Probability Theory Lecture 06

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October 19, 2020

Overview

Definition

Moments

Recall that the i^{th} moment of a random variable X is defined as $E(X^i)$.

Moment Generating Function

The moment generating function for a random variable X is defined as $M_X(t) = E(e^{Xt})$, where $t \in \mathbb{R}$.

Note that this is computed as $\sum p(x)e^{xt}$ in the discrete case and as $\int f(x)e^{xt}dx$ in the continuous case.

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Continuous

Let X be a random variable with density $f_X(x) = 2x, x \in [0, 1]$ and 0 otherwise. The MGF of X is
$\int_0^1 2xe^{xt}dx = \frac{2}{t}e^t - \frac{2}{t^2}e^t + \frac{2}{t^2}$

What's in a name?

- $M_X(t) = E(e^{Xt}) = E(1 + Xt + \frac{X^2t^2}{2!} + \frac{X^3t^3}{3!} + \ldots)$ which, by Linearity of expectation, is
  $= 1 + E(X)t + \frac{E(X^2)t^2}{2!} + \frac{E(X^3)t^3}{3!} + \ldots)$
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- $M_X^{(1)}(0) = E(X)$.
- $M_X^{(2)}(0) = E(X^2)$.

So by differentiating i times, and plugging in $t = 0$, we can obtain the i^{th} moment of X.

Bernoulli distribution

MGF for Bernoulli distribution

Let X be a random variable with distribution $1(p)$, that is, a Bernoulli random variable with probability of success p. Then,
$M_X(t) = E(e^{Xt}) = e^{0t}(1-p) + e^{1t}p = 1 - p + pe^t$

Binomial distribution

MGF for Binomial distribution

Let X be a random variable with distribution $Bin(n, p)$, that is, a Binomial random variable with parameters n, p. Then,
$M_X(t) = E(e^{Xt}) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} e^{it} =$
$\sum_{i=0}^{n} \binom{n}{i} (pe^t)^i (1-p)^{n-i} = (1-p+pe^t)^n$

Poisson distribution

MGF for Poisson distribution

Let X be a random variable with distribution $Pois(\lambda)$, that is, a Poisson random variable with parameter λ. Then,
$M_X(t) = E(e^{Xt}) = \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} e^{it} = \sum_{i=0}^{\infty} \frac{(\lambda e^t)^i}{i!} e^{-\lambda} = e^{\lambda(e^t - 1)}$

Geometric distribution

MGF for Geometric distribution

Let X be a random variable with distribution $Geo(p)$, that is, a Geometric random variable with probability of success p. Then,
$M_X(t) = E(e^{Xt}) = \sum_{i=1}^{\infty}(1-p)^{i-1}pe^{it} =$
$pe^t \sum_{i=1}^{\infty}((1-p)e^t)^{i-1} = pe^t \sum_{i=0}^{\infty}((1-p)e^t)^i = \frac{pe^t}{1-(1-p)e^t}$

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Note: It is important to note that the final step could only have been performed when $(1-p)e^t < 1$, otherwise the series does not converge. So we get the condition that $t < -ln(1-p)$.

Uniform distribution

MGF for Uniform distribution

Let X be a random variable with distribution $U(a, b)$, that is, a uniformly distributed random variable in the interval (a, b). Then, $M_X(t) = E(e^{Xt}) = \int_a^b \frac{1}{(b-a)} e^{xt} dx = [\frac{e^{xt}}{t(b-a)}]_a^b = \frac{e^{bt}-e^{at}}{t(b-a)}$

Exponential distribution

MGF for Exponential distribution

Let X be a random variable with distribution $Exp(\lambda)$, that is, a exponential distribution with parameter λ. Then,

$M_X(t) = E(e^{Xt})$
$= \int_0^\infty \lambda e^{-\lambda x} e^{xt} dx = \int_0^\infty \lambda e^{-(\lambda - t)x} dx = [\frac{-\lambda}{\lambda - t} e^{-(\lambda - t)x}]_0^\infty = \frac{\lambda}{\lambda - t}$

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Note: It is important to note that the definite integral is defined only when $\lambda - t > 0$. So we get the condition that $t < \lambda$.

Properties of MGFs

Positivity

$M_X(t) \geq 0 \forall t \in \mathbb{R}$. In words, since MGFs are the expected value of exponential function, and exponential functions are non-negative functions, so MGFs are also non-negative.

Value at 0

$M_X(0) = 1$.

Properties of MGFs cont.

Given a random variable X, what can we say about the moment generating function of $X + b$ where $b \in \mathcal{R}$?

Translation

$M_{X+b}(t) = E(e^{(X+b)t}) = E(e^{Xt+bt}) = E(e^{bt}e^{Xt})$
$= e^{bt}E(e^{Xt}) = e^{bt}M_X(t)$
So, $M_{X+b}(t) = E(e^{(X+b)t}) = e^{bt}M_X(t)$

Properties of MGFs cont.

Given a random variable X, what can we say about the moment generating function of aX where $a \in \mathcal{R}$?

Scaling

$M_{aX}(t) = E(e^{(aX)t}) = E(e^{X(at)}) = M_X(at)$

So, $M_{aX}(t) = M_X(at)$

Properties of MGFs cont.

Given two **independent** random variables X, Y what can we say about the moment generating function of their convolution $X + Y$?

Sum of Random Variables

$M_{X+Y}(t) = E(e^{(X+Y)t}) = E(e^{Xt+Yt}) = E(e^{Xt}e^{Yt}) =$
$E(e^{Xt})E(e^{Yt}) = M_X(t)M_Y(t)$
So, $M_{X+Y}(t) = M_X(t)M_Y(t)$

Properties of MGFs cont.

MGFs uniquely determine the distribution

For two given random variable, X and Y, if $M_X(t) = M_Y(t) \forall t$ then,
$F_X(x) = F_Y(y)$

Example

Given a random variable X with MGF $\frac{2}{3} + \frac{1}{3}e^t$, what is P(X=1)?

Properties of MGFs cont.

Limits of MGFs (Informally)

For a sequence of random variables X_n and another variable X, if $M_{X_n}(t) \to M_X(t)$ then,
$f_{X_n} \to f_X$

Poisson Approximation

Let X_n be a sequence of independent random variables with distribution $Bin(n, \frac{\lambda}{n})$, where λ is a constant. Then, $\lim_{n\to\infty} M_{X_n} = \lim_{n\to\infty}(1 + \frac{\lambda}{n}(e^t - 1))^n =$ $\lim_{n\to\infty} e^{ln((1+\frac{\lambda}{n}(e^t-1))^n)} = \lim_{n\to\infty} e^{n(ln(1+\frac{\lambda}{n}(e^t-1)))} =$ $\lim_{n\to\infty} e^{n(\frac{\lambda}{n}(e^t-1))}$ here we have omitted other terms in the log Taylor series expansion, as those terms vanish when we take the limit. $= e^{(\lambda(e^t-1))}$

And this is the MGF of Poisson distribution. So the pmfs of these Binomial variables converge to a Poisson distribution.

Central Limit Theorem (simplified version)

Let $X_1, X_2, ...$ be a sequence of independent, identically distributed random variables with mean 0 and $E(X_i^2) = 1$. Then, if we take $Z_n = \frac{X_1+X_2+..+X_n}{\sqrt{n}}$, then

$$\lim_{n\to\infty} M_{Z_n}(t) \to e^{\frac{1}{2}t^2}$$

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Proof: Let $M_{X_i} = M(t)$. Note that $M_{Z_n}(t) = (M(\frac{t}{\sqrt{n}}))^n$. We want to then show that,

$$\lim_{n\to\infty} (M(\frac{t}{\sqrt{n}}))^n \to e^{\frac{1}{2}t^2}$$

. but this is equivalent to showing that,

$$\lim_{n\to\infty} n \ln(M(\frac{t}{\sqrt{n}})) \to \frac{1}{2}t^2$$

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