Extra practice problems for the Final.

1. Let X and Y be independent random variables, where $X \sim U(0,1)$ and

$$f_Y : y \mapsto \begin{cases} \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} & \text{if } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is $\mathbb{P}(X^2 < Y < 1)$?

2. Let the following be the joint density function of X and Y for some $a, b \in \mathbb{R}$:

$$f_{X,Y} : (x,y) \mapsto \begin{cases} a(4x+y) + bxy + \frac{2}{5} & \text{if } 0 < x < 1, 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

For what values of a and b are X and Y independent?

3. A machine on the North Pole fills bags with gifts. The width of the bags has standard normal distribution with mean 40cm and standard deviation 1.5cm. (Suppose that the bags are not rotated, their width is well defined.) A bag is chimney-compatible if its width is at most 42cm. What is the probability that from two randomly picked bags, at least one is chimney-compatible?

4. In a TV show, short interviews of guests are aired live one after the other, where each guest is asked to talk for m minutes. Past experience shows that the expected length of the interviews is m, and the standard deviation is 10% of the expected value. What should be m for a group of 10 guests, if the probability of exceeding the total show time of 100 minutes should be around 3%?

5. A lecture is attended by 500 students. Assume that the students attend (or don't attend) the lecture independently of each other. If there is a test from another course on that day, then the probability of showing up is $\frac{1}{6}$, otherwise $\frac{1}{4}$.
(a) What is the probability of at most 100 students showing up on a day when there is a test from another course?
(b) The probability of there being a test from another course is 0.2. If at most 100 students showed up, what is the probability that there was a test?

6. At an insurance company, each insured client has a probability of 0.9 of not incurring any damage during the insured time. In this case the company does not have to pay the client anything. If there is a damage, then the payment due to the client is 20 with probability 0.9 and 200 with probability 0.1. (The probability of damage is independent for each client.) What should be the insurance fee if there are $10,000$ clients and the company wants to make a profit with a probability of 0.99?

7. The following is the joint density function of X and Y:

$$f_{X,Y} : (x,y) \mapsto \begin{cases} \frac{1}{4}\left(1 + xy(x^2+y^2)\right) & \text{if } |x| < 1, |y| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

What is $\text{Cov}(3X, 4Y)$? (Hint: Simplify the work using properties of odd functions)

8. Let (X,Y) have a uniform distribution in the triangle determined by the points $(-1,0)$, $(1,0)$ and $(0,1)$. What is the linear regression of Y in terms of X?

9. Let $X, Y \sim Bin(3, \frac{1}{3})$ be independent. Find the regression function $\mathbb{E}\left((Y-X)^2 \mid X\right)$.

10. Let the joint density function of X and Y be $f_{X,Y}(x,y) = \alpha(xy^2 + xy)$ if $0 < x < 2$ and $0 < y < 1$, and 0 otherwise.
(a) Determine α.
(b) Find the linear regression of $2X + 3Y + 1$ in terms of Y.
(c) Find the regression function $\mathbb{E}(2X + 3Y + 1 \mid Y)$.

11. We toss a coin that lands on heads with probability p. If we obtained a heads, then we roll a fair dice until we get a 6, and if we obtained a tails, we roll the fair dice until we get an even number. Let X denote the number of times the dice was rolled. Given that $\mathbb{E}(5X) = \mathbb{E}(X^2)$, what is p?

12. Let X and Y be random variables for which $\mathbb{E}(X) = 10$, and $Y \sim \text{Exp}(3)$. We further know that $\mathbb{E}(X^2 \mid Y = y) = \frac{1}{\sqrt{y}}$. Determine the variance of X.

13. Let $Y \sim U(0, 1)$ and X be uniformly distributed on the interval $[Y^2, Y]$. What is the probability $\mathbb{P}(X < 0.5)$?

14. Let L denote the total volume of shipments at a delivery service on a given day, and let A be the probability that a certain package scheduled to be delivered that day is delivered on time. Assume that $f_L(x) = \frac{1}{2}\sin(x)$ if $x \in [0, \pi]$ and 0 otherwise. Also, if $L = x$, then the probability that the package arrives on time is $\frac{1}{2}\big(\cos(x) + 1\big)$. What is $\mathbb{P}(A)$?

15. Let $\boldsymbol{X}$ have the standard 2D normal distribution, and

$$\boldsymbol{A} = \begin{bmatrix} 2 & v \\ 0 & 3 \end{bmatrix}$$

for some scalar $v \in \mathbb{R}$. Let $\boldsymbol{Y} = (Y_1, Y_2) = \boldsymbol{A} \cdot \boldsymbol{X}$. If $\rho(Y_1, Y_2) = 0.5$, then what is v? What is the determinant of the covariance matrix of $\boldsymbol{Y}$?

16. Let $\boldsymbol{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ have the following density function

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi} e^{-5(x_1+1)^2+7(x_1+1)(x_2+2)-\frac{5}{2}(x_2+2)^2} \qquad (x_1, x_2 \in \mathbb{R})$$

Find $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

17. Let $X, Y \sim N(0, 1)$ be independent, and let $U = 3X + 2Y$ and $V = 2X - Y$. Find $E(U|V)$.

18. Find $f_{X|Y}(x|y)$ given that $f_{X,Y}(x, y) = \frac{1}{\pi\sqrt{3}} e^{-\frac{2}{3}(x^2 - xy + y^2)}$.

Final Answers.

1. 0.3139
2. $a = \frac{1}{5}, b = \frac{2}{5}$ or $a = -\frac{6}{5}, b = \frac{72}{5}$.
3. 0.9916.
4. $m \approx 9.44$.
5. (a) 0.9773, (b) 0.98.
6. 4.282
7. 1.6
8. $y = \frac{1}{3}$
9. $\frac{5}{3} - 2X + X^2$
10. (a) $\frac{3}{5}$, (b) $3y + \frac{11}{3}$ (c) $3Y + \frac{11}{3}$.
11. $p = 0.1$
12.

13. 0.6130
14. $\frac{1}{2}$
15. 36
16. $\boldsymbol{\mu} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $\boldsymbol{\Sigma} = \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix}$.
17. $y = \frac{4}{5}x$.
18. $\sqrt{\frac{2}{3\pi}} e^{-\frac{2}{3}(x^2 - xy + \frac{y^2}{4})}$