Pis'ma v ZhETF

# SLE martingales in coset conformal field theory

*Anton Nazarov*[+*1)]

[+] *Department of High-Energy and Elementary Particle Physics, Faculty of Physics and Chebyshev Laboratory, Faculty of Mathematics and Mechanics, SPb State University 198904, Saint-Petersburg, Russia*

Submitted 26 April 2012

Scharmm-Loewner evolution (SLE) and conformal field theory (CFT) are popular and widely used instruments to study critical behavior of two-dimensional models, but they use different objects. While SLE has natural connection with lattice models and is suitable for strict proofs, it lacks computational and predictive power of conformal field theory. To provide a way for the concurrent use of SLE and CFT we consider CFT correlation functions which are martingales with respect to SLE. We establish connection between parameters of Schramm-Loewner evolution on coset space and algebraic data of coset conformal field theory. Then we check the consistency of our approach with the behaviour of parafermionic and minimal models. Coset models are connected with off-critical massive field theories and we discuss implications for SLE.

## 1. INTRODUCTION

Schramm-Loewner evolution [1] is a stochastic process satisfying stochastic differential equation (1). A solution of equation (1) can be seen as probability measure on random curves (called SLE traces). SLE is useful for the study of critical behavior [2, 3], since for many lattice models the convergence of domain walls to SLE traces can be proved [4, 5]. Practical computations with SLE rely on Ito calculus while proofs use discrete complex analysis [6].

Conformal field theory [7] is formulated in terms of families of quantum fields which usually has no direct microscopic interpretation. On the other hand CFT has efficient computational tools such as Virasoro symmetry, Knizhnik-Zamolodchikov equations, fusion algebra etc and provides numerous predictions, such as celebrated Cardy formula [8, 9] which gives crossing probability for critical percolation.

Combination of these two approaches is immensely powerful, so it is important to find objects which can be studied from both points of view.

We present a new class of such observables in coset conformal field theory – correlation functions which are martingales with respect to Schramm-Loewner evolution with additional Brownian motion on factor-space of Lie group $G$ by subgroup $A$ [10]. We derive relations connecting SLE and CFT parameters. We then check the consistency of these results with the earlier results for parafermionic observables [11].

All minimal unitary models can be obtained by coset construction. Coset structure leads to specific off-critical behavior. Massive excitations of coset CFTs are given by affine Toda field theories [12, 13, 14]. Recent experimental study confirming this result attracted a lot of attention [15]. In conclusion we discuss possible connections of massive CFT perturbations with off-critical SLEs (containing additional drift term [16]).

[1)] e-mail: anton.nazarov@hep.phys.spbu.ru

## 2. MARTINGALE CONDITIONS

Consider some critical lattice model on the upper half-plane $\mathbb{H}$ with the cut along a critical interface $\gamma_t$ (domain wall up to some length $t$). We denote this slit domain by $\mathbb{H}_t = \mathbb{H} \setminus \gamma_t$. Assume that $\gamma_t$ satisfies restriction property and is conformally invariant [2]. Then conformal map $g_t : \mathbb{H}_t \to \mathbb{H}$ satisfies stochastic differential equation [1]:

$$\frac{\partial g_t(z)}{\partial t} = \frac{2}{g_t(z) - \sqrt{\kappa}\xi_t}, \tag{1}$$

where $\xi_t$ is the Brownian motion. The dynamic of the tip $z_t$ of critical curve $\gamma_t$ (tip of SLE trace) is given by the law $z_t = g_t^{-1}(\sqrt{\kappa}\xi_t)$.

For us it is convenient to use the map $w_t(z) = g_t(z) - \sqrt{\kappa}\xi_t$, so the equation (1) becomes

$$dw_t = \frac{2dt}{w_t} - \sqrt{\kappa}\xi_t \tag{2}$$

Consider $N$-point correlation function of boundary conformal field theory with boundary condition changing operators sitting at the tip of SLE trace and at the infinity:

$$\mathcal{F}(\{z_i, \bar{z}_i\}_{i=1}^N)_{\mathbb{H}_t} = \langle \varphi(z_t)\phi_1(z_1, \bar{z}_1)\dots\phi_n(z_n, \bar{z}_n)\varphi(\infty)\rangle, \tag{3}$$

where $\phi_i$ are bulk primary fields with conformal weights $h_i$ and $\varphi(z_t)$ is boundary condition changing operator.

Let $\mathfrak{g}$ be a (semisimple) Lie algebra of a Lie group $G$ and $\mathfrak{a}$ – Lie algebra of a subgroup $A \subset G$. Denote by $x_e$ embedding index for $\mathfrak{a} \to \mathfrak{g}$.

Primary fields in coset models are labeled by pairs $(\lambda, \eta)$ of weights for $\mathfrak{g}$- and $\mathfrak{a}$-representations. For each $\lambda$ the set of possible weights $\eta$ includes those which appear in the decomposition $L^{(\mu)}_{\mathfrak{g}} = \bigoplus_\eta H^\lambda_\eta \otimes L^{(\eta)}_{\mathfrak{a}}$. (Some pairs are equivalent, see [17, 18]). We assume that boundary field $\varphi$ is also primary and is labeled by weights $(\mu, \nu)$.

We use the conformal map $w(z) : \mathbb{H} \setminus \gamma_t \to \mathbb{H}$ to rewrite expression (3) in the whole upper half plane:

$$\mathcal{F}(\{z_i, \bar{z}_i\})_{\mathbb{H}_t} = = \prod \left(\frac{\partial w(z_i)}{\partial z_i}\right)^{h_i} \prod \left(\frac{\partial \bar{w}(\bar{z}_i)}{\partial \bar{z}_i}\right)^{\bar{h}_i} \mathcal{F}(\{w_i, \bar{w}_i\})_{\mathbb{H}} \quad (4)$$

We can extend the definition of $\mathcal{F}$ to the whole plane by doubling the number of fields $\phi_i$ [19, 20] and by considering $w_i, \bar{w}_i$ as independent variables. To simplify notations we will write $\mathcal{F}(\{w_i\}_{i=1}^{2N})$.

$G/A$-coset conformal field theory can be realized as a WZNW-model (with gauge group $G$) interacting with pure gauge fields of gauge group $A \subset G$ [21, 22]. The action is written in terms of fields $\gamma : \mathbb{C} \to G$ and $\alpha, \bar{\alpha} : \mathbb{C} \to A$:

$$\begin{aligned} S_{G/A}(\gamma, \alpha) = &-\frac{k}{8\pi} \int_{S^2} d^2x\, \mathcal{K}(\gamma^{-1}\partial^\mu \gamma, \gamma^{-1}\partial_\mu \gamma) - \\ &- \frac{k}{24\pi} \int_B \epsilon_{ijk} \mathcal{K}\left(\tilde{\gamma}^{-1}\frac{\partial \tilde{\gamma}}{\partial y^i}, \left[\tilde{\gamma}^{-1}\frac{\partial \tilde{\gamma}}{\partial y^j} \tilde{\gamma}^{-1}\frac{\partial \tilde{\gamma}}{\partial y^k}\right]\right) d^3y + \\ &+ \frac{k}{4\pi} \int_{S^2} d^2z \left(\mathcal{K}(\alpha, \gamma^{-1}\bar{\partial}\gamma) - \mathcal{K}(\bar{\alpha}, (\partial\gamma)\gamma^{-1})\right. \\ &\left. + \mathcal{K}(\alpha, \gamma^{-1}\bar{\alpha}\gamma) - \mathcal{K}(\alpha, \bar{\alpha})\right). \quad (5) \end{aligned}$$

Here we denote by $\mathcal{K}$ the Killing form of a Lie algebra $\mathfrak{g}$ corresponding to a Lie group $G$.

If we fix $A$-gauge we have $G/A$ gauge symmetry. Then we add random gauge transformations to Schramm-Loewner evolution [10] similar to the case of WZNW-models [23]. Denote by $t^a_i$ ($\tilde{t}^b_i$) the generators of $\mathfrak{g}$-representation ($\mathfrak{a}$-representation) corresponding to the primary field $\phi_i$.

Now we need to consider the evolution of SLE trace $\gamma_t$ from $t$ to $t + dt$. First factor in the right-hand-side of equation (4) gives us

$$-\frac{2h_i}{w_i^2}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}.$$

We denote by $\mathcal{G}_i$ the generator of infinitesimal transformation of primary field $\phi_i$:$d\phi_i(w_i) = \mathcal{G}_i \phi_i(w_i)$. We normalize additional ($\dim \mathfrak{g}$)-dimensional Brownian motion as $\mathbb{E}\left[d\theta^a\ d\theta^b\right] = \mathcal{K}(t^a, t^b)dt$. The generator of field transformation is equal to

$$\mathcal{G}_i = \left(\frac{2dt}{w_i} - \sqrt{\kappa} d\xi_t\right)\partial_{w_i} + \frac{\sqrt{\tau}}{w_i}\left(\sum_{a:\mathcal{K}(t^a, \tilde{t}^b)=0} (d\theta^a t^a_i)\right). \quad (6)$$

So we have fixed $A$-gauge by allowing random walk only in direction orthogonal to subalgebra $\mathfrak{a}$.

Ito formula gives us an expression for the differential of $\mathcal{F}$ which should be zero due to martingale condition:

$$d\mathcal{F}_{\mathbb{H}_t} = \left(\prod_{i=1}^{2N}\left(\frac{\partial w_i}{\partial z_i}\right)^{h_i}\right)\left(-\sum_{i=1}^{2N}\frac{2h_i dt}{w_i^2} + \left[\sum_{i=1}^{2N}\mathcal{G}_i + \frac{1}{2}\sum_{i,j}\mathcal{G}_i\mathcal{G}_j\right]\right)\mathcal{F}_{\mathbb{H}} = 0 \quad (7)$$

Substituting definition (6) we get

$$\left(-2\mathcal{L}_{-2} + \frac{1}{2}\kappa\mathcal{L}_{-1}^2 + \frac{\tau}{2}\left(\sum_a \mathcal{J}^a_{-1}\mathcal{J}^a_{-1} - \sum_b \tilde{\mathcal{J}}^b_{-1}\tilde{\mathcal{J}}^b_{-1}\right)\right)\mathcal{F}_{\mathbb{H}} = 0, \quad (8)$$

with

$$\mathcal{L}_{-n} = \sum_i \left(\frac{(n-1)h_i}{(w_i - z)^n} - \frac{\partial_{w_i}}{(w_i - z)^{n-1}}\right),$$

$$\mathcal{J}^a_{-n} = -\sum_i \frac{t^a_i}{(w_i - z)^n};\ \tilde{\mathcal{J}}^b_{-n} = -\sum_i \frac{\tilde{t}^b_i}{(w_i - z)^n}.$$

This equation is equivalent to the following algebraic condition on the boundary state $\varphi(0)\left|0\right\rangle$:

$$\left\langle 0\right|\varphi(\infty)\phi_1(w_1)\dots\phi_{2N}(w_{2N}) \left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J^a_{-1}J^a_{-1} - \sum_{b=1}^{\dim\mathfrak{a}}\tilde{J}^b_{-1}\tilde{J}^b_{-1}\right)\right) \varphi(0)|0\rangle = 0. \quad (9)$$

Since the set $\{\phi_i\}$ consists of arbitrary primary fields we conclude that

$$|\psi> = \left(-2L_{-2} + \frac{1}{2}\kappa L_{-1}^2 + \frac{1}{2}\tau\left(\sum_{a=1}^{\dim\mathfrak{g}} J^a_{-1}J^a_{-1} - \sum_{b=1}^{\dim\mathfrak{a}}\tilde{J}^b_{-1}\tilde{J}^b_{-1}\right)\right) \cdot \varphi(0)|0> \quad (10)$$

is a null-state. Now we act on $\psi$ by raising operators to get equations on $\kappa, \tau$. Since in a coset theory commutation relations of full chiral algebra are

$$\begin{aligned} &L_n = L^{\mathfrak{g}}_n - L^{\mathfrak{a}}_n, \\ &[L^{\mathfrak{g}}_n, J^a_m] = -mJ^a_{n+m}, \\ &\left[L^{\mathfrak{g}}_n, \tilde{J}^b_m\right] = -m\tilde{J}^b_{n+m}, \\ &[L^{\mathfrak{g}}_n, L_m] = [L^{\mathfrak{g}}_n, L^{\mathfrak{g}}_m] - [L^{\mathfrak{g}}_n, L^{\mathfrak{a}}_m] = \\ &\quad = (n-m)L_{m+n} + \tfrac{c}{12}(n^3 - n)\delta_{m+n,0}, \end{aligned} \quad (11)$$

it is more convenient to act with $L_2^{\mathfrak{g}}$ and $\left(L_1^{\mathfrak{g}}\right)^2$. Applying $L_2^{\mathfrak{g}}$ we get

$$L_2^{\mathfrak{g}}\psi = \left(-8L_0 - c + 3\kappa L_0 + \frac{1}{2}\tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a})\right)\varphi_{(\mu,\nu)} = 0.$$

We use $L_0\varphi_{(\mu,\nu)} = h_{(\mu,\nu)}\varphi_{(\mu,\nu)}$, with the conformal weight $h_{(\mu,\nu)} = \left(\frac{(\mu,\mu+2\rho)}{2(k+h^\vee_{\mathfrak{g}})} - \frac{(\nu,\nu+2\rho_{\mathfrak{a}})}{2(kx_e+h^\vee_{\mathfrak{a}})}\right)$ and the central charge $c = \frac{k\dim\mathfrak{g}}{k+h^\vee_{\mathfrak{g}}} - \frac{x_e k\dim\mathfrak{a}}{x_e k+h^\vee_{\mathfrak{a}}}$. This leads to the relation on $\kappa,\tau$:

$$(3\kappa-8)h_{(\mu,\nu)} - c + \tau(k\dim\mathfrak{g} - x_e k\dim\mathfrak{a}) = 0. \quad (12)$$

The second relation appears as a result of the $L_1^{\mathfrak{g}}$-action:

$$-12h_{(\mu,\nu)} + 2\kappa h_{(\mu,\nu)}(2h_{(\mu,\nu)}+1) + \tau(C_\mu - \tilde{C}_\nu) = 0, \quad (13)$$

where $C_\mu = (\mu,\mu+2\rho)$ and $\tilde{C}_\nu = (\nu,\nu+2\rho_{\mathfrak{a}})$ are the eigenvalues of the quadratic Casimir operators $\sum_a t^a t^a$ and $\sum_b \tilde{t}^b\tilde{t}^b$ of Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$. Relations (12),(13) are the necessary conditions for CFT correlation functions to be SLE martingales.

From equations (12),(13) we immediately get $\kappa,\tau$ for each pair $(\mu,\nu)$ of $\mathfrak{g}$ and $\mathfrak{a}$-weights.

### 21 Examples

As an example consider $\frac{\hat{su}(2)_N}{\hat{u}(1)_N}$-coset models which are equivalent to $Z_N$-parafermions. The central charge is $c = \frac{3N}{N+2} - 1 = \frac{2N-2}{N+2}$. Conformal weights of primary fields with Dynkin indeces $(k,l)$ are $h_{(k,l)} = \frac{k(k+2)}{4(N+2)} - \frac{l^2}{4N}$.

Case $N=2$, $c=\frac{1}{2}$ corresponds to the Ising model, we have two non-trivial primary fields with conformal weights $h_{(2,0)} = 1/2, h_{(1,1)} = 1/16$. Substituting the field $\varphi_{(2,0)}$ into equations (12,13) we get: $3\kappa - 9 + 4\tau = 0$; $-3+\kappa+4\tau = 0$. The solution is $\kappa = 3, \tau = 0$. For the field $\varphi_{(1,1)}$ the relations are $3\kappa - 16 + 64\tau = 0$, $-64 + 9\kappa + 64\tau = 0, \kappa = 16/3, \tau = 0$. So we have no additional motion for the Ising model and two possible values for SLE parameter $\kappa$ coinciding with the well-known results [5].

For $N=3$ the parafermionic model central charge is $c = \frac{4}{5}$. Conformal weights are $h_{(0,0)} = 0$, $h_{(0,2)} = h_{(0,-2)} = \frac{2}{3}$ mod 1, $h_{(2,0)} = \frac{2}{5}$, $h_{(2,2)} = h_{(2,-2)} = \frac{1}{15}$. The corresponding values of $\kappa,\tau$ are: $(\frac{208}{25},\frac{242}{225}),(\frac{10}{3},0),(\frac{80}{19},\frac{14}{171})$. As it was mentioned in [11] the field $\varphi_{(2,0)}$ with the conformal weight $h_{(2,0)} = \frac{2}{5}$ constitutes the $Z_3$-singlet, so an additional random walk does not appear and $\tau = 0$. The form of equations (9) is similar to that of [11], but the normalization of $\tau$ differs.

It is easy to see that for the $\frac{\hat{su}(2)_N \oplus \hat{su}(2)_1}{\hat{su}(2)_{N+1}}$-coset realization of minimal unitary models with $c = \frac{3N}{N+2} + 1 - \frac{3(N+1)}{N+3} = 1 - \frac{6}{(N+2)(N+3)}$ the system of equations (12,13) is always consistent for $\tau = 0$ and we get a standard connection between the SLE-parameter $\kappa$ and the central charge $c = \frac{(6-\kappa)(3\kappa-8)}{2\kappa}$.

## 3. CONCLUSION AND OUTLOOK

In [23] a connection between WZNW-models and Schramm-Loewner evolution with additional Brownian motion on group manifold was established. The authors also stated a problem of a possible connection of SLE parameters for martingales in WZNW, coset and minimal models. In present letter we used the method proposed in [24] to obtain necessary conditions on SLE martingales. This method allowed us to compare our results with parafermionic results presented in [11, 25].

The coset structure of minimal models is manifest in field theory perturbed by an external magnetic field [12, 13, 14]. This theory is supported by experimental data [15]. Relations between correlation functions in coset theory and SLE observables can be a starting point in studies of domain walls in lattice models away from the critical point.

Massive perturbations of $G/A$-coset theory are realized as an affine Toda field theory and are classified by simple roots of the Lie algebra $\mathfrak{g}$.

It is possible to get the perturbed theory from the action [26, 27, 28]

$$S_{\text{pert}} = S_{G/A}(\gamma,\alpha) - \frac{k\lambda}{2\pi}\int \mathcal{K}(\gamma T, \gamma^{-1}\bar{T}), \quad (14)$$

where $T,\bar{T} \in \mathfrak{g}$ are specially chosen Lie algebra elements.

For example consider $\frac{\hat{su}(2)_N \oplus \hat{su}(2)_1}{\hat{su}(2)_{N+1}}$-coset realization of minimal unitary models. Then we get perturbation by $\phi_{(2,1)}$ primary field from the action

$$S = S_{\frac{\hat{su}(2)_N \oplus \hat{su}(2)_1}{\hat{su}(2)_{N+1}}}(\gamma_1,\gamma_2,\alpha,\bar{\alpha}) - \frac{k\lambda}{2\pi}\int Tr(\gamma_1^{-1}\gamma_2 + \gamma_2^{-1}\gamma_1) \quad (15)$$

This perturbation leads to insertion of perturbing operator in all correlation functions

Massive off-critical SLEs have additional drift term in driving Brownian motion [16, 29]. Inclusion of a drift term in (2) leads to the appearance of new terms in the martingale condition (7).

On the other hand the perturbation can be seen as an insertion of certain primary field in all the correlation functions [14]. The question is whether the interaction of this primary field with the $\tau$-term of the equation (8) leads to the same contribution as the addition of a massive drift to SLE.

Domain walls in the Ising model perturbed by a random Gaussian external field were studied numerically using Schramm-Loewner evolution in a recent paper [30]. In the forthcoming studies we will compare this results with massive perturbations.

## ACKNOWLEDGEMENTS

I am grateful to A. Bytsko, K. Izyurov, D. Chelkak and R. Santachiara for the discussions on SLE and WZNW models and to V.D. Lyakhovsky for the careful reading of the text. This work is supported by the Chebyshev Laboratory (Department of Mathematics and Mechanics, Saint-Petersburg State University) under the grant 11.G34.31.0026 of the Government of the Russian Federation.

---

1. Schramm O 2000 *Israel Journal of Mathematics* **118** 221–288
2. Cardy J L 2005 *Annals Phys.* **318** 81–118 (*Preprint* `cond-mat/0503313`)
3. Rohde S and Schramm O 2005 *Annals of mathematics* 883–924
4. Bauer M and Bernard D 2006 *Physics reports* **432** 115–221
5. Schramm O 2006 *Arxiv preprint math/0602151*
6. Duminil-Copin H and Smirnov S 2011 *Arxiv preprint arXiv:1109.1549*
7. Belavin A, Polyakov A and Zamolodchikov A 1984 *Nuclear Physics* **241** 333–380
8. Cardy J 1992 *Journal of Physics A: Mathematical and General* **25** L201
9. Smirnov S 2001 *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics* **333** 239–244
10. Nazarov A 2012 *Journal of Physics: Conference Series* **343** 012085 (*Preprint* `1112.4354`) URL `http://stacks.iop.org/1742-6596/343/i=1/a=012085`
11. Santachiara R 2008 *Nuclear Physics B* **793** 396–424
12. Fateev V and Zamolodchikov A 1990 *Physics and mathematics of strings: memorial volume for Vadim Knizhnik* 245
13. Eguchi T and Yang S 1989 *Physics Letters B* **224** 373–378
14. Hollowood T and Mansfield P 1989 *Physics Letters B* **226** 73–79
15. Coldea R, Tennant D, Wheeler E, Wawrzynska E, Prabhakaran D, Telling M, Habicht K, Smeibidl P and Kiefer K 2010 *Science* **327** 177–180
16. Makarov N, Smirnov S *et al.* 2010 Off-critical lattice models and massive sles *XVITH INTERNATIONAL CONGRESS ON MATHEMATICAL PHYSICS. Held 3-8 August 2009 in Prague, Czech Republic. Edited by Pavel Exner (Doppler Institute, Prague, Czech Republic). Published by World Scientific Publishing Co. Pte. Ltd., 2010. ISBN# 9789814304634, pp. 362-371* vol 1 pp 362–371
17. Fuchs J, Schellekens B and Schweigert C 1996 *Nuclear Physics B* **461** 371–404
18. Schellekens A and Yankielowicz S 1990 *Nuclear Physics B* **334** 67–102
19. Cardy J 2004 *Arxiv preprint hep-th/0411189*
20. Cardy J 1984 *Nuclear Physics B* **240** 514–532
21. Gawdzki A *et al.* 1988 *Physics Letters B* **215** 119–123
22. Figueroa-OFarrill J *ITP Stony Brook preprint ITP-SB-89-41*
23. Bettelheim E, Gruzberg I, Ludwig A and Wiegmann P 2005 *Physical review letters* **95** 251601
24. Alekseev A, Bytsko A and Izyurov K 2010 *Letters in Mathematical Physics* 1–19
25. Picco M and Santachiara R 2008 *Physical review letters* **100** 15704
26. Bakas I, Park Q and Shin H 1996 *Physics Letters B* **372** 45–52
27. Hollowood T, Miramontes J and Park Q 1995 *Nuclear Physics B* **445** 451–468
28. Park Q 1994 *Physics Letters B* **328** 329–336
29. Bauer M, Bernard D and Cantini L 2009 *Journal of Statistical Mechanics: Theory and Experiment* **2009** P07037
30. Stevenson J and Weigel M 2011 *EPL (Europhysics Letters)* **95** 40001