The purpose of the scoring guide is to provide a uniform evaluation of the tests by the graders. Therefore, the guide gives you the main ideas (and at least one possible solution) for each problem and the associated scores. This guide is not intended to be a detailed description of a complete solution to each problem; the steps described are sketches of a solution that achieves maximum score.
The partial scores given in the guide are only awarded to the solver if the related idea is included in the test as a step in a clearly described and reasoned solution. Thus, for example, the mere description of statements or definitions in the material without its application (even if any of the facts described do indeed play a role in the solution) is not worth the point.
A partial score is given for any idea or part solution from which the correct solution of the problem could be obtained with the appropriate addition of the reasoning described in the test. If a solver starts several solutions that are significantly different, then you can give score for at most one. If each described solution or part of solution is correct or can be extended to be correct, then the solution with the most points will be evaluated. However, if there are correct and incorrect solutions between several solution attempts and it is not clear from the test which one the solver considered to be correct, then the solution with fewer points will be evaluated (even if this score is 0). The scores in this guide may be further subdivided as necessary. Of course, a good solution other than that described in this guide is also worth the maximum score.
In the event of an arithmetic error, one point is deducted from each exercise. An exception to this is when computing the numbers significantly simplifies or modifies the structure of the task. In these cases, no points are awarded to the steps which follow the computation and are missing from the solution of the student.

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1. Write the following definition or statement:
   (a) What is the Law of Total Probability (discrete case for events)? Also state clearly the assumptions for which the law holds.
   (b) Let (X, Y) be a jointly continuous random variable vector. How do we define the conditional density function of Y given X?

Solution:
(a)
(5 points) Given events $A_1, ..., A_n$ that form a **partition** of the sample space Ω (that is, they are pairwise mutually exclusive and their union is Ω)
(5 points) then, for a given event B, $P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$
(if limits are not specified then -1 point, for wrong terms like $\mathbb{P}(A_i|B)$, maximum 2 points for this part.)

(b)
(8 points)

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{f_{X,Y}(x,y)}{\int_{-\infty}^{\infty} f_{X,Y}(x,u)\mathrm{d}u},$$

(2 points) for such points $x, y \in \mathbb{R}$, where $f_X(x) \neq 0$, and $f_{Y|X}(y \mid x) = 0$ if $f_X(x) = 0$.

2. We want to make a call from a phone booth, and are waiting for the person in front of us to finish the conversation. Assume that the duration of the person's call is measured in minutes, is continuous, has memoryless property and has an expected value of 3.
   (a) What is the probability that the total duration of the conversation will be more than 6 minutes, given that the person has been talking for more than 4 minutes?
   (b) Our satisfaction, on a scale of zero to ten, decreases over time according to the function $10 \cdot e^{-t}$, where t is the time we wait for. what is the probability that our satisfaction falls below 5? What is the expected value of our satisfaction?

Solution:
(0 points) Let X denote the duration of the telephone conversation
(1 point) X is continuous and has the memoryless property, so $X \sim Exp(\lambda)$ for some parameter $\lambda > 0$
(1 point) $\mathbb{E}(X) = \frac{1}{\lambda} = 3$, so $\lambda = \frac{1}{3}$

(a)
(2 points) $\mathbb{P}(X > t+2 | X > t) = \mathbb{P}(X > 2)$ because of the memoryless property
(2 points) $\mathbb{P}(X > 2) = 1 - \mathbb{P}(X < 2) = 1 - F_X(2)$
(2 points) $= e^{-2/3} \approx 0.5134$

(b)
(0 points) Let Y denote our satisfaction
(1 point) $Y = 10 \cdot e^{-X}$
(2 points) $\mathbb{P}(Y < 5) = \mathbb{P}\left(10 \cdot e^{-X} < 5\right) = \mathbb{P}\left(e^{-X} < 0.5\right) = \mathbb{P}(X > -\ln 0.5)$
(2 points) $= 1 - F_X(-\ln 0.5) = e^{-(-\ln 0.5)/3}$
(1 point) $= 0.5^{\frac{1}{3}} \approx 0.7937$

(2 points) $\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\,\mathrm{d}x$
(2 points) $\mathbb{E}(Y) = \mathbb{E}(10 \cdot e^{-X}) = \int_0^{\infty} 10 \cdot e^{-x} \cdot \frac{1}{3} e^{-\frac{1}{3}x}\,\mathrm{d}x$
(0 points) $= \int_0^{\infty} \frac{10}{3} e^{-\frac{4}{3}x}\,\mathrm{d}x$
(1 point) $= \left[\frac{10}{3}(-\frac{3}{4}) e^{-\frac{4}{3}x}\right]_0^{\infty}$
(1 point) $= \frac{10}{4} = 2.5$

3. There are 60 multiple choice questions in an exam. Each question has four possible answers of which exactly one is correct. Assume that an examinee completes the questions without any prior knowledge and for each question independently and randomly guesses the answer. What is the approximate probability that the examinee gives more than 20 correct answers? How much does this probability change if he can eliminate one of the four possible answers for each question but guesses uniformly randomly from the rest?

Solution:
(0 points) Let X be the number of correct answers
(3 points) $X \sim Bin\left(60, \dfrac{1}{4}\right)$
(1 point) $\mathbb{E}(X) = 60 \cdot \dfrac{1}{4} = 15$
(1 point) $\sigma_X = \sqrt{60 \cdot \dfrac{1}{4} \cdot \dfrac{3}{4}} = \dfrac{3\sqrt{5}}{2} \approx 3.3541$
(0 points) We have to find $\mathbb{P}(X > 20) =?$
(2 points) standardizing,

$$\mathbb{P}\left(\frac{X-15}{3\sqrt{5}/2} > \frac{20-15}{3\sqrt{5}/2}\right) = \mathbb{P}\left(\frac{X-15}{3\sqrt{5}/2} > \frac{2\sqrt{5}}{3}\right)$$

(2 points) Using the de Moivre–Laplace theorem
(2 points) $\dfrac{X-15}{3\sqrt{5}/2}$ has approximately the standard normal distribution.
(2 points) So the required amount: $1 - \Phi\left(\dfrac{2\sqrt{5}}{3}\right) = 1 - \Phi(1.4907)$
(1 point) $\approx 1 - \Phi(1.49) = 1 - 0.9319 = 0.0681$
(2 points) if he can eliminate one option from the four given for each question, then, $X \sim Bin\left(60, \dfrac{1}{3}\right)$
(1 point) $\mathbb{E}(X) = 60 \cdot \dfrac{1}{3} = 20$
(3 points) which after standardization will be exactly $1 - \Phi(0) = 0.5$

4. Let X and Y be two random variables. Suppose that the linear regression of X in terms of Y is $\frac{3}{2}Y + 2$, while the linear regression of Y in terms of X is $\frac{1}{2}X - 1$. Determine the expected value of both the variables X and Y and find their correlation coefficient.

Solution: (2 points) The linear regression line of X in terms of Y is $\beta_1 Y + \alpha_1$, where

$$\beta_1 = \frac{Cov(X,Y)}{\sigma_Y^2} = \frac{3}{2}, \qquad \alpha_1 = \mathbb{E}(X) - \beta_1\,\mathbb{E}(Y) = \mathbb{E}(X) - \frac{3}{2}\,\mathbb{E}(Y) = 2,$$

(2 points) and the linear regression line of Y in terms of X is $\beta_2 X + \alpha_2$

$$\beta_2 = \frac{Cov(X,Y)}{\sigma_X^2} = \frac{1}{2}, \qquad \alpha_2 = \mathbb{E}(Y) - \beta_2\,\mathbb{E}(X) = \mathbb{E}(Y) - \frac{1}{2}\,\mathbb{E}(X) = -1,$$

(2 points) from these, we get the following system of equations:

$$\mathbb{E}(X) - \frac{3}{2}\,\mathbb{E}(Y) = 2,$$
$$\mathbb{E}(Y) - \frac{1}{2}\,\mathbb{E}(X) = -1.$$

(1 point) (adding the half of the first equation to the second,) $\frac{1}{4}\,\mathbb{E}(Y) = 0$
(1 point) so $\mathbb{E}(Y) = 0$
(2 points) and so, $\mathbb{E}(X) - \frac{3}{2}\,\mathbb{E}(Y) = \mathbb{E}(X) = 2$.

(2 points) $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$
(2 points) $\rho_{X,Y} = \beta_2 \cdot \frac{\sigma_X}{\sigma_Y}$
(2 points) since $\frac{\beta_1}{\beta_2} = \frac{\sigma_X^2}{\sigma_Y^2}$
(2 points) so, $\frac{\sigma_X}{\sigma_Y} = \sqrt{3}$
(2 points) $\rho_{X,Y} = \frac{1}{2} \cdot \sqrt{3} = \frac{\sqrt{3}}{2} = 0.8660$ (points are awarded if not computed here to the decimal)

5. Let the joint probability density function of the variables X and Y be as given below:

$$f_{X,Y}(x,y) = \begin{cases} \alpha(x + y + x^2 y) & \text{if } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

for some $\alpha \in \mathbb{R}$.

(a) Determine the value of α and the expected value $\mathbb{E}(Y)$.
(b) Determine the probability $\mathbb{P}(X^2 < Y)$.

Solution:
(a)
(2 points) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y = 1$
(2 points) So, $1 = \int_0^1 \int_0^1 \alpha(x + y + x^2 y)\mathrm{d}x\,\mathrm{d}y = \int_0^1 \alpha\left(\frac{1}{2} + \frac{4}{3}y\right)\,\mathrm{d}y = \frac{7}{6}$
(1 point) $\alpha = \frac{6}{7}$
(2 points) $\mathbb{E}(Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$
(2 points) $= \int_0^1 \int_0^1 y \cdot \frac{6}{7}(x + y + x^2 y)\mathrm{d}x\,\mathrm{d}y =$
(2 points) $= \int_0^1 \frac{6}{7}\left(\frac{1}{2}y + \frac{4}{3}y^2\right)\,\mathrm{d}y = \frac{25}{42}$

(b)
(2 points) For any $H \subseteq \mathbb{R}^2$, $\mathbb{P}((X,Y) \in H) = \int\int_H f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y$

(1 point) Let $H = \{(x, y) \in \mathbb{R}^2 \mid x^2 < y\}$. (If the definition of H is implicit in the previous statement then also points are awarded)
(3 points) Since f is non-zero only in the region, $0 < x < 1$ and $0 < y < 1$, so,

$$\mathbb{P}(X^2 < Y) = \int_H f_{X,Y}(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_{x^2}^1 \frac{6}{7}(x + y + x^2 y)\,\mathrm{d}y\,\mathrm{d}x$$

(3 points)

$$= \int_0^1 \frac{6}{7}\left[x(1 - x^2) + \frac{1}{2}(1 + x^2)(1 - x^4)\right]\mathrm{d}x = \int_0^1 \frac{3}{7}(2x - 2x^3 + 1 + x^2 - x^4 - x^6)\,\mathrm{d}x = \frac{3}{7}\left[1 - \frac{2}{4} + 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7}\right]_0^1 = 0.6388$$

6.* In a meadow $N \geq 1$ deer graze unsuspectingly. Unknown to the deers and to each other, N hunters sneak into the clearing and fire on the deer simultaneously. Assume that each shot hits a deer and is fatal. (In principle, multiple hunters can shoot at the same deer, and each hunter independently and randomly picks a deer to shoot at.)
(a) What is the probability of a given deer surviving?
(b) What is the expected number of deer that will survive?

Solution:
(a) (5 points)
Every shot independently has a probability of $\frac{N-1}{N}$ of missing the given deer, because the hunters independently and uniformly pick a deer to shoot at. So, the probability that a given deer is not shot is: $\left(\frac{N-1}{N}\right)^N$

The final answer is only awarded points with a sound reasoning
(b) (15 points)
Let X_i be the indicator random variable of the event that the i-th deer survives, where $i = 1..N$
Then $X_i = 1$ with probability found in part (a).
The expected value we are supposed to find is $\mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{X_i}\right) = \sum_{i=1}^N \mathbb{E}\left(\mathbf{1}_{X_i}\right) = \sum_{i=1}^N \mathbb{P}\left(X_i\right) = N \cdot \left(\frac{N-1}{N}\right)^N$
If the distribution is said to be Binomial then 0 points are awarded.