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SLE martingales in coset conformal field theory

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Scharmm-Loewner evolution (SLE) and conformal field theory (CFT) are popular instruments to study critical behavior of physical models in two dimensions, but they use different objects. While SLE has natural connection with lattice models and is suitable for strict proofs, it lacks computational and predictive power of conformal field theory. To provide a way for the concurrent use of SLE and CFT we consider CFT correlation functions which are martingales with respect to SLE. We establish connection between parameters of Schramm-Loewner evolution on coset space and algebraic data of coset conformal field theory. Then we check the consistency of our approach with the behaviour of parafermionic and minimal models. Coset models can be connected to off-critical massive field theories and we discuss possible implications for SLE.

1. INTRODUCTION

Schramm-Loewner evolution [1] is a stochastic process satisfying stochastic differential equation (1). Solution of equation (1) can be seen as probability measure on random curves called SLE traces. SLE is useful for the study of critical behavior [2, 3], since for many lattice models the convergence of domain walls to SLE traces can be proved [4, 5]. Practical computations with SLE rely on Ito calculus.

Conformal field theory [6] is formulated in terms of families of quantum fields which usually has no natural interpretation in microscopic terms. On the other hand CFT has efficient computational tools such as Virasoro symmetry, Knizhnik-Zamolodchikov equations, fusion algebra etc and provides numerous predictions, such as celebrated Cardy formula [7, 8] which gives crossing probability for critical percolation.

Combination of these two approaches should be immensely powerful, so it is important to find objects which can be studied from both points of view. We consider correlation functions of CFT which are martingales with respect to Schramm-Loewner evolution are such objects.

We present new class of such observables in coset conformal field theory [9]. They are correlation functions in G/A-coset conformal field theory which are martingales with respect to Schramm-Loewner evolution with additional Brownian motion on factor-space of Lie group G by subgroup A [10]. We derive relations connecting parameters of SLE and CFT. We then check the consistency of these results with the earlier results obtained for parafermionic observables [11].

All minimal unitary models can be obtained by coset construction. This coset structure leads to specific off-critical behavior such as classification of massive excitations by affine Toda field theory [12, 13, 14]. In conclusion we discuss possible connections of these CFT perturbation with massive off-critical SLEs (which contain additional drift term [15]).

2. MARTINGALE CONDITIONS

Consider some critical lattice model on the upper half-plane $\mathbb{H}$ with the cut along a critical interface γ_t. We denote this slit domain by $\mathbb{H}_t = \mathbb{H} \setminus \gamma_t$. Assume that γ_t satisfies restriction property and is conformally invariant [2]. Then conformal map $g_t : \mathbb{H}_t \to \mathbb{H}$ satisfies stochastic differential equation [1]:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}\xi_t}, \tag{1}$$

where ξ_t is the Brownian motion. The dynamic of the tip z_t of critical curve γ_t (tip of SLE trace) is given by the law $z_t = g_t^{-1}(\sqrt{\kappa}\xi_t)$.

For us it is more convenient to use map $w_t(z) = g_t(z) - \sqrt{\kappa}\xi_t$, so the equation (1) becomes

$$dw_t = \frac{2dt}{w_t} - \sqrt{\kappa}\xi_t \tag{2}$$

Consider N-point correlation function of boundary conformal field theory with boundary condition changing operators sitting at the tip of SLE trace and at the infinity:

$$\mathcal{F}(\{z_i, \bar{z}_i\}_{i=1}^N)_{\mathbb{H}_t} = \langle \varphi(z_t)\phi_1(z_1, \bar{z}_1)\dots\phi_n(z_n, \bar{z}_n)\varphi(\infty)\rangle, \tag{3}$$

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where ϕ_i are bulk primary fields and $\varphi(z_t)$ is boundary condition changing operator.

Let $\mathfrak{g}$ be (semisimple) Lie algebra of Lie group G and $\mathfrak{a}$ – Lie algebra of subgroup $A \subset G$. Denote by x_e index of embedding $\mathfrak{a} \to \mathfrak{g}$.

Primary fields in coset models are labeled by pairs (λ, η) of weights $\mathfrak{g}$ and $\mathfrak{a}$-representations. For each λ possible values of η are those which appear in decomposition of representation $L_{\mathfrak{g}}^{(\mu)} = \bigoplus_\eta H_\eta^\lambda \otimes L_{\mathfrak{a}}^{(\eta)}$. Some pairs are equivalent, we discuss and use this equivalence relation later. We assume that boundary field φ is also primary and labeled by weights (μ, ν).

We use the conformal map $w(z) : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ to rewrite expression (3) in the whole upper half plane:

$$\mathcal{F}(\{z_i, \bar{z}_i\})_{\mathbb{H}_t} = \\ = \prod \left(\frac{\partial w(z_i)}{\partial z_i}\right)^{h_i} \prod \left(\frac{\partial \bar{w}(\bar{z}_i)}{\partial \bar{z}_i}\right)^{\bar{h}_i} \mathcal{F}(\{w_i, \bar{w}_i\})_{\mathbb{H}} \quad (4)$$

We can extend the definition of $\mathcal{F}$ to the whole plane by doubling the number of fields ϕ_i [16, 17] and consider $w_i, \bar{w}_i$ as independent variables. To simplify notation we will write $\mathcal{F}(\{w_i\}_{i=1}^{2N})$.

G/A-coset conformal field theory can be realized as WZNW-model with gauge group G interacting with pure gauge field of gauge group $A \subset G$ [18, 19]. So we have G/A gauge symmetry and we add random gauge transformations to Schramm-Loewner evolution [10] similar to the case of WZNW-models [20]. Denote by $\mathcal{K}$ Killing form of Lie algebra $\mathfrak{g}$ corresponding to Lie group G and by t_i^a ($\tilde{t}_i^b$) the generators of $\mathfrak{g}$-representation ($\mathfrak{a}$-representation) corresponding to primary field ϕ_i.

Now we need to consider the evolution of SLE trace γ_t from t to $t + dt$. First factor in right-hand-side of equation (3) gives us

$$-\frac{2h_i}{w_i^2}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}.$$

We denote by $\mathcal{G}_i$ generator of infinitesimal transformation of primary field ϕ_i:$d\phi_i(w_i) = \mathcal{G}_i\phi_i(w_i)$. We normalize additional $\dim \mathfrak{g}$-dimensional Brownian motion as $\mathbb{E}\left[d\theta^a \, d\theta^b\right] = \mathcal{K}(t^a, t^b)dt$, and the generator of field transformation is equal to

$$\mathcal{G}_i = \left(\frac{2dt}{w_i} - \sqrt{\kappa}d\xi_t\right)\partial_{w_i} + \frac{\sqrt{\tau}}{w_i}\left(\sum_{a:\mathcal{K}(t^a,\tilde{t}^b)=0} (d\theta^a t_i^a)\right). \quad (5)$$

Ito formula gives us the expression for the differential of $\mathcal{F}$ which should be zero due to martingale condition:

$$d\mathcal{F}_{\mathbb{H}_t} = \\ \left(\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}\right)\left(-\sum_{i=1}^{2N}\frac{2h_i dt}{w_i^2} + \left[\sum_{i=1}^{2N}\mathcal{G}_i + \frac{1}{2}\sum_{i,j}\mathcal{G}_i\mathcal{G}_j\right]\right)\mathcal{F}_{\mathbb{H}} \\ = 0 \quad (6)$$

Substitute definition (5) and get

$$\left(-2\mathcal{L}_{-2} + \frac{1}{2}\kappa\mathcal{L}_{-1}^2 + \frac{\tau}{2}\left(\sum_a \mathcal{J}_{-1}^a\mathcal{J}_{-1}^a - \sum_b \tilde{\mathcal{J}}_{-1}^b\tilde{\mathcal{J}}_{-1}^b\right)\right)\mathcal{F}_{\mathbb{H}} = 0, \quad (7)$$

where

$$\mathcal{L}_{-n} = \sum_i \left(\frac{(n-1)h_i}{(w_i - z)^n} - \frac{\partial_{w_i}}{(w_i - z)^{n-1}}\right)$$
$$\mathcal{J}_{-n}^a = -\sum_i \frac{t_i^a}{(w_i - z)^n}; \; \tilde{\mathcal{J}}_{-n}^b = -\sum_i \frac{\tilde{t}_i^b}{(w_i - z)^n}$$

This equation is equivalent to following algebraic condition on boundary state $\varphi(0)\left|0\right\rangle$:

$$\langle 0\left|\varphi(\infty)\phi_1(w_1)\ldots\phi_{2N}(w_{2N})\right. \\ \left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a - \sum_{b=1}^{\dim\mathfrak{a}} \tilde{J}_{-1}^b\tilde{J}_{-1}^b\right)\right) \\ \varphi(0)|0\rangle = 0 \quad (8)$$

Since the set $\{\phi_i\}$ consists of arbitrary primary fields we conclude that

$$|\psi>= \left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J_{-1}^a J_{-1}^a - \sum_{b=1}^{\dim\mathfrak{a}} \tilde{J}_{-1}^b\tilde{J}_{-1}^b\right)\right) \\ \cdot\varphi(0)|0> \quad (9)$$

is null-state. Now we act on ψ by raising operators to get equations on κ, τ. Since in coset theory

$$\begin{aligned} &L_n = L_n^{\mathfrak{g}} - L_n^{\mathfrak{a}} \\ &[L_n^{\mathfrak{g}}, J_m^a] = -mJ_{n+m}^a \\ &\left[L_n^{\mathfrak{g}}, \tilde{J}_m^b\right] = -m\tilde{J}_{n+m}^b \\ &[L_n^{\mathfrak{g}}, L_m] = [L_n^{\mathfrak{g}}, L_m^{\mathfrak{g}}] - [L_n^{\mathfrak{g}}, L_m^{\mathfrak{a}}] = \\ &\quad = (n-m)L_{m+n} + \tfrac{c}{12}(n^3 - n)\delta_{m+n,0} \end{aligned} \quad (10)$$

it is more convenient to act by $L_2^{\mathfrak{g}}$ and $\left(L_1^{\mathfrak{g}}\right)^2$. Acting with $L_2^{\mathfrak{g}}$ we get

$$L_2^{\mathfrak{g}}\psi = \left(-8L_0 - c + 3\kappa L_0 + \frac{1}{2}\tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a})\right)\varphi_{(\mu,\nu)} = 0$$

We use $L_0\varphi_{(\mu,\nu)} = h_{(\mu,\nu)}\varphi_{(\mu,\nu)}$, where conformal weight is $h_{(\mu,\nu)} = \left(\frac{(\mu,\mu+2\rho)}{2(k+h^\vee_{\mathfrak{g}})} - \frac{(\nu,\nu+2\rho_{\mathfrak{a}})}{2(kx_e+h^\vee_{\mathfrak{a}})}\right)$ and central charge is $c = \frac{k\dim\mathfrak{g}}{k+h^\vee_{\mathfrak{g}}} - \frac{x_e k\dim\mathfrak{a}}{x_e k+h^\vee_{\mathfrak{a}}}$ and get relation on κ,τ:

$$(3\kappa-8)h_{(\mu,\nu)} - c + \tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a}) = 0 \quad (11)$$

Second relation appears as the result of $L_1^{\mathfrak{g}}$-action:

$$-12h_{(\mu,\nu)} + 2\kappa h_{(\mu,\nu)}(2h_{(\mu,\nu)}+1) + \tau(C_\mu - \tilde{C}_\nu) = 0, \quad (12)$$

where $C_\mu = (\mu,\mu+2\rho), \tilde{C}_\nu = (\nu,\nu+2\rho_{\mathfrak{a}})$ is the eigenvalue of quadratic Casimir operators $\sum_a t^a t^a, \sum_b \tilde{t}^b\tilde{t}^b$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$. Relations (11),(12) are necessary conditions for CFT correlation functions to be SLE martingales.

From equations (11),(12) we immediately get κ,τ for each pair (μ,ν) of $\mathfrak{g}$ and $\mathfrak{a}$-weights.

21 Examples

As an example consider $\frac{\hat{su}(2)_N}{\hat{u}(1)_N}$-coset models which are equivalent to Z_N-parafermions. Central charge is $c = \frac{3N}{N+2} - 1 = \frac{2N-2}{N+2}$. Conformal weights of primary fields with Dynkin indeces (k,l) are $h_{(k,l)} = \frac{k(k+2)}{4(N+2)} - \frac{l^2}{4N}$.

Case $N=2$, $c=\frac{1}{2}$ corresponds to Ising model, we have two non-trivial primary fields with conformal weights $h_{(2,0)} = 1/2, h_{(1,1)} = 1/16$. Substituting field $\varphi_{(2,0)}$ into equations (11,12) we get $3\kappa - 9 + 4\tau = 0; \quad -3+\kappa+4\tau = 0$. Then $\kappa = 3, \tau = 0$. For field $\varphi_{(1,1)}$ we obtain $3\kappa - 16 + 64\tau = 0, \quad -64 + 9\kappa + 64\tau = 0$, $\kappa = 16/3, \tau = 0$. So we have no additional motion for Ising model and two possible values for SLE parameter κ, coinciding with well-known results.

For $N=3$ parafermionic model central charge is $c = \frac{4}{5}$. Conformal weights are $h_{(0,0)} = 0$, $h_{(0,2)} = h_{(0,-2)} = \frac{2}{3} \bmod 1$, $h_{(2,0)} = \frac{2}{5}$, $h_{(2,2)} = h_{(2,-2)} = \frac{1}{15}$. Corresponding values of κ,τ are: $(\frac{208}{25},\frac{242}{225}),(\frac{10}{3},0),(\frac{80}{19},\frac{14}{171})$. As was mentioned in [11] field $\varphi_{(2,0)}$ with conformal weight $h_{(2,0)} = \frac{2}{5}$ constitutes Z_3-singlet, so additional random walk does not appear and $\tau = 0$. The form of equations (8) is similar to that of [11], but normalization of τ differs.

It is easy to see that for $\frac{\hat{su}(2)_N \oplus \hat{su}(2)_1}{\hat{su}(2)_{N+1}}$-coset realization of minimal unitary models with $c = \frac{3N}{N+2} + 1 - \frac{3(N+1)}{N+3} = 1 - \frac{6}{(N+2)(N+3)}$ system of equations (11,12) is always consistent for $\tau = 0$ and we get standard connection between SLE-parameter κ and central charge $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$.

3. CONCLUSION AND OUTLOOK

In excellent paper [20] connection between WZNW-models and Schramm-Loewner evolution with additional Brownian motion on group manifold was established. This paper also stated a problem of the connection of SLE parameters for martingales in WZNW, coset and minimal models. In present letter we used the method to obtain necessary and sufficient conditions on SLE martingales proposed in paper [21]. This method allowed us to compare our results with parafermionic results of the papers [11, 22].

Coset structure of minimal models is manifest in field theory perturbed by external magnetic field [12, 13, 14]. This theory is supported by experimental data [23]. Relation between correlation functions in coset theory and SLE observables can be a starting point for the study of domain walls in lattice models away of critical point.

Massive perturbations of G/A-coset theory are realized as affine Toda field theory and classified by simple roots of Lie algebra $\mathfrak{g}$.

Massive off-critical SLEs have additional drift term in driving Brownian motion [15]. Addition of drift term to (2) leads to appearance of new terms in martingale condition (6).

On the other hand perturbation can be seen as the insertion of certain primary field in all correlation functions [14]. The question is whether the interaction of this primary field with the τ-term of the equation (7) lead to the same contribution as the addition of massive drift to SLE.

Domain walls in Ising model perturbed by random Gaussian external were studied numerically using Schramm-Loewner evolution in recent paper [24]. Further task is to compare this results with massive perturbations.

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