colimits. By Proposition 3.7, $\mathfrak{A}_f$ is just $C^\infty(U_f)$, with the canonical map being restriction. As such, an $a \in A$ with $\eta(a) = 0$ is simply a smooth function vanishing on U_f. Now by [12, Thm 2.29], U_f^c may be realized as the zeroset of some *characteristic function* $b \in \mathfrak{A}$, whereby $ab = 0$. Furthermore, $\eta(b)$ is indeed a unit, as it vanishes nowhere in U_f. $\square$

Corollary 3.12. *If $\mathfrak{A}$ is a C^∞-domain and $\mathfrak{A}_f \neq 0$, then $\eta \colon \mathfrak{A} \to \mathfrak{A}_f$ is an injection and $\mathfrak{A}_f$ is a C^∞-domain.*

Proof. For injectivity, assume that $\eta(a) = 0$, so that the previous proposition furnishes a $b \in A$ with $ab = 0$. But since $\eta(b)$ is a unit, $b \neq 0$, whereby integrality forces $a = 0$. To see integrality, let $a, b \in \mathfrak{A}_f$ be such that $ab = 0$. Then by Proposition 3.10, we may write $a = \eta(r_1)/\eta(s_1)$, $b = \eta(r_2)/\eta(s_2)$ for $r_i, s_i \in \mathfrak{A}$, $\eta(s_i)$ units. It follows that $\eta(r_1 r_2) = 0$, whence $r_1 r_2 = 0$ by injectivity. But $\mathfrak{A}$ is integral, whereby $r_1 \neq 0 \implies \eta(a) \neq 0$ without loss of generality. $\square$

Remark. Proposition 3.10 and 3.11 packages algebraically about as much actual analysis as is required for understanding C^∞-localizations, and in the sequel we will capitalize off of this and work almost exclusively behind this layer of abstraction so as to enjoy arguments closer in spirit to ones from algebraic geometry.

As with rings, localizations of arbitrary multiplicative submonoids can be defined by taking a filtered colimit of various $\mathfrak{A}_s$:

Definition 3.13. For S a multiplicative submonoid of a C^∞-ring $\mathfrak{A}$, the **C^∞-localization of $\mathfrak{A}$ with respect to S** is the filtered colimit $S^{-1}\mathfrak{A} := \varinjlim_{s \in S} \mathfrak{A}_s$, where $s \leq t$ iff there exists some $r \in A$ with $t = rs$. For $\mathfrak{p}$ a prime ideal of A, we denote by $\mathfrak{A}_\mathfrak{p}$ the localization $(A \setminus \mathfrak{p})^{-1}\mathfrak{A}$.

With the basic machinery of C^∞-localizations in place, we are ready to deduce the pointwise constitution of the C^∞-spectrum. This requires a thorough review of the steps involved in proving the Γ-Spec adjunction in commutative algebra, which we succinctly summarize in the following roadmap:

$\mathscr{O}_{\mathrm{Spec}^\infty \mathfrak{A}}$ is well-defined: For $f, g \in \mathfrak{A}$ with $D_\infty(f) = D_\infty(g)$, $A_f \cong A_g$.
$\mathrm{Spec}^\infty \mathfrak{A}$ is spectral: Each $D^\infty(f)$ is quasicompact, irreducible subsets have unique generic points.
$\mathscr{O}_{\mathrm{Spec}^\infty \mathfrak{A}}$ is a sheaf: Locality and gluability on the base $\{ D(f) \}$ via a partitions of unity argument.
Functoriality: Action of Spec^∞ on C^∞-morphisms.
Adjunction: Construct morphism of local C^∞-ringed spaces from morphisms of C^∞-rings.

Of the five steps, it is the proof of the last that is of the most use to us: indeed, even for rings the adjunction is possible only by virtue of the *locality* condition on morphisms in LRS. If one were to conceive of an adjunction between Ring and RS, a trivial functor would lurk in the place of Spec. The usefulness of locality is due entirely to the following:

Proposition 3.14. *Fix a morphism of ringed spaces $\pi\colon X \to Y$. Then the following diagram commutes*

$$\begin{tikzcd} \Gamma(Y, \mathscr{O}_Y) \arrow[r, "\pi^\sharp"] \arrow[d] & \Gamma(X, \mathscr{O}_X) \arrow[d] \\ \mathscr{O}_{Y,\pi(x)} \arrow[r, "\pi_x^\sharp"] & \mathscr{O}_{X,x}. \end{tikzcd}$$

Lemma 3.15. *For any LRS morphism $\pi\colon X \to \operatorname{Spec} A$, $\pi(x)$ is the inverse image of $\mathfrak{m}_x$ under the composition $\alpha_x\colon A \xrightarrow{\pi^\sharp} \mathscr{O}_X(X) \xrightarrow{i} \mathscr{O}_{X,x}$.*

Proof. By the above proposition, we have a diagram

$$\begin{tikzcd} A \arrow[r, "\pi^\sharp"] \arrow[d] & \mathscr{O}_X(X) \arrow[d] \\ A_{\mathfrak{m}_{\pi(x)}} \arrow[r, "\pi_x^\sharp"] & \mathscr{O}_{X,x}. \end{tikzcd}$$

By the locality of $\pi_x^\sharp$, $\mathfrak{m}_x$ pulls back along $\pi_x^\sharp$ to $\mathfrak{m}_{\pi(x)}$. Now clearly $\mathfrak{m}_{\pi(x)}$ pulls back along the left localization map to the prime ideal corresponding to $\pi(x)$. □

Evidently both results continue to hold when we move to $\mathsf{L}C^\infty\text{-}\mathsf{RS}$. Assuming the adjunction, we arrive at a complete description of the points of $\operatorname{Spec}^\infty \mathfrak{A}$:

Proposition 3.16. *For any C^∞-ring $\mathfrak{A}$, we have*

$$\operatorname{Spec}^\infty(\mathfrak{A}) = \left\{ \varphi^{-1}(\mathfrak{m})\colon \varphi\colon \mathfrak{A} \to \mathfrak{B} \text{ a } C^\infty\text{-morphism to a local } C^\infty\text{-ring} \right\}.$$

Proof. Let us assume that $\operatorname{Spec}^\infty A$ is a local C^∞-ringed space. Applying the preceding lemma to identity morphism on $\operatorname{Spec}^\infty A$, we obtain the backwards inclusion, i.e. every $\mathfrak{p} \in \operatorname{Spec}^\infty(\mathfrak{A})$ is of the form $\varphi^{-1}(\mathfrak{m})$. To show the forwards inclusion, let us assume the Γ-Spec adjunction. Fix a C^∞-morphism $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ to a local C^∞-ring with maximal ideal $\mathfrak{m}$. Consider the one-point local C^∞-ringed space $X := (\{\ast\}, \mathscr{O}_X)$ with $\mathscr{O}_X(\{\ast\}) = B$. We may thus view φ as a morphism $\mathfrak{A} \to \Gamma^\infty(X, \mathscr{O}_X)$, which corresponds under the adjunction to a unique morphism $\Phi\colon X \to \operatorname{Spec}^\infty \mathfrak{A}$. By the preceding lemma, $\alpha_\ast^{-1}(\mathfrak{m}) = \Phi(\ast) \in \operatorname{Spec}^\infty \mathfrak{A}$. But since $\mathscr{O}_{X,\ast} = B$, $\alpha_\ast$ is precisely φ. □

We now seek a better description for points of $\operatorname{Spec}^\infty(\mathfrak{A})$. It turns out that it suffices to directly transplant the definition of a *radical ideal*, replacing ring localization with C^∞-localization (recall that $\sqrt{I}$ can be defined, albeit somewhat unconventionally, as $\{ r \in A\colon (A/I)_r = 0 \}$).

Definition 3.17. Fix $\mathfrak{A}$ a C^∞-ring, I an ideal. The **C^∞-radical of** I, denoted $\sqrt[\infty]{I}$, consists precisely of those $a \in \mathfrak{A}$ for which there exists some $b \in I$ that is a unit in $\mathfrak{A}_a$, or equivalently, $(\mathfrak{A}/I)_{[a]} = 0$. Thus I is said to be **C^∞-radical** if $I = \sqrt[\infty]{I}$.
A C^∞-ring $\mathfrak{A}$ is said to be **C^∞-reduced** if 0 is C^∞-radical. This is tantamount to that for every $a \in \mathfrak{A}$, $a = 0 \iff \mathfrak{A}_a = 0$.

As a first sign that C^∞-radicality is indeed equivalent to the previous condition, we note the following:

Proposition 3.18. *Every maximal ideal of a C^∞-ring is C^∞-radical.*

Proof. Let J be a maximal ideal. Then $(\mathfrak{A}/J)_1 = \mathfrak{A}/J \neq 0 \implies 1 \notin \sqrt[\infty]{J}$, whereby $\sqrt[\infty]{J}$ is a proper ideal, hence contained in J. □

We now present one direction of the equivalence:

Proposition 3.19. *Let $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ be a C^∞-morphism to a local C^∞-ring with maximal ideal $\mathfrak{m}$. Then the prime ideal $\varphi^{-1}(\mathfrak{m})$ is C^∞-radical.*

Proof. Put $\mathfrak{p} := \varphi^{-1}(\mathfrak{m})$, and write $\psi\colon \mathfrak{A}/\mathfrak{p} \to \mathfrak{B}/\mathfrak{m}$ for the injective C^∞-morphism induced from $\pi_{\mathfrak{B}} \circ \varphi$. Let $a \in \sqrt[\infty]{\mathfrak{p}}$, so that $(\mathfrak{A}/\mathfrak{p})_{[a]} = 0$. If the image of $[a]$ under ψ were a unit, then ψ would factor through $(\mathfrak{A}/\mathfrak{p})_{[a]}$ and thus be the zero map. This contradicts the injectivity of ψ, so $\psi([a]) = 0 \implies a \in \mathfrak{p}$, as desired. $\square$

The converse follows from the following more general condition on when $\mathfrak{A}_\mathfrak{p}$ is local:

Proposition 3.20. *For $\mathfrak{p}$ a prime ideal, $\mathfrak{A}_\mathfrak{p}$ is local iff the set of C^∞-radical prime ideals in A contained in $\mathfrak{p}$ admits a unique maximal element $\mathfrak{m}$, in which case the unique unique maximal ideal of $\mathfrak{A}_\mathfrak{p}$ is $\mathfrak{m}\mathfrak{A}_\mathfrak{p}$. Hence for all $a \in A$, $\eta(a)$ is a unit iff $a \notin \mathfrak{m}$.*

Proof. This follows readily from Proposition 3.30 (the reader is welcome to verify the absence of circular reasoning). Indeed, given any proper ideal I in $\mathfrak{A}_\mathfrak{p}$, Zorn's lemma furnishes an ideal $\mathfrak{m}$ containing it that is maximal, hence C^∞-radical prime, by Proposition 3.18. As such, to show an ideal is maximal it suffices to check maximality among all C^∞-radical prime.
The implication follows immediately from that $\mathfrak{m} = \eta^{-1}(\mathfrak{m}A_\mathfrak{p})$, which is guaranteed by the bijection of Proposition 3.30. $\square$

Example 3.21. Contrary to the circumstance in Ring, prime ideals in general need not be C^∞-radical! As an example, consider $I_\infty(0) \subseteq C_0^\infty(\mathbb{R})$ from Example 3.3, which is prime as the ring of formal power series is integral. Since the set of non-units in a local ring forms an ideal, namely the unique maximal ideal, it will suffice to exhibit non-units in $C_0^\infty(\mathbb{R})_{I_\infty(0)}$ summing to 1. Now $\eta\colon C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R})_{I_\infty(0)}$ evidently factors $C_0^\infty(\mathbb{R})_{\mathrm{id}_\mathbb{R}} \cong C_0^\infty(\mathbb{R} \setminus \{\, 0 \,\})$, whereby the non-units may be taken to be the image of

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x > 0, \end{cases} \qquad g(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0. \end{cases}$$

We now show that the image of f, $\overline{f} \in C_0^\infty(\mathbb{R})_{I_\infty(0)}$, is not invertible; the argument for g is analogous. Assume towards a contradiction that $\overline{f}$ is invertible. It will suffice to exhibit a $b \in C_0^\infty(\mathbb{R}) \setminus I_\infty(0)$ whose zeroset contains that of f, for then b vanishes on $(-\infty, 0)$, whence smoothness forces it to be flat at 0. Indeed, Proposition 3.10 furnishes $a, b \in C^\infty(\mathbb{R})$ for which $1/\overline{f} = \eta(a)/\eta(b)$ and $\eta(b)$ a unit. In particular, $b \notin I_\infty(0)$. But then $\eta(b) = \overline{f}\eta(a)$, and restricting to $\mathbb{R} \setminus \{\, 0 \,\}$, we see that f vanishing implies that $b|_{\mathbb{R} \setminus \{\, 0 \,\}}$, hence b vanishes.

Example 3.22. To concoct a concrete example of when locality does not imply C^∞-radicality, we begin with a prime ideal $\mathfrak{q}$ in $\mathfrak{A}$ that is not C^∞-radical. Pick a prime ideal $\mathfrak{p}$ such that $\sqrt[\infty]{\mathfrak{p}} \subsetneq \mathfrak{q}$. Then consider $(\mathfrak{A}/\mathfrak{p})_\mathfrak{q}$, which is integral, hence local by Proposition 3.60. But $\mathfrak{q}$ remains non C^∞-radical when viewed as an ideal of $\mathfrak{A}/\mathfrak{p}$. consider $C^\infty(\mathbb{R})/$.

Notice how the pathological behavior displayed in the previous example hinges on our taking a quotient of $C^\infty(\mathbb{R})$. Indeed, it cannot happen when $\mathfrak{A}$ is the ring of smooth functions over a manifold:

Proposition 3.23. *Fix a smooth manifold M. Then for $\mathfrak{p}$ a prime ideal of $C^\infty(M)$, $C^\infty(M)_\mathfrak{p}$ is local iff $\mathfrak{p}$ is C^∞-radical.*

Proof. As this result will not be required in the sequel and is proven with hard analysis, we shall be content with referring the reader to [14, Thm 1.14] for details. $\square$

Remark. A retrospective rationalization of the critical role played by C^∞-radicals is the following: points of spectra, regardless of the specific category at hand, are supposed to be *loci for evaluating functions* by way of projecting onto the residue field. As such, they cannot leave room for any sort of "nilpotent" function, i.e. one that is non-zero but should be thought of vanishing at the point. Of course, nilpotence is a phenomenon intrinsic to the category, as witnessed by its definition using category-specific localization.

Proposition 3.19 and 3.20 together show that it is correct, at least on points, to define the C^∞-spectrum as follows:

Definition 3.24. For $\mathfrak{A}$ a C^∞-ring, we define the C^∞**-spectrum** $\operatorname{Spec}^\infty(\mathfrak{A})$ to be the set of C^∞-radical prime ideals, equipped with the **Zariski topology** generated by the base of **distinguished opens**

$$D^\infty(f) := \{\, \mathfrak{p} \in \operatorname{Spec}^\infty(A) \colon a \notin \mathfrak{p} \,\}.$$

The **vanishing sets** of a subset S of $\mathfrak{A}$ is

$$V^\infty(S) := \{\, \mathfrak{p} \in \operatorname{Spec}^\infty(\mathfrak{A}) \colon S \subseteq \mathfrak{p} \,\}.$$

The **vanishing ideal** of a subset $T \subseteq \operatorname{Spec}^\infty \mathfrak{A}$ is $I^\infty(T) := \bigcap_{\mathfrak{p}\in T} \mathfrak{p}$. The **structure sheaf** $\mathscr{O}_{\operatorname{Spec}^\infty(A)}$ on $\operatorname{Spec}^\infty(A)$ is given on each basic open set $D^\infty(f)$ as $\mathfrak{A}_f$.

We immediately import some basic properties of the operators of I and V from algebraic geometry, whose proofs are almost verbatim.

Proposition 3.25. *Fix a C^∞-ring $\mathfrak{A}$.*

1. *Let $S \subseteq A$. Then $V^\infty(S) = V^\infty((S))$. Dually, for $T \subseteq \operatorname{Spec}^\infty A$, $I^\infty(T) = I^\infty(\overline{T})$.*
2. *Given a set of ideals $\{\, I_\alpha \,\}$, $\cap_\alpha V^\infty(I_\alpha) = V^\infty(\sum_\alpha I_\alpha)$.*
3. *Given ideals I, J, $V^\infty(I) \cup V^\infty(J) = V^\infty(IJ) = V^\infty(I \cap J)$. Dually, for subsets $S, T \subseteq \operatorname{Spec}^\infty A$, $I^\infty(S \cup T) = I^\infty(S) \cap I^\infty(T)$.*
4. *For $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ a C^∞-morphism, $S \subseteq A$, ${\varphi^*}^{-1}(V^\infty(S)) = V^\infty(\varphi(S))$, and thus for any $f \in A$, ${\varphi^*}^{-1}(D^\infty(f)) = D^\infty(\varphi(f))$.*

Proof. For (1), the backwards inclusion is obvious, and the forwards follows from that $S \subseteq \mathfrak{p} \implies (S) \subseteq \mathfrak{p}$, an elementary property of ideals. Since $T \subseteq \overline{T}$, the dual statement would follow from that $I^\infty(T) \subseteq I^\infty(\overline{T})$. Now $T \subseteq V^\infty(I^\infty(T))$, so $\overline{T} \subseteq V^\infty(I^\infty(T))$, and it suffices to show that $I^\infty(T) \subseteq I^\infty(V^\infty(I^\infty(T)))$. Indeed, given $f \in I^\infty(T)$, we have that for any $[\mathfrak{q}] \in V^\infty(I^\infty(T)) \implies \mathfrak{q} \supseteq I^\infty(T)$, $f \in \mathfrak{q}$. For (2), suppose that $\mathfrak{p}$ lies in all $V^\infty(I_\alpha)$, namely that $I_\alpha \subseteq \mathfrak{p}$ for all α. Then $\sum I_\alpha \subseteq \mathfrak{p}$ follows from that $\mathfrak{p}$ is a subgroup. The converse is trivial, as individual elements of I_α are 1-fold finite sums. For (3), $\mathfrak{p} \in V^\infty(I) \cup V^\infty(J) \iff I \subseteq \mathfrak{p} \vee J \subseteq \mathfrak{p}$. That $IJ \subseteq \mathfrak{p}$ again follows from closure. The converse is primality: if there are $a \in I$, $b \in J$ both not in $\mathfrak{p}$, then $ab \notin \mathfrak{p}$. The second equality follows from a chain of inclusion: $V^\infty(I \cap J) \supseteq V^\infty(I) \cup V^\infty(J)$, as $I \cap J \subseteq I_k \implies V^\infty(I_k) \subseteq V^\infty(I \cap J)$, and $V^\infty(I \cap J) \subseteq V^\infty(IJ)$, with $IJ \subseteq I \cap J$. The dual statement is immediate from the commutativity of intersections. For (4), note that ${\varphi^*}^{-1}(V^\infty(S)) = \{\, \mathfrak{p} \in \operatorname{Spec}^\infty B \colon \varphi^{-1}(\mathfrak{p}) \in V^\infty(S) \,\} = \{\, \mathfrak{p} \colon S \subseteq \varphi^{-1}(\mathfrak{p}) \,\} = \{\, \mathfrak{p} \colon \varphi(S) \subseteq \mathfrak{p} \,\} = V^\infty(\varphi(S))$. ☐

Corollary 3.26. *The Zariski topology on $\operatorname{Spec}^\infty \mathfrak{A}$ has as closed sets vanishing sets of all subsets of A.*

Proof. Fix a closed $V \subseteq \operatorname{Spec}^\infty \mathfrak{A}$. Then $V^c = \bigcup_\alpha D^\infty(f_\alpha) \implies V = \bigcap_\alpha D^\infty(f_\alpha)^c = \bigcap_\alpha V^\infty(f_\alpha) = \bigcap_\alpha V^\infty((f_\alpha)) = V^\infty(\sum_\alpha (f_\alpha))$. ☐

Proposition 3.27. *For $\mathfrak{A}$ a C^∞-ring, $\operatorname{Spec}^\infty \mathfrak{A}$ is quasicompact.*

Proof. The classical proof again transfers verbatim. Since $\{\, D^\infty(f) \colon f \in A \,\}$ forms a base for A, it suffices to show that any cover consisting of distinguished open sets has a finite subcover. This amounts to proving the equivalence

$$\{\, D^\infty(f_\alpha) \,\} \text{ covers } \operatorname{Spec}^\infty \mathfrak{A} \iff (\{\, f_\alpha \,\}) = \mathfrak{A} \iff \exists \text{ finitely-supported } (a_\alpha)\colon \sum a_\alpha f_\alpha = 1.$$

For the distinguished open sets of the finitely many f_i whose corresponding a_i is non-zero would then cover $\operatorname{Spec}^\infty \mathfrak{A}$: if there were some $\mathfrak{p} \in \operatorname{Spec}^\infty \mathfrak{A}$ such that $\mathfrak{p} \notin D^\infty(f_i) \iff f_i \in \mathfrak{p}$ for all i, then $\sum a_i f_i = \sum a_\alpha f_\alpha = 1$ would vanish at $\mathfrak{p}$.

The second equivalence is immediate: the third condition is tantamount to that $1 \in (\{\, f_\alpha \,\})$, and an ideal contains a unit iff it is the entire ring. For the first equivalence, notice that $\{\, D^\infty(f_\alpha) \,\}$ covers iff for all $\mathfrak{p} \in \operatorname{Spec}^\infty \mathfrak{A}$, there exists an α such that $\mathfrak{p} \in D^\infty(f_\alpha) \iff f_\alpha \notin \mathfrak{p}$, iff no C^∞-radical prime ideal contains $(\{ f_\alpha \})$. But any proper ideal is contained in a maximal, hence C^∞-radical prime ideal, so $(\{ f_\alpha \})$ must be A. ☐

Theorem 3.28. $\operatorname{Spec}^\infty$ *defines a contravariant functor* C^∞-Ring $\to$ Top *sending a* C^∞*-morphism* $\varphi\colon \mathfrak{A} \to \mathfrak{B}$ *to a map* $\varphi^*\colon \operatorname{Spec}^\infty \mathfrak{B} \to \operatorname{Spec}^\infty \mathfrak{A}$ *sending* $\mathfrak{p} \rightsquigarrow \varphi^{-1}(\mathfrak{p})$.

Proof. Since ring Spec is known to be topologically functorial, it suffices to show that C^∞-radicality pulls back. Indeed, if $\mathfrak{p} \in \operatorname{Spec}^\infty \mathfrak{B}$ is C^∞-radical, then $\mathfrak{p} = \eta_{\mathfrak{B}}^{-1}(\mathfrak{m})$, whereby $\varphi^{-1}(\mathfrak{p}) = (\eta \circ \varphi)^{-1}(\mathfrak{m})$ is also C^∞-radical. Continuity of φ^* is immediate from Proposition 3.25 (4). ☐

These elementary observations, which may be collectively referred to as an *I-V calculus*, already enable us to formulate descriptions of $\operatorname{Spec}^\infty \mathfrak{A}/I$ and $\operatorname{Spec}^\infty \mathfrak{A}_\mathfrak{p}$ in terms of subspaces of $\operatorname{Spec}^\infty \mathfrak{A}$, just as one does with affine schemes.

Proposition 3.29. *For a surjective* C^∞*-morphism* $\varphi\colon \mathfrak{A} \to \mathfrak{B}$, $\operatorname{Spec}^\infty \varphi$ *is a homeomorphism onto* $V^\infty(\ker\varphi)$. *Hence for any ideal* I *of* $\mathfrak{A}$, $\operatorname{Spec}^\infty \mathfrak{A}/I$ *is naturally a closed subspace of* $\operatorname{Spec}^\infty \mathfrak{A}$.

Proof. We proceed by restricting the classical homeomorphism, whereby it suffices to show that $\operatorname{Spec}^\infty \varphi$ surjects onto $V^\infty(\ker\varphi)$. Let $\mathfrak{p} \in \operatorname{Spec}^\infty(\mathfrak{A})$ be such that $\ker\varphi \subseteq \mathfrak{p}$. By the surjectivity of φ, $\mathfrak{q} := \varphi(\mathfrak{p})$ is prime, and $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. To see that $\mathfrak{q}$ is C^∞-radical, observe that $\mathfrak{A}/\mathfrak{p} \cong \mathfrak{B}/\mathfrak{q}$, again due to the surjectivity of φ, whereby if $\varphi(a) \in \mathfrak{q}$ is such that $(\mathfrak{B}/\mathfrak{q})_{\varphi(a)} = 0$, we have $(\mathfrak{A}/\mathfrak{q})_a = 0 \implies a \in \mathfrak{p}$, whence $\varphi(a) \in \mathfrak{q}$. ☐

Proposition 3.30. *Let* S *be a multiplicative submonoid of a* C^∞*-ring* $\mathfrak{A}$. *For* $\eta\colon \mathfrak{A} \to S^{-1}\mathfrak{A}$ *the canonical map,* η^* *is a homeomorphism onto* $\operatorname{im}\eta^* = \{ \mathfrak{p} \in \operatorname{Spec}^\infty \mathfrak{A} \colon \mathfrak{p} \cap S = \varnothing \}$.

Proof. We first show that η^* is well-defined. Indeed, given $\mathfrak{q} \in \operatorname{Spec}^\infty S^{-1}\mathfrak{A}$ and $a \in A$ such that $(\mathfrak{A}/\eta^{-1}(\mathfrak{q}))_a = 0$, we have the diagram

$$\begin{array}{ccc} \mathfrak{A}/\eta^{-1}(\mathfrak{q}) & \longrightarrow & S^{-1}\mathfrak{A}/\mathfrak{q} \\ \downarrow & & \downarrow \\ (\mathfrak{A}/\eta^{-1}(\mathfrak{q}))_a & \longrightarrow & (S^{-1}\mathfrak{A}/\mathfrak{q})_{\eta(a)}. \end{array}$$

Since a gets mapped to a unit along the upper path and to zero along the lower one, $(S^{-1}\mathfrak{A}/\mathfrak{q})_{\eta(a)} = 0 \implies \eta(a) \in \mathfrak{q} \implies a \in \eta^{-1}(\mathfrak{q})$. Hence $\eta^{-1}(\mathfrak{q})$ is C^∞-radical. Furthermore, $\eta^{-1}(\mathfrak{q}) \cap S = \varnothing$, lest $\mathfrak{q}$ contain a unit.

We now show that the putative inverse to η^*, extension of ideals, is well-defined. Fix a $\mathfrak{p} \in \operatorname{Spec}^\infty \mathfrak{A}$ with $\mathfrak{p} \cap S = \varnothing$. For each $s \in S$, we have by Proposition 3.8 that $\mathfrak{A}_s/(\eta_s(\mathfrak{p})) \cong (\mathfrak{A}/\mathfrak{p})_{[s]}$. Then

$$S^{-1}(\mathfrak{A}/\mathfrak{p}) = \varinjlim_{s \in S} (\mathfrak{A}/\mathfrak{p})_{[s]} \cong \varinjlim_{s \in S} \mathfrak{A}_s/(\eta_s(\mathfrak{p})) \cong S^{-1}\mathfrak{A}/(\eta(\mathfrak{p})).$$

If $1 \in (\eta(\mathfrak{p}))$, then $S^{-1}(\mathfrak{A}/\mathfrak{p}) \cong S^{-1}\mathfrak{A}/\eta(\mathfrak{p}) = 0$. But this is impossible, as with $\mathfrak{A}/\mathfrak{p}$ a C^∞-domain and $S \cap \mathfrak{p} \neq 0$, by Corollary 3.12 $(\mathfrak{A}/\mathfrak{p})_s \neq 0$ for all $s \in S$, and furthermore all natural maps $(\mathfrak{A}/\mathfrak{p})_{[s]} \to (\mathfrak{A}/\mathfrak{p})_{[st]}$ are injective. Primality is immediate: $S^{-1}\mathfrak{A}/(\eta(\mathfrak{p})) \cong S^{-1}(\mathfrak{A}/\mathfrak{p})$ is integral as a directed colimit of C^∞-domains. Finally, assume that $S^{-1}(\mathfrak{A}/\mathfrak{p})_{[a]} = 0$ for some $[a] \in \mathfrak{A}/\mathfrak{p}$. Then there exists some $s \in S$ for which $sa \in \sqrt[\infty]{\mathfrak{p}} = \mathfrak{p}$, for otherwise $(\mathfrak{A}/\mathfrak{p})_t$ would be integral, and the canonical maps injective for all $t \in S \cup \{ [a] \}$, whence $S^{-1}(\mathfrak{A}/\mathfrak{p})_{[a]}$ cannot be zero. But as $\mathfrak{p}$ is prime and $s \notin \mathfrak{p}$, $a \in \mathfrak{p}$.

To see that the two maps are mutually inverse, let $\mathfrak{p} \in \operatorname{im}\eta^*$, $\mathfrak{q} \in \operatorname{Spec}^\infty S^{-1}\mathfrak{A}$. We must show that $\eta^{-1}((\eta(\mathfrak{p}))) \subseteq \mathfrak{p}$ and $\mathfrak{q} \subseteq (\eta(\eta^{-1}(\mathfrak{q})))$. On the one hand, let $a \in \eta^{-1}((\eta(\mathfrak{p})))$, so that $\eta(a) \in (\eta(\mathfrak{p}))$. That is, $\eta(a) = \sum r_i \eta(p_i)$ for $r_i \in S^{-1}\mathfrak{A}$, $p_i \in \mathfrak{p}$. Setting $r_i = \eta(a_i)/\eta(b_i)$ via Proposition 3.10, we have $\eta(ba) = \eta(p)$ for $p \in \mathfrak{p}$, $b \in A$ with $\eta(b)$ a unit. Then Proposition 3.11 furnishes some $c \in A$ for which $c(ba - p) = 0 \implies cba \in \mathfrak{p}$ and $\eta(c)$ is a unit. Since $cb \notin \mathfrak{p}$ (lest $(\eta(p))$ contain a unit), the

primality of $\mathfrak{p}$ forces $a \in \mathfrak{p}$. On the other hand, let $x \in \mathfrak{q}$, so that by Proposition 3.10 $x = \eta(r)/\eta(s)$ for $r, s \in \mathfrak{A}$, $\eta(s)$ a unit. Then $r \in \eta^{-1}(\mathfrak{q})$, whereby $\eta(r) \in \eta(\eta^{-1}(\mathfrak{q})) \implies x \in (\eta(\eta^{-1}(\mathfrak{q})))$.
Since η^* is continuous by Theorem 3.28, it remains to show the continuity of $\mathfrak{p} \rightsquigarrow (\eta(\mathfrak{p}))$. Let $D^\infty(f) \subseteq \operatorname{Spec}^\infty S^{-1}(\mathfrak{A})$. By Proposition 3.10, f may be written as $\eta(a)/\eta(b)$ for $a, b \in \mathfrak{A}$, $\eta(b)$ a unit. Then $D^\infty(f) = D^\infty(\eta(a))$. We claim that this set pulls back to $D^\infty(a)$. That is, we must show that for $\mathfrak{p} \in \operatorname{Spec}^\infty \mathfrak{A}$ with $S \cap \mathfrak{p} = \varnothing$, $f \notin \mathfrak{p} \iff \eta(f) \notin (\eta(\mathfrak{p}))$. But we have shown earlier that $\eta^{-1}((\eta(\mathfrak{p}))) = \mathfrak{p}$. □

We now prove a C^∞-version of the Nullstellensatz. For this we require a preliminary result about C^∞-radicals:

Lemma 3.31. *Fix $\mathfrak{A}$ a C^∞-ring, $\{ J_\alpha \}$ a set of ideal, one has*

(1) $\sqrt[\infty]{\bigcap J_\alpha} = \bigcap \sqrt[\infty]{J_\alpha}$*;*
(2) $\sqrt[\infty]{J}$ *is the intersection of all C^∞-radical ideals containing J; hence $\sqrt[\infty]{-}$ is idempotent.*

Proof. We first show (1). Put $J := \bigcap J_\alpha$. Let $a \in \sqrt[\infty]{J}$, so that there exists $b \in J$ invertible in $\mathfrak{A}_a$. Then $b \in J_\alpha \implies a \in \sqrt{J_\alpha} = J_\alpha$ for all α, as needed. For the reverse inclusion, let $a \in \sqrt[\infty]{J_\alpha}$ for all α, so that there exists for every α some $b_\alpha \in J_\alpha$ invertible in $\mathfrak{A}_a$. Now simply consider $\prod b_\alpha$.
For (2), let I be any C^∞-radical ideal containing J. For any $a \in \sqrt[\infty]{J}$, to show that $a \in I$ it suffices to exhibit a $b \in I$ that is a unit in $\mathfrak{A}_a$. Of course, the $b \in J$ furnished by the definition of $\sqrt[\infty]{J}$ suffices. To see the other inclusion, it suffices to show that $\sqrt[\infty]{J}$ is itself C^∞-radical. Let $a \in \mathfrak{A}$, $b \in \sqrt[\infty]{J}$ such that b is a unit in $\mathfrak{A}_a$, yielding a C^∞-morphism $\mathfrak{A}_b \to \mathfrak{A}_a$. Then there is a $c \in J$ that is a unit in $\mathfrak{A}_b$, which in turn induces a C^∞-morphism $\mathfrak{A}_c \to \mathfrak{A}_b$. Composing the two, we obtain a C^∞-morphism $\mathfrak{A}_c \to \mathfrak{A}_a$ sending c to a unit, whereby $a \in \sqrt[\infty]{J}$. □

Theorem 3.32 (C^∞-Nullstellensatz)**.** *$I^\infty(-)$ and $V^\infty(-)$ give an inclusion-reversing bijection*

$$\{ \textit{closed subsets of } \operatorname{Spec}^\infty \mathfrak{A} \} \underset{V^\infty(-)}{\overset{I^\infty(-)}{\rightleftarrows}} \{ C^\infty\textit{-radical ideals of } \mathfrak{A} \},$$

where in particular $I^\infty(V^\infty(J)) = \sqrt[\infty]{J}$ and $V^\infty(I^\infty(S)) = \overline{S}$. Furthermore, it restricts to a bijection between irreducible closed subsets and C^∞-radical primes.

Proof. We first note that $I^\infty(-)$ is well-defined, as the intersection of C^∞-radical ideals remains C^∞-radical by the preceding lemma. Thus it makes sense to show that $I^\infty(V^\infty(J)) = \sqrt[\infty]{J}$. By (2) of the preceding lemma, the backwards inclusion is immediate. For the forward inclusion, we must show that if $a \notin \sqrt[\infty]{J}$, then there exists a $\mathfrak{p} \in V^\infty(J)$ with $a \notin \mathfrak{p}$. Indeed, since $(\mathfrak{A}/J)_a \neq 0$, $\mathfrak{p}$ can be taken to be the preimage of any $\mathfrak{q} \in \operatorname{Spec}^\infty(\mathfrak{A}/J)_a$ under the composition $\mathfrak{A} \to \mathfrak{A}/J \to (\mathfrak{A}/J)_a$. Indeed, by Proposition 3.29 and 3.30, necessarily $J \subseteq \mathfrak{p}$ and $a \notin \mathfrak{p}$.
We now show that $V^\infty(I^\infty(S)) = \overline{S}$. Since the left hand side is closed, it suffices to show that for any closed set $V^\infty(J)$ containing S, $V^\infty(I^\infty(S)) \subseteq V^\infty(J)$. By construction, for all $\mathfrak{q} \in S$, $J \subseteq \mathfrak{p}$, so $J \subseteq \bigcap_{\mathfrak{q} \in S} \mathfrak{q} = I^\infty(S)$. Then for any $\mathfrak{p} \in \operatorname{Spec}^\infty A$, $I^\infty(S) \subseteq \mathfrak{p} \implies J \subseteq \mathfrak{p}$, as needed.
As it is clear that the maps are inclusion-reversing, it remains to show the statement about restriction to irreducibles/primes. But this follows immediately from the same statement for the classical I-V correspondence. □

Corollary 3.33. *For J an ideal of a C^∞-ring $\mathfrak{A}$, $V^\infty(\sqrt[\infty]{J}) = V^\infty(J)$.*

Proof. Indeed, $V^\infty(\sqrt[\infty]{J}) = V^\infty(I^\infty(V^\infty(J))) = \overline{V^\infty(J)} = V^\infty(J)$. □

That is, vanishing sets cannot distinguish between a C^∞-ideal and its C^∞-radical. The C^∞-Nullstellensatz also allows us to show the converse, i.e. to establish the C^∞-radical as an indicator for how much functions are determined by their vanishing sets. A C^∞-ring is then C^∞-reduced precisely when this difference is zero, namely, when there is no nilpotence in the categorical sense.

Proposition 3.34. *For functions $f, g \in \mathfrak{A}$, the following are equivalent:*

(1) $V^\infty(f) \subseteq V^\infty(g)$*;*
(2) f *is invertible in* $\mathfrak{A}_g$*;*
(3) $g \in \sqrt[\infty]{(f)}$*.*

Hence $D^\infty(f) = \varnothing \iff f \in \sqrt[\infty]{0}$*.*

Proof. For (1) $\implies$ (2), suppose for contradiction that $\eta(f)$ is not invertible in $\mathfrak{A}_g$. Then $(\eta(f)) \subseteq \mathfrak{A}_g$ is proper, hence contained in a maximal ideal $\mathfrak{m}$. By Proposition 3.30, $\mathfrak{p} := \eta^{-1}(\mathfrak{m})$ is a C^∞-radical prime of $\mathfrak{A}$ with $g \notin \mathfrak{p} \implies \mathfrak{p} \in D^\infty(g)$. Then $\mathfrak{p} \in D^\infty(f)$, so $f \notin \mathfrak{p}$. But this is a contradiction, as $\eta(f) \in \mathfrak{m} \implies f \in \eta^{-1}(\mathfrak{m})$. (2) $\implies$ (3) is obvious. For (3) $\implies$ (1), we have $g \in \sqrt[\infty]{(f)} = I^\infty(V^\infty(f))$ iff $g \in \mathfrak{p}$ for all $\mathfrak{p} \in V^\infty(f)$, iff $f \in \mathfrak{p}$ implies $g \in \mathfrak{p}$, iff $V^\infty(f) \subseteq V^\infty(g)$. □

Proposition 3.35. *Two functions in a* C^∞*-ring* $\mathfrak{A}$ *that agree on a subset* $S \subseteq \operatorname{Spec}^\infty A$ *differ by an element of* $I^\infty(\overline{S})$*. Hence functions are uniquely determined by their values on points iff the ring is* C^∞*-reduced.*

Proof. If $f, g \in A$ agree on S, then $f - g$ vanishes on S, i.e. $f - g \in \bigcap_{\mathfrak{p} \in S} \mathfrak{p} = I^\infty(S) = I^\infty(\overline{S})$. To see the implication, we note that by *uniquely determine* we understand $f = 0 \iff D^\infty(f) = \varnothing$. By the first claim, functions f with $D^\infty(f) = \varnothing$ are precisely elements of $I^\infty(\operatorname{Spec}^\infty \mathfrak{A}) = I^\infty(V^\infty(0)) = \sqrt[\infty]{0}$, which is a singleton precisely when $\mathfrak{A}$ is C^∞-reduced. □

Remark. It follows from Theorem 3.32 and Proposition 3.27 that $\operatorname{Spec}^\infty \mathfrak{A}$ is a *spectral space* [3], as by Proposition 3.30 the basic open sets are affine, hence quasicompact, and the unique generic point of each closed irreducible subset is simply its associated C^∞-radical prime ideal. As a consequence, one may realize $\operatorname{Spec}^\infty \mathfrak{A}$ as the prime spectrum of a commutative ring A', and it would be of interest to study in general how this A' relates the the base ring of $\mathfrak{A}$. We do recognize, however, that the structure sheaf on A needs not coincide with that of $\mathfrak{A}$, whereby a possible research direction is whether there are conditions on $\mathfrak{A}$ ensuring they do align. There is also the inverse problem of deciding whether an arbitrary ring can be the C^∞-spectrum of some C^∞-ring.

This medley of topological results sets the stage for our study of the structure sheaf.

Proposition 3.36. $\mathscr{O}_{\operatorname{Spec}^\infty \mathfrak{A}}$ *forms a sheaf.*

Proof. We first show that the presheaf is indeed well-defined, i.e. that $\mathfrak{A}_f \cong \mathfrak{A}_g$ if $D(f) = D(g)$. Given our preparatory work, this is straightforward: Proposition 3.34 furnishes natural C^∞-morphisms $\mathfrak{A}_f \to \mathfrak{A}_g$, $\mathfrak{A}_g \to \mathfrak{A}_f$ which are necessarily mutually inverse by uniqueness.
We now verify the sheaf axioms. To this end, fix a distinguished base $\{ D^\infty(f_i) \}$. We shall denote $D^\infty(f_i)$ by U_i and $D^\infty(f_i) \cap D^\infty(f_j) = D^\infty(f_i f_j)$ by U_{ij}. To show locality, let $s \in \mathfrak{A}$ be such that $s|_{U_i} = 0$ for all i. Proposition 3.11 then furnishes for each i some $u_i \in A$ with $su_i = 0$ and $u_i|_{U_i}$ a unit. It follows from Proposition 3.34 that $D^\infty(f_i) \subseteq D^\infty(u_i)$, whereby $\{ D^\infty(u_i) \}$ also covers and, as in the proof of quasicompactness, there exist $r_i \in \mathfrak{A}$ with $1 = \sum r_i u_i$. Thus $s = \sum a_i(r_i s) = 0$. This extends readily to local sections, as each distinguished open $D(f_i)$ is affine by Proposition 3.30, whence we need only apply the above to $\mathscr{O}_{\operatorname{Spec}^\infty \mathfrak{A}_{f_i}}$.
For gluability, fix $s_i \in \mathfrak{A}_{f_i}$ with $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j. Proposition 3.10 supplies a representation of each s_i as $a_i|_{U_i}/b_i|_{U_i}$, with $a_i, b_i \in \mathfrak{A}$, $b_i|_{U_i}$ a unit. Agreement on intersections then amounts to that $a_i b_j|_{U_{ij}} = a_j b_i|_{U_{ij}}$. Furthermore, since $\{ D^\infty(b_i) \}$ covers, there exist $r_i \in \mathfrak{A}$ with $1 = \sum_i r_i b_i$. Put $s := \sum_i r_i a_i$. Then for each j, $sb_j|_{U_{ij}} = \sum_i r_i|_{U_{ij}} a_i b_j|_{U_{ij}} = \sum_i r_i b_i|_{U_{ij}} a_j|_{U_{ij}} = a_j|_{U_{ij}}$, whence $s|_{U_{ij}} = a_j|_{U_{ij}}/b_j|_{U_{ij}} = s_j|_{U_{ij}} = s_i|_{U_{ij}}$. It follows from locality that $s|_{U_i} = s_i$, as needed. The generalization to local sections is identical to the above. □

In particular, $(\operatorname{Spec}^\infty \mathfrak{A}, \mathscr{O}_{\operatorname{Spec}^\infty \mathfrak{A}})$ is a C^∞-ringed space. As such, the titular objects may be put on formal footing:

Definition 3.37. An **affine C^∞-scheme** is a C^∞-ringed space $(X, \mathscr{O})$ that is isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some ring A. A **C^∞-scheme** is a C^∞-ringed space $(X, \mathscr{O}_X)$ such that every point x admits an **affine neighborhood**, i.e. a neighborhood $(U, \mathscr{O}_X|_U)$ isomorphic to some affine scheme.

But for the $\operatorname{Spec}^\infty$ functor to be well-defined, the stalks of $\mathscr{O}_{\operatorname{Spec}^\infty \mathfrak{A}}$ must be local C^∞-rings. This is implied by the following

Proposition 3.38. *For $\mathfrak{p} \in \operatorname{Spec}^\infty \mathfrak{A}$, there is a natural isomorphism $(\mathscr{O}_{\operatorname{Spec}^\infty \mathfrak{A}})_\mathfrak{p} \cong \mathfrak{A}_\mathfrak{p}$. Hence affine C^∞-schemes are local C^∞-ringed spaces.*

Proof. This is evident from the construction of $\mathfrak{A}_\mathfrak{p}$ as a directed colimit: both directed systems consists of $\mathfrak{A}_s$ with $\mathfrak{p} \in D(s) \iff s \notin \mathfrak{p}$ and the canonical inclusions. $\square$

It follows that C^∞-schemes are local C^∞-ringed spaces. As such, the category they constitute should be subordinate to $\mathsf{L}C^\infty\text{-}\mathsf{RS}$, not $C^\infty\text{-}\mathsf{RS}$.

Definition 3.39. The **category of affine C^∞-schemes** is the full subcategory of $\mathsf{L}C^\infty\text{-}\mathsf{RS}$ of affine C^∞-schemes, denoted $\mathsf{Aff}C^\infty\text{-}\mathsf{Sch}$. Likewise, the **category of C^∞-schemes** is the full subcategory denoted $C^\infty\text{-}\mathsf{Sch}$.

We are now in a position to prove the Γ^∞-$\operatorname{Spec}^\infty$ adjunction.

Theorem 3.40 (c.f. [14, Thm 1.7])**.** *The global sections functor $\Gamma^\infty \colon \mathsf{L}C^\infty\text{-}\mathsf{RS} \to C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}$ is left adjoint to the affine C^∞-spectrum functor $\operatorname{Spec}^\infty \colon C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}} \to \mathsf{L}C^\infty\text{-}\mathsf{RS}$. Hence $\Gamma^\infty \colon \mathsf{L}C^\infty\text{-}\mathsf{RS} \to C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}$ is represented by $R := \operatorname{Spec}^\infty C^\infty(\mathbb{R})$, and $\operatorname{Spec}^\infty$ defines an equivalence $C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}} \cong \mathsf{Aff}C^\infty\text{-}\mathsf{Sch}$.*

Proof. Fix X a local C^∞-ringed space, $\mathfrak{A}$ a C^∞-ring. Consider the map Φ sending each morphism $\pi \colon X \to \operatorname{Spec}^\infty \mathfrak{A}$ to $\Gamma^\infty(\pi) \colon \mathfrak{A} \to \Gamma^\infty(X, \mathscr{O}_X)$. This is manifestly natural in X. We claim that the inverse is given by $\Psi := \eta^* \circ \operatorname{Spec}^\infty$, where $\eta \colon X \to \operatorname{Spec}^\infty \Gamma^\infty(X, \mathscr{O}_X)$ is the *affinization* map sending each point to the preimage in $\Gamma^\infty(X, \mathscr{O}_X)$ of the maximal ideal of its stalk and with pullback given by natural maps $\Gamma^\infty(X, \mathscr{O}_X)_f \to \Gamma^\infty(D^\infty(f), \mathscr{O}_X)$ (in particular, the global leg is the identity). Note that this will turn out to be the unit of the putative adjunction. That $\Phi \circ \Psi = \mathrm{id}$ is evident, and that $\Psi \circ \Phi = \mathrm{id}$ follows from the C^∞-version of Lemma 3.15: π and $\Psi \circ \Phi(\pi)$ have the same global leg, hence act identically pointwise, and the local legs are determined by the global one by the universal property of C^∞-localization.
The natural identification $\Gamma^\infty(X, \mathscr{O}_X) \cong C^\infty\text{-}\mathsf{Ring}(C^\infty(\mathbb{R}), \Gamma^\infty(X, \mathscr{O}_X))$ yields the first implication, while the second one follows from that

$$\mathsf{L}C^\infty\text{-}\mathsf{RS}(\operatorname{Spec}^\infty \mathfrak{A}, \operatorname{Spec}^\infty \mathfrak{B}) \cong C^\infty\text{-}\mathsf{Ring}(\mathfrak{B}, \Gamma^\infty \operatorname{Spec}^\infty \mathfrak{A}) \cong C^\infty\text{-}\mathsf{Ring}(\mathfrak{B}, \mathfrak{A}). \qquad \square$$

Our proof of the adjunction justifies a brief detour into functorial geometry, whose viewpoint resolves around the philosophy that copresheaves on $C^\infty\text{-}\mathsf{Ring}$ are preferred objects of study than C^∞-schemes, defining which requires three layers of data. In particular, the fact that $\operatorname{Spec}^\infty$ is an anti-equivalence allows us to embed $\mathsf{L}C^\infty\text{-}\mathsf{RS}$ in $\widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}$:

Lemma 3.41. *The functor $\iota \colon \mathsf{L}C^\infty\text{-}\mathsf{RS} \to \widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}$ sending $(X, \mathscr{O}_X) \rightsquigarrow \mathsf{L}C^\infty\text{-}\mathsf{RS}(\operatorname{Spec}^\infty -, X)$ is fully faithful.*

Proof. This is entirely analogous to Proposition 2.1. Indeed, ι evidently factors as

$$\mathsf{L}C^\infty\text{-}\mathsf{RS} \xrightarrow{h_-} \widehat{\mathsf{L}C^\infty\text{-}\mathsf{RS}} \xrightarrow{r} \widehat{\mathsf{Aff}C^\infty\text{-}\mathsf{Sch}} \xrightarrow{(\operatorname{Spec}^\infty)^*} \widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}},$$

where r simply restricts the domain of each presheaf $\mathsf{L}C^\infty\text{-}\mathsf{RS}^{\mathrm{op}} \to \mathsf{Set}$. We need only show the fully faithfulness of r on the essential image of h_-, with that of h_- being tautological and that of $(\operatorname{Spec}^\infty)^*$ a consequence of the fact that $\operatorname{Spec}^\infty$ is an anti-equivalence. That is, for any local C^∞-ringed spaces X, Y, we must show that the map $\operatorname{Hom}(h_X, h_Y) \to \operatorname{Hom}(rh_X, rh_Y)$ induced by r

is a bijection. But this is merely that morphisms glue: given a morphism $\varphi\colon rh_X \to rh_Y$, we define $\psi\colon h_X \to h_Y$ on any local C^∞-ringed space U as the map sending a morphism $\pi\colon U \to X$ to the unique morphism $U \to Y$ glued from $\varphi_{U_i}(\pi|_{U_i})$, for $\{ U_i \}$ an affine cover of U. □

It follows from $\operatorname{Spec}^\infty$ being an anti-equivalence that $\iota \operatorname{Spec}^\infty \mathfrak{A} \cong h^{\mathfrak{A}}$: this is nothing more than the assertion that $\mathsf{LC}^\infty\text{-}\mathsf{RS}(\operatorname{Spec}^\infty -, \operatorname{Spec}^\infty \mathfrak{A}) \cong C^\infty\text{-}\mathsf{Ring}(\mathfrak{A}, -)$. This hints at the existence of a more general adjoint pair extending $\Gamma^\infty \dashv \operatorname{Spec}^\infty$ in which the right adjoint is the Yoneda embedding. Indeed, it suffices to take the left adjoint as once again being represented by ιR:

Theorem 3.42. *The representable functor $h_{\iota R}\colon \widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}} \to C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}$ is left adjoint to the Yoneda embedding $h_-\colon C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}} \to \widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}$.*

Proof. We first show this for representables, i.e. exhibit isomorphisms $C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}(h_{\iota R}(h^{\mathfrak{A}}), \mathfrak{B}) \cong \widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}(h^{\mathfrak{A}}, h^{\mathfrak{B}})$. Indeed, $h_{\iota R}(h^{\mathfrak{A}}) \cong \widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}(h^{\mathfrak{A}}, \iota R) \cong \iota R(\mathfrak{A}) = \mathsf{LC}^\infty\text{-}\mathsf{RS}(\operatorname{Spec}^\infty \mathfrak{A}, R) \cong \operatorname{Spec}^\infty \mathfrak{A}$, and $\widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}(h^{\mathfrak{A}}, h^{\mathfrak{B}}) \cong h^{\mathfrak{B}}(\mathfrak{A}) = C^\infty\text{-}\mathsf{Ring}(\mathfrak{B}, \mathfrak{A})$. Naturality is clear as this is essentially a Yoneda isomorphism. For the full adjunction, we appeal to the fact that every copresheaf is a colimit of representables: for each $F\colon C^\infty\text{-}\mathsf{Ring} \to \mathsf{Set}$, we have

$$\begin{aligned}
\widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}(F, h^{\mathfrak{B}}) &\cong \widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}\left(\varinjlim_{(\mathfrak{A},x)\in \int F} h^{\mathfrak{A}}, h^{\mathfrak{B}} \right) \\
&\cong \varprojlim_{(\mathfrak{A},x)\in \int F} \widehat{C^\infty\text{-}\mathsf{Ring}}^{\mathrm{op}}(h^{\mathfrak{A}}, h^{\mathfrak{B}}) \\
&\cong \varprojlim_{(\mathfrak{A},x)\in \int F} C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}(h_{\iota R}(h^{\mathfrak{A}}), \mathfrak{B}) \\
&\cong C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}\left(\varinjlim_{(\mathfrak{A},x)\in \int F} h_{\iota R}(h^{\mathfrak{A}}), \mathfrak{B} \right) \\
&\cong C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}}(h_{\iota R}(F), \mathfrak{B}).
\end{aligned}$$
□

We conclude the section by tying the affine C^∞-spectrum of manifolds back to the original, Hausdorff version of the space. While in algebraic geometry analytification is a delicate process that dates back to Serre's seminal paper *Géométrie Algébrique et Géométrie Analytique* (GAGA), the inherently analytic nature of our objects renders this unnecessary. Indeed, even to a fully-fleged affine C^∞-scheme one can attach a homeomorphic space that is of a more "analytic" flavor once a presentation is fixed. There is little ingenuity involved in the construction, but some lattice theory is required for us to state it precisely:

Definition 3.43. A poset $(L, \leq)$ is a **(bounded) lattice** if when viewed as a category it admits all finite products, called **joins**, and coproducts, called **meets**. That is, given a finite subset $\{ x_i \} \subseteq L$,

$$\bigvee_i x_i := \sup\{ x_i \} \in L \quad \text{and} \quad \bigwedge_i x_i := \inf\{ x_i \} \in L.$$

A **lattice homomorphism** is but a functor between the respective underlying categories that preserves finite produdcts and coproducts. The resultant category is denoted Lat.

Examples of lattices abound; the primordial one is perhaps given by the ordering on a power set given by inclusion.

Definition 3.44. Fix a lattice $(L, \leq)$. A non-empty subset F of L is a **filter** if it is

(1) Downward Directed: for all $x, y \in F$, there exists $z \in F$ with $x \leq z$ and $y \leq z$;
(2) Upward Closed: for all $x \in F$, $y \in L$, $x \leq y \implies y \in F$.

I is **proper** if it is not the whole of L, and furthermore **prime** if $x \vee y \in F \implies x \in F \vee y \in F$. It is **maximal** if it is not properly contained in any other filter, in which case it is said to be an **ultrafilter**.

Theorem 3.45. *Fix $\mathfrak{A}$ a C^∞-ring with presentation $C^\infty(\mathbb{R}^X)/I$. To each $J \subseteq C^\infty(\mathbb{R}^X)$, we associate the set $\hat{J} := \{ Z(f) \colon f \in J \}$. In particular, $L := \widehat{C^\infty(\mathbb{R}^X)}$ is a lattice. Then there is a homeomorphism $\pi \colon \operatorname{Spec}^\infty \mathfrak{A} \to \Pi_I$, where $\Pi_I := \left\{ P \colon P \text{ is a prime filter in } L \wedge \hat{I} \subseteq P \right\}$ is endowed with the topology generated by $\left\{ O_f := \{ P \colon Z(f) \notin P \} \colon f \in C^\infty(\mathbb{R}^X) \text{ lifts an element of } \mathfrak{A} \right\}$, called the **Stone topology**.*

Proof. We first verify that L is a lattice: indeed, $Z(f) \cup Z(g) = Z(fg)$, $Z(f) \cap Z(g) = Z(f^2 + g^2)$, and its lower and upper bounds are given by $Z(1)$ and $Z(0)$, respectively. We may then define π as sending $\mathfrak{p} \rightsquigarrow \hat{\mathcal{P}}$, where $\mathcal{P}$ is the prime ideal of $C^\infty(\mathbb{R}^X)$ (which contains I) that corresponds to $\mathfrak{p}$; that $\hat{\mathcal{P}}$ is a prime filter follows readily from that $\mathfrak{p}$ is a prime ideal. π is also continuous: the preimage of O_f is $D(f)$, as $f \notin \mathfrak{p} \iff Z(f) \notin \hat{\mathcal{P}}$.
For the bijectivity of π, consider the map ψ sending a prime filter P to $\{ f \in \mathfrak{A} \colon Z(f) \in P \}$, which is an ideal as $Z(f+g) \supseteq Z(f) \cap Z(g) \in P$ (and the latter is in P by downward directedness) and $Z(fg) = Z(f) \cup Z(g) \supseteq Z(f)$. The primality of $\psi(P)$ follows from that of P. To see that it is C^∞-radical, let $f \in \mathfrak{A}$ be such that there exists $g \in \psi(P)$ with g a unit in $\mathfrak{A}_f$. Since $\mathfrak{A}_f \cong C^\infty(U_f)$, $Z(g) \cap U_f = \varnothing \implies Z(g) \subseteq Z(f)$; $f \in \psi(P)$ then follows by upward closure. ψ is inverse to π essentially by construction.
It remains to show that π is an open map, namely that $\pi(D(f)) = O_f$. This is now straightforward, as we have shown that every prime filter is of the form $\hat{\mathcal{P}}$. $\square$

By sending each $p \in Z(I)$ to its vanishing ideal in $C^\infty(\mathbb{R}^n)$ (which necessarily contains I), we obtain a canonical injection $\iota \colon Z(I) \to \operatorname{Spec}^\infty \mathfrak{A}$ (injectivity is by virtue of the smooth Urysohn's lemma). Composing this map with the above homeomorphism, we obtain an injection $Z(I) \to \Pi_I$ sending $p \rightsquigarrow \widehat{I^\infty(p)}$. As $I^\infty(p)$ is maximal, our map factors through $\operatorname{Spec}_m \mathfrak{A}$ to yield

$$\tilde{\iota} \colon Z(I) \xrightarrow{\quad \iota \quad} \operatorname{Spec}_m \mathfrak{A} \xrightarrow{\pi|_{\operatorname{Spec}_m \mathfrak{A}}} \beta_I,$$

where β_I denotes the space of ultrafilters in L which contain $\hat{I}$, which is evidently the image of $\pi|_{\operatorname{Spec}_m \mathfrak{A}}$. Note that $\widehat{I^\infty(p)}$ is in fact the *principal ultrafilter* $P_p := \{ Z(f) \colon p \in Z(f) \}$ generated by p. With the map looking strikingly similar to the ultrafilter construction for the Stone-Čech compactification of $Z(I)$ (modulo the minor quibble that we are working with smooth functions as opposed to the continuous, bounded ones employed classically), we are on pace to establish $\tilde{\iota}$ as a putative compactification map. Now recall that such a map must

(1) have a compact Hausdorff codomain,
(2) have its image be dense in the codomain,
(3) satisfy a certain universal property.

$\beta_I \cong \operatorname{Spec}_m \mathfrak{A}$ is already known to be compact, being a closed subspace of the compact space $\operatorname{Spec}^\infty \mathfrak{A}$. For Hausdorffness, we appeal to the general theory of lattice representations: the space of ultrafilters of a lattice is Hausdorff iff the lattice L is normal, in the sense that for all $a, b \in L$ with $a \wedge b = 0$, there exists $c, d \in L$ with $a \wedge c = 0$, $b \wedge d = 0$, and $c \vee d = 1$. [cite] $Z(f) \colon f \in C^\infty(\mathbb{R}^X)$. Indeed, given such a, b

The initial premise of analytification is now to be realized through a characterization of the image of $Z(I)$ inside $\operatorname{Spec}_m \mathfrak{A}$. To this end we foreground the notion of an *archimedean* local C^∞-ring, the set of which stalks does admit an inclusion into $\operatorname{Spec}_m \mathfrak{A}$ by virtue of the following

Proposition 3.46. *Every C^∞-radical prime ideal $\mathfrak{p}$ in a C^∞-ring $\mathfrak{A}$ with $\mathfrak{A}_\mathfrak{p}$ archimedean is maximal.*

Proof. Since the composite $\pi \circ \eta$, for $\eta\colon \mathfrak{A} \to \mathfrak{A}_\mathfrak{p}$ the inclusion and $\pi\colon \mathfrak{A}_\mathfrak{p} \to \kappa(\mathfrak{A}_\mathfrak{p}) = \mathbb{R}$ the projection, is an $\mathbb{R}$-algebra homomorphism onto $\mathbb{R}$, it is necessarily surjective, whence $\ker \pi \circ \eta = \mathfrak{p}$ is maximal. □

Remark. Evidently the converse does not hold; with every affine C^∞-scheme being quasicompact, the points with residue field distinct from $\mathbb{R}$ arise as overhead from compactification in the sense that we have described above.

The archimdean points of a C^∞-ring can also be extracted via a relative construction:

Definition 3.47. An **$\mathbb{R}$-point** of a C^∞-ring $\mathfrak{A}$ is a C^∞-morphism $\mathfrak{A} \to \mathbb{R}$, and $\mathfrak{A}$ is said to be **pointed** if it admits an $\mathbb{R}$-point.

Remark. This terminology arises from the following observation: under the adjunction, an $\mathbb{R}$-point corresponds to a C^∞-morphism $\operatorname{Spec}^\infty \mathbb{R} \to \operatorname{Spec}^\infty \mathfrak{A}$, whose image is necessarily a point. This is analogous to how one extracts points from sets using maps $\{\,*\,\} \to X$. Nevertheless, $\operatorname{Spec}^\infty \mathbb{R}$ is not a separator in $\mathsf{L}C^\infty\text{-}\mathsf{RS}$; for one, it cannot detect non-archimedean loci in a natural way.

Proposition 3.48. *For $\mathfrak{A}$ a C^∞-ring, there is a bijection*

$$C^\infty\text{-}\mathsf{Ring}(\mathfrak{A}, \mathbb{R}) \underset{-\circ\eta}{\overset{\ker}{\rightleftarrows}} \{\, \mathfrak{p} \in \operatorname{Spec}^\infty \mathfrak{A} \colon \mathfrak{A}_\mathfrak{p} \text{ is archimedean} \,\}.$$

Proof. We first show that both maps are well-defined. As in the proof of the above proposition, the kernel $\mathfrak{p}$ of each $\mathbb{R}$-point $\varphi\colon \mathfrak{A} \to \mathbb{R}$ is maximal, hence C^∞-radical prime, and by surjectivity $\mathbb{R} \cong \mathfrak{A}/\mathfrak{p} \cong \mathfrak{A}_\mathfrak{p}/(\mathfrak{p})$. On the other hand, precomposing $\mathfrak{A}_\mathfrak{p} \to \kappa(\mathfrak{p}) = \mathbb{R}$ with $\eta\colon \mathfrak{A} \to \mathfrak{A}_\mathfrak{p}$ evidently yields an $\mathbb{R}$-point. The two are manifestly mutually inverse: one amounts to φ factoring through $\mathfrak{A}_{\ker\varphi}$, the other the above isomorphism of quotients. □

The claim, then, is that the subspace of $\mathbb{R}$-points is precisely the analytification, in the sense that under the choice of a presentation $C^\infty(\mathbb{R}^X)/I$, it is homeomorphic to $Z(I)$. The key to this identification will be the following

Lemma 3.49. *For $\mathfrak{A}$ a finitely-generated C^∞-ring with presentation $C^\infty(\mathbb{R}^n)/I$, every $\mathbb{R}$-algebra homomorphism $\varphi\colon \mathfrak{A} \to \mathbb{R}$ is of the form ev_p for some $p \in Z(I)$.*

Proof. We begin by reducing the problem to when $I = 0$. Notice, if $\varphi \circ \pi = \mathrm{ev}_p$, where $\pi\colon C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)/I$ is the projection, then for φ to be well-defined as evaluation at p it suffices to show that $p \in Z(I)$. Indeed, given $f \in I$, $\pi(f) = 0$, so $f(p) = \varphi(\pi(f)) = \varphi(0) = 0$. It thus remains to show in the free case that $\ker\varphi = I(p)$ for some $p \in \mathbb{R}^n$. For then $f - f(p) \in I(p) \implies \varphi(f - f(p)) = 0 \implies \varphi(f) = \varphi(f(p)) = f(p)$. This in turn amounts to that $\bigcap \widehat{\ker\varphi} \neq \varnothing$, as then for any $p \in \bigcap \widehat{\ker\varphi}$, $\ker\varphi \subseteq I(p)$, and equality follows from maximality. To this end, it suffices to exhibit a compact set K in $\widehat{\ker\varphi}$, for then with $\left\{\, K \cap Z \colon Z \in \widehat{\ker\varphi} \,\right\} \subseteq \widehat{\ker\varphi}$ satisfying the finite intersection property ($\widehat{\ker\varphi}$ is closed under finite intersections, and being a proper filter, does not contain the empty set), compactness ensures that $\bigcap K \cap Z \neq \varnothing \implies \bigcap Z \neq \varnothing$. But $\widehat{\ker\varphi}$ even contains an $(n-1)$-sphere: define $g \in C^\infty(\mathbb{R}^n)$ as sending $x \rightsquigarrow \|x\|^2$; then $\varphi(g - \varphi(g)) = 0$, whence $Z(g - \varphi(g)) = \left\{\, x \colon \|x\|^2 = \varphi(g) \,\right\}$ is an element of $\widehat{\ker\varphi}$. Note that here necessarily $\varphi(g) \geq 0$, as otherwise $Z(g - \varphi(g)) = \varnothing$, contradicting the properness of $\widehat{\ker\varphi}$. □

Corollary 3.50.

Definition 3.51. A local C^∞-ringed space is **archimedean** if every stalk thereof is an archimedean local C^∞-ring. The resultant full subcategory of $\mathsf{L}C^\infty\text{-}\mathsf{RS}$ is denoted $\mathsf{L}C^\infty\text{-}\mathsf{RS}^{\mathsf{ar}}$.

The fact that focusing on archimedeans indeed discards some information provides some hope that archimedeans are truly the points capturing analytic information, i.e. $Z(I)$. The compactness requirement in Theorem 1.8 may be lifted if we restrict our attention to them.

As an aside, we recover a theorem promised long ago in §1.

Corollary 3.52. *For all* $\mathfrak{A} \in C^\infty\text{-}\mathsf{Ring}$*,* $C^\infty\text{-}\mathsf{Ring}(\mathfrak{A}, \mathbb{R}) = \mathbb{R}\text{-}\mathsf{Alg}(\mathfrak{A}, \mathbb{R})$*. Hence the embedding* $\mathsf{Man}^\infty \to C^\infty\text{-}\mathsf{Ring}$ *composed with the forgetful functor* $C^\infty\text{-}\mathsf{Ring} \to \mathbb{R}\text{-}\mathsf{Alg}$ *remains fully faithful.*

Proof. Show every $\mathbb{R}$-algebra homomorphism F is actually precomposition. By the above, every $F \circ \mathrm{ev}_p$ is of the form ev_q; define φ as sending $p \rightsquigarrow q$. Then $F(f) = f \circ \varphi$, and φ is smooth as all its components $F(\pi_i)$ are. □

Proposition 3.53. *For* $\mathfrak{A}$ *a finitely-generated* C^∞*-ring with presentation* $C^\infty(\mathbb{R}^n)/I$*, there is a bijection* $Z(I) \cong C^\infty\text{-}\mathsf{Ring}(\mathfrak{A}, \mathbb{R})$*.*

Proof. □

An entirely formal argument yields a description for $\mathbb{R}$-points of arbitrary C^∞-rings:

Corollary 3.54. *For* $\mathfrak{A}$ *a* C^∞*-ring with presentation* $\varinjlim C^\infty(\mathbb{R}^n)/I$*, there is a bijection* $\varinjlim Z(I) \cong C^\infty\text{-}\mathsf{Ring}(\mathfrak{A}, \mathbb{R})$*.*

This alone is good reason we restrict our purview to finitely-generated C^∞-rings; it is unwieldy to constantly deal with colimits, and besides the study of infinite-dimensional spaces really falls under functional analysis.

Proposition 3.55. *The inclusion* $\iota\colon \mathsf{L}C^\infty\text{-}\mathsf{RS}^{\mathsf{ar}} \to \mathsf{L}C^\infty\text{-}\mathsf{RS}$ *admits a right adjoint* $\mathrm{Ar}\colon \mathsf{L}C^\infty\text{-}\mathsf{RS} \to \mathsf{L}C^\infty\text{-}\mathsf{RS}^{\mathsf{ar}}$ *sending each* $(X, \mathscr{O}_X)$ *to* $(X', \mathscr{O}_X|_{X'})$*, where* X' *is the subspace of* X *of all points with archimedean stalks.*

Proof. The claim amounts to the existence of isomorphisms $\mathsf{L}C^\infty\text{-}\mathsf{RS}(\iota(X), Y) \cong \mathsf{L}C^\infty\text{-}\mathsf{RS}(X, \mathrm{Ar}(Y))$, for $X \in \mathsf{L}C^\infty\text{-}\mathsf{RS}^{\mathsf{ar}}$, $Y \in \mathsf{L}C^\infty\text{-}\mathsf{RS}$, natural in X. Evidently restriction will do: □

Corollary 3.56. *There is an adjoint pair* $\begin{tikzcd} \mathsf{L}C^\infty\text{-}\mathsf{RS}^{\mathsf{ar}} \arrow[r, bend left, "\Gamma^\infty \circ \iota"] \arrow[r, phantom, "\perp"] & C^\infty\text{-}\mathsf{Ring}^{\mathrm{op}} \arrow[l, bend left, "\mathrm{Ar} \circ \mathrm{Spec}^\infty"] \end{tikzcd}$*.*

The functor $\mathrm{Ar} \circ \mathrm{Spec}^\infty$ will thus be said to be the **Archimedean spectrum**. As promised, it admits a nifty concrete description:

Theorem 3.57. *For* $\mathfrak{A} = C^\infty(\mathbb{R}^X)/I$*. Then* $\mathrm{Ar} \circ \mathrm{Spec}^\infty(\mathfrak{A})$ *is homeomorphic to* $Z(I)$*.*

Proof. □

Proposition 3.58. *Fix* $\mathfrak{A}$ *a local* C^∞*-ring. Then the following are equivalent:*

(1) $\mathfrak{A}$ *is pointed;*
(2) $\mathfrak{A}$ *is Archimedean;*
(3) if $\mathfrak{A} = C^\infty(\mathbb{R}^X)/I$*, then* $Z(I)$ *is non-empty.*

Proof. For (1) $\implies$ (2), note that every $\mathbb{R}$-algebra homomorphism $\varphi\colon \mathfrak{A} \to \mathbb{R}$ is surjective, whence $\ker \varphi$ is the unique maximal ideal, and $\mathbb{R}$ the residue field. □

the functor preserves transversal pullbacks (discuss this somewhere)

As an aside, we may now prove the locality of C^∞-domains.

Lemma 3.59. *For* $\mathfrak{p}$ *a prime ideal of* $\mathfrak{A}$*,* $\sqrt[\infty]{\mathfrak{p}}$ *is also prime.*

Proof. $\mathfrak{A}/\mathfrak{p}_{fg}$ is C^∞-domain,

$\sqrt[\infty]{\mathfrak{p}}$ consists of $a \in \mathfrak{A}$ for which there exists $b \in \mathfrak{p}$ invertible in $\mathfrak{A}_a$, i.e. $V^\infty(b) \subseteq V^\infty(a)$. Let $fg \in \sqrt[\infty]{\mathfrak{p}}$. Then there exist $b \in \mathfrak{p}$ with b a unit in $\mathfrak{A}_{fg}$. $V^\infty(b) \subseteq V^\infty(fg) = V^\infty(f) \cup V^\infty(g)$. That is, if $b \in \mathfrak{q}$, then $fg \in \mathfrak{q}$, i.e. one of f and g in $\mathfrak{q}$. □

Proposition 3.60. *Every C^∞-domain is local.*

Proof. Let $\mathfrak{A}$ be a C^∞-domain. We must show that the set of non-units forms an ideal. As the absorption property clearly holds, this amounts to $f+g$ being invertible implies f or g is invertible. Write $\mathfrak{A} := C^\infty(\mathbb{R}^X)/I$, and choose lifts $\overline{f}, \overline{g} \in C^\infty(\mathbb{R}^X)$. Then $\overline{f+g}$ is invertible in $D^\infty(\overline{f+g})$. U_f, U_g. □

4. Smooth Functors and Synthetic Differential Geometry

It can be argued that the C^∞-scheme is not so alluring an object of study in and of itself. For despite the evident wealth of objects subsumed under its scope, the flexibility it offers is arguably more excessive than what was initially sought, which is sufficiently addressed by affine C^∞-schemes alone. Unlike in algebraic geometry, projective spaces are C^∞-affine, and singular spaces can be constructed without resorting to extraneous gluing. Even then, C^∞-schemes still fail to give rise to a Cartesian closed category.

From these considerations, it is clear that the C^∞-scheme is a provisional construction (in the same way triangulated categories are precursors to stable ∞-categories), an alternative to which must be sought. Thus we choose to refrain from a study of quasicoherent sheaves over C^∞-schemes, as is done in algebraic geometry. Of course, even if we chose to give one, our account can hardly surpass the lucid exposition found in [].

The teased alternative has fortunately already made an appearance—they are precisely the co-presheaves over C^∞-Ring, which, as shown in Lemma 3.41, faithfully encompass $\mathsf{L}C^\infty$-RS. This is hardly new to C^∞-geometry; the choice to work with a presheaf category reflects the perspectival shift in algebraic geometry championed by Grothendieck in the 70s. Anyone familiar with this viewpoint in algebraic geometry would recall the introduction of a Grothendieck topology on AffSch; we will do the same. Before this, we give a review of basic notions from topos theory to be read at the reader's leisure.

Definition 4.1. A **Grothendieck topology** on a category C consists of a set J of families of morphisms, called **coverings**, satisfying the following axioms:

(1) For every isomorphism $\varphi \colon x \to y$, $\{\varphi\} \in J$;
(2) Closure under composition: if $\{U_i \to U\} \in J$ and $\{U_{ij} \to U_i\} \in J$ for each i, then the composites $\{U_{ij} \to U\} \in J$;
(3) Closure under base change: if $\{U_i \to U\} \in J$ and $V \to U$, then $U_i \times_U V$ exists and $\{U_i \times_U V \to V\} \in J$.

The pair (C, J) is a **site**. A presheaf $\mathscr{F} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is a **sheaf** on (C, J) if for all $U \in \mathsf{C}$, $\{U_i \to U\} \in J$, $\mathscr{F}(U)$ equalizes $\prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \times_U U_j)$. The full subcategory of $\mathrm{Psh}(\mathsf{C})$ of such sheaves is the **category of sheaves on** (C, J), denoted $\mathrm{Sh}(\mathsf{C})$. Such a category is said to be a **Grothendieck topos**.

Grothendieck topoi can be described axiomatically:

Proposition 4.2. *A category D is a Grothendieck topos iff it admits a fully faithful functor $U \colon \mathsf{D} \to \mathrm{Psh}(\mathsf{C})$, for C a category, which in turn admits a left exact left adjoint, known as* ***sheafification****.*

Proof. Omitted. □

As shown in [], a Grothendieck topos enjoy a number of good properties (practically every property of a presheaf category that one can conceive of), but only ones pertinent to C^∞-geometry will be listed here. For one, Grothendieck topoi are (elementary) topoi in the following sense:

Definition 4.3. A category C is a **topos** if it is finitely bicomplete, Cartesian closed, and has a subobject classifier, i.e. an object Ω and a morphism $\top\colon 1 \to \Omega$ such that for all monomorphisms $U \hookrightarrow X$, there exists a unique morphism $\chi_U\colon X \to \Omega$ making the following diagram Cartesian:

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\top"] \\ X \arrow[r, "\chi_U"] & \Omega. \end{tikzcd}$$

Remark. The quintessential subobject classifier is $\{0,1\}$ in Set, with the image of $1 \to \Omega$ being $\{1\}$ and the map χ_U associated to each inclusion sending x to 1 if $x \in U$, 0 otherwise. U can then be understood as the preimage of 1 under χ_U. This intuition carries over flawlessly to the general case, especially since topoi are, informally speaking, categories that behave like Set.

It will be helpful to spell out these constructions explicitly in a Grothendieck topos.

Proposition 4.4. *Let* $\mathrm{Sh}(\mathsf{C})$ *be a Grothendieck topos. Then given* $F, G \in \mathrm{Sh}(\mathsf{C})$*, the exponential object* G^F *sends* $c \in \mathsf{C}$ *to* $\mathrm{Hom}(h_c \times F, G)$.

Proof. We first exhibit adjunction isomorphisms $\mathrm{Hom}(F \times G, H) \cong \mathrm{Hom}(F, H^G)$; the action of G^F on morphisms is then uniquely determined. Indeed, to each map $F(c) \times G(c) \to H(c)$ the Cartesian closed structure of Set assigns a unique map $F(c) \to H(c)^{G(c)}$, $F(c) \to \mathrm{Hom}(h_c \times G, H)$. □

Corollary 4.5. *power object*

Other than the high-brow, categorical reasons,

(1) Define Weil algebras and other infinitesimal objects
(2) Define smooth functors
(3) Introduce SDG axioms
(4) Prove the axioms for $\widehat{\mathsf{Aff}C^\infty\text{-}\mathsf{Sch}}$.

Throughout the section, we denote by L the category $\mathsf{Aff}C^\infty\text{-}\mathsf{Sch}$.

The observant reader will notice that we have not shown L to be Cartesian closed in the preceding sections, as should befit a proper enlarging of Man^∞, at least according to our argument at the outset. The solution to this issue will be take the free cocompletion *once more*:

Definition 4.6. A **smooth functor** is a sheaf $\mathsf{L} \to \mathsf{Set}$, i.e. an element of $\widehat{\mathsf{L}}$.

Remark. Here $\widehat{\mathsf{L}}$ is in fact a *Grothendieck topos*: it is the category of sheaves over the site (L, J), where J is the trivial coverage. Replacing J with a non-trivial coverage, one is led to various smooth topoi that not only satisfy the axiomatics of synthetic differential geometry but also enjoy better formal properties than $\widehat{\mathsf{L}}$. Unfortunately, there is no space for such investigations in this paper, and the reader is referred to [].

Definition 4.7. A **model for SDG** is a topos consisting of a **line object** R, an infinitesimal object D,

Kock-Lawvere Axiom: For any $g\colon D \to R$, there exists a unique $b \in R$ such that $g(d) = g(0) + db$ for all $d \in D$.

:

It is our duty to immediately provide interpretations for these axioms. Kock-Lawvere can be unerstood by nothing more than spelling it with words: it literally asserts that g is a linear map, or a line $y = ax + b$. Of course, since D is a sort of infinitesimal object, which evidently starts at $g(0)$,

5. Conclusion

This paper lays the *foundation* for the theory of C^∞-schemes and synthetic differential geometry. Recognizing several categorical flaws of Man^∞, we proposed C^∞-rings as an alternative that accounts for singular spaces and gives rise to a bicomplete category. But as C^∞-Ring is still not Cartesian closed, we proceeded as in functorial algebraic geometry and considered categories of sheaves over sites on the category of affine C^∞-schemes, from which the axioms of SDG were deduced. Through this process, Man^∞ has been faithfully embedded in a *vastly* larger category, albeit one in which a surprising amount of classical results still hold, and objects that arise from smooth manifolds can be detected with ease.

We must admit, however, to have made no effort in developing the concomitant *concrete* theory of differential geometry that should now admit simpler proofs in this synthetic framework. This work is undertaken in the SDG literature, and part of the aim of this paper is to introduce this field to a wider crowd who may in turn choose to participate in this collective effort. We also take this chance to acknowledge the possibility of *other* models for SDG, and the exceptional textbook [11] that has presented in an elementary way the wealth of results that follow from axioms alone.

Let us end by describing a direction for future work that has sprung forth during writing. In finding the correct notion of an affine C^∞-scheme we seem to have come across an intrinsic property of points of *any* sort of affine scheme, namely Proposition 3.16, that is overlooked in the classical setting as it is automatically satisfied by prime ideals. In our case this amounts to C^∞-radicality, whence the theory unfolded entirely parallel to the classical one, with C^∞-localization replacing ordinary localization.

It is therefore of interest to determine the most general setting in which such an “affine geometry” may be developed. For instance, is it possible whenever one has a category C admitting a forgetful functor to Ring? From here, will the notion of radicality arising from C-localization always be equivalent to Proposition 3.30, and can results such as Proposition 3.30 and Nullstellensatz always be shown? Having done this, is it possible to extract an axiomatic framework for this type of duality that is independent of Ring, similar to how Grothendieck Galois theory generalizes the classical one?

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