The purpose of the scoring guide is to provide a uniform evaluation of the tests by the graders. Therefore, the guide gives you the main ideas (and at least one possible solution) for each problem and the associated scores. This guide is not intended to be a detailed description of a complete solution to each problem; the steps described are sketches of a solution that achieves maximum score.
The partial scores given in the guide are only awarded to the solver if the related idea is included in the test as a step in a clearly described and reasoned solution. Thus, for example, the mere description of statements or definitions in the material without its application (even if any of the facts described do indeed play a role in the solution) is not worth the point.
A partial score is given for any idea or part solution from which the correct solution of the problem could be obtained with the appropriate addition of the reasoning described in the test. If a solver starts several solutions that are significantly different, then you can give score for at most one. If each described solution or part of solution is correct or can be extended to be correct, then the solution with the most points will be evaluated. However, if there are correct and incorrect solutions between several solution attempts and it is not clear from the test which one the solver considered to be correct, then the solution with fewer points will be evaluated (even if this score is 0). The scores in this guide may be further subdivided as necessary. Of course, a good solution other than that described in this guide is also worth the maximum score.
In the event of an arithmetic error, one point is deducted from each exercise. An exception to this is when computing the numbers significantly simplifies or modifies the structure of the task. In these cases, no points are awarded to the steps which follow the computation and are missing from the solution of the student.

1. Bob has two unfair coins. The first lands on heads with probability $\frac{1}{3}$, while the second has tails on both sides. He randomly picks one coin and tosses it four times. Given that the number of tails he observed was even, what is the probability that he selected the first coin?
   (2 points) $A_1 = \{\text{the first coin is picked }\}$, $A_2 = \{\text{the second coin is picked}\}$,
   $B = \{\text{number of tails is even}\}$ (If the events are not clearly written as here, but they are used accordingly in the next steps, then also points are awarded.)
   (1 points) $\mathbb{P}(A_1|B) = ?$
   (4 points) $\mathbb{P}(B|A_1) = \left(\frac{1}{3}\right)^4 + \binom{4}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^4 = \frac{41}{81} \approx 0.5062$
   (2 points) $\mathbb{P}(A_1) = \mathbb{P}(A_2) = \frac{1}{2}$ and $\mathbb{P}(B|A_2) = 1$
   (1 points) Since A_1 and A_2 for a partition of the sample space,
   (1 points) so using the law of Total Probability, (if Bayes theorem is directly applied, then also points are awarded.)
   (4 points) $\mathbb{P}(B) = \mathbb{P}(B|A_1)\,\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\,\mathbb{P}(A_2) = \frac{61}{81} \approx 0.7531$
   (2 points) So, using the Bayes theorem
   (2 points) $\mathbb{P}(A_1|B) = \frac{\mathbb{P}(B|A_1)\,\mathbb{P}(A_1)}{\mathbb{P}(B)}$
   (1 points) $\mathbb{P}(A_1|B) = \frac{41}{122} \approx \underline{\underline{0.3361}}$

2. A point (U, V) is picked uniformly at random from the square $[0, 1] \times [0, 1]$. Let $X = \sqrt{U}$ and $Y = \sqrt{V}$. Determine
   (a) the distribution of X, (b) the density function of $X + Y$.
   (1 points) $\mathrm{Ran}(X) = \mathrm{Ran}(Y) = [0, 1]$
   (3 points) If $0 < x < 1$, then $F_X(x) = \mathbb{P}(X < x) = \mathbb{P}(\sqrt{U} < x) = \mathbb{P}(U < x^2) = x^2$,
   (0 points) and $F_X(x) = 0$, if $x \leq 0$ and $F_X(x) = 1$ if $x \geq 1$. (If the distribution function is determined but completely but $\mathrm{Ran}(X) = [0, 1]$ is not mentioned, still the first point is awarded.)
   (3 points) Hence, $f_X(x) = 2x$, if $0 < x < 1$, and 0 otherwise. Y has the same distribution by similar reasoning. (If Y is not mentioned but is used with same distribution in the following steps, then also points are awarded.)
   (1 points) $\mathrm{Ran}(X + Y) = [0, 2]$ (If later its mentioned that $f_{X+Y}(z) = 0$, if $z \leq 0$ or $z > 2$, then also points are awarded)
   (3 points) $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\mathrm{d}x$ (No points for mistakes in writing the formula.)
   (2 points) $= \int_{\max(0,z-1)}^{\min(1,z)} 2x \cdot 2(z - x)\mathrm{d}x =$ (The two cases, when $z < 1$ and $z > 1$, if written as seperate integrals, then also the points are awarded.)
   (1 points) $\left[2zx^2 - \frac{4}{3}x^3\right]_{\max(0,z-1)}^{\min(1,z)}$
   (2 points) If $0 < z < 1$, then, $\left[2zx^2 - \frac{4}{3}x^3\right]_0^z = \frac{2}{3}z^3$
   (2 points) If $1 < z < 2$, then $\left[2zx^2 - \frac{4}{3}x^3\right]_{z-1}^1 = -\frac{2}{3}(z^3 - 6z + 4)$

(2 points) So,

$$f_{X+Y}(z) = \begin{cases} \frac{2}{3}z^3 & \text{if } 0 < z < 1, \\ -\frac{2}{3}(z^3 - 6z + 4) & \text{if } 1 < z < 2, \\ 0 & \text{otherwise.} \end{cases}$$

3. A certain institution ships in n bags of an item 'T' every week, the number of bags ordered being the same every week. The quantity of T in each bag is independently and identically distributed, with mean 10 kg and standard deviation 2 kg. In any given week, if the amount of T supplied is more than the amount ordered by at least 20 kg (i.e. supply is more than $10n + 20$ kg), then they keep the amount ordered and 20 kg of extra T, but throw the remaining surplus in the harbor. (If the extra T is less than 20 kg, then they don't throw anything). Further, it was observed that the probability of some T being thrown in the harbor was 0.0336. What is the number of bags of T ordered every week?
(2 points) Let $X_1, \dots, X_n$ be the weight of the n bags.
(1 points) $\mathbb{E}(X_i) = 10$, $\sigma_{X_i} = 2$
(2 points) $\mathbb{P}(\sum_{i=1}^n X_i > 10n + 20) = 0.0336$
(3 points) $\mathbb{P}(\sum_{i=1}^n X_i > 10n + 20) = \mathbb{P}\left(\frac{\sum_{i=1}^n X_i - 10n}{2\sqrt{n}} > \frac{20}{2\sqrt{n}}\right)$
(2 points) And because of the Central Limit Theorem,
(3 points) $\frac{\sum_{i=1}^n X_i - 10n}{2\sqrt{n}} \sim N(0,1)$.
(3 points) So we get, $0.0336 \approx 1 - \Phi\left(\frac{20}{2\sqrt{n}}\right)$
(1 points) $\left(\Phi^{-1}(0.9664)\right) \approx 1.83$,
(1 points) so, $\sqrt{n} \approx \frac{10}{1.83} \approx 5.46$
(2 points) $n \approx \underline{\underline{30}}$.

4. In the imaginary land of Surrealia, the amount of delay in the arrival of any train has the memoryless property. Delays are measured in the units of quarter of an hour. The average delay is known to be 3 quarter hours. Each time a train is delayed, the train company receives letters of complaint. For a fixed delay of x quarter hours, assume that the letters of complaint are all independent, identical, but small probability events. If a train is x quarter hours late, then, the probability that the company does not receive any complaint letters is e^{-100x}.
(a) What is the conditional expected number of letters of complaint for a fixed x quarter hours of delay?
(b) What is the expected number of letters received due to the delay of a given single train?
(2 points) X: delay of a given train, Y: number of letters of complaint, $\mathbb{E}(Y) =?$
(Optional definition: Z is the number of letters of complaint conditioned on $X = x$)
(2 points) For a fixed x, Y has Poisson distribution (or $Z \sim \text{Pois}(\mu)$)
(3 points) $\mathbb{P}(Y = 0 \mid X = x) = e^{-100x}$ (or $\mathbb{P}(Z = 0) = e^{-100x}$)
(3 points) $\mathbb{E}(Y \mid X = x) = 100x$, because for fixed delay x, $Y \sim \text{Pois}(100x)$
(or $\mathbb{E}(Z) = 100x$, because $Z \sim \text{Pois}(100x)$)
(1 points) $X \sim \text{Exp}(\lambda)$
(1 points) $\mathbb{E}(X) = 3 \Rightarrow \lambda = \frac{1}{3}$
(1 points) Using the law of Total Expectation,
(3 points) $\mathbb{E}(Y) = \int_{-\infty}^{\infty} \mathbb{E}(Y \mid X = x) f_X(x) \mathrm{d}x$ (or $\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y \mid X))$)
(2 points) $= \int_0^{\infty} 100x \cdot \lambda e^{-\lambda x} \mathrm{d}x$ (or $\mathbb{E}(Y \mid X) = 100X$ hence, $= \mathbb{E}(100X) = 100\,\mathbb{E}(X)$)
(2 points) $= 100\lambda\left(\left[x\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-\lambda x}}{-\lambda}\mathrm{d}x\right) = 100\lambda\left[\frac{e^{-\lambda x}}{-\lambda^2}\right]_0^{\infty} = \frac{100}{\lambda} = \underline{\underline{300}}$
If the random variable Y is used for both a fixed $X = x$, and also without, then 3 points must be deducted.

5. Let $(X, Y) \sim N(\mathbf{0}, \mathbf{\Sigma})$ be a vector random variable with bivariate (2D) normal distribution. Assume that $Var(X) = Var(Y)$, that the correlation of X and Y is 0.5, and that $\det(\mathbf{\Sigma}) = 12$.
(a) What is the probability that X is greater than 3.6?
(b) What is the density function of (X, Y)?
(2 points) $\mathbf{\Sigma} = \begin{bmatrix} Var(X) & \text{Cov}(X,Y) \\ \text{Cov}(Y,X) & Var(Y) \end{bmatrix}$
(2 points) $\det(\mathbf{\Sigma}) = Var(X)Var(Y) - \text{Cov}(X,Y)^2$
(2 points) $\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$, so,
(1 points) $\det(\mathbf{\Sigma}) = Var(X)Var(Y) \cdot \left(1 - \rho(X,Y)^2\right) = \sigma_X^4(1 - 0.5^2)$
(if particular steps vary but the relationship between Covariance matrix and standard deviation is made, then

also points are awarded.)
(1 points) $\Rightarrow \sigma_X = 2$
(1 points) $\text{Cov}(X,Y) = \rho(X,Y) \cdot \sigma_X \sigma_Y = 0.5 \cdot 2 \cdot 2 = 2$
(1 points) So, $\mathbf{\Sigma} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$
(1 points) $\mathbb{P}(X > 3.6) = 1 - \mathbb{P}(X < 3.6)$
(1 points) Components of a bivaraite Normal distribution are normal, so,
(2 points) $= 1 - \Phi\left(\frac{3.6 - \mathbb{E}(X)}{\sigma_X}\right)$
(1 points) $\approx \underline{0.0361} = 3.61\%$
(2 points) $\mathbf{\Sigma}^{-1} = \frac{1}{12}\begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix}$
(3 points) $f_{(X,Y)}(x,y) = \frac{1}{(2\pi)\sqrt{12}} e^{-\frac{1}{2}(x,y)^T \mathbf{\Sigma}^{-1}(x,y)}$, where the exponent $-\frac{1}{2}\frac{1}{12}(4x^2 - 4xy + 4y^2) = -\frac{1}{6}(x^2 - xy + y^2)$

6.* Let (U,V) be a random variable vector. Assume that $Var(U)$, $Var(V)$ and $\text{Cov}(U,V)$ are positive real numbers. Let the linear regression of V in terms of U be $\beta_1 U + \alpha_1$, and the linear regression of U in terms of $-V$ be $\beta_2(-V) + \alpha_2$. Further assume that the line, $e = \{(x,y) \in \mathbb{R}^2 \mid y = \beta_1 x + \alpha_1\}$, is perpendicular to the line, $f = \{(x,y) \in \mathbb{R}^2 \mid y = \beta_2 x + \alpha_2\}$. If U has exponential distribution, then what is the distribution of V?
If we also assume that $\alpha_1 = \alpha_2$, then what can you say about V?
(3 points) $\beta_1 = \frac{\text{Cov}(U,V)}{Var(U)}$, and $\beta_2 = \frac{\text{Cov}(U,-V)}{Var(-V)}$
(2 points) The lines are perpendicular if $\beta_1 \cdot \beta_2 = -1$
(2 points) $\beta_1 \cdot \beta_2 = -1$ is true only if $-\frac{\text{Cov}(U,V)^2}{Var(U)Var(V)} = -1$.
(2 points) $\Rightarrow \rho(U,V) = \pm 1$
(1 points) $\text{Cov}(U,V) > 0$, so, $\rho(U,V) > 0$, so $= +1$.
(3 points) $\rho(U,V) = 1 \Rightarrow V = bU + a$ for some $b, a \in \mathbb{R}$ and where $b > 0$. (Correlation ± 1 implies linear relationship between the random variables).
(2 points) If U has exponential distribution, then V is a linear transformation of the exponential distribution.
(2 points) A multiple of a random variable with exponential distribution is also exponential.
(1 points) $\alpha_1 = \mathbb{E}(V) - \frac{\text{Cov(U,V)}}{Var(U)}\mathbb{E}(U) = b\,\mathbb{E}(U) + a - b\,\mathbb{E}(U) = a$
(1 points) $\alpha_2 = \mathbb{E}(U) - \frac{\text{Cov(U,−V)}}{Var(-V)}\mathbb{E}(-V) = \mathbb{E}(U) + \frac{1}{b}(-b\,\mathbb{E}(U) - a) = -\frac{a}{b}$
(1 points) If $\alpha_1 = \alpha_2$, then $b \neq -1$ so, $a = 0$, so $V \sim \text{Exp}\left(\frac{\lambda}{b}\right)$, if $U \sim \text{Exp}(\lambda)$.

Name	Range	$P(X=i)$ or $F_X(x)$	f_X	$E(X)$	σ_X
Indicator $1(p)$	{0,1}	$P(X=1)=p$		p	$\sqrt{pq}$
Binomial $Bin(n,p)$	{0,1,...,n}	$\binom{n}{i}p^i(1-p)^{n-i}$		np	$\sqrt{np(1-p)}$
Poisson $Pois(\lambda)$	{0,1,...}	$\frac{\lambda^i}{i!}e^{-\lambda}$		λ	$\sqrt{\lambda}$
Geometric $Geo(p)$	{1,2,...}	$=(1-p)^{i-1}p$		$\frac{1}{p}$	$\frac{\sqrt{1-p}}{p}$
Uniform $U(a,b)$	(a,b)	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{b-a}{2\sqrt{3}}$
Exponential $Exp(\lambda)$	$\mathbf{R}^+$	$1-e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$
Normal $N(\mu,\sigma^2)$	$\mathbf{R}$	$\Phi(\frac{x-\mu}{\sigma})$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ
Multivariate Normal $N(\boldsymbol{\mu},\mathbf{\Sigma})$	$\mathbf{R}^n$	$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}det(\mathbf{\Sigma})^{\frac{1}{2}}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\mathbf{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$		$\boldsymbol{\mu}$	Covariance matrix $\mathbf{\Sigma}$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998