Please note: The duration of the exam is **100** minutes. You may use a calculator. Please give a numeric answer (rounded to 4 decimal places). You are expected to **write all steps** taken in getting the final answer along with a **mention of properties/theorems used** in these steps. You are not allowed to leave the examination hall during the first 30 minutes of the exam.

1. Bob has two unfair coins. The first lands on heads with probability $\frac{1}{3}$, while the second has tails on both sides. He randomly picks one coin and tosses it four times. Given that the number of tails he observed was even, what is the probability that he selected the first coin?

2. A point (U, V) is picked uniformly at random from the square $[0, 1] \times [0, 1]$. Let $X = \sqrt{U}$ and $Y = \sqrt{V}$. Determine
   (a) the distribution of X, (b) the density function of $X + Y$.

3. A certain institution ships in n bags of an item 'T' every week, the number of bags ordered being the same every week. The quantity of T in each bag is independently and identically distributed, with mean 10 kg and standard deviation 2 kg. In any given week, if the amount of T supplied is more than the amount ordered by at least 20 kg (i.e. supply is more than $10n + 20$ kg), then they keep the amount ordered and 20 kg of extra T, but throw the remaining surplus in the harbor. (If the extra T is less than 20 kg, then they don't throw anything). Further, it was observed that the probability of some T being thrown in the harbor was 0.0336. What is the number of bags of T ordered every week?

4. In the imaginary land of Surrealia, the amount of delay in the arrival of any train has the memoryless property. Delays are measured in the units of quarter of an hour. The average delay is known to be 3 quarter hours. Each time a train is delayed, the train company receives letters of complaint. For a fixed delay of x quarter hours, assume that the letters of complaint are all independent, identical, but small probability events. If a train is x quarter hours late, then, the probability that the company does not receive any complaint letters is e^{-100x}.
   (a) What is the conditional expected number of letters of complaint for a fixed x quarter hours of delay?
   (b) What is the expected number of letters received due to the delay of a given single train?

5. Let $(X, Y) \sim N(\mathbf{0}, \mathbf{\Sigma})$ be a vector random variable with bivariate (2D) normal distribution. Assume that $Var(X) = Var(Y)$, that the correlation of X and Y is 0.5, and that $\det(\mathbf{\Sigma}) = 12$.
   (a) What is the probability that X is greater than 3.6?
   (b) What is the density function of (X, Y)?

6.* Let (U, V) be a random variable vector. Assume that $Var(U)$, $Var(V)$ and $\mathrm{Cov}(U, V)$ are positive real numbers. Let the linear regression of V in terms of U be $\beta_1 U + \alpha_1$, and the linear regression of U in terms of $-V$ be $\beta_2(-V) + \alpha_2$. Further assume that the line, $e = \{(x, y) \in \mathbb{R}^2 \mid y = \beta_1 x + \alpha_1\}$, is perpendicular to the line, $f = \{(x, y) \in \mathbb{R}^2 \mid y = \beta_2 x + \alpha_2\}$. If U has exponential distribution, then what is the distribution of V?
   If we also assume that $\alpha_1 = \alpha_2$, then what can you say about V?

Name	Range	$P(X=i)$ or $F_X(x)$	f_X	$E(X)$	σ_X
Indicator $1(p)$	{0,1}	$P(X=1)=p$		p	$\sqrt{pq}$
Binomial $Bin(n,p)$	{0,1,...,n}	$\binom{n}{i}p^i(1-p)^{n-i}$		np	$\sqrt{np(1-p)}$
Poisson $Pois(\lambda)$	{0,1,...}	$\frac{\lambda^i}{i!}e^{-\lambda}$		λ	$\sqrt{\lambda}$
Geometric $Geo(p)$	{1,2,...}	$=(1-p)^{i-1}p$		$\frac{1}{p}$	$\frac{\sqrt{1-p}}{p}$
Uniform $U(a,b)$	(a,b)	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{b-a}{2\sqrt{3}}$
Exponential $Exp(\lambda)$	$\mathbf{R}^+$	$1-e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$
Normal $N(\mu,\sigma^2)$	$\mathbf{R}$	$\Phi(\frac{x-\mu}{\sigma})$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ
Multivariate Normal $N(\boldsymbol{\mu},\boldsymbol{\Sigma})$	$\mathbf{R}^n$	$f_{\boldsymbol{X}}(\boldsymbol{x})=\frac{1}{(2\pi)^{\frac{n}{2}}det(\boldsymbol{\Sigma})^{\frac{1}{2}}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$		$\boldsymbol{\mu}$	Covariance matrix $\boldsymbol{\Sigma}$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998