Probability Theory Lecture 01

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Overview

Birthday Paradox

Will you bet?

Consider the following bet: if two people in this class have their birthday on the same day, then every student has to give me 1 dollar, while if no two students have a birthday on the same day, then I will give the class 1000 dollars.

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Consider the following bet: if two people in this class have their birthday on the same day, then every student has to give me 1 dollar, while if no two students have a birthday on the same day, then I will give the class 1000 dollars.
There are 365 days in a year and 53 students in this class. Do you think its a fair bet? Who do you think will win? How high is the probability that I will win?

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- Suppose we pick a point at random from the square $[-1,1]\times[-1,1]$. What is the probability it will lie in a unit circle centered at the origin?

Basic definitions

Experiment

Eg. Rolling of a dice, tossing a coin, buying potatoes.

Outcome

A possible result of the experiment.

Sample Space Ω (Omega)

The set of all possible outcomes.

Event

A subset of the sample space $A \subset \Omega$.

Examples

Experiment

Example 1: single dice roll, Example 2: weight of potatoes, Example 3: randomly picking a point.

Outcome

Six possible outcomes: $1, 2, 3, 4, 5, 6$, Infinitely many outcomes: any real number in the interval $[0, 2]$, Infinitely many outcomes: any point in the square.

Sample space Ω

$$\{1, 2, 3, 4, 5, 6\}, [0, 2], [-1, 1] \times [-1, 1]$$

Examples

Events

Even dice roll $A = \{2, 4, 6\}$
Odd dice roll $B = \{1, 3, 5\}$
Prime dice roll $C = \{2, 3, 5\}$
Dice roll of 1 $D = \{1\}$
Dice roll of a number bigger than 10 $E = \phi$ (Empty event).
Weight of potatoes is exactly 1 kg $F = \{1\}$
Weight of potatoes is less than 1 kg $G = [0, 1]$

Set operations, Mutually exclusive events

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De Morgan's laws

For any two events A and B, $\overline{A \cup B} = \overline{A} \cap \overline{B}$. etc.

Intuitive definition of probability

It is the fraction of favorable outcomes out of all possible outcomes of an experiment. (Also called classical probability)

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- If I roll the dice 6 times, will the outcome be 1 exactly once?

Relative Frequency

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The relative frequency is the fraction of times we get a favorable result out of the total number of times an experiment is performed.

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$$\frac{\text{number of 1 outcomes}}{\text{number of dice rolls}}$$

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This number converges to our intuitive definition of probability for a large number of experiments. (Central Limit Theorem, Lecture 10)

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When did the above intuitive definition of Probability get formulated?
Intuitive and easy as it may seem, this definition along with the first systematic treatment of probability was published (posthumously) by Cardano in 1663.
Compare this with Pythagorus theorem whose oldest axiomatic proof dates from Euclid's elements from 300 BC!

Two probability problems: revisited

The intuitive definition of Probability is not useful when the Sample space Ω is infinite.

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Definition of Probability

Probability is a measure (a function), $P : \mathcal{F} \mapsto [0,1]$, on a 'proper' collection of subsets (events), $\mathcal{F}$, over the sample space Ω. The following are the Axioms of Probability.

- $P(\Omega) = 1$
- (Sigma additivity) For a collection $A_1, A_2, A_3, \ldots, A_i, \ldots$ of mutually exclusive events, that is, $A_i \cap A_j = \phi$, $\forall i \neq j$,

$$P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$$

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Examples of different measures

Fair coin - Uniform measure

$P(H) = P(T) = \frac{1}{2}$, here our previous intuition of probability, that it is number of favorable outcomes divided by total number of outcomes, holds.

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Unfair coin - Non-uniform measure

$P(H) = \frac{1}{3}, P(T) = \frac{2}{3}$. Note, here also $P(\Omega) = 1$ and sigma additivity holds. But now, probability of an outcome is not number of favorable outcomes divided by total number of outcomes.

Principle of Inclusion-Exclusion (Poincare's formula)

Case of two events

Given two events A_1, A_2 and a probability measure P, the following always holds: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.
Proof: Exercise, use Venn diagram and sigma additivity.

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Proof: Exercise, use Venn diagram and sigma additivity.

Case of three events

Given three events A_1, A_2, A_3 and a probability measure P, the following always holds: $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$.
Proof: Exercise, use Venn diagram and sigma additivity.

Example of Inclusion-Exclusion

Let $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{3}$. What is $P(A \cup B)$?

Example of Inclusion-Exclusion

Let $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(A \cap B) = \frac{1}{3}$. What is $P(A \cup B)$?
So, $P(A \cup B) = \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = \frac{2}{3}$

Union bound or Boole's inequality

Union bound

Given n events $A_1, ..., A_n$ and a probability measure P, the following holds:

$$P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$$

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Proof: Use new events $B_1, \ldots B_n$, where $B_1 = A_1$ and $B_i = A_i \setminus (\cup_{j=1}^{i-1} A_j)$, $\forall i > 1$.

Limit Properties

Property 1

Given events $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$, and a probability measure P, the following holds:

$$P(\cup_i A_i) = \lim_{n\to\infty} P(A_n)$$

Proof

Describe new events $B_1 = A_1$ and $B_i = A_i \setminus A_{i-1}$. Then use sigma additivity for B_i's that $P(\cup_i B_i) = \sum_i P(B_i)$.

Limit Properties

Property 2

Given events $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots$, and a probability measure P, the following holds:

$$P(\cap_i A_i) = \lim_{n\to\infty} P(A_i)$$

Geometric probability

Example 1

Given a point in the plane with both coordinates in the interval $[0,1]$, what is the probability that the point is below the curve $y = x^2$?

Example 2

Given a point in a square of side 1, what is the probability that its distance to the center of the square is less than $\frac{1}{2}$?

Combinatorics

Picking k out of n items.

Experiment	With replacement	Without replacement
Ordered	n^k	$n(n-1)...(n-k+1)$
Unordered	$\binom{n+k-1}{k}$	$\binom{n}{k}$

Table: Some basic ways of counting

Combinatorics

Examples

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You want to randomly distribute 6 identical candies among 3 children. What is the probability the first child will get all 6 candies?

Summary

- Classical probability: fraction of favorable outcomes out of all possible outcomes of an experiment.
- (Sigma additivity) For mutually exclusive A_i's, $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.
- Inclusion-Exclusion: $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$.
- Union bound: For any A_i's, $P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$
- Limit Property: For increasing events A_i, $P(\cup_i A_i) = \lim_{n\to\infty} P(A_n)$

The End