The purpose of the scoring guide is to provide a uniform evaluation of the tests by the graders. Therefore, the guide gives you the main ideas (and at least one possible solution) for each problem and the associated scores. This guide is not intended to be a detailed description of a complete solution to each problem; the steps described are sketches of a solution that achieves maximum score.
The partial scores given in the guide are only awarded to the solver if the related idea is included in the test as a step in a clearly described and reasoned solution. Thus, for example, the mere description of statements or definitions in the material without its application (even if any of the facts described do indeed play a role in the solution) is not worth the point.
A partial score is given for any idea or part solution from which the correct solution of the problem could be obtained with the appropriate addition of the reasoning described in the test. If a solver starts several solutions that are significantly different, then you can give score for at most one. If each described solution or part of solution is correct or can be extended to be correct, then the solution with the most points will be evaluated. However, if there are correct and incorrect solutions between several solution attempts and it is not clear from the test which one the solver considered to be correct, then the solution with fewer points will be evaluated (even if this score is 0). The scores in this guide may be further subdivided as necessary. Of course, a good solution other than that described in this guide is also worth the maximum score.
In the event of an arithmetic error, one point is deducted from each exercise. An exception to this is when computing the numbers significantly simplifies or modifies the structure of the task. In these cases, no points are awarded to the steps which follow the computation and are missing from the solution of the student.

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1. An exit to a highway has toll booths with three lanes for cars to pass through. Assume that the number of cars that go through the toll booth in lane $i$ in one minute has $Pois(i)$ distribution, where $i = 1, 2, 3$. A person distributing pamphlets randomly picks a lane and offers pamphlets to all cars passing through that lane. Assume that all cars that are offered a pamphlet take it. If we know that he gave 2 pamphlets in one minute, what is the probability that he picked lane $i = 1$? How about $i = 2$ and $i = 3$?
(1 point) Let $A_i, i = 1, 2, 3$ be the events of picking lane $1, 2, 3$
(1 point) Let $B$ be the event of distributing exactly 2 pamphlets in a minute
(1 points) $P(A_i) = \frac{1}{3}, i = 1, 2, 3$ since the lane is uniformly randomly picked.
(3 points) $P(B|A_1) = e^{-1}\frac{1^2}{2}$, $P(B|A_2) = e^{-2}\frac{2^2}{2}$, $P(B|A_3) = e^{-3}\frac{3^2}{2}$
(1 points) Since $A_1, A_2, A_3$ form a partition of the sample space,
(1 points) Using Bayes Theorem,
(4 points) $P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1)+P(B|A_2)P(A_2)+P(B|A_3)P(A_3)}$
(3 points) substituting, $P(A_1|B) =$ (here the points are awarded for substituting the correct values)
(1 point) $= 0.271$
(2 points) Similarly, $P(A_2|B) = 0.3988$
(2 points) and $P(A_3|B) = 0.3301$

2. A certain professional fast-chess player plays 100 online chess games every day. For each game, he has an 80% chance of winning, 10% chance of losing and 10% chance of the game ending in a draw. In the month of December, there is a grand tournament in which there are cash prizes for those who win 2040 games or more by 25th of December (Christmas day). If he plays every day in December until (and including) Christmas day, estimate the approximate probability that he will win a cash prize.
(2 points) Total number of games played $n = 100 * 25 = 2500$.
(1 points) Probability of winning a single game $p = 0.8$
(1 points) Let $X$ be the number of games won.
(2 points) Then $X \sim Bin(2500, 0.8)$
(1 points) And we want to find $P(X \geq 2040)$
(2 points) We can approximate it using the De-Moivre Laplace theorem (points are also awarded if it is instead stated as the Central Limit Theorem).
(3 points) where $\frac{X-np}{\sqrt{np(1-p)}} \sim N(0, 1)$
(1 points) $np = 2000$

(2 points) and $\sqrt{np(1-p)} = 20$
(3 points) so, $P(X \geq 2040) = P(\frac{X-2000}{20} \geq \frac{2040-2000}{20})$
(1 points) $= 1 - \Phi(2)$
(1 points) $= 0.0227$

3. A virtual night sky simulator generates shooting stars at random intervals, where the time in minutes for the next shooting star to appear, $Y$, has $Exp(X)$ distribution, where

$$f_X = \begin{cases} bx & 1 < x < a \\ 0 & \text{otherwise} \end{cases}$$

We know that on an average we have to wait $\frac{1}{3}$ of a minute for the next shooting star to appear. Find $a, b$ and the probability that the next shooting star will not appear for a minute.
(1 points) Since $f_X$ is a density function, $\int_{-\infty}^{\infty} f_X(x)dx = 1$
(1 points) or, $\int_1^a bxdx = 1$
(1 points) so, $\frac{b}{2}(a^2 - 1) = 1$ or, $b(a-1)(a+1) = 2$ – Equation 1.
(1 points) We are given $E(Y) = \frac{1}{3}$.
(1 points) We know $Y \sim Exp(X)$, so $E(Y|X = x) = \frac{1}{x}$
(1 points) Using Law of Total Expectation, $E(Y) = E(E(Y|X))$
(2 points) $= \int_{-\infty}^{\infty} E(Y|X = x)f_X(x)dx$
(1 points) $= \int_1^a \frac{1}{x} bxdx = b(a-1)$
(1 points) So, $b(a-1) = \frac{1}{3}$ – Equation 2
(1 points) Substituting this in Equation 1, $a + 1 = 6$ or, $a = 5$.
(1 points) So, $b = \frac{1}{12}$.
( points) Let $A$ be the event the next shooting star will not appear for a minute.
(1 points) $P(A|X = x) = e^{-x}$ be the event the next shooting star will not appear for a minute.
(1 points) Using Law of Total Probability,
(2 points) $P(A) = \int_{-\infty}^{\infty} P(A|X = x)f_X(x)dx$
(1 points) $= \int_1^5 e^{-x} \frac{x}{12} dx$
(2 points) using Integration by parts, $= [-\frac{x}{12}e^{-x} - \frac{1}{12}e^{-x}]_1^5$
(1 points) $= 0.05794$

4. Alice needs to use a standard normal distribution in her project and her two friends, Betty and Claude write a computer program each to simulate it. Lets call Betty's standard normal distribution $X$, and Claude's, $Y$. To not disappoint any of her friends, Alice writes a code that simulates a coin toss (probability of Heads = $p$), and each time the coin lands on Heads, her program takes the value from Betty's code for $X$ and outputs it, and if Tails, then it takes the value from Claude's code for $Y$. Let the random variable generated by Alice's program be $Z$. What is the distribution of $Z$? What is $E(XZ)$? Are $X$ and $Z$ independent? (Note: None of the answers are numerical).
(1 points) $F_Z(z) = P(Z \leq z)$
(1 points) Let $A$ be the event that the coin toss landed in Heads.
(1 points) Then $A$ and $\overline{A}$ form a partition of the sample space.
(1 points) Using Law of Total probability,
(3 points) $P(Z \leq z) = P(Z \leq z|A)P(A) + P(Z \leq z|\overline{A})P(\overline{A})$
(1 points) $pP(X \leq z) + (1-p)P(Y \leq z)$
(1 points) $p\Phi(z) + (1-p)\Phi(z) = \Phi(z)$
(1 points) so, $Z \sim N(0, 1)$.
(1 points) Using Law of Total Expectation,
(3 points) $E(XZ) = pE(X^2) + (1-p)E(XY)$
(3 points) $= p$ since $E(X^2) = 1$ and $E(XY) = E(X)E(Y) = 0$
(2 points) Since $Cov(X, Z) = E(XZ) - E(X)E(Z) = p \neq 0$, (alternately one could say since $E(XZ) \neq E(X)E(Z) = 0$),
(1 points) $X$ and $Z$ are not independent. (points are not awarded if the student writes this sentence without any reason supporting their claim).

5. Let $X, Y \sim Exp(1)$ be independent random variables and let $Z = X + Y$. Use Markov's, Chebyshev's and Chernoff's inequalities to find bounds for $P(Z \geq 5)$.
(1 points) $E(Z) = 1 + 1 = 2$
(1 points) $Var(Z) = Var(X) + Var(Y)$ since they are independent
(1 points) $= 2$
(2 points) Markov's inequality is $P(X \geq a) \leq \frac{E(X)}{a}$
(1 points) this gives us $P(Z \geq 5) \leq \frac{2}{5} = 0.4$
(2 points) Chebyshev's inequality is $P(|X - \mu_X| \geq a) \leq \frac{Var(X)}{a^2}$
(1 points) $P(Z \geq 5) \leq P(|Z - 2| \geq 3)$
(1 points) and using Chebyshev's inequality, $\leq \frac{2}{9} = 0.2222$
(2 points) Chernoff's inequality is $P(X \geq a) \leq \frac{M_X(t)}{e^{at}}, t > 0$
(1 points) $M_Z(t) = M_X(t)M_Y(t)$, since $X, Y$ are independent.
(1 points) $= \frac{1}{(1-t)^2}, t < 1$
(1 points) substituting this, $P(X \geq 5) \leq \frac{e^{-5t}}{(1-t)^2}, 0 < t < 1$
(1 points) To optimize over the value of $t$, the derivative of RHS must be 0.
(1 points) So, $\frac{-5e^{-5t}}{(1-t)^2} + \frac{2e^{-5t}}{(1-t)^3} = 0$ where $0 < t < 1$
(1 points) or, $-5(1-t) + 2 = 0$ where $0 < t < 1$
(1 points) so, $t = \frac{3}{5}$
(1 points) Substituting this value, $P(X \geq 5) \leq \frac{25e^{-3}}{4} = 0.3112$

6.* Let $Y$ be the number of times a fair coin is tossed until (and including) the first time it lands on Heads. Let $X$ be a random variable that is 1 if the number of tails is even, and is 0 otherwise. Write the joint probability mass function table of $X$ and $Y$ (for the first 6 values in the range of $Y$). What is the conditional probability mass function $p(Y|X = 1)$? Use this table to recognize $(Y|X = 1)$ as a linear transform of a standard random variable (from the formula sheet). Determine $E(Y|X = 1)$ and $Cov(X, Y)$.

(5 points)

| $X \backslash Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 0 | 0 | $\frac{1}{4}$ | 0 | $\frac{1}{16}$ | 0 | $\frac{1}{64}$ |
| 1 | $\frac{1}{2}$ | 0 | $\frac{1}{8}$ | 0 | $\frac{1}{32}$ | 0 |

(1 points) $P(X = 0) = \frac{1}{4} + \frac{1}{16} + ... = \sum_{i=1}^{\infty} \frac{1}{4^i} = \frac{1}{4}\frac{1}{1-\frac{1}{4}} = \frac{1}{3}$
(1 points) Then $P(X = 1) = 1 - P(X = 0) = \frac{2}{3}$.
(If $P(X = 1)$ is directly found then also points are awarded.
(2 points)

| $Y$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $p_{Y\|X}(y\|1)$ | $\frac{3}{4}$ | 0 | $\frac{3}{16}$ | 0 | $\frac{3}{64}$ | 0 |

(1 points) We can see that $p_{Y|X}(y|1) = \frac{3}{4}(\frac{1}{4})^{y-1}$, so, if $Z \sim Geo(\frac{3}{4})$, then $Y|X = 1$ has the same distribution as $2Z - 1$.
(1 points) $E(Y|X = 1) = E(2Z - 1) = \frac{2}{\frac{3}{4}} - 1 = \frac{5}{3}$
(1 points) Using Law of Total Expectation,
(2 points) $E(XY) = E(E(XY|X))$
(2 points) $= \frac{2}{3}E(XY|X = 1) + \frac{1}{3}E(XY|X = 0) = \frac{10}{9}$
(2 points) $E(Y) = 2$, $E(X) = \frac{2}{3}$.
(1 points) Since, $Cov(X, Y) = E(XY) - E(X)E(Y)$,
(1 points) so, $Cov(X, Y) = \frac{10}{9} - \frac{4}{3} = \frac{-2}{9}$.
$E(Y|X = 1)$ and $E(XY)$ can be found by simply summing the terms in the table without using Law of Total Expectation. Then also full points are awarded.