The Game of Nim

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These are some notes on the game of Nim, a basic (but interesting!) game in which two players remove objects from piles. First, we'll talk about the game itself and how to play it, then try it a few times. Next, we'll talk about strategy and the notion of impartial games, and end off with a cool result relating Nim and other impartial games.

1 Nimtroduction

According to Wikipedia, Nim is a very old game, and was first played in ancient times. No clue what else to put here, except for maybe this really great segway into the rules of Nim.

1.1 Rules

The rules are quite simple.

1. Two players, and some piles of things (we'll commonly refer to the things as stones) in any amount.
2. The players alternate turns, and on each turn, a player can select a pile and remove any nonzero number of stones from the pile.
3. Whichever player removes the last stone wins.

There are many, many variations on Nim, which may change the rules slightly, but the above rules are for the basic Nim game.

1.2 Example Games

For these examples, we'll have the players A and B, where A always does the first move.

2 Nimtresting Stuff

While Nimming, you may have come up with some of the following observations.

Proposition 2.1. *All games of Nim end in finitely many moves, and never in a tie.*

Proof. Crux is monovariants. Let's look at the number of stones in total: after every move, this number decreases and nothing can make it stay the same or increase. However, the number also cannot be negative, and when it's zero the game ends, after some player removes the last stones. Hence, the game must end by some player winning. □

Proposition 2.2. *In one pile Nim, the first player always wins unless there's no stones in the pile.*

Proof. Self-evident. □

Problem 1. Prove that given the starting configuration of a Nim game, one could determine if the first or second player has winning strategy (a guaranteed win).

2.1 Nimming and Winning

Let's do some definitions to analyze Nim a bit further.

For the sake of notation, we will denote Nim games as $(a_1, a_2, \cdots, a_n)$, where a_i are natural numbers denoting the number of stones in each pile (so order doesn't matter).

Definition 2.3. Define an *N-position* to be a position in which the next player to move is guaranteed to win, and a *P-position* to be a position in which the previous player to move is guaranteed a win.

Remark. I say "position," but I really mean "game." In general, once a move is done in a game, one could interpret it as the game becoming a different game. For example, in one pile Nim, I could move 3 stones off of a 7 stone starting pile, and it's the same as if it we decided to play a 4 stone Nim game with the other player starting first.

The result of Problem 1 tells us that we can tell which player will win for any starting conditions, but how can we tell? We already have Proposition 2.2 but what if there's 2 piles? 3 piles?

Proposition 2.4. *In two pile Nim, (n, n) is a P-position, and (a, b) is an N-position.*

Proof. (N, N) is a P-position, because any move done to (n, n) can only affect one pile, and then the player can simply mirror the action done in the turn to the other pile. In otherwords, the game is sorta like this: $(n, n) \xrightarrow{N} (n', n) \xrightarrow{P} (n', n')$

This will repeat until the P player moves it to $(0, 0)$, thereby winning.

For (a, b) and $a \neq b$, we say WLOG $a < b$, and the game turns out like $(a, b) \xrightarrow{N} (a, a)$, and as we saw before, this guarantees that the next player to play (or, the P player) will lose. $\square$

Problem 2. Determine which games are P-winning or N-winning in 3 pile Nim. Any patterns?

3 Game Theory, or the Theory of Nim

Our goal is primarily to find a way to determine which games of Nim are P or N games, however, in doing so we will also discover something pretty darn cool.

First, the definitions:

Definition 3.1. A game is *impartial* if it satisfies the following;

1. Two player, alternating game
2. The moves available on a given turn is not dependent on the player (in other words, the move set available to one player is identical to the move set available to the other)
3. The game terminates in finite time
4. The winner is the last player to make a valid move
5. Game is deterministic (no random chance)
6. Complete information (no unknowns)

Problem 3. Suppose that you can determine if every move from a position in an impartial game is a P or N position. Is this enough information to tell you the type of position the starting position is?

3.1 Game Arithmetic

We can add games together to get other games! For instance, if I take a game of single pile Nim, and add another single pile Nim, then I get a game of two pile Nim. Let's generalize:

Definition 3.2. Suppose I have two combinatorical games, G_1 and G_2. Then the game $G_1 + G_2$ is the game G_1 next to the game G_2, where on a player's turn, they may choose which game to perform a move on.

As one can tell, the sum of two impartial games is impartial.

Problem 4 (*)**.** Show that this operation is associative and commutative.

Problem 5. Determine the outcome of $G + G$ for some impartial game G.

Definition 3.3. Two games G, H are *equivalent* if $G + A$ is a P-position iff $H + A$ is a P-position. Symbolically, this is $G \approx H$.

This definition is a bit weird. We can think of it like this: doing the same thing to the game G and the game H will guarantee that their outcome is the same.

Problem 6 (*)**.** Show that $\approx$ is reflexive, symmetric, and transitive.

Proposition 3.4. *A game G starts in a P game iff $G \approx 0$, where 0 denotes the game in which there are no valid moves to be made.*

Proof. This is true because the 0 game is by definition a P position, and if G is equivalent to a game of P position only, then it too is a P game. $\square$

3.2 Sprague-Grundy

This is where everything prior has built up to.

Definition 3.5. Let the game of Nim with a single pile of n objects to be $*n$. This is called the nth *nimber*.

So, with the above, a game of Nim with a pile of 4 and a pile of 6 would be $*4 + *6$.

Definition 3.6. Let the *Sprague-Grundy* value of a position G (or game, remember position = game) be the smallest nonnegative integer that isn't a Sprague-Grundy value of any of the possible positions that can be accessed validly from G.

I'll probably refer to these as Skrundy values, because it sounds funny.

Corollary 3.7. *When a game has Skrundy value 0, it is a P game.*

Proof. Suppose the Skrundy value of G is 0. This means that, by definition, the Skrundy value of any move from G will *not* be zero. Then, the next player simply does the move that makes the Skrundy value 0, and repeats this. Eventually, the game will reduce to the 0 game and the last player who goes is the person who is always making the Skrundy value 0. □

Definition 3.8. The *nim-sum* of two numbers a, b is denoted $a \oplus b$ and is calculated by taking the number in binary and for every digits position, the resultant will have a 1 iff there is exactly one 1 among the corresponding digit of a and b.

Theorem 3.9. *The Skrundy value of the game of Nim $*n_1 + *n_2 + \cdots + *n_i$ is the same as*

$$n_1 \oplus n_2 \oplus \cdots \oplus n_i,$$

where $\oplus$ is the Nim-sum operator.

Too lazy to prove this, so here's a link: https://en.wikipedia.org/wiki/Nim#Proof_of_the_winning_formula

Now here's the cool result:

Theorem 3.10 (Sprague-Grundy)**.** *Suppose G has Skrundy value n. Then, G is equivalent to the nth nimber.*

Proof. This proof relies on induction. This is classically referred to as structural induction, which inducts on a graph basically. However, this proof is translated to basically only use strong induction.

Suppose that for all $k < n$, a game with Skrundy value k is the same as the nimber $*k$.

By definition 3.3, we want to show that $G + *n \approx *n + *n \approx 0$.

Suppose a valid move here on G is to make it into G'. Then, on the game $G + *n$, there are several things that could happen.

First, a move can be done to the $*n$ component, changing it into $*m$ for some $m < n$. Well, by definition of Skrundy values, there exists a move G' such that the Skrundy value of G' is m. Since $G' + *n \approx 0$ by induction hypothesis, this case is resolved.

Next, a move can be done to G to make it G', with Skrundy value $m < n$ and then the next player simply makes $*n$ into $*m$. Then, as before, we are done.

The last case is the most annoying. By definition of Skrundy values, no move exists that sends G to G' with Skrundy value of G' being equal to n. However, this doesn't mean that a move can't exist sending G' value to be $m > n$. In this case, however, by definition we can simply choose the move that sends G' to G'', which has Skrundy value n. By definition of impartial games, this game $G + *n$ must move in finite time, thus, this cannot repeat forever and this case is thus resolved.

□