Probability Theory Lecture 04

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Overview

1. Continuous Random Variables
2. Expected Value
3. Transformations of Random Variables

Historical Introduction

Problem

Suppose you are waiting for a bus. Let the number of minutes it is late be denoted by X. Further assume that X is equally likely to be any number in $[0, 1]$.
Since there are infinitely many equally likely points in the sample space, $P(X = a) = 0$ for all a. But we can instead understand the CDF.
What can you say about $F_X(x) = P(X \leq x)$, the cumulative distribution function?

Historical Introduction cont.

Derivative of CDF

The derivative of the CDF of the above problem is $y = \frac{1}{10}$. What does it mean?

Historical Introduction cont.

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Historical Introduction cont.

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Probability Density Function

The derivative of the CDF is called the Probability Density Function (PDF) of the random variable and intuitively denotes the relative probability of points of the sample space.

Continuous Random Variable

Definition

X is said to be a **continuous random variable**, if there exists a Riemann integrable function $f_X : \mathbb{R} \to [0, \infty)$ such that $\int_{-\infty}^{\infty} f_X(x)dx = 1$.
f_X is said to be the **probability density function** of the random variable X.

CDF

The **cumulative distribution function**, denoted $F_X(a)$ is defined as $F_X(a) = \int_{-\infty}^{a} f_X(x)dx = P(X \leq a)$

Properties of CDF

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- It is continuous from right, i.e., $\lim_{x \to a^+} F_X(x) = F(a)$.
- Limits: $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.

Properties of CDF cont.

Proof of Property 2

Consider any sequence $x_n \to a^+$. Then, $\lim_{x_n \to a^+} F_X(x_n) = \lim_{x_n \to a^+} P(X \leq x_n) = P(\cap_n(-\infty, x_n]) = P((-\infty, a]) = F_X(a)$. Notice that above we are using the Limit Property for a reducing sequence of events.

Pdf for finding probabilities.

Probability of an interval

Theorem: $P(a < x < b) = \int_a^b f_X(x)dx$.

For proof, use CDF.

Examples

Uniform Distribution

$X \sim U(a, b)$ is said to have uniform distribution if its pdf is given by $f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

What is the CDF of $U(a, b)$?

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What is the CDF of $U(a, b)$?
What is the probability $P(X > \frac{a+b}{2})$?

Examples

Example 1a from book

$$f_X(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

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What is c?
What is $P(X > 1)$?

Expected value of a continuous random variable

Definition

Given a random variable X with a pdf $f_X(x)$, if $\int_{-\infty}^{\infty} |x| f_X(x) dx = L$ for some real number L, then the Expected value of X is given by,

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

Expected value of a continuous random variable

Examples

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Expected value of a continuous random variable

Examples

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$$E(X) =$$

The pdf $f_X(x) = \frac{1}{\pi}\frac{1}{1+x^2}$ is continuous integrable, but $E(X)$ does not exist.

Properties of Expected Value

We have the same properties as in the discrete case.

- $E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$ (Law of the Unconscious Statistician).

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- $E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x) dx$ (Law of the Unconscious Statistician).
- $E(X + Y) = E(X) + E(Y)$ (Linearity of Expectation).

Examples

Let $f_X(x) = \begin{cases} c(4x - 2x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$, as before.

What is $E(X^2)$?

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What is $E(X^2)$?

What is $E(X + X^2)$?

Simulation using Uniform distribution

Let $X \sim U(0,1)$. We know that all computer languages have a random number generator which can simulate X.
Suppose Y is a random variable such that
$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^2 & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases}$. How can we make a computer simulate it?

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Suppose Y is a random variable such that

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How can we make a computer simulate it?
Let $Y = \sqrt{X}$. Then,
$F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = y^2$.
Hence, to make a computer simulate Y, use the usual random number generator and take its square root.
Here Y is called a transform of the random variable X.

Transformation

Definition

Given a random variable X, $Y = g(X)$ is called a transformation of X. It is called a linear transformation if $g(X) = aX + b$ for some constants a, b.

In general, you will be given the PDF of X and your task will be to find the PDF of Y. To do this, one must **always start with the CDF**.

$F_Y(y) = P(Y \leq y) = P(g(X) \leq y)$

From here, we try to derive bounds for X and use its CDF to find the CDF of Y. Be careful when taking the inverse of g!

Transformation cont.

Example 1

Example 1: $Y = 2X$
Then, $F_Y(y) = P(Y \leq y) = P(2X \leq y) = P(X \leq \frac{y}{2}) = F_X(\frac{y}{2})$.
If we differentiate with respect to y, we get, $f_Y(y) = \frac{1}{2} f_X(\frac{y}{2})$.
So, for example if $f_X(x) = e^{-x}$, for $x \geq 0$, then $f_Y(y) = \frac{1}{2} e^{-y/2}$ for $y \geq 0$.

Transformation cont.

Example 2

Example 2: $Y = e^X$

Then,

$F_Y(y) = P(Y \leq y) = P(e^X \leq y) = P(X \leq ln(y)) = F_X(ln(y))$.

If we differentiate with respect to y, we get, $f_Y(y) = \frac{1}{y} f_X(ln(y))$.

So, for example if $f_X(x) = 3x^2$, for $0 \leq x \leq 1$, then

$f_Y(y) = \frac{1}{y} 3(ln(y))^2$ for $1 \leq y \leq e$.

Bertrand's Paradox

Problem

Consider a unit circle and an equilateral triangle inscribed in it. We pick a chord of the circle at random. What is the probability that the length of this chord is larger than the side of the equilateral triangle?

Assume that the chord passes through a fixed point P on the circumference.

Solution 1

Uniform distance: Pick a number in $[0, 1]$ and draw a concentric circle, and draw a tangent to the circle from P.

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Solution 1

Uniform distance: Pick a number in $[0, 1]$ and draw a concentric circle, and draw a tangent to the circle from P. It is easy to see that this chord has the required length with probability $\frac{1}{2}$.

Bertrand's Paradox cont.

Solution 2

Uniform angle: Pick an angle uniformly between $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and draw a chord from P with that angle.

Bertrand's Paradox cont.

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Uniform angle: Pick an angle uniformly between $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and draw a chord from P with that angle. It is easy to see that this chord has the required length with probability $\frac{1}{3}$.

Bertrand's Paradox cont.

Solution 2

Uniform angle: Pick an angle uniformly between $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and draw a chord from P with that angle. It is easy to see that this chord has the required length with probability $\frac{1}{3}$.

Solution 3

Uniform point: Pick another point Q at random from the interior of the unit circle. Draw the circle with OQ as radius and take the tangent to that from P.

Bertrand's Paradox cont.

Solution 2

Uniform angle: Pick an angle uniformly between $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and draw a chord from P with that angle. It is easy to see that this chord has the required length with probability $\frac{1}{3}$.

Solution 3

Uniform point: Pick another point Q at random from the interior of the unit circle. Draw the circle with OQ as radius and take the tangent to that from P. It is easy to see that this chord has the required length with probability $\frac{1}{4}$.

The End