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Overview

- Distribution of two RVs
- 2 Independence
- 3 Covariance
- 4 Linear Algebra

Definitions

Distribution of two RVs

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Joint Probability Mass Function

Given two discrete random variables X and Y, the **joint probability mass function** is a function, $p_{X,Y}: \mathbb{R}^2 \to [0,1]$, such that.

$$\sum_{x}\sum_{y}p_{X,Y}(x,y)=1$$

Here we think of $p_{X,Y}(x,y)$ as the probability that X=x and Y = v.

Where its obvious, the subscript X, Y is dropped and it is written as p(x, y).

Definitions contd.

Distribution of two RVs

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Marginal Probability Mass Functions

The probability mass functions of X, $p_X(x)$ and of Y, $p_Y(y)$ are called the marginal probability mass functions.

Notice that we can obtain the marginal pmfs from the joint pmf as follows:

$$p_X(x) = \sum_{y} p_{X,Y}(x,y)$$

$$p_Y(y) = \sum_{x} p_{X,Y}(x,y)$$

Examples

Suppose that a letter is randomly chosen from the set $\{e, x, a, m\}$. We define three random variables as follows:

$$X(letter) = \begin{cases} 1 & \text{letter is e or m} \\ 0 & \text{otherwise} \end{cases}$$

$$Y(letter) = \begin{cases} 1 & \text{letter is e} \\ 2 & \text{letter is x} \\ 3 & \text{letter is a} \\ 4 & \text{letter is m} \end{cases}$$

$$Z(letter) = \begin{cases} 1 & \text{letter is x or m} \end{cases}$$

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be three random variables.

Examples

		Value of	$Y \mid p_Y(y)$
Value of $X \mid p_X$	$\chi(x)$	1	$\frac{1}{4}$
0	$\frac{1}{2}$	2	$\frac{1}{4}$
1	$\frac{1}{2}$	3	$\frac{1}{4}$
		4	$\left \frac{1}{4} \right $
Value of $Z \mid p_Z(z) \mid$			
	0	$\frac{1}{2}$	
	-1	1	

Table: Marginal pmfs of X, Y and Z

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Examples

X	1	2	3	4	$p_X(x)$
0	0	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{2}$
1	$\frac{1}{4}$	0	0	$\frac{1}{4}$	$\frac{1}{2}$
$p_Y(y)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1

$$\begin{array}{|c|c|c|c|c|c|} \hline X & Z & 0 & 1 & p_X(x) \\ \hline 0 & |\frac{1}{4}| & \frac{1}{4}| & \frac{1}{2} \\ \hline 1 & |\frac{1}{4}| & \frac{1}{4}| & \frac{1}{2} \\ \hline |p_Z(z)| & |\frac{1}{2}| & |\frac{1}{2}| & 1 \\ \hline \end{array}$$

Table: Joint pmfs of X, Y and X, Z

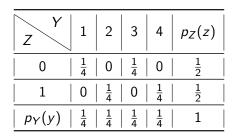


Table: Joint pmf of Z, Y

Expected Value

Distribution of two RVs 000000000

The expected value of any function g(X, Y) is given by

$$\sum_{x}\sum_{y}g(x,y)p_{X,Y}(x,y)$$

For example, if we wanted to compute E(Y) and E(XY), we must employ the following sums:

$$E(Y) = \sum_{x} \sum_{y} y p_{X,Y}(x,y)$$

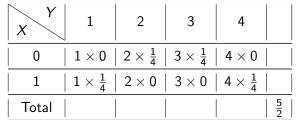
$$E(XY) = \sum_{x} \sum_{y} xyp_{X,Y}(x,y)$$

Expected Value

Distribution of two RVs

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For E(Y),



For E(XY),

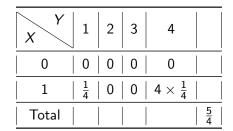


Table: Finding E(XY)

Independence

Definition

Two random variables X and Y are said to be independent if $p_{X,Y}(x,y) = p_X(x)p_Y(y), \ \forall x,y \in \mathcal{R}.$

In the case of discrete random variables, when two variables are independent, the joint pmf table's rows and columns are multiples of the marginal pmfs of the variables. This is because, if X and Y are independent, then for any constant c,

 $p_{X,Y}(x,c) = p_X(x)p_Y(c)$. So in the corresponding column for Y = c, the probabilities in the table for $p_{X,Y}$ would be a multiple of the column containing the marginal probability mass function p_X .



Independence - Example

Which of the variables X, Y, Z are independent?

Expected value of XY

Let X and Y be two independent random variables. Then,

$$E(XY) = E(X)E(Y)$$

Proof: Recall, $E(1_A) = P(A)$. We also note that, $1_A 1_B = 1_{A \cap B}$ and if A, B are independent,

 $E(1_A 1_B) = E(1_{A \cap B}) = P(A \cap B) = P(A)P(B) = E(1_A)E(1_B)$. So the above property holds for independent indicator random variables. Hence, the only thing left to do is to see that any discrete finite random variable can be written as a linear combination of indicator random variables. This can be done as follows:



Expected value of XY

We will use $1_{X=k}$ to denote the indicator random variable of the event X = k. Similarly, we have $1_{Y=1}$. We note that:

$$X = \sum_{k} k 1_{X=k}$$
 and similarly $Y = \sum_{l} / 1_{Y=l}$.

Exercise: Please finish the proof.



Examples

$$E(X) = \frac{1}{2}$$
, $E(Y) = \frac{5}{2}$, $E(Z) = \frac{1}{2}$.
Is $E(XY) = E(X)E(Y)$? Are they independent?

Independence

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Examples

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, $E(Y) = \frac{5}{2}$, $E(Z) = \frac{1}{2}$.
Is $E(XY) = E(X)E(Y)$? Are they independent?
IMPORTANT: if $E(XY) = E(X)E(Y)$ it does not imply that X, Y are independent. The above example is one such example.

Definition

Let X, Y be two random variables. Then, the covariance of X and Y, denoted by Cov(X, Y) is defined as $E((X - \mu_X)(Y - \mu_Y))$. Because of Linearity of expectation, this is $E(XY) - \mu_X \mu_Y$.

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What is the covariance of two independent variables?



Distribution of two RVs

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What is the covariance of two independent variables? If two random variables X, Y are independent, then $E(XY) = \mu_X \mu_Y$, so Cov(X, Y) = 0.

Is the opposite true? If X, Y are two random variables such that Cov(X, Y) = 0, does it imply that X, Y are independent?



Distribution of two RVs

Definition

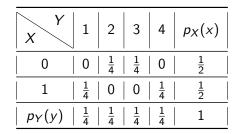
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What is the covariance of two independent variables? If two random variables X, Y are independent, then $E(XY) = \mu_X \mu_Y$, so Cov(X, Y) = 0.

Is the opposite true? If X, Y are two random variables such that Cov(X, Y) = 0, does it imply that X, Y are independent? No.

Covariance - Examples

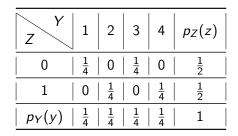
Covariance of X, Y?



$$E(X) = \frac{1}{2}$$
, $E(Y) = \frac{1}{4}(1+2+3+4) = \frac{5}{2}$
 $E(XY) = 1 \times \frac{1}{4} + 4 \times \frac{1}{4} = \frac{5}{4}$
So $Cov(X, Y) = 0$.

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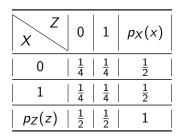
Covariance Examples cont.



$$E(Z) = \frac{1}{2}, E(Y) = \frac{1}{4}(1+2+3+4) = \frac{5}{2}$$

 $E(ZY) = 2 \times \frac{1}{4} + 4 \times \frac{1}{4} = \frac{6}{4}$
So $Cov(Z, Y) = \frac{1}{4}$.

Covariance Examples cont.



$$E(X) = \frac{1}{2}$$
, $E(Z) = \frac{1}{2}$
 $E(XZ) = 1 \times \frac{1}{4} = \frac{1}{4}$
So $Cov(X, Z) = 0$ (as it should be since they are independent).

• (Commutative)
$$Cov(X, Y) = Cov(Y, X)$$
.

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- Cov(aX + b, Y) = aCov(X, Y)
- Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)

Variance: Revisited

Distribution of two RVs

Recall that we defined the Variance of a random variable X as $Var(X) = E((X - \mu_X)^2)$. This we can see is $E((X - \mu_X)(X - \mu_X)) = Cov(X, X)$.

Standard deviation

We define the standard deviation of a random variable X as $\sqrt{Var(X)}$, and is denoted by σ_X (pronounced: Sigma X).



Examples

	Value of Y	$p_Y(y)$
Value of $X \mid p_X(x) \mid$	1	$\frac{1}{4}$
$0 \qquad \left \frac{1}{2} \right $	2	$\frac{1}{4}$
$\left \begin{array}{c c} 1 & \left \begin{array}{c c} \frac{1}{2} \end{array} \right \right $	3	$\frac{1}{4}$
	4	$\frac{1}{4}$
Value of Z	$p_Z(z)$	
0	$\frac{1}{2}$	

Table: Marginal pmfs of X, Y and Z

Examples

$$\begin{array}{l} \textit{Var}(X) = \textit{Var}(Z) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \\ \textit{Var}(Y) = \frac{1}{4}(1 + 4 + 9 + 16) - \frac{25}{4} = \frac{5}{4} \\ \textit{And so, } \sigma_X = \sigma_Z = \frac{1}{2}, \; \textit{while } \sigma_Y = \frac{\sqrt{5}}{2}. \end{array}$$

• $Var(X) \ge 0$. Also, if Var(X) = 0, then it is 'almost surely' a constant.

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- $Var(X) \ge 0$. Also, if Var(X) = 0, then it is 'almost surely' a constant.
- $Var(aX + b) = a^2 Var(X)$. Then, $\sigma_{aX+b} = |a|\sigma_X$.
- Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y). In particular, if X, Y are independent, then Var(X + Y) = Var(X) + Var(Y).
- $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$.

Correlation coefficient

Definition

Distribution of two RVs

Given two random variables, X, Y, the correlation coefficient, denoted by $\rho(X, Y)$ (pronounced 'Row' X, Y), is defined as $\frac{Cov(X,Y)}{\sigma_X\sigma_Y}$.

Range of ρ

$$-1 \le \rho(X, Y) \le 1.$$

Proof: Consider $Z_1 = \frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}$ and $Z_2 = \frac{X}{\sigma_X} - \frac{Y}{\sigma_Y}$. Now we note that $Var(Z_i) > 0$.

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Examples

For the random variables of previous examples, $Cov(Y, Z) = \frac{1}{4}$ and $\rho(X, Y) = \frac{1}{\sqrt{5}}$, since $\sigma_Y = \frac{\sqrt{5}}{2}$, $\sigma_Z = \frac{1}{2}$.

Random Variable Vectors

All the discussion of multiple random variables is much more succintly expressed as vectors. Again consider X_1, X_2 as two random variables with joint pmf p_{X_1,X_2} . We could instead think of them as a random variable vector, $m{X} = \left(egin{array}{c} X_1 \\ X_2 \end{array}
ight)$. The joint pmf will be the same, only denoted by the vector $p_{\mathbf{X}}$.

Expectation

$$\mu_{\mathbf{X}} = E(\mathbf{X}) = E\left(\begin{array}{c} X_1 \\ X_2 \end{array}\right) = \left(\begin{array}{c} E(X_1) \\ E(X_2) \end{array}\right).$$



Random Variable Vectors cont.

Covariance Matrix

Distribution of two RVs

The Covariance Matrix for X is denoted by Σ and is defined as

follows:
$$\Sigma = Cov(X, X) = E((X - \mu_X)(X - \mu_X)^T)$$

 $= E\begin{pmatrix} (X_1 - \mu_1)^2 & (X_1 - \mu_1)(X_2 - \mu_2) \\ (X_1 - \mu_1)(X_2 - \mu_2) & (X_2 - \mu_2)^2 \end{pmatrix}$
 $= \begin{pmatrix} Var(X_1) & Cov(X_1, X_2) \\ Cov(X_1, X_2) & Var(X_2) \end{pmatrix}$

Random Variable Vectors cont.

Properties

Distribution of two RVs

Let X be a random variable vector and A a matrix of constants.

Then $\Sigma_{\mathbf{A}} = Cov(\mathbf{A}\mathbf{X}, \mathbf{A}\mathbf{X}) = \mathbf{A}Cov(\mathbf{X}, \mathbf{X})\mathbf{A}^T = \mathbf{A}\Sigma\mathbf{A}^T$.

Where Σ is the Covariance matrix of \boldsymbol{X} and $\Sigma_{\boldsymbol{A}}$ is the Covariance matrix of $\boldsymbol{A}\boldsymbol{X}$.

Example: Suppose X and Y are two random variables with Var(X) = Var(Y) = 2 and Cov(X, Y) = 1. What can we say about Cov(2X + 3Y, 3X - Y)?

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Random Variable Vectors cont.

Properties

Distribution of two RVs

Let **X** be a random variable vector and **A** a matrix of constants.

Then $\Sigma_{\mathbf{A}} = Cov(\mathbf{A}\mathbf{X}, \mathbf{A}\mathbf{X}) = \mathbf{A}Cov(\mathbf{X}, \mathbf{X})\mathbf{A}^T = \mathbf{A}\Sigma\mathbf{A}^T$.

Where Σ is the Covariance matrix of X and Σ_A is the Covariance matrix of AX.

Example: Suppose X and Y are two random variables with Var(X) = Var(Y) = 2 and Cov(X, Y) = 1. What can we say about Cov(2X + 3Y, 3X - Y)?

We just note that here if $\boldsymbol{X} = \begin{pmatrix} X \\ Y \end{pmatrix}$, then

$$\begin{pmatrix} 2X+3Y \\ X-2Y \end{pmatrix} = AX$$
, where, $A = \begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix}$

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The End

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