The purpose of the scoring guide is to provide a uniform evaluation of the tests by the graders. Therefore, the guide gives you the main ideas (and at least one possible solution) for each problem and the associated scores. This guide is not intended to be a detailed description of a complete solution to each problem; the steps described are sketches of a solution that achieves maximum score.
The partial scores given in the guide are only awarded to the solver if the related idea is included in the test as a step in a clearly described and reasoned solution. Thus, for example, the mere description of statements or definitions in the material without its application (even if any of the facts described do indeed play a role in the solution) is not worth the point.
A partial score is given for any idea or part solution from which the correct solution of the problem could be obtained with the appropriate addition of the reasoning described in the test. If a solver starts several solutions that are significantly different, then you can give score for at most one. If each described solution or part of solution is correct or can be extended to be correct, then the solution with the most points will be evaluated. However, if there are correct and incorrect solutions between several solution attempts and it is not clear from the test which one the solver considered to be correct, then the solution with fewer points will be evaluated (even if this score is 0). The scores in this guide may be further subdivided as necessary. Of course, a good solution other than that described in this guide is also worth the maximum score.
In the event of an arithmetic error, one point is deducted from each exercise. An exception to this is when computing the numbers significantly simplifies or modifies the structure of the task. In these cases, no points are awarded to the steps which follow the computation and are missing from the solution of the student.

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1. Shrek wants to buy a birthday gift for Fiona. Assume that the amount of time spent deciding what to get for her takes X days, where the distribution of the continuous random variable X has the memoryless property and its range is $[0, \infty)$. The probability that he takes more than 3 days to decide what to buy is $e^{-6} \approx 0.0025$. When he has decided to get a certain gift, then the amount of time taken to procure it takes an additional e^X days. What is the expected amount of days it takes Shrek to get Fiona's gift (including both the decision making time and the time required for procuring it)?
(1 point) $\mathbb{E}(X + e^X) =?$
(2 points) $X \sim \text{Exp}(\lambda)$ for some real number $\lambda > 0$
(1 point) $\mathbb{P}(X > 3) = e^{-6}$
(1 point) $\mathbb{P}(X > 3) = 1 - \mathbb{P}(X < 3)$
(1 points) $= 1 - (1 - e^{-\lambda \cdot 3}) = e^{-\lambda \cdot 3}$
(1 point) So, $-3\lambda = -6$, or, $\lambda = 2$
(1 point) $\mathbb{E}(X) = \frac{1}{\lambda} = \frac{1}{2}$
(1 point) Law of Unconscious Statistician (or Expected value of transform of a random variable):
(2 points) $\mathbb{E}\left(g(X)\right) = \int_{\infty}^{\infty} g(x) f_X(x)\,\mathrm{d}x$
(1 point) $f_X(x) = 2e^{-2x}$
(2 points) $\mathbb{E}(e^X) = \int_0^{\infty} e^x \cdot 2e^{-2x}\,\mathrm{d}x$
(2 points) $= \int_0^{\infty} 2e^{x-2x}\,\mathrm{d}x = 2\int_0^{\infty} e^{-x}\,\mathrm{d}x = 2 \cdot 1 = 2$
(1 point) Using Linearity of Expectation,
(2 point) $\mathbb{E}(X + e^X) = \mathbb{E}(X) + \mathbb{E}(e^X)$
(1 points) $= \frac{1}{2} + 2 = \underline{\underline{2.5}}$

2. The 'This week's 49' news show comprises of the discussion of 49 most important news items of the week. Assume that the amount of time spent on discussing each news item has independent, identical distribution, each with an expected value of 2 minutes. The probability that the total show time is between 98 and 100 minutes is 0.17. Determine the (approximate) standard deviation of the time taken in discussing a single news item. (Assume that there is no time lag between two consecutive news items).
(0 points) Let $X_1, \ldots, X_{49}$ denote the time taken by individual news items.
(1 point) $\sigma_{X_1} =?$
(1 point) $\mathbb{P}(98 < \sum_{i=1}^{49} X_i < 100) = 0.17$
(4 points) Standardizing:
$$\mathbb{P}\left(\frac{98 - 49 \cdot 2}{\sqrt{49} \cdot \sigma_{X_1}} < \frac{\sum_{i=1}^{49} X_i - 49 \cdot 2}{\sqrt{49} \cdot \sigma_{X_1}} < \frac{100 - 49 \cdot 2}{\sqrt{49} \cdot \sigma_{X_1}}\right) = 0.17$$

(2 points) Let $Z = \frac{\sum_{i=1}^{49} X_i - 49\cdot 2}{\sqrt{49}\cdot \sigma_{X_1}}$. Then the above probability:

$$= \mathbb{P}\left(0 < Z < \frac{2}{7\cdot \sigma_{X_1}}\right)$$

(2 points)

$$= \mathbb{P}\left(Z < \frac{2}{7\cdot \sigma_{X_1}}\right) - \mathbb{P}(Z \le 0)$$

(2 points) Using the Central Limit Theorem,
(3 points) Z has approximately standard normal distribution
(1 points) Then by the above equation,

$$0.17 = \Phi\left(\frac{2}{7\sigma_{X_1}}\right) - \Phi(0)$$

(1 point) $\Phi(0) = \frac{1}{2}$
(1 point) So, $\Phi\left(\frac{2}{7\sigma_{X_1}}\right) = 0.67$
(1 point) $\frac{2}{7\sigma_{X_1}} = 0.44$
(1 point) $\sigma_{X_1} = \frac{2}{7\cdot 0.44} = \frac{50}{77} \approx 0.6494$.

3. The joint probability mass function of random variables X and Y is given below:

Y \ X	0	2	5
2	α	0.2	0.1
3	0.2	0.25	β

for some real numbers $\alpha, \beta \in [0,1]$. We know that the events $\{X = 0\}$ and $\{Y = 2\}$ are independent. Determine α and β. Compute the covariance of X and Y.

(2 points) Because of independence, $\mathbb{P}(X = 0, Y = 2) = \mathbb{P}(X = 0)\cdot \mathbb{P}(Y = 2)$
(0 points) where $\mathbb{P}(X = 0, Y = 2) = \alpha$
(2 points) and $\mathbb{P}(X = 0) = \alpha + 0.2$ and $\mathbb{P}(Y = 2) = \alpha + 0.2 + 0.1$
(1 point) So, $\alpha = (\alpha + 0.2)\cdot(\alpha + 0.3)$
(1 point) Solving for α: $\alpha = 0.2$ or $\alpha = 0.3$.
(2 points) Sum of the six probabilities in the table must be 1.
(1 point) So, $\alpha \ne 0.3$, since then the sum will be greater than 1.
(1 point) So, $\alpha = 0.2$ and $\beta = 0.05$.
(2 points) $\mathrm{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\cdot\mathbb{E}(Y)$
(2 points) So:

$$\mathbb{E}(XY) = 0\cdot 2\cdot 0.2 + 2\cdot 2\cdot 0.2 + 5\cdot 2\cdot 0.1 + 0\cdot 3\cdot 0.2 + 2\cdot 3\cdot 0.25 + 5\cdot 3\cdot 0.05$$

(Points are awarded even if the above is written without explicit values of α and β.)
(1 point) $\mathbb{E}(XY) = 4.05$
(1 point) $\mathbb{E}(X) = 0\cdot\mathbb{P}(X = 0) + 2\cdot\mathbb{P}(X = 2) + 5\cdot\mathbb{P}(X = 5)$
(If $\mathbb{E}(X)$ is written only with values as below then also points are awarded.)
(1 point) $= 0 + 2\cdot 0.45 + 5\cdot 0.15 = 1.65$
(1 point) Similarly, $\mathbb{E}(Y) = 2\cdot 0.5 + 3\cdot 0.5$
(1 point) $= 2.5$
(1 point) So, $\mathrm{cov}(X,Y) = 4.05 - 1.65\cdot 2.5 = \underline{\underline{-0.075}}$

4. Let the moment generating function of random variables X and Y be $e^{\frac{t^2}{2}}$ and e^{2t^2} respectively. Let $Z = X + Y$. We further know that X and Y are independent. Find an estimate for $P(Z > a)$ using Chernoff's inequality

(note that the answer is not numerical). For $a = \sqrt{10}$, find the exact probability of $P(Z > a)$ and compare it with the estimate.
(4 points) Chernoff's inequality is $P(X \geq a) \leq \frac{M_X(t)}{e^{at}}, t > 0$
(2 points) $M_Z(t) = M_X(t)M_Y(t)$, since X, Y are independent.
(1 point) $= e^{\frac{5t^2}{2}}$
(1 point) substituting this, $P(X > a) \leq e^{(\frac{5t^2}{2} - at)}, t > 0$
(1 point) To optimize over the value of t, the derivative of RHS must be 0.
(2 points) $e^{(\frac{5t^2}{2} - at)}(5t - a) = 0$, where, $t > 0$
(1 point) or, $t = \frac{a}{5}$ if $a > 0$.
(1 point) Substituting this value, $P(X > a) \leq e^{\frac{-a^2}{10}}$ for $a > 0$.
(1 point) Evaluating this for $a = \sqrt{10}$, we get $P(X > \sqrt{10}) \leq e^{-1} = 0.3679$
(2 points) From MGF of Z we can conclude that $Z \sim N(0, 5)$.
(2 points) So, standardizing, $P(Z > \sqrt{10}) = 1 - \Phi(\frac{\sqrt{10}}{\sqrt{5}})$
(1 point) $= 1 - \Phi(\sqrt{2}) = 1 - 0.9207 = 0.0793$
(1 point) Chernoff's bound gave 0.3679 when the exact value was 0.0793.

5. In a horse riding competition, competitors are each given a random horse and they have to clear 12 hurdles with it. Any one horse clears each of the 12 hurdles with identical probability, and clearing different hurdles are independent events. Depending on the horse, let this probability of clearing a hurdle be denoted by the continuous random variable V.

   (a) Assume that $V = p$ for a given horse. What is the probability, in terms of p, that a competitor with this horse cannot clear at most 1 of the 12 hurdles?

   (b) V has the following distribution function:

$$F_V(x) = \begin{cases} 3x^2 - 2x^3 & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x \leq 0 \\ 1 & \text{otherwise} \end{cases}$$

   Determine the probability that a competitor will not clear at most 1 of the 12 hurdles.

(0 points) let X be the number of hurdles **not** cleared: (where the probability of clearing a hurdle is p)
(1 point) $\mathbb{P}(X \leq 1) =?$
(2 points) $X \sim B(12, 1-p)$
(1 point) $\mathbb{P}(X \leq 1) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1)$
(2 points) $= \binom{12}{0}(1-p)^0 p^{12} + \binom{12}{1}(1-p)^1 p^{11}$
(1 point) $= p^{12} + 12(1-p)p^{11} = 12p^{11} - 11p^{12}$
(0 points) let Y be the number of hurdles **not** cleared, where probability of clearing a hurdle is V, $\mathbb{P}(Y \leq 1) =?$
(4 points) Using Law of Total Probability:

$$\mathbb{P}(Y \leq 1) = \int_{-\infty}^{\infty} \mathbb{P}(Y \leq 1 \mid V = x) f_V(x) \mathrm{d}x$$

(2 points) $\mathbb{P}(Y \leq 1 \mid V = p) = \mathbb{P}(X \leq 1)$
(Full points are awarded if the notation is different but correct.)
(2 points) $f_V = F_V'$
(2 points) $f_V(x) = 6(x - x^2)$ if $0 < x < 1$ and 0 otherwise.
(1 point) Substituting this:

$$\mathbb{P}(Y \leq 1) = \int_0^1 (12x^{11} - 11x^{12}) \cdot 6(x - x^2)\, \mathrm{d}x$$

(2 points)

$$= \int_0^1 72x^{12} - 72x^{13} - 66x^{13} + 66x^{14}\, \mathrm{d}x = \left[\frac{72}{13}x^{13} - \frac{72}{14}x^{14} - \frac{66}{14}x^{14} + \frac{66}{15}x^{15}\right]_0^1 = \frac{37}{455} \approx 0.0813$$

6.* Bob conducts independent identical experiments one after another, each with a $\frac{1}{2}$ probability of success. Let X denote the number of experiments conducted until (and including) the second success. Determine the distribution ($P(X = k)$) and the expected value of X.
(3 points) $\mathbb{P}(X = k) = \mathbb{P}(X > k - 1) - \mathbb{P}(X > k)$
(2 points) $\mathbb{P}(X > k) = \mathbb{P}(\text{at most 1 success in the first k experiments })$
(2 points) $= \binom{k}{0}\frac{1}{2^k} + \binom{k}{1}\frac{1}{2^k} = (k+1)\frac{1}{2^k}$
(1 point) Substituting $k - 1$: $\mathbb{P}(X > k - 1) = k \cdot \frac{1}{2^{k-1}}$
(1 point) So, $\mathbb{P}(X = k) = k \cdot \frac{1}{2^{k-1}} - (k+1)\frac{1}{2^k}$
(2 points) $= (k-1)\frac{1}{2^k}$ for every $k \geq 1$.

1. First solution for finding expected value:
(3 points) $\mathbb{E}(X) = \sum_{k=1}^{\infty} k \cdot \mathbb{P}(X = k)$
(2 points) $= \sum_{k=1}^{\infty} k \cdot (k-1)\frac{1}{2^k}$
(2 points) $= \mathbb{E}(Z^2) - \mathbb{E}(Z)$ where $Z \sim \text{Geo}\left(\frac{1}{2}\right)$
(1 point) $\mathbb{E}(Z^2) = Var(Z) + \mathbb{E}(Z)^2 = 2 + 2^2 = 6$ and $\mathbb{E}(Z) = 2$
(1 point) $\mathbb{E}(X) = 6 - 2 = \underline{\underline{4}}$

2. Second solution for finding expected value:
(3 points) let Y_1 be the number of experiments until (and including) the first success and let Y_2 be the number of experiments conducted after the first success and until the second success (excluding the first success and including the second.)
(2 points) $\mathbb{E}(X) = \mathbb{E}(Y_1 + Y_2) = \mathbb{E}(Y_1) + \mathbb{E}(Y_2)$
(2 points) $Y_1 \sim \text{Geo}\left(\frac{1}{2}\right)$
(1 point) $Y_2 \sim \text{Geo}\left(\frac{1}{2}\right)$
(1 point) So, $\mathbb{E}(X) = \mathbb{E}(Y_1 + Y_2) = 2 + 2 = \underline{\underline{4}}$