Please note: The duration of the exam is 90 minutes. You may use a calculator. Please give a numeric answer (rounded to 4 decimal places). You are expected to **write all steps** taken in getting the final answer along with a **mention of properties/theorems used** in these steps. You are not allowed to leave the examination hall during the first 30 minutes of the exam.

1. Let X and Y be independent random variables, where $X \sim U(0,1)$ and

$$f_Y : y \mapsto \begin{cases} \frac{1}{y} & \text{if } 1 < y < e, \\ 0 & \text{otherwise.} \end{cases}$$

   What is $\mathbb{P}(e^X < Y)$?
   Solution: (5 points for picture and shading the region whose probability youhave to compute). Integral for the required probability is $\int_1^e \int_0^{\ln y} \frac{1}{y} dx dy$ OR $\int_0^1 \int_{e^x}^e \frac{1}{y} dy dx$, evaluates to $\frac{1}{2}$.

2. While driving on the highway, you noticed that on average there was an exit to a restaurant every kilometer. Assume that the distance to the next restaurant has exponential distribution. If you record the distance you travelled until the 100^{th} restaurant, what is the probability that it is greater than 120 kilometers?
   Solution: If X_i is the distance to the i^{th} restaurant from the $(i-1)^{th}$, then $X_i \sim Exp(1)$. Using CLT, $\frac{\sum X_i - n\mu}{\sigma\sqrt{n}} \sim N(0,1)$. Here $n\mu = 100, \sigma\sqrt{n} = 10$. The required probability is $1 - \Phi(2) = 0.0228$.

3. Leia and Han Solo are trying to dodge the imperial spaceships, when Han suggests flying into the asteroid belt. Leia informs him that the number of asteroids X that could hit a spaceship when flying through the asteroid belt has Poisson distribution with an average of 2 hits per minute. Han retorts that the number of hits Y by the fires coming from the imperial squad also has Poisson distribution with an average of 4 hits per minute. Leia replies, correctly, that the imperial squad will follow them into the belt and they'll be hit by both.
   (a) Assuming that Y is unchanged when driving in the asteroid belt and that X and Y are independent, what is the distribution of $Z = X + Y$?
   (b) C3PO at this point cannot contain itself and computes $p_{X|Z}$ and $\mathbb{E}(X|Z)$. What are they? (Note: none of the answers are numeric.)

   Solution: $X \sim Pois(2)$, $Y \sim Pois(4)$. We require the probability mass function of the convolution, $P(X+Y = z) = \sum_y p_X(z-y)p_Y(y) = \sum_{0 \le y \le z} \frac{2^{z-y}}{(z-y)!} e^{-2} \frac{4^y}{y!} e^{-4}$. Solving we see this is $Pois(6)$. (b) $X|Z \sim Bin(z, \frac{1}{3})$ and so $E(X|Z) = \frac{z}{3}$.

4. Four friends are playing bridge. In bridge, a standard deck of cards is evenly distributed (uniformly and randomly) to the four players (so, each player gets 13 cards). Use the following steps to compute the probability that each player has exactly one Ace.
   (a) Let A be the event that the Ace of hearts is not with the person with Ace of spades. What is $\mathbb{P}(A)$?
   (b) Let B be the event that the Ace of diamonds is not with the person who has Ace of spades or hearts. What is $\mathbb{P}(B|A)$?
   (c) Define event C which follows the sequence of definitions of A, B and compute $\mathbb{P}(C|AB)$. Use these steps to find the probability that each person has exactly one Ace.
   (d) If the friends play 10 games of bridge, what is the expected number of games in which all of them have an Ace?

   Solution: $P(A) = \frac{39}{51}, P(B|A) = \frac{26}{50}, P(C|AB) = \frac{13}{49}$. Multiplication rule for probability of each person having an Ace gives the product of results of (a),(b) and (c), giving 0.1055. For (d), the distribution is $Bin(10, 0.1055)$, so expected value is 1.055.

5. A coin, with probability p of turning up heads, is tossed until the first time heads appears. Let X be the number of tosses required and let Y be 1 if the first toss is heads, 0 otherwise.
   (a) What is $\mathbb{E}(X^2|Y = 1)$? What is $\mathbb{E}(X)$?
   (b) Write $\mathbb{E}(X^2|Y = 0)$ in terms of $\mathbb{E}(X^2)$ (and some additional terms). (Hint: think of how many tosses would you need if the first toss was Tails).
   (c) Use the result in the previous parts to find a recurrence for $\mathbb{E}(X^2)$. Use this to find $Var(X)$. State the theorem you are using. (No points for using the formula sheet to find $Var(X)$!)

(Note: none of the answers are numeric)
Solution: $1, \frac{1}{p}, \mathbb{E}(X^2|Y=0) = E((X+1)^2) = E(X^2) + \frac{2}{p} + 1$. Using Law of Total Expectation, $E(X^2) = E((X+1)^2)(1-p) + p, \mathbb{E}(X^2) = \frac{2-p}{p^2}, Var(X) = \frac{1-p}{p^2}$.

6. A disease control division specializing in mosquito related diseases, analysed data near a pond. Denote the temperature on a certain day in June by X (recorded in Celsius), and the population of mosquitoes that day by Y (recorded in millions). They found that X and Y could be modelled with a bivariate (2D) normal distribution. The division further found that $y = \frac{x}{2} - 18$ was the regression of Y in terms of X and $x = \frac{y}{2} + 39$ was the regression of X in terms of Y.
   (a) Find $\rho(X,Y), \mu_X$ and μ_Y.
   (b) If it is known that $\sigma_X = 1$, what can you say about σ_Y?
   (c) What is the joint density function of (X,Y)?
   (d) On a particular day, if the temperature was 46 Celsius, what is expected population of mosquitoes that day?

   Solution: $\rho = \frac{1}{2}, \mu_X = 40, \mu_Y = 2, \sigma_X = \sigma_Y = 1$. Joint density function is $\frac{1}{\pi\sqrt{3}} e^{-\frac{2}{3}(x^2 - 80x + y^2 - 4y - xy + 1520)}$, (d) 5 million.