Probability Theory Lecture 05

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Overview

1. Two related distributions
   - Poisson distribution
   - Exponential distribution
2. Memoryless Property
3. Poisson Approximation

Introduction

Example

Suppose at a call center, on an average 5 calls arrive every 10 minutes. What is the probability that there will be 10 calls in the next 10 minutes?
If we assume that occurance of one call is independent of another, i.e., that occurance of one call does not affect the probability of another, then such a problem can be modelled with **Poisson distribution**.

Introduction

History

First introduced by French mathematician, Siméon Denis Poisson in his book, *Recherches sur la probabilité des jugements en matière criminelle et en matière civile (Investigations into the Probability of Verdicts in Criminal and Civil matters)*.
He is quoted to have said: "Life is good for only two things: doing mathematics and teaching it." He is one of the 72 names inscribed on the Eiffel tower. Sources: Wikipedia and book of Sheldon Ross.

Poisson distribution

Definition

A random variable X that takes on values $0, 1, 2, \ldots$ has Poisson distribution $Pois(\lambda)$, with parameter $\lambda > 0$ (read Lambda), if

$$P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}, i = 0, 1, 2, \ldots$$

Where λ denotes the average number of occurances in a given time interval, while in $P(X = i)$, we are interested in i number of occurrances in the **same** time interval.

Validity

To see that it is a valid probability distribution, we must check $\sum_{i=0}^{\infty} P(X = i) = 1$.

Poisson distribution

Expected Value

$E(X) = \sum_{i=0}^{\infty} iP(X = i) =$

Poisson distribution

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Variance

$Var(X) = \sum_{i=0}^{\infty} i^2 P(X = i) - \lambda^2 =$

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Examples

Example cont.

Probability of 10 calls in 10 minutes is

Examples

Example cont.

Probability of 10 calls in 10 minutes is $e^{-5}\frac{5^{10}}{10!}$.

What is the probability of 5 calls in the next 5 minutes?

Examples

Example cont.

Probability of 10 calls in 10 minutes is $e^{-5}\frac{5^{10}}{10!}$.
What is the probability of 5 calls in the next 5 minutes? $e^{-2.5}\frac{2.5^5}{5!}$.

Examples

Example 7a from book

Suppose that the number of typographical errors on a single page of this book has a Poisson distribution with parameter $\lambda = \frac{1}{2}$. Calculate the probability that there is at least one error on this page.

Example 7c from book

Consider an experiment that consists of counting the number of α particles given off in a 1-second interval by 1 gram of radioactive material. If we know from past experience that, on the average, 3.2 such α particles are given off, what is a good approximation to the probability that no more than 2 α particles will appear?

Exponential distribution

Exponential as dual of Poisson distribution

The probability distribution of the **time** between two consecutive events in a Poisson process has exponential distribution. So, events occur independently and continuously at a constant average rate.

Example

In the call center example, if we are interested in the time before the next call arrives, it can be modelled with Exponential distribution.

Exponential distribution

Definition

An exponential random variable $Exp(\lambda)$ is a continuous random variable with the probability density function given by,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for some $\lambda > 0$. Here λ denotes the average number of occurances in unit time.

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CDF

The cumulative distribution function for the Exponential distribution is given by

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}.$$

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Variance

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Call center example

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What is the probability that the next call will arrive in 1 minute? Since our unit of time is minutes, we must find the average number of calls in one minute. This is $\frac{1}{2}$. So $\lambda = \frac{1}{2}$. Then, what we are interested in is $P(X \leq 1) = F_X(1) = 1 - e^{-\frac{1}{2}}$.
What is the probability no call will arrive in the next one minute?

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What is the probability no call will arrive in the next one minute?
$e^{-\frac{1}{2}}$.

Examples cont.

Example 5b from book

Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait (a) more than 10 minutes; (b) between 10 and 20 minutes.
(a) We are interested in $P(X \geq 10) =$

Examples cont.

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(a) We are interested in $P(X \geq 10) = 1 - F_X(10) = e^{-1}$.

(b) Here we want $P(10 \leq X \leq 20) =$

Examples cont.

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Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait (a) more than 10 minutes; (b) between 10 and 20 minutes.
(a) We are interested in $P(X \geq 10) = 1 - F_X(10) = e^{-1}$.
(b) Here we want $P(10 \leq X \leq 20) =$
$F_X(20) - F_X(10) = e^{-1} - e^{-2}$.

Introduction

Example

Suppose I have the 13 spades shuffled and I draw one card at a time without replacement. Let X denote the number of cards drawn until I draw the Ace of Spades. What can you say about $P(X > 10|X > 5)$? Should it be same as $P(X > 5)$ or different?

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Introduction

Definition of Memoryless Property

A random variable X (continuous or discrete) is said to have memoryless property if the following is true:

$$P(X > t + s | X > s) = P(X > t)$$

Examples

Geometric distribution

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What is $P(X > i)$?

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Geometric distribution

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This will occur if the first i tosses were Tails. The probability of this is $(1-p)^i$.
Then, $P(X > t+s | X > s) = \frac{P((X>t+s) \cap (X>s))}{P(X>s)} = \frac{P(X>t+s)}{P(X>s)} = \frac{(1-p)^{t+s}}{(1-p)^s} = (1-p)^t = P(X > t)$

Examples cont.

Exponential distribution

Let X be a random variable with distribution $Exp(\lambda)$.
What is $P(X > t)$?

Examples cont.

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Let X be a random variable with distribution $Exp(\lambda)$.
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Memoryless distributions

Are there any other distributions with memoryless property?

Memoryless distributions

Are there any other distributions with memoryless property?
Let us assume X is discrete and has memoryless property. Then,
$P(X > t + s | X > s) = P(X > t) \implies \frac{P(X > t+s)}{P(X > s)} = P(X > t)$
$\implies P(X > t + s) = P(X > s)P(X > t)$.
Let us set $P(X = 1) = p$, then $P(X > 1) = (1 - p)$. Using the equation above, $P(X > i) = (1 - p)^i$ and so
$P(X = i) = P(X > i - 1) - P(X > i) = (1 - p)^{i-1}p$.
Similar proof can be used to show that any continuous distribution that has the memoryless property is necessarily Exponential (upto a translation).

Memoryless distributions

Memoryless Distributions

The distribution of a discrete (continuous) RV is memoryless if and only if it is Geometric (Exponential).

Poisson Approximation of Binomial Distribution

Let X have $Bin(n, p)$ distribution where n is large and the parameter p is small, so that $\lambda = np$ is moderate. Then,

$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i} = \frac{\binom{n}{p}}{n^i} \lambda^i (1 - \frac{\lambda}{n})^{n-i}$

If we consider $np_n \to \lambda$ as $n \to \infty$, then the above tends to $e^{-\lambda} \frac{\lambda^i}{i!}$.

Poisson Approximation

Given $Bin(n, p)$, where $np = \lambda$ can be considered as a constant average while n and p are respectively large and quite small, we can approximate this with $Pois(\lambda)$.

Examples

Example 1

Number of people who survive to the age of 100. Assume population of Budapest to be 1 million and that the probability of a person to survive to the age of 100 is 10^{-5}. What is the probability that at a given time there are exactly 5 people of age greater than or equal to 100?

Example 2

Number of misprints in a page: Suppose there are 500 words on an average on a page of a book and each word has a probability of 10^{-3} of being misspelt. What is the probability that a given page has at most 2 misspellings?

The End