Tyler Nevell

101039497

tyler.nevell@colorado.edu

Problem 1

1A. For all n natural numbers, $8^n - 3^n$ is divisible by 5.

$Proof$: by weak induction

Base Case: Let $n = 0$. $8^0 - 3^0 = 1 - 1 = 0$. 0 is divisible by 5. Base case holds.

Induction Hypothesis: If $8^n - 3^n$ is divisible by 5, then $8^{n+1} - 3^{n+1}$ is divisible by 5.

$$8^{n+1} - 3^{n+1} \mid 5 = 8 * (8^k) - 3 * (3^k) \mid 5$$

$$\begin{aligned} &= (5+3)8^k - 3 * 3^k \\ &= 5 * 8^k + 3 * 8^k - 3 * 3^k \\ &= 5(8^k) + 3(8^k - 3^k) \\ 8^k - 3^k &= 0 \\ &= 5(8^k) + 3(0) \\ &= 5(8^k) \\ 8^k &= l \\ &= 5l \\ 5l &\mid 5 \\ boomshakalaka \end{aligned}$$

1B. For all n natural numbers $10^n mod 3 = 1$

$Proof$: by weak induction

Base Case: $n = 0$, $10^0 mod 3 = 1$ holds

Inductive Hypothesis: If $10^n mod 3 - 1$, then $10^{n+1} mod 3 = 1$

$$
\begin{aligned}
1 = 10^{n+1} mod 3 \\
1 = (10^n mod 3 * 10 mod 3) mod 3 \\
1 = (1 * 1) mod 3 \\
1 = 1 \\
boomshakalaka
\end{aligned}
$$

1C. For all n natural numbers $n \geq 1$, prove that

$$1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4}$$

$Proof$: by weak Induction

Base Case: $LHS = 1^3 = 1 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$

Induction Hypothesis: Assume $n = k$ and $n = k + 1$

$$1^3 + 2^3 + \ldots + k^3 = \frac{k^2(k+1)^2}{4}$$

Then

$$1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

Manipulate LHS

$$1^3 + 2^3 + \ldots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$\frac{k^2(k+1)^2 + 4(k+1)^3}{4} = \frac{(k+1)^2(k+2)^2}{4}$$

Both sides have common denominators, so eliminate those and use algebra to distribute both sides and equate them to each other.

$$k^2(k+1)^2 + 4(k+1)^3 = (k+1)^2(k+2)^2$$
$$k^4 + 6k^3 + 13k^2 + 12k + 4 = k^4 + 6k^3 + 13k^2 + 12k + 4$$

$boomshakalaka$

Problem 2

2A. For all n natural number, $F_n \geq 3^n$

$Proof$: by strong induction

Base Case: $n = 0$, $F_0 = 1 \leq 3^0 = 1$ Base case holds

Induction Hypothesis: If for all $k \leq n$, $F_k \leq 3^k$ then $F_{n+1} \leq 3^{n+1}$

$Case\ 1 : n + 1 = 1$

$F_1 = 1 \leq 3^1 = 3$

$Case\ 2 : n + 1 \geq 1$

$$\begin{gathered} F_{n+1} = F_n + F_{n-1} \\ \leq 3^n + 3^{n-1} \\ \leq 3^n(1 + 1/3) \\ \leq 3^n + 1 \\ F_{n+1} \leq 3^{n+1} \end{gathered}$$

$boomshakalaka$

2B. For all n natural numbers, $n \leq 2$, n can be written as the product of a prime number.

$Proof$: by strong induction

Base Case: $n = 2$, $2 * 1 = 2$. 2 is a prime number and 2 is product of prime number 2 and 1.

Induction Hypothesis: If $k \geq 2$, and k is the product of a prime number, then $P(n+1)$ holds.

Assume that for all $2 \leq k \leq n$, k is the product of 1 or more prime numbers.

$Case\ 1 : (n+1)$ we assume is prime. Therefore, it is divisible by itself, a prime number.

$Case\ 2 : (n+1)$ is a composite of one or more prime numbers

$(n+1) = p * q$ where $2 \leq p < n+1$ such that $(n+1)$ is divisible by p, p is divisible by some prime number l. Therefore, $(n+1)$ is divisible by prime number l.

Problem 3

3A. Because $i = 0$ and $n = 100$, then $2(100) + 0 = 200$ and the loop variant $2n + i = 200$ holds.

3B. For the loop to be executed, i must be less than n. Because $i = 0$ and $n = 100$, the loop executes. For this is to continue to be true, we have to explore i' and n' once the loop executes.

3C. Once the loop executes once, (i, n) becomes (i', n'). We know the following:

$$\begin{aligned} i' &= i + 2 \\ n' &= n - 1 \\ i &< n \\ 2n + i &= 200 \end{aligned}$$

Consider the new equation once the loop executes:

$$2n' + i' = 200$$

To prove the loop invariant through induction, we plug in what we know for the prime variables.

$$2(n-1) - i + 2 = 200$$
$$2n - 2 + i + 2 = 200$$
$$2n + i = 200$$

Therefore the loop invariant holds through multiple executions of the loop while $i < n$