SLE martingales in coset conformal field theory

Anton Nazarov

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1 Introduction

Generalizing results of papers [1, 2, 3] we consider Schramm-Loewner evolution SLE_κ with additional Brownian motion on coset space G/A characterised by parameter τ proposed in our paper [4]. We consider correlation functions in coset models of conformal field theory which satisfy martingale conditions with respect to this stochastic process. Since all minimal unitary models of conformal field theory can be obtained from coset construction for certain choice of Lie group G and subgroup A [5] these correlation functions correspond to SLE observables appearing as the scaling limit of lattice observables. Trace of Schramm-Loewner γ_t evolution describes critical domain wall in lattice models.

Consider upper half-plane $\mathbb{H}$ with the cut along a critical interface γ_t. We denote this slit domain by $\mathbb{H}_t = \mathbb{H}\setminus\gamma_t$. Conformal map from $\mathbb{H}_t$ to $\mathbb{H}$ is denoted by $g_t : \mathbb{H}_t \to \mathbb{H}$ and satisfies stochastic differential equation [6]:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}\xi_t}, \tag{1}$$

where ξ_t is the Brownian motion. The dynamic of the tip z_t of critical curve γ_t (tip of SLE trace) is given by the law $z_t = g_t^{-1}(\sqrt{\kappa}\xi_t)$.

For us it is more convenient to use map $w_t(z) = g_t(z) - \sqrt{\kappa}\xi_t$, so the equation (1) becomes

$$dw_t = \frac{2dt}{w_t} - \sqrt{\kappa}\xi_t \tag{2}$$

It is natural to study the problem in upper half-plane and consider correlation functions of boundary conformal field theory with boundary condition changing operators sitting at the tip of SLE trace and at the infinity.

So we are to consider N-point boundary correlation functions on the upper half-plane of the form

$$\mathcal{F}(\{z_i, \bar{z}_i\}_{i=1}^N)_{\mathbb{H}_t} = \langle \varphi(z_t)\phi_1(z_1, \bar{z}_1)\ldots\phi_n(z_n, \bar{z}_n)\varphi(\infty)\rangle\,, \tag{3}$$

where ϕ_i are bulk primary fields and $\varphi(z_t)$ is boundary condition changing operator. We assume conformal invariance of boundary conditions so boundary states satisfy Cardy conditions [7, 8, 9].

We consider Schramm-Loewner evolution with additional Brownian motion on G/A-coset space [4]. Denote by $\mathcal{K}$ Killing form of Lie algebra $\mathfrak{g}$ corresponding to Lie group G and by t_i^a ($\tilde{t}_i^b$) the generators of $\mathfrak{g}$-representation ($\mathfrak{a}$-representation) corresponding to primary field ϕ_i. Primary fields in coset models are labeled by pairs (λ, η) of weights $\mathfrak{g}$ and $\mathfrak{a}$-representations. For each λ possible values of η are those which appear in decomposition of representation $L_{\mathfrak{g}}^{(\mu)} = \bigoplus_{\eta} H_{\eta}^{\lambda} \otimes L_{\mathfrak{a}}^{(\eta)}$. Some pairs are equivalent, we discuss and use this equivalence relation later. We assume that boundary field φ is also primary and labeled by weights (μ, ν).

We use the conformal map $w(z) : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ to rewrite expression (3) in the whole upper half plane:

$$\mathcal{F}(\{z_i, \bar{z}_i\})_{\mathbb{H}_t} = \prod \left(\frac{\partial w(z_i)}{\partial z_i}\right)^{h_i} \prod \left(\frac{\partial \bar{w}(\bar{z}_i)}{\partial \bar{z}_i}\right)^{\bar{h}_i} \mathcal{F}(\{w_i, \bar{w}_i\})_{\mathbb{H}} \tag{4}$$

We can extend the definition of $\mathcal{F}$ to the whole plane by doubling the number of fields ϕ_i [10, 9] and consider $w_i, \bar{w}_i$ as independent variables. To simplify notation we will write $\mathcal{F}(\{w_i\}_{i=1}^{2N})$.

Now we need to consider the evolution of SLE trace γ_t from t to $t+dt$. First factor in right-hand-side of equation (3) gives us

$$-\frac{2h_i}{w_i^2}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}.$$

We denote by $\mathcal{G}_i$ generator of infinitesimal transformation of primary field ϕ_i:$d\phi_i(w_i) = \mathcal{G}_i \phi_i(w_i)$. Then normalization condition of additional $\dim \mathfrak{g}$-dimensional Brownian motion (15) to $\mathbb{E}\left[d\theta^a\, d\theta^b\right] = \mathcal{K}(t^a, t^b)dt$, and the generator of field transformation (3) is

$$\mathcal{G}_i = \left(\frac{2dt}{w_i} - \sqrt{\kappa} d\xi_t\right)\partial_{w_i} + \frac{\sqrt{\tau}}{w_i}\left(\sum_{a:\mathcal{K}(t^a, \tilde{t}^b)=0} (d\theta^a t_i^a)\right). \tag{5}$$

Ito formula gives us the expression for the differential of $\mathcal{F}$, it should be zero due to martingale condition:

$$d\mathcal{F}_{\mathbb{H}_t} = \left(\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}\right)\left(-\sum_{i=1}^{2N}\frac{2h_i dt}{w_i^2} + \left[\sum_{i=1}^{2N}\mathcal{G}_i + \frac{1}{2}\sum_{i,j}\mathcal{G}_i\mathcal{G}_j\right]\right)\mathcal{F}_{\mathbb{H}} = 0 \tag{6}$$

Substitute definition (5) and get

$$\left(-2\mathcal{L}_{-2} + \frac{1}{2}\kappa\mathcal{L}_{-1}^2 + \frac{\tau}{2}\left(\sum_a \mathcal{J}_{-1}^a\mathcal{J}_{-1}^a - \sum_b \tilde{\mathcal{J}}_{-1}^b\tilde{\mathcal{J}}_{-1}^b\right)\right)\mathcal{F}_{\mathbb{H}} = 0, \tag{7}$$

where

$$\mathcal{L}_{-n} = \sum_i\left(\frac{(n-1)h_i}{(w_i-z)^n} - \frac{\partial_{w_i}}{(w_i-z)^{n-1}}\right); \quad \mathcal{J}_{-n}^a = -\sum_i \frac{t_i^a}{(w_i-z)^n}; \; \tilde{\mathcal{J}}_{-n}^b = -\sum_i \frac{\tilde{t}_i^b}{(w_i-z)^n}$$

This equation is equivalent to following algebraic condition on boundary state $\varphi(0)\left|0\right\rangle$:

$$< 0|\varphi(\infty)\phi_1(w_1)\dots\phi_{2N}(w_{2N})\left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a - \sum_{b=1}^{\dim\mathfrak{a}} \tilde{J}_{-1}^b\tilde{J}_{-1}^b\right)\right)\varphi(0)|0 >= 0 \tag{8}$$

Since the set $\{\phi_i\}$ consists of arbitrary primary fields we conclude that $|\psi>= \left(-2L_{-2}+\frac{1}{2}\kappa L_{-1}^2+\frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a-\sum_{b=1}^{\dim\mathfrak{a}} \tilde{J}_{-1}^b\tilde{J}_{-1}^b\right)\right)\varphi(0)|0>$ is null-state. Now we act on ψ by raising operators to get equations on κ,τ. Since in coset theory

$$\begin{array}{l} L_n = L_n^{\mathfrak{g}} - L_n^{\mathfrak{a}} \\ [L_n^{\mathfrak{g}}, J_m^a] = -mJ_{n+m}^a \\ \left[L_n^{\mathfrak{g}}, \tilde{J}_m^b\right] = -m\tilde{J}_{n+m}^b \\ [L_n^{\mathfrak{g}}, L_m] = [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{g}}, L_m^{\mathfrak{a}}] = (n-m)L_{m+n} + \frac{c}{12}(n^3-n)\delta_{m+n,0} \end{array} \tag{9}$$

it is more convenient to act by $L_2^{\mathfrak{g}}$ and $\left(L_1^{\mathfrak{g}}\right)^2$. Acting with $L_2^{\mathfrak{g}}$ we get

$$L_2^{\mathfrak{g}}\psi = \left(-8L_0 - c + 3\kappa L_0 + \frac{1}{2}\tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a})\right)\varphi_{(\mu,\nu)} = 0$$

We use $L_0\varphi_{(\mu,\nu)} = h_{(\mu,\nu)}\varphi_{(\mu,\nu)}$ where conformal weight $h_{(\mu,\nu)} = \left(\frac{(\mu,\mu+2\rho)}{2(k+h_{\mathfrak{g}}^\vee)} - \frac{(\nu,\nu+2\rho_{\mathfrak{a}})}{2(kx_e+h_{\mathfrak{a}}^\vee)}\right)$ and central charge $c = \frac{k\dim\mathfrak{g}}{k+h_{\mathfrak{g}}^\vee} - \frac{x_e k\dim\mathfrak{a}}{x_e k+h_{\mathfrak{a}}^\vee}$ and get relation on κ,τ:

$$(3\kappa-8)h_{(\mu,\nu)} - c + \tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a}) = 0 \tag{10}$$

Second relation appears as the result of $L_1^{\mathfrak{g}}$-action:

$$-12h_{(\mu,\nu)} + 2\kappa h_{(\mu,\nu)}(2h_{(\mu,\nu)}+1) + \tau(C_\mu - \tilde{C}_\nu) = 0, \tag{11}$$

where $C_\mu = (\mu,\mu+2\rho), \tilde{C}_\nu = (\nu,\nu+2\rho_{\mathfrak{a}})$ is the eigenvalue of quadratic Casimir operators $\sum_a t^a t^a, \sum_b \tilde{t}^b\tilde{t}^b$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$.

From equations (27),(29) we immediately get κ,τ for each pair (μ,ν) of $\mathfrak{g}$ and $\mathfrak{a}$-weights.

It is also possible to consider coset theory as $G\times A$ gauge theory and impose constraints on physical states later (See [11, 12, 13, 14]). In this case second term in equation (5) will include sum over $\mathfrak{g}$ and $\mathfrak{a}$ generators and two parameters of Brownian motion $\tau_{\mathfrak{g}}$ and $\tau_{\mathfrak{a}}$.

2 Algebraic equations

3 Examples

4 Conclusion

References

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