Approximative Lowerbound of Prime Pairs

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Abstract

This paper establishes a formula that estimates an approximative lowerbound of prime pairs for even numbers, and proves that Goldbach's Conjectures are true.

Concepts

The Even Number $2n$

Let n be an arbitrary positive integer greater than 2, so $2n$ is an even number greater than 4.

Integral Array in $[0, 2n]$

Mathematica code:

```
In:n=10;Range[0, 2n]
```

```
Out:{0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20}
```

Obviously, there are total of $2n + 1$ integers in $[0, 2n]$.

Number Pairs

Definition 1. A *number pair* is two integers, whose sum is equal to $2n$.

```
In:Map[{#, 2n - #}&, Range[0, n]]
```

```
Out:{{0,20}, {1,19}, {2,18}, {3,17}, {4,16}, {5,15}, {6,14}, {7,13},
{8,12}, {9,11}, {10,10}}
```

```
In:Grid[{Range[0, n], Table[20 - x, {x, 0, n}]}]
```

Out:

0	1	2	3	4	5	6	7	8	9	10
20	19	18	17	16	15	14	13	12	11	10

Obviously, there are total of n number pairs in $[0, 2n]$.

Prime Pairs

Definition 2. A *prime pair* is a number pair, for which both members are prime.

For example, here is how many prime pairs there are for even number 20:

```
In:Map[If[PrimeQ[#], Framed[#], #]&, Range[0, n], Table[20 - x, {x,
0, n}], {2}];
Grid[%]
```

Out:

0	1	$\boxed{2}$	$\boxed{3}$	4	$\boxed{5}$	6	$\boxed{7}$	8	9	10
20	$\boxed{19}$	18	$\boxed{17}$	16	15	14	$\boxed{13}$	12	$\boxed{11}$	10

```
In:Map[{#, 2n - #}&, Range[0, n]];
Select[%, PrimeQ [#[[1]]] && PrimeQ [#[[2]]]&]
Out:{{3,17},{7,13}}
```

As the result shows, there are 2 prime pairs which sum to the even number 20.

However, how many prime pairs are there for an arbitrary even number?

Distribution of Prime Factors

Lemma 1. *Any integer greater than 1 can be considered a composition of one or more prime factors. The number of same prime factors is irrelevant.*

For example, to get all prime factors of number 20:

```
In:FactorInteger[20]
Out:{{2,2},{5,1}}
```

The number of same prime factors is irrelevant to this problem. Merging the same items into one, we get:

```
In:Map[#[[1]]&, FactorInteger[20]]
Out:{2,5}
```

Applied to all integers:

```
In:Range[2n];FactorInteger[%];Map[#[[1]]&, %, 2]
```

```
Out:{{1},{2},{3},{2},{5},{2,3},{7},{2},
{3},{2,5},{11},{2,3},{13},{2,7},{3,5},
{2},{17},{2,3},{19},{2,5}}
```

The integral array will become like the one above.

From the above result, we see that:

Lemma 2. *All prime factors, with phase starting from 0, are cyclic with constantly equidistant spacing and are distributed in an integral array.*

Lemma 3. *In $[0, 2n]$, there are a total of $Floor(2n/p)$ integers with prime factor p.*

Counting, there are a total of $2n/2$ integers with prime factor 2, $Floor(2n/3)$ integers with prime factor 3, etc.

Prime Factors for the Sieve of Eratosthenes

Lemma 4. *There is only a need for every prime factor less than or equal to $\sqrt{2n}$ to sieve in $[0, 2n]$. Any prime factor greater than $\sqrt{2n}$ should be ignored or considered as a hole in the array.*

For example, to sieve integral array $[0, 100]$, one only needs prime factors {2,3,5,7} ($\leq \sqrt{100}$).

Integers Remaining after One Sieve Step

Lemma 5. *After sieving away prime factor p, by which $2n$ is divisible, there will be exactly $Round(2n/pq)$ integers with prime factor q among the remainders.*

For example, after sieving away all integers with prime factor 2 in the integral array $[0, 2n]$, if $2n$ is divisible by 3, there will be exactly $n/3$ integers with prime factor 3 among the remainders. Otherwise, if $2n$ is not divisible by 3, there will be $Round(n/3)$.

Let p be the prime number for sieving. After one sieve step, the remainders will be approximately:

$$2n - 2n/p \tag{1}$$

Number Pairs Remaining after One Sieve Step

A number pair is sieved away when one of its member is sieved away. So after one sieve step:

If $2n$ is divisible by p, this many number pairs will remain:

$$n[(p-1)/p] \tag{2}$$

If $2n$ is not divisible by p, it is going to eliminate two pairs at each interval of the prime factor. So this many number pairs will remain:

$$n[(p-2)/p] \tag{3}$$

Prime Numbers Remaining after All Sieve Steps

Deducing from Formula (1) , by simultaneously sieving by all prime factors less than or equal to $\sqrt{2n}$, the following is the formula which estimates the number of prime numbers remaining in (p, 2n], where p is the last prime factor for sieving, which is less than or equal to $\sqrt{2n}$:

$$2n[(2-1)/2][(3-1)/3]\dots[(p-1)/p]-1 \tag{4}$$

Prime Pairs Remaining after All Sieve Steps

Deducing from Formula (2) and (3), by simultaneously sieving by all prime factors less than or equal to $\sqrt{2n}$, the following is the formula which estimates the least number of pairs of holes possibly remaining, where p is the last prime factor for sieving, which is less than or equal to $\sqrt{2n}$:

$$n[(2-1)/2][(3-2)/3]\dots[(p-2)/p] \tag{5}$$

Note. Formula (5) does not permit any number pair to remain, for which either member is less than or equal to p, and it does not expect $2n$ to be divisible by any prime factor other than 2.

Goldbach Conjectures [1]

Deducing from Lemma 1, 2 and 4, it follows easily that

Lemma 6. *After applying the Sieve of Eratosthenes, every prime factor less than or equal to* $\sqrt{2n}$ *can not eliminate all integers in (p, 2n].*

If all integers in (p, 2n] are eliminated, it is impossible and contradicts Lemma 1 and 2.

Let $E(n)$ be Formula (4). It is easily seen that

$$\lim_{n\to\infty}\frac{\pi(2n)-\pi(\sqrt{2n})}{E(n)}=1$$

Lemma 7. *After applying the Sieve of Eratosthenes, every prime factor less than or equal to* $\sqrt{2n}$ *can not eliminate all number pairs in (p, 2n]. When* $2n \to \infty$*, Formula* (5) $\to \infty$*.*

The same proof works for Lemma 7, hence the first Goldbach conjecture is true.

Every odd integer greater than 5 can be a sum of an even integer and 3, every even integer greater than 5 can be a sum of an even integer and 2, hence the second Goldbach conjecture is consequently true.

Conclusion

We conclude from the above that all Goldbach's Conjectures are true.

Appendix

The following codes visualize Formula (4) and $\pi(2n) - \pi(\sqrt{2n})$:

```
EstimatedPrimeCount = Function[n2, Round[(n2 Times @@ ((# - 1)/# &
/@ Table[Prime[n], {n, 1, PrimePi[Sqrt[n2]]}]))] - 1 ]; t1 = Table[
PrimePi[2 n] - PrimePi[Sqrt[2 n]], {n, Prime[2], Prime[100]}]; t2 =
Table[EstimatedPrimeCount[2 n], {n, Prime[2], Prime[100]}]; ListPlot
[{t1, t2}]
```

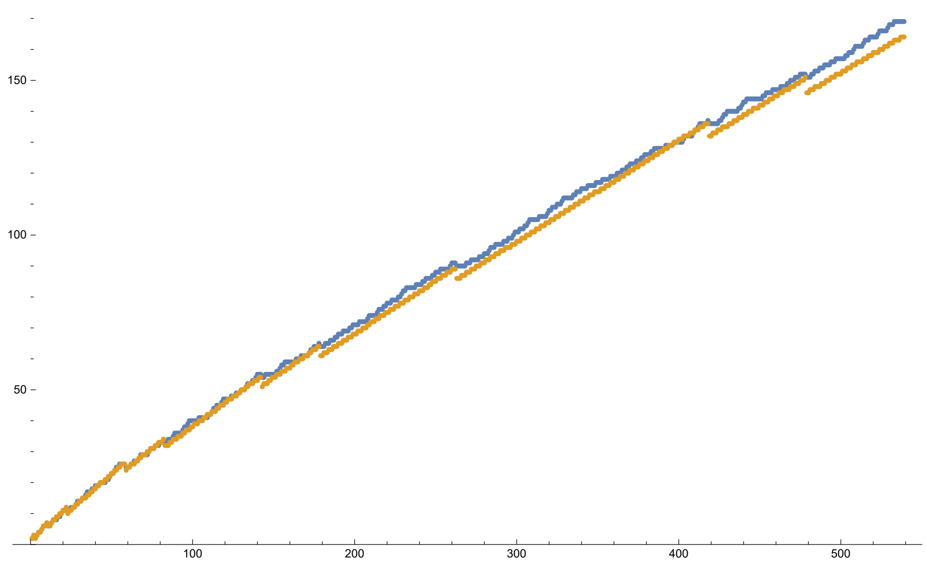

Figure 1:

The following codes visualize Formula (5) and the actual counts of prime pairs:

```
NumberPairs = Function[n2, Module[{n = n2/2}, Map[{n - #, n + #} &,
Range[0, n]]] ]; PrimePairs = Function[n2, Select[NumberPairs[n2],
PrimeQ[#[[1]]] && PrimeQ[#[[2]]] &] ]; LowerBoundofPrimePairCount =
Function[n2, Floor[ (n2/2) 1/ 2 Times @@ ((# - 2)/# & /@ Table[ Prime[
```

```
n ], {n, 2, PrimePi[Sqrt[n2]]}]) ] ]; t1 = Table[Length[PrimePairs[2
n]], {n, Prime[2], Prime[100]}]; t2 = Table[ LowerBoundofPrimePairCount[
2 n], {n, Prime[2], Prime[100]}]; ListPlot[{t1, t2}]
```

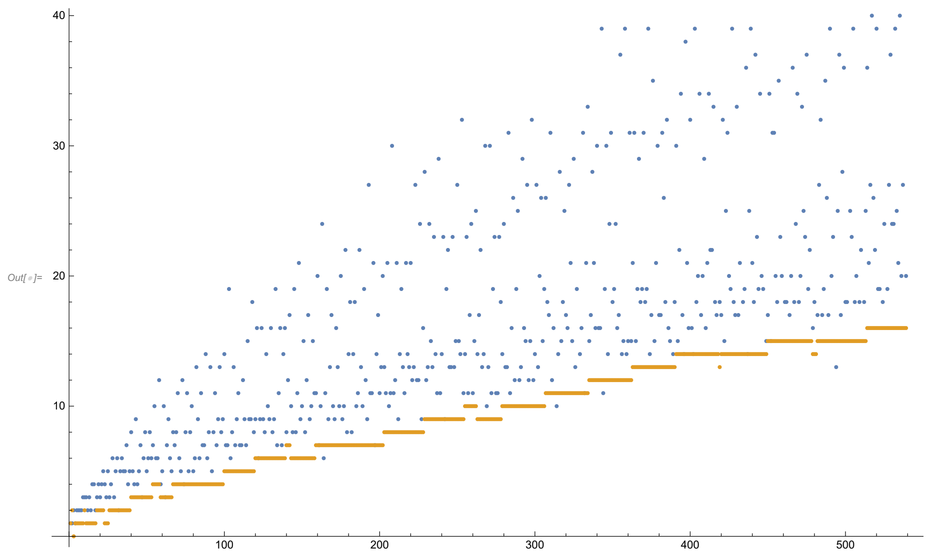

Figure 2:

References

[1] L. E. Dickson. *History of the theory of numbers*. Vol. 1. New York, 1934.