The purpose of the scoring guide is to provide a uniform evaluation of the tests by the graders. Therefore, the guide gives you the main ideas (and at least one possible solution) for each problem and the associated scores. This guide is not intended to be a detailed description of a complete solution to each problem; the steps described are sketches of a solution that achieves maximum score.
The partial scores given in the guide are only awarded to the solver if the related idea is included in the test as a step in a clearly described and reasoned solution. Thus, for example, the mere description of statements or definitions in the material without its application (even if any of the facts described do indeed play a role in the solution) is not worth the point.
A partial score is given for any idea or part solution from which the correct solution of the problem could be obtained with the appropriate addition of the reasoning described in the test. If a solver starts several solutions that are significantly different, then you can give score for at most one. If each described solution or part of solution is correct or can be extended to be correct, then the solution with the most points will be evaluated. However, if there are correct and incorrect solutions between several solution attempts and it is not clear from the test which one the solver considered to be correct, then the solution with fewer points will be evaluated (even if this score is 0). The scores in this guide may be further subdivided as necessary. Of course, a good solution other than that described in this guide is also worth the maximum score.
In the event of an arithmetic error, one point is deducted from each exercise. An exception to this is when computing the numbers significantly simplifies or modifies the structure of the task. In these cases, no points are awarded to the steps which follow the computation and are missing from the solution of the student.

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1. In a bag of roasted salted pistachios, it might happen that some pistachios are not split but are completely closed and are hard to open. Assume that a bag has many pistachios, and the pistachios each have an identical, independent small probability of being closed. Denote the number of closed pistachios in a bag with X. We know that $P(X = 0 \mid X \leq 2) = 0.4$. Determine $\mathrm{Cov}(1 + 2X, 3X)$.
(2 points) $X \sim \mathrm{Pois}(\lambda)$
(1 point) where $\lambda > 0$
(If $\lambda > 0$ condition appears later in the solution, then also points are awarded.)
(2 points) $\mathbb{P}(X = 0 \mid X \leq 2) = \frac{\mathbb{P}(X=0)}{\mathbb{P}(X\leq 2)}$, using $\{X = 0\} \cap \{X \leq 2\} = \{X = 0\}$
(2 points) $\mathbb{P}(X \leq 2) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) + \mathbb{P}(X = 2)$
(1 point) $\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$
(2 points) So, $\mathbb{P}(X \leq 2) = \left(1 + \lambda + \frac{\lambda^2}{2}\right)e^{-\lambda}$,
(0 points) and $\mathbb{P}(X = 0) = e^{-\lambda}$
(1 point) Substituting it back: $\mathbb{P}(X = 0 \mid X \leq 2) = \frac{e^{-\lambda}}{\left(1+\lambda+\frac{\lambda^2}{2}\right)e^{-\lambda}}$
(2 points) Using the condition $0.4 = \mathbb{P}(X = 0 \mid X \leq 2) = \frac{1}{1+\lambda+\frac{\lambda^2}{2}}$, solving which we get: $\lambda \in \{-3, 1\}$
(1 point) but $\lambda > 0$, so, $\lambda = 1$.
(1 point) $\mathrm{Cov}(1 + 2X, 3X) = \mathrm{Cov}(2X, 3X)$
(2 points) $= 2 \cdot 3 \cdot \mathrm{Cov}(X, X)$
(1 point) $= 6 \cdot \mathrm{Var}(X)$
(1 point) $\mathrm{Var}(X) = \lambda = 1$
(1 point) So, $\mathrm{Cov}(1 + 2X, 3X) = \underline{\underline{6}}$

2. A firm buys 90 light bulbs for their office space. Assume that the life span, measured in years, of each light bulb has $\mathrm{Exp}(\frac{1}{\sqrt{10}})$ distribution, and that they are independent of each other.
   (a) Consider the total life span of all the 90 light bulbs. For what positive real number z can we say that this total life span is more than z with probability approximately 3%?
   (b) How would the above value of z change if the expected life span of individual light bulbs was half a year more, but the standard deviation remained the same as before. (The distribution will not necessarily be exponential.)

(0 points) Denote the life span of the light bulbs with $X_1, \ldots, X_{90}$
(1 point) $\mathbb{P}\left(\sum_{i=1}^{90} X_i \geq z\right) = 0.03$

(2 points) Taking its complement: $\mathbb{P}\left(\sum_{i=1}^{90} X_i < z\right) = 1 - \mathbb{P}\left(\sum_{i=1}^{90} X_i \geq z\right) = 1 - 0.03 = 0.97$
(1 point) $\mathbb{E}(X_1) = \frac{1}{\frac{1}{\sqrt{10}}} = \sqrt{10}$
(1 point) $\mathrm{Var}(X_1) = \frac{1}{\left(\frac{1}{\sqrt{10}}\right)^2} = 10$, so $\sigma_{X_1} = \sqrt{10}$
(3 points) Standardizing:

$$\mathbb{P}\left(\frac{\sum_{i=1}^{90} X_i - 90\cdot\sqrt{10}}{\sqrt{90}\sqrt{10}} < \frac{z - 90\cdot\sqrt{10}}{\sqrt{90}\sqrt{10}}\right) = 0.97$$

(0 points) Since X_i's are independent identically distributed random variables, so,
(2 points) using the Central Limit Theorem,
(3 points) $\frac{\sum_{i=1}^{90} X_i - 90\cdot\sqrt{10}}{\sqrt{90}\sqrt{10}}$ has approximately standard normal distribution.
(1 point) So, $0.97 = \Phi\left(\frac{z-90\cdot\sqrt{10}}{\sqrt{90}\sqrt{10}}\right)$
(1 point) Substituting the value from the Φ table: $\frac{z-90\cdot\sqrt{10}}{\sqrt{90}\sqrt{10}} = 1.88$
(1 point) So $z = 1.88\cdot\sqrt{90}\sqrt{10} + 90\cdot\sqrt{10}$
(1 point) $\approx \underline{\underline{341.005}}$
(1 point) If the expected value were to be more by 0.5, then the standardization would look like:

$$\mathbb{P}\left(\frac{\sum_{i=1}^{90} X_i - 90\cdot(\sqrt{10}+0.5)}{\sqrt{90}\sqrt{10}} < \frac{z - 90\cdot(\sqrt{10}+0.5)}{\sqrt{90}\sqrt{10}}\right)$$

(1 point) Using similar steps as in part (a), we conclude that, $z_{\text{new}} = z_{\text{old}} + 90\cdot 0.5$
(1 point) $\approx \underline{\underline{386.005}}$
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3. The joint probability density function of random variables X and Y is given as follows:

$$f_{X,Y} : (x,y) \mapsto \begin{cases} \alpha x + \beta y & \text{if } 0 < x < 2 \text{ and } 0 < y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $\mathrm{Cov}(X,Y) = -\frac{1}{36}$. Determine α and β. What is $P(X<1)$?
(2 points) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$
(2 points) So, $\int_0^2\int_0^2(\alpha x+\beta y)\mathrm{d}x\,\mathrm{d}y = \int_0^2(\alpha\cdot 2+\beta\cdot 2y)\,\mathrm{d}y = 4(\alpha+\beta) = 1$
(1 point) $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$
(2 points) $\mathbb{E}(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\cdot f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$
(1 point) So, $\mathbb{E}(XY) = \int_0^2\int_0^2 xy\cdot(\alpha x+\beta y)\,\mathrm{d}x\,\mathrm{d}y$
(1 point) $= \int_0^2(\frac{2^3}{3}\alpha y + 2\beta y^2)\,\mathrm{d}y = \frac{16}{3}(\alpha+\beta)$
(1 point) $= \frac{4}{3}$ using $4(\alpha+\beta)=1$.
(2 points) $\mathbb{E}(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\cdot f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$ and we can similarly find $\mathbb{E}(Y)$
(1 point) So, $\mathbb{E}(X) = \int_0^2\int_0^2 x\cdot(\alpha x+\beta y)\,\mathrm{d}x\,\mathrm{d}y = \frac{16}{3}\alpha + 4\beta$
(0 points) $= \frac{4}{3}\alpha + 1$ again using that $4(\alpha+\beta)=1$.
(If computation of $\mathbb{E}(X)$ involves the substitution that $4(\alpha+\beta)=1$, but the computation of $\mathbb{E}(XY)$ doesn't, then 1 point.)
(1 point) Similarly, $\mathbb{E}(Y) = 4\alpha + \frac{16}{3}\beta = -\frac{4}{3}\alpha + \frac{4}{3}$
(1 point) Substituting these back: $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y) = \frac{4}{3} - \left(\frac{4}{3}\alpha+1\right)\cdot\left(-\frac{4}{3}\alpha+\frac{4}{3}\right)$
(1 point) Since $\mathrm{Cov}(X,Y) = -\frac{1}{36}$, so, $\alpha = \frac{1}{8}$.
(1 point) and so, $\beta = \frac{1}{4} - \alpha = \frac{1}{8}$
(2 points) $\mathbb{P}(X<1) = \int_0^2\int_0^1(\frac{1}{8}x+\frac{1}{8}y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^2\left(\frac{1}{16}+\frac{1}{8}y\right)\,\mathrm{d}y$
(1 point) $= \frac{3}{8} = 0.375$

4. A bug walks an integer number of steps on the X-axis starting at $x=0$. Let right and left directions denote increasing and decreasing values of x respectively. The bug first moves Y steps to the right, where the moment generating function of Y is given by $\frac{1}{3}(e^{-t}+1+e^{t})$. It then moves Z steps to the right where the moment

generating function of Z is given by $\frac{e^t}{2-e^t}$, where $t < \ln 2$. Assume Y and Z are independent. Let X denote the position of the bug after it has moved Y and Z steps. What is the range of X? What is $P(X = k)$ for k in the range of X?
(2 points) Since MGF uniquely determines the distribution of a random variable,
(2 points) we know that $Range(Y) = \{-1, 0, 1\}$ and it takes all these values with probability $\frac{1}{3}$.
(1 point) similarly we can deduce that $Z \sim Geo(\frac{1}{2})$
(1 point) so, $Range(Z) = \{1, 2, ...\}$
(2 points) combining these, $Range(X) = \{0, 1, 2, ..\}$
(2 points) since $X = Y + Z$.
(3 points) We need to use the convolution formula $p_X(x) = \sum_a p_Y(x-a)p_Z(a)$
(1 point) so, $p_X(0) = p_Y(-1)p_Z(1) = \frac{1}{3} \cdot \frac{1}{2}$
(2 points) $p_X(1) = p_Y(0)p_Z(1) + p_Y(-1)p_Z(2) = \frac{1}{3}(\frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2})$
(2 points) and for $x > 1$, $p_X(x) = p_Y(-1)p_Z(x+1) + p_Y(0)p_Z(x) + p_Y(1)p_Z(x-1) = \frac{1}{3}(\frac{1}{2^{x+1}} + \frac{1}{2^x} + \frac{1}{2^{x-1}})$
(2 points) $= \frac{7}{3 \cdot 2^{x+1}}$

5. We toss a coin four times. We say that we win if there are no two consecutive Heads in the four tosses, otherwise we lose. Assume that the outcomes of individual tosses are independent.

   (a) If the probability of the coin landing on Tails on a single coin toss is p, then what is the probability of winning?

   (b) An adversary tries to tamper with the coin, but is only partially successful in changing it. Assume that the probability of landing on Tails for this tampered coin is uniformly distributed in the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$. What is the probability of winning?

(3 points) $\mathbb{P}$(there are no two consecutive Heads) $=$ $\mathbb{P}$(there are no Heads)+$\mathbb{P}$(there is only one Head)+$\mathbb{P}$($HTHT$ or $HTTH$
(the above computed in terms of p)
(3 points) $= p^4 + 4p^3(1-p) + 3p^2(1-p)^2 = 3p^2 - 2p^3$
(If the expression is not simplified to the rightmost form, then also points awarded.)
(3 points) $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A \mid U = x) f_U(x)\,\mathrm{d}x$
(3 points) $f_U(x) = 2$ if $\frac{1}{4} < x < \frac{3}{4}$ and 0 otherwise
(2 points) $\mathbb{P}(\text{win} \mid U = x) = 3x^2 - 2x^3$ using part (a)
(3 points) $\mathbb{P}(\text{win}) = \int_{\frac{1}{4}}^{\frac{3}{4}} (3x^2 - 2x^3) \cdot 2\,\mathrm{d}x$
(2 points) $= \left[2x^3 - x^4\right]_{\frac{1}{4}}^{\frac{3}{4}} = 2\left(\frac{3}{4}\right)^3 - \left(\frac{3}{4}\right)^4 - 2\left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4$
(1 point) $= \underline{\underline{\frac{1}{2}}}$

6.* Bob is very particular about the number of followers he has on a certain online platform. Assume that he currently has 1000 followers. Based on his posts in a given week, the number of his followers might increase by a factor of 1.1, or stay the same or decrease by a factor of 0.9. Each of these events has probability $\frac{1}{3}$. Assume that changes in the number of followers in different weeks are independent.

   (a) Determine the expected number of followers Bob has after 104 weeks (approximately two years).

   (b) What is the approximate probability that Bob has fewer followers after 104 weeks than he has now?

(1 point) Denote by $X_1, \ldots, X_{104}$ the multiplication factor incurred in each week. (so, X_i is 1.1 or 1 or 0.9)
(1 point) $\mathbb{E}(1000 \cdot X_1 \cdot X_2 \cdot \ldots X_{104}) =?$
(2 points) $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$ if X and Y are independent
(3 points) Iterating this: $\mathbb{E}(X_1 \cdot X_2 \cdot \ldots X_{104}) = \mathbb{E}(X_1)\,\mathbb{E}(X_2)\ldots\mathbb{E}(X_{1000})$
(1 points) $\mathbb{E}(X_1) = \sum_{\mathrm{Ran}(X_1)} k \cdot \mathbb{P}(X_1 = k) = 1.1 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0.9 \cdot \frac{1}{3} = 1$
(1 points) So the required value is: $1000 \cdot 1 \cdot \ldots \cdot 1 = \underline{\underline{1000}}$
(1 points) $\mathbb{P}(1000 \cdot X_1 \cdot X_2 \ldots X_{104} < 1000) =?$
(Points are awarded if even the factor of 1000 is missing.)
(2 points) $\mathbb{P}(1000 \cdot X_1 \cdot X_2 \cdot \ldots X_{104} < 1000) = \mathbb{P}(\ln X_1 + \ln X_2 + \ldots + \ln X_{104} < 0)$
(0 points) Let $Y_i = \ln X_i$

(2 points) Standardizing:

$$\mathbb{P}\left(\frac{\sum_{i=1}^{104} Y_i - 104\,\mathbb{E}(Y_1)}{\sqrt{104}\sigma_{Y_1}} < \frac{-104\,\mathbb{E}(Y_1)}{\sqrt{104}\sigma_{Y_1}}\right)$$

(0 points) Since Y_i are independent identically distributed random variables, so,
(2 points) using the Central Limit Theorem, $\frac{\sum_{i=1}^{104} Y_i - 104\,\mathbb{E}(Y_1)}{\sqrt{104}\sigma_{Y_1}}$ can be approximated with the standard normal distribution.
(1 points) So the required probability is: $\Phi\left(\frac{-104\,\mathbb{E}(Y_1)}{\sqrt{104}\sigma_{Y_1}}\right)$
(1 points) $\mathbb{E}(Y_1) = \frac{1}{3}(\ln 0.9 + \ln 1 + \ln 1.1) = -0.00335$
(1 points) $\mathrm{Var}(Y_1) = \frac{1}{3}((\ln 0.9)^2 + (\ln 1)^2 + (\ln 1.1)^2) - (-0.00335)^2 = -0.0034 = 0.006717$
(0 points) $\sigma_{Y_1} = 0.082$
(1 points) $\Phi\left(\frac{-104\cdot(-0.00335)}{\sqrt{104}\cdot 0.082}\right) \approx \Phi(0.42) \approx \underline{\underline{0.6628}}$ (from looking up the appropriate value from the table)