Probability Theory Lecture 02

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Overview

1. Conditional Probability
2. Independence
   - Total Independence
3. Law of Total Probability and Bayes Theorem

Conditional Probability

Definition

Given events A and B the conditional probability $P(A|B)$ (said A **given** B) is the probability of event A, **given** that the event B **has occured** (by assumption or evidence).

We can think of B as the new **updated** sample space, and we want to know how the probability of A changes in this new space.

Conditional Probability

Examples

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How about probability of rolling an even number given that we rolled 2?

Conditional Probability

Formula for conditional probability

For two events A, B and a probability measure P, if $P(B) > 0$, then the condition probability of A given B is defined as follows:

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Using the above formula, $P(A|B) = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$, as before.

Independence

Independence: definition

An event A is said to be independent of another event B, if conditioning on B **does not** change its probability. Intuitively, this is when $P(A|B) = P(A)$. This definition is problematic when $P(B) = 0$ and the conditional probabilities are not defined. So, we instead say two events are independent if

$$P(A \cap B) = P(A)P(B)$$

Independence

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Example 1: Is drawing an Ace in a standard deck of cards independent of the event of drawing a Spade?

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Example 2: Is drawing two Aces in a standard deck of cards independent of the event of drawing two Spades? (drawn without replacement).

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Example 2: Is drawing two Aces in a standard deck of cards independent of the event of drawing two Spades? (drawn without replacement).

Example 3: Let us roll two fair dice and let A be the event that the first dice rolled a 1, let B be the event that the sum of the two dice is 7. Are they independent?

Independence

Commutativity

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So we can say that events A and B are independent.

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Example 2: In a class, 50% of the students are computer science majors and the rest are electrical engineering majors. We also know that 30% of students are girls. If we know that the major of a student and their gender are independent, what is the probability that a random student is a girl with computer science major?

Multiplication Rule

Multiplication Rule: two events

If A_1, A_2 are two events that are **not** necessarily independent, and $P(A_2|A_1)$ is defined, then

$$P(A_1 \cap A_2) = P(A_2|A_1) \cdot P(A_1)$$

Note: it is good to think of A_1, A_2 as chronologically occuring in that order.

Multiplication Rule

Multiplication Rule: many events

If $A_1, A_2, \ldots, A_n$ are n events that are **not** necessarily independent and if $\forall 1 < i \leq n, P(A_i | A_1 \cap \ldots \cap A_{i-1})$ is defined, then

$$P(\cap_{i=1}^{n} A_i) =$$

$$P(A_n|A_1 \cap A_2 \ldots \cap A_{n-1}).P(A_{n-1}|A_1 \cap A_2 \ldots \cap A_{n-2}) \ldots P(A_2|A_1).P(A_1)$$

Note: it is good to think of $A_1, A_2, \ldots A_n$ as chronologically occuring in that order.

Multiplication Rule

Examples of multiplicaiton Rule

Example 0: What is the probability that two cards drawn without replacement are both Aces?
Let A_1 be the event that the first card is an Ace and A_2 that the second card is an Ace.

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Example 1: Birthday Paradox
Let A_i be the event that the i^{th} person's birthday is not the same as people $1, 2, ..., i-1$.
Example 2: (2019 exam problem): What is the probability that the four aces are with four different players in a bridge game?

Independence of many events

Pairwise Independence

Events $A_1, ..., A_n$ are said to be pairwise independent if $\forall i \neq j$, A_i, A_j are independent, or $P(A_i \cap A_j) = P(A_i)P(A_j)$.

Total Independence

Events $A_1, ..., A_n$ are said to be totally independent if every subset of events are independent.

$$P(A_{j_1} \cap A_{j_2} .. \cap A_{j_k}) = P(A_{j_1})P(A_{j_2})...P(A_{j_k})$$

where all $j_1, j_2, .., j_k$ are all distinct indices.

Are these two equivalent?

Example

Two fair coins are tossed. Let A_i be the event that the i^{th} coin resulted in Heads. Further let B be the event that there are an even number of heads in the two tosses.

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Are A_1, A_2 independent? A_1, B? A_2, B?
Are A_1, A_2, B totally independent?

Law of Total Probability

Partition

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Given a parition $A_1, ...A_n$ of the sample space Ω such that $P(A_i) > 0, \forall i$, and another event B,

$$P(B) = \sum_i P(B|A_i)P(A_i)$$

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Always draw a tree!!!

Law of Total Probability

Example

Consider a certain factory with three machines manufacturing the same product. The machines A_1, A_2, A_3 manufacture $0.1, 0.1, 0.8$ of the total items produced. If the machines have a probability of $4\%, 4\%$ and 1% of producing a defective item, what is the probability that a randomly chosen product will be defective?

Bayes Theorem

Bayes Theorem: informally

Suppose we are given an urn with 10 dice. Our adversary randomly picks a few of them and rolls them and counts the sum. We are only told what this sum is. Can we make an intelligent guess about how many dice he picked out of the urn?

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Given a partition of the sample space $A_1, .., A_n$ and a property (event) B. If we get to know that B did happen, can we make a better guess at the probabilities of the partition?

Bayes Theorem

Bayes Theorem

Given a parition $A_1, ...A_n$ of the sample space Ω such that $P(A_i) > 0, \forall i$, and another event B such that $P(B) > 0$, then,

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_i P(B|A_i)P(A_i)}$$

Bayes Theorem: Examples

Example 1

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Bayes Theorem: Examples

Example 2

A certain rare disease occurs approximately in 0.1% of the population of a certain country. A screening test for this disease is said to be 90% specific and 90% sensitive. If a certain individual's test came positive, what is the probability they have the disease? In other words, is it a good reliable test or a bad one?

Summary

- Conditional Probability: $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
- Independence of two events: $P(A \cap B) = P(A)P(B)$.
- Pairwise Independence: $\forall i \neq j$, $P(A_i \cap A_j) = P(A_i)P(A_j)$.
- Total Independence:
  $P(A_{j_1} \cap A_{j_2} .. \cap A_{j_k}) = P(A_{j_1})P(A_{j_2})...P(A_{j_k})$ for all subsets.
- Law of Total Probability: $P(B) = \sum_i P(B|A_i)P(A_i)$.
- Bayes Theorem: $P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_i P(B|A_i)P(A_i)}$.

The End