The purpose of the scoring guide is to provide a uniform evaluation of the tests by the graders. Therefore, the guide gives you the main ideas (and at least one possible solution) for each problem and the associated scores. This guide is not intended to be a detailed description of a complete solution to each problem; the steps described are sketches of a solution that achieves maximum score.
The partial scores given in the guide are only awarded to the solver if the related idea is included in the test as a step in a clearly described and reasoned solution. Thus, for example, the mere description of statements or definitions in the material without its application (even if any of the facts described do indeed play a role in the solution) is not worth the point.
A partial score is given for any idea or part solution from which the correct solution of the problem could be obtained with the appropriate addition of the reasoning described in the test. If a solver starts several solutions that are significantly different, then you can give score for at most one. If each described solution or part of solution is correct or can be extended to be correct, then the solution with the most points will be evaluated. However, if there are correct and incorrect solutions between several solution attempts and it is not clear from the test which one the solver considered to be correct, then the solution with fewer points will be evaluated (even if this score is 0). The scores in this guide may be further subdivided as necessary. Of course, a good solution other than that described in this guide is also worth the maximum score.
In the event of an arithmetic error, one point is deducted from each exercise. An exception to this is when computing the numbers significantly simplifies or modifies the structure of the task. In these cases, no points are awarded to the steps which follow the computation and are missing from the solution of the student.

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1. Write the following definition or statement:
   (a) How, and under what conditions, do we define the correlation of two random variables X and Y, using the covariance and standard deviation of X and Y?
   (b) Let X be a simple (discrete) probability variable, and $g : \mathbb{R} \to \mathbb{R}$ be a function of $\mathbb{E}\left(g(X)\right)$. Express the value of $\mathbb{E}\left(g(X)\right)$ using the distribution of X.

   Solution:
   (a)
   (8 points) The correlation of two random variables X, Y is defined as $\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$
   (2 points) if σ_X and σ_Y are both not 0.

   (b)
   (10 points)
   $$\mathbb{E}(g(X)) = \sum_{x \in \text{Range}(X)} g(x) \cdot p_X(x)$$
   (2 points are for writing $x \in \text{Range}(X)$.)

2. Ben buys 3 lottery tickets every week. He always chooses between two different types, A and B, of lottery tickets, but once he decides the type, he buys all 3 tickets of that type. For type A tickets, on an average, every 12-th ticket wins, while a B ticket has a 10% chance of winning. To decide which ticket to buy in a given week, Ben rolls a fair dice. If he rolls 1 or 2, then he chooses type A, otherwise he chooses type B.
   (a) What is the probability that Ben wins on at least one lottery ticket in a given week?
   (b) Assuming that he won in a given week, what is the probability that he bought tickets of type A?

   Solution:
   (a)
   Consider the following events:
   (0 points) A = Ben rolls 1 or 2 with the dice that week, W = Ben wins at least one lottery ticket that week
   (1 point) $\mathbb{P}(A) = \frac{1}{3}$
   (2 points) The events A and $\bar{A}$ form a partition of the sample space

(1 point) Therefore, using the Law of Total Probability:
(2 points)

$$\mathbb{P}(W) = \mathbb{P}(W \mid A)\,\mathbb{P}(A) + \mathbb{P}(W \mid \bar{A})\,\mathbb{P}(\bar{A})$$

(1 point) $\mathbb{P}(W \mid A)$ is the probability of winning for 3 tickets of type A
(1 point) $\mathbb{P}(W \mid A) = 1 - \mathbb{P}(\bar{W} \mid A)$
(1 point) $\mathbb{P}(\bar{W} \mid A) = \mathbb{P}(\text{none of the three tickets of type A wins}) = \left(\frac{11}{12}\right)^3$
(1 point) $\mathbb{P}(W \mid A) = 1 - \left(\frac{11}{12}\right)^3 = \frac{397}{1728} \approx 0.2297$
(2 points) Similarly, $\mathbb{P}(W \mid \bar{A}) = 1 - \left(\frac{9}{10}10\right)^3 = 0.271$
(1 point) $\mathbb{P}(W) = \mathbb{P}(W \mid A)\frac{1}{3} + \mathbb{P}(W \mid \bar{A})\frac{2}{3} \approx 0.2297 \cdot \frac{1}{3} + 0.271 \cdot \frac{2}{3}$
(1 point) ≈ 0.2572

(b)
(1 point) The question $\mathbb{P}(A \mid W) =?$
(1 point) Using Bayes' theorem
(2 points)

$$\mathbb{P}(A \mid W) = \frac{\mathbb{P}(W \mid A)\,\mathbb{P}(A)}{\mathbb{P}(W)}$$

(1 point) $\approx \dfrac{0.2297 \cdot \frac{1}{3}}{0.2572}$
(1 point) ≈ 0.2977

3. Captain Shark has fished with his boat every year since he was 20. At the end of each year, he and his fellow fishermen compared the total weight of fish caught on their boats during that year. They found that the average total weight of fish caught on a fishing boat was 150 tons, with a standard deviation of 30 tons. The distribution has not changed over the years and amount of fish caught in two different years is independent of each other. Using his knowledge of probability, Captain Shark concluded that, the standard deviation for the total weight of all the fish caught by a given vessel in all the years since he started fishing, is 150 tons.
(a) How old is Captain Shark?
(b) What is the probability that the total weight of fish caught by a fishing boat over the years (since Captain Shark started fishing) exceeds the expected value of the total catch weight of a boat by more than 3%?

Solution:
(a)
(0 points) Let X_i denote the total annual catch of a given vessel in the ith year, n the number of years elapsed
(1 point) $\sigma_{X_i} = 30$
(1 point) Let $X = \left(\sum_{i=1}^{n} X_i\right)$, then $\sigma_X = 150$
(1 point) X_i's are independent and identically distributed
(2 points) therefore $150^2 = Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) = n \cdot 30^2$
(1 point) $n = 25$
(1 point) The captain is therefore $20 + 25 = 45$ years old ($20 + 24 = 44$ is also worth a point)

(b) Again using notation that $X = \sum_{1}^{25} X_i$,
(1 point) $\mathbb{P}\left(X - \mathbb{E}\left(X\right) > 0.03 \cdot \mathbb{E}\left(X\right)\right) =?$
/or $\mathbb{P}\left(X > 1.03 \cdot \mathbb{E}\left(X\right)\right) =?$/
(1 point) $\mathbb{E}(X) = 25 \cdot \mathbb{E}(X_1) = 3750$
(2 points) Standardizing:

$$\mathbb{P}\left(\frac{X - \mathbb{E}(X)}{\sigma_X} > \frac{0.03 \cdot \mathbb{E}(X)}{\sigma_X}\right) =?$$

(1 point) so the question is

$$\mathbb{P}\left(\frac{X - 3750}{150} > \frac{0.03 \cdot 3750}{150}\right) = \mathbb{P}\left(\frac{X - 3750}{150} > 0.75\right)$$

(1 point) $= 1 - \mathbb{P}\left(\frac{X-3750}{150} < 0.75\right)$
(1 point) $X_1, \ldots, X_{25}$ are identically distributed, jointly independent probability variables
(1 point) hence using the central limit theorem,
(2 points) $\frac{X-3750}{150} = \frac{sum_{i=1}^{25} X_i - 3750}{150}$ is approximately normally distributed.
(2 points) So, the quantity we are looking for is $1 - \Phi(0.75)$
(1 point) $\approx 1 - 0.7734 = 0.2266$.

4. When Santa arrives at the last home where he has to deliver christmas gifts, he realizes that all he is left with is 2 (not identical) chocolates, while there are stockings of 4 children hanging at the fireplace. He puts each chocolate in a stocking that he picks independently and randomly and then leaves. Unknown to Santa, two of the children Alice and Bob are arch enemies. Let X denote the number of chocolates Alice gets, while Y is the indicator random variable of whether Alice and Bob get the same number of chocolates.
(a) Write the joint probability mass function table of X and Y. Are they independent?
(b) Compute $Cov(X, Y)$ and the standard deviation of $Z = -2X + 3$.

Solution:
(a)
(3 points) The joint distribution of the probability variables X and Y is described in the table below:

Y \ X	0	1	2	p_Y
0	$\frac{5}{16}$	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{5}{8}$
1	$\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{3}{8}$
p_X	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$	

(2 points) they are not independent because, for example, $0 = \mathbb{P}(X = 2, Y = 1) \neq \mathbb{P}(X = 2) \cdot \mathbb{P}(Y = 1) = \frac{1}{16} \cdot \frac{3}{8}$
(other good justification also deserves 2 points)
(b)
(2 points) $\text{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X) \cdot \mathbb{E}(Y)$
(1 point) know the definition of expected value
(1 point) $\mathbb{E}(X) = \frac{1}{2}$, $\mathbb{E}(Y) = \frac{3}{8}$
(2 points) $\mathbb{E}(XY) = \sum_{i,j} i \cdot j \cdot \mathbb{P}(X = i, Y = j)$
(if you don't write it this way, but use it well in this particular case, you still get a point)
(1 point) $\mathbb{E}(XY) = \frac{1}{8}$
(1 point) So $\text{cov}(X, Y) = -\frac{1}{16}$
(2 points) $Var(Z) = Var(-2X + 3) = 4Var(X)$
(or $\sigma_Z = \sigma_{-2X+3} = 2\sigma_X$, but if it is multipled by -2 then 0 points are awarded)
(2 points) $Var(X) = \mathbb{E}(X^2) - (\mathbb{E}\, X)^2$
(1 point) $\mathbb{E}(X^2) = \frac{5}{8}$
(1 point) $Var(X) = \frac{3}{8}$
(1 point) $\sigma_Z = \sqrt{\frac{3}{8}}$

5. Let (X, Y) be a continuous probability random variable vector whose density function for some $\alpha \in \mathbb{R}$ is

$$f_{X,Y}(x, y) = \begin{cases} \alpha \cdot e^{-3x-2y} & \text{if } 0 < x \text{ and } 0 < y, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the value of α. (b) Determine the regression $\mathbb{E}(X \cdot Y + 2 \cdot X^2 \frac{1}{Y} + 1 | Y)$.

Solution:
(a)
(2 points) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, \mathrm{d}x\, \mathrm{d}y = 1$
(1 point) So, $1 = \int_0^{\infty} \int_0^{\infty} \alpha \cdot e^{-3x-2y} \mathrm{d}x\, \mathrm{d}y = \int_0^{\infty} \alpha \cdot e^{-2y} \cdot \left[\frac{e^{-3x}}{-3}\right]_0^{\infty} \mathrm{d}y =$

(1 point) $= \int_0^\infty \alpha \cdot e^{-2y} \cdot \frac{1}{3}\,\mathrm{d}y = \alpha \cdot \frac{1}{3} \cdot \left[\frac{e^{-2y}}{-2}\right]_0^\infty = \alpha \cdot \frac{1}{6}$
(1 point) $\alpha = 6$

(b)
(1 point) the joint density function can be written in the multiplicative form: $f_{X,Y}(x,y) = (3 \cdot e^{-3x}) \cdot (2 \cdot e^{-2y})$
(1 point) so, X and Y are independent
(2 points) $f_X(x) = 3 \cdot e^{-3x}$, if $x > 0$, $\longrightarrow X \sim Exp(3)$, and $f_Y(y) = 2 \cdot e^{-2y}$, if $y > 0 \longrightarrow Y \sim Exp(2)$
(1 point) $\mathbb{E}(X \cdot Y + 2 \cdot X^2 \frac{1}{Y} + 1 \mid Y) = \mathbb{E}(X \cdot Y \mid Y) + \mathbb{E}(2 \cdot X^2 \frac{1}{Y} \mid Y) + 1$
(1 point) because...
(2 points) $= Y \cdot \mathbb{E}(X \mid Y) + 2 \cdot \frac{1}{Y} \cdot \mathbb{E}(X^2 \mid Y) + 1$
(1 point) because... (reference to the corresponding property of regression)
(1 point) $\mathbb{E}(X \mid Y) = \mathbb{E}(X)$, $\mathbb{E}(X^2 \mid Y) = \mathbb{E}(X^2)$
(2 points) because they are independent
(1 point) $\mathbb{E}(X) = \frac{1}{3}$, $Var(X) = \frac{1}{9}$
(1 point) $\mathbb{E}(X^2) = Var(X) + (\mathbb{E}(X))^2 = \frac{2}{9}$
(1 point) So the regression in question is $= Y \cdot \frac{1}{3} + 2 \cdot \frac{1}{Y} \cdot \frac{2}{9} + 1 = \frac{1}{3}Y + \frac{4}{9}\frac{1}{Y} + 1$

6.* Whenever Uncle George breaks a stick, the breaking point is evenly distributed in the middle third of the stick. Uncle George now takes a 15 centimetre long stick in his hand, breaks it, and then breaks the piece in his left hand again. Give the density function of the length of the piece remaining in his left hand after the second break.

Solution:
(1 point) Denote with X the distance of the first breakpoint from the left end of the stick, where $X \sim U(5,10)$
(2 points) Mark the distance of the second breakpoint from the left end of the stick Y, where $Y \sim U\left(\frac{X}{3}, \frac{2X}{3}\right)$
(0 points) we have to find $f_Y(y)$
(3 points) Using the Law of total probability: $f_Y(y) = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x)\,\mathrm{d}x$
(1 point) $f_X(x) = \frac{1}{5}$ if $x \in (5,10)$, otherwise 0
(2 points) $f_{Y|X}(y|x) = \frac{3}{x}$ if $5 < x < 10$ and $\frac{x}{3} < y < \frac{2x}{3}$, otherwise 0
(3 points) $f_Y(y) = \int_{\max(5,\frac{3}{2}y)}^{\min(10,3y)} \frac{3}{x} \cdot \frac{1}{5}\,\mathrm{d}x$
(2 points) $= \frac{3}{5} \cdot [\ln x]_{\max(5,\frac{3}{2}y)}^{\min(10,3y)}$,
(2 points) if $\frac{5}{3} < y < \frac{20}{3}$, otherwise 0
(2 points) if $\frac{5}{3} < y < \frac{10}{3}$, then, $f_Y(y) = \frac{3}{5} \cdot [\ln x]_5^{3y} = \frac{3}{5} \cdot \ln \frac{3y}{5}$
(2 points) if $\frac{10}{3} \le y < \frac{20}{3}$, then, $f_Y(y) = \frac{3}{5} \cdot [\ln x]_{\frac{3}{2}y}^{10} = \frac{3}{5} \cdot \ln \frac{20}{3y}$