# Fan, splint and branching rules.

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**Abstract**

Splint of root system of simple Lie algebra appears naturally in the study of (regular) embeddings of reductive subalgebras. It can be used to derive branching rules. We show that application of splint properties drastically simplifies calculations of branching coefficients.

## 1 Introduction

Splint $\phi$ of a root system $\Delta_1$ into a root system $\Delta$ is a bijective map of roots of $\Delta_1$ to (proper) subset of $\Delta$ which commutes with vector composition law in $\Delta_1$ and $\Delta$.

$$\phi : \Delta_1 \longrightarrow \Delta$$

$$\phi \circ (\alpha + \beta) = \phi \circ \alpha + \phi \circ \beta, \;\; \alpha, \beta \in \Delta_1$$

Note that the image $Im(\phi)$ must not inherit the root system properties except the addition rules equivalent to the addition rules in $\Delta_1$ (for pre-images). The term *splint* was introduced by D. Richter in [1] where the classification of splints for simple Lie algebras was obtained. There was also mentioned that the notion of splint must have tight connections with the injection fan approach. The fan $\Gamma \subset \Delta$ was introduced in [2] as the subset of root system describing recurrent properties of branching coefficients for maximal embeddings . Injection fan is an efficient tool to study branching rules. Later this construction was generalized to non-maximal embeddings and infinite-dimensional Lie algebras in [3, 4].

In the present paper we study the connection of splint with injection fan for regular embeddings of reductive subalgebras $\mathfrak{a} \longrightarrow \mathfrak{g}$ of simple Lie algebra $\mathfrak{g}$. We show that (under certain conditions described in section 3) splint is a natural tool to study reduction properties of $\mathfrak{g}$-modules with respect to a subalgebra $\mathfrak{a} \longrightarrow \mathfrak{g}$. Using this tool we obtain the main result of the present paper – the one-to-one correspondence between weight multiplicities in irreducible modules of splint and branching coefficients for a reduced module $L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}}$.

# 2 Injections and splints

Consider a simple Lie algebra $\mathfrak{g}$ and its regular subalgebra $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$. Let $\mathfrak{a}^{\mathfrak{s}}$ be a semisimple summand of $\mathfrak{a}$, this means that $\mathfrak{a} = \mathfrak{a}^{\mathfrak{s}} \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \ldots$. We shall consider $\mathfrak{a}^{\mathfrak{s}}$ to be a proper regular subalgebra and $\mathfrak{a}$ to be the maximal subalgebra with $\mathfrak{a}^S$ fixed that is the rank $r$ of $\mathfrak{a}$ is equal to that of $\mathfrak{g}$.

We use the following notations: $r$ , $(r_{\mathfrak{a}^{\mathfrak{s}}})$ — the rank of $\mathfrak{g}$ (resp.$\mathfrak{a}^{\mathfrak{s}}$) ;

$\Delta$ $(\Delta_{\mathfrak{a}})$— the root system; $\Delta^+$ (resp.$\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$S$ $(S_{\mathfrak{a}})$ — the system of simple roots (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

$\alpha_i$ , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the $i$-th (resp. $j$-th) simple root for $\mathfrak{g}$ (resp.$\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}^S})$; $\omega_i$ , $\left(omega_{(\mathfrak{a})j}\right)$ — the $i$-th (resp. $j$-th) fundamental weight for $\mathfrak{g}$ (resp.$\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}^S})$;

$W$ , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

$C$ , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}$, $\left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

$\epsilon(w) := (-1)^{\mathrm{length}(w)}$;

$\rho$ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

$L^{\mu}$ $(L^{\nu}_{\mathfrak{a}})$ — the integrable module of $\mathfrak{g}$ with the highest weight $\mu$ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight $\nu$ );

$\mathcal{N}^{\mu}$ , $(\mathcal{N}^{\nu}_{\mathfrak{a}})$ — the weight diagram of $L^{\mu}$ (resp. $L^{\nu}_{\mathfrak{a}}$ );

$P$ (resp. $P_{\mathfrak{a}}$) — the weight lattice;

$P^+$ (resp. $P^+_{\mathfrak{a}}$) — the dominant weight lattice;

$\mathcal{E}$ (resp. $\mathcal{E}_{\mathfrak{a}}$) — the formal algebra;

$m^{(\mu)}_{\xi}$ , $\left(m^{(\nu)}_{\xi}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in \mathrm{P}_{\mathfrak{a}}$) in the module $L^{\mu}$ , (resp. $\xi \in L^{\nu}_{\mathfrak{a}}$);

$ch\,(L^{\mu})$ (resp. $\mathrm{ch}\,(L^{\nu}_{\mathfrak{a}})$)— the formal character of $L^{\mu}$ (resp. $L^{\nu}_{\mathfrak{a}}$);

$ch\,(L^{\mu}) = \frac{\sum_{w\in W}\epsilon(w)e^{w\circ(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}$ — the Weyl formula;

$R := \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})$ (resp. $R_{\mathfrak{a}} := \prod_{\alpha\in\Delta^+_{\mathfrak{a}}}(1-e^{-\alpha})$ ) — the Weyl denominator.

Let $L^{\mu}$ be completely reducible with respect to $\mathfrak{a}$,

$$L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}} = \bigoplus_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}}.$$

$$\pi_{\mathfrak{a}} ch\,(L^{\mu}) = \sum_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu} ch\,(L^{\nu}_{\mathfrak{a}})\,. \tag{1}$$

For the modules we are interested in the Weyl formula for $\mathrm{ch}\,(L^{\mu})$ can be written in terms of singular elements [5]

$$\Psi^{(\mu)} := \sum_{w\in W}\epsilon(w)e^{w(\mu+\rho)-\rho},$$

namely,

$$\mathrm{ch}\,(L^{\mu}) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}. \tag{2}$$

The same is true for submodules $\mathrm{ch}\,(L^{\nu}_{\mathfrak{a}})$ in (1)

$$\mathrm{ch}\,(L^{\nu}_{\mathfrak{a}}) = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{\Psi^{(0)}_{\mathfrak{a}}} = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{R_{\mathfrak{a}}},$$

with

$$\Psi^{(\nu)}_{\mathfrak{a}} := \sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}.$$

Applying formula (2) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{\mathfrak{a}}^{(\nu)}$ :

$$\begin{aligned}
\frac{\sum_{w\in W}\epsilon(w)e^{w(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^{+}}(1-e^{-\alpha})} &= \sum_{\nu\in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)}\frac{\sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta_{\mathfrak{a}}^{+}}(1-e^{-\beta})},\\
\frac{\Psi^{(\mu)}}{R} &= \sum_{\nu\in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)}\frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}}.
\end{aligned} \tag{3}$$

In [3] it was proven that branching coefficients $b_{\xi}^{(\mu)}$ corresponding to the injection $\mathfrak{a}\hookrightarrow\mathfrak{g}$ are subject to the set of recurrent relations:

$$\begin{aligned}
b_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)}\Big(&\sum_{u\in W/W_{\perp}}\epsilon(u)\ \dim\left(L_{\mathfrak{a}_{\perp}}^{\mu_{\mathfrak{a}_{\perp}}(u)}\right)\delta_{\xi-\gamma_0,\pi_{\widetilde{\mathfrak{a}}}(u(\mu+\rho)-\rho)}+\\
&+\sum_{\gamma\in\Gamma_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}} s\left(\gamma+\gamma_0\right) b_{\xi+\gamma}^{(\mu)}\Big).
\end{aligned} \tag{4}$$

where $\mathfrak{a}_{\perp}$ is the subalgebra determined by the roots of $\mathfrak{g}$ orthogonal to roots of $\mathfrak{a}$, $W_{\perp}$ is a Weyl group of $\mathfrak{a}_{\perp}$

$$\Delta_{\mathfrak{a}_{\perp}} \ : \ = \left\{\beta\in\Delta_{\mathfrak{g}}|\forall h\in\mathfrak{h}_{\mathfrak{a}};\beta\left(h\right)=0\right\}, \tag{5}$$

$$\widetilde{\mathfrak{a}_{\perp}} := \mathfrak{a}_{\perp}\oplus\mathfrak{h}_{\perp} \qquad \widetilde{\mathfrak{a}} := \mathfrak{a}\oplus\mathfrak{h}_{\perp} \tag{6}$$

and $\pi$ is the projection operator. When the injection is maximal the projection becomes trivial and the relation (4) is simplified:

$$\begin{aligned}
b_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)}\Big(&\sum_{u\in W}\epsilon(u)\delta_{\xi-\gamma_0,u(\mu+\rho)-\rho}+\\
&+\sum_{\gamma\in\Gamma_{\mathfrak{a}\to\mathfrak{g}}} s\left(\gamma+\gamma_0\right) b_{\xi+\gamma}^{(\mu)}\Big).
\end{aligned} \tag{7}$$

The recursion is goverened by the set $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ called the injection fan. The latter is defined by the carrier set $\{\xi\}_{\mathfrak{a}\to\mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\mathfrak{a}\to\mathfrak{g}} := \{\xi\in P_{\mathfrak{a}}|s(\xi)\neq 0\}$$

appearing in the expansion

$$\prod_{\alpha\in\Delta^{+}\setminus\Delta_{\mathfrak{a}}^{+}}\left(1-e^{-\alpha}\right) = -\sum_{\gamma\in P_{\mathfrak{a}}} s(\gamma)e^{-\gamma}; \tag{8}$$

Now we remind two definitions introduced in [1]

**Definition 2.1.** *Suppose $\Delta_0$ and $\Delta$ are root systems with the corresponding weight lattices $P_0$ and $P$. Then $\phi$ is an "embedding",*

$$\phi : \begin{cases} \Delta_0 \hookrightarrow \Delta, \\ P_0 \hookrightarrow P, \end{cases} \tag{9}$$

*if*
*(a) it injects $\Delta_0$ in $\Delta$, and*
*(b) acts homomorphically with respect to the vector groups in $P_0$ and $P$:*

$$\phi(\gamma) = \phi(\alpha) + \phi(\beta)$$

*for any triple $\alpha, \beta, \gamma \in P_0$ such that $\gamma = \alpha + \beta$.*

$\phi$ induces an injection of formal algebras : $\mathcal{E}_0 \longrightarrow \mathcal{E}$ and for the image $\mathcal{E}_i = Im_\phi(\mathcal{E}_0)$ one can consider its inverse $\phi^{-1} : \mathcal{E}_i \longrightarrow \mathcal{E}_0$.

Notice that one must distinguish two classes of embeddings: when the scalar product (defined by the Killing form) in the root space $P_0$ is invariant with respect to $\phi$ and when it is not $\phi$-invariant. The first embedding is called "metric" , the second – "nonmetric".

**Definition 2.2.** *A root system $\Delta$ "splinters" as $(\Delta_1, \Delta_2)$ if there are two embeddings $\phi_1 : \Delta_1 \hookrightarrow \Delta$ and $\phi_2 : \Delta_2 \hookrightarrow \Delta$ where (a) $\Delta$ is the disjoint union of the images of $\phi_1$ and $\phi_2$ and (b) neither the rank of $\Delta_1$ nor the rank of $\Delta_2$ exceeds the rank of $\Delta$.*

It is equivalent to say that $(\Delta_1, \Delta_2)$ is a "splint" of $\Delta$ and we shall denote this by $\Delta \approx (\Delta_1, \Delta_2)$. Each component $\Delta_1$ and $\Delta_2$ is a "stem" of the splint $(\Delta_1, \Delta_2)$.

In [1] it was indicated that the injection fan technique used in [2] is close to that of splints.

To demonstrate the relations between these instruments consider the case where one of the stems $\Delta_1 = \Delta_{\mathfrak{a}}$ is a root subsystem in $\Delta$.

Splint $\Delta \approx (\Delta_1, \Delta_2)$ is called "injective" if one of its stems, say $\Delta_1 = \Delta_{\mathfrak{a}}$ , is a root subsystem in $\Delta$ corresponding to a regular reductive subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$.

In case of injective splint the second stem $\Delta_{\mathfrak{s}} := \Delta_2 = \Delta \setminus \Delta_{\mathfrak{a}}$ can be translated into a product (8) and it defines an injection fan $\Gamma_{\mathfrak{a} \hookrightarrow \mathfrak{g}}$. Denote by $\Delta_{\mathfrak{s}0}$ the coimage of the second embedding $\phi : \Delta_{\mathfrak{s}0} \to \Delta_{\mathfrak{g}}$.

Thus we see that

**Conjecture 2.3.** *Each injective splint $\Delta \approx (\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$ defines an injection fan with the carrier $\{\xi\}_{\mathfrak{a}\to\mathfrak{g}}$ fixed by the product*

$$\prod_{\beta\in\Delta_{\mathfrak{s}}^{+}} \left(1-e^{-\beta}\right) = -\sum_{\gamma\in P} s(\gamma)e^{-\gamma} \tag{10}$$

In this case we say that the subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ splinters $\Delta$ (and call $\mathfrak{a}$ the "splinting subalgebra" of $\mathfrak{g}$). In [1] splints are classified (see Appendix there) and the first three types of them are injective.

# 3 How stems define multiplicity functions

In this Section we study properties of injective splints $\Delta \approx (\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$. It will be demonstrated that in this case to find branching coefficients for a splinting injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ means to find weight multiplicities of an irreducible $\mathfrak{s}$-module $L_{\mathfrak{s}}^{\nu}$ with fixed highest weight $\nu$. Notice that $\mathfrak{s}$ must not be a subalgebra of $\mathfrak{g}$.

Let us return to relation (3) and multiply both sides by $R_{\mathfrak{a}}$:

$$\frac{1}{\prod_{\beta\in\Delta_{\mathfrak{s}}^{+}}(1-e^{-\beta})}\Psi_{\mathfrak{g}}^{(\mu)} = \sum_{\nu\in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)}\Psi_{\mathfrak{a}}^{(\nu)}. \tag{11}$$

Here the first factor in the l.h.s. is the inverse of the fan $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$. Consider the highest weight module $L_{\mathfrak{s}}^{\nu}$. The embedding $\phi : \Delta_{\mathfrak{s}0} \longrightarrow \Delta_{\mathfrak{g}}$ sends the singular element $\Psi_{\mathfrak{s}}^{(\nu)}$ into $\Psi_{\mathfrak{g}}^{(\mu)}$. Applying the inverse morphism $\phi^{-1}$ to the product $\left(\prod_{\beta\in\Delta_{\mathfrak{s}}^{+}}(1-e^{-\beta})\right)^{-1} \phi\left(\Psi_{\mathfrak{s}}^{(\nu)}\right)$ one gets the character of the module $L_{\mathfrak{s}}^{\nu}$,

$$\phi^{-1}\left(\frac{1}{\prod_{\beta\in\Delta_{\mathfrak{s}}^{+}}(1-e^{-\beta})}\phi\left(\Psi_{\mathfrak{s}}^{(\nu)}\right)\right) = \frac{1}{\prod_{\beta\in\Delta_{\mathfrak{s}0}^{+}}(1-e^{-\beta})}\Psi_{\mathfrak{s}}^{(\nu)} = \mathrm{ch}\left(L_{\mathfrak{s}}^{\nu}\right). \tag{12}$$

Our task is to prove that the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ contains the element $\Psi_{\mathfrak{s}}^{(\xi)}$ for a module $L_{\mathfrak{s}}^{\xi}$ uniquely defined by $L_{\mathfrak{g}}^{\mu}$ and that the branching coefficients $b_{\nu}^{(\mu)}$ in the r.h.s. of (11) coincide with multiplicities $m_{\zeta}^{(\xi)}$ of the corresponding weights in $\mathcal{N}_{\mathfrak{s}}^{\xi}$ .

For a highest weight irreducible module $L_{\mathfrak{g}}^{\mu}$ the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ is an element of $\mathcal{E}$ corresponding to the shifted Weyl-orbit of the weight

$(\mu+\rho) \in P^{+}$ with the sign function $\epsilon(w)$. It is convenient to use also unshifted singular elements

$$\Phi^{(\mu)} := \Psi^{(\mu)} e^{\rho}. \tag{13}$$

In these terms the relation (11) looks like

$$\frac{e^{\rho_{\mathfrak{g}}-\rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{s}}^{+}}\left(1-e^{-\beta}\right)} \Phi_{\mathfrak{g}}^{(\mu)}=\sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \Phi_{\mathfrak{a}}^{(\nu)}. \tag{14}$$

The orbit related to $\Phi_{\mathfrak{g}}^{(\mu)}$ is completely defined by the set of edges $\{\lambda_i\}_{i=1,\ldots,r}$ adjusted to the end of the highest weight vector $\mu+\rho$. Let $\mu=\sum m_i \omega_i$ then these edges are

$$\lambda_i=-\left(m_i+1\right) \alpha_i. \quad i=1, \ldots, r \tag{15}$$

Each formal exponent $e^{\mu+\rho+\lambda_i}$ in $\Phi_{\mathfrak{g}}^{(\mu)}$ bears the sign coefficient $\epsilon=(-)$. The defining property of $\Phi_{\mathfrak{g}}^{(\mu)}$ is as follows. Consider any pair of edges $\lambda_i, \lambda_j$ and the corresponding weights $\mu+\rho$, $\mu+\rho+\lambda_i$ and $\mu+\rho+\lambda_j$. Apply the reflection $s_{\alpha_i}$ (or $s_{\alpha_j}$),

$$s_{\alpha_i} \circ \left\{\begin{array}{l}(\mu+\rho) \\ (\mu+\rho+\lambda_i) \\ (\mu+\rho+\lambda_j)\end{array}\right.=\left\{\begin{array}{l}(\mu+\rho+\lambda_i) \\ (\mu+\rho) \\ (\mu+\rho+\lambda_i-(m_j+1) s_{\alpha_i} \circ \alpha_j)\end{array}\right. \tag{16}$$

**Property 3.1.** *The edge $\lambda_{i,j}$ of $\Phi_{\mathfrak{g}}^{(\mu)}$ starting at the weight $(\mu+\rho+\lambda_i)$ along the root $-s_{\alpha_i} \circ \alpha_j$ has the same length in $(s_{\alpha_i} \circ \alpha_j)$ as $\lambda_j$ has in $\alpha_j$. (The same is true for the edge $\lambda_{j,i}$, its length in $(s_{\alpha_j} \circ \alpha_i)$ is equal to the length of $\lambda_i$ in $\alpha_i$.)*

In $\Phi_{\mathfrak{g}}^{(\mu)}$ the elements $e^{\left(\mu+\rho+\lambda_i-(m_j+1) s_{\alpha_i} \circ \alpha_j\right)}$ and $e^{\left(\mu+\rho+\lambda_j-(m_i+1) s_{\alpha_j} \circ \alpha_i\right)}$ have the sign coefficient $\epsilon=(+)$.

Remember that only the first three types in Richter classification are injective splints and thus are naturally connected with branching. Below we reproduce the part of the splints table from [1] corresponding to injective splints:

| type | $\Delta$ | $\Delta_{\mathfrak{a}}$ | $\Delta_{\mathfrak{s}}$ |
|---|---|---|---|
| (i) | $G_2$ | $A_2$ | $A_2$ |
| | $F_4$ | $D_4$ | $D_4$ |
| (ii) | $B_r (r \geq 2)$ | $D_r$ | $\oplus^r A_1$ |
| | $C_r (r \geq 3)$ | $D_r$ | $\oplus^r A_1$ |
| (iii) | $A_r (r \geq 2)$ | $A_{r-1} \oplus u(1)$ | $\oplus^r A_1$ |
| | $B_2$ | $A_1 \oplus u(1)$ | $A_2$ |

Each row in the table gives a splint $(\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$ of the simple root system $\Delta$. In the first two types both $\Delta_{\mathfrak{a}}$ and $\Delta_{\mathfrak{s}}$ are embedded metrically. Stems in the first type splints are equivalent and in the second are not. In the third type splints only $\Delta_{\mathfrak{a}}$ is embedded metrically. The summands $u(1)$ are added to keep $r_{\mathfrak{a}} = r$ . This does not change the principle properties of branching but makes it possible to use the multiplicities of $\mathfrak{s}$ -modules without further projecting their weights.

Splints induce a decomposition of the set $S = S_{\mathfrak{c}} \cup S_{\mathfrak{d}}$ with $S_{\mathfrak{c}} = S \cap S_{\mathfrak{a}}$ and $S_{\mathfrak{d}} = S \cap S_{\mathfrak{s}}$. It is easy to check that for all the injective splints the subset $S_{\mathfrak{d}}$ is nonempty. It follows that in the set $\{\lambda_i\}_{i=1,\ldots,r}$ one can always find simple roots $\beta_k \in \Delta_{\mathfrak{s}}$ and that the orbit corresponding to $\Phi_{\mathfrak{g}}^{(\mu)}$ contains the edges

$$\lambda_k = -\left(m_k + 1\right)\beta_k \tag{17}$$

attached to the weight $\mu + \rho$. As far as $\Delta_{\mathfrak{a}}$ is a root system and for any pair of simple roots from $S_{\mathfrak{c}}$ the property 3.1 is fulfilled, the element $\Phi_{\mathfrak{g}}^{(\mu)}$ being a singular element for a set of $\mathfrak{a}$-modules. Consider $\beta_l \in \Delta_{\mathfrak{s}}$ whose coimage in $\Delta_{\mathfrak{s}0}$ is simple. In Appendix it is shown that for any such $\beta_l$ there exists a root $\alpha_l \in S_{\mathfrak{c}}$ such that $\beta_l = \alpha_l + \beta_k$. It is easily seen that the corresponding edge intersects the boundary plane of the fundamental chamber $\bar{C}_{\mathfrak{a}}$ orthogonal to the root $\alpha_l$,

$$s_{\alpha_l}\left(\mu + \rho - p\beta_l\right) = s_{\alpha_l}\left(\mu + \rho\right) - p s_{\alpha_l}\beta_l = \mu + \rho - p\beta_l \tag{18}$$

$$\mu + \rho - s_{\alpha_l}\left(\mu + \rho\right) = \left(m_l + 1\right)\alpha_l = \left(m_l + 1\right)\beta_l - \left(m_l + 1\right)\beta_k = p\beta_l - p s_{\alpha_l}\beta_l \tag{19}$$

It follows that $p = (m_l + 1)$ and $s_{\alpha_l}\beta_l = \beta_k$. Now apply the operator $s_{\beta_k}$ and find that the edge along the root $s_{\beta_k}\alpha_l$ attached at the weight $s_{\beta_k}(\mu+\rho)$ is also equal to $-p s_{\beta_k}\alpha_l$. This means that for the triple of roots $\beta_k, \beta_l$ and $s_{\beta_k}\alpha_l$ in $\Delta_{\mathfrak{s}}$ the edges $\lambda_k = -\left(m_k + 1\right)\beta_k$, $\lambda_l = -\left(m_l + 1\right)\beta_l$ and $\lambda_{kl} = -\left(m_l + 1\right) s_{\beta_k}\alpha_l$ demonstrate the property 3.1. One can continue this procedure further in the 2-dimensional subspace fixed by the roots $\beta_k$ and $\beta_l$ and find the set of formal exponents that being supplied with the corresponding sign factors compose the coimage of the singular element of a module for the subalgebra in $\mathfrak{s}$ (this subalgebra has rank $r = 2$).

The same can be proven for any positive root $\beta_l \in \Delta$ that is simple in $\Delta_{\mathfrak{s}0}$ and correspondingly for any $r = 2$ subalgebra in $\mathfrak{s}$. The latter means that to "find" a singular element of $\mathfrak{s}$-module in $\Phi_{\mathfrak{g}}^{(\mu)}$ it is necessary to incorporate in it additional formal elements $\left\{-e^{\mu+\rho-(m_l+1)\beta_l} | \beta_l \in S_{\mathfrak{c}}\right\}$ . This fixes the starting edges of the diagram $\phi^{-1}\left(\Phi_{\mathfrak{s}}^{\widetilde{\mu}}\right)$. As it follows from the reconstruction procedure the highest weight $\widetilde{\mu}$ is totally defined by the weight $\mu$, they have the same Dynkin numbers:

$$\mu = \sum m_k \omega_k \qquad \Longrightarrow \qquad \widetilde{\mu} = \sum m_k \widetilde{\omega}_k \tag{20}$$

The next step is to construct the full $W_{\mathfrak{s}}$-orbit $\Phi_{\mathfrak{s}}^{(\widetilde{\mu})}$ in $P_{\mathfrak{s}}$. It is easily seen that its coimage belongs to $\bar{C}_{\mathfrak{a}}$. and that the set $\phi^{-1}\left(\Phi_{\mathfrak{s}}^{\widetilde{\mu}}\right) \setminus \Phi_{\mathfrak{g}}^{(\mu)}|_{\bar{C}_{\mathfrak{a}}}$ corresponds to the weights belonging to the boundary $\bar{C}_{\mathfrak{a}}$ (including the subset $\left\{-e^{\mu+\rho-(m_l+1)\beta_l} | \beta_l \in S_{\mathfrak{c}}\right\}$). Thus we have constructed all the formal elements with the appropriate sign factors that after being added to $\Phi_{\mathfrak{g}}^{(\mu)}|_{\bar{C}_{\mathfrak{a}}}$ form the diagram $\phi^{-1}\left(\Phi_{\mathfrak{s}}^{\widetilde{\mu}}\right)$ in $\bar{C}_{\mathfrak{a}}$.

Now let us return to the relation (14). One can add to $\Phi_{\mathfrak{g}}^{(\mu)}$ pairs of additional formal elements constructed above with the opposite signs: $\epsilon(w)|_{w\in W_{\mathfrak{s}}}$ and $-\epsilon(w)|_{w\in W_{\mathfrak{s}}}$. Attribute the signs $\epsilon(w)|_{w\in W_{\mathfrak{s}}}$ to the elements whose weights we shall consider beloning to $\bar{C}_{\mathfrak{a}}$. The same elements with the opposite signs are to be referred to the neighbor Weyl chambers of $\bar{C}_{\mathfrak{a}}^{(l)}$ (the latter are connected with the main one via simple reflections $s_{\alpha_l}$ so the signes $-\epsilon(w)|_{w\in W_{\mathfrak{s}}}$ are natural for them). In fact one can repeat the procedure of finding additional singular weights in any Weyl chamber $\bar{C}_{\mathfrak{a}}^{(m)}$ and in them additional singular weights always have the signs opposite to that in their nearest neighbors. Thus without changing in fact the element $\Phi_{\mathfrak{g}}^{(\mu)}$ one can present it as a sum

$$\Phi_{\mathfrak{g}}^{(\mu)} = \sum_{w\in W_{\mathfrak{a}}} w \circ \left(e^{\rho_{\mathfrak{a}}} \Psi^{\widetilde{\mu}+\rho_{\mathfrak{s}}}\right) \tag{21}$$

where the weight $\widetilde{\mu} = \sum m_k \omega_{\mathfrak{s}}^k$ was defined above. The decomposition (21) provides the possibility to apply the factor $\left(\prod_{\beta\in\Delta_{\mathfrak{s}}^+}(1-e^{-\beta})\right)^{-1}$ to each summand of the singular element $\Phi_{\mathfrak{g}}^{(\mu)}$ separately because the sets of weights from different Weyl summands do not intersect. Taking into account the isomorphism $\phi$ one can see that in the main Weyl chamber $\bar{C}_{\mathfrak{a}}$ the set of weights generated by the factor $\left(\prod_{\beta\in\Delta_{\mathfrak{s}}^+}(1-e^{-\beta})\right)^{-1}$ is isomorphic to the weight diagram $\mathcal{N}_{\mathfrak{s}}^{\widetilde{\mu}}$ of the $\mathfrak{s}$-module $L_{\mathfrak{s}}^{\widetilde{\mu}}$. Finally one must restrict relation (14) to $\bar{C}_{\mathfrak{a}}$ and obtain the main result:

**Property 3.2.**

$$\frac{e^{\rho_{\mathfrak{g}}}}{\prod_{\beta\in\Delta^+_{\mathfrak{s}}}(1-e^{-\beta})}\left(\Psi^{\widetilde{\mu}+\rho_{\mathfrak{s}}}\right) = \sum_{\widetilde{\nu}\in\mathcal{N}^{\widetilde{\mu}}_{\mathfrak{s}}} M^{\widetilde{\mu}}_{(\mathfrak{s})\widetilde{\nu}} e^{(\mu-\phi(\widetilde{\mu}-\widetilde{\nu}))} = \sum_{\nu\in P^{++}_{\mathfrak{a}}} b^{(\mu)}_{\nu} e^{\nu}. \tag{22}$$

*Any weight with nonzero multiplicity in the r. h. s. is equal to one of the highest weights in the decomposition. The multiplicity $M^{\widetilde{\mu}}_{(\mathfrak{s})\widetilde{\nu}}$ of the weight $\widetilde{\nu}\in\mathcal{N}^{\widetilde{\mu}}_{\mathfrak{s}}$ defines the branching coefficient $b^{(\mu)}_{\nu}$ for the highest weight $\nu = (\mu-\phi(\widetilde{\mu}-\widetilde{\nu}))$:*

$$b^{(\mu)}_{(\mu-\phi(\widetilde{\mu}-\widetilde{\nu}))} = M^{\widetilde{\mu}}_{(\mathfrak{s})\widetilde{\nu}}.$$

# 4 Examples

**Example 4.1.** *Consider Lie algebra $A_2(\mathbf{sl(3)})$ and branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra $A_1\oplus u(1)$ with root system spanned by first simple root of $A_2$. Singular element of $L^{(3,2)}$ is decomposed into the sum of splint images of singular elements of $A_1\oplus A_1$-modules and branching coefficients coincide with weight multiplicities of $A_1\oplus A_1$-module (see Fig. 1).*

**Example 4.2.** *Now consider Lie algebra $B_2(\mathbf{so(5)})$ and branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra $A_1\oplus u(1)$ with root system spanned by first simple root of $B_2$. Singular element of $L^{(3,2)}$ is decomposed into the sum of splint images of singular elements of $A_2$-modules and branching coefficients coincide with weight multiplicities of $A_2$-module (see Fig. 2).*

**Example 4.3.** *Lie algebra $G_2$ has regular subalgebra $A_2$ with root system built on long roots of $G_2$. Consider branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra $A_2$. Singular element of $L^{(3,2)}$ is decomposed into the sum of splint images of singular elements of $A_2$-modules and branching coefficients coincide with weight multiplicities of $A_2$-module (see Fig. 3).*

# 5 Conclusions

It is explicitly demonstrated that splint presents a very effective tool to find branching coefficients. In particular injective splints provide a possibility to

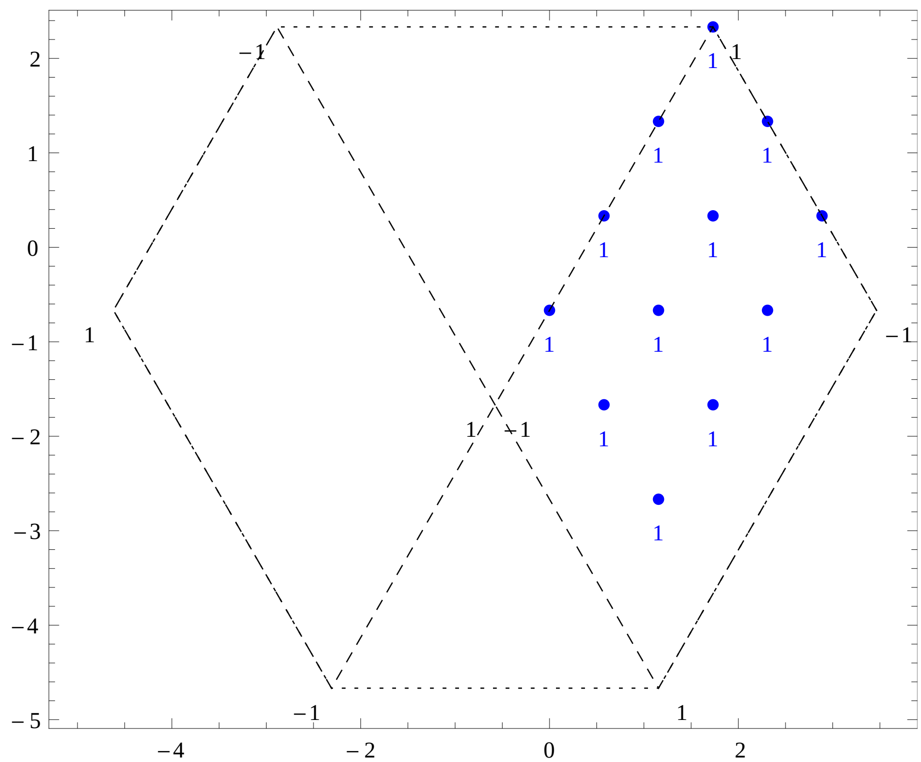


Figure 1: Weyl group orbit (dotted) producing singular element of $A_2$-module $L^{(3,2)}$ and its decomposition into the sum of splint images of singular elements of $A_1 \oplus A_1$-modules (dashed). Weight multiplicities of $A_1 \oplus A_1$-module coincide with branching coefficients for the reduction $L^{(3,2)}_{A_2 \downarrow A_1 \oplus u(1)}$.

reduce branching rules calculations for highest weight modules to a determination of weight multiplicities for a module with the same Dynkin labels referred to a different Lie algebra. This algebra $\mathfrak{s}$ must not be a subalgebra in the initial $\mathfrak{g}$ , it has the same rank $r_{\mathfrak{s}} = r$ , but as a rule is less complicated than $\mathfrak{g}$ .

It is significant that for the injections $D_r \hookrightarrow B_r$ , $D_r \hookrightarrow C_r$ and $A_{r-1} \oplus u(1) \hookrightarrow A_r$ splint technique shows immediately Gelfand-Tzeytlin rules for branching: the nonzero branching coefficients are equal to 1, the reduction is multiplicity free. Here it is an immediate consequence of the structure of the second stem being a direct sum of $A_1$ algebras and the fact that the corresponding module $L^{\mu}_{\mathfrak{s}}$ is irreducible.

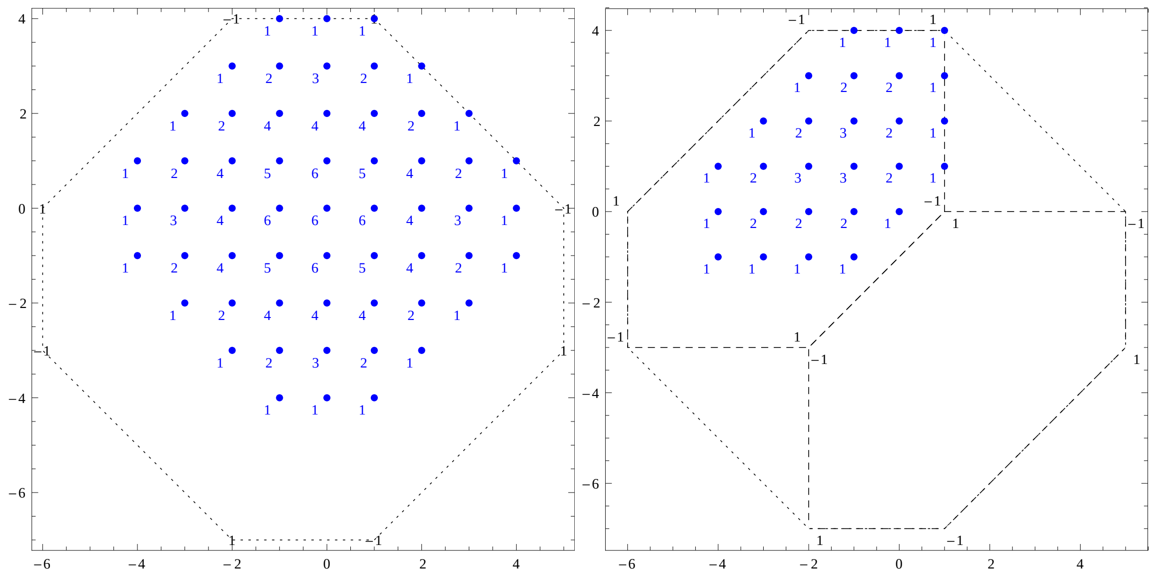

Figure 2: $B_2$-module $L^{(3,2)}$ is shown on the left. Weight multiplicities are indicated. Contour of singular element is shown by dotted line. Right figure represents the decomposition of $L^{(3,2)}$-singular element into the sum of splint images of singular elements of $A_2$-modules (dashed). Weight multiplicities of $A_2$-module coincide with branching coefficients for the reduction $L^{(3,2)}_{B_2\downarrow A_1\oplus u(1)}$.

# Acknowledgements

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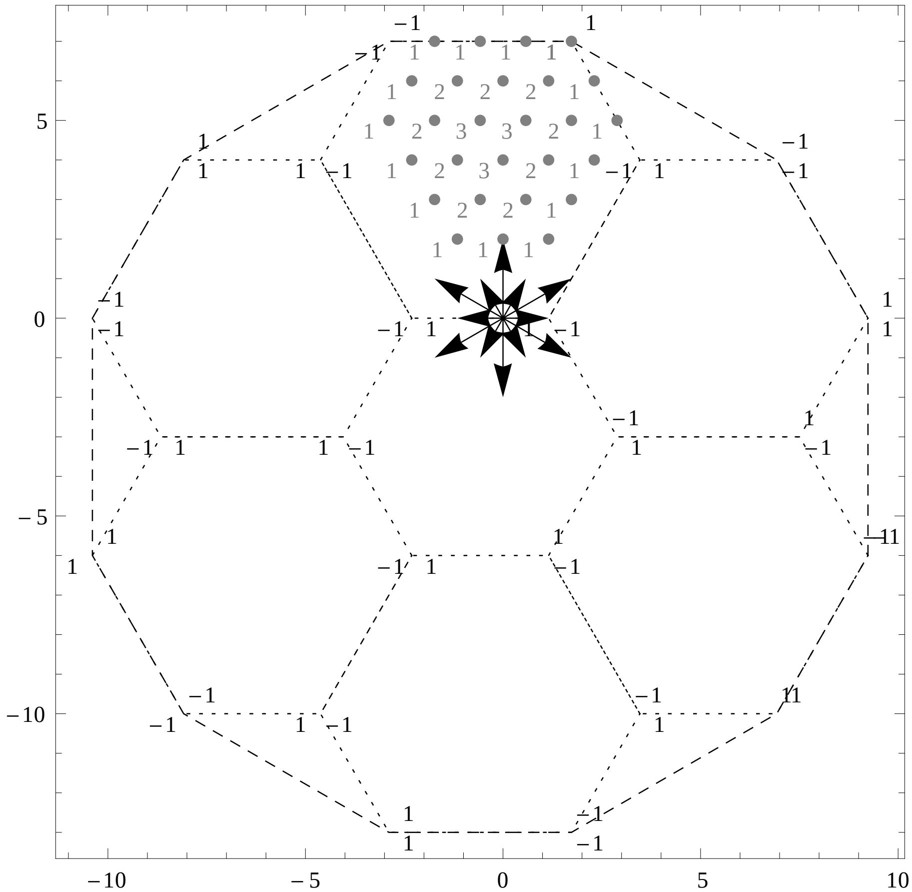


Figure 3: Weyl group orbit (dotted) producing singular element of $G_2$-module $L^{(3,2)}$ and its decomposition into the sum of splint images of singular elements of $A_2$-modules (dashed). Weight multiplicities of $A_2$-module coincide with branching coefficients for the reduction $L^{(3,2)}_{G_2 \downarrow A_2}$.

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# Appendix

Let us demonstrate that for injective splints of classical Lie algebras the following property is valid:

Let $\Delta \approx (\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$ be an injective splint with the decomposition of simple roots $S = S_{\mathfrak{c}} \cup S_{\mathfrak{d}}$ with $S_{\mathfrak{c}} = S \cap S_{\mathfrak{a}}$ and $S_{\mathfrak{d}} = S \cap S_{\mathfrak{s}}$.

Then for any simple root $\beta \in S_{\mathfrak{s}}$ there exists the pair of roots ( $\alpha$ ,$\beta'$) with $\alpha \in S_{\mathfrak{c}}, \beta' \in S_{\mathfrak{s}}$ such that $\alpha = \beta - \beta'$

Type 1. $\Delta_{G_2} \approx (\Delta_{A_2}, \Delta_{A_2})$.

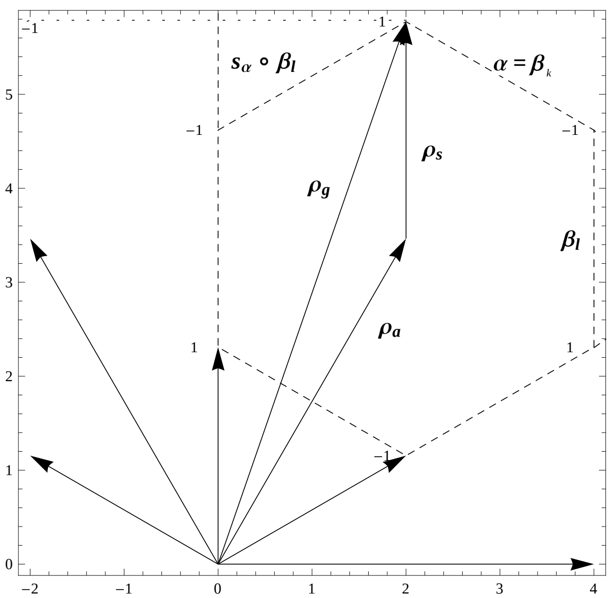


Figure 4: Positive roots of $G_2$ and formation of singular element $\Phi_{\mathfrak{s}}^{(0)}$ in the main Weyl chamber of $\mathfrak{a} = A_2$.

Here both stems are metric and the corresponding root systems are equivalent. In Figure 4 a part of the singular element $\Phi_{G_2}^{(0)}$ is presented. The boundaries of $\bar{C}_{\mathfrak{a}}$ are the dashed lines starting in the center of the singular element. It contains the edge $\lambda_2 = -\alpha_2 = -\beta_2$ and the roots $-\beta_1 = -s_{\alpha_2} \circ \beta_3$, $-\beta_3$ ($\beta_3$ is indicated as $\beta_l$). For this root $\beta_1$ the necessary pair is $(\alpha_1, \beta_2)$: $\alpha_1 = \beta_1 - \beta_2$. The $\lambda_{2,3}^{\mathfrak{s}} = \beta_3$ edge is equal to $\lambda_1^{\mathfrak{s}} = \beta_1 = s_{\alpha_2} \circ \beta_3$ and

$m_1$ index is aquired by the $\mathfrak{s}$-module that inherit the second indice. In this particular case they are $m_1 = m_2 = 0$. The general case with the initial module $L^{\mu}$ and $\mu = m_1\omega_1 + m_2\omega_2$.can be treated in the same way: one finds an edge $\lambda_2 = -(m_2+1)\beta_2$ and put $\lambda_1^{\mathfrak{s}} = -(m_1+1)\beta_1$, its end belongs to the boundary $\bar{C}_{\mathfrak{a}}$ . The reflection $s_{\beta_2}$ sends $\beta_1$ to $\beta_3$ and the corresponding edge $\lambda_{2,3}^{\mathfrak{s}} = -(m_1+1)\beta_3$ has the same length. Now consider $\lambda_1^{\mathfrak{s}}$ (or $\lambda_{2,3}^{\mathfrak{s}}$) and $\lambda_{1,3}^{\mathfrak{s}}$ (or $\lambda_{2,3,1}^{\mathfrak{s}}$) edges to find that they belong to the boundary $\bar{C}_{\mathfrak{a}}$ and the Weyl symmetry predicts that $\lambda_{1,3}^{\mathfrak{s}} = -(m_2+1)\beta_3$ $(\lambda_{2,3,1}^{\mathfrak{s}} = -(m_2+1)\beta_1)$ . Finally the edge $\lambda_{1,3,2}^{\mathfrak{s}} = -(m_1+1)\beta_2$ close the polytope. The latter corresponds to the singular element $\Phi_{\mathfrak{s}}^{(\widetilde{\mu})} = \sum_{w\in W_{\mathfrak{s}}} \varepsilon(w)\, e^{w\circ(\widetilde{\mu}+\rho_{\mathfrak{s}})}$ of the module $L_{\mathfrak{s}}^{(\widetilde{\mu})}$ with $\widetilde{\mu} = m_1\widetilde{\omega}_1 + m_2\widetilde{\omega}_2$ . Notice that in this case the sign factors can be obtained directly in the initial weight system as far as the stem is metric.

Type 1. $\Delta_{F_4} \approx (\Delta_{D_4}, \Delta_{D_4})$.

Both stems are metric and the corresponding root systems are equivalent. The system $\Delta_{D_4}$ of the subalgebra $\mathfrak{a} = D_4$ is formed by the set $\{\pm e_i, \pm e_j\}_{|i,j=1,\ldots 4,\, i\neq j}$ . The simple roots $S_{\mathfrak{c}}$ are $\{e_2-e_3, e_3-e_4\}$ and $S_{\mathfrak{d}} = \left\{e_4, \frac{1}{2}(e_1-e_2-e_3-e_4)\right\}$.

For a module $L^{\mu}$ with $\mu = \sum m_k\omega_k$ consider the edge $\lambda_3 = -(m_3+1)e_4 = -(m_3+1)\beta_3$ Compose an edge $\lambda_2^{\mathfrak{s}} = -(\widetilde{m}_2+1)\beta_2$ . Now the necessary pair is $(\alpha_2 = e_3 - e_4, \beta_3)$. The intersection of $\lambda_2^{\mathfrak{s}}$ with the $\alpha_2$ -boundary of $\bar{C}_{\mathfrak{a}}$ fixes its length to be $\lambda_2^{\mathfrak{s}} = -(m_2+1)\beta_2$ and the length of the edge $\lambda_{3,2}^{\mathfrak{s}}$ is equal to that of $\lambda_2^{\mathfrak{s}}$. Next consider the edge $\lambda_2^{\mathfrak{s}} = -(m_2+1)\beta_2$ and the pair $(\alpha_1 = e_2 - e_3, \beta_1 = e_2)$ The length of the edge $\lambda_1^{\mathfrak{s}}$ becomes equal to $\lambda_1^{\mathfrak{s}} = -(m_1+1)\beta_1$ and so on. The edges along the roots like $\alpha_4 = \beta_4 = \frac{1}{2}(e_1-e_2-e_3-e_4)$ are treated similarly and finally the singular element $\Phi_{\mathfrak{s}}^{(\widetilde{\mu})} = \sum_{w\in W_{\mathfrak{s}}} \varepsilon(w)\, e^{w\circ(\widetilde{\mu}+\rho_{\mathfrak{s}})}$ of the module $L_{\mathfrak{s}}^{(\widetilde{\mu})}$ with $\widetilde{\mu} = \sum m_k\widetilde{\omega}_k$ is formed in $\bar{C}_{\mathfrak{a}}$.

Type 2. $\Delta_{B_r} \approx (\Delta_{D_r}, \Delta_{\oplus^r A_1})$.

Both stems are metric. The injection is fixed by the stem $\Delta_{D_r}$ with simple roots $S_{\mathfrak{a}} = \{e_1-e_2, e_2-e_3, \ldots, e_{r-1}-e_r, e_{r-1}+e_r\}$ . The second stem corresponds to a direct sum of algebras $A_1$ and has the simple roots $S_{\mathfrak{s}} = \{e_1, e_2, \ldots, e_{r-1}, e_r\}$. Consider the edge $\lambda_r = -(m_r+1)\beta_r$ (here $\beta_r = e_r$) and $\lambda_{r-1} = -(\widetilde{m}_{r-1}+1)\beta_{r-1}$ attached to it (here $\beta_{r-1} = e_{r-1}$). The corresponding pair is $(\alpha_{r-1} = e_{r-1} - e_r, \beta_{r-1} = e_{r-1})$ . The intersection condition fixes the second edge to be $\lambda_{r-1} = -(m_{r-1}+1)\beta_{r-1}$ , it is orthogonal to $\beta_r$ so the opposite edge has the same length. The Dynkin index $m_{r-1}$ now refers also to the simple root $\beta_{r-1}$ . Next consider the obtained edge $\lambda_{r-1} = -(m_{r-1}+1)\beta_{r-1}$ and $\lambda_{r-2} = -(\widetilde{m}_{r-2}+1)\beta_{r-2}$ to obtain the index $\widetilde{m}_{r-2} = m_{r-2}$ and the edge $\lambda_{r-2} = -(m_{r-2}+1)\beta_{r-2}$ and so on till all the pairs of edges are properly fixed. Finally in $\bar{C}_{D_r}$ the element $\Phi^{(\widetilde{\mu})}_{\oplus^r A_1} = \sum_{w \in W_{\oplus^r A_1}} \varepsilon(w) e^{w \circ \left(\widetilde{\mu} + \frac{1}{2}\sum e_k\right)}$ can be constructed for the module $L^{(\widetilde{\mu})}_{\oplus^r A_1}$ with $\widetilde{\mu} = \sum m_k \frac{1}{2} e_k$ .

Type 2. $\Delta_{C_r} \approx (\Delta_{D_r}, \Delta_{\oplus^r A_1})$.

The situation in this case is eqivalent to the previous one and the additional edges are constructed similarly.

Type 3 $\Delta_{A_r} \approx (\Delta_{A_{r-1} \oplus u_1}, \Delta_{\oplus^r A_1})$.

Here only the first stem is metric and it fixes the injection with simple roots $S_{\mathfrak{a}} = \{e_1 - e_2, e_2 - e_3, \ldots, e_{r-1} - e_r\}$ . The second stem corresponding to a direct sum of $r$ copies of $A_1$ has the simple roots $S_{\mathfrak{s}} = \{e_1 - e_{r+1}, e_2 - e_{r+1}, \ldots, e_r - e_{r+1}\}$. Consider the edge $\lambda_r = -(m_r+1)\beta_r$ with $\beta_r = e_r - e_{r+1}$ and $\lambda_{r-1} = -(\widetilde{m}_{r-1}+1)\beta_{r-1}$ with $\beta_{r-1} = e_{r-1} - e_{r+1}$ attached to it . Then the corresponding pair is $(\alpha_{r-1} = e_{r-1} - e_r, \beta_{r-1} = e_{r-1} - e_{r+1})$ . The intersection with the boundary of $\bar{C}_{A_{r-1}}$ orthogonal to $\alpha_{r-1}$ fixes the second edge to be $\lambda_{r-1} = -(m_{r-1}+1)\beta_{r-1}$. The Dynkin index $m_{r-1}$ is to be used for the fundamental weight $\omega_{r-1}$. The reflection $s_{\beta_r}$ sends $\lambda_{r-1} = -(m_{r-1}+1)\beta_{r-1}$ to $\lambda_{r,r-1} = -(m_{r-1}+1)\beta_{r-1}$. Next consider the obtained edge $\lambda_{r-1} = -(m_{r-1}+1)\beta_{r-1}$ and $\lambda_{r-2} = -(\widetilde{m}_{r-2}+1)\beta_{r-2}$ with $\beta_{r-2} = e_{r-2} - e_{r+1}$ to obtain the index $\widetilde{m}_{r-2} = m_{r-2}$ and the edge $\lambda_{r-2} = -(m_{r-2}+1)\beta_{r-2}$ and so on till all the pairs of edges are properly fixed. Finally in $\bar{C}_{D_r}$ the element $\Phi^{(\widetilde{\mu})}_{\oplus^r A_1} = \sum_{w \in W_{\oplus^r A_1}} \varepsilon(w) e^{w \circ (\widetilde{\mu} + \widetilde{\rho})}$ can be constructed for the module $L^{(\widetilde{\mu})}_{\oplus^r A_1}$ with $\widetilde{\mu} = \sum m_k \beta_k$. The simplest case $\Delta_{A_2} \approx (\Delta_{A_1 \oplus u_1}, \Delta_{A_1 \oplus A_1})$ is presented in Example 4.1 and Figure 1.

Type 3 $\Delta_{B_2} \approx (\Delta_{A_1}, \Delta_{A_2})$.

The necessary elements are presented in Example 4.1 and Figure 1 , $S_{A\_1} = \{e_1 - e_2\}$ , $S_{A\_2} = \{e_1, e_2\}$.The edge $\lambda_{\alpha_2} = \lambda_{\beta_2} = -(m_2+1)\beta_2$ has an edge $\lambda_{\beta_1} = -(\widetilde{m}_1+1)\beta_1$ attached to it. Consider the pair $(\alpha_1 = e_1 - e_2, \beta_1 = e_1)$ . The end of the edge $\lambda_{\beta_1}$ is to define a weight invariant under the reflexion $s_{\alpha_1}$. This defines its length: $\lambda_{\beta_1} = -(m_1+1)\beta_1$ . In the coimage of the second stem, that is in the $A_2$ root system, the reflection $s_{\beta_2}$ sends $\lambda_{\beta_1} = -(m_1+1)\beta_1$ to $\lambda_{2,3}$, thus the latter edge has the same length in $\beta_3 = e_1 + e_3$ , we have $\lambda_{2,3} = -(m_1+1)\beta_3$ with $\beta_3 = e_1 + e_3$ . The irreducible $\mathfrak{s}$-module has the highest weight $\widetilde{\mu} = m_1\widetilde{\omega}_1 + m_2\widetilde{\omega}_2$ . In Figure 1 we see the details of these relations in a particular case where $L^{[3,2]}_{B_2}$ is reduced to a subalgebra $A_1 \oplus u(1)$ and the corresponding highest weights with their multiplicities form the diagram $\mathcal{N}_{A\_2}^{[3,2]}$ .