1. Ben started a weight loss program where he goes running once a week. To keep things interesting, he picks a random workday of the week to run, such that the probability of a day being picked is proportional to the number of vowels in the name of that day. Also, the number of kilometers he then runs on that particular day is equal to the number of consonants in the name of that day. (He doesn't run on the other workdays or weekends.) What is the expected value and variance of the amount he runs in a week?
Please use the following table:

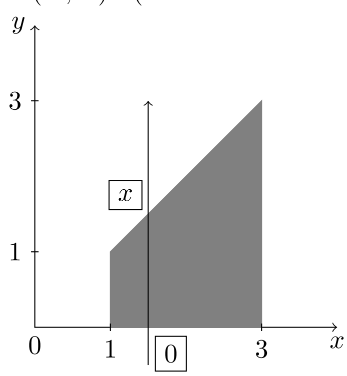
Day	Monday	Tuesday	Wednesday	Thursday	Friday	$\sum$
Consonants	4	4	6	6	4	24
Vowels	2	3	3	2	2	12

Solution: (2 points) X: the number of kilometers Ben ran in a week
(2 points) Probability of the different days of being picked: $\frac{2}{12}, \frac{3}{12}, \frac{3}{12}, \frac{2}{12}, \frac{2}{12}$.
(1 points) $\mathbb{P}(X=4) = \frac{2}{12} + \frac{3}{12} + \frac{2}{12} = \frac{7}{12}$
(1 points) $\mathbb{P}(X=6) = \frac{3}{12} + \frac{2}{12} = \frac{5}{12}$
(1 points) $\mathrm{Ran}(X) = \{4, 6\}$ (If this is implicitly used in the sum, then also points are awarded)
(2 points) $\mathbb{E}(X) = \sum_{k \in \mathrm{Ran}(X)} k \cdot \mathbb{P}(X=k)$ (If the sum runs over values 4 and 6, then also 3 points.)
(3 points) $= 4 \cdot \frac{7}{12} + 6 \cdot \frac{5}{12}$
(1 points) $= \underline{\underline{4.833}}$
(2 points) $Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$,
(2 points) $\mathbb{E}(X^2) = \sum_{k \in \mathrm{Ran}(X)} k^2 \cdot \mathbb{P}(X=k)$
(2 points) $= 4^2 \cdot \frac{7}{12} + 6^2 \cdot \frac{5}{12} = 24.333$
(1 points) $Var(X) = 24.333 - 4.833^2 = \underline{\underline{0.9751}}$

2. Let (X, Y) be a random variable vector with the following density function for some $\alpha \in \mathbb{R}$:

$$f_{X,Y} : (x, y) \mapsto \begin{cases} \frac{\alpha}{2x} & \text{in the region bounded by } x = 1, x = 3, y = 0, y = x, \\ 0 & \text{otherwise.} \end{cases}$$

What is $\mathrm{Cov}(X, Y)$? (Please draw a clear picture of the domain with limits of the integrals.)



Solution:
(5 points) Drawing axes, and bounding lines (2 points), shading the domain (1 point), line of integration (1 point), writing limits of integration (1 point).
(2 points) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) \mathrm{d}x\mathrm{d}y = 1$
(1 points) $= \int_1^3 \int_0^x \frac{\alpha}{2x} \mathrm{d}y\mathrm{d}x$
(2 points) $= \frac{\alpha}{2} \int_1^3 \int_0^x x^{-1} \mathrm{d}y\mathrm{d}x = \frac{\alpha}{2} \int_1^3 \left[yx^{-1}\right]_0^x \mathrm{d}x = \frac{\alpha}{2} \int_1^3 1\mathrm{d}x = \alpha$
(1 points) so, $\alpha = 1$
(2 points) $\mathbb{E}(XY) = \int_1^3 \int_0^x \frac{1}{2} xyx^{-1} \mathrm{d}y\mathrm{d}x = \int_1^3 \left[\frac{y^2}{4}\right]_0^x \mathrm{d}x = \int_1^3 \frac{x^2}{4} \mathrm{d}x = \left[\frac{x^3}{12}\right]_1^3 = \frac{27}{12} - \frac{1}{12} = \frac{13}{6}$
(2 points) $\mathbb{E}(X) = \int_1^3 \int_0^x \frac{1}{2} xx^{-1} \mathrm{d}y\mathrm{d}x = \int_1^3 \left[\frac{1}{2}\right]_0^x \mathrm{d}x = \int_1^3 \frac{x}{2} \mathrm{d}x = \left[\frac{x^2}{4}\right]_1^3 = \frac{9}{4} - \frac{1}{4} = 2$
(at most one point for incorrect marginal function or its expected value)
(2 points) $\mathbb{E}(Y) = \int_1^3 \int_0^x \frac{1}{2} yx^{-1} \mathrm{d}y\mathrm{d}x = \int_1^3 \left[\frac{y^2}{4x}\right]_0^x \mathrm{d}x = \int_1^3 \frac{x}{4} \mathrm{d}x = \left[\frac{x^2}{8}\right]_1^3 = \frac{9}{8} - \frac{1}{8} = 1$
(at most one point for incorrect marginal function or its expected value)
(2 points) $\mathrm{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$
(1 points) $= \frac{13}{6} - 2 \cdot 1 = \frac{1}{6} \approx \underline{\underline{0.1667}}$

3. A machine needs to execute 100 commands consecutively, where the execution time of the commands are independent, identically distributed random variables. It is further known that their expected value is twice their standard deviation. What is the probability that the total execution time of all the 100 commands will be more than 1.1 times the total expected execution time?
Solution: (1 points) Let the execution times be: $X_1, X_2, \ldots, X_{100}$.
(1 points) If $m = \mathbb{E}(X_1)$ and $s = \sigma_{X_1}$ then $m = 2 \cdot s$
(2 points) $\mathbb{E}(\sum_{i=1}^{100} X_i) = \sum_{i=1}^{100} \mathbb{E}(X_i) = 100m$
(2 points) $\mathbb{P}(\sum_{i=1}^{100} X_i > 1.1 \cdot 100m) =?$ (If in the question $\mathbb{E}(\sum_{i=1}^{100} X_i)$ appears, then also points are awarded.)
(1+1 points) $= \mathbb{P}(\sum_{i=1}^{100} X_i - 100m > 10m) = \mathbb{P}\left(\frac{\sum_{i=1}^{100} X_i - 100m}{s\sqrt{100}} > \frac{10m}{10s}\right)$
(1+1 points) $= \mathbb{P}\left(\frac{\sum_{i=1}^{100} X_i - 100m}{10s} > \frac{20s}{10s}\right) = \mathbb{P}\left(\frac{\sum_{i=1}^{100} X_i - 100m}{10s} > 2\right)$
(1 points) $= 1 - \mathbb{P}\left(\frac{\sum_{i=1}^{100} X_i - 100m}{10s} \leq 2\right)$
(2 points) By the Central Limit Theorem,
(3 points) $\frac{\sum_{i=1}^{100} X_i - 100m}{10s} \sim N(0, 1)$.
(2 points) So the required probability is $\approx 1 - \Phi(2)$
(2 points) $\approx 1 - 0.9772 = \underline{\underline{0.0228}}$

4. Let $f_{X,Y}(x, y) = e^{-y}$ if $y \geq x \geq 0$ and 0 otherwise. Determine the $\mathbb{E}(Y|X)$ regression function. (Note: The answer is not numeric!)
Solution: (2 points) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$
(2 points) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\mathrm{d}y$ (If the two formulas are written in one, then also points are awarded. If the limits are x and ∞, then also points are awarded.)
(2 points) $f_X(x) = \int_x^{\infty} e^{-y}\mathrm{d}y = [-e^{-y}]_x^{\infty} = 0 - (-e^{-x}) = e^{-x}$
(1 points) if $x > 0$ and 0 otherwise.
(2 points) $f_{Y|X}(y|x) = \frac{e^{-y}}{e^{-x}} = e^{x-y}$
(2 points) if $y \geq x > 0$ and 0 otherwise.
(3 points) $\mathbb{E}(Y \mid X = x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\mathrm{d}y$
(2 points) $= \int_x^{\infty} y \cdot e^{x-y}\mathrm{d}y$
(2 points)$= e^x\Big([y \cdot (-e^{-y})]_x^{\infty} - \int_x^{\infty}(-e^{-y})\mathrm{d}y\Big) = e^x\left(0 - (-xe^{-x})\right) - e^x\left[e^{-y}\right]_x^{\infty} = x + 1$
(2 points) and so, $\mathbb{E}(Y \mid X) = \underline{\underline{X + 1}}$

5. Let X be a number picked uniformly at random from the interval $\left[0, \frac{1}{3}\right]$. We then construct an unfair dice, such that, every even number has a probability X of being rolled, while every odd number has a probability of $\frac{1}{3} - X$ of being rolled. Please write, as a table, the probability mass function for one dice roll, given $X = x$. Given that we roll this dice twice, compute the following probabilities:
(a) The probability of rolling the same number twice for a fixed $\{X = x\}, 0 < x < \frac{1}{3}$.
(b) The probability of rolling the same number twice, when we don't know the value of X.
Solution: (2 points) The probability mass function for one dice roll given $X = x$ is as follows:

Roll	1	2	3	4	5	6
Probability	$1 - x$	x	$1 - x$	x	$1 - x$	x

(1 points) Y_1: first roll, Y_2: second roll, assuming $\{X = x\}$. (If this is not written but used in the following steps consistently, then also points are awarded.)
(1 points) $\mathbb{P}(Y_1 = Y_2) = \sum_{k=1}^{6} \mathbb{P}(Y_1 = k, Y_2 = k)$
(1 points) $= \sum_{k=1}^{6} \mathbb{P}(Y_1 = k)\,\mathbb{P}(Y_2 = k)$, since they are independent.
(2 points) $= \left(\frac{1}{3} - x\right)^2 + x^2 + \left(\frac{1}{3} - x\right)^2 + x^2 + \left(\frac{1}{3} - x\right)^2 + x^2$.
(1 points) So, $\mathbb{P}(Y_1 = Y_2) = \underline{\underline{3x^2 + 3\left(\frac{1}{3} - x\right)^2}} = 6x^2 - 2x + \frac{1}{3}$
If conditional probability terms are used, then also points are awarded. If incorrectly used, then the points for definition of Y_1, Y_2 cannot be awarded.
(1 points) Let A denote the event that the two rolls are the same. (If it is not explicitly defined but used correctly in the next few steps, then also points are awarded.) We want: $\mathbb{P}(A) =?$
(1 points) $\mathbb{P}(A|X = x) = 3x^2 + 3\left(\frac{1}{3} - x\right)^2$
(1 points) Using the Law of Total Probability,

(3 points) $\mathbb{P}(A) = \int_{-\infty}^{\infty} \mathbb{P}(A|X=x) f_X(x)\mathrm{d}x$
(2 points) $f_X(x) = 3$ if $0 < x < \frac{1}{3}$ and 0 otherwise.
(2 points) $= \int_0^{\frac{1}{3}} \left(3x^2 + 3\left(\frac{1}{3} - x\right)^2\right) \cdot 3\,\mathrm{d}x$
(1 points) $= \left[x^3 - \left(\frac{1}{3} - x\right)^3\right]_0^{\frac{1}{3}} \cdot 3 = \left(\left(\frac{1}{27} - 0\right) - \left(0 - \frac{1}{27}\right)\right) \cdot 3$
(1 points) $= \frac{2}{9} \approx \underline{\underline{0.2222}}$

6. Let (X, Y) be a random variable vector with the following density function:

$$f_{X,Y}(x,y) = \frac{1}{8\pi} e^{-\frac{1}{8}\left((x-3)^2 + (y-3)^2\right)}$$

Determine $Var(XY)$.
Solution: (2 points) (X,Y) is bivariate (2D) normal: $(X,Y) \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
(1 points) where $\boldsymbol{\mu} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ and
(2 points) $\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$
(2 points) Then, $\det(\boldsymbol{\Sigma}) = 4 \cdot 4$ and $\boldsymbol{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$
(3 points) X and Y are independent, since $\text{cov}(X,Y) = 0$
OR:
(4 points) $f_{X,Y}(x,y) = \frac{1}{8\pi} e^{-\frac{1}{8}\left((x-3)^2+(y-3)^2\right)} = \frac{1}{\sqrt{2\pi}\sqrt{4}} e^{-\frac{1}{8}(x-3)^2} \cdot \frac{1}{\sqrt{2\pi}\sqrt{4}} e^{-\frac{1}{8}(y-3)^2}, \forall x, y \in \mathbb{R}$,
(3 points) and so, $X, Y \sim N(3,4)$,
(3 points) and so X and Y are independent because $f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y), \forall x, y \in \mathbb{R}$
(2 points) $Var(XY) = \mathbb{E}(X^2Y^2) - E(XY)^2$
(1 points) Because of independence:
(2 points) $= \mathbb{E}(X^2)\,\mathbb{E}(Y^2) - \mathbb{E}(X)^2\,\mathbb{E}(Y)^2$
(2 points) $\mathbb{E}(X^2) = Var(X) + \mathbb{E}(X)^2$
(1 points) $= 4 + 3^2 = 13$,
(1 points) Similarly, $\mathbb{E}(Y^2) = 13$
(1 points) So, $Var(XY) = 13 \cdot 13 - 3^2 \cdot 3^2 = \underline{\underline{88}}$

Name	Range	$P(X=i)$ or $F_X(x)$	f_X	$E(X)$	σ_X
Indicator $1(p)$	{0,1}	$P(X=1)=p$		p	$\sqrt{pq}$
Binomial $Bin(n,p)$	{0,1,...,n}	$\binom{n}{i}p^i(1-p)^{n-i}$		np	$\sqrt{np(1-p)}$
Poisson $Pois(\lambda)$	{0,1,...}	$\frac{\lambda^i}{i!}e^{-\lambda}$		λ	$\sqrt{\lambda}$
Geometric $Geo(p)$	{1,2,...}	$= (1-p)^{i-1}p$		$\frac{1}{p}$	$\frac{\sqrt{1-p}}{p}$
Uniform $U(a,b)$	(a,b)	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{b-a}{2\sqrt{3}}$
Exponential $Exp(\lambda)$	$\mathbf{R}^+$	$1-e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$
Normal $N(\mu,\sigma^2)$	$\mathbf{R}$	$\Phi(\frac{x-\mu}{\sigma})$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ
Multivariate Normal $N(\boldsymbol{\mu},\boldsymbol{\Sigma})$	$\mathbf{R}^n$	$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}} det(\boldsymbol{\Sigma})^{\frac{1}{2}}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$		$\boldsymbol{\mu}$	Covariance matrix $\boldsymbol{\Sigma}$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998