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SLE martingales in coset conformal field theory

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Scharmm-Loewner evolution (SLE) and conformal field theory (CFT) are popular instruments to study critical behavior of physical models in two dimensions, but they use different objects. While SLE has natural connection with lattice models and is suitable for strict proofs, it lacks computational and predictive power of conformal field theory. To provide a way for the concurrent use of SLE and CFT we consider CFT correlation functions which are martingales with respect to SLE. We establish necessary and sufficient conditions on such observables for coset models of conformal field theory and Schramm-Loewner evolution on group manifolds and check the consistency with the behaviour of parafermionic models. Coset models are connected with off-critical massive field theories and we discuss possible implications for SLE.

1. INTRODUCTION

Schramm-Loewner evolution [1] is a stochastic process satisfying stochastic differential equation (1). Solution of equation (1) can be seen as probability measure on random curves called SLE traces. SLE is useful for the study of critical behavior [2, 3], since for many lattice models the convergence of domain walls to SLE traces can be proved [4, 5]. Practical computations with SLE rely on Ito calculus.

Conformal field theory [6] is formulated in terms of families of quantum fields which usually has no natural interpretation in microscopic terms. On the other hand CFT has efficient computational tools such as Virasoro symmetry, Knizhnik-Zamolodchikov equations, fusion algebra etc and provides numerous predictions, such as celebrated Cardy formula [7, 8] which gives crossing probability for critical percolation.

Combination of these two approaches should be immensely powerful, so it is important to find objects which can be studied from both points of view. We consider correlation functions of CFT which are martingales with respect to Schramm-Loewner evolution are such objects.

We present new class of such observables in coset conformal field theory [9]. They are correlation functions in G/A-coset conformal field theory which are martingales with respect to Schramm-Loewner evolution with additional Brownian motion on factor-space of Lie group G by subgroup A [10]. Using Knizhnik-Zamolodchikov equations we establish necessary and sufficient conditions on such functions and derive relations connecting parameters of SLE and CFT. We then check the consistency of these results with the earlier results obtained for parafermionic observables [11].

All minimal unitary models can be obtained by coset construction. This coset structure leads to specific off-critical behavior such as classification of massive excitations by affine Toda field theory [12, 13, 14]. In conclusion we discuss possible connections of these CFT perturbation with massive off-critical SLEs (which contain additional drift term [15]).

2. MARTINGALE CONDITIONS

Consider some critical lattice model on the upper half-plane $\mathbb{H}$ with the cut along a critical interface γ_t. We denote this slit domain by $\mathbb{H}_t = \mathbb{H} \setminus \gamma_t$. Assume that γ_t satisfies restriction property and is conformally invariant [2]. Then conformal map $g_t : \mathbb{H}_t \to \mathbb{H}$ satisfies stochastic differential equation [1]:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}\xi_t}, \tag{1}$$

where ξ_t is the Brownian motion. The dynamic of the tip z_t of critical curve γ_t (tip of SLE trace) is given by the law $z_t = g_t^{-1}(\sqrt{\kappa}\xi_t)$.

For us it is more convenient to use map $w_t(z) = g_t(z) - \sqrt{\kappa}\xi_t$, so the equation (1) becomes

$$dw_t = \frac{2dt}{w_t} - \sqrt{\kappa}\xi_t \tag{2}$$

Consider N-point correlation function of boundary conformal field theory with boundary condition changing operators sitting at the tip of SLE trace and at the infinity:

$$\mathcal{F}(\{z_i, \bar{z}_i\}_{i=1}^{N})_{\mathbb{H}_t} = \langle \varphi(z_t)\phi_1(z_1, \bar{z}_1)\dots\phi_n(z_n, \bar{z}_n)\varphi(\infty)\rangle, \tag{3}$$

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where ϕ_i are bulk primary fields and $\varphi(z_t)$ is boundary condition changing operator.

Primary fields in coset models are labeled by pairs (λ, η) of weights $\mathfrak{g}$ and $\mathfrak{a}$-representations. For each λ possible values of η are those which appear in decomposition of representation $L_{\mathfrak{g}}^{(\mu)} = \bigoplus_\eta H_\eta^\lambda \otimes L_{\mathfrak{a}}^{(\eta)}$. Some pairs are equivalent, we discuss and use this equivalence relation later. We assume that boundary field φ is also primary and labeled by weights (μ, ν).

We use the conformal map $w(z) : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ to rewrite expression (3) in the whole upper half plane:

$$\mathcal{F}(\{z_i, \bar{z}_i\})_{\mathbb{H}_t} = \\ = \prod \left(\frac{\partial w(z_i)}{\partial z_i}\right)^{h_i} \prod \left(\frac{\partial \bar{w}(\bar{z}_i)}{\partial \bar{z}_i}\right)^{\bar{h}_i} \mathcal{F}(\{w_i, \bar{w}_i\})_{\mathbb{H}} \quad (4)$$

We can extend the definition of $\mathcal{F}$ to the whole plane by doubling the number of fields ϕ_i [20, 19] and consider $w_i, \bar{w}_i$ as independent variables. To simplify notation we will write $\mathcal{F}(\{w_i\}_{i=1}^{2N})$.

G/A-coset conformal field theory can be realized as WZNW-model with gauge group G interacting with pure gauge field of gauge group $A \subset G$ [23, 24]. So we have G/A gauge symmetry and we add random gauge transformations to Schramm-Loewner evolution [10] similar to the case of WZNW-models [25]. Denote by $\mathcal{K}$ Killing form of Lie algebra $\mathfrak{g}$ corresponding to Lie group G and by t_i^a ($\tilde{t}_i^b$) the generators of $\mathfrak{g}$-representation ($\mathfrak{a}$-representation) corresponding to primary field ϕ_i.

Now we need to consider the evolution of SLE trace γ_t from t to $t + dt$. First factor in right-hand-side of equation (3) gives us

$$-\frac{2h_i}{w_i^2}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}.$$

We denote by $\mathcal{G}_i$ generator of infinitesimal transformation of primary field ϕ_i:$d\phi_i(w_i) = \mathcal{G}_i\phi_i(w_i)$. We normalize additional $\dim \mathfrak{g}$-dimensional Brownian motion as $\mathbb{E}\left[d\theta^a\ d\theta^b\right] = \mathcal{K}(t^a, t^b)dt$, and the generator of field transformation is equal to

$$\mathcal{G}_i = \left(\frac{2dt}{w_i} - \sqrt{\kappa}d\xi_t\right)\partial_{w_i} + \frac{\sqrt{\tau}}{w_i}\left(\sum_{a:\mathcal{K}(t^a,\tilde{t}^b)=0} (d\theta^a t_i^a)\right). \quad (5)$$

Ito formula gives us the expression for the differential of $\mathcal{F}$ which should be zero due to martingale condition:

$$d\mathcal{F}_{\mathbb{H}_t} = \\ \left(\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}\right)\left(-\sum_{i=1}^{2N}\frac{2h_i dt}{w_i^2} + \left[\sum_{i=1}^{2N}\mathcal{G}_i + \frac{1}{2}\sum_{i,j}\mathcal{G}_i\mathcal{G}_j\right]\right)\mathcal{F}_{\mathbb{H}} \\ = 0 \quad (6)$$

Substitute definition (5) and get

$$\left(-2\mathcal{L}_{-2} + \frac{1}{2}\kappa\mathcal{L}_{-1}^2 + \frac{\tau}{2}\left(\sum_a \mathcal{J}_{-1}^a\mathcal{J}_{-1}^a - \sum_b \tilde{\mathcal{J}}_{-1}^b\tilde{\mathcal{J}}_{-1}^b\right)\right)\mathcal{F}_{\mathbb{H}} = 0, \quad (7)$$

where

$$\mathcal{L}_{-n} = \sum_i\left(\frac{(n-1)h_i}{(w_i - z)^n} - \frac{\partial_{w_i}}{(w_i - z)^{n-1}}\right)$$

$$\mathcal{J}_{-n}^a = -\sum_i \frac{t_i^a}{(w_i - z)^n};\ \tilde{\mathcal{J}}_{-n}^b = -\sum_i \frac{\tilde{t}_i^b}{(w_i - z)^n}$$

This equation is equivalent to following algebraic condition on boundary state $\varphi(0)\left|0\right>$:

$$\langle 0\left|\varphi(\infty)\phi_1(w_1)\dots\phi_{2N}(w_{2N})\right. \\ \left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a - \sum_{b=1}^{\dim\mathfrak{a}} \tilde{J}_{-1}^b\tilde{J}_{-1}^b\right)\right) \\ \varphi(0)|0\rangle = 0 \quad (8)$$

Since the set $\{\phi_i\}$ consists of arbitrary primary fields we conclude that

$$|\psi> = \left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a - \sum_{b=1}^{\dim\mathfrak{a}} \tilde{J}_{-1}^b\tilde{J}_{-1}^b\right)\right) \\ \cdot\varphi(0)|0> \quad (9)$$

is null-state. Now we act on ψ by raising operators to get equations on κ, τ. Since in coset theory

$$\begin{aligned} &L_n = L_n^{\mathfrak{g}} - L_n^{\mathfrak{a}} \\ &[L_n^{\mathfrak{g}}, J_m^a] = -mJ_{n+m}^a \\ &\left[L_n^{\mathfrak{g}}, \tilde{J}_m^b\right] = -m\tilde{J}_{n+m}^b \\ &[L_n^{\mathfrak{g}}, L_m] = [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{g}}, L_m^{\mathfrak{a}}] = \\ &\quad = (n-m)L_{m+n} + \tfrac{c}{12}(n^3 - n)\delta_{m+n,0} \end{aligned} \quad (10)$$

it is more convenient to act by $L_2^{\mathfrak{g}}$ and $\left(L_1^{\mathfrak{g}}\right)^2$. Acting with $L_2^{\mathfrak{g}}$ we get

$$L_2^{\mathfrak{g}}\psi = \left(-8L_0 - c + 3\kappa L_0 + \frac{1}{2}\tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a})\right)\varphi_{(\mu,\nu)} = 0$$

We use $L_0\varphi_{(\mu,\nu)} = h_{(\mu,\nu)}\varphi_{(\mu,\nu)}$ where conformal weight $h_{(\mu,\nu)} = \left(\frac{(\mu,\mu+2\rho)}{2(k+h_{\mathfrak{g}}^\vee)} - \frac{(\nu,\nu+2\rho_{\mathfrak{a}})}{2(kx_e+h_{\mathfrak{a}}^\vee)}\right)$ and central charge $c = \frac{k\dim\mathfrak{g}}{k+h_{\mathfrak{g}}^\vee} - \frac{x_e k\dim\mathfrak{a}}{x_e k+h_{\mathfrak{a}}^\vee}$ and get relation on κ, τ:

$$(3\kappa - 8)h_{(\mu,\nu)} - c + \tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a}) = 0 \quad (11)$$

Second relation appears as the result of $L_1^{\mathfrak{g}}$-action:

$$-12h_{(\mu,\nu)} + 2\kappa h_{(\mu,\nu)}(2h_{(\mu,\nu)} + 1) + \tau(C_\mu - \tilde{C}_\nu) = 0, \quad (12)$$

where $C_\mu = (\mu, \mu + 2\rho), \tilde{C}_\nu = (\nu, \nu + 2\rho_{\mathfrak{a}})$ is the eigenvalue of quadratic Casimir operators $\sum_a t^a t^a, \sum_b \tilde{t}^b \tilde{t}^b$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$. Relations (11),(12) are necessary conditions for CFT correlation functions to be SLE martingales.

From equations (11),(12) we immediately get κ, τ for each pair (μ, ν) of $\mathfrak{g}$ and $\mathfrak{a}$-weights.

21 Examples

Minimal models $su(2)/u(1)$, $su(2) \oplus su(2)/su(2)$.

Consider standard coset realization of minimal models: $su(2)_N/u(1)_N$. Central charge is $c = \frac{3N}{N+2} - 1 = \frac{2N-2}{N+2}$. Conformal weights of primary fields with Dynkin indeces (k,l) are $h_{(k,l)} = \frac{k(k+2)}{4(N+2)} - \frac{l^2}{4N}$. $C_\mu = k(k+2)/2, C_\nu = l^2/2$ For Ising model $N = 2, c = 1/2$, $h_{(2,0)} = 1/2, h_{(1,1)} = 1/16$. We get for $\varphi_{(2,0)}$: $3\kappa - 9 + 4\tau = 0, -3 + \kappa + 4\tau = 0$, $\kappa = 3, \tau = 0$. For $\varphi_{(1,1)}$: $3\kappa - 16 + 64\tau = 0, -64 + 9\kappa + 64\tau = 0$, $\kappa = 16/3, \tau = 0$. For $N = 3, c = 4/5$. Conformal weights $h_{(0,0)} = 0$, $h_{(0,2)} = 2/3$, $h_{(0,-2)} = 2/3$, $h_{(1,3)} = 2/5$, $h_{(1,1)} = 1/15$, $h_{(1,-1)} = 1/15$. Corresponding values of κ, τ are: $(208/25, 242/225), (460/33, 116/33), (260/53, 28/477)$

Now use another coset realization of minimal unitary models: $\frac{\hat{su}(2)_N \oplus \hat{su}(2)_1}{\hat{su}(2)_{N+1}}$, $c = \frac{3N}{N+2} + 1 - \frac{3(N+1)}{N+3} = 1 - \frac{6}{(N+2)(N+3)}$

three-state Potts model $\frac{\hat{su}(3)_1 \oplus \hat{su}(3)_1}{\hat{su}(3)_2}$

But correlation functions in coset conformal field theory must satisfy (generalized) Knizhnik-Zamolodchikov equations [21] which are of the form

$$\left[\frac{1}{2} \frac{\partial}{\partial w_i} + \sum_{j \neq i} \left(\frac{\sum_a t_i^a t_j^a}{k + h^\vee} - \frac{\sum_b \tilde{t}_i^b \tilde{t}_j^b}{x_e k + h_{\mathfrak{a}}^\vee} \right) \right] \mathcal{F}_{\mathbb{H}} = 0 \quad (13)$$

Rewrite the equation as

$$\left[\frac{1}{2} \frac{\partial}{\partial w_i} + \sum_{j \neq i} \left(\frac{\sum_a t_i^a t_j^a}{k + h^\vee} - \frac{\sum_b \tilde{t}_i^b \tilde{t}_j^b}{x_e k + h_{\mathfrak{a}}^\vee} \right) \right] \mathcal{F}_{\mathbb{H}} = 0 \quad (14)$$

From this relations and equation (6) we can write sufficient conditions for SLE martingale to be CFT correlation function:

$$Mx = 0 \quad (15)$$

It is also possible to consider coset theory as $G \times A$ gauge theory and impose constraints on physical states later (See [22, 21, 23, 24]). In this case second term in equation (5) will include sum over $\mathfrak{g}$ and $\mathfrak{a}$ generators and two parameters of Brownian motion $\tau_{\mathfrak{g}}$ and $\tau_{\mathfrak{a}}$.

3. CONCLUSION AND OUTLOOK

In excellent paper [25] connection between WZNW-models and Schramm-Loewner evolution with additional Brownian motion on group manifold was established. This paper also stated a problem of the connection of SLE parameters for martingales in WZNW, coset and minimal models. In present letter we used the method to obtain necessary and sufficient conditions on SLE martingales proposed in paper [26]. This method allowed us to compare our results with parafermionic results of the papers [11, 27].

Coset structure of minimal models is manifest in field theory perturbed by external magnetic field [12, 13, 14]. This theory is supported by experimental data [28]. Relation between correlation functions in coset theory and SLE observables can be a starting point for the study of domain walls in lattice models away of critical point.

Massive off-critical SLEs has additional drift term in driving Brownian motion [15].

Schramm-Loewner evolution was applied for the description of domain walls in Ising model with external field in recent paper [29], although the perturbation was not compatible with affine field theory structure.

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