End term - Report

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1 Introduction

Options are fundamental financial derivatives whose value depends on an underlying asset such as a stock. Among the most influential models for option pricing is the Black–Scholes model, introduced by Fischer Black and Myron Scholes in 1973. The model provides a closed-form analytical expression for pricing European options under a set of idealized assumptions.

The objective of this project was to understand the Black–Scholes model both theoretically and computationally. Over the course of the WiDS Winter Program, I studied the mathematical foundations of the model and implemented it in Python. In addition to analytical pricing, I explored numerical methods such as Monte Carlo simulation and implied volatility estimation. This report documents my learnings, implementations, and experiments conducted during the project.

2 Model Assumptions

The Black–Scholes model relies on several simplifying assumptions:

- The underlying asset price follows a Geometric Brownian Motion (GBM).
- Markets are frictionless with no transaction costs or taxes.
- Trading is continuous and arbitrage opportunities do not exist.
- The risk-free interest rate and volatility are constant.
- The option is European, meaning it can only be exercised at maturity.
- The underlying asset may pay a continuous dividend yield.

While these assumptions are not perfectly satisfied in real markets, they allow for analytical tractability and form the basis of modern quantitative finance.

3 Stock Price Dynamics

Under the Black–Scholes framework, the stock price S_t is modeled as a stochastic process following the stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where:

- μ is the drift rate,

- σ is the volatility,
- W_t is a standard Brownian motion.

Under the risk-neutral measure, the drift μ is replaced by the risk-free rate r, leading to:

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}$$

This change of measure allows option prices to be expressed as discounted expectations under the risk-neutral probability measure.

4 Black–Scholes Pricing Formula

The payoff of a European call option at maturity T with strike price K is given by:

$$\max(S_T - K, 0)$$

The Black–Scholes formula for the price of a European call option is:

$$C = S_0 e^{-qT}\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

Similarly, the price of a European put option is:

$$P = Ke^{-rT}\Phi(-d_2) - S_0 e^{-qT}\Phi(-d_1)$$

where:

$$d_1 = \frac{\ln(S_0/K) + (r - q + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$$

Here, $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

5 Put–Call Parity

An important consistency check for European options is put–call parity, given by:

$$C - P = S_0 e^{-qT} - Ke^{-rT}$$

In the implemented code, this relationship was numerically verified to ensure correctness of the pricing functions.

6 Option Greeks

Greeks measure the sensitivity of option prices to various parameters.

6.1 Delta

Delta measures sensitivity with respect to the underlying asset price:

$$\Delta_{\text{call}} = e^{-qT}\Phi(d_1)$$

6.2 Gamma

Gamma measures the rate of change of Delta:

$$\Gamma = \frac{e^{-qT}\phi(d_1)}{S_0\sigma\sqrt{T}}$$

6.3 Vega

Vega measures sensitivity to volatility:

$$\text{Vega} = S_0 e^{-qT}\phi(d_1)\sqrt{T}$$

These Greeks were computed analytically and used in later experiments.

7 Monte Carlo Simulation

In addition to analytical pricing, Monte Carlo simulation was used to estimate option prices numerically. Under the risk-neutral measure, the terminal stock price is simulated as:

$$S_T = S_0 \exp\left((r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}Z\right)$$

where $Z \sim \mathcal{N}(0, 1)$.

By averaging discounted payoffs over many simulated paths, numerical estimates of option prices were obtained. A convergence study was conducted by varying the number of simulations.

8 Implied Volatility Estimation

Implied volatility is the volatility value that, when substituted into the Black–Scholes formula, matches the observed market price.

Since no closed-form solution exists for implied volatility, a numerical bisection method was implemented. This inverse problem highlights the practical use of the Black–Scholes model in real markets.

9 Parameter Experiments

To study the effect of volatility and time to maturity, parameter sweeps were performed. Option prices and Greeks were computed for multiple values of volatility and maturity. The experiments confirmed expected monotonic behavior:

- Option prices increase with volatility.
- Vega increases with maturity.
- Gamma is highest near at-the-money options.

These experiments helped build intuition about option sensitivities.

10 Learnings and Observations

Through this project, I gained:

- A solid understanding of risk-neutral pricing.
- Practical experience implementing analytical and numerical methods.
- Insight into the limitations of the Black–Scholes model.
- Appreciation for numerical methods used in real-world option pricing.

11 Conclusion

This project provided a comprehensive introduction to option pricing using the Black–Scholes framework. By combining theory, implementation, and experimentation, I developed a deeper understanding of derivative pricing. Future work could include volatility surface modeling, stochastic volatility models, and American option pricing.