SLE martingales in coset conformal field theory

Anton Nazarov

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Abstract

The problem is to provide a way for the concurrent use of Scramm-Loewner evolution (SLE) and conformal field theory (CFT). Both SLE and CFT are popular instruments to study critical behavior of physical models in two dimensions, but they use different objects. While Scharmm-Loewner evolution has natural connection with lattice models and is suitable for strict proofs, it lacks computational and predictive power of conformal field theory. In order to connect two approaches we consider CFT correlation functions which are martingales with respect to SLE. We establish necessary and sufficient conditions on such observables for coset models of conformal field theory and Schramm-Loewner evolution on group manifolds and check the consistency with the behaviour of parafermionic models.

1 Introduction

SLE describes critical behavior. It is convenient for strict mathematical proofs. For many lattice models the convergence of domain walls to SLE was proved.

CFT on the other hand is formulated in terms of families of quantum fields which usually has no natural interpretation in microscopic terms. On the other hand CFT has efficient computational tools such as Virasoro symmetry, Knizhnik-Zamolodchikov equations, fusion algebra etc and provides numerous predictions, such as celebrated Cardy formula which gives crossing probability for critical percolation.

Combination of these two approaches should be immensely powerful, so it is important to find objects which can be studied from both points of view. Correlation functions of CFT which are martingales with respect to Schramm-Loewner evolution are such objects. They are actively studied.

We present new class of such observables. We consider correlation functions in coset conformal field theory which are martingales with respect to Schramm-Loewner evolution with additional random motion on factor of group G by subgroup A. Using Knizhnik-Zamolodchikov equations we establish necessary and sufficient conditions on such functions and derive relations connecting parameters of SLE and CFT. We then check the consistency of these results with the earlier results obtained for parafermionic observables and general Cardy conditions on boundary states in conformal field theory.

2 Martingale conditions

We consider Schramm-Loewner evolution SLE_κ with additional Brownian motion on coset space G/A characterised by parameter τ proposed in our paper [1]. We consider correlation functions in coset models of conformal field theory which satisfy martingale conditions with respect to this stochastic process. Since all minimal unitary models of conformal field theory can be obtained from coset construction for certain choice of Lie group G and subgroup A [2] these correlation functions correspond to SLE observables appearing as the scaling limit of lattice observables. Trace of Schramm-Loewner γ_t evolution describes critical domain wall in lattice models.

Consider upper half-plane $\mathbb{H}$ with the cut along a critical interface γ_t. We denote this slit domain by $\mathbb{H}_t = \mathbb{H} \setminus \gamma_t$. Conformal map from $\mathbb{H}_t$ to $\mathbb{H}$ is denoted by $g_t : \mathbb{H}_t \to \mathbb{H}$ and satisfies stochastic differential equation [3]:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}\xi_t}, \tag{1}$$

where ξ_t is the Brownian motion. The dynamic of the tip z_t of critical curve γ_t (tip of SLE trace) is given by the law $z_t = g_t^{-1}(\sqrt{\kappa}\xi_t)$.

For us it is more convenient to use map $w_t(z) = g_t(z) - \sqrt{\kappa}\xi_t$, so the equation (1) becomes

$$dw_t = \frac{2dt}{w_t} - \sqrt{\kappa}\xi_t \tag{2}$$

It is natural to study the problem in upper half-plane and consider correlation functions of boundary conformal field theory with boundary condition changing operators sitting at the tip of SLE trace and at the infinity.

So we are to consider N-point boundary correlation functions on the upper half-plane of the form

$$\mathcal{F}(\{z_i, \bar{z}_i\}_{i=1}^{N})_{\mathbb{H}_t} = \langle \varphi(z_t)\phi_1(z_1, \bar{z}_1)\dots\phi_n(z_n, \bar{z}_n)\varphi(\infty)\rangle\,, \tag{3}$$

where ϕ_i are bulk primary fields and $\varphi(z_t)$ is boundary condition changing operator. We assume conformal invariance of boundary conditions so boundary states satisfy Cardy conditions [4, 5, 6].

We consider Schramm-Loewner evolution with additional Brownian motion on G/A-coset space [1]. Denote by $\mathcal{K}$ Killing form of Lie algebra $\mathfrak{g}$ corresponding to Lie group G and by t_i^a ($\tilde{t}_i^b$) the generators of $\mathfrak{g}$-representation ($\mathfrak{a}$-representation) corresponding to primary field ϕ_i. Primary fields in coset models are labeled by pairs (λ, η) of weights $\mathfrak{g}$ and $\mathfrak{a}$-representations. For each λ possible values of η are those which appear in decomposition of representation $L_{\mathfrak{g}}^{(\mu)} = \bigoplus_\eta H_\eta^\lambda \otimes L_{\mathfrak{a}}^{(\eta)}$. Some pairs are equivalent, we discuss and use this equivalence relation later. We assume that boundary field φ is also primary and labeled by weights (μ, ν).

We use the conformal map $w(z) : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ to rewrite expression (3) in the whole upper half plane:

$$\mathcal{F}(\{z_i, \bar{z}_i\})_{\mathbb{H}_t} = \prod \left(\frac{\partial w(z_i)}{\partial z_i}\right)^{h_i} \prod \left(\frac{\partial \bar{w}(\bar{z}_i)}{\partial \bar{z}_i}\right)^{\bar{h}_i} \mathcal{F}(\{w_i, \bar{w}_i\})_{\mathbb{H}} \tag{4}$$

We can extend the definition of $\mathcal{F}$ to the whole plane by doubling the number of fields ϕ_i [7, 6] and consider $w_i, \bar{w}_i$ as independent variables. To simplify notation we will write $\mathcal{F}(\{w_i\}_{i=1}^{2N})$.

Now we need to consider the evolution of SLE trace γ_t from t to $t+dt$. First factor in right-hand-side of equation (3) gives us

$$-\frac{2h_i}{w_i^2}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}.$$

We denote by $\mathcal{G}_i$ generator of infinitesimal transformation of primary field ϕ_i:$d\phi_i(w_i) = \mathcal{G}_i\phi_i(w_i)$. We normalize additional $\dim\mathfrak{g}$-dimensional Brownian motion as $\mathbb{E}\left[d\theta^a\, d\theta^b\right] = \mathcal{K}(t^a, t^b)dt$, and the generator of field transformation (3) is equal to

$$\mathcal{G}_i = \left(\frac{2dt}{w_i} - \sqrt{\kappa}d\xi_t\right)\partial_{w_i} + \frac{\sqrt{\tau}}{w_i}\left(\sum_{a:\mathcal{K}(t^a,\tilde{t}^b)=0}(d\theta^a t_i^a)\right). \tag{5}$$

Ito formula gives us the expression for the differential of $\mathcal{F}$, it should be zero due to martingale condition:

$$d\mathcal{F}_{\mathbb{H}_t} = \left(\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}\right)\left(-\sum_{i=1}^{2N}\frac{2h_i dt}{w_i^2} + \left[\sum_{i=1}^{2N}\mathcal{G}_i + \frac{1}{2}\sum_{i,j}\mathcal{G}_i\mathcal{G}_j\right]\right)\mathcal{F}_{\mathbb{H}} = 0 \tag{6}$$

Substitute definition (5) and get

$$\left(-2\mathcal{L}_{-2} + \frac{1}{2}\kappa\mathcal{L}_{-1}^2 + \frac{\tau}{2}\left(\sum_a \mathcal{J}_{-1}^a\mathcal{J}_{-1}^a - \sum_b \tilde{\mathcal{J}}_{-1}^b\tilde{\mathcal{J}}_{-1}^b\right)\right)\mathcal{F}_{\mathbb{H}} = 0, \tag{7}$$

where

$$\mathcal{L}_{-n} = \sum_i\left(\frac{(n-1)h_i}{(w_i-z)^n} - \frac{\partial_{w_i}}{(w_i-z)^{n-1}}\right);\quad \mathcal{J}_{-n}^a = -\sum_i\frac{t_i^a}{(w_i-z)^n};\ \tilde{\mathcal{J}}_{-n}^b = -\sum_i\frac{\tilde{t}_i^b}{(w_i-z)^n}$$

This equation is equivalent to following algebraic condition on boundary state $\varphi(0)\,|0\rangle$:

$$<0|\varphi(\infty)\phi_1(w_1)\dots\phi_{2N}(w_{2N})\left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a - \sum_{b=1}^{\dim\mathfrak{a}}\tilde{J}_{-1}^b\tilde{J}_{-1}^b\right)\right)\varphi(0)|0>=0 \tag{8}$$

Since the set $\{\phi_i\}$ consists of arbitrary primary fields we conclude that $|\psi>= \left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a - \sum_{b=1}^{\dim\mathfrak{a}}\tilde{J}_{-1}^b\tilde{J}_{-1}^b\right)\right)\varphi(0)|0>$ is null-state. Now we act on ψ by raising operators to get equations on κ, τ. Since in coset theory

$$\begin{aligned}
&L_n = L_n^{\mathfrak{g}} - L_n^{\mathfrak{a}}\\
&[L_n^{\mathfrak{g}}, J_m^a] = -mJ_{n+m}^a\\
&\left[L_n^{\mathfrak{g}}, \tilde{J}_m^b\right] = -m\tilde{J}_{n+m}^b\\
&[L_n^{\mathfrak{g}}, L_m] = [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{g}}, L_m^{\mathfrak{a}}] = (n-m)L_{m+n} + \tfrac{c}{12}(n^3-n)\delta_{m+n,0}
\end{aligned} \tag{9}$$

it is more convenient to act by $L_2^{\mathfrak{g}}$ and $\left(L_1^{\mathfrak{g}}\right)^2$. Acting with $L_2^{\mathfrak{g}}$ we get

$$L_2^{\mathfrak{g}}\psi = \left(-8L_0 - c + 3\kappa L_0 + \frac{1}{2}\tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a})\right)\varphi_{(\mu,\nu)} = 0$$

We use $L_0\varphi_{(\mu,\nu)} = h_{(\mu,\nu)}\varphi_{(\mu,\nu)}$ where conformal weight $h_{(\mu,\nu)} = \left(\frac{(\mu,\mu+2\rho)}{2(k+h_{\mathfrak{g}}^\vee)} - \frac{(\nu,\nu+2\rho_{\mathfrak{a}})}{2(kx_e+h_{\mathfrak{a}}^\vee)}\right)$ and central charge $c = \frac{k\dim\mathfrak{g}}{k+h_{\mathfrak{g}}^\vee} - \frac{x_e k\dim\mathfrak{a}}{x_e k+h_{\mathfrak{a}}^\vee}$ and get relation on κ, τ:

$$(3\kappa - 8)h_{(\mu,\nu)} - c + \tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a}) = 0 \tag{10}$$

Second relation appears as the result of $L_1^{\mathfrak{g}}$-action:

$$-12h_{(\mu,\nu)} + 2\kappa h_{(\mu,\nu)}(2h_{(\mu,\nu)} + 1) + \tau(C_\mu - \tilde{C}_\nu) = 0, \tag{11}$$

where $C_\mu = (\mu, \mu+2\rho), \tilde{C}_\nu = (\nu, \nu+2\rho_{\mathfrak{a}})$ is the eigenvalue of quadratic Casimir operators $\sum_a t^a t^a, \sum_b \tilde{t}^b\tilde{t}^b$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$.

From equations (10),(11) we immediately get κ, τ for each pair (μ,ν) of $\mathfrak{g}$ and $\mathfrak{a}$-weights.

But correlation functions in coset conformal field theory must satisfy (generalized) Knizhnik-Zamolodchikov equations [8] which are of the form

$$\left[\frac{1}{2}\frac{\partial}{\partial w_i} + \sum_{j\neq i}\left(\frac{\sum_a t_i^a t_j^a}{k+h^\vee} - \frac{\sum_b \tilde{t}_i^b\tilde{t}_j^b}{x_e k + h_{\mathfrak{a}}^\vee}\right)\right]\mathcal{F}_{\mathbb{H}} = 0 \tag{12}$$

Rewrite the equation as

$$\left[\frac{1}{2}\frac{\partial}{\partial w_i} + \sum_{j\neq i}\left(\frac{\sum_a t_i^a t_j^a}{k+h^\vee} - \frac{\sum_b \tilde{t}_i^b\tilde{t}_j^b}{x_e k + h_{\mathfrak{a}}^\vee}\right)\right]\mathcal{F}_{\mathbb{H}} = 0 \tag{13}$$

From this relations and equation (6) we can write sufficient conditions for SLE martingale to be CFT correlation function:

$$Mx = 0 \tag{14}$$

It is also possible to consider coset theory as $G \times A$ gauge theory and impose constraints on physical states later (See [9, 8, 10, 11]). In this case second term in equation (5) will include sum over $\mathfrak{g}$ and $\mathfrak{a}$ generators and two parameters of Brownian motion $\tau_{\mathfrak{g}}$ and $\tau_{\mathfrak{a}}$.

2.1 Examples

Minimal models $su(2)/u(1)$.

3 Outlook

In excellent paper [12] connection between WZNW-models and Schramm-Loewner evolution with additional Brownian motion on group manifold was established. This paper also stated a problem of the connection of SLE parameters for martingales in WZNW, coset and minimal models. Here we used the method to obtain necessary and sufficient conditions on SLE martingales proposed in paper [13]. This method allowed us to compare our results with parafermionic results of the papers [14, 15].

4 Conclusion

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