An Erasure Queue-Channel with Feedback: Optimal Transmission Control to Maximize Capacity

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***Abstract*—A queue-channel is a model that captures waiting time-dependent degradation of information bits—a scenario motivated by quantum communications and delay-sensitive streaming. Recent work has characterised the capacity of the erasure queue-channel [1], and other noise models encountered in quantum communications . In this paper, we study an erasure queue-channel with feedback, and ask after the optimal transmission strategy to minimize waiting-induced erasures. Specifically, we assume that instantaneous feedback of queue-length (or of the queue-channel output) is available at the transmitter, which can modulate the rate of Poisson transmissions into the queue-channel. We pose an optimal control problem using HJB-style equations to maximize the information capacity, when the transmitter can choose from a bounded set of transmission rates. We show (under a numerically verifiable condition) that the optimal transmission policy is a *single-threshold policy of the bang-bang* type. In other words, transmitting at the maximum (minimum) possible rate when the queue is below (above) a threshold, maximizes the information capacity of the erasure queue-channel with feedback.**

***Index Terms*—Erasure Queue-Channel, feedback, optimal control**

I. Introduction

A 'queue-channel' is a model that combines information-theoretic and queueing-theoretic aspects to capture the degradation of information bits, as they wait in buffer to be processed. Such a scenario arises naturally in quantum communications [2], where the inevitable buffering of quantum bits at intermediate nodes causes them to suffer a waiting time-dependent decoherence — the longer a qubit waits in a buffer, the more likely it is to suffer errors/erasures. Similar scenarios also arise in delay-sensitive applications such as multimedia communication, and stream computing, where information bits become useless after a certain time.

Our present work takes off from recent papers [1], [2] that model waiting time-dependent degradation of information bits as a queue-channel. Specifically, the authors consider sequential processing of (qu)bits, and model the probability of error/erasure of a (qu)bit as an explicit function its waiting time. They derive expressions for the capacity of the queue-channel for erasures and other 'additive' noise models. An important and relevant result from [1] is that for any stationary and ergodic queue, the information capacity in (bits/second) of the Erasure Queue Channel (EQC) is given by the limiting *average rate of unerased bits* at the receiver.

A. Our Contributions

In the present paper, we study an EQC with feedback, in a setting where the source can *control the transmission rate* of bits based on instantaneous feedback from the queue-channel. Specifically, we assume that instantaneous feedback of queue-length (or of the queue-channel output) is available at the transmitter, which can modulate the instantaneous rate of Poisson transmissions into the queue-channel.

Suppose that the transmitter can choose the instantaneous rate of Poisson transmission from a bounded set $[\lambda_{\min}, \lambda_{\max}]$. Intuitively, when the queue is heavily backlogged, waiting times are also large, and this typically leads to a burst of erasures. Therefore, it seems prudent to reduce the transmission rate when the queue length is large, so as to prevent further erasures. Conversely, when the queue length is small, it seems reasonable to increase the transmission rate, so as to increase the number of channel uses. Such a trade-off between the channel quality and number of channel uses is somewhat reminiscent of water-filling in power allocation over fading wireless channels [3]. Considering that the information capacity (in bits/second) is simply the limiting average rate of unerased bits, how exactly should the transmitter control the instantaneous rate of transmission into the EQC, in order to maximize the capacity?

We answer this question by posing it mathematically as an optimal control problem—specifically we derive Hamilton-Jacobi-Bellman (HJB)-type equations, and obtain conditions under which these equations admit an optimal solution of the single-threshold type. We derive capacity expressions for single-threshold policies and show that under a numerically verifiable condition, the optimal policy is of the single-threshold 'bang-bang' type. Thus, when the queue length is (below) (above) a threshold, it is optimal to transmit at the (largest) (smallest) possible rate ($\lambda_{\max}$) ($\lambda_{\min}$), and the intermediate rates are not used. We also show that the capacity of the EQC increases in the presence of feedback, compared to an 'open-loop' EQC.

B. Related Work

Gallager and Telatar initiated the area of multiple access queues in [4], which is perhaps the earliest work at the intersection of queuing and information theory. Around the same time, Anantharam and Verdu considered timing channels where information is encoded in the times between consecutive information packets, and these packets are subsequently processed according to some queuing discipline [5]. In contrast to [5], we are not concerned with information encoded in the timing between packets — in our work, all the information is in the bits.

Information theoretic notions of of reliable service in queues is studied in [6], where the capacity for such systems is defined as the number of bits reliably processed per unit time. They characterize capacity in terms of queuing system parameters and also characterize the distributions of the arrival and service processes that maximize and minimize the capacity in a discrete-time setting.

A congestion control problem was studied in [7] wherein the queue length information is used to optimally control the transmission rate under congestion constraints. Their work assumes queue length as a continuous real-valued variable, and they show that a 'bang-bang' control policy is optimal. [8] studies the role of queue length information in a single-server queue with congestion-based flow control, where they analyzed the optimality of two-threshold policies.

II. System model

A. Erasure Queue Channel

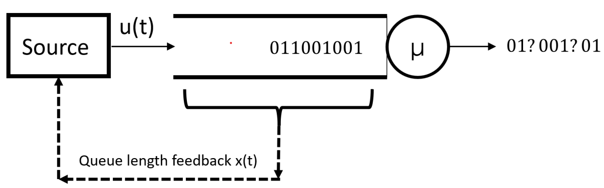

Fig. 1. Queue Erasure Channel with feedback

The queue erasure channel depicted in Figure 1 has three components, the source which pumps bits, the queue, where the bits wait for their service and the finally the server which serves or processes the bits. The arrival process of the bits from the source is modelled as a possibly non-homogeneous Poisson process of rate $u(t)$. The server follows a FIFO policy with independent and exponentially distributed service times for each bit with service rate μ.

Next we model the erasure probability of a bit as an explicit function of its waiting time W in the queue. To be more precise the erasure probability for waiting time W is given by $p(W) = 1 - e^{\kappa W}$, where κ is termed as the decoherence parameter in previous work [1], [2]. Using the general formula for capacity [5], the capacity for the EQC under a constant transmission policy is derived in [1] as given the theorem below.

Theorem 1. *For the EQC defined in* (II-A)*, the capacity is* $\lambda\mathbf{E}[1-p(W)] = \lambda\mathbf{E}[e^{-\kappa W}]$ *bits/sec, where the expectation is with respect to the stationary waiting time W in the queue.*

B. Queue-Channel feedback

We consider either one of the following feedback paradigms (which turn out to be equivalent in the erasure case): (i) The number of bits waiting in the queue at any given time is instantaneously available to the transmitter, or (ii) The queue-channel output is made available instantaneously to the transmitter. Note that (ii) is more aligned with the notion of feedback in information theory. Moreover, it is easy to see that the instantaneous queue length can be inferred by the transmitter from the channel feedback in (ii), by simply subtracting the number of bits received from the number transmitted. For erasures, the channel output symbols themselves turn out to be irrelevant, although for other channel models (ii) could be a strictly 'richer' feedback paradigm compared to (i). We do not pursue this further, and simply assume in the sequel that instantaneous queue length is available at the transmitter.

At time t, we define $x(t)$ to the number of bits waiting in the queue, or the queue length of the system. Next we will consider control policies $u(x(t))$ measurable on the filtration $\mathcal{F}_t = \sigma\{x(s) : s \leq t\}$ from the family of σ-algebras $\mathcal{F} = \sigma\{\mathcal{F}_s : s \in \mathbb{R}^{0+}\}$, so that $\mathcal{F}_t \in \mathcal{F}$. Next, we define the information capacity of the EQC, which is limiting average rate of unerased bits at the receiver for the above policy (for service rate μ and decoherence parameter κ) as:

$$C_u(\mu,\kappa) = \lim_{t \to \infty} \frac{1}{t}\,\mathbb{E}[\int_0^t r_u(x(\tau))\,d\tau],$$

where $r_u(x)$ is defined as:

$$r_u(x) = u(x)\,\mathbb{E}[1 - p(W)|x(t) = x]. \tag{1}$$

The term $\mathbb{E}[1 - p(W)|x(t) = x]$ denotes the fraction of unerased bits.

Lemma 1. *For the EQC defined in* (II-A) *with service rate μ and decoherence parameter κ, we have*

$$\mathbb{E}[1 - p(W)|x(t) = x] = (\frac{\mu}{\mu+\kappa})^{x+1}$$

Proof. This proof uses properties of memory-less Poisson arrivals and independent and exponentially distributed service times with service rate μ to relate the probability of erasure to Laplace transform of $p(W)$. Thus the waiting of a bit given the queue length ahead of it is x is the sum of $x + 1$ i.i.d exponential service times. Therefore,

$$\mathbb{E}[1 - p(W)|x(t) = x] = \mathbb{E}[e^{\kappa W}|x(t) = x] = (\frac{\mu}{\mu+\kappa})^{x+1}.$$

□

This simplifies $r_u(x)$ in (1) as

$$r_u(x(t)) = u(x(t))(\frac{\mu}{\mu+\kappa})^{x(t)+1}.$$

III. A Single-threshold Control Policy

Consider the control policy

$$u(x(t)) = \begin{cases} \lambda_{\max}, & x(t) < m \\ \lambda_{\min}, & x(t) \geq m \end{cases} \tag{2}$$

This induces birth-death Markov chain, and we get the stationary distribution in time using local balance equations:

$$\pi_m(x) = \begin{cases} \pi_m(0)\rho_{\max}^x, & x(t) < m \\ \pi_m(0)\rho_{\max}^m \rho_{\min}^{x-m}, & x(t) \geq m \end{cases}$$

where $\rho_{\max} = \frac{\lambda_{\max}}{\mu}$ and $\rho_{\min} = \frac{\lambda_{\min}}{\mu}$ and since sum of probabilities is 1,

$$\pi_m(0) = \frac{1}{\frac{1-\rho_{\max}^m}{1-\rho_{\max}} + \frac{\rho_{\max}^m}{1-\rho_{\min}}}. \tag{3}$$

Therefore the information capacity can be written as

$$C_u(\mu,\kappa) = \sum_{x=0}^{m-1} \lambda_{\max}\left(\frac{\mu}{\mu+\kappa}\right)^{x+1}\pi_m(x) + \sum_{x=m}^{\infty} \lambda_{\min}\left(\frac{\mu}{\mu+\kappa}\right)^{x+1}\pi_m(x),$$

which simplifies to give the following Theorem.

Theorem 2. *The information capacity of the EQC under the single-threshold policy defined in* (2) *is*

$$C_u(\mu,\kappa) = \pi_m(0)\frac{\mu}{\mu+k}[\lambda_{\max}\frac{1-c_{\max}^m}{1-c_{\max}} + \lambda_{\min}\frac{c_{\max}^m}{1-c_{\min}}],$$

where $c_{\max} = \frac{\lambda_{\max}}{\mu+\kappa}$, $c_{\min} = \frac{\lambda_{\min}}{\mu+\kappa}$ *and* $\pi_m(0)$ *defined in* (3).

Under this control policy, the average rate of bits transmission by the source becomes

$$\lambda_{avg} = \sum_{x=0}^{m-1} \lambda_{\max}\pi_m(x) + \sum_{x=m}^{\infty}\lambda_{\min}\pi_m(x) = \lambda_{\max}\left(1 - \frac{\pi_m(0)\rho_{\max}^m}{1-\rho_{\min}}\right) + \lambda_{\min}\frac{\pi_m(0)\rho_{\max}^m}{1-\rho_{\min}}.$$

The maximum information rate over all possible thresholds gives us the optimal single-threshold policy of the queue erasure channel.

$$C(\mu,\kappa) = \max_u C_u(\mu,\kappa)$$

$$C(\mu,\kappa) = \max_m \pi_m(0)\frac{\mu}{\mu+k}[\lambda_{\max}\frac{1-c_{\max}^m}{1-c_{\max}} + \lambda_{\min}\frac{c_{\max}^m}{1-c_{\min}}] \tag{4}$$

Corollary 1. *The information capacity of the EQC for a single-threshold ON/OFF policy i.e* $\lambda_{\max} = \lambda$ *and* $\lambda_{\min} = 0$ *is given as*

$$C_u(\mu,\kappa) = \frac{\lambda(\mu-\lambda)}{\mu+\kappa-\lambda}\frac{1-(\frac{\lambda}{\mu+\kappa})^m}{1-(\frac{\lambda}{\mu})^{m+1}},$$

where m *is the threshold.*

In the limit $m \to \infty$, when we transmit at constant rate λ throughout, we recover the expression in [2]

$$C = \frac{\lambda(\mu-\lambda)}{\mu+\kappa-\lambda},$$

which is the constant rate policy when we don't have any feedback.

Corollary 2. *In the limit* $\lambda \to \infty$, *the optimal threshold* $m^* = 1$ *and the information capacity is*

$$\lim_{\lambda\to\infty} \frac{\lambda(\mu-\lambda)}{\mu+\kappa-\lambda}\frac{1-\frac{\lambda}{\mu+\kappa}}{1-(\frac{\lambda}{\mu})^2} = \frac{\mu^2}{\mu+\kappa}$$

Intuitively, $\lambda \to \infty$ means that when we have a source that instantaneous pump bits into the queue without delay. So using feedback the source can optimally transmit only when the queue is empty to minimise the waiting time of the bits in the queue. This ensures that the server is always occupied, which implies $\lambda_{avg} = \mu$. In this case, erasure occurs only due to the time spent in service, which is unavoidable. Thus, $\mathbb{E}[1 - p(W)|x(t) = 1] = \frac{\mu}{\mu+\kappa}$, and we can derive the following upper bound for information capacity which is achieved asymptotically:

$$C_u(\mu,\kappa) \leq \lambda_{avg}\frac{\mu}{\mu+\kappa}. \tag{5}$$

Fig (2) illustrates that feedback combined with transmission rate control leads to higher information capacity for the same average rate of transmission of bits at the source compared to the constant rate policy.

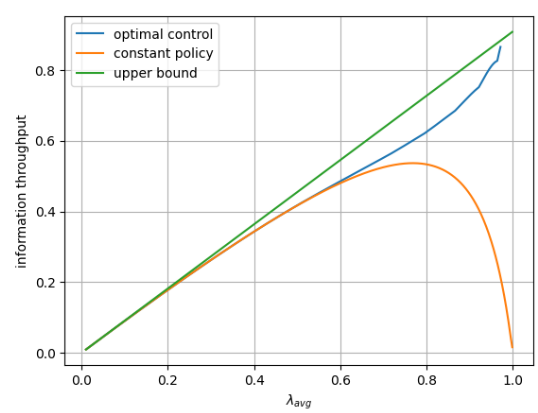

Fig. 2. Comparison of information capacity among constant policy, optimal feedback control and upper bound (5) for $\mu = 1, \kappa = 0.1$ for ON/OFF policy

IV. Optimal feedback Policy

A. Problem Formulation

The optimal policy that maximises the information capacity over all control policies $u(x(t))$ measurable on the filtration $\mathcal{F}_t$ can be formulated as maximising the following objective function (the average rate of unerased bits):

$$J_u = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}[\int_0^t u(\tau)(\frac{\mu}{\mu+k})^{x(\tau)+1}\,d\tau].$$

Next, define

$$J(x,t,T) := \max_u \mathbb{E}[\int_t^T u(\tau)(\frac{\mu}{\mu+k})^{x(\tau)+1}\, d\tau | x(t)=x].$$

Now for differential δ and $x>0$, we have

$$\begin{aligned} J(x,t,T) = \max_u u(x)(\frac{\mu}{\mu+k})^{x+1}\delta \\ + J(x,t+\delta,T)(1-(u(x)+\mu)\delta) \\ + J(x+1,t+\delta,T)u(x)\delta \\ + J(x-1,t+\delta,T)\mu\delta. \end{aligned}$$

Taking derivative w.r.t to t we get

$$\begin{aligned} -\frac{\partial J}{\partial t} = \max_u u(x)(\frac{\mu}{\mu+k})^{x+1} - J(x,t,T)(u(x)+\mu) \\ + J(x+1,t,T)u(x) + J(x-1,t,T)\mu. \end{aligned} \quad (6)$$

Define differential value function

$$W(x) := \lim_{T\to\infty} J(x,t,T) - (T-t)J^*.$$

Since we have an irreducible and ergodic Markov process, the queue asymptotically in the limit $T\to\infty$ reaches stationary distribution given $x(t)=x$, and $W(x)$ is independent of t (see [7]). Therefore, we have

$$-\frac{\partial W}{\partial t} = 0 \implies -\frac{\partial J}{\partial t} = J^*, \quad \text{as } T\to\infty \quad (7)$$

where $J^* = \max_u J_u$. So taking the limit $T\to\infty$ and combining (6) and (7) yields

$$\begin{aligned} J^* = \max_u u(x)[(\frac{\mu}{\mu+k})^{x+1} - W(x) + W(x+1)] \\ + \mu[W(x-1) - W(x)]. \end{aligned}$$

In the case of $x=0$, since queue is empty, the service rate $\mu=0$, and we can similarly obtain

$$J^* = \max_u u(x)[(\frac{\mu}{\mu+k})^{x+1} - W(x) + W(x+1)].$$

Therefore, if $u(t)\in[\lambda_{\min},\lambda_{\max}]$, where $\lambda_{\min}<\mu$, then one can expect the optimal control to have the bang-bang form:

$$u(t) = \begin{cases} \lambda_{\max}, & (\frac{\mu}{\mu+k})^{x+1} - W(x) + W(x+1) > 0 \\ \lambda_{\min}, & (\frac{\mu}{\mu+k})^{x+1} - W(x) + W(x+1) \le 0 \end{cases} \quad (8)$$

Suppose there exists a stationary control policy and differential value function $W(x)$ that satisfies the conditions in (8); then the policy is optimal.

B. Optimality of single-threshold Policies

In [7], the authors use this approach to show the optimality of the bang-bang policy in congestion control. The main technical difference is that, the queue length (congestion) in their problem is modeled as a real-valued variable with continuous-time dynamics. This leverages the use HJB equations to solve the problem. In our problem setting, we restrict ourselves to an integer-valued queue. Consequently, we have a continuous-time system with discrete state variables, which renders the system dynamics too complex to solve in closed form. However, we exploit the fact that we have a finite state birth-death Markov chain to find the stationary distribution, enabling us to compute the information capacity.

In this section, we analyze single-threshold policies of the form:

$$u(x(t)) = \begin{cases} \lambda_{\max}, & x(t) < m \\ \lambda_{\min}, & x(t) \ge m \end{cases}$$

and characterize the conditions under which they are optimal. We need the following lemma before we proceed.

Lemma 2.

$$\lim_{T\to\infty} J(x,t,T) - J(x,t+\delta,T) = \delta J^*$$

Proof. From (7) we have

$$\lim_{T\to\infty} \frac{\partial J(x,t,T)}{\partial t} = -J^* \quad \forall x$$

Therefore

$$\lim_{T\to\infty} J(x,t+\delta,T) - J(x,t,T) = \int_t^{t+\delta} \lim_{T\to\infty} \frac{\partial J(x,t,T)}{\partial t} = -\delta J^* \qquad \square$$

In light of Lemma 2, our goal is to reduce

$$\lim_{T\to\infty} J(x+1,t,T) - J(x,t,T)$$

to a closed-form expression which can be evaluated easily. To do this, we take limit $T\to\infty$ and using a Bellman-type expansion, we get

$$\begin{aligned} J(x+1,t,T) = \\ \max_u \mathbb{E}[\int_t^T u(\tau)(\frac{\mu}{\mu+k})^{x(\tau)+1}\, d\tau | x(t)=x+1] \\ = \max_u \mathbb{E}[\int_t^{t+T_u(x+1,t)} u(\tau)(\frac{\mu}{\mu+k})^{x(\tau)+1}\, d\tau | x(t)=x+1] \\ + \mathbb{E}[\int_{t+T_u(x+1,t)}^T u(\tau)(\frac{\mu}{\mu+k})^{x(\tau)+1}\, d\tau | x(t+T_u(x+1,t))=x] \\ = g_u(x+1,t) + J(x,t+T_u(x+1,t),T) \end{aligned}$$

where $T_u(x+1,t)$ is defined as

$$T_u(x+1,t) = \min_\tau\{\tau : x(t+\tau)\le x \ | x(t)=x+1\},$$

and $g_u(x+1,t)$ is defined as

$$\begin{aligned} g_u(x+1,t) = \\ \mathbb{E}[\int_t^{t+T_u(x+1,t)} u(\tau)(\frac{\mu}{\mu+k})^{x(\tau)+1}\, d\tau | x(t)=x+1]. \end{aligned}$$

Next, consider

$$\begin{aligned} J(x,t+T_u(x+1,t),T) - J(x,t,T) = \\ \mathbb{E}[\int_t^{t+T_u(x+1,t)} \frac{\partial J(x(t),t,T)}{\partial t}\, d\tau | x(t)=x]. \end{aligned}$$

The event $\{x(t) = x; T_u(x+1,t) = \tau_u\} \in \mathcal{F}_{t+\tau_u}$ as $T_u(x+1,t) = \tau_u$ is a stopping rule under $\mathcal{F}_{t+\tau_u}$. Since $\mathcal{F}_t \subseteq \mathcal{F}_{t+\tau_u}$, using the tower law of expectation, we get

$$\begin{aligned}
&J(x, t + T_u(x+1,t), T) - J(x,t,T) = \\
&\qquad \mathbb{E}[\ \mathbb{E}[\int_t^{t+\tau_u} \frac{\partial J(x(t),t,T)}{\partial t} d\tau | x(t) = x \\
&\qquad\qquad ; T_u(x+1,t) = \tau_u]\ | x(t) = x]
\end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}[\ \mathbb{E}[\tau_u J^* | x(t) = x; T_u(x+1,t) = \tau_u]\ | x(t) = x] \qquad (9)\\
&= -\mathbb{E}[T_u(x+1,t)|x(t) = x] J^* \\
&:= -t_u(x+1) J^* \qquad (10)
\end{aligned}$$

The equality (9) comes from the previous Lemma 2 as limit $T \to \infty$. Therefore we can now replace

$$\begin{aligned}
&(\frac{\mu}{\mu+k})^{x+1} - W(x) + W(x+1) = \\
&\qquad (\frac{\mu}{\mu+k})^{x+1} - J(x,t,T) + J(x+1,t,T) = \beta(x),
\end{aligned}$$

where $\beta(x)$ is defined as

$$\beta(x) = (\frac{\mu}{\mu+k})^{x+1} + g_u(x+1) - t_u(x+1)J^*,$$

and $t_u(x+1)$ as defined in (10).

Now, following the methodology in [7], let

$$J^* = \pi_{m^*}(0)\frac{\mu}{\mu+k}[\lambda_{\max}\frac{1-c_{\max}^{m^*}}{1-c_{\max}} + \lambda_{\min}\frac{c_{\max}^{m^*}}{1-c_{\min}}],$$

where m^* denote the threshold value of the optimal single-threshold policy obtained from (4), then it is sufficient to show

$$\begin{cases} \beta(x) \geq 0, & x < m^* \\ \beta(x) < 0, & x \geq m^* \end{cases}$$

for the policy to be optimal. To do this, we calculate the $t_u(.)$ and $g_u(.)$ using the following recursion equations using properties of competing exponentials of Poisson process.

$$t_u(x) = \frac{1}{\mu+u(x)} + \frac{u(x)}{\mu+u(x)}(t_u(x) + t_u(x+1)) \qquad (11)$$

$$g_u(x) = \frac{u(x)(\frac{\mu}{\mu+\kappa})^{x+1}}{\mu+u(x)} + \frac{u(x)}{\mu+u(x)}(g_u(x)+g_u(x+1)) \qquad (12)$$

To obtain the boundary conditions, we exploit the fact that for $x > m^*$, the queue behaves like an M/M/1 system with a constant arrival rate $\lambda_{\min}$. Therefore,

$$t_u(m^* + \tilde{x} + 1) = t_u(m^* + \tilde{x}) \qquad (13)$$

$$g_u(m^* + \tilde{x} + 1) = (\frac{\mu}{\mu+\kappa}) g_u(m^* + \tilde{x}) \qquad (14)$$

Therefore, for all $\tilde{x} > 0$. Solving the recursive equations (11)-(14), we get:

$$t_u(x) = \begin{cases} \frac{1}{\mu}\frac{1-\rho_{\max}^{m^*-x}}{1-\rho_{\max}} + \frac{1}{\mu-\lambda_{\min}}\rho_{\max}^{m^*-x}, & x < m^* \\ \frac{1}{\mu-\lambda_{\min}}, & x \geq m^* \end{cases}$$

$$g_u(x) = \begin{cases} c_{\max}\frac{1-c_{\max}^{m^*-x}}{1-c_{\max}}\gamma^x + g_u(m^*)\rho_{\max}^{m^*-x}, & x < m^* \\ \frac{\lambda_{\min}}{\mu+k-\lambda_{\min}}\gamma^x, & x \geq m^* \end{cases}$$

where $\gamma = \frac{\mu}{\mu+\kappa}$. These closed form expressions help us to numerically evaluate $\beta(.)$ and by showing that $\beta(x) < 0$ for $x \geq m^*$, we have proven optimality. Also for $x \geq m^*$

$$\beta(x) = \gamma^{x+1} + \frac{\lambda_{\min}}{\mu+k-\lambda_{\min}}\gamma^{x+1} - \frac{J^*}{\mu-\lambda_{\min}},$$

since $\beta(x)$ only decreases because of the geometrically decaying terms, showing

$$\begin{cases} \beta(x) \geq 0, & x < m^* \\ \beta(x) < 0, & x = m^* \end{cases} \qquad (15)$$

is sufficient for optimality. See Fig (3) for illustration.

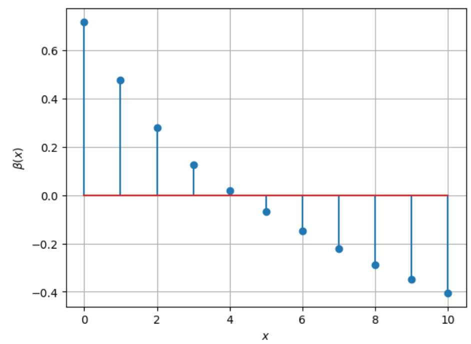

Fig. 3. Numerical evaluation of $\beta(x)$ for the optimal single-threshold bang-bang policy with $\mu = 1, \kappa = 0.1, \lambda_{\max} = 0.8, \lambda_{\min} = 0.4, m^* = 5$

To summarise, we have shown that if optimal single-threshold policy satisfies (15), then it is optimal among all policies measurable on the filtration $\mathcal{F}_t$. Since queue length is discrete valued, we do not have closed form expressions for m^* and J^*, we numerically verify that the optimality conditions are satisfied.

V. Concluding Remarks

In this paper, we studied an EQC with instantaneous queue feedback, and and analysed the structure of optimal transmission policy at the source. We showed conditions under which a single-threshold 'bang-bang' policy is optimal and derived expression for the information capacity. We also showed that the capacity increases in the presence of optimal control based on feedback, compared to an 'open-loop' EQC.

Constructing capacity achieving coding schemes for the EQC in the presence of feedback is an open problem. It is also interesting to consider queue-channels with feedback under other noise models including Binary Symmetric channel, and characterise information capacity and optimal control. Future work can also study the impact of feedback being delayed by a random duration, or possibly through another queue-channel.

VI. Acknowledgement

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