Quantitative Analysis for Prime Numbers, Prime Pairs and Prime Gaps in Moiré Pattern Space

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Abstract

In this paper, it estimates the count of prime number, lower-bounds the count of prime pairs by quantitative analysis from sieving approach. It defines Moiré pattern space for prime-factor-sieving. It upper-bounds the gap in an extended and sieved number-array and number-pair-array.

Prime-Factorizing Arrays

Definition 1. A *number pair* is two integers, whose sum is equal to $2n$.

For example, there is a number-pair-array in $[0, 2n]$:

$$\begin{array}{ccccccc} 0 & 1 & 2 & \dots & n-2 & n-1 & n \\ 2n & 2n-1 & 2n-2 & \dots & n+2 & n+1 & n \end{array}$$

Obviously, there are total of $n+1$ number pairs in $[0, 2n]$.

Definition 2. A *prime pair* is a number pair, for which both members are prime.

For example, suppose $n = 10$, so there is 2 prime pairs in number-pair-array:

$$\begin{array}{cc} 3 & 7 \\ 17 & 13 \end{array}$$

Lemma 1. *Any integer greater than 1 can be considered a composition of one or more prime factors. The number of same prime factors is irrelevant.*

Lemma 2. *All prime factors are cyclic with constantly equidistant spacing and are distributed in an integral array.*

Lemma 3. *In $[0, 2n]$, there are a total of $Floor(2n/p)$ integers with prime factor p.*

Counting, there are a total of $2n/2$ integers with prime factor 2, $Floor(2n/3)$ integers with prime factor 3, etc.

Lemma 4. *There is only a need for every prime factor less than or equal to $\sqrt{2n}$ to sieve in $[0, 2n]$. Any prime factor greater than $\sqrt{2n}$ should be considered as a hole in the array.*

For example, to sieve integral array $[0, 100]$, one only needs prime factors {2,3,5,7} ($\leq \sqrt{100}$).

Let p_n be the last prime factor for sieving, which is less than or equal to $\sqrt{2n}$. Let p be an arbitrary prime factor less than or equal to p_n.

Integers Remaining after One Sieve Step

Lemma 5. *After sieving away prime factor p, by which $2n$ is divisible, there will be exactly $Round(2n/pq)$ integers with prime factor q among the remainders.*

For example, after sieving away all integers with prime factor 2 in the integral array $[0, 2n]$, if $2n$ is divisible by 3, there will be exactly $n/3$ integers with prime factor 3 among the remainders. Otherwise, if $2n$ is not divisible by 3, there will be $Round(n/3)$.

Let p be the prime number for sieving. After one sieve step, the remainders will be approximately:

$$2n - 2n/p \tag{1}$$

Number Pairs Remaining after One Sieve Step

A number pair is sieved away when one of its member is sieved away. So after one sieve step:

If $2n$ is divisible by p, this many number pairs will remain:

$$n[(p-1)/p] \tag{2a}$$

If $2n$ is not divisible by p, it is going to eliminate two pairs at each interval of the prime factor. So this many number pairs will remain:

$$n[(p-2)/p] \tag{2b}$$

Prime Numbers Remaining after All Sieve Steps

Deducing from Formula (1) , by simultaneously sieving by all prime factors less than or equal to $\sqrt{2n}$, the following is the formula which estimates the number of prime numbers remaining in $(p_n, 2n]$:

$$f_1(n) = 2n[(2-1)/2][(3-1)/3]\ldots[(p_n-1)/p_n] \tag{3}$$

Prime Pairs Remaining after All Sieve Steps

Deducing from Formula (2a) and (2b), by simultaneously sieving by all prime factors less than or equal to $\sqrt{2n}$, the following is the formula which estimates the least number of pairs of holes possibly remaining:

$$f_2(n) = n[(2-1)/2][(3-2)/3]\ldots[(p_n-2)/p_n] \tag{4}$$

Note. Formula (4) does not permit any number pair to remain, for which either member is less than or equal to p, and it does not expect $2n$ to be divisible by any prime factor other than 2.

Deducing from all lemmas above, it follows easily that

Lemma 6. *After applying the Sieve of Eratosthenes, every prime factor less than or equal to $\sqrt{2n}$ can not eliminate all integers in $(p_n, 2n]$. When $n > 50$, Formula (3) $\gg 1$.*

If all integers in $(p_n, 2n]$ are eliminated, it is impossible and contradicts Lemma 1 and 2.

Lemma 7. *After applying the Sieve of Eratosthenes, every prime factor less than or equal to $\sqrt{2n}$ can not eliminate all number pairs in $(p_n, 2n]$. When $n > 50$, Formula (4) $\gg 1$.*

The same proof works for Lemma 7.

Patternizing to Moiré Pattern

Lemma 8. *Sieving number or number pairs with prime factor p will generate a pattern repeating itself in each interval of length p.*

Lemma 9. *Sieving with all prime factors will generate a one-dimension multiple-layer moiré pattern, a super-pattern which combined by all elementary sub-patterns, which repeating itself in every $p_n\#$ length interval.*

So the length of full period of the super-pattern pattern is

$$p_n\# \tag{5}$$

Expanding to Length $p_n\#$

There are not enough elements to complete the full super-pattern except $p_n \leq 3$. In order to do that we need to expand a number-array or a number-pair-array to $p_n\#$ length long. For a number-pair-array it will be like:

$$\begin{array}{ccccc} n & n+1 & n+2 & \ldots & n+(p_n\#-1) \\ n & n-1 & n-2 & \ldots & n-(p_n\#-1) \end{array}$$

Note. A number pair that either member is negative, which is virtual one, not actual one, only used as a placeholder to continue and complete the super-pattern.

Deducing from Formula (3) the total of remained elements in extended and sieved number-array is

$$\prod_{i=1}^{n} p_i - 1 \tag{6}$$

Deducing from Formula (4), the total of remained elements in extended and sieved number-pair-array is

$$\prod_{i=2}^{n} p_i - 2 \tag{7}$$

Let $\pi_1(x)$ be the prime-number-counting function and $\pi_2(x)$ be the prime-pair-counting function. From above, now we are confident to say that

$$\pi_1(2n) - \pi_1(\sqrt{2n}) \sim f_1(n) - 1 \tag{8}$$

and that

$$\pi_2(2n) = \Omega\left(f_2(n) - 1\right) \tag{9}$$

Note. Here integer division must be used in order to get absolutely lower bound.

Longest Gap in Moiré Pattern Space

Definition 3. A *Moiré pattern space* is an extended and sieved number-array or number-pair-array, a full combination space which contains every combination of every remainder of division of a number by every prime factor.

Lemma 10. *After sieving the prime factor 2, the least difference between two remained elements both with prime factor p is $2p$.*

Deducing from Formula (3), we have

$$\begin{aligned}
&2p_n \prod_{i=1}^{n} \frac{p_i - 1}{p_i} \\
&= 2p_n \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{p_n - 1}{p_n} \\
&= \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdots \frac{p_n - 1}{1} \\
&= 2 \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{p_n - 1}{p_{n-1}} \\
&> 2
\end{aligned}$$

By calculating most conservative quantitative formula

$$\left\lfloor \cdots \left\lfloor \left\lfloor \left\lfloor 2p_n \frac{1}{2} \right\rfloor \frac{2}{3} \right\rfloor \frac{4}{5} \right\rfloor \cdots \frac{p_n - 1}{p_n} \right\rfloor$$

, it assures that

Lemma 11. *In arbitrary $2p_n \geq 6$ length long array-segment, there are at least 2 elements eventually remained at last.*

For example, suppose $p_n = 5$, there are {25,29,31,35} in a row remained after sieving prime factor 3. 29 and 31 will be eventually remained, but 25 and 35 which hamburgered them will be eventually eliminated.

For another example, suppose $p_n = 7$, there are {203,209,211,217} in a row remained after sieving prime factor 5. 209 and 211 will be eventually remained, but 203 and 217 which hamburgered them will be eventually eliminated.

Definition 4. A *gap* is a continuous sieved-away array segment in pattern space between 2 remained elements.

Let G_n^+ be the length of the longest gap in an extended and sieved number-array. Deducing from Lemma 11, we have:

$$p_n \leq G_n^+ < 2p_n \tag{10}$$

So, we finally have

Theorem 1. *The length of the longest gap in an extended and sieved number-array must be less than $2p_n$.*

Mapping to N-dimension Chessboard Space

A one-dimension pattern space can be one-to-one mapped to a 2 by 3 by 5 .... by p_n n-dimension space which is a parameter space of remainder set.

An extended and sieved number-array is a space where has removed any element whose remainder of division of that element by certain prime factor p is equal to certain predefined parameter value. Metaphorically it is equivalent to a 2 by 3 by 5 ....

by p_n n-dimension chessboard where a rook stands at some position on it and removes every element or piece on the rook's sight (movable space or kill zone) include the rook itself. For simplicity we only consider that the rook is always standing at position {0,0,0,...}, since the spacial symmetry that it doesn't matter the length of the longest gap based on where the only rook stand at.

An extended and sieved number-pair-array is equivalent to a situation that 2 rooks on one chessboard. For simplicity we only consider that one rook is always standing at position {0,0,0,...} and the other one is anywhere but out of the first one's sight. The second rook's position does affect the length of the longest gap significantly. So the second rook has to be moved everywhere in order to check and find the longest gap in overall situation.

Deducing from Formula (4), we have

$$\begin{aligned}
&4p_n \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdots \frac{p_n - 2}{p_n} \\
&= 2 \cdot \frac{1}{3} \cdot \frac{3}{5} \cdot \frac{5}{7} \cdots \frac{p_n - 2}{1} \\
&= 2 \cdot \frac{3}{3} \cdot \frac{5}{5} \cdots \frac{p_n - 2}{p_{n-1}} \\
&> 2
\end{aligned}$$

Similarly, we have

Theorem 2. *The length of the longest gap in an extended and sieved number-pair-array must be less than* $4p_n$.

Conclusion

In further research, it should take a closer look at for more detail in Moiré pattern space.

Acknowledgments

Thanks for ... donation. Thanks for comments from redditors ThereRNoFkingNmsleft, columbus8myhw and edderiofer.