Chapter 4.2 Area

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January 2, 2023

Objectives

- Use sigma notation to write and evaluate a sum
- Understand the concept of area
- Approximate the area of a plane region
- Find the area of a plane region using limits

Definition 4.4 – Sigma Notation The sum of n terms $a_1, a_2, a_3, ..., a_n$ is written as:

$$\Sigma_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$$

Where i is the **index of summation**, a_i is the **ith term** of the sum, and the **upper and lower bounds of summation** are n and 1.

Problem 4.10 – Examples of Sigma Notation The upper and lower bounds must be constant with respect to the index of summation. However, the lower bound doesn't have to be 1. Any integer less than or equal to the upper bound is legitimate.

- $\Sigma_{i=1}^{6}\ i$
- $\Sigma_{i=0}^{5}\ (i+1)$
- $\Sigma_{j=3}^{7}\ j^2$
- $\Sigma_{j=1}^{5}\ \frac{1}{\sqrt{j}}$
- $\Sigma_{k=1}^{n}\ \frac{1}{n}(k^2+1)$
- $\Sigma_{i=1}^{n}\ f(x_i)\Delta x$

Theorem 4.2 – Summation formulas Provide proofs.

- $\Sigma_{i=1}^{n} c = cn$, c is a constant
- $\Sigma_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\Sigma_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\Sigma_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Problem 4.11 – Evaluating a Sum Evaluate $\Sigma_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000, 10,000$.

Upper and Lower Sums

The procedure used in Example 3 can be generalized as follows. Consider a plane region bounded above by the graph of a nonnegative, continuous function

$$y = f(x)$$

as shown in Figure 4.9. The region is bounded below by the x-axis, and the left and right boundaries of the region are the vertical lines $x = a$ and $x = b$.

To approximate the area of the region, begin by subdividing the interval $[a, b]$ into n subintervals, each of width

$$\Delta x = \frac{b-a}{n}$$

as shown in Figure 4.10. The endpoints of the intervals are

$$\overbrace{a + 0(\Delta x)}^{a = x_0} < \overbrace{a + 1(\Delta x)}^{x_1} < \overbrace{a + 2(\Delta x)}^{x_2} < \cdots < \overbrace{a + n(\Delta x)}^{x_n = b}.$$

Because f is continuous, the Extreme Value Theorem guarantees the existence of a minimum and a maximum value of $f(x)$ in *each* subinterval.

$f(m_i)$ = Minimum value of $f(x)$ in ith subinterval
$f(M_i)$ = Maximum value of $f(x)$ in ith subinterval

Next, define an **inscribed rectangle** lying *inside* the ith subregion and a **circumscribed rectangle** extending *outside* the ith subregion. The height of the ith inscribed rectangle is $f(m_i)$ and the height of the ith circumscribed rectangle is $f(M_i)$. For *each i*, the area of the inscribed rectangle is less than or equal to the area of the circumscribed rectangle.

$$\begin{pmatrix}\text{Area of inscribed}\\ \text{rectangle}\end{pmatrix} = f(m_i)\,\Delta x \le f(M_i)\,\Delta x = \begin{pmatrix}\text{Area of circumscribed}\\ \text{rectangle}\end{pmatrix}$$

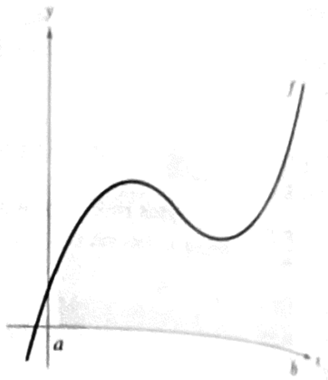
The sum of the areas of the inscribed rectangles is called a **lower sum,** and the sum of the areas of the circumscribed rectangles is called an **upper sum.**

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x \qquad \text{Area of inscribed rectangles}$$

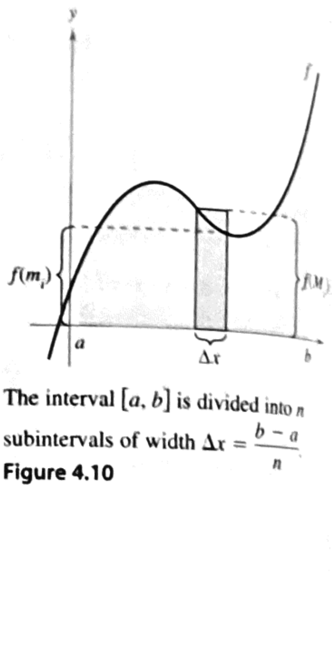
$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x \qquad \text{Area of circumscribed rectangles}$$

From Figure 4.11, you can see that the lower sum $s(n)$ is less than or equal to the upper sum $S(n)$. Moreover, the actual area of the region lies between these two sums.

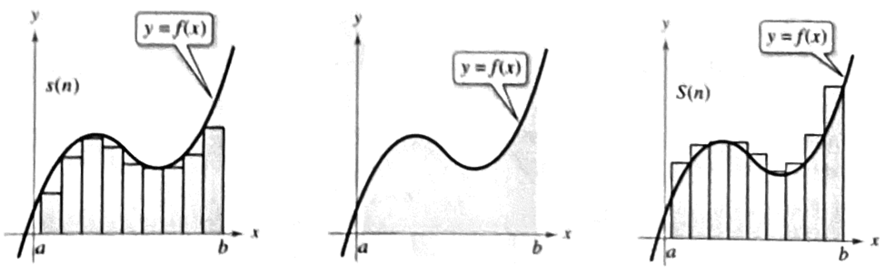
$$s(n) \le (\text{Area of region}) \le S(n)$$



The region under a curve
Figure 4.9



The interval $[a, b]$ is divided into n subintervals of width $\Delta x = \frac{b-a}{n}$
Figure 4.10



Area of inscribed rectangles is less than area of region.

Area of region

Area of circumscribed rectangles is greater than area of region.

Figure 4.11

Finding Upper and Lower Sums for a Region Notes

Take notes from pg. 294 from textbook.

Problem 4.12 – Approximating the Area of a Plane Region Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x-axis between x = 0 and x = 2.

Problem 4.13 – Finding Upper and Lower Sums for a Region Find the upper and lower sums for the region bounded by the graph of $f(x) = x^2$ and the x=axis between x = 0 and x = 2.

Theorem 4.3 – Limits of the Lower and Upper Sums Let f be continuous and nonnegative on the interval [a,b]. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other.

That is, $\lim_{n\to\infty} s(n) = \lim_{n\to\infty} \Sigma_{i=1}^{n} f(m_i \Delta x)$
$= \lim_{n\to\infty} \Sigma_{i=1}^{n} f(M_i \Delta x)$
$= \lim_{n\to\infty} S(n)$

where $\Delta x = (b-a)/n$ and $f(m_i) and f(M_i)$ are the minimum and maximum values of f on the subinterval.

Definition 4.5 – Definition of the Area of a Region in the Plane Let f be continuous and nonnegative on the interval [a,b]. The area of the region bounded by the graph of f, the x-axis, and the vertical lines x = a and x = b is

$Area = \lim_{n\to\infty} \Sigma_{i=1}^{n} f(c_i \Delta x)$
where, $x_{i-1} <= c_i <= x_i$ and
$\Delta x = \frac{b-a}{n}$

Problem 4.14 – Finding Area by the Limit Definition Find the area of the region bounded by the graph $f(x) = x^3$, the x-axis, and the vertical lines $x = 0$ and $x = 1$.

Problem 4.15 – Finding Area by the Limit Definition Find the area of the region bounded by the graph $f(x) = 4 - x^2$, the x-axis, and the vertical lines $x = 1$ and $x = 2$.

Problem 4.16 – A Region Bounded by the y-axis Find the area of the region bounded by the graph $f(y) = y^2$, the x-axis, and the vertical lines $y = 0$ and $y = 1$.

Problem 4.17 – Approximating Area with the Midpoint Rule Use the Midpoint Rule with n = 4 to approximate the area of the region bounded by the graph of $f(x) = sin(x)$ and the x-axis for $0 <= x <= \pi$.