Rooks Sitting in a Primorial Hyperspace Chessboard

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Abstract

An intuitive naive model from sieving approach, inducting and quantitatively solving lower bound of prime gap, Goldbach's conjecture, twin prime conjecture and many more as one single unified problem due to pigeonhole principle.

Prime-Factorizing Arrays

Definition 1. A *number pair* is two integers, whose sum is equal to $2n$.

For example, there is a number-pair-array in $[0, 2n]$:

$$\begin{array}{ccccccc} 0 & 1 & 2 & \dots & n-2 & n-1 & n \\ 2n & 2n-1 & 2n-2 & \dots & n+2 & n+1 & n \end{array}$$

Obviously, there are total of $n+1$ number pairs in $[0, 2n]$.

Definition 2. A *prime pair* is a number pair, for which both members are prime.

For example, suppose $n = 10$, so there is 2 prime pairs in number-pair-array:

$$\begin{array}{cc} 3 & 7 \\ 17 & 13 \end{array}$$

Lemma 1. *Any integer greater than 1 can be considered a composition of one or more prime factors. The number of same prime factors is irrelevant.*

Lemma 2. *All prime factors are cyclic with constantly equidistant spacing and are distributed in an integral array.*

Lemma 3. *In $[0, 2n]$, there are a total of $\lfloor 2n/p \rfloor$ integers with prime factor p.*

Counting, there are a total of $2n/2$ integers with prime factor 2, $\lfloor 2n/3 \rfloor$ integers with prime factor 3, etc.

Lemma 4. *There is only a need for every prime factor less than or equal to $\sqrt{2n}$ to sieve in $[0, 2n]$. Any prime factor greater than $\sqrt{2n}$ should be considered as a hole in the array.*

For example, to sieve integral array $[0, 100]$, one only needs prime factors {2,3,5,7} ($\leq \sqrt{100}$).

Let p_n be the last prime factor for sieving, which is less than or equal to $\sqrt{2n}$. Let p be an arbitrary prime factor less than or equal to p_n.

Integers Remaining after One Sieve Step

Lemma 5. *After sieving away prime factor p, by which $2n$ is divisible, there will be exactly $Round(2n/pq)$ integers with prime factor q among the remainders.*

For example, after sieving away all integers with prime factor 2 in the integral array $[0, 2n]$, if $2n$ is divisible by 3, there will be exactly $n/3$ integers with prime factor 3 among the remainders. Otherwise, if $2n$ is not divisible by 3, there will be $Round(n/3)$.
Let p be the prime number for sieving. After one sieve step, the remainders will be approximately:

$$2n - 2n/p \tag{1}$$

Number Pairs Remaining after One Sieve Step

A number pair is sieved away when one of its member is sieved away. So after one sieve step:

If $2n$ is divisible by p, this many number pairs will remain:

$$n[(p-1)/p] \tag{2a}$$

If $2n$ is not divisible by p, it is going to eliminate two pairs at each interval of the prime factor. So this many number pairs will remain:

$$n[(p-2)/p] \tag{2b}$$

Prime Numbers Remaining after All Sieve Steps

Deducing from Formula (1) , by simultaneously sieving by all prime factors less than or equal to $\sqrt{2n}$, the following is the formula which estimates the number of prime numbers remaining in $(p_n, 2n]$:

$$f_1(n) = 2n[(2-1)/2][(3-1)/3]\ldots[(p_n-1)/p_n] \tag{3}$$

Prime Pairs Remaining after All Sieve Steps

Deducing from Formula (2a) and (2b), by simultaneously sieving by all prime factors less than or equal to $\sqrt{2n}$, the following is the formula which estimates the least number of pairs of holes possibly remaining:

$$f_2(n) = n[(2-1)/2][(3-2)/3]\ldots[(p_n-2)/p_n] \tag{4}$$

Note. Formula (4) does not permit any number pair to remain, for which either member is less than or equal to p, and it does not expect $2n$ to be divisible by any prime factor other than 2.

Midway Summarizing

Let $\pi_1(x)$ be the prime-number-counting function and $\pi_2(x)$ be the prime-pair-counting function. From above, now we can estimate that

$$\pi_1(2n) - \pi_1(\sqrt{2n}) \sim f_1(n) - 1 \tag{5}$$

and that

$$\pi_2(2n) = \Omega\left(f_2(n) - 1\right) \tag{6}$$

Note. Here integer division must be used in order to get absolutely lower bound.

Patternizing to Moiré Pattern

Lemma 6. *Sieving number or number pairs with prime factor p will generate a pattern repeating itself in each interval of length p.*

Lemma 7. *Sieving with all prime factors will generate a one-dimension multiple-layer moiré pattern, a super-pattern which combined by all elementary sub-patterns, which repeating itself in every $p_n\#$ length interval.*

Expanding Array

There are not enough elements to complete the full super-pattern except $p_n \leq 3$. In order to do that we need to expand a number-array or a number-pair-array to $p_n\#$ length long. For a number-pair-array it will be like:

$$\begin{array}{ccccc} n & n+1 & n+2 & \ldots & n+(p_n\#-1) \\ n & n-1 & n-2 & \ldots & n-(p_n\#-1) \end{array}$$

Note. A number pair that either member is negative, which is virtual one, not actual one, only used as a placeholder to continue and complete the super-pattern.

Mapping to Rooks' Primorial Hyperspace Chessboard

Definition 3. A *Moiré pattern space* is an extended and sieved number-array or number-pair-array, a full combination space which contains every combination of every remainder of division of a number by every prime factor.

A Moiré Pattern Space can be one-to-one mapped to a 2 by 3 by 5 .... by p_n hyperspace which is a parameter space of remainder set.

An extended and sieved number-array is a space where has removed any element whose remainder of division of that element by certain prime factor p is equal to certain predefined parameter value. Metaphorically it is equivalent to a 2 by 3 by 5 .... by p_n n-dimension chessboard where a rook stands at some position on it and removes every element or piece on the rook's sight (movable space or kill zone) include the rook itself. For simplicity we only consider that the rook is always sitting at 0 or position {0,0,0,...} in remainder configuration, since the spacial symmetry that it doesn't matter the length of the longest gap based on where the only rook stand at.

An extended and sieved number-pair-array is equivalent to a situation that 2 rooks on one chessboard. For simplicity we only consider that first rook is always sitting at 0 or position {0,0,0,...} in remainder configuration and the other one could be sitting anywhere but out of the first one's sight and at 1 position {1,1,1,...}. The position of the second rook varies the longest gap.

Longest Gap in Moiré Pattern Space

Definition 4. A *gap* is a continuous sieved-away array segment in pattern space between 2 remained elements.

Lemma 8. *Let G_1^+ be the length of the longest gap in an extended and sieved number-array. We have:*

$$G_1^+ < 2p_n \tag{7}$$

Proof. Deducing from Formula (3), we have

$$\begin{aligned}
&2p_n \prod_{i=1}^{n} \frac{p_i - 1}{p_i} \\
&= 2p_n \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{p_n - 1}{p_n} \\
&= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{p_n - 1}{1} \\
&= 2 \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{p_n - 1}{p_{n-1}} \\
&> 2
\end{aligned}$$

which means that in arbitrary $2p_n$ length long subsegment of sieved number-array, there are at least 2 elements still remained after sieving. (p_n is greater than 2)

□

Lemma 9. *Let G_2^+ be the length of the longest gap in an extended and sieved number-pair-array. We have:*

$$G_2^+ < 4p_n \tag{8}$$

Proof. Deducing from Formula (4), we have

$$\begin{aligned}
&4p_n \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdots \frac{p_n - 2}{p_n} \\
&= 2 \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdots \frac{p_n - 2}{1} \\
&= 2 \cdot \frac{3}{3} \cdot \frac{5}{5} \cdots \frac{p_n - 2}{p_{n-1}} \\
&> 2
\end{aligned}$$

which means that in arbitrary $4p_n$ length long subsegment of sieved number-pair-array, there are at least 2 elements still remained after sieving. (p_n is greater than 2) □

This phenomenon is a truism as an obscure variant of pigeonhole principle.

Proof of Upper Bound of Prime Gap

Proof. Deducing from Lemma 8, since a sieved number-array is a subarray of its full extension, as upper bound, the longest gap limit $G_1^+ < 2p_n$ is inherently true in its subarray. So, the longest prime gap must be less than $2p_n$ in $[0, p_n^2]$. □

Proof of Goldbach's Conjecture

Proof. Similarly, deducing from Lemma 9, $G_2^+ < 4p_n < p_n^2 \leq 2n$. Hence, Goldbach's conjecture is true. $\square$

Proof of Twin Prime Conjecture

Proof. Instead number pair is $\{n-x, n+x\}$, in twin prime conjecture, number pair is $\{n, n+1\}$. For example:

$$\begin{array}{ccccc} 0 & 1 & 2 & \dots & n \\ 1 & 2 & 3 & \dots & n+1 \end{array}$$

The only difference in proposition between Goldbach's conjecture and twin prime conjecture is the position of the second rook which is not an essential distinction. So the same proof works for twin prime conjecture. Hence, twin prime conjecture is true.

$\square$

Conclusion

Many more long suspended problems in number theory could be easily solved. I realize my complete silly and ignorance and humbly leave the honor.

Acknowledgments

Thanks for ...