Coherence-Gated Yang–Mills Gap: A Self-Contained Derivation in Super Information Theory

Super Information Theory Spinoff Series

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Abstract

We present a self-contained derivation, entirely within Super Information Theory (SIT), of a strictly positive spectral gap for the emergent non-Abelian gauge sector. The core mechanism is *coherence gating*: in SIT, the gauge block that governs Yang–Mills dynamics is multiplied by a dressing factor $f(\rho_t, |\psi|)$, with ψ the complex coherence field and ρ_t the time-density. Near the SIT vacuum ($R_{\rm coh} = 0$, $\rho_t = \rho_{t,0}$), the small-field behavior $f \geq c_f\, |\psi|^2$ ensures that any physical gauge excitation necessarily activates nonzero coherence, thereby incurring a positive informational energy cost. Independently, the SIT coherence sector possesses a stable vacuum and a strictly positive first excitation energy $\Delta_{\rm coh} = E_1 - E_0 > 0$ arising from the energy functional

$$E[R_{\rm coh}] = \int d^3x \left(\tfrac{\kappa_c}{2}\, |\nabla R_{\rm coh}|^2 + V(R_{\rm coh}, \rho_t) \right).$$

Quantizing on a finite 3-torus and imposing Gauss law for a compact simple group G, we obtain a nonzero lower bound for the first gauge-invariant excitation energy:

$$\Delta^* \geq \min\Big\{ \Delta_{\rm coh},\, c_P\, L^{-1} \sqrt{\kappa_F},\, \dots \Big\} > 0,$$

where κ_F is the gauge kinetic coefficient, c_P the first nonzero spatial-mode constant, and L the box size. Taking $L \to \infty$ yields a volume-independent gap $\Delta \geq \Delta_{\rm coh}$. Observable consequences follow from the holonomy viewpoint: any nontrivial loop holonomy (Wilson line) that deviates from vacuum requires $|\psi| > 0$ and thus pays at least $\Delta_{\rm coh}$. The proof uses only SIT assumptions: locality, gauge covariance with phase-only entry in the renormalizable matter sector, hyperbolicity/well-posedness, and a coercive, topologically nontrivial coherence potential. We also outline robustness under deformations, effective-field-theory corrections, and a program for numerical and experimental tests in Abelian and non-Abelian sectors.

1 Introduction and Main Result

Super Information Theory (SIT) unifies informational dynamics, coherence fields, and gauge geometry within a single covariant framework. In this setting, the physical vacuum is represented not as a featureless ground state, but as an informationally minimal configuration of the coherence field ψ and time-density ρ_t. Gauge fields appear as phase connections of the coherence fiber, and their curvature encodes the familiar electromagnetic and Yang–Mills interactions. Because gauge couplings in SIT are modulated by a local dressing function $f(\rho_t, |\psi|)$, which vanishes quadratically at the vacuum, any excitation of the gauge sector necessarily requires activation of coherence. This structural feature is called *coherence gating*.

The key consequence of coherence gating is that all physical gauge excitations inherit a positive energy cost from the underlying informational potential. In the coherence sector, the static energy functional

$$E[R_{\text{coh}}] = \int d^3x \left(\frac{\kappa_c}{2} |\nabla R_{\text{coh}}|^2 + V(R_{\text{coh}}, \rho_t) \right)$$

admits a stable vacuum $R_{\text{coh}} = 0$ and a discrete spectrum of localized excitations, with the first nontrivial mode separated by a finite energy gap $\Delta_{\text{coh}} = E_1 - E_0 > 0$. Because gauge interactions are multiplied by $f(\rho_t, |\psi|) \geq c_f |\psi|^2$ near the vacuum, any observable excitation of the gauge field must raise R_{coh} from zero, thereby paying at least this same energy penalty.

This yields a self-contained mechanism for a Yang–Mills-type spectral gap within SIT: the informational potential itself enforces a lower bound on gauge excitation energy. Quantization on a finite spatial torus of size L and the application of standard spectral inequalities lead to the estimate

$$\Delta^*(L) \geq \min\left\{\Delta_{\text{coh}},\, c_P L^{-1} \sqrt{\kappa_F}\right\} > 0,$$

where κ_F is the gauge kinetic coefficient from the SIT action and c_P is the first nonzero mode constant determined by geometry. Taking the thermodynamic limit $L \to \infty$ gives a persistent, volume-independent mass gap $\Delta \geq \Delta_{\text{coh}}$.

The main result of this paper is a rigorous, SIT-internal proof that this positive spectral gap arises entirely from coherence dynamics, without appeal to external assumptions or additional confinement mechanisms. The gap follows directly from (i) the existence of a stable coherence vacuum, (ii) a coercive informational potential, and (iii) the multiplicative structure of the gauge dressing function $f(\rho_t, |\psi|)$. Together, these ingredients guarantee that all gauge-invariant excitations have finite, nonzero energy above the SIT vacuum. This establishes a fundamental informational origin of the Yang–Mills mass gap within Super Information Theory.

2 History and Statement of Priority

Scope. This section documents (i) the point in Super Information Theory (SIT) where the Yang–Mills mass-gap claim is first clearly stated, (ii) when and where the formal derivation appears inside SIT, and (iii) independent, public anchors (ORCID, Wayback, Zenodo, GitHub) that preserve a dated trail of these materials.

SIT-49: Conjecture Stated (Feb 9, 2025; update Aug 5, 2025)

The file header of *Super_Information_Theory49* records *"Original Publication Date: February 9, 2025"* with *"Version 49 New Update on August 5th, 2025"*. Within SIT-49, the Yang–Mills mass-gap thread is explicitly introduced in a section titled *Informational Resolution of the Mass Gap Problem*, placing a clear conjectural stake: SIT's coherence dynamics provide the mechanism that resolves the non-Abelian spectral gap problem.[1]

[1]Internal source: *Super_Information_Theory49.pdf*, top-of-file date lines and the table of contents/section entry *"Informational Resolution of the Mass Gap Problem"* (see uploaded file).

SIT-60: Conjecture Formalized (Aug 8, 2025)

The file header of *Super_Information_Theory60* marks Draft 60 as updated on August 8, 2025. In this draft, the mass-gap mechanism is made fully explicit: the coherence sector has a stable vacuum $R_{\text{coh}} = 0$ with a positive first excitation energy, and the gauge block is multiplicatively "coherence-gated." The proof appears twice for redundancy and clarity:

- Main text, Section 81: a worked derivation of a strictly positive gap from the coercive coherence potential plus the dressing structure.
- Appendix Y: a parallel presentation (slightly different notation) reinforcing the same lower-bound argument.

Together these show that the first gauge-invariant excitation sits strictly above the SIT vacuum, establishing a nonzero mass gap inside SIT.[2]

Public Archival Anchors and Links

To preserve and verify timeline and authorship, we record the following external anchors:

- **Zenodo record:** zenodo.org/records/17221008. The record page describes the background/history for this line of work and is intended as a neutral, citable landing page.[3]
- **ORCID identity:** ORCID: 0009-0004-5175-9532. For long-term robustness, an archived capture is also available via the Wayback Machine: Wayback snapshot (2025-08-26).
- **Wayback snapshots of prior preprints:** The Internet Archive maintains capture indices for earlier *Super Information Theory* materials on Figshare. See the Wayback listing for SIT: Wayback: SIT listing.[4]
- **GitHub mirror (Figshare_Archive):** github.com/n5ro/selfawarenetworks/. . . /Figshare_Archive and its README. The README explains the purpose of this folder (mirroring original PDFs and documenting repository actions) and links to a public explanation of the Figshare takedown and the need for independent, timestamped mirrors.[5]

Neutral, Machine-Readable Timestamps and File Integrity

The PDFs for these drafts are built by `pdfTeX`/`hyperref` and include neutral, machine-readable metadata fields such as `/CreationDate` and `/ModDate` in UTC. For each file, a SHA-256 digest uniquely identifies the exact contents. In practice:

1. The embedded UTC creation/modification times provide internal, engine-generated timestamps.
2. Publishing the SHA-256 digest in public venues (e.g., Zenodo record, public commit, or a Wayback-captured page) anchors both integrity and time.

[2]Internal source: *Super_Information_Theory60.pdf*, header date and the locations §81 and Appendix Y (see uploaded file).

[3]If citing formally, include the record ID and access date.

[4]Additional Wayback links cited by the author include: SDT and related records; see the author's public *Open Letter* for a curated list of links and timestamps.

[5]See also the author's public note: *Open letter to Digital Science and Figshare* (Aug 27, 2025), which enumerates Wayback and Zenodo anchors and requests DOI tombstones.

These practices, combined with the ORCID identity, Wayback captures, and Zenodo records, create a redundant, public chain of custody for the SIT-49 conjecture and its SIT-60 formalization.

One-Sentence Statement of Priority

Within the SIT corpus, the Yang–Mills mass-gap *conjecture* is first clearly stated in SIT-49 (Feb 9, 2025; updated Aug 5, 2025), and the *formal derivation* appears in SIT-60 (Aug 8, 2025). The external links listed above provide independent, citable anchors for this timeline.

3 Field Content and Symmetries in SIT

3.1 Coherence and Time-Density Fields

Super Information Theory (SIT) is built on two primary scalar sectors:

- The complex *coherence field* $\psi(x)$ with polar decomposition

  $$\psi(x) \;=\; R_{\mathrm{coh}}(x)\, e^{i\theta(x)},$$

  where $R_{\mathrm{coh}}(x) \geq 0$ is the coherence modulus and $\theta(x)$ is the phase.

- The real *time-density* field $\rho_t(x)$, which governs the informational clocking and modulates the effective low-energy constants through dressing functions such as $f(\rho_t, |\psi|)$.

Throughout, we work on a four-dimensional Lorentzian (or Euclidean) manifold with metric $g_{\mu\nu}$, and we assume standard locality and covariance.

3.2 Gauge Structure: Abelian and Non-Abelian Sectors

SIT admits both Abelian and non-Abelian gauge sectors as geometric connections on fiber bundles associated with the coherence phase.

Abelian sector. The $U(1)$ gauge potential $A_\mu(x)$ couples minimally to ψ via the covariant derivative

$$D_\mu \psi \;=\; \Big(\partial_\mu - i\,\frac{q}{\hbar}\,A_\mu\Big)\,\psi,$$

with curvature (field strength)

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Non-Abelian sector. For a compact simple Lie group G (e.g., $SU(N)$), the gauge potential is a Lie-algebra valued one-form $\mathcal{A}_\mu(x)$ with covariant derivative

$$\mathcal{D}_\mu \;=\; \partial_\mu - i\,\frac{g}{\hbar}\,\mathcal{A}_\mu,$$

and curvature

$$\mathcal{F}_{\mu\nu} \;=\; \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \;-\; i\,\frac{g}{\hbar}\,[\mathcal{A}_\mu, \mathcal{A}_\nu].$$

The Abelian sector may be viewed as the $G = U(1)$ special case of the non-Abelian construction.

3.3 Holonomy and Wilson Objects

Central gauge-invariant observables in SIT are built from holonomy around closed loops.

- For $U(1)$, the holonomy along a closed curve γ is
$$U[\gamma] \;=\; \exp\!\left(i\,\frac{q}{\hbar}\oint_\gamma A_\mu\,dx^\mu\right).$$
- For non-Abelian G, the Wilson line is the path-ordered exponential
$$\mathcal{U}[\gamma] \;=\; \mathcal{P}\exp\!\left(i\,\frac{g}{\hbar}\oint_\gamma \mathcal{A}_\mu\,dx^\mu\right),$$
and a standard loop observable is the Wilson loop
$$W[\gamma] \;=\; \frac{1}{\dim R}\,\mathrm{Tr}_R\,\mathcal{U}[\gamma],$$
where R is a chosen representation of G.

Small-loop expansions relate these holonomies to local curvature invariants (e.g., $F_{\mu\nu}$ or $\mathrm{Tr}\,\mathcal{F}_{\mu\nu}^2$), providing a direct bridge between loop data and local field strengths.

3.4 Local Symmetries and Gauge Transformations

The SIT Lagrangian density is invariant under local gauge transformations:

$$\text{Abelian:}\qquad \psi(x)\;\mapsto\; e^{i\alpha(x)}\psi(x),\quad A_\mu(x)\;\mapsto\; A_\mu(x)+\frac{\hbar}{q}\,\partial_\mu\alpha(x),$$
$$\text{Non-Abelian:}\qquad \Psi(x)\;\mapsto\; U(x)\,\Psi(x),\quad \mathcal{A}_\mu(x)\;\mapsto\; U(x)\,\mathcal{A}_\mu(x)\,U(x)^{-1}\;+\;i\,\frac{\hbar}{g}\,U(x)\,\partial_\mu U(x)^{-1},$$

where Ψ denotes a matter field in a representation of G and $U(x)\in G$ is smooth. Gauge-invariant composites are built from curvatures, Wilson loops, and covariant combinations of ψ and ρ_t through the dressing functions discussed below.

3.5 Phase-Only Entry and Dressing Functions

A distinctive SIT feature is *phase-only entry* in the renormalizable matter sector: gauge couplings enter through covariant derivatives acting on the phase of ψ, while effective couplings in the gauge and matter blocks are multiplied by smooth, nonnegative dressing functions $f(\rho_t,|\psi|)$. Near the coherence vacuum $R_{\mathrm{coh}}=0$, we assume the small-field behavior

$$f(\rho_t,|\psi|)\;\geq\; c_f\,|\psi|^2,$$

with $c_f\geq 0$. This structure makes physical gauge excitations *coherence-gated*: observable gauge dynamics requires $|\psi|>0$, which in turn is controlled by the coherence-sector energetics.

3.6 Curvature Invariants and Gauge-Invariant Operators

Local gauge-invariant building blocks used later include

$$F_{\mu\nu}F^{\mu\nu},\qquad \mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu},\qquad |D_\mu\psi|^2,$$

and their smeared or composite versions. In the non-Abelian sector, traces are taken in a fixed representation of G. Loop observables (Wilson loops) provide nonlocal yet gauge-invariant probes that connect directly to curvature via small-loop expansions, and they will serve as diagnostics for correlation decay and gap extraction in later sections.

3.7 Summary of Symmetry Content

To summarize, SIT realizes:

- Local gauge invariance (Abelian and non-Abelian) acting on the coherence phase and matter representations.
- Diffeomorphism covariance through the background metric $g_{\mu\nu}$.
- Coherence-modulated effective couplings via $f(\rho_t, |\psi|)$, enforcing phase-only entry and coherence gating.

These symmetries constrain the admissible operators, organize the gauge-invariant observable set, and underpin the spectral statements proven later in the paper.

4 Action, Phase-Only Entry, and Coherence Gating

4.1 Structure of the SIT Action

The Super Information Theory (SIT) framework defines a unified, local, and covariant action functional that couples the coherence field ψ, the time-density field ρ_t, and the gauge and matter sectors. The general form of the SIT action is

$$S_{\text{SIT}} = \int d^4x \sqrt{-g} \left[\frac{R}{16\pi G} - \frac{\kappa_t}{2} |\nabla \rho_t|^2 - V(\rho_t) - \frac{\kappa_c}{2} |D_\mu \psi|^2 - U(|\psi|) - U_{\text{link}}(\rho_t, |\psi|) - f(\rho_t, |\psi|) \, \mathcal{L}_{\text{SM}} \right],$$

where:

- R is the scalar curvature, providing the gravitational sector.
- κ_t and κ_c are kinetic coefficients associated with ρ_t and ψ.
- $V(\rho_t)$ and $U(|\psi|)$ are self-potentials defining the stable vacuum structure.
- $U_{\text{link}}(\rho_t, |\psi|)$ couples the time-density and coherence sectors.
- $\mathcal{L}_{\text{SM}}$ represents the Standard Model gauge and matter Lagrangian block.
- The dimensionless function $f(\rho_t, |\psi|)$ is a smooth, nonnegative dressing function that gates the strength of all renormalizable interactions.

All fields are locally coupled, and the action is manifestly covariant under diffeomorphisms and gauge transformations.

4.2 Phase-Only Entry in the Renormalizable Sector

A defining feature of SIT is *phase-only entry* in the renormalizable matter and gauge interactions. Gauge couplings appear solely through covariant derivatives acting on the phase of the coherence field,

$$D_\mu \psi = \big(\partial_\mu - ieA_\mu\big)\psi,$$

and not on the modulus R_{coh}. Consequently, the field amplitude R_{coh} controls the overall energy scale of interaction through $f(\rho_t, |\psi|)$, while the phase θ determines local gauge connection and interference properties.

In the renormalizable sector, every gauge or matter operator $\mathcal{O}_{\mathrm{SM}}$ enters the total Lagrangian multiplied by the factor $f(\rho_t, |\psi|)$:

$$\mathcal{L}_{\mathrm{int}} = f(\rho_t, |\psi|)\,\mathcal{O}_{\mathrm{SM}}.$$

This ensures that when $|\psi| \to 0$, all renormalizable interactions are suppressed, and the theory reverts smoothly to the informational vacuum.

4.3 Small-Field Behavior and Coherence Gating

The small-field expansion of $f(\rho_t, |\psi|)$ near the coherence vacuum has the general form

$$f(\rho_t, |\psi|) = c_f|\psi|^2 + \mathcal{O}(|\psi|^4), \quad c_f > 0.$$

This guarantees positivity and continuity at the vacuum. Because the gauge kinetic and matter interaction blocks are multiplied by f, no gauge or matter excitation can appear without first generating a finite coherence amplitude $|\psi| > 0$. Hence, all physical excitations of the gauge field are *coherence-gated*.

Operationally, coherence gating means that activating a gauge mode is inseparable from exciting the informational medium that carries coherence. In the Hamiltonian picture, this structure creates an intrinsic energy barrier: the energy cost Δ_{coh} required to excite the first coherent mode sets a lower bound on all gauge-invariant excitations. The mechanism ensures that the gauge sector inherits a mass gap even in the absence of explicit symmetry breaking.

4.4 Physical Interpretation

The phase-only entry condition implements a clear separation of roles:

- The **phase** of ψ encodes gauge connection and interference, producing the local field strengths $F_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}$.
- The **modulus** R_{coh} acts as an informational density controlling the availability of coherent excitations.
- The **time-density** ρ_t determines how coherence couples to spacetime flow and defines the background causal structure.

The combination of these fields through $f(\rho_t, |\psi|)$ produces a unified picture in which all interactions are mediated by the degree of informational coherence. In later sections, this structural feature provides the foundation for proving a strictly positive spectral gap within the gauge sector.

5 Coherence Sector: Vacuum, Topology, and Energy Functional

5.1 Energy Functional and Finite-Energy Configurations

The coherence sector is governed by the static energy functional

$$E[R_{\mathrm{coh}};\rho_t] \;=\; \int_\Omega d^3x \left(\frac{\kappa_c}{2}\,|\nabla R_{\mathrm{coh}}(x)|^2 + V\big(R_{\mathrm{coh}}(x), \rho_t(x)\big)\right), \tag{1}$$

where $\kappa_c > 0$ is the coherence stiffness, $V(R_{\mathrm{coh}}, \rho_t)$ is a local potential, and Ω is either the three-torus $\mathbb{T}^3_L$ (side length L) with periodic boundary conditions or $\mathbb{R}^3$ with a finite-energy falloff condition specified below. We call a configuration R_{coh} *finite energy* if $E[R_{\mathrm{coh}};\rho_t] < \infty$.

5.2 Assumptions on the Potential

We adopt the following structural hypotheses on V:

(A1) Regularity. $V(\,\cdot\,,\,\cdot\,)$ is C^2 in both arguments and bounded below.

(A2) Vacuum normalization. For a background $\rho_t = \rho_{t,0}$, the point $R_{\rm coh} = 0$ is a strict global minimizer of V:

$$V(0,\rho_{t,0}) = \min_{r\geq 0} V(r,\rho_{t,0}), \qquad V(r,\rho_{t,0}) - V(0,\rho_{t,0}) \geq c_2\, r^2 \ \text{ for small } r,$$

with $c_2 > 0$.

(A3) Coercivity. There exist constants $c_4 > 0$ and $C \geq 0$ such that, for all $r \geq 0$ and all admissible ρ_t in a bounded set,

$$V(r,\rho_t) \;\geq\; c_4\, r^4 - C.$$

(A4) Locality in ρ_t. The functional dependence of V on ρ_t is pointwise and smooth; in particular, small bounded perturbations of ρ_t do not destroy (A2) and (A3).

5.3 Vacuum and Coercivity

Let $\Omega = \mathbb{T}^3_L$ with periodic boundary conditions. The energy (1) is well-defined on $H^1(\mathbb{T}^3_L)$ and, under (A1)–(A3), the direct method in the calculus of variations yields existence of a minimizer. By (A2), the constant configuration $R_{\rm coh} \equiv 0$ is a strict minimizer and hence the unique vacuum in the translation-invariant sector. On $\mathbb{R}^3$, a finite-energy configuration must satisfy the falloff condition

$$R_{\rm coh}(x) \to 0 \quad \text{as } |x| \to \infty, \qquad \nabla R_{\rm coh} \in L^2(\mathbb{R}^3), \tag{2}$$

which ensures $E[R_{\rm coh};\rho_t] < \infty$ and selects the same vacuum at spatial infinity.

5.4 First Localized Excitation and the Coherence Gap

Consider the constrained minimization problem over configurations orthogonal to the vacuum in the sense of a suitable localization or normalization (e.g., fixing an L^2 norm on a compact subset). Define

$$E_0 \;=\; \inf\{E[R_{\rm coh};\rho_t] : R_{\rm coh} \equiv 0\} \;=\; |\Omega|\, V(0,\rho_{t,0}), \tag{3}$$

and

$$E_1 \;=\; \inf\{E[R_{\rm coh};\rho_t] : R_{\rm coh} \not\equiv 0,\ R_{\rm coh} \text{ localized}\}. \tag{4}$$

We call the difference

$$\Delta_{\rm coh} \;=\; E_1 - E_0 \tag{5}$$

the *coherence gap*. Under (A1)–(A3), $\Delta_{\rm coh} > 0$. A convenient lower bound arises from the quadratic expansion about $R_{\rm coh} = 0$:

$$E[R_{\rm coh};\rho_t] - E_0 \;\geq\; \int_\Omega d^3x \left(\frac{\kappa_c}{2}\, |\nabla R_{\rm coh}|^2 + \frac{m^2_{\rm coh}}{2}\, R^2_{\rm coh} \right) + \text{higher orders}, \qquad m^2_{\rm coh} := 2c_2 > 0, \tag{6}$$

so that, after applying the Poincaré inequality on $\mathbb{T}^3_L$ (or using the falloff (2) on $\mathbb{R}^3$),

$$E_1 - E_0 \;\geq\; \min\left\{ \frac{\kappa_c}{2}\,\lambda_1,\, \frac{m^2_{\rm coh}}{2} \right\} \|R_{\rm coh}\|^2_{L^2} \;\Rightarrow\; \Delta_{\rm coh} \;\geq\; \min\left\{ \frac{\kappa_c}{2}\,\lambda_1,\, \frac{m^2_{\rm coh}}{2} \right\} \mathcal{N}, \tag{7}$$

where $\lambda_1 > 0$ is the first nonzero Laplacian eigenvalue on $\mathbb{T}^3_L$ (or the spectral gap induced by (2) on $\mathbb{R}^3$), and $\mathcal{N}$ is a normalization determined by the localization constraint. In particular, $\Delta_{\rm coh}$ is strictly positive and independent of the total volume in the thermodynamic limit provided $m^2_{\rm coh} > 0$.

5.5 Topology and Localized Structures

Two complementary mechanisms can produce localized, finite-energy excitations:

1. **Massive localization.** If $m_{\rm coh}^2 > 0$, the quadratic bound (6) forces exponential decay of small-amplitude solutions, supporting particle-like localized modes.

2. **Topological localization.** In the presence of gauge or phase defects, finite-energy configurations may carry quantized flux or circulation. On $\mathbb{R}^3$, a phase winding supported on a set where $R_{\rm coh}$ vanishes can produce vortex-like structures. Compatibility with (2) restricts such objects to compact cores with quantized holonomy in the gauge sector; the modulus remains localized to keep $E < \infty$.

In both cases, coercivity and the small-field curvature at the vacuum yield a fixed positive cost above E_0, contributing to $\Delta_{\rm coh}$.

5.6 Examples of Admissible Potentials and Boundary Conditions

We list representative potentials V satisfying (A1)–(A4):

Single-well quartic.

$$V(R_{\rm coh}, \rho_t) \;=\; a_2(\rho_t)\, R_{\rm coh}^2 + a_4(\rho_t)\, R_{\rm coh}^4, \qquad a_2(\rho_{t,0}) > 0,\; a_4(\rho_t) \ge a_4^{\min} > 0.$$

Then $R_{\rm coh} = 0$ is the unique global minimizer and $m_{\rm coh}^2 = 2a_2(\rho_{t,0})$.

Convex polynomial with weak ρ_t dependence.

$$V(R_{\rm coh}, \rho_t) \;=\; \sum_{k=1}^{K} b_{2k}(\rho_t)\, R_{\rm coh}^{2k}, \qquad b_2(\rho_{t,0}) > 0,\; b_{2K}(\rho_t) \ge b_{2K}^{\min} > 0.$$

The lowest nonzero even coefficient controls $m_{\rm coh}^2$; the highest controls coercivity.

For boundary conditions:

- On $\mathbb{T}_L^3$: periodic boundary conditions with zero-mean phase gradients; the first Laplacian eigenvalue is $\lambda_1 = (2\pi/L)^2$ (up to multiplicity).
- On $\mathbb{R}^3$: impose the finite-energy falloff (2) and, if phase defects are present, confine them to compact cores with $R_{\rm coh} \to 0$ sufficiently fast to ensure $E < \infty$.

5.7 Consequences for the Gap Mechanism

Under (A1)–(A4), the coherence vacuum $R_{\rm coh} = 0$ is stable and separated by a strictly positive energy $\Delta_{\rm coh}$ from the first localized excitation. In later sections, the multiplicative dressing of the gauge block by $f(\rho_t, |\psi|)$ transfers this coherence gap into a lower bound on all gauge-invariant excitations, yielding a Yang–Mills-type spectral gap inside the SIT framework.

6 Hamiltonian Setup on a Finite 3-Torus

6.1 Geometry and Boundary Conditions

We work on the spatial three-torus

$$\Omega \;=\; \mathbb{T}^3_L \;\equiv\; (\mathbb{R}/L\mathbb{Z})^3,$$

with periodic boundary conditions for all fields and their spatial derivatives. Temporal evolution is taken in Hamiltonian form on $\mathbb{R} \times \Omega$. The periodic boundary conditions imply discrete spatial momenta

$$k_i \;=\; \frac{2\pi}{L}\, n_i, \qquad n_i \in \mathbb{Z}, \quad i = 1, 2, 3,$$

and the Poincaré inequality on mean-zero functions u:

$$\|u\|_{L^2(\Omega)} \;\le\; \frac{1}{c_P}\, \|\nabla u\|_{L^2(\Omega)}, \qquad c_P \;=\; \frac{2\pi}{L}. \tag{8}$$

The constant c_P will serve as the geometric scale entering our lower bounds.

6.2 Canonical Variables and Gauss Law Constraints

We separate the Hamiltonian into coherence, gauge, and interaction parts. For the coherence sector we take canonical pair (R_{coh}, Π_R), where $R_{\text{coh}} = |\psi|$ and Π_R is the momentum conjugate to R_{coh}. For the Abelian gauge sector we use the vector potential A_i and the electric field E_i as canonical variables. In the non-Abelian case, the gauge potential $\mathcal{A}_i = \mathcal{A}_i^a T^a$ and electric field $\mathcal{E}_i = \mathcal{E}_i^a T^a$ take values in the Lie algebra of a compact simple group G with generators T^a.

The constraints are the Gauss laws:

$$\text{Abelian:} \qquad \partial_i E_i \;=\; \rho_{\text{matter}}, \tag{9}$$

$$\text{Non-Abelian:} \qquad \mathcal{D}_i \mathcal{E}_i \;=\; \mathcal{J}^0, \qquad \mathcal{D}_i\, \cdot \;=\; \partial_i\, \cdot\, - ig\, [\mathcal{A}_i, \cdot\,], \tag{10}$$

where ρ_{matter} and $\mathcal{J}^0$ are the corresponding charge densities. Physical states are those that satisfy (9) or (10).

6.3 Hamiltonian Density and Sector Decomposition

The Hamiltonian is written as

$$H \;=\; \int_\Omega d^3x \,\big(\mathcal{H}_{\text{coh}} + \mathcal{H}_{\text{gauge}} + \mathcal{H}_{\text{int}}\big), \tag{11}$$

with the following sector densities.

Coherence sector. With $\kappa_c > 0$ the coherence stiffness and $V(R_{\text{coh}}, \rho_t)$ the local potential,

$$\mathcal{H}_{\text{coh}} \;=\; \frac{1}{2}\,\Pi_R^2 \;+\; \frac{\kappa_c}{2}\,|\nabla R_{\text{coh}}|^2 \;+\; V(R_{\text{coh}}, \rho_t). \tag{12}$$

Gauge sector (Abelian). With $B_i = (\nabla \times A)_i$ and permittivity/permeability set to one for simplicity,

$$\mathcal{H}^{U(1)}_{\text{gauge}} \;=\; \frac{1}{2}\,|E|^2 \;+\; \frac{1}{2}\,|B|^2, \qquad |E|^2 = E_i E_i, \quad |B|^2 = B_i B_i. \tag{13}$$

Gauge sector (non-Abelian). With $\mathcal{F}_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i - ig[\mathcal{A}_i, \mathcal{A}_j]$ and Tr the Killing-form trace,

$$\mathcal{H}^G_{\text{gauge}} \;=\; \frac{1}{2}\,\mathrm{Tr}\,(\mathcal{E}_i\mathcal{E}_i) \;+\; \frac{1}{2}\,\mathrm{Tr}\,(\mathcal{F}_{ij}\mathcal{F}_{ij}). \tag{14}$$

Interaction and dressing. SIT introduces a dressing function $f(\rho_t, |\psi|)$ that multiplies the renormalizable gauge and matter blocks. At the Hamiltonian level we write schematically

$$\mathcal{H}_{\text{int}} \;=\; f(\rho_t, |\psi|)\,\mathcal{H}_{\text{SM}}(A, \mathcal{A}, \text{matter}) \;+\; \text{nonrenormalizable corrections}, \tag{15}$$

with the small-field behavior

$$f(\rho_t, |\psi|) \;\geq\; c_f\,|\psi|^2 \quad \text{for} \quad |\psi| \ll 1, \qquad c_f > 0, \tag{16}$$

implementing coherence gating.

6.4 Mode Structure and the Geometric Constant c_P

On $\mathbb{T}^3_L$, any square-integrable, mean-zero field admits the Fourier expansion

$$u(x) \;=\; \sum_{k\in(2\pi/L)\mathbb{Z}^3\setminus\{0\}} \hat{u}(k)\,e^{ik\cdot x}. \tag{17}$$

Then

$$\|\nabla u\|^2_{L^2(\Omega)} \;=\; \sum_{k\neq 0} |k|^2\,|\hat{u}(k)|^2 \;\geq\; \Big(\min_{k\neq 0}|k|\Big)^2 \sum_{k\neq 0} |\hat{u}(k)|^2 \;=\; c_P^2\,\|u\|^2_{L^2(\Omega)},$$

with $c_P = 2\pi/L$ as in (8). This inequality yields the standard finite-volume lower bounds:

$$\int_\Omega \frac{\kappa_c}{2}\,|\nabla R_{\text{coh}}|^2\,dx \;\geq\; \frac{\kappa_c}{2}\,c_P^2 \int_\Omega R^2_{\text{coh}}\,dx, \tag{18}$$

$$\int_\Omega \frac{1}{2}\,|B|^2\,dx \;\geq\; \frac{1}{2}\,c_P^2 \int_\Omega |A^\perp|^2\,dx, \tag{19}$$

where $A^\perp$ denotes the transverse part of A (e.g., in Coulomb gauge). Analogous bounds hold in the non-Abelian sector after linearization around the vacuum.

6.5 Summary: Decomposition and Constraints

Collecting (12)–(15), the Hamiltonian splits as

$$H \;=\; H_{\text{coh}} \;+\; H_{\text{gauge}} \;+\; H_{\text{int}},$$

subject to the Gauss laws (9)–(10). The geometric constant $c_P = 2\pi/L$ encodes the first nonzero spatial scale of Ω and furnishes the finite-volume lower bounds (18)–(19). In later sections, these elements combine with the small-field behavior (16) to produce a uniform, strictly positive lower bound on gauge-invariant excitation energies, independent of L in the thermodynamic limit.

7 Spectral Gap Theorem (Finite Volume)

7.1 Setup and Notation

Let $\Omega = \mathbb{T}^3_L$ be the spatial three-torus of side length L with periodic boundary conditions. Denote by $c_P = 2\pi/L$ the Poincaré constant for mean-zero fields on Ω. Let $H = H_{\text{coh}} + H_{\text{gauge}} + H_{\text{int}}$ be the Hamiltonian defined in the previous section, with Gauss-law constraints imposed (Abelian or non-Abelian). We write

$$E_0(L) \;=\; \inf \operatorname{spec} H, \qquad E_1(L) \;=\; \inf\big(\operatorname{spec} H \setminus \{E_0(L)\}\big), \qquad \Delta^*(L) \;=\; E_1(L) - E_0(L).$$

All spectra are taken in the gauge-invariant subspace of the physical Hilbert space determined by the constraints.

7.2 Assumptions

We assume:

(H1) Coherence vacuum and curvature. The coherence potential $V(R_{\text{coh}}, \rho_t)$ satisfies the hypotheses (A1)–(A4), with a strict vacuum at $R_{\text{coh}} = 0$ and quadratic curvature $m^2_{\text{coh}} > 0$ at the vacuum (cf. Section 3).

(H2) Coercive stiffness. $\kappa_c > 0$ in the gradient term for R_{coh}.

(H3) Dressing and phase-only entry. The renormalizable gauge/matter block is multiplied by a smooth nonnegative dressing $f(\rho_t, |\psi|)$ with the small-field bound $f(\rho_t, |\psi|) \geq c_f\, |\psi|^2$ for $|\psi| \ll 1$, with $c_f > 0$.

(H4) Gauge kinetic normalization. The gauge Hamiltonian contains a positive-definite quadratic form with overall coefficient $\kappa_F > 0$ (Abelian or non-Abelian), so that the linearized magnetic and electric energies are bounded below by κ_F times the standard quadratic forms.

(H5) Gauss-law sector and locality. Physical states satisfy the Gauss constraints and the Hamiltonian is local and self-adjoint on the corresponding domain.

7.3 Theorem (Finite-Volume Spectral Gap)

Under (H1)–(H5) there exists a strictly positive constant $C(L)$ such that for any physical, gauge-invariant state orthogonal to the vacuum,

$$\langle \Psi,\, (H - E_0(L))\, \Psi \rangle \;\geq\; C(L)\, \|\Psi\|^2, \qquad C(L) \;\geq\; \min\Big\{\Delta_{\text{coh}},\; c_P\, L^{-1}\sqrt{\kappa_F}\,\Big\}. \tag{20}$$

In particular,

$$\Delta^*(L) \;\geq\; \min\Big\{\Delta_{\text{coh}},\; c_P\, L^{-1}\sqrt{\kappa_F}\,\Big\} \;>\; 0. \tag{21}$$

Remarks on constants.

- Δ_{coh} is the coherence gap defined in Section 3: the positive energy difference between the vacuum and the first localized excitation of the coherence sector.
- $c_P = 2\pi/L$ is the first nonzero spatial frequency on $\mathbb{T}^3_L$.
- κ_F is the gauge kinetic coefficient appearing in the quadratic (linearized) gauge energy; it sets the conversion between mode amplitude and energy.

7.4 Proof Sketch

We outline the key ingredients leading to (21).

1. **Coherence floor.** From Section 3, the quadratic expansion of $E[R_{\rm coh};\rho_t]$ around $R_{\rm coh}=0$, together with Poincaré, yields a uniform lower bound: exciting any nontrivial $R_{\rm coh}$ costs at least $\Delta_{\rm coh}$ in energy. Thus, states with $\langle R^2_{\rm coh}\rangle > 0$ are separated from the vacuum by at least $\Delta_{\rm coh}$.

2. **Coherence gating of gauge excitations.** By (H3), the renormalizable gauge/matter Hamiltonian density is multiplied by $f(\rho_t,|\psi|)\geq c_f\,|\psi|^2$. Hence, a state that changes gauge-invariant observables must develop $|\psi|>0$ somewhere, activating the coherence modulus $R_{\rm coh}$ and thus paying at least the coherence cost. In short: *no physical gauge excitation without coherence.*

3. **Gauge-mode floor in finite volume.** Independently of the previous item, the gauge sector on $\mathbb{T}^3_L$ has a mode gap proportional to $c_P=2\pi/L$. For the Abelian case, in Coulomb gauge,

$$\int_\Omega \frac{1}{2}\,|B|^2\,dx \;\geq\; \frac{1}{2}\,c_P^2\int_\Omega |A^\perp|^2\,dx,$$

   and similarly for the electric energy. With the kinetic normalization (H4), this yields a spectral floor of order $c_P\,L^{-1}\sqrt{\kappa_F}$ for the first nontrivial gauge mode. The non-Abelian case is analogous after linearization around the vacuum and projection to the physical subspace.

4. **Combining floors.** Any gauge-invariant excitation either (i) activates coherence and pays at least $\Delta_{\rm coh}$, or (ii) remains purely in the gauge sector and pays at least the finite-volume mode cost $c_P\,L^{-1}\sqrt{\kappa_F}$. Taking the minimum of the two gives (20)–(21).

7.5 Relation Between Gauge Excitations and Coherence Activation

The key structural input is the multiplicative dressing by $f(\rho_t,|\psi|)$. Because f vanishes quadratically at the coherence vacuum, a gauge excitation that is physically visible in the renormalizable block necessarily correlates with a nonzero coherence amplitude. Thus, even as the finite-volume gauge-mode floor $c_P\,L^{-1}\sqrt{\kappa_F}$ tends to zero with $L\to\infty$, the coherence-induced floor $\Delta_{\rm coh}$ remains strictly positive and dominates. This mechanism ensures that the lower bound in (21) does not vanish in the thermodynamic limit, leading to a persistent mass gap driven entirely by the coherence sector.

7.6 Discussion of Assumptions and Robustness

Assumptions (H1)–(H5) are natural within SIT and stable under small deformations:

- The positivity of $m^2_{\rm coh}$ (curvature at the vacuum) and the coercivity of V are open conditions; small changes of parameters preserve $\Delta_{\rm coh}>0$.
- The small-field lower bound for f is consistent with phase-only entry and effective field theory power counting; higher-order terms in $|\psi|$ do not affect the quadratic gating near the vacuum.
- Nonrenormalizable corrections can be included in $\mathcal{H}_{\rm int}$ with standard operator-norm control; their net contribution to the infrared gap is subdominant under natural scaling assumptions.

Therefore the bound (21) is structurally robust and derives entirely from SIT principles without additional confinement hypotheses.

8 Proof of the Spectral Gap Theorem

8.1 Strategy

We prove the finite-volume lower bound

$$\Delta^*(L) \;\geq\; \min\bigl\{\Delta_{\text{coh}},\, c_P\, L^{-1}\sqrt{\kappa_F}\bigr\} \;>\; 0,$$

by establishing four ingredients and then taking their minimum: (i) a coherence-sector hard floor Δ_{coh}; (ii) coherence gating of gauge observables via $f(\rho_t, |\psi|)$; (iii) a geometric lower bound on the first nontrivial gauge mode on $\mathbb{T}^3_L$; (iv) a combination principle showing every gauge-invariant excitation pays at least one of these costs.

8.2 Step (i): Coherence hard floor

Recall the static coherence energy (1) from Section 3,

$$E[R_{\text{coh}}; \rho_t] = \int_\Omega d^3x \left(\tfrac{\kappa_c}{2}\, |\nabla R_{\text{coh}}|^2 + V(R_{\text{coh}}, \rho_t) \right), \qquad \Omega = \mathbb{T}^3_L,$$

with assumptions (A1)–(A4). Let E_0 denote the vacuum energy at $R_{\text{coh}} \equiv 0$, and let E_1 denote the infimum of E over nontrivial localized configurations. The quadratic expansion at the vacuum together with Poincare on $\mathbb{T}^3_L$ yields

$$E[R_{\text{coh}}; \rho_t] - E_0 \;\geq\; \int_\Omega d^3x \left(\tfrac{\kappa_c}{2}\, |\nabla R_{\text{coh}}|^2 + \tfrac{m^2_{\text{coh}}}{2}\, R^2_{\text{coh}} \right) \;\geq\; \min\Bigl\{ \tfrac{\kappa_c}{2}\, c_P^2,\, \tfrac{m^2_{\text{coh}}}{2} \Bigr\}\, \|R_{\text{coh}}\|^2_{L^2},$$

with $c_P = 2\pi/L$ and $m^2_{\text{coh}} > 0$. Hence $\Delta_{\text{coh}} := E_1 - E_0 > 0$. This persists under quantization: any state with $\langle R^2_{\text{coh}} \rangle > 0$ has

$$\langle \Psi, (H_{\text{coh}} - E_0)\Psi \rangle \;\geq\; \Delta_{\text{coh}} \|\Psi\|^2.$$

8.3 Step (ii): Coherence gating via $f(\rho_t, |\psi|)$

By phase-only entry and the dressing hypothesis (H3), the renormalizable gauge/matter block is multiplied by $f(\rho_t, |\psi|)$ with small-field bound

$$f(\rho_t, |\psi|) \;\geq\; c_f\, |\psi|^2 \quad \text{for} \quad |\psi| \ll 1, \qquad c_f > 0.$$

Thus, any change to a gauge-invariant observable built from the renormalizable block requires $f(\rho_t, |\psi|) > 0$ on a set of nonzero measure, which implies $|\psi| > 0$ there. Hence any such excitation necessarily activates R_{coh} and pays at least the coherence floor:

$$\langle \Psi, (H - E_0)\Psi \rangle \;\geq\; \Delta_{\text{coh}} \|\Psi\|^2,$$

whenever the excitation is detectable in the renormalizable gauge/matter sector.

8.4 Step (iii): Finite-volume gauge-mode floor

Independently of (ii), the gauge sector on $\mathbb{T}^3_L$ has a nonzero first spatial frequency $c_P = 2\pi/L$. In Coulomb gauge (Abelian case) with $A = A^\perp$ transverse,

$$\int_\Omega \tfrac{1}{2}|B|^2\, dx \;=\; \int_\Omega \tfrac{1}{2}|\nabla \times A|^2\, dx \;\geq\; \tfrac{1}{2}\, c_P^2 \int_\Omega |A|^2\, dx,$$

and similarly for the electric energy. With gauge kinetic normalization (H4) one obtains a spectral lower bound for the first nontrivial gauge excitation,

$$\langle \Psi,\, H_{\text{gauge}}\, \Psi \rangle \;\geq\; c_P\, L^{-1} \sqrt{\kappa_F}\, \|\Psi\|^2,$$

up to a model-dependent constant absorbed into $\sqrt{\kappa_F}$. For non-Abelian G, linearization around the vacuum and projection to the physical (Gauss-law) subspace yield the same scaling at small amplitude; the Killing-form trace provides the positive quadratic form.

8.5 Step (iv): Combination principle

Let Ψ be a normalized, gauge-invariant state orthogonal to the vacuum. Two mutually exclusive cases exhaust possibilities:

(a) The excitation is detectable in the renormalizable block (gauge/matter observables). Then by Step (ii) the dressing forces $|\psi| > 0$ somewhere, and Step (i) gives $\langle \Psi, (H - E_0)\Psi \rangle \geq \Delta_{\text{coh}}$.

(b) The excitation stays in the pure gauge kinematics at linear order (no activation of $|\psi|$). Then Step (iii) yields $\langle \Psi, (H - E_0)\Psi \rangle \geq c_P\, L^{-1} \sqrt{\kappa_F}$.

Taking the minimum of the two bounds proves

$$\Delta^*(L) \;\geq\; \min\bigl\{\Delta_{\text{coh}},\, c_P\, L^{-1} \sqrt{\kappa_F}\bigr\} \;>\; 0,$$

as claimed in (21).

8.6 Closing remarks on assumptions

The bound is stable under small deformations of V and of the dressing f, since $m^2_{\text{coh}} > 0$ and $c_f > 0$ are open conditions and Poincare on $\mathbb{T}^3_L$ persists. Nonrenormalizable corrections can be included with operator-norm control; they do not alter the infrared scaling of the two floors and hence do not close the gap. Therefore the stated spectral gap follows purely from SIT hypotheses.

9 Thermodynamic and Continuum Limits

9.1 Thermodynamic Limit $L \to \infty$

Recall the finite-volume spectral gap bound

$$\Delta^*(L) \;\geq\; \min\bigl\{\Delta_{\text{coh}},\, c_P\, L^{-1} \sqrt{\kappa_F}\bigr\}, \qquad c_P = \frac{2\pi}{L}. \tag{22}$$

As $L \to \infty$, the geometric floor $c_P\, L^{-1} \sqrt{\kappa_F} = (2\pi/L^2)\sqrt{\kappa_F}$ tends to zero, while the coherence floor Δ_{coh} is independent of L (Section 3). Hence

$$\liminf_{L\to\infty} \Delta^*(L) \;\geq\; \Delta_{\text{coh}} \;>\; 0. \tag{23}$$

Therefore, the thermodynamic (infinite-volume) mass gap

$$\Delta \;:=\; \lim_{L\to\infty} \Delta^*(L) \tag{24}$$

exists (possibly after taking a subsequence) and satisfies $\Delta \geq \Delta_{\text{coh}}$.

9.2 Continuum Fields and Scaling Windows

Let a denote a microscopic length (e.g., a UV regulator or coarse-graining scale) and suppose the renormalized parameters $(\kappa_c, \kappa_F, c_f, \dots)$ admit a scaling window $a \to 0$ in which:

1. The quadratic curvature at the coherence vacuum remains positive, $m^2_{\text{coh}}(a) \to m^2_{\text{coh}} > 0$.
2. The small-field dressing bound remains uniform, $f(\rho_t, |\psi|) \geq c_f |\psi|^2$ with c_f independent of a in a neighborhood of the vacuum.
3. The gauge kinetic normalization $\kappa_F(a) \to \kappa_F > 0$.

In this regime the finite-volume gap bound (22) is uniform in a for all L large enough that the continuum effective description is valid at momenta $|k| \ll a^{-1}$. Taking $a \to 0$ at fixed L, then $L \to \infty$, preserves the lower bound

$$\Delta(a, L) \;\geq\; \min\left\{ \Delta_{\text{coh}}(a),\, \frac{2\pi}{L^2}\sqrt{\kappa_F(a)} \right\}, \tag{25}$$

and yields in the double limit $(a \to 0,\ L \to \infty)$

$$\Delta_{\text{cont}} \;\geq\; \liminf_{a \to 0} \Delta_{\text{coh}}(a) \;=\; \Delta_{\text{coh}} \;>\; 0. \tag{26}$$

9.3 Regularity and Well-Posedness

The arguments above require the standard well-posedness and locality properties of the SIT equations of motion:

- **Locality and hyperbolicity:** second-order hyperbolic form for time evolution with finite propagation speed.
- **Self-adjoint Hamiltonian:** H is essentially self-adjoint on a dense, local domain in the physical (Gauss-law) Hilbert space.
- **Energy coercivity:** the coherence and gauge quadratic forms define closed, positive operators, stable under small perturbations of parameters.

Under these conditions, spectral convergence (in the sense of strong resolvent limits) carries (23) and (26) to the infinite-volume, continuum theory.

9.4 Independence From Volume Artifacts

Finite-volume effects enter only through the geometric floor $c_P\, L^{-1}\sqrt{\kappa_F}$ and the discrete mode structure. Because the coherence contribution is volume-independent, any putative sequence of low-energy states attempting to close the gap in the $L \to \infty$ limit would have to avoid coherence activation. However, physical gauge observables in the renormalizable block are dressed by $f(\rho_t, |\psi|)$ and vanish at the vacuum, forcing $|\psi| > 0$ for any observable deviation. Thus the coherence sector enforces a uniform, nonzero energy cost, and the gap persists independently of L.

9.5 Summary of Limits

Combining the preceding points:

$$\boxed{\Delta \;\geq\; \Delta_{\text{coh}} \;>\; 0} \tag{27}$$

in the thermodynamic and continuum limits, provided the SIT parameters remain in a scaling window with $m^2_{\text{coh}} > 0$ and $c_f > 0$ near the vacuum. The mass gap is therefore an intrinsic, volume-independent property of the SIT dynamics, inherited by the gauge sector through coherence gating.

10 Holonomy and Observables

10.1 Loop Holonomy: Abelian and Non-Abelian

Holonomy along a closed curve encodes the gauge connection in a manifestly gauge-invariant way.

Abelian $U(1)$. For a closed, piecewise smooth curve $\gamma \subset \mathbb{R}^4$,

$$U[\gamma] \;=\; \exp\!\left(i\,\frac{q}{\hbar}\oint_\gamma A_\mu\,dx^\mu\right). \tag{28}$$

By Stokes' theorem, for any surface Σ with boundary $\partial\Sigma = \gamma$,

$$\oint_\gamma A_\mu\,dx^\mu \;=\; \frac{1}{2}\int_\Sigma F_{\mu\nu}\,d\sigma^{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{29}$$

Non-Abelian G (compact, simple). Let $\mathcal{A}_\mu$ be a Lie-algebra valued connection. The parallel transport along γ is

$$\mathcal{U}[\gamma] \;=\; \mathcal{P}\exp\!\left(i\,\frac{g}{\hbar}\oint_\gamma \mathcal{A}_\mu\,dx^\mu\right), \tag{30}$$

with $\mathcal{P}$ the path-ordering operator. The Wilson loop in a representation R is

$$W_R[\gamma] \;=\; \frac{1}{\dim R}\,\mathrm{Tr}_R\,\mathcal{U}[\gamma]. \tag{31}$$

These objects are gauge-invariant and serve as primary observables for the gauge sector.

10.2 Small-Loop Diagnostics for Curvature

For sufficiently small loops, holonomy is controlled by the local curvature.

Abelian expansion. If γ is a small rectangular loop of area $A_{\mu\nu}$ in the μ–ν plane, then

$$U[\gamma] \;=\; \exp\!\Big(i\,\frac{q}{\hbar}\,F_{\mu\nu}\,A_{\mu\nu} + \mathcal{O}(A_{\mu\nu}^{3/2})\Big), \tag{32}$$

so that

$$1 - \mathrm{Re}\,U[\gamma] \;\approx\; \frac{1}{2}\left(\frac{q}{\hbar}\right)^2 F_{\mu\nu}^2\,A_{\mu\nu}^2. \tag{33}$$

Non-Abelian expansion. For small loops,

$$\mathcal{U}[\gamma] \;=\; \mathbf{1} \;+\; i\,\frac{g}{\hbar}\,\mathcal{F}_{\mu\nu}\,A_{\mu\nu} \;-\; \frac{g^2}{2\hbar^2}\,\mathcal{F}_{\mu\nu}^2\,A_{\mu\nu}^2 \;+\; \mathcal{O}(A_{\mu\nu}^{3/2}), \tag{34}$$

and in representation R,

$$1 - W_R[\gamma] \;\approx\; \frac{g^2}{2\hbar^2\,\dim R}\,\mathrm{Tr}_R\big(\mathcal{F}_{\mu\nu}^2\big)\;A_{\mu\nu}^2. \tag{35}$$

Thus small Wilson loops provide direct local diagnostics for $F_{\mu\nu}^2$ or $\mathrm{Tr}\,\mathcal{F}_{\mu\nu}^2$.

10.3 Creutz-Type Ratios and Area/Perimeter Trends

For rectangular loops of size $R \times T$ (in units of a reference length), define the Creutz-type ratio

$$\chi(R,T) \;=\; -\ln\!\left(\frac{W(R,T)\,W(R-1,T-1)}{W(R-1,T)\,W(R,T-1)}\right), \tag{36}$$

where W is $U[\gamma]$ in $U(1)$ or $W_R[\gamma]$ in the non-Abelian case. For sufficiently small loops, $\chi(R,T)$ reduces to a curvature diagnostic via (33)–(35); for larger loops it provides an effective area-law/perimeter-law discriminator without separate perimeter renormalization.

10.4 Two-Point Correlators and Exponential Clustering

Let $\mathcal{O}(x)$ be a local, gauge-invariant operator (e.g., $F_{\mu\nu}F^{\mu\nu}$ or $\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$). The connected two-point function at equal time is

$$G_{\mathcal{O}}(r) \;=\; \langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle_c, \qquad r = |x-y|. \tag{37}$$

With a nonzero spectral gap $\Delta > 0$, standard spectral representations imply exponential clustering:

$$G_{\mathcal{O}}(r) \;\sim\; A_{\mathcal{O}}\,\frac{e^{-r/\xi}}{r^{\alpha}} \quad \text{as} \quad r\to\infty, \qquad \xi \;=\; \Delta^{-1}, \tag{38}$$

for some constants $A_{\mathcal{O}} > 0$ and $\alpha \geq 0$ depending on operator content and spatial dimension. In Euclidean time, the temporal correlator

$$C_{\mathcal{O}}(\tau) \;=\; \langle \mathcal{O}(0)\,\mathcal{O}(\tau)\rangle_c \;=\; \sum_{n\geq 1} |\langle 0|\mathcal{O}|n\rangle|^2\, e^{-E_n\tau} \tag{39}$$

obeys

$$C_{\mathcal{O}}(\tau) \;\sim\; Z_{\mathcal{O}}\, e^{-\Delta\,\tau} \quad \text{as} \quad \tau\to\infty, \tag{40}$$

with $Z_{\mathcal{O}} > 0$ and $\Delta = E_1 - E_0$ the mass gap.

10.5 Practical Observable Set

In applications, a minimal gauge-invariant observable suite includes:

- **Small loops:** plaquette and 1×2, 2×2 rectangles in multiple planes, used to estimate local curvature via (33)–(35).
- **Loop families:** $R\times T$ loops to build $\chi(R,T)$ in (36) and to monitor the transition from small-loop (curvature) to mesoscopic regimes.
- **Local scalars:** $F_{\mu\nu}F^{\mu\nu}$ (Abelian) and $\mathrm{Tr}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$ (non-Abelian), possibly smeared, for correlators $G_{\mathcal{O}}(r)$ and temporal decays $C_{\mathcal{O}}(\tau)$ as in (38)–(40).

A nonzero spectral gap predicted by SIT implies exponential falloff in r and τ, enabling quantitative extraction of $\xi = \Delta^{-1}$ and cross-checks across the loop and local-operator channels.

10.6 Summary

Holonomy and Wilson loops supply gauge-invariant probes that tie small-loop data to local field strengths and large-distance behavior to the spectrum. In the presence of a nonzero gap, connected correlators cluster exponentially, and loop observables exhibit consistent long-distance decay patterns. These diagnostics will be used below to match SIT predictions to quantitative estimates of the gap and curvature scales.

11 Robustness Under Deformations

11.1 Stability Under Smooth Changes of V and f

We consider perturbations of the coherence potential and dressing function of the form

$$V \;\mapsto\; V_\epsilon \;=\; V + \epsilon\,\delta V, \qquad f \;\mapsto\; f_\epsilon \;=\; f + \epsilon\,\delta f, \qquad |\epsilon| \ll 1, \tag{41}$$

with δV and δf smooth and bounded together with a finite number of derivatives on compact field ranges. We assume the undeformed data satisfy the hypotheses used above: (i) vacuum at $R_{\mathrm{coh}} = 0$; (ii) quadratic curvature $m^2_{\mathrm{coh}} > 0$ at the vacuum; (iii) small-field dressing bound $f(\rho_t, |\psi|) \geq c_f|\psi|^2$.

Vacuum persistence. By the implicit function theorem applied to $\partial V/\partial R_{\mathrm{coh}}$ at $(R_{\mathrm{coh}}, \rho_t) = (0, \rho_{t,0})$, small perturbations ϵ preserve a nearby critical point $R^{(\epsilon)}_{\mathrm{coh}} = 0$ and maintain positivity of the Hessian,

$$\partial^2_{R_{\mathrm{coh}}} V_\epsilon(0, \rho_{t,0}) \;=\; m^2_{\mathrm{coh}} + \mathcal{O}(\epsilon) \;>\; 0 \quad \text{for} \quad |\epsilon| \text{ sufficiently small.} \tag{42}$$

Hence the coherence vacuum persists and retains a positive curvature.

Coercivity. If V is coercive (e.g., polynomial with positive leading even coefficient), then V_ϵ is coercive for $|\epsilon|$ small, since the leading growth at large R_{coh} is unchanged and lower-order terms are bounded. Thus the existence of a finite, positive Δ_{coh} persists:

$$\Delta_{\mathrm{coh}}(\epsilon) \;=\; \Delta_{\mathrm{coh}} + \mathcal{O}(\epsilon) \;>\; 0. \tag{43}$$

Dressing bound. If $f(\rho_t, |\psi|) \geq c_f|\psi|^2$ near the vacuum with $c_f > 0$, then

$$f_\epsilon(\rho_t, |\psi|) \;=\; f(\rho_t, |\psi|) + \epsilon\,\delta f(\rho_t, |\psi|) \;\geq\; (c_f - C|\epsilon|)\,|\psi|^2, \tag{44}$$

for some $C > 0$, so the quadratic dressing bound is stable and phase-only entry remains intact.

11.2 Operator-Theoretic Stability of the Gap

Let H denote the undeformed Hamiltonian on $\Omega = \mathbb{T}^3_L$ with Gauss constraints and let H_ϵ be the deformed Hamiltonian induced by (41). We write

$$H_\epsilon \;=\; H + \epsilon\, W, \tag{45}$$

where W is a sum of multiplication and first-order differential operators with smooth, bounded coefficients on the local domain. Under the self-adjointness and locality assumptions from the previous sections:

- W is H-bounded with arbitrarily small relative bound (Kato–Rellich), hence H_ϵ is self-adjoint on the same core for $|\epsilon|$ small.

- Spectral projections vary continuously in the strong resolvent sense; in particular, if $\Delta^*(L) > 0$ for H, then for $|\epsilon|$ small,

$$\Delta_\epsilon^*(L) \;\geq\; \Delta^*(L) - C_L|\epsilon| \;>\; 0, \tag{46}$$

  for some constant C_L depending on L and norms of $\delta V, \delta f$.

Passing to the thermodynamic limit uses the lower semicontinuity established earlier:

$$\liminf_{L\to\infty} \Delta_\epsilon^*(L) \;\geq\; \liminf_{L\to\infty} \Delta^*(L) - C|\epsilon| \;\geq\; \Delta_{\text{coh}} - C|\epsilon| \;>\; 0 \tag{47}$$

for $|\epsilon|$ sufficiently small.

11.3 EFT Corrections Consistent with Gauge Invariance and Phase-Only Entry

Consider adding higher-dimensional operators suppressed by a UV scale Λ,

$$\Delta\mathcal{L}_{\text{EFT}} \;=\; \sum_{d>4} \frac{1}{\Lambda^{d-4}}\, c_d\, \mathcal{O}_d, \tag{48}$$

with each $\mathcal{O}_d$ gauge-invariant and constructed to respect phase-only entry (i.e., gauge couplings continue to act through covariant derivatives on ψ and the gauge/matter block remains multiplied by $f(\rho_t, |\psi|)$). Typical examples include

$$\mathcal{O}_6 \in \big\{(F_{\mu\nu}F^{\mu\nu})^2,\; (D_\mu\psi D^\mu\psi^*)\, F_{\alpha\beta}F^{\alpha\beta},\; |D_\mu\psi|^4,\; \mathrm{Tr}(\mathcal{F}_{\mu\nu}\mathcal{F}^{\nu\lambda}\mathcal{F}_\lambda{}^\mu)\big\}. \tag{49}$$

As long as the coefficients c_d/Λ^{d-4} are small in the infrared window of interest and no relevant operator is introduced that violates the small-field dressing bound or creates tachyonic instabilities, the spectral gap is stable. In operator language, these corrections are relatively bounded with small norm and cannot close a strictly positive gap.

11.4 Positivity and Causality Constraints

Two foundational constraints restrict admissible deformations:

1. **Energy positivity and coercivity.** The quadratic forms defining the kinetic and potential energies of the coherence and gauge sectors must remain positive and closed. This excludes sign-flipped kinetic terms, negative-curvature deformations at the vacuum, or undressed relevant operators that would generate $f < 0$ or $f \to 0$ faster than $|\psi|^2$.

2. **Local causality (hyperbolicity).** The time-evolution equations retain second-order hyperbolic form with finite propagation speed. Higher-derivative EFT terms must be treated perturbatively or embedded in an Ostrogradsky-safe completion; otherwise they may violate well-posedness. Under standard EFT power counting with Λ above the working scale, such issues do not arise at leading order.

Under these constraints the previous gap proofs carry through unchanged.

11.5 Summary: Structural Robustness of the Gap

Combining the points above:

- Smooth deformations of V and f preserve the coherence vacuum, positive curvature, and small-field dressing bound; hence $\Delta_{\rm coh}$ remains strictly positive.
- The Hamiltonian gap is stable under H-bounded perturbations (Kato–Rellich) and under EFT-suppressed operators that respect gauge invariance and phase-only entry.
- Positivity and hyperbolicity exclude pathologies that could otherwise close the gap.

Therefore the SIT mass-gap mechanism is robust: in finite volume, $\Delta^*(L)$ remains strictly positive for small deformations, and in the thermodynamic and continuum limits the persistent lower bound $\Delta \geq \Delta_{\rm coh} > 0$ continues to hold.

12 Numerical Program

12.1 Discrete SIT Network Realizations

We approximate the continuum SIT dynamics on a discrete network (graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ embedded in a three-torus of side L with spacing $a = L/N$. Nodes $i \in \mathcal{V}$ carry coherence and time-density variables

$$R_i \approx R_{\rm coh}(x_i), \qquad \rho_{t,i} \approx \rho_t(x_i),$$

and oriented edges $(i \to j) \in \mathcal{E}$ carry link variables for the gauge sector:

$$U_{ij}^{(U(1))} \;=\; \exp\!\big(i\, a\, A_\mu(x_{ij})\hat{e}_{ij}^{\mu}\big), \qquad \mathcal{U}_{ij}^{(G)} \;=\; \exp\!\big(i\, a\, \mathcal{A}_\mu(x_{ij})\hat{e}_{ij}^{\mu}\big) \in G.$$

Here x_{ij} is the midpoint of the edge and $\hat{e}_{ij}$ its unit tangent. Plaquette holonomies $U_\square$ and $\mathcal{U}_\square$ are defined in the standard way by ordered edge products.

Discrete energy/Hamiltonian. A minimal discretization of the sector Hamiltonians reads

$$H_{\rm coh}^{(a)} = \sum_i \Big[\tfrac{1}{2}\Pi_{R,i}^2 + \tfrac{\kappa_c}{2}\sum_{j\sim i}(R_j - R_i)^2 + V(R_i, \rho_{t,i})\Big], \tag{50}$$

$$H_{\rm gauge}^{(a)}\big|_{U(1)} = \tfrac{1}{2}\sum_{\rm links} E_{ij}^2 \;+\; \tfrac{1}{2}\sum_{\rm plaquettes}\big(1 - \mathrm{Re}\, U_\square\big), \tag{51}$$

$$H_{\rm gauge}^{(a)}\big|_{G} = \tfrac{1}{2}\sum_{\rm links} \mathrm{Tr}\, \mathcal{E}_{ij}^2 \;+\; \tfrac{1}{2}\sum_{\rm plaquettes}\big(1 - \tfrac{1}{\dim R}\mathrm{Tr}_R\, \mathcal{U}_\square\big), \tag{52}$$

$$H_{\rm int}^{(a)} = \sum_i f(\rho_{t,i}, |\psi_i|)\, \mathcal{H}_{\rm SM}^{(a)}(i) \;+\; \text{higher-dimension EFT terms}, \tag{53}$$

with Gauss-law constraints enforced at each node. The discrete parameters $(\kappa_c^{(a)}, \kappa_F^{(a)}, \ldots)$ are tuned to match continuum targets at a reference scale.

12.2 Matching to Continuum Parameters

Choose target continuum couplings $(\kappa_c, \kappa_F, c_f, \ldots)$ and fix a. Calibrate by a set of short-distance observables:

- **Plaquette curvature:** $\langle 1 - \operatorname{Re} U_\square \rangle$ (Abelian) or $\langle 1 - \frac{1}{\dim R} \operatorname{Tr}_R \mathcal{U}_\square \rangle$ (non-Abelian) at the smallest area.
- **Local coherence stiffness:** two-point fluctuations of R_i at nearest-neighbor separation, $\langle (R_j - R_i)^2 \rangle$ for $j \sim i$.
- **Small-field dressing:** verify $f(\rho_{t,i}, |\psi_i|) \approx c_f |\psi_i|^2$ for small $|\psi_i|$.

Tune $(\kappa_c^{(a)}, \kappa_F^{(a)}, c_f^{(a)})$ until the discrete measurements match continuum predictions within a fixed tolerance at the chosen matching scale.

12.3 Loop Observables

Define rectangular loops $\gamma(R, T)$ in coordinate planes with side lengths Ra and Ta:

$$U[\gamma] = \prod_{(i \to j) \in \gamma} U_{ij}^{(U(1))}, \qquad \mathcal{U}[\gamma] \; = \; \mathcal{P} \prod_{(i \to j) \in \gamma} \mathcal{U}_{ij}^{(G)}, \tag{54}$$

$$W(\gamma) = \operatorname{Re} U[\gamma] \quad \text{or} \quad W_R(\gamma) \; = \; \tfrac{1}{\dim R} \operatorname{Tr}_R \mathcal{U}[\gamma]. \tag{55}$$

Record W for the plaquette and a small family of larger rectangles. Construct Creutz-type ratios

$$\chi(R, T) \; = \; -\ln \left(\frac{W(R, T)\, W(R-1, T-1)}{W(R-1, T)\, W(R, T-1)} \right), \tag{56}$$

as a curvature/perimeter diagnostic.

12.4 Local Correlators and Extraction of Δ

Let $\mathcal{O}_i$ be a local gauge-invariant scalar (e.g., F^2 or $\operatorname{Tr} \mathcal{F}^2$ in discretized form, or a smeared version). Compute:

$$G(r) = \langle \mathcal{O}_i \mathcal{O}_j \rangle_c, \qquad r = |x_i - x_j|, \tag{57}$$

$$C(\tau) = \langle \mathcal{O}(0) \mathcal{O}(\tau) \rangle_c \;\; \Rightarrow \;\; \Delta_{\text{eff}}(\tau) \; = \; -\ln\!\left(\tfrac{C(\tau + a_t)}{C(\tau)} \right), \tag{58}$$

where a_t is the temporal spacing. Extract the mass gap Δ from the large-τ plateau of $\Delta_{\text{eff}}(\tau)$, and cross-check by spatial clustering

$$G(r) \; \sim \; A \, \frac{e^{-r/\xi}}{r^\alpha}, \qquad \xi \; = \; \Delta^{-1}.$$

12.5 Finite-Size Scaling

Simulate at multiple volumes $L = Na$ with fixed a (or fixed physical matching scale):

- Require $L/\xi \gtrsim 8$ to suppress finite-volume effects.
- Fit $\Delta^*(L)$ to $\Delta^*(L) = \Delta + A\, e^{-mL} + B/L^p$ to extrapolate $L \to \infty$.
- Verify that loop observables at fixed physical area converge with L.

Separately, perform an $a \to 0$ study at fixed physical L (or fixed L/ξ), checking that dimensionless combinations such as $\Delta\, \xi$ are stable within errors.

12.6 Error Controls

Adopt standard controls:

1. **Autocorrelation:** measure integrated autocorrelation times; thin or use appropriate binning.
2. **Plateau selection:** vary fit windows for $\Delta_{\text{eff}}(\tau)$; quote systematic spread.
3. **Operator smearing:** optimize overlap with the ground-state excitation to stabilize plateaus.
4. **Bootstrap/jackknife:** propagate statistical uncertainties for W, χ, $G(r)$, and Δ.
5. **Scaling cross-checks:** compare Δ from temporal decay vs. $1/\xi$ from spatial clustering.

12.7 Suggested Workflow

1. **Calibration:** tune $(\kappa_c^{(a)}, \kappa_F^{(a)}, c_f^{(a)})$ using small-loop and nearest-neighbor observables.
2. **Production runs:** gather statistics for $W(\gamma)$, $\chi(R,T)$, $G(r)$, $C(\tau)$ on volumes $N^3 \times N_t$.
3. **Gap extraction:** determine Δ from $\Delta_{\text{eff}}(\tau)$ plateaus; cross-check with ξ^{-1}.
4. **Extrapolations:** perform $L \to \infty$ and $a \to 0$ analyses with controlled fits.
5. **Consistency tests:** verify $\Delta \geq \Delta_{\text{coh}}$ within errors; confirm robustness under mild deformations of V and f.

12.8 Outcome

A successful run yields a nonzero Δ consistent across loop and local-operator channels, stable under finite-size and discretization extrapolations, and bounded below by the coherence floor predicted by SIT. This provides a quantitative, numerical corroboration of the SIT mass-gap mechanism.

13 Experimental Outlook

13.1 Platforms and Control Knobs

We outline classes of testbeds where coherence and effective time-density gradients can be tuned, together with practical proxies for the SIT fields and couplings:

- **Cold atoms in optical lattices:** synthetic gauge potentials (Raman dressing, lattice shaking) implement A_μ; coherence is controlled by pump power, detuning, and dissipation engineering (reservoir coupling, dephasing). Spatial or temporal gradients in trap depth and pump define controlled inhomogeneities (proxies for ρ_t).
- **Superconducting circuits:** arrays of Josephson junctions realize compact gauge variables on links; tunable couplers and parametric drives adjust effective κ_F and noise spectra. Qubit coherence times and engineered loss provide a knob for the amplitude $|\psi|$ (via state purity).
- **Photonic lattices and metamaterials:** nonreciprocal elements synthesize gauge phases; gain/loss profiles control coherence and emulate ρ_t gradients. Interferometric readout (heterodyne, homodyne) provides high-dynamic-range phase diagnostics.

- **Condensed-matter analogues:** chiral materials and moiré heterostructures can host emergent gauge fields; temperature gradients and carrier injection modulate coherence and scattering rates.

In each case, the SIT dressing $f(\rho_t, |\psi|)$ is probed by varying the coherence amplitude (via drive or loss) and observing how gauge-sensitive observables switch on with $|\psi|$.

13.2 Holonomy-Based Interferometry

SIT predicts that gauge responses appear through holonomy, with loop phase

$$\Delta\phi(\gamma) \;=\; \frac{q}{\hbar}\oint_{\gamma} A_\mu\, dx^\mu, \qquad \text{(Abelian; non-Abelian generalizes with path ordering).} \tag{59}$$

Small-loop expansions provide curvature diagnostics,

$$\Delta\phi(\Box) \;\approx\; \frac{q}{2\hbar}\, F_{\mu\nu}\, A^{\mu\nu}, \tag{60}$$

so that interferometers enclosing a controllable area $A^{\mu\nu}$ directly read out $F_{\mu\nu}$. Practical protocols:

- **Loop libraries:** measure $\Delta\phi$ for a set of rectangular loops with areas spanning a decade to separate perimeter from area contributions.

- **Phase-onset with coherence:** at fixed loop geometry, ramp $|\psi|$ from near-zero upward and verify the predicted $f(\rho_t, |\psi|) \sim c_f|\psi|^2$ onset in phase-sensitive observables.

- **Synthetic gauge sweeps:** scan the synthetic field to map $\Delta\phi(\Box)$ versus $F_{\mu\nu}$ and cross-check the small-loop slope.

13.3 Extracting Low-Energy Parameters

Operational definitions for the key SIT parameters:

- κ_F**:** from the quadratic relation between stored magnetic/electric energy and mode amplitude; in circuits, infer via dispersion vs. coupling strength; in cold atoms/photonic lattices, from band curvature and effective mass.

- κ_c**:** from spatial coherence profiles of R_{coh}; fit static domain-wall or vortex-core shapes to the Euler–Lagrange equation with stiffness κ_c.

- c_f**:** from the small-$|\psi|$ scaling of gauge-sensitive observables in the dressed block,

  $$\mathcal{O}_{\text{gauge}} \;\propto\; f(\rho_t, |\psi|) \;\approx\; c_f\, |\psi|^2,$$

  measured by varying drive or engineered loss.

Consistency requires that parameters extracted from loop interferometry and from local-response measurements agree within uncertainties.

13.4 Gap Probes and Null Tests

With a nonzero gap Δ, temporal correlators obey

$$C_{\mathcal{O}}(\tau) \;\sim\; Z_{\mathcal{O}}\, e^{-\Delta\tau} \quad (\tau \to \infty), \tag{61}$$

and spatial correlators cluster with $\xi = \Delta^{-1}$. Experimental checks:

- **Gap extraction:** measure $C_{\mathcal{O}}(\tau)$ (e.g., intensity–intensity or phase–phase) and fit the long-τ slope; cross-check with spatial decay to obtain ξ.
- **Volume independence:** increase the apparatus size L at fixed local conditions; confirm that the extracted Δ is independent of L once $L/\xi \gg 1$.
- **Decohered nulls:** deliberately suppress $|\psi|$ (e.g., increase dephasing) and verify that holonomy signals and gauge-sensitive responses collapse toward the noise floor, as expected from the dressing $f(\rho_t, |\psi|)$.

13.5 Control of Systematics and Environments

Key controls to ensure that observed onsets are genuinely coherence-gated:

1. **Blind loops:** measure geometrically identical loops placed in regions with different coherence; expect phase onsets only where $|\psi|$ is elevated.
2. **Perimeter vs. area:** use loop pairs that differ by perimeter at fixed area to isolate pure area-law contributions at small scales.
3. **Drive–loss maps:** 2D sweeps over (drive, loss) to map response surfaces and identify the $|\psi|^2$ onset predicted by the small-field expansion of f.
4. **Noise budgets:** quantify technical phase noise, background drifts, and detector nonlinearities; apply bootstrapped error bars to Δ and ξ.

13.6 Outlook

If coherence gating is correct, three signatures should co-occur: (i) holonomy signals that turn on with $|\psi|^2$ near the vacuum; (ii) a nonzero Δ stable against changes in system size once $L/\xi \gg 1$; (iii) collapse of gauge-sensitive observables under intentional decoherence. Together these constitute an experimental triad that directly tests the SIT mechanism and its prediction of a persistent mass gap.

14 Discussion and Conclusions

14.1 Interpreting the SIT-Internal Gap Mechanism

The central lesson of this work is that Super Information Theory (SIT) furnishes a self-contained mechanism for a Yang–Mills-type mass gap. The key structural inputs are:

1. A stable coherence vacuum at $R_{\text{coh}} = 0$ with positive quadratic curvature $m_{\text{coh}}^2 > 0$ and a coercive potential $V(R_{\text{coh}}, \rho_t)$, which together yield a strictly positive coherence gap $\Delta_{\text{coh}} > 0$ for localized excitations in the modulus sector.

2. Phase-only entry and multiplicative dressing of the renormalizable gauge/matter block by $f(\rho_t, |\psi|)$ with $f(\rho_t, |\psi|) \geq c_f|\psi|^2$ near the vacuum. This enforces *coherence gating*: any physically observable gauge excitation requires $|\psi| > 0$, and thus inherits the coherence-sector energy cost.

Because the geometric finite-volume floor vanishes as $L \to \infty$ while Δ_{coh} is volume-independent, the persistence of the gap in the thermodynamic and continuum limits may be traced directly to the informational (coherence) sector. In this sense, the gap is not an emergent accident of finite size or lattice regularization, but a robust, dynamical property of the SIT vacuum and its gating of gauge interactions.

14.2 Scope, Limitations, and Assumptions

Our analysis is confined to the following regime:

- **Locality and hyperbolicity.** The field equations are second-order hyperbolic with a well-posed Cauchy problem; the Hamiltonian is self-adjoint on a natural local core.
- **Stability of the coherence vacuum.** The potential V possesses a strict minimum at $R_{\text{coh}} = 0$ with $m^2_{\text{coh}} > 0$, and is coercive at large amplitudes.
- **Phase-only entry and quadratic dressing.** The renormalizable gauge/matter block is multiplied by $f(\rho_t, |\psi|)$ with $f \geq c_f|\psi|^2$ near the vacuum; relevant operators that would circumvent this structure are absent.

If any of these assumptions are violated (e.g., $m^2_{\text{coh}} \leq 0$, noncoercive V, or introduction of undressed relevant operators), the proof strategy would have to be revised and the gap could, in principle, close. Within the stated hypotheses, however, the lower bound is structurally stable.

14.3 Open Problems

Several natural directions remain:

1. **Sharpness of the bound.** Determine whether $\Delta = \Delta_{\text{coh}}$ exactly or whether interaction effects lift the gap above the coherence floor in specific models.
2. **Nonperturbative spectra.** Compute the low-lying spectrum beyond the one-gap statement, including multiplet structure and selection rules for gauge-invariant operators.
3. **Topological sectors.** Classify topological excitations (defects, vortices, flux tubes) in the full SIT setting and quantify their contributions to the infrared spectrum.
4. **Dynamical ρ_t backgrounds.** Analyze space- and time-dependent ρ_t profiles, including domain walls and gradients, and their impact on $f(\rho_t, |\psi|)$ and on the effective low-energy constants.
5. **Rigorous continuum limits.** Strengthen the functional-analytic control of the ($a \to 0$, $L \to \infty$) double limit and characterize the strong-resolvent convergence of the Hamiltonians with EFT corrections.

14.4 Extensions: Gauge Groups and Matter Content

The mechanism is not tied to a single gauge group or field representation:

- **Multiple gauge factors.** For $G = G_1 \times \cdots \times G_n$ with independent kinetic normalizations, the proof carries through componentwise; any gauge-invariant excitation in any factor still requires $|\psi| > 0$ in the dressed block and therefore pays at least Δ_{coh}.
- **Chiral matter and fermions.** Provided the renormalizable couplings appear inside the dressed block and the Dirac (or Weyl) sector preserves positivity and locality, the coherence gating argument remains intact. The primary changes are in operator mixing and in the details of observable construction.
- **Higgs-like sectors.** Additional scalar sectors can be included as long as their interactions do not destabilize the coherence vacuum or introduce relevant operators that remove the dressing by $f(\rho_t, |\psi|)$. In models where extra scalars are heavy, the gap mechanism persists unchanged in the infrared.

14.5 Conclusions

We have shown that SIT entails a strictly positive spectral gap in the gauge sector, arising from coherence gating and the existence of a nonzero energy barrier Δ_{coh} in the coherence modulus. The result holds in finite volume, survives the thermodynamic and continuum limits, and is robust under smooth deformations consistent with gauge invariance, positivity, and causality. From a conceptual standpoint, the gap is an informational property of the vacuum: gauge dynamics become physically accessible only insofar as the underlying coherence can be activated, and that activation costs a fixed, nonzero amount of energy. This provides a clear, internal explanation for a Yang–Mills-type mass gap within Super Information Theory and sets the stage for numerical tests and experimental probes based on holonomy, correlation decay, and controlled coherence modulation.

Appendix A: Mathematical Preliminaries

A.1 Hyperbolic PDE Well-Posedness (Energy Method)

Consider a second-order hyperbolic system on $\mathbb{R} \times \Omega$,

$$\partial_t^2 u - \mathcal{A}u + \mathcal{B}\partial_t u = \mathcal{F}(u, \partial u, x, t), \qquad u : \mathbb{R} \times \Omega \to \mathbb{R}^m, \tag{62}$$

with $\Omega = \mathbb{T}_L^3$ and periodic boundary conditions. Assume:

- $\mathcal{A}$ is a uniformly elliptic, self-adjoint, positive operator on $L^2(\Omega)^m$ with domain $H^2(\Omega)^m$ (e.g., $-\kappa\Delta$ plus lower-order symmetric terms).
- $\mathcal{B}$ is bounded and symmetric on L^2 (damping allowed).
- $\mathcal{F}$ is locally Lipschitz in $(u, \partial u)$ from $H^1 \times L^2$ to L^2 on bounded sets.

Define the energy

$$E[u](t) \;=\; \frac{1}{2}\Big(\|\partial_t u(t)\|_{L^2}^2 + \langle \mathcal{A}u(t), u(t)\rangle\Big). \tag{63}$$

A standard energy estimate yields

$$\frac{d}{dt}E[u](t) \;\le\; C\,E[u](t) + C\,\|\mathcal{F}(u, \partial u)\|_{L^2}\,\|\partial_t u\|_{L^2}, \tag{64}$$

whence, by Gronwall and local Lipschitz continuity of $\mathcal{F}$, local-in-time existence and uniqueness

$$(u, \partial_t u) \in C\big([0,T]; H^1(\Omega)^m \times L^2(\Omega)^m\big)$$

for initial data $(u_0, u_1) \in H^1 \times L^2$. If $\mathcal{B}$ is dissipative and $\mathcal{F}$ is subcritical, solutions extend globally with a-priori energy bounds.

Remark (Hamiltonian formulation). For systems derived from a coercive Hamiltonian density $\mathcal{H}$ and local constraints (e.g., Gauss law), the same energy method applies on the physical subspace; self-adjointness of the generator follows from standard symplectic or Dirac-constraint constructions on a dense core.

A.2 Basic Spectral Theory (Self-Adjoint Operators)

Let H be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$ (e.g., the gauge-invariant subspace with Gauss constraints imposed). The spectral theorem provides a projection-valued measure $E_H(\cdot)$ such that

$$H \;=\; \int_{\mathbb{R}} \lambda \, dE_H(\lambda).$$

Write

$$E_0 \;=\; \inf \operatorname{spec} H, \qquad \Delta^* \;=\; \inf\big(\operatorname{spec} H \setminus \{E_0\}\big) - E_0,$$

so that $\Delta^* > 0$ defines a spectral gap above the ground energy.

Perturbation stability (Kato–Rellich sketch). If W is H-bounded with relative bound $a < 1$ (i.e., $\|W\psi\| \le a\|H\psi\| + b\|\psi\|$), then $H + W$ is self-adjoint on $D(H)$ and $\operatorname{spec}(H + W)$ depends continuously on W in the strong resolvent sense. In particular, a strictly positive gap is stable under sufficiently small relatively bounded perturbations:

$$\Delta^*(H + W) \;\ge\; \Delta^*(H) - C\|W\|_{\mathrm{rel}} \;>\; 0$$

for $\|W\|_{\mathrm{rel}}$ small enough.

Spectral representations and clustering. If $\mathcal{O}$ is a bounded (or suitably controlled) local operator with $\langle 0|\mathcal{O}|0\rangle = 0$, then its Euclidean two-point function admits the spectral representation

$$C_{\mathcal{O}}(\tau) \;=\; \sum_{n\ge 1} |\langle 0|\mathcal{O}|n\rangle|^2 e^{-(E_n - E_0)\tau},$$

so a gap $\Delta^* > 0$ implies $C_{\mathcal{O}}(\tau) \sim Z_{\mathcal{O}} e^{-\Delta^*\tau}$ as $\tau \to \infty$. Analogous statements hold for spatial clustering via the transfer-operator formalism.

A.3 Poincaré-Type Inequalities on $\mathbb{T}^3_L$

Let $\Omega = \mathbb{T}^3_L = (\mathbb{R}/L\mathbb{Z})^3$ and denote by $\bar{u}$ the spatial average of u. For $u \in H^1(\Omega)$ with $\bar{u} = 0$,

$$\|u\|_{L^2(\Omega)} \;\le\; \frac{1}{c_P}\, \|\nabla u\|_{L^2(\Omega)}, \qquad c_P \;=\; \frac{2\pi}{L}. \tag{65}$$

Sketch. Expand u in Fourier modes $u(x) = \sum_{k\in(2\pi/L)\mathbb{Z}^3\setminus\{0\}} \hat{u}(k) e^{ik\cdot x}$. Then

$$\|u\|^2_{L^2} = \sum_{k\ne 0} |\hat{u}(k)|^2, \qquad \|\nabla u\|^2_{L^2} = \sum_{k\ne 0} |k|^2 |\hat{u}(k)|^2 \;\ge\; \Big(\min_{k\ne 0} |k|\Big)^2 \sum_{k\ne 0} |\hat{u}(k)|^2,$$

and $\min_{k\ne 0} |k| = 2\pi/L$ gives (65).

Vector and gauge fields. For a vector field A in Coulomb gauge ($\nabla \cdot A = 0$ and zero average), the transverse Poincaré inequality yields

$$\|A\|_{L^2(\Omega)} \;\le\; \frac{1}{c_P}\,\|\nabla \times A\|_{L^2(\Omega)}, \tag{66}$$

so that $\int |B|^2\,dx \ge c_P^2 \int |A|^2\,dx$ with $B = \nabla \times A$. Linearized non-Abelian fields obey analogous bounds with the Killing-form trace.

A.4 Consequences for Energy Bounds

Combining coercivity of the coherence potential near the vacuum with (65) gives, for mean-zero R_{coh},

$$\int_\Omega \Big(\tfrac{\kappa_c}{2}|\nabla R_{\mathrm{coh}}|^2 + \tfrac{m_{\mathrm{coh}}^2}{2} R_{\mathrm{coh}}^2\Big)\,dx \;\ge\; \min\Big\{\tfrac{\kappa_c}{2}c_P^2,\,\tfrac{m_{\mathrm{coh}}^2}{2}\Big\}\,\|R_{\mathrm{coh}}\|_{L^2}^2,$$

which underlies the coherence-sector floor. Similarly, (66) produces a finite-volume floor for gauge excitations in the transverse sector. These inequalities are the geometric inputs to the spectral gap bounds used in the main text.

Appendix B: Gauge Bundle Notes

B.1 Principal Bundles and Associated Bundles

Let $P \xrightarrow{\pi} M$ be a principal G-bundle over a smooth d-manifold M with right G-action $R_g : P \to P$, $R_g(p) = p \cdot g$, and compact, simple Lie group G. A local trivialization over an open set $U \subset M$ is a G-equivariant diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times G$. On overlaps $U \cap V \neq \emptyset$, transition functions $t_{UV} : U \cap V \to G$ satisfy the cocycle condition $t_{UV}t_{VW}t_{WU} = \mathbf{1}$.

Given a representation $R : G \to \mathrm{GL}(V)$, the associated vector bundle is $E = P \times_R V$, obtained by identifying $(p, v) \sim (p \cdot g, R(g^{-1})v)$. Sections of E transform as matter fields under gauge transformations.

B.2 Connections and Curvature

A *connection* on P is a $\mathfrak{g}$-valued one-form $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying:

1. Equivariance: $(R_g)^*\omega = \mathrm{Ad}_{g^{-1}}\,\omega$ for all $g \in G$.
2. Reproduction: $\omega(\xi_P) = \xi$ for all *fundamental* vector fields ξ_P generated by $\xi \in \mathfrak{g}$.

The *curvature* two-form is $\Omega = d\omega + \frac{1}{2}[\omega, \omega] \in \Omega^2(P; \mathfrak{g})$, which obeys the Bianchi identity

$$D\Omega \;=\; d\Omega + [\omega, \Omega] \;=\; 0. \tag{67}$$

In a local trivialization $U \subset M$ with a section $s : U \to P$, the *gauge potential* is $A := s^*\omega \in \Omega^1(U; \mathfrak{g})$ and the *field strength* is

$$F \;=\; s^*\Omega \;=\; dA + A \wedge A \;\in\; \Omega^2(U; \mathfrak{g}). \tag{68}$$

Gauge transformations. On U, a gauge transformation $g : U \to G$ acts by

$$A \;\mapsto\; A^g \;=\; gAg^{-1} + g\,dg^{-1}, \tag{69}$$

$$F \;\mapsto\; F^g \;=\; gFg^{-1}. \tag{70}$$

Thus any gauge-invariant local scalar built from F uses a G-invariant bilinear form on $\mathfrak{g}$, e.g. the (negative) Killing form or Tr in a faithful representation.

B.3 Holonomy and Parallel Transport

For a smooth path $\gamma : [0,1] \to U \subset M$, the *parallel transport* is the solution of

$$\frac{d}{dt} U_\gamma(t) \;=\; -\,A_\mu(\gamma(t))\,\dot{\gamma}^\mu(t)\,U_\gamma(t), \qquad U_\gamma(0) = \mathbf{1}, \tag{71}$$

so that the *path-ordered exponential* is

$$\mathcal{U}[\gamma] \;=\; \mathcal{P}\exp\Big(-\int_\gamma A\Big). \tag{72}$$

For a closed loop γ based at $x \in U$, the *holonomy* $\mathcal{U}[\gamma] \in G$ is gauge covariant and its conjugacy class is gauge invariant. In a representation R, the *Wilson loop* is

$$W_R[\gamma] \;=\; \frac{1}{\dim R}\,\mathrm{Tr}_R\,\mathcal{U}[\gamma], \tag{73}$$

a gauge-invariant observable. For small loops bounding an area element $A^{\mu\nu}$,

$$\mathcal{U}[\gamma] \;=\; \mathbf{1} \;-\; \int_\Sigma F \;+\; \mathcal{O}(A^{3/2}), \qquad 1 - W_R[\gamma] \;\approx\; \frac{1}{2\,\dim R}\,\mathrm{Tr}_R(F_{\mu\nu}F^{\mu\nu})\,A^2, \tag{74}$$

which underlies small-loop curvature diagnostics.

B.4 Compact Simple Groups G

Let G be compact and simple with Lie algebra $\mathfrak{g}$. Choose generators T^a in a faithful representation with

$$[T^a, T^b] \;=\; i f^{abc} T^c, \qquad \mathrm{Tr}(T^a T^b) \;=\; \frac{1}{2}\delta^{ab}, \tag{75}$$

where f^{abc} are the real structure constants and the normalization is conventional (e.g., for $SU(N)$ in the fundamental). Write $A_\mu = A^a_\mu T^a$ and $F_{\mu\nu} = F^a_{\mu\nu} T^a$ with

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{76}$$

The gauge-covariant derivative in the adjoint is $(D_\mu X)^a = \partial_\mu X^a + f^{abc} A^b_\mu X^c$, and the Bianchi identity becomes $D_{[\mu} F_{\nu\rho]} = 0$.

Invariant quadratic forms and positivity. For compact G, the negative Killing form is positive definite on $\mathfrak{g}$, ensuring that

$$\mathcal{E}_{\mathrm{mag}} \;=\; \frac{1}{2}\int \kappa_F\,\langle F_{ij}, F_{ij}\rangle\, d^3x \tag{77}$$

is positive for $\kappa_F > 0$, where $\langle \cdot, \cdot \rangle$ is any chosen invariant inner product (e.g., $-\,\mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)$ or Tr in a faithful rep). This underlies the coercivity of the gauge quadratic form used in gap estimates.

B.5 Local Trivializations, Overlaps, and Gauge Patching

On overlaps $U \cap V$, the local potentials are related by the transition function $t_{UV} : U \cap V \to G$:

$$A_U = t_{UV} A_V t_{UV}^{-1} + t_{UV}\, d t_{UV}^{-1}, \tag{78}$$

$$F_U = t_{UV} F_V t_{UV}^{-1}. \tag{79}$$

Global gauge-invariant functionals (e.g., Wilson loops, Chern classes) are insensitive to the choice of trivialization. On $M = \mathbb{T}^d$, nontrivial holonomies around noncontractible cycles encode topological sectors even when $F = 0$ locally (flat connections).

B.6 Remarks on SU(N) and Casimirs

For $G = SU(N)$, the quadratic Casimir $C_2(R)$ in representation R defines

$$\mathrm{Tr}_R(T^a T^b) \;=\; \kappa_R\, \delta^{ab}, \qquad C_2(R)\, \mathbf{1}_R \;=\; T^a T^a, \tag{80}$$

with κ_R a normalization constant (e.g., $\kappa_{\text{fund}} = 1/2$). In Wilson-loop observables, $C_2(R)$ controls the small-loop scaling and enters the matching of local curvature to representation-dependent coefficients.

B.7 Summary

Principal bundles encode the global gauge structure; connections and curvature encode local gauge data; holonomy ties the two by integrating the connection along loops. For compact simple G, invariant inner products ensure positivity of gauge quadratic forms, enabling energy coercivity and spectral bounds. These geometric ingredients are the backbone of the gauge-sector constructions used throughout the main text.

Appendix C: Coherence Potentials and Topological Sectors

C.1 Potentials Ensuring $\Delta_{\text{coh}} > 0$

We recall the static energy functional for the coherence modulus R_{coh} on $\Omega = \mathbb{T}^3_L$ or $\mathbb{R}^3$:

$$E[R_{\text{coh}}; \rho_t] \;=\; \int_\Omega d^3x \left(\tfrac{\kappa_c}{2}\, |\nabla R_{\text{coh}}|^2 + V(R_{\text{coh}}, \rho_t) \right), \qquad \kappa_c > 0. \tag{81}$$

A strictly positive coherence gap $\Delta_{\text{coh}} > 0$ follows if:

1. $R_{\text{coh}} = 0$ is a strict global minimizer of $V(\,\cdot\,, \rho_{t,0})$,
2. the Hessian at the vacuum is positive, $m^2_{\text{coh}} := \partial^2_R V(0, \rho_{t,0}) > 0$,
3. V is coercive for large R, e.g. $V(R, \rho_t) \ge c_4 R^4 - C$.

Under these conditions and the Poincare inequality on $\mathbb{T}^3_L$ (or suitable falloff on $\mathbb{R}^3$), one obtains the quadratic lower bound

$$E[R_{\text{coh}}] - E[0] \;\ge\; \int_\Omega d^3x \left(\tfrac{\kappa_c}{2}\, |\nabla R_{\text{coh}}|^2 + \tfrac{m^2_{\text{coh}}}{2}\, R^2_{\text{coh}} \right), \tag{82}$$

which implies a positive separation $E_1 - E_0 = \Delta_{\text{coh}} > 0$ between the vacuum and the first localized excitation.

Example C.1 (single-well quartic).

$$V(R, \rho_t) \;=\; a_2(\rho_t)\, R^2 + a_4(\rho_t)\, R^4, \qquad a_2(\rho_{t,0}) > 0, \quad a_4(\rho_t) \ge a_4^{\min} > 0. \tag{83}$$

Then $m^2_{\text{coh}} = 2a_2(\rho_{t,0}) > 0$ and coercivity is set by $a_4^{\min}$.

Example C.2 (convex polynomial).

$$V(R,\rho_t) \;=\; \sum_{k=1}^{K} b_{2k}(\rho_t)\,R^{2k}, \qquad b_2(\rho_{t,0})>0, \quad b_{2K}(\rho_t)\ge b_{2K}^{\min}>0. \tag{84}$$

The smallest nonzero even coefficient fixes $m_{\rm coh}^2 = 2b_2(\rho_{t,0})$, while $b_{2K}^{\min}$ ensures coercivity.

Example C.3 (weakly ρ_t-dependent convex well).

$$V(R,\rho_t) \;=\; \alpha(\rho_t)\,R^2 + \beta(\rho_t)\,R^4, \quad \alpha'(\rho_{t,0}),\beta'(\rho_{t,0}) \text{ bounded}, \quad \alpha(\rho_{t,0})>0,\ \beta(\rho_t)\ge\beta_{\min}>0. \tag{85}$$

Uniform bounds in a neighborhood of $\rho_{t,0}$ preserve $m_{\rm coh}^2>0$ and coercivity.

C.2 Uniformity in ρ_t and Stability of the Gap

Suppose ρ_t varies in a compact set $\mathcal{I}$ containing $\rho_{t,0}$ and the maps $\rho_t \mapsto a_2(\rho_t), a_4(\rho_t)$ (or the corresponding polynomial coefficients) are C^1-bounded on $\mathcal{I}$. Then there exist uniform constants

$$m_{\rm coh,\,min}^2 \;=\; \inf_{\rho_t\in\mathcal{I}} \partial_R^2 V(0,\rho_t) \;>\; 0, \qquad c_{4,\min} \;=\; \inf_{\rho_t\in\mathcal{I}} c_4(\rho_t) \;>\; 0, \tag{86}$$

and the bound (82) holds with $m_{\rm coh}^2$ replaced by $m_{\rm coh,\,min}^2$. Consequently, $\Delta_{\rm coh}$ is bounded away from zero uniformly for all admissible backgrounds $\rho_t\in\mathcal{I}$.

C.3 Defects, Winding, and Finite Energy

Although the vacuum sits at $R=0$, localized excitations can support phase structure if $R>0$ on a compact region while $R\to 0$ at its boundary, so that gradients of the phase do not produce long-range energy costs. A schematic construction on $\mathbb{R}^3$:

- Choose a compact domain $D\subset\mathbb{R}^3$ and a smooth profile $R(x)$ with $R(x)>0$ for $x\in D$ and $R(x)=0$ on ∂D and outside a thin collar of ∂D.
- On D, define a phase $\theta(x)$ with quantized winding along closed curves in D, while avoiding singularities by keeping $R>0$ where $\nabla\theta$ is nonzero.
- Outside D (where $R=0$), phase is irrelevant and does not contribute to energy.

In Abelian language, the circulation

$$\oint_{\gamma\subset D} \nabla\theta\cdot d\ell \;=\; 2\pi n, \qquad n\in\mathbb{Z}, \tag{87}$$

defines an integer winding that can be detected by loop holonomy when the gauge sector is present. Finite energy follows because R vanishes where θ would otherwise be singular, and $R\to 0$ outside D suppresses gradient and potential terms.

Vortex-like tubes and rings. In 3D one may realize line defects (tubes) or ring defects (closed loops) by taking D to be a solid torus and endowing the core with nontrivial winding. Energy remains finite provided $R\to 0$ sufficiently fast across the core boundary and at spatial infinity.

C.4 Stability Criteria: Energetic and Topological

Two complementary mechanisms support the existence and persistence of localized excitations:

1. **Energetic stability.** If $m_{\text{coh}}^2 > 0$ and V is coercive, small perturbations of a localized solution increase energy (to quadratic order), preventing decay into the vacuum sector. This provides a local minimum of E constrained by localization.

2. **Topological stability.** If a configuration carries nontrivial winding on some compact support of $R > 0$, continuity and the vanishing of R on the boundary prevent unwinding without passing through $R = 0$ within D. When coupled to the gauge sector, such winding can be read out via holonomy. Topological charge is then an integer that cannot change under smooth variations preserving the support structure.

These criteria are compatible: energetic stability supplies a gap above the vacuum, while topology prevents decay channels that would bypass the energy barrier.

C.5 Illustrative Model Potentials

We list concrete choices useful in analysis and numerics.

C.5.1 Quartic with soft ρ_t modulation.

$$V(R, \rho_t) \;=\; \big(\alpha_0 + \alpha_1(\rho_t - \rho_{t,0})\big)\, R^2 \;+\; \big(\beta_0 + \beta_1(\rho_t - \rho_{t,0})\big)\, R^4, \tag{88}$$

with $\alpha_0 > 0$ and $\beta_0 > 0$, and $|\alpha_1|, |\beta_1|$ small. Then $m_{\text{coh}}^2 = 2\alpha_0 + \mathcal{O}(\rho_t - \rho_{t,0})$ and coercivity persists.

C.5.2 Sextic stabilizer.

$$V(R, \rho_t) \;=\; a_2(\rho_t)R^2 + a_4(\rho_t)R^4 + a_6R^6, \qquad a_2(\rho_{t,0}) > 0,\; a_6 > 0, \tag{89}$$

used when a_4 is small or mildly negative in some range; positivity of a_6 restores coercivity while $a_2(\rho_{t,0}) > 0$ secures the vacuum at $R = 0$.

C.5.3 Piecewise-smooth wells for compact support. For constructing compactly supported R, one may use a well with a steep wall near a target radius $R_\star$ inside D and a steep rise for $R > R_\star$, together with a strong penalty near the boundary to enforce $R \to 0$ across ∂D. Smoothing these features yields a C^2 potential that still satisfies the hypotheses above.

C.6 Consequences for Δ_{coh} and Observables

Under any of the examples above, one has $m_{\text{coh}}^2 > 0$ and coercivity, hence $\Delta_{\text{coh}} > 0$. In coupled SIT, loop holonomies measured over paths threading the support of a localized excitation detect phase winding and, indirectly, the presence of coherence through the dressing of gauge observables. Thus:

- Local measurements (e.g., two-point correlators of R) exhibit exponential clustering controlled by m_{coh} and by the first nonzero spatial mode on $\mathbb{T}_L^3$.
- Loop observables register nontrivial phases only when $R > 0$ along the enclosed support, consistent with coherence gating.

These features link the potential-level gap Δ_{coh} to experimentally or numerically accessible diagnostics.

Appendix D: Discrete-to-Continuum SIT

D.1 Graph Geometry and Variables

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a regular cubic graph embedded in the three-torus $\Omega = \mathbb{T}^3_L$ with lattice spacing $a = L/N$ and $|\mathcal{V}| = N^3$. For $i \in \mathcal{V}$ denote the spatial position by x_i, and for an oriented edge $(i \to j) \in \mathcal{E}$ write $x_{ij} = (x_i + x_j)/2$ and $\hat{e}_{ij} = (x_j - x_i)/a$. Discrete SIT variables:

$$R_i \approx R_{\text{coh}}(x_i), \qquad \rho_{t,i} \approx \rho_t(x_i), \tag{90}$$

$$U^{(U(1))}_{ij} = \exp\big(ia\, A_\mu(x_{ij})\, \hat{e}^\mu_{ij}\big), \qquad \mathcal{U}^{(G)}_{ij} = \exp\big(ia\, \mathcal{A}_\mu(x_{ij})\, \hat{e}^\mu_{ij}\big) \in G, \tag{91}$$

with $U_{ji} = (U_{ij})^{-1}$ and similarly for $\mathcal{U}$. Gauge-electric variables E_{ij} and $\mathcal{E}_{ij}$ live on links and satisfy discrete Gauss constraints.

D.2 Discrete Differential Operators

For a scalar f_i and link field A_{ij},

$$(\nabla_a f)_i := \frac{1}{a} \sum_{j\sim i} \alpha_{ij}\, (f_j - f_i), \qquad \alpha_{ij} = \frac{1}{6} \text{ on a cubic graph}, \tag{92}$$

$$(\Delta_a f)_i := \frac{1}{a^2} \sum_{j\sim i} (f_j - f_i), \tag{93}$$

so that $\Delta_a \to \Delta$ as $a \to 0$ for smooth fields. Magnetic field is encoded by plaquette holonomies. For an elementary square $\square = \{i \to j \to k \to l \to i\}$,

$$U_\square = U_{ij} U_{jk} U_{kl} U_{li} \;=\; \exp\big(ia^2 F_{\mu\nu}(x_\square)\, \hat{e}^\mu \wedge \hat{e}^\nu + O(a^3)\big), \tag{94}$$

$$\mathcal{U}_\square = \mathcal{U}_{ij} \mathcal{U}_{jk} \mathcal{U}_{kl} \mathcal{U}_{li} \;=\; \mathcal{P} \exp\big(ia^2 \mathcal{F}_{\mu\nu}(x_\square)\, \hat{e}^\mu \wedge \hat{e}^\nu + O(a^3)\big). \tag{95}$$

D.3 Discrete Hamiltonian and Gauss Law

A minimal nearest-neighbor Hamiltonian consistent with SIT reads

$$H^{(a)}_{\text{coh}} = \sum_i \Big[\tfrac{1}{2} \Pi^2_{R,i} + \tfrac{\kappa^{(a)}_c}{2} \sum_{j\sim i} (R_j - R_i)^2 + V^{(a)}(R_i, \rho_{t,i}) \Big], \tag{96}$$

$$H^{(a)}_{\text{gauge}}\Big|_{U(1)} = \tfrac{1}{2} \sum_{\text{links}} E^2_{ij} + \tfrac{\kappa^{(a)}_F}{2} \sum_{\text{plaquettes}} \big(1 - \operatorname{Re} U_\square\big), \tag{97}$$

$$H^{(a)}_{\text{gauge}}\Big|_{G} = \tfrac{1}{2} \sum_{\text{links}} \operatorname{Tr} \mathcal{E}^2_{ij} + \tfrac{\kappa^{(a)}_F}{2} \sum_{\text{plaquettes}} \Big(1 - \tfrac{1}{\dim R} \operatorname{Tr}_R \mathcal{U}_\square\Big), \tag{98}$$

$$H^{(a)}_{\text{int}} = \sum_i f^{(a)}(\rho_{t,i}, |\psi_i|)\, \mathcal{H}^{(a)}_{\text{SM}}(i) + \text{EFT corrections}. \tag{99}$$

Discrete Gauss constraints at node i:

$$\text{Abelian:} \quad \sum_{j\sim i} E_{ij} \;=\; \rho^{\text{matter}}_i, \tag{100}$$

$$\text{Non-Abelian:} \quad \sum_{j\sim i} \Big(\mathcal{E}_{ij} - \mathcal{U}_{ij} \mathcal{E}_{ji} \mathcal{U}^{-1}_{ij} \Big) \;=\; \mathcal{J}^0_i. \tag{101}$$

D.4 Coarse-Graining and Block Spins

Introduce a block scale $b \in \mathbb{N}$ and partition $\mathcal{V}$ into disjoint cubes of side ba. Define block-averaged variables

$$\bar{R}_I = \frac{1}{b^3}\sum_{i\in I} R_i, \qquad \bar{\rho}_{t,I} = \frac{1}{b^3}\sum_{i\in I}\rho_{t,i}, \tag{102}$$

$$\bar{U}_{IJ} = \mathrm{Proj}\Big(\prod_{(i\to j)\in\Gamma_{IJ}} U_{ij}\Big), \qquad \bar{\mathcal{U}}_{IJ} = \mathrm{Proj}\Big(\mathcal{P}\prod_{(i\to j)\in\Gamma_{IJ}} \mathcal{U}_{ij}\Big), \tag{103}$$

where Γ_{IJ} is a shortest-edge path between block centers and Proj returns the closest group element (for $U(1)$, the phase reduced modulo 2π). Iterating $a \to a' = ba$ generates a flow of couplings

$$(\kappa_c^{(a)}, \kappa_F^{(a)}, c_f^{(a)}, \ldots) \;\mapsto\; (\kappa_c^{(a')}, \kappa_F^{(a')}, c_f^{(a')}, \ldots), \tag{104}$$

which is used to match to continuum targets at a reference physical scale.

D.5 Practical Parameter Matching

Fix a physical matching scale μ_{match} (e.g., a small loop area and one lattice spacing for nearest-neighbor correlators). Choose tolerances $\varepsilon_{\text{loop}}$, ε_{coh}, ε_f and solve:

$$\Big|\langle 1-\mathrm{Re}\,U_\square\rangle^{(a)} - \langle 1-\mathrm{Re}\,U_\square\rangle_{\text{cont}}\Big| \le \varepsilon_{\text{loop}}, \tag{105}$$

$$\Big|\langle (R_j-R_i)^2\rangle^{(a)} - \langle(\partial_\mu R_{\text{coh}})^2\rangle_{\text{cont}}\Big| \le \varepsilon_{\text{coh}}, \tag{106}$$

$$\Big|\langle \mathcal{O}_{\text{gauge}}\rangle^{(a)} - c_f\,\langle|\psi|^2\rangle^{(a)}\Big| \le \varepsilon_f \quad \text{for small } |\psi|. \tag{107}$$

Tune $(\kappa_c^{(a)}, \kappa_F^{(a)}, c_f^{(a)})$ until all three conditions hold simultaneously. As a cross-check, verify that the effective mass gap extracted from temporal correlators satisfies

$$\Delta^{(a)} \;\ge\; \Delta_{\text{coh}}^{(a)} \quad \text{and} \quad \Delta^{(a)} \to \Delta_{\text{cont}} \;\text{ within errors as }\; a\to 0. \tag{108}$$

D.6 Error Budgets and Scaling Windows

Discretization and finite-volume effects obey the canonical estimates on a smooth background:

$$\text{Discretization:}\quad \delta_{\text{disc}} \;=\; O(a^p), \quad p\ge 2 \text{ for central differences}, \tag{109}$$

$$\text{Finite volume:}\quad \delta_{\text{FV}} \;=\; O(e^{-L/\xi}) \;\text{ for gapped spectra with } \xi = \Delta^{-1}. \tag{110}$$

A safe scaling window requires $a \ll \xi \ll L$ and stability of dimensionless composites (e.g., $\Delta\xi$, small-loop slopes, and ratios of correlator amplitudes). When performing block coarse-graining, keep b modest (e.g., 2 or 3) to avoid oversmoothing of small-loop diagnostics.

D.7 Continuum Consistency Checks

At fixed physical area A and separation r,

$$\langle 1-\mathrm{Re}\,U_\square(A)\rangle^{(a)} \to c_A\,A^2 + O(A^{5/2}), \tag{111}$$

$$G^{(a)}(r) \to A_{\mathcal{O}}\,\frac{e^{-r/\xi}}{r^\alpha}, \tag{112}$$

with c_A proportional to $\langle F_{\mu\nu}^2\rangle$ (or its non-Abelian trace), and $\xi = \Delta^{-1}$. Agreement of these limits across several a and L establishes correct discrete-to-continuum matching of SIT and supports the persistence of the gap.

D.8 Summary

A regular cubic network with link holonomies, plaquette phases, and node-based coherence fields provides a faithful discretization of SIT. Coarse-graining defines a coupling flow used to match discrete parameters to continuum targets. Within a scaling window $a \ll \xi \ll L$, loop and correlator observables converge with controlled errors, and the discrete gap saturates the coherence floor predicted by the continuum theory.

Appendix E: Notation and Conventions

Spacetime and Metric

- Spacetime indices are Greek letters: $\mu, \nu, \rho, \sigma = 0, 1, 2, 3$.
- Spatial indices are Latin letters: $i, j, k = 1, 2, 3$.
- We use the mostly-minus Minkowski metric:

$$g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1), \qquad g = \det(g_{\mu\nu}).$$

- For Euclidean calculations we use the flat metric $\delta_{\mu\nu}$ with all positive signs.
- Raising and lowering of indices is performed with $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

Derivatives and Covariant Structures

- Partial derivatives: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$.
- Covariant derivative acting on the coherence field ψ:

$$D_\mu \psi \equiv (\partial_\mu - iqA_\mu)\, \psi \quad \text{(Abelian)}, \qquad D_\mu \psi \equiv (\partial_\mu - ig\mathcal{A}_\mu)\, \psi \quad \text{(non-Abelian)}.$$

- Abelian field strength:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.$$

- Non-Abelian field strength:

$$\mathcal{F}_{\mu\nu} \equiv \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - ig\, [\mathcal{A}_\mu, \mathcal{A}_\nu]\, .$$

- Holonomy (Wilson line) along a path γ:

$$U[\gamma] \equiv \mathcal{P} \exp\!\left(i \int_\gamma A_\mu \, dx^\mu \right) \quad \text{(Abelian)}, \qquad U[\gamma] \equiv \mathcal{P} \exp\!\left(i \int_\gamma \mathcal{A}_\mu \, dx^\mu \right) \quad \text{(non-Abelian)}.$$

  Here $\mathcal{P}$ denotes path ordering.

Gauge Groups and Generators

- For Abelian sectors we denote the gauge field by A_μ and the coupling by q.
- For non-Abelian sectors we take a compact simple group G with generators T^a in the chosen representation:

$$\left[T^a, T^b\right] = i f^{abc} T^c, \qquad \mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}\delta^{ab}.$$

- The non-Abelian gauge field is $\mathcal{A}_\mu = \mathcal{A}^a_\mu T^a$ with coupling g.

Levi-Civita, Epsilon, and Contractions

- The Levi-Civita symbol is defined by $\epsilon^{0123} = +1$ and $\epsilon^{\mu\nu\rho\sigma}$ is totally antisymmetric.
- Contractions follow the metric signature; for example $X^\mu Y_\mu = g_{\mu\nu} X^\mu Y^\nu$.

Fourier and Integration Conventions

- Spacetime integration: $\int d^4x \sqrt{-g}\,\mathcal{L}$ in Minkowski space, $\int d^4x\,\mathcal{L}_E$ in Euclidean space.
- Spatial integration on a torus of side L: $\int_{[0,L]^3} d^3x$.
- A common Fourier convention is

$$\tilde{\phi}(p) = \int d^4x\, e^{-ip\cdot x}\phi(x), \qquad \phi(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{+ip\cdot x}\tilde{\phi}(p).$$

Units and Dimensions

- Natural units are used unless stated otherwise: $\hbar = c = 1$.
- The gauge couplings q (Abelian) and g (non-Abelian) are dimensionless in $3+1$ dimensions.
- The coherence field ψ and its modulus $R_{\text{coh}} = |\psi|$ carry mass dimensions consistent with the kinetic term $|D\psi|^2$.

Sign Conventions

- The Lagrangian density for an Abelian gauge field is taken as

$$\mathcal{L}_{\text{U(1)}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

- For non-Abelian fields:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{2}\text{Tr}\left(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right).$$

- The kinetic term for the coherence field uses the mostly-minus metric:

$$\mathcal{L}_{\psi,\text{kin}} = \frac{\kappa_c}{2}\, g^{\mu\nu}(D_\mu\psi)^*(D_\nu\psi).$$

- The potential terms $V(R_{\text{coh}}, \rho_t)$ enter with a minus sign in the action:

$$\mathcal{L}_{\text{pot}} = -V(R_{\text{coh}}, \rho_t).$$

Miscellaneous

- Throughout, repeated indices are summed unless noted otherwise.
- "c.c." denotes complex conjugation; h.c. denotes Hermitian conjugation.
- Boldface symbols are not used for tensors; all indices are explicit.