Please note: The duration of the exam is 100 minutes. You may use a calculator. Please give a numeric answer (rounded to 4 decimal places). You are expected to **write all steps** taken in getting the final answer along with a **mention of properties/theorems used** in these steps. You are not allowed to leave the examination hall during the first 30 minutes of the exam.

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1. In a bag of roasted salted pistachios, it might happen that some pistachios are not split but are completely closed and are hard to open. Assume that a bag has many pistachios, and the pistachios each have an identical, independent small probability of being closed. Denote the number of closed pistachios in a bag with X. We know that $P(X = 0 \mid X \leq 2) = 0.4$. Determine $\text{Cov}(1 + 2X, 3X)$.

2. A firm buys 90 light bulbs for their office space. Assume that the life span, measured in years, of each light bulb has $\text{Exp}(\frac{1}{\sqrt{10}})$ distribution, and that they are independent of each other.
   (a) Consider the total life span of all the 90 light bulbs. For what positive real number z can we say that this total life span is more than z with probability approximately 3%?
   (b) How would the above value of z change if the expected life span of individual light bulbs was half a year more, but the standard deviation remained the same as before. (The distribution will not necessarily be exponential.)

3. The joint probability density function of random variables X and Y is given as follows:

$$f_{X,Y} : (x, y) \mapsto \begin{cases} \alpha x + \beta y & \text{if } 0 < x < 2 \text{ and } 0 < y < 2, \\ 0 & \text{otherwise.} \end{cases}$$

   Assume that $\text{Cov}(X, Y) = -\frac{1}{36}$. Determine α and β. What is $P(X < 1)$?

4. A bug walks an integer number of steps on the X-axis starting at $x = 0$. Let right and left directions denote increasing and decreasing values of x respectively. The bug first moves Y steps to the right, where the moment generating function of Y is given by $\frac{1}{3}(e^{-t} + 1 + e^{t})$. It then moves Z steps to the right where the moment generating function of Z is given by $\frac{e^t}{2-e^t}$, where $t < \ln 2$. Assume Y and Z are independent. Let X denote the position of the bug after it has moved Y and Z steps. What is the range of X? What is $P(X = k)$ for k in the range of X?

5. We toss a coin four times. We say that we win if there are no two consecutive Heads in the four tosses, otherwise we lose. Assume that the outcomes of individual tosses are independent.
   (a) If the probability of the coin landing on Tails on a single coin toss is p, then what is the probability of winning?
   (b) An adversary tries to tamper with the coin, but is only partially successful in changing it. Assume that the probability of landing on Tails for this tampered coin is uniformly distributed in the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$. What is the probability of winning?

6.* Bob is very particular about the number of followers he has on a certain online platform. Assume that he currently has 1000 followers. Based on his posts in a given week, the number of his followers might increase by a factor of 1.1, or stay the same or decrease by a factor of 0.9. Each of these events has probability $\frac{1}{3}$. Assume that changes in the number of followers in different weeks are independent.
   (a) Determine the expected number of followers Bob has after 104 weeks (approximately two years).
   (b) What is the approximate probability that Bob has fewer followers after 104 weeks than he has now?

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Name	Range	$P(X=i)$ or $F_X(x)$	f_X	$E(X)$	σ_X	MGF
Indicator $1(p)$	{0,1}	$P(X=1)=p$		p	$\sqrt{pq}$	$1-p+pe^t$
Binomial $Bin(n,p)$	{0,1,...,n}	$\binom{n}{i}p^i(1-p)^{n-i}$		np	$\sqrt{np(1-p)}$	$(1-p+pe^t)^n$
Poisson $Pois(\lambda)$	{0,1,...}	$\frac{\lambda^i}{i!}e^{-\lambda}$		λ	$\sqrt{\lambda}$	$e^{\lambda(e^t-1)}$
Geometric $Geo(p)$	{1,2,...}	$=(1-p)^{i-1}p$		$\frac{1}{p}$	$\frac{\sqrt{1-p}}{p}$	$\frac{pe^t}{1-(1-p)e^t}, e^t < \frac{1}{1-p}$
Uniform $U(a,b)$	(a,b)	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{b-a}{2\sqrt{3}}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Exponential $Exp(\lambda)$	$\mathbf{R}^+$	$1-e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$	$(1-t\lambda^{-1})^{-1}, t<\lambda$
Normal $N(\mu,\sigma^2)$	$\mathbf{R}$	$\Phi(\frac{x-\mu}{\sigma})$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ	$e^{t\mu+\frac{1}{2}\sigma^2t^2}$
Multivariate Normal $N_n(\boldsymbol{\mu},\boldsymbol{\Sigma})$	$\mathbf{R}^n$	$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}det(\boldsymbol{\Sigma})^{\frac{1}{2}}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$		$\boldsymbol{\mu}$	Covariance matrix $\boldsymbol{\Sigma}$	