1.

$$\ddot{y} + 7\dot{y} + 10y = \dot{u} + 3u$$

a)

Zero at $t = 0$

Laplace transform:

$$s^2Y(s) + 7sY(s) + 10Y(s) = sU(s) + 3U(s)$$
$$\Rightarrow Y(s)(s^2 + 7s + 10) = U(s)(s + 3)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{s+3}{s^2 + 7s + 10}$$

b)

Unit step in laplace space is $U(s) = \frac{1}{s}$

$$Y(s) = \frac{s+3}{s^2 + 7s + 10} \cdot \frac{1}{s}$$

Final value theorem:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

$$sY(s) = \frac{s+3}{s^2 + 7s + 10}$$

$$\lim_{s\to 0} \frac{s+3}{s^2 + 7s + 10} = \frac{3}{10}$$

$$y(\infty) = 0.3$$

c)

$$Y(s) = \frac{s+3}{s(s^2 + 7s + 10)} = \frac{s+3}{s(s+2)(s+5)}$$

$$= \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5}$$

$$\Rightarrow s+3 = A(s+2)(s+5) + Bs(s+5) + Cs(s+2)$$

$$\begin{cases} s=0: \;\; 3 = A\cdot 2\cdot 5 & \Rightarrow A = \frac{3}{10} \\ s=-2: \;\; 3-2 = B\cdot(-2)\cdot 3 & \Rightarrow B = -\frac{1}{6} \\ s=-5: \;\; 3-5 = C\cdot(-5)\cdot(-3) & \Rightarrow C = -\frac{2}{15} \end{cases}$$

$$\Rightarrow Y(s) = \frac{3/10}{s} - \frac{1/6}{s+2} - \frac{2/15}{s+5}$$

$$\Rightarrow \; y(t) = \frac{3}{10} - \frac{1}{6}e^{-2t} - \frac{2}{15}e^{-5t}, \;\; t \geq 0$$

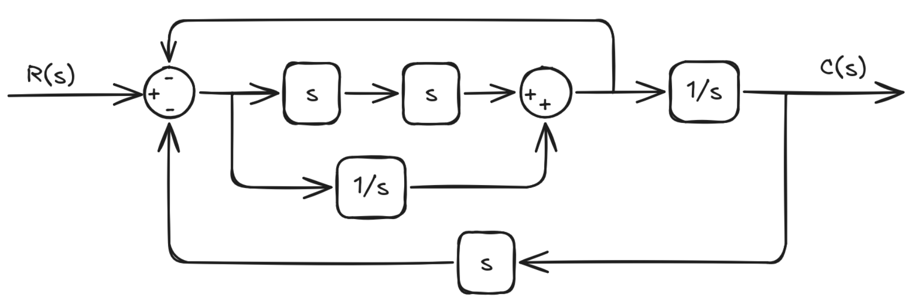
d)

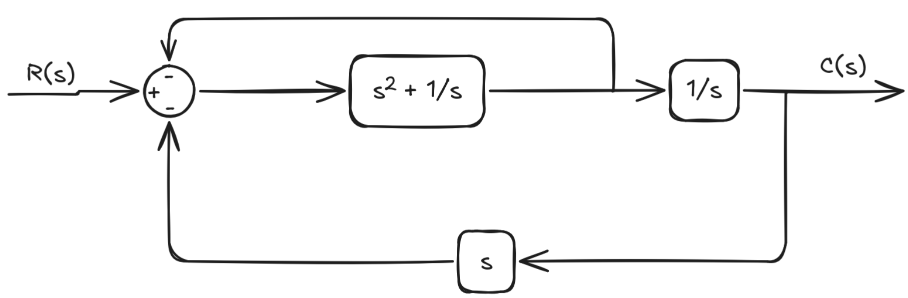
$$\lim_{t\to\infty} y(t) = \frac{3}{10} - \frac{1}{6}\cdot 0 - \frac{2}{15}\cdot 0 = \frac{3}{10}$$

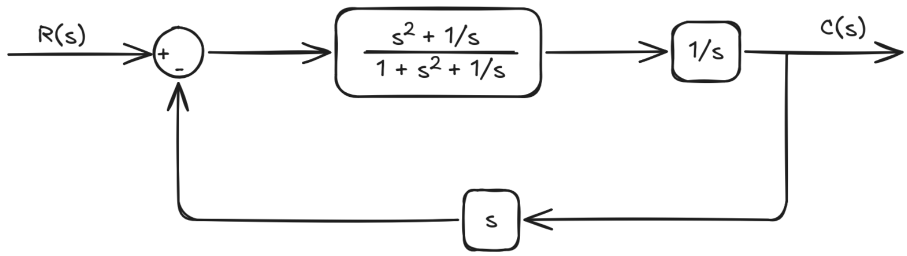
Matches with final value theorem

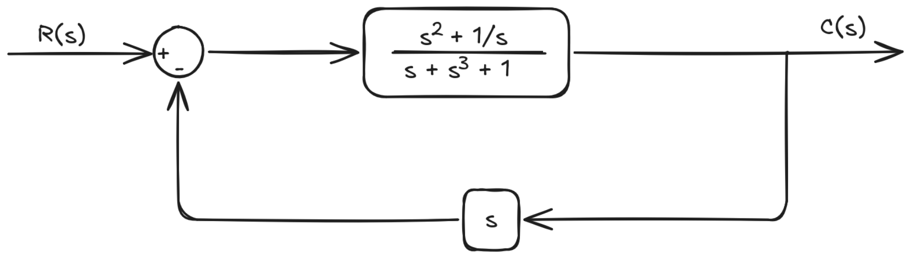
2.

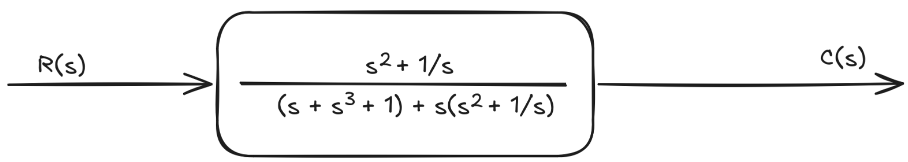
a)











Simplified:

$$\frac{C(s)}{R(s)} = \frac{s^2 + \frac{1}{s}}{2s^3 + s + 2} = \frac{s^3 + 1}{2s^4 + s^2 + 2s}$$

b)

Characteristic equation:

$$2s^4 + s^2 + 2s = 0$$

$$\Rightarrow s(2s^3 + s + 2) = 0$$

Pole at $s = 0$

Poles from $2s^3 + s + 2 = 0$:

Numeric solver on calculator gives root at $s \approx -0.84$

$$\Rightarrow (s + 0.84)(2s^2 - 1.67s + 2.39) = 0$$

solving for roots in $2s^2 - 1.67s + 2.39 = 0$ gives:

$$s \approx 0.42 \pm \sqrt{0.42^2 - 1.20} \approx 0.42 \pm 1.17j$$

Since the system roots have positive real parts the system is **Unstable**.

3.

Open-loop transfer function:

$$G(s) = \frac{K(s+20)}{s(s+2)(s+3)}$$

Closed loop denominatior is $1 + G(s)$ which gives the characteristic equation:

$$1 + \frac{K(s+20)}{s(s+2)(s+3)} = 0$$

$$\Rightarrow s(s+2)(s+3) + K(s+20) = s^3 + 5s^2 + (6+K)s + 20K = 0$$

For the system to be stable, all poles from the cubic must have negative real parts. We get negative real parts if all coefficients are > 0

$$\begin{cases} s^3: & 1 > 0 \\ s^2: & 5 > 0 \\ s^1: & 6 + K > 0 \\ s^0: & 20K > 0 \end{cases}$$

From that we get the lower limit $K > 0$.

Find where K becomes to big and crosses over in to the right half-plane ($s = j\omega$ crosses the imaginary axis)

$$(j\omega)^3 + 5(j\omega)^2 + (6+K)(j\omega) + 20K = 0$$

$$\Rightarrow -j\omega^3 - 5\omega^2 + j(6+K)\omega + 20K = 0$$

from the real part:

$$-5\omega^2 + 20K = 0 \Rightarrow K = \frac{\omega^2}{4}$$

imaginary part:

$$\omega(-\omega^2 + 6 + K) = 0$$

for $\omega \neq 0$:

$$-\omega^2 + 6 + K = 0 \Rightarrow \omega^2 = 6 + K$$

ω^2 in $K = \frac{\omega^2}{4}$ gives:

$$K = \frac{6 + K}{4} \Rightarrow 3K = 6 \Rightarrow K = 2$$

which means the real part of the pole becomes positive, and the system becomes unstable, when $K > 2$

Answer: $0 < K < 2$

4.

$$\frac{h_1 - h_2}{R_1} = q_1$$

$$C_1 \frac{dh_1}{dt} = q - q_1$$

$$\frac{h_2}{R_2} = q_2$$

$$C_2 \frac{dh_2}{dt} = q_1 - q_2$$

Laplace transform:

$$Q_1(s) = \frac{H_1(s) - H_2(s)}{R}$$

$$C_1 s H_1(s) = Q(s) - Q_1(s)$$

$$Q_2(s) = \frac{H_2(s)}{R_2}$$

$$C_2 s H_2(s) = Q_1(s) - Q_2(s)$$

$$H_2(s) = Q_2(s)R_2 \quad \Rightarrow \quad C_2 s Q_2(s)R_2 = Q_1(s) - Q_2(s)$$

$$\Rightarrow Q_1(s) = Q_2(s)[C_2R_2s + 1]$$

From laplace-transformed (1):

$$H_1(s) = Q_1(s)R_1 + H_2(s) = Q_1(s)R_1 + Q_2(s)R_2$$

$$\Rightarrow C_1 s[R_1Q_1(s) + R_2Q_2(s)] = Q(s) - Q_1(s)$$

$$\Rightarrow Q(s) = Q_1(s)[C_1R_1s + 1] + Q_2(s)C_1sR_2$$

subsitute in $Q_1(s) = Q_2(s)[C_2R_2s + 1]$:

$$Q(s) = Q_2(s)[(C_2R_2s + 1)(C_1R_1s + 1) + C_1R_2s]$$

$$\Rightarrow \frac{Q_2(s)}{Q(s)} = \frac{1}{R_1C_1R_2C_2s^2 + (R_1C_1 + R_2C_2 + R_2C_1)s + 1}$$

Q.E.D.

5.

$G(s)$ with $K = 30$:

$$G(s) = \frac{K}{s^3 + 6s^2 + 20s}$$

a)

$s = j\omega$:

$$G(j\omega) = \frac{K}{(j\omega)^3 + 6(j\omega)^2 + 20j\omega} = \frac{K}{-6\omega^2 + j\omega(20 - \omega^2)}$$

$$G(j\omega) = \frac{K(-6\omega^2 - j\omega(20 - \omega^2))}{(-6\omega^2)^2 + (\omega(20 - \omega^2))^2}$$

Crosses the real axis when $\mathfrak{Im}[G(j\omega)] = 0$

$$\Rightarrow K\omega(20-\omega^2)=0$$

for $\omega \neq 0$:

$$20 - omega^2 = 0 \;\Rightarrow\; \omega = \sqrt{20}$$

$$\mathfrak{Re}[G(j\omega)] = \frac{K(-6\omega^2)}{(-6\omega^2)^2 + (\omega(20-\omega^2))^2}$$

at $\omega = \sqrt{20}$ we get $20 - \omega^2 = 20 - \left(\sqrt{20}\right)^2 = 0$

$$\Rightarrow \mathfrak{Re}[G(j\sqrt{20})] = \frac{K\left(-6\left(\sqrt{20}\right)^2\right)}{\left(-6\left(\sqrt{20}\right)^2\right)^2} = -\frac{K}{6\left(\sqrt{20}\right)^2} = -\frac{K}{120} = -\frac{30}{120} = -0.25$$

Crossing $G(j\omega) = -0.25$ at $\omega = \sqrt{20} \approx 4.472$ rad/s $\Rightarrow \angle - 90°$

b)

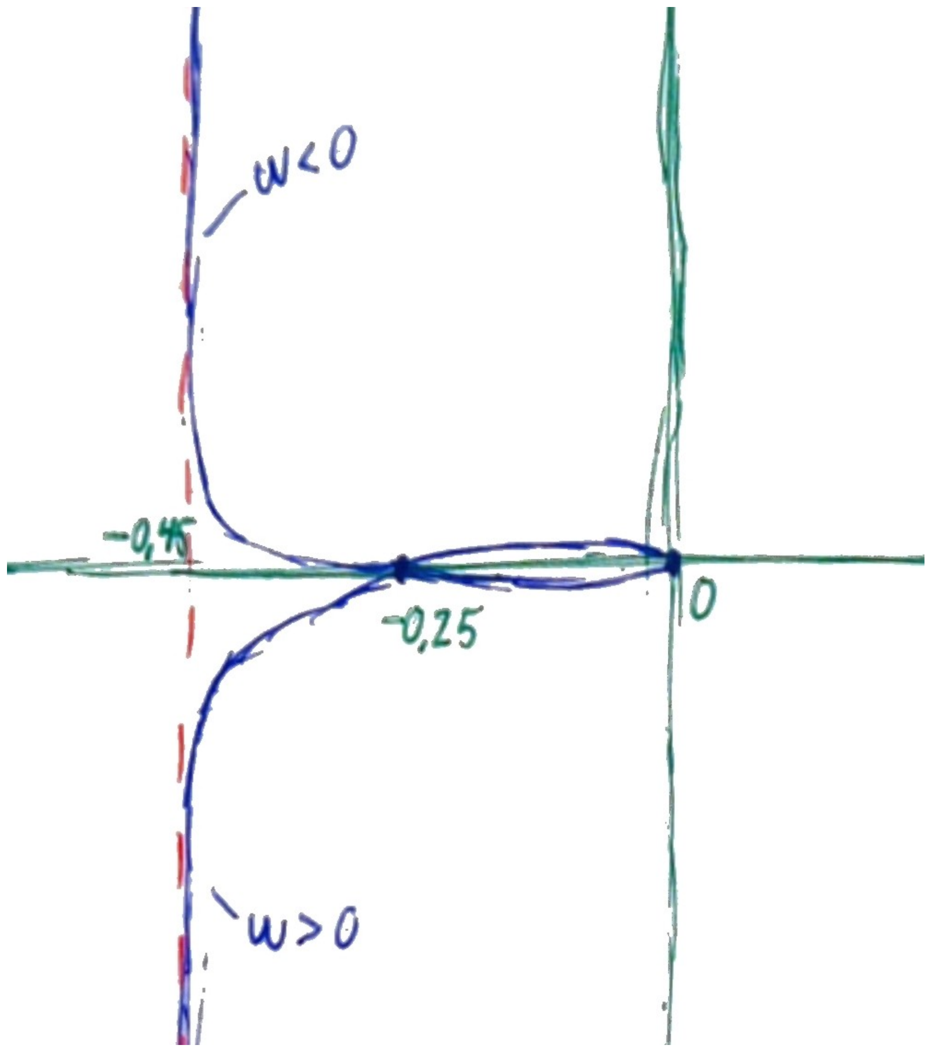
$$G(j\omega) = -\frac{180\omega^2}{36\omega^4 + \omega^2(20-\omega^2)^2} - j\frac{30\omega(20-\omega^2)}{36\omega^4 + \omega^2(20-\omega^2)^2}$$

$$\Rightarrow G(j\omega) = -\frac{180}{36\omega^2 + (20-\omega^2)^2} - j\frac{30(20-\omega^2)}{36\omega^3 + \omega(20-\omega^2)^2}$$

at $\omega \to \infty$ we get $G(j\infty) = 0$

at $\omega \to 0^+$ we get $G(j0^+) = -\frac{180}{20^2} - j\infty = -0.45 - j\infty$

at $\omega \to 0^-$ we get $G(j0^-) = -0.45 + j\infty$



c)

$Z = N + P$ where

- Z: Unstable poles of the closed loop system
- N: Encirclements of $(-1, j0)$
- P: Unstable poles of the open loop system

P:

$$s^3 + 6s^2 + 20s = 0$$

$\Rightarrow$ pole at $s = 0$

$$s^2 + 6s + 20 = 0 \Rightarrow s = -3 \pm \sqrt{-11}$$

which means $P = 0$.

For stable system no poles should lie in RHP wich means $N = 0$.

N becomes 1 when the plot crosses the real axis to the left of -1.

From a) we have the crossing point:

$$G(j\omega) = -\frac{K}{120}$$

Since $G(j\omega) > -1$ we get $K < 120$ we get $K < 120$. The crossing point must also lay in the LHP which gives $K > 0$.

Answer: $0 < K < 120$

6.

$$G(s) = \frac{K}{(s^2 + 13s + 40)(s^2 + 2s + 5)}$$

a)

from $s^2 + 13s + 40 = 0$:

$$s = \frac{-13}{2} \pm \sqrt{\left(\frac{13}{2}\right)^2 - 40} = -6.5 \pm 1.5$$

$$s_1 = -5, \; s_2 = -8$$

from $s^2 + 2s + 5 = 0$:

$$s = -1 \pm \sqrt{1 - 5} = -1 \pm j2$$

$$s_3 = -1 + j2, \; s_4 = -1 - j2$$

Since s_3 and s_4 lie closer to the imaginary axis (real part closer to zero) they are the dominant poles.

b)

When poles have real parts closer to zero their exponetial decay becomes slower, eg. e^{-1t} vs e^{-8t}. Therefore they determine the long term characteristics of the system more than poles that lie further from the imaginary axis.

7.

$$G(s) = \frac{K(s+5)}{s(s+2)(s+4)}$$

a)

$$K = 1 \quad \Rightarrow \quad G(s) = \frac{s+5}{s(s+2)(s+4)}$$

in Bode form:

$$G(j\omega) = \frac{j\omega + 5}{j\omega(j\omega+2)(j\omega+4)}$$

normalised:

$$G(j\omega) = \frac{1 + \frac{j\omega}{5}}{j\omega(1+\frac{j\omega}{2})(1+\frac{j\omega}{4})}$$

$$\begin{cases} [1] : (1+\frac{j\omega}{5})^1 & \text{First order term with real root} \\ [2] : (j\omega)^{-1} & \text{Root at origo} \\ [3] : (1+\frac{j\omega}{2})^{-1} & \text{First order term with real root} \\ [4] : (1+\frac{j\omega}{4})^{-1} & \text{First order term with real root} \end{cases}$$

Break points ($1/\tau$):

$$\begin{cases} [1] : & \omega = 5 \\ [2] : & - \\ [3] : & \omega = 2 \\ [4] : & \omega = 4 \end{cases}$$

Behaves like (ω above breakpoint) [dB]:

$$\begin{cases} [1]: & +20\log_{10}\omega \\ [2]: & -20\log_{10}\omega \\ [3]: & -20\log_{10}\omega \\ [4]: & -20\log_{10}\omega \end{cases}$$

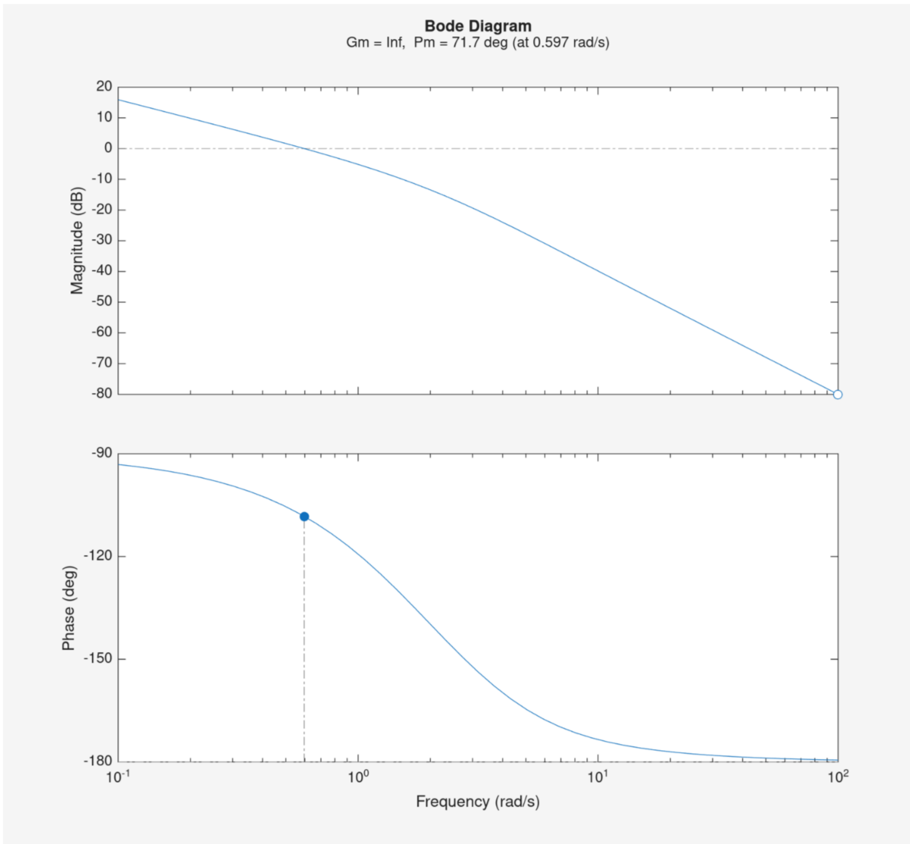
Phase:

$$\begin{cases} [1]: & +90^\circ \\ [2]: & -90^\circ \\ [3]: & -90^\circ \\ [4]: & -90^\circ \end{cases}$$

b)

From the plot we get a phase margin of around 60° and no gain margin.

c)


Bode Diagram
Gm = Inf, Pm = 71.7 deg (at 0.597 rad/s)
Magnitude (dB)
20
10
0
-10
-20
-30
-40
-50
-60
-70
-80
Phase (deg)
-90
-120
-150
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)