Fan, splint and branching rules.

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Abstract

Splint of root system of simple Lie algebra appears naturally in the study of (regular) embedding of reductive subalgebra. It can be used to derive branching rules. We show that the use of splint drastically simplifies the derivation of branching functions.

1 Introduction

Splint ϕ of root system Δ_1 to root system Δ is a bijective map of roots of Δ_1 to (proper) subset of Δ which commutes with the addition in Δ_1 and Δ.

$$\phi : \Delta_1 \longrightarrow \Delta$$

$$\phi \circ \alpha + \beta = \phi \circ \alpha + \phi \circ \beta, \ \ \alpha, \beta \in \Delta_1$$

Note that image $Im(\phi)$ is not required to have the properties of root system except the addition rules equivalent to the addition rules in Δ_1 (for pre-images). The term *splint* was introduce in paper [1] where the classification of splints for simple Lie algebras was obtained. The conjecture of the connection splint with injection fan was stated in the same paper. The fan $\Gamma \subset \Delta$ as the subset of root system describing recurrent properties of branching coefficients for maximal embeddings was introduced in [2]. Injection fan is an efficient tool for the study of branching rules. This construction was generalized to non-maximal embeddings and infinite-dimensional Lie algebras in [3, 4].

In present paper we study the connection of splint with the injection fan of regular embedding of reductive subalgebra $\mathfrak{a}$. We show that (under certain conditions described in section 3) splint is a natural tool for the reduction of simple Lie algebra $\mathfrak{g}$-modules to the modules of subalgebra $\mathfrak{a} \to \mathfrak{g}$. The use of this tool allows us to state the main result of the present paper – the one-to-one relationship between weight multiplicities of irreducible modules of splint and branching coefficients for the reduction $L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}}$.

2 Injections and splints

Consider a simple Lie algebra $\mathfrak{g}$ and its regular subalgebra $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$. Let $\mathfrak{a}^S$ be a semisimple summand of $\mathfrak{a}$, this means that $\mathfrak{a} = \mathfrak{a}^S \oplus \mathfrak{a}(1) \oplus \mathfrak{a}(1) \oplus \ldots$. We shall consider $\mathfrak{a}^S$ to be a proper regular subalgebra and $\mathfrak{a}$ to be the maximal subalgebra with $\mathfrak{a}^S$ fixed that is the rank r of $\mathfrak{a}$ is equal to that of $\mathfrak{g}$.

r , $(r_{\mathfrak{a}^S})$ — the rank of $\mathfrak{g}$ $\left(\text{resp.}\mathfrak{a}^{\mathrm{S}}\right)$;

Δ $(\Delta_{\mathfrak{a}})$— the root system; Δ^+ (resp.$\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

S $(S_{\mathfrak{a}})$ — the system of simple roots (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) simple root for $\mathfrak{g}$ (resp.$\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}^S})$;

W , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

C , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

$\epsilon(w) := (-1)^{\mathrm{length}(w)}$;

ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

L^{μ} $(L^{\nu}_{\mathfrak{a}})$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

$\mathcal{N}^{\mu}$, $(\mathcal{N}^{\nu}_{\mathfrak{a}})$ — the weight diagram of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$);

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

P^+ (resp. $P^+_{\mathfrak{a}}$) — the dominant weight lattice;

$\mathcal{E}$ (resp. $\mathcal{E}_{\mathfrak{a}}$) — the formal algebra;

$m_{\xi}^{(\mu)}$, $\left(m_{\xi}^{(\nu)}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in \mathrm{P}_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right)$ (resp. $\mathrm{ch}\left(L_{\mathfrak{a}}^{\nu}\right)$)— the formal character of L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$);

$ch\left(L^{\mu}\right)=\frac{\sum_{w \in W} \epsilon(w) e^{w \circ(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^{+}}\left(1-e^{-\alpha}\right)}$ — the Weyl formula;

$R:=\prod_{\alpha \in \Delta^{+}}\left(1-e^{-\alpha}\right)$ (resp. $R_{\mathfrak{a}}:=\prod_{\alpha \in \Delta_{\mathfrak{a}}^{+}}\left(1-e^{-\alpha}\right)$) — the Weyl denominator.

Let L^{μ} be completely reducible with respect to $\mathfrak{a}$,

$$L_{\mathfrak{g} \downarrow \mathfrak{a}}^{\mu}=\bigoplus_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} L_{\mathfrak{a}}^{\nu}.$$

$$\pi_{\mathfrak{a}} ch\left(L^{\mu}\right)=\sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} ch\left(L_{\mathfrak{a}}^{\nu}\right). \tag{1}$$

For the modules we are interested in the Weyl formula for $\mathrm{ch}\left(L^{\mu}\right)$ can be written in terms of singular elements [5]

$$\Psi^{(\mu)}:=\sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho},$$

namely,

$$\mathrm{ch}\left(L^{\mu}\right)=\frac{\Psi^{(\mu)}}{\Psi^{(0)}}=\frac{\Psi^{(\mu)}}{R}. \tag{2}$$

The same is true for submodules $\mathrm{ch}\left(L_{\mathfrak{a}}^{\nu}\right)$ in (1)

$$\mathrm{ch}\left(L_{\mathfrak{a}}^{\nu}\right)=\frac{\Psi_{\mathfrak{a}}^{(\nu)}}{\Psi_{\mathfrak{a}}^{(0)}}=\frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}},$$

with

$$\Psi_{\mathfrak{a}}^{(\nu)}:=\sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w\left(\nu+\rho_{\mathfrak{a}}\right)-\rho_{\mathfrak{a}}}.$$

Applying formula (2) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{\mathfrak{a}}^{(\nu)}$:

$$\begin{aligned}
\frac{\sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^{+}}\left(1-e^{-\alpha}\right)} &= \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \frac{\sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w\left(\nu+\rho_{\mathfrak{a}}\right)-\rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{a}}^{+}}\left(1-e^{-\beta}\right)}, \\
\frac{\Psi^{(\mu)}}{R} &= \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}}.
\end{aligned} \tag{3}$$

In [3] it was proven that branching coefficients $b_{\xi}^{(\mu)}$ corresponding to the injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ are subject to the set of recurrent relations:

$$\begin{array}{c} b_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)} \left(\sum_{u \in U} \epsilon(u) \ \dim \left(L_{\mathfrak{a}_{\perp}}^{\mu_{\mathfrak{a}_{\perp}}(u)} \right) \delta_{\xi - \gamma_0, \pi_{\widetilde{\mathfrak{a}}}(u(\mu+\rho)-\rho)} + \right. \\ \left. + \sum_{\gamma \in \Gamma_{\widetilde{\mathfrak{a}} \to \mathfrak{g}}} s\left(\gamma + \gamma_0\right) b_{\xi+\gamma}^{(\mu)} \right). \end{array} \tag{4}$$

where $\mathfrak{a}_{\perp}$ is the subalgebra determined by the roots of $\mathfrak{g}$ orthogonal to roots of $\mathfrak{a}$

$$\Delta_{\mathfrak{a}_{\perp}} \ \ : \ = \left\{ \beta \in \Delta_{\mathfrak{g}} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \beta\left(h\right) = 0 \right\}, \tag{5}$$

$$\widetilde{\mathfrak{a}_{\perp}} := \mathfrak{a}_{\perp} \oplus \mathfrak{h}_{\perp} \qquad \widetilde{\mathfrak{a}} := \mathfrak{a} \oplus \mathfrak{h}_{\perp} \tag{6}$$

and π is the projection operator. When the injection is maximal the projection becomes trivial and the relation (4) is simplified:

$$\begin{array}{c} b_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)} \left(\sum_{u \in W} \epsilon(u) \delta_{\xi - \gamma_0, u(\mu+\rho)-\rho} + \right. \\ \left. + \sum_{\gamma \in \Gamma_{\mathfrak{a} \to \mathfrak{g}}} s\left(\gamma + \gamma_0\right) b_{\xi+\gamma}^{(\mu)} \right). \end{array} \tag{7}$$

The recursion is goverened by the set $\Gamma_{\mathfrak{a} \to \mathfrak{g}}$ called the injection fan. The latter is defined by the carrier set $\{\xi\}_{\mathfrak{a} \to \mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\mathfrak{a} \to \mathfrak{g}} := \{\xi \in P_{\mathfrak{a}} | s(\xi) \neq 0\}$$

appearing in the expansion

$$\prod_{\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{a}}^+} \left(1 - e^{-\alpha}\right) = -\sum_{\gamma \in P_{\mathfrak{a}}} s(\gamma) e^{-\gamma}; \tag{8}$$

Now we remind two definitions introduced in [1]

Definition 2.1. *Suppose Δ_0 and Δ are root systems with the corresponding weight lattices P_0 and P. Then ϕ is an "embedding",*

$$\phi : \left\{ \begin{array}{l} \Delta_0 \hookrightarrow \Delta, \\ P_0 \hookrightarrow P, \end{array} \right. \tag{9}$$

if
(a) it injects Δ_0 in Δ, and
(b) acts homomorphically with respect to the vector groups in P_0 and P:

$$\phi(\gamma) = \phi(\alpha) + \phi(\beta)$$

for any triple $\alpha, \beta, \gamma \in P_0$ such that $\gamma = \alpha + \beta$.

ϕ induces an injection of formal algebras : $\mathcal{E}_0 \longrightarrow \mathcal{E}$ and for the image $\mathcal{E}_i = Im_\phi(\mathcal{E}_0)$ one can consider its inverse $\phi^{-1} : \mathcal{E}_i \longrightarrow \mathcal{E}_0$.

Notice that one must distinguish two classes of embeddings: when the scalar product (defined by the Killing form) in the root space P_0 is invariant with respect to ϕ and when it is not ϕ-invariant. The first embedding is called "metric" , the second – "nonmetric".

Definition 2.2. *A root system Δ "splinters" as (Δ_1, Δ_2) if there are two embeddings $\phi_1 : \Delta_1 \hookrightarrow \Delta$ and $\phi_2 : \Delta_2 \hookrightarrow \Delta$ where (a) Δ is the disjoint union of the images of ϕ_1 and ϕ_2 and (b) neither the rank of Δ_1 nor the rank of Δ_2 exceeds the rank of Δ.*

It is equivalent to say that (Δ_1, Δ_2) is a "splint" of Δ and we shall denote this by $\Delta \approx (\Delta_1, \Delta_2)$. Each component Δ_1 and Δ_2 is a "stem" of the splint (Δ_1, Δ_2).

In [1] it was indicated that the injection fan technique used in [2] is close to that of splints.

To demonstrate the relations between these instruments consider the case where one of the stems $\Delta_1 = \Delta_{\mathfrak{a}}$ is a root system of a regular reductive subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$. Then the second stem is $\Delta_{\mathfrak{s}} := \Delta_2 = \Delta \setminus \Delta_{\mathfrak{a}}$ can be translated into the product (8) and defines the injection fan $\Gamma_{\mathfrak{a} \hookrightarrow \mathfrak{g}}$.

Thus we see that

Conjecture 2.3. *Each splint of the type $\Delta \approx (\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$ defines an injection fan with the carrier $\{\xi\}_{\mathfrak{a} \to \mathfrak{g}}$ fixed by the product*

$$\prod_{\beta \in \Delta_{\mathfrak{s}}^+} \left(1 - e^{-\beta}\right) = -\sum_{\gamma \in P} s(\gamma) e^{-\gamma} \tag{10}$$

In this case we say that the subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ splinters Δ (and call $\mathfrak{a}$ the "splinting subalgebra" of $\mathfrak{g}$). In [1] the splints are classified (see Appendix) and in the first three types one can easily see that one of the stems corresponds to a splinting subalgebra.

3 How stems define multiplicity functions

In this Section we consider the situation where the splinting subalgebra is "larger" than the Lie algebra $\mathfrak{s}$ defined by the root system $\Delta_{\mathfrak{s}\,0} = CoIm_\phi(\Delta_{\mathfrak{s}})$

(the details will be given below). In such a case to find branching coefficients for a splinting injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ means to find weight multiplicities for an irreducible $\mathfrak{s}$-module $L_{\mathfrak{s}}^{\nu}$ with the fixed highest weight ν. Notice that $\mathfrak{s}$ must not be a subalgebra of $\mathfrak{g}$.

Let us return to relation (3) and multiply both sides by $R_{\mathfrak{a}}$:

$$\frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}}^{+}} (1 - e^{-\beta})} \Psi_{\mathfrak{g}}^{(\mu)} = \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \Psi_{\mathfrak{a}}^{(\nu)}. \tag{11}$$

Here the first factor in the l.h.s. is the inverse of the fan $\Gamma_{\mathfrak{a} \to \mathfrak{g}}$. Consider the highest weight module $L_{\mathfrak{s}}^{\nu}$. An embedding $\phi : \Delta_{\mathfrak{s}\,0} \longrightarrow \Delta_{\mathfrak{g}}$ sends the singular element $\Psi_{\mathfrak{s}}^{(\nu)}$ into $\Psi_{\mathfrak{g}}^{(\mu)}$. Applying the inverse morphism ϕ^{-1} to the product $\left(\prod_{\beta \in \Delta_{\mathfrak{s}}^{+}} (1 - e^{-\beta})\right)^{-1} \phi\left(\Psi_{\mathfrak{s}}^{(\nu)}\right)$ one obtains the character of the module $L_{\mathfrak{s}}^{\nu}$,

$$\phi^{-1}\left(\frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}}^{+}} (1 - e^{-\beta})} \phi\left(\Psi_{\mathfrak{s}}^{(\nu)}\right)\right) = \frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}(0)}^{+}} (1 - e^{-\beta})} \Psi_{\mathfrak{s}}^{(\nu)} = \mathrm{ch}\left(L_{\mathfrak{s}}^{\nu}\right). \tag{12}$$

Our task is to prove that the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ contains the element $\Psi_{\mathfrak{g}}^{(\xi)}$ for a module $L_{\mathfrak{s}}^{\xi}$ uniquely defined by $L_{\mathfrak{g}}^{\mu}$ and that the branching coefficients $b_{\nu}^{(\mu)}$ in the r.h.s. of (11) coincide with multiplicities $m_{\zeta}^{(\xi)}$ of the weights of $L_{\mathfrak{s}}^{\xi}$.

For a highest weight irreducible module $L_{\mathfrak{g}}^{\mu}$ the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ is an element of $\mathcal{E}$ corresponding to the shifted Weyl-orbit of the weight $(\mu + \rho) \in P^{+}$ and the sign function $\epsilon(w)$. It is convenient to use also unshifted singular elements

$$\Phi^{(\mu)} := \Psi^{(\mu)} e^{\rho}. \tag{13}$$

In these terms the relation (11) looks like

$$\frac{e^{\rho_{\mathfrak{g}} - \rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{s}}^{+}} (1 - e^{-\beta})} \Phi_{\mathfrak{g}}^{(\mu)} = \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \Phi_{\mathfrak{a}}^{(\nu)}. \tag{14}$$

The orbit that corresponds to $\Phi_{\mathfrak{g}}^{(\mu)}$ is completely defined by the set of edges $\{\lambda_i\}_{i=1,\ldots,r}$ adjusted to the end of the highest weight vector $\mu + \rho$. Let $\mu = \sum m_i \omega_i$ then

$$\lambda_i = -(m_i + 1)\, \alpha_i. \quad i = 1, \ldots, r \tag{15}$$

Each weight $\mu + \rho + \lambda_i$ bears the sign coefficient $(-)$ in the singular element. The defining property of $\Phi_{\mathfrak{g}}^{(\mu)}$ is as follows. Consider any pair of edges λ_i, λ_j

and the corresponding weights $\mu+\rho$, $\mu+\rho+\lambda_i$ and $\mu+\rho+\lambda_j$. Apply the reflection s_{α_i} (or s_{α_j}),

$$s_{\alpha_i}\circ\left\{\begin{array}{l}(\mu+\rho)\\(\mu+\rho+\lambda_i)\\(\mu+\rho+\lambda_j)\end{array}\right. = \left\{\begin{array}{l}(\mu+\rho+\lambda_i)\\(\mu+\rho)\\(\mu+\rho+\lambda_i-(m_j+1)s_{\alpha_i}\circ\alpha_j)\end{array}\right. \tag{16}$$

Property 3.1. *The edge $\lambda_{i,j}$ of $\Phi_{\mathfrak{g}}^{(\mu)}$ starting at the weight $(\mu+\rho+\lambda_i)$ along the root $-s_{\alpha_i}\circ\alpha_j$ has the same length in $(s_{\alpha_i}\circ\alpha_j)$ as λ_j has in α_j. (The same is true for the edge $\lambda_{i,j}$, its length in $(s_{\alpha_j}\circ\alpha_i)$ is equal to the length of λ_i in α_i.)*

In $\Phi_{\mathfrak{g}}^{(\mu)}$ the elements $e^{\left(\mu+\rho+\lambda_i-(m_j+1)s_{\alpha_i}\circ\alpha_j\right)}$ and $e^{\left(\mu+\rho+\lambda_j-(m_i+1)s_{\alpha_j}\circ\alpha_i\right)}$ have the sign $(+)$.

From now on we consider splints for simple Lie algebras of the first three types according to the classification in [1]:

type	Δ	$\Delta_{\mathfrak{a}}$	$\Delta_{\mathfrak{s}}$
(i)	G_2	A_2	A_2
	F_4	D_4	D_4
(ii)	$B_r(r\geq 2)$	D_r	rA_1
	$C_r(r\geq 3)$	D_r	rA_1
(iii)	$A_r(r\geq 2)$	A_{r-1}	rA_1
	B_2	A_1	A_2

Each row in the table gives a splint $(\Delta_{\mathfrak{a}},\Delta_{\mathfrak{s}})$ of the simple root system Δ. For these types: (i) Both $\Delta_{\mathfrak{a}}$ and $\Delta_{\mathfrak{s}}$ are embedded metrically, and $\Delta_{\mathfrak{a}}\cong\Delta_{\mathfrak{s}}$. (ii) Both $\Delta_{\mathfrak{a}}$ and $\Delta_{\mathfrak{s}}$ are embedded metrically, but $\Delta_{\mathfrak{a}}$ and $\Delta_{\mathfrak{s}}$ are not isomorphic. (iii) Only $\Delta_{\mathfrak{a}}$ is embedded metrically.

Splints induce a decomposition of the set $S=S_{\mathfrak{c}}\cup S_{\mathfrak{d}}$ where $S_{\mathfrak{c}}=S\cap S_{\mathfrak{a}}$ and $S_{\mathfrak{d}}=S\cap S_{\mathfrak{s}}$. It is easy to check that for all the splints we are interested in the subset $S_{\mathfrak{d}}$ is nonempty. It follows that in the set $\{\lambda_i\}_{i=1,\ldots,r}$ one can always find simple roots $\beta_k\in\Delta_{\mathfrak{s}}$ and that the orbit corresponding to $\Phi_{\mathfrak{g}}^{(\mu)}$ contains edges

$$\lambda_k=-\left(m_k+1\right)\beta_k \tag{17}$$

attached to the weight $\mu+\rho$. As far as $\Delta_{\mathfrak{a}}$ is a root system and for any pair of simple roots from $S_{\mathfrak{c}}$ the singular element $\Phi_{\mathfrak{g}}^{(\mu)}$ is a singular element for a set of $\mathfrak{a}$-modules the property 3.1 is fulfilled. Consider $\alpha_l\in S_{\mathfrak{c}}$ with the corresponding edge $\lambda_l = -(m_l+1)\,\alpha_l$ and a root $\beta_l \in \Delta_{\mathfrak{s}}$ whose coimage in $\Delta_{\mathfrak{s}0}$ is simple. The root β_l is not simple in Δ. Firstly suppose that $\beta_l = \alpha_l + \beta_k$. It is easily seen that the corresponding edge intersects the boundary plane of the fundamental chamber $\bar{C}_{\mathfrak{a}}$ orthogonal to the root α_l:

$$s_{\alpha_l}(\mu+\rho-p\beta_l) = s_{\alpha_l}(\mu+\rho) - ps_{\alpha_l}\beta_l = \mu+\rho-p\beta_l \tag{18}$$

$$\mu+\rho-s_{\alpha_l}(\mu+\rho) = (m_l+1)\,\alpha_l = (m_l+1)\,\beta_l - (m_l+1)\,\beta_k = p\beta_l - ps_{\alpha_l}\beta_l \tag{19}$$

It follows that $p = (m_l+1)$ and $s_{\alpha_l}\beta_l = \beta_k$. Now apply the operator s_{β_k} and find that the edge along the root $s_{\beta_k}\alpha_l$ attached at the weight $s_{\beta_k}(\mu+\rho)$ is also equal to $-ps_{\beta_k}\alpha_l$. This means that for the triple of roots β_k, β_l and $s_{\beta_k}\alpha_l$ in $\Delta_{\mathfrak{s}}$ the edges $\lambda_k = -(m_k+1)\,\beta_k$, $\lambda_l = -(m_l+1)\,\beta_l$ and $\lambda_{kl} = -(m_l+1)\,s_{\beta_k}\alpha_l$ have the property 3.1. One can continue this procedure further in the 2-dimensional subspace fixed by the roots β_k and $beta_l$ and find the set of formal exponents that being supplied with the corresponding sign factors compose the coimage of the singular element of a module for the subalgebra in $\mathfrak{s}$ (this subalgebra has rank $r=2$).

The same can be proven for any positive root $\beta_l \in \Delta$ that is simple in $\Delta_{\mathfrak{s}0}$ and correspondingly for any $r=2$ subalgebra in $\mathfrak{s}$. The latter means that to "find" a singular element of $\mathfrak{s}$-module in $\Phi_{\mathfrak{g}}^{(\mu)}$ it is necessary to incorporate in it additional formal elements corresponding to the weights belonging to the boundary of the chamber $\bar{C}_{\mathfrak{a}}$.

Now let us return to the relation (14). One can add to $\Phi_{\mathfrak{g}}^{(\mu)}$ pairs of necessary formal elements with the opposite signs. Notice that all the additional weights belong to the boundaries of $\bar{C}_{\mathfrak{a}}$. To form a singular element for an $\mathfrak{s}$-module one must attribute to these weights the fixed signs. The same elements with the opposite signs are to be referred to the neighbor Weyl chambers of $\bar{C}_{\mathfrak{a}}^{(l)}$ (the latter are connected with the main one by simple reflections s_{α_l}). In fact one can repeat the procedure of finding additional singular weights in any Weyl chamber $\bar{C}_{\mathfrak{a}}^{(m)}$ and in them additional singular weights always have the signs opposite to that in their neighbors. Thus without changing in fact the element $\Phi_{\mathfrak{g}}^{(\mu)}$ one can present it as a sum

$$\Phi_{\mathfrak{g}}^{(\mu)} = \sum_{w\in W_{\mathfrak{a}}} w \circ \left(e^{\rho_{\mathfrak{a}}}\Psi^{\widetilde{\mu}+\rho_{\mathfrak{s}}}\right) \tag{20}$$

where the weight $\widetilde{\mu} = \sum m_k\omega_{\mathfrak{s}}^k$ has the same Dynkin labels m_k as in the corresponding decomposition $\mu = \sum m_k\omega_{\mathfrak{g}}^k$. The decomposition (20) provides the possibility to apply the factor $\left(\prod_{\beta\in\Delta_{\mathfrak{s}}^+}(1-e^{-\beta})\right)^{-1}$ to each summand of the singular element $\Phi_{\mathfrak{g}}^{(\mu)}$ separately because the sets of weights from different Weyl summands do not intersect. Taking into account the isomorphism ϕ one can see that in the main Weyl chamber $\bar{C}_{\mathfrak{a}}$ the set of weights generated by the factor $\left(\prod_{\beta\in\Delta_{\mathfrak{s}}^+}(1-e^{-\beta})\right)^{-1}$ is isomorphic with the weight diagram of the $\mathfrak{s}$-module $L_{\mathfrak{s}}^{\widetilde{\mu}}$. Finally one must restrict relation (14) to $\bar{C}_{\mathfrak{a}}$ and obtain the main result:

Property 3.2.

$$\frac{e^{\rho_{\mathfrak{g}}}}{\prod_{\beta\in\Delta_{\mathfrak{s}}^+}(1-e^{-\beta})}\left(\Psi^{\widetilde{\mu}+\rho_{\mathfrak{s}}}\right)=e^{\rho_{\mathfrak{g}}}\left(\phi^{-1}\left(\mathcal{N}_{\mathfrak{a}}^{\widetilde{\mu}}\right)\right)=\sum b_{\nu}^{(\mu)}e^{\nu}. \qquad (21)$$

4 Examples

Example 1. *Consider Lie algebra $A_2(\mathbf{sl}(\mathbf{3}))$ and branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra $A_1\oplus u(1)$ with root system spanned by first simple root of A_2. Singular element of $L^{(3,2)}$ is decomposition into the sum of splint images of singular elements of $A_1\oplus A_1$-modules and branching coefficients coincide weight multiplicities of $A_1\oplus A_1$-module (see Fig. 1).*

Example 2. *Now consider Lie algebra $B_2(\mathbf{so}(\mathbf{5}))$ and branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra $A_1\oplus u(1)$ with root system spanned by first simple root of B_2. Singular element of $L^{(3,2)}$ is decomposition into the sum of splint images of singular elements of A_2-modules and branching coefficients coincide weight multiplicities of A_2-module (see Fig. 2).*

Example 3. *Lie algebra G_2 has regular subalgebra A_2 with root system built on long roots of G_2. Consider branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra A_2. Singular element of $L^{(3,2)}$ is decomposition into the sum of splint images of singular elements of A_2-modules and branching coefficients coincide weight multiplicities of A_2-module (see Fig. 3).*

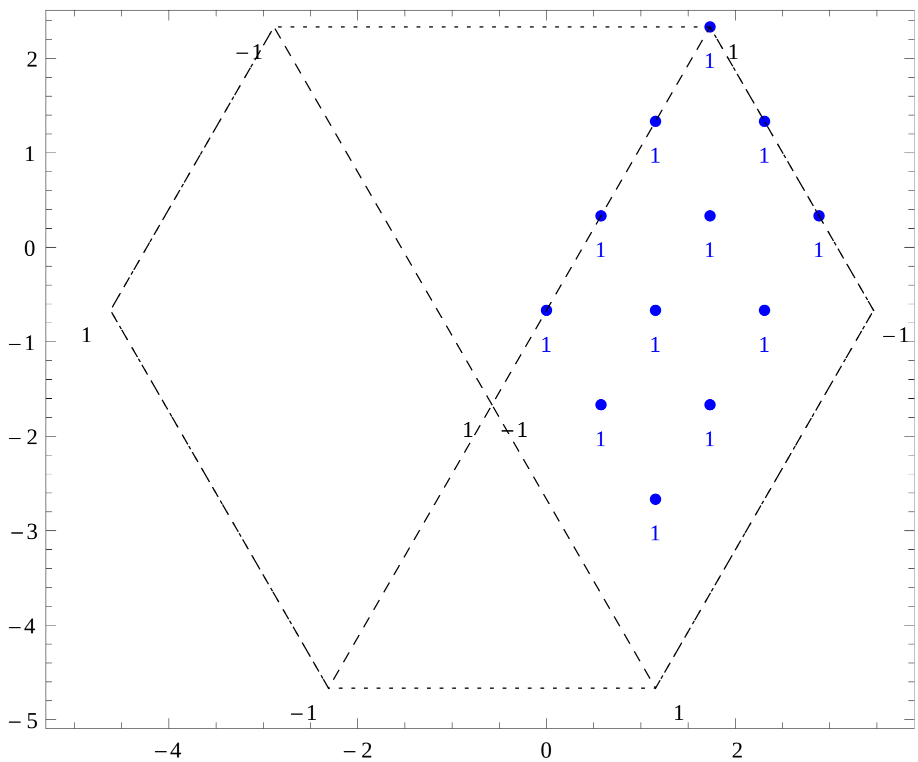

Figure 1: Weyl group orbit (dotted) producing singular element of A_2-module $L^{(3,2)}$ and its decomposition into the sum of splint images of singular elements of $A_1 \oplus A_1$-modules (dashed). Weight multiplicities of $A_1 \oplus A_1$-module coincide with branching coefficients for the reduction $L^{(3,2)}_{A_2 \downarrow A_1 \oplus u(1)}$.

5 Conclusions

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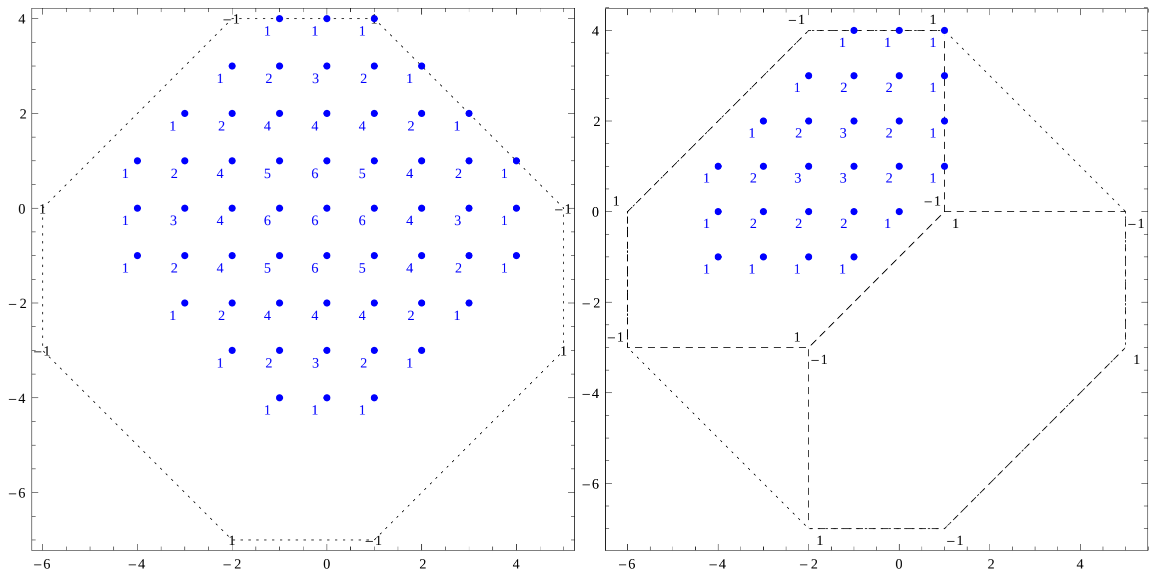

Figure 2: B_2-module $L^{(3,2)}$ is shown on the left. Weight multiplicities are indicated. Contour of singular element is shown by dotted line. Right figure represents the decomposition of $L^{(3,2)}$-singular element into the sum of splint images of singular elements of A_2-modules (dashed). Weight multiplicities of A_2-module coincide with branching coefficients for the reduction $L^{(3,2)}_{B_2\downarrow A_1\oplus u(1)}$.

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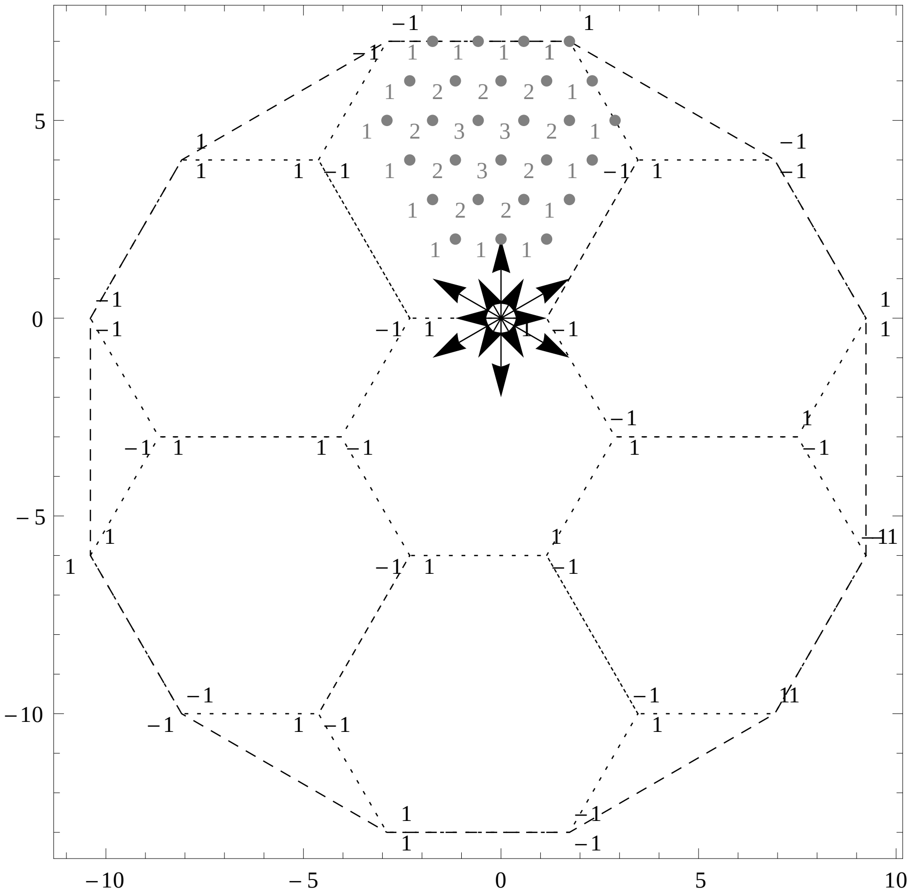

Figure 3: Weyl group orbit (dotted) producing singular element of G_2-module $L^{(3,2)}$ and its decomposition into the sum of splint images of singular elements of A_2-modules (dashed). Weight multiplicities of A_2-module coincide with branching coefficients for the reduction $L^{(3,2)}_{G_2\downarrow A_2}$.

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