Please note: The duration of the exam is **100** minutes. You may use a calculator. Please give a numeric answer (rounded to 4 decimal places). You are expected to **write all steps** taken in getting the final answer along with a **mention of properties/theorems used** in these steps. You are not allowed to leave the examination hall during the first 30 minutes of the exam.

1. Troll-tree is under threat from Chef again, who has planted a troll catching device under the tree. Of the trolls that the device catches, Chef finds only those trolls delectable whose height is at most 2cm. You know that the heights of the trolls has probability density function $\frac{x}{4}$, if $1 < x < 3$ and 0 otherwise (where x is measured in cm). Assume that the device can catch one troll at a time and heights of trolls caught on different occasions are totally independent. Let A, B, C be the events that Chef finds the first, second and the third troll caught by the device delectable. Determine $\mathbb{P}(A \cup B \cup C)$.

2. Let $c > 0$ be a real number, and let X be a random variable with the following distribution function:

$$F_X : x \mapsto \begin{cases} 0 & \text{if } x \leq c, \\ \ln(x) - \ln(c) & \text{if } c < x \leq ce, \\ 1 & \text{if } ce < x, \end{cases}$$

   where e is Euler's constant, ≈ 2.718.
   (a) Given that $\mathbb{E}(X) = e - 1$, determine c.
   (b) Determine the variance of X.

3. Barny works at the IT hardware support division of a firm where he is responsible for resolving exactly 18 problems every day. A given problem is said to be 'relevant' if the restart of the machine does not resolve the problem. Let all problems, independent of each other, have a probability of $\frac{1}{3}$ of being relevant.
   (a) What is the distribution of the number of relevant problems resolved during a workday of Barny?
   (b) Approximately what is the probability that there are more than 580 and less than 620 relavant problems in a given period of 100 days? (Assume that problems arising on different days are also independent.)

4. Let $X \sim \text{Exp}(1)$ be a random variable. After the value of X is determined, we pick a number Y uniformly from the interval $[0, e^X]$.
   (a) Determine $\mathbb{P}(Y < X \mid X = x)$.
   (b) What is $\mathbb{P}(Y < X)$?

5. Let $(X, Y) \sim N(\mathbf{0}, \mathbf{\Sigma})$ be a random variable vector with a bivariate (2D) normal distribution, such that,

$$\mathbf{\Sigma} = \begin{bmatrix} 9 & 1 \\ 1 & 4 \end{bmatrix}$$

   (a) For what value of $c > 0$ are the random variables $X + cY$ and $X - 2cY$ uncorrelated?
   Use the value of c obtained in part (a) for the following two parts.
   (b) Are $X + cY$ and $X - 2cY$ independent?
   (c) Determine $\mathbb{P}(X + cY > 0, X - 2cY < 0)$.

6.* Let $X, Y \sim Exp(1)$ be independent random variables. Further let $Z = X + Y$. Determine $\mathbb{E}(X + 2Y \mid Z)$.

Name	Range	$P(X=i)$ or $F_X(x)$	f_X	$E(X)$	σ_X
Indicator $1(p)$	{0,1}	$P(X=1)=p$		p	$\sqrt{pq}$
Binomial $Bin(n,p)$	{0,1,...,n}	$\binom{n}{i}p^i(1-p)^{n-i}$		np	$\sqrt{np(1-p)}$
Poisson $Pois(\lambda)$	{0,1,...}	$\frac{\lambda^i}{i!}e^{-\lambda}$		λ	$\sqrt{\lambda}$
Geometric $Geo(p)$	{1,2,...}	$=(1-p)^{i-1}p$		$\frac{1}{p}$	$\frac{\sqrt{1-p}}{p}$
Uniform $U(a,b)$	(a,b)	$\frac{x-a}{b-a}$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{b-a}{2\sqrt{3}}$
Exponential $Exp(\lambda)$	$\mathbf{R}^+$	$1-e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda}$
Normal $N(\mu,\sigma^2)$	$\mathbf{R}$	$\Phi(\frac{x-\mu}{\sigma})$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ
Multivariate Normal $N(\boldsymbol{\mu},\boldsymbol{\Sigma})$	$\mathbf{R}^n$	$f_{\boldsymbol{X}}(\boldsymbol{x})=\frac{1}{(2\pi)^{\frac{n}{2}}det(\boldsymbol{\Sigma})^{\frac{1}{2}}}e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$		$\boldsymbol{\mu}$	Covariance matrix $\boldsymbol{\Sigma}$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998