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PROBLEM SET: WEEK 11

P1.

Prove that the product of any k ≥ 2 consecutive natural numbers is divisible by k! using a combinatorial (counting) argument.

Proof: Let $k \geq 2$, and the product of our k natural numbers $n(n-1)*...*(n-k+1) \mid k!$.

Base Case: $k = 2$, product of consecutive natural numbers $(3, 2) = 6.\ 2! = 2.\ 6 \mid 2.\ 6$ is divisible by 2 and provides an integer result.

Case:

$$k! = k * (k-1)...*(k-k+1)$$

Permutation of $n, k = \frac{n!}{(n-k)!}$ and as stated in the hint the permutation of n, k also $= n(n-1)*...*(n-k+1)$

So...

$$\frac{n!}{(n-k)!} = n(n-1)*...*(n-k+1)$$

We divide this expression by k!.

$$\frac{\frac{n!}{(n-k)!}}{k!}$$

We know that this is the same as combinations of n, k which can be rewritten as:

$$\frac{n!}{(n-k)!k!}$$

Since combinations can be represented by the binomial coefficient n, k which is known to result in an integer, the product of of $P_{n,k}$ for any $k \geq 2$ consecutive natural numbers is divisible by $k!$.

P2

Theorem: Prove the following identity for $n \geq 1$

$$\binom{2n}{n+1} + \binom{2n}{n} = \frac{1}{2}\binom{2n+2}{n+1}$$

Proof:

Our LHS can be rewritten as:

$$\frac{(2n)!}{(2n-(n+1))!(n+1)!} + \frac{(2n)!}{(2n-n)!n!}$$

$$\frac{(2n)}{(n-1)!(n+1)!} + \frac{(2n)!}{(n)!(n)!}$$

Mulitply by 1 to each side:

$$\frac{n}{n} * \frac{(2n)!}{(n-1)!(n+1)!} + \frac{(n+1)}{(n+1)} * \frac{(2n)!}{(n)!(n)!}$$

Simplify to equate denominators:

$$\frac{(n)*(2n)!}{(n)!(n+1)!} + \frac{(n+1)*(2n)!}{(n)!(n+1)!}$$

$$\frac{(n+n+1)*(2n)!}{(n)!(n+1)!}$$

Then we simplify further and get:

$$\frac{(2n+1)!}{(n)!(n+1)!}$$

Which can be written as

$$\binom{2n+1}{n+1}$$

Now, we have to break down our RHS of

$$\frac{1}{2}\binom{2n+2}{n+1}$$

We will take the LHS argument out of the Binomial Coeff. notation to compare to our RHS once it is broken down.

Which becomes:

$$\frac{(2n+2)!}{2(2n+2-n-1)!(n+1)!}$$

simplify:

$$\frac{(2n+2!)}{2(n+1)!(n+1)!}$$

$$\frac{(2n+2)(2n+1)!}{2(n+1)n!(n+1)!}$$

$$\frac{(2n+2)(2n+1)!}{(2n+2)n!(n+1)!}$$

Which simplifies to:

$$\frac{(2n+1)!}{n!(n+1)!}$$

Now we compare our LHS in fractional notation to our RHS in fractional notation.

$$\frac{(2n+1)!}{n!(n+1)!} = \frac{(2n+1)!}{n!(n+1)!}$$

From here, we see that both LHS and RHS are now equal to each other.

Therefore:

$$\binom{2n}{n+1} + \binom{2n}{n} = \frac{1}{2}\binom{2n+2}{n+1}$$

QED

P3

Base Case:

$M_0 = 1$

$M_1 = 1$

$M_2 = 1$

$M_3 = 1$

Used python to print out mature cells at each hour. Then looked for a pattern in the cell growth and found that every hour after hour n = 4 recursion is found to be $M_n = M_{n-1} + M_{n-4}$