Fan, splint and branching rules.

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December 17, 2011

Abstract

Splint of root system for simple Lie algebra appears naturally in studies of (regular) embeddings of reductive subalgebras. Splint can be used to construct branching rules. We demonstrate that splint properties implementation drastically simplify calculations of branching coefficients.

1 Introduction

Embedding ϕ of a root system Δ_1 into a root system Δ is a bijective map of roots of Δ_1 to a (proper) subset of Δ that commutes with vector composition law in Δ_1 and Δ.

$$\phi : \Delta_1 \longrightarrow \Delta$$

$$\phi \circ (\alpha + \beta) = \phi \circ \alpha + \phi \circ \beta, \ \ \alpha, \beta \in \Delta_1$$

Note that the image $Im(\phi)$ must not inherit the root system properties except the addition rules equivalent to the addition rules in Δ_1 (for pre-images). Two embeddings ϕ_1 and ϕ_2 can splinter Δ when the latter can be presented as a disjoint union of images $Im(\phi_1)$ and $Im(\phi_2)$. The term *splint* was introduced by D. Richter in [1] where the classification of splints for simple Lie algebras was obtained. There was also mentioned that splint must have tight connections with the injection fan construction. The fan $\Gamma \subset \Delta$ was introduced in [2] as a subset of root system describing recurrent properties of branching coefficients for maximal embeddings. Injection fan is an efficient tool to study branching rules. Later this construction was generalized to non-maximal embeddings and infinite-dimensional Lie algebras in [3, 4].

In the present paper we study connections between splint and injection fan for regular embedding of reductive subalgebras $\mathfrak{a}$ in simple Lie algebra $\mathfrak{g}$. We show that (under certain conditions described in section 3) splint is a natural tool to study reduction properties of $\mathfrak{g}$-modules with respect to a subalgebra $\mathfrak{a} \longrightarrow \mathfrak{g}$. Using this tool we obtain the main result – the one-to-one correspondence between weight multiplicities in irreducible modules of splint and branching coefficients for a reduced module $L^{\mu}_{\mathfrak{g}\downarrow\mathfrak{a}}$.

2 Injections and splints

Consider a simple Lie algebra $\mathfrak{g}$ and its regular subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ such that $\mathfrak{a}$ is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}^*_{\mathfrak{a}} \subset \mathfrak{h}^*_{\mathfrak{g}}$. Let $\mathfrak{a}^{\mathfrak{s}}$ be a semisimple summand of $\mathfrak{a}$, this means that $\mathfrak{a} = \mathfrak{a}^{\mathfrak{s}} \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \ldots$. We shall consider $\mathfrak{a}^{\mathfrak{s}}$ to be a proper regular subalgebra and $\mathfrak{a}$ to be the maximal subalgebra with $\mathfrak{a}^{\mathfrak{s}}$ fixed that is the rank r of $\mathfrak{a}$ is equal to that of $\mathfrak{g}$.

The following notations are used:

r , $(r_{\mathfrak{a}^{\mathfrak{s}}})$ — the rank of $\mathfrak{g}$ (resp.$\mathfrak{a}^{\mathfrak{s}}$) ;

Δ $(\Delta_{\mathfrak{a}})$— the root system; Δ^+ (resp.$\Delta^+_{\mathfrak{a}}$)— the positive root system (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

S , $(S_{\mathfrak{a}})$ — the system of simple roots (of $\mathfrak{g}$ and $\mathfrak{a}$ respectively);

α_i , $\left(\alpha_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) simple root for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}^s})$;

ω_i , $\left(\omega_{(\mathfrak{a})j}\right)$ — the i-th (resp. j-th) fundamental weight for $\mathfrak{g}$ (resp. $\mathfrak{a}$); $i = 0, \ldots, r$, $(j = 0, \ldots, r_{\mathfrak{a}^s})$;

W , $(W_{\mathfrak{a}})$— the corresponding Weyl group;

C , $(C_{\mathfrak{a}})$— the fundamental Weyl chamber;

$\bar{C}, \left(\bar{C}_{\mathfrak{a}}\right)$ — the closure of the fundamental Weyl chamber;

$\epsilon(w) := (-1)^{\mathrm{length}(w)}$;

ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;

L^{μ} $(L^{\nu}_{\mathfrak{a}})$ — the integrable module of $\mathfrak{g}$ with the highest weight μ ; (resp. integrable $\mathfrak{a}$ -module with the highest weight ν);

$\mathcal{N}^{\mu}$, $(\mathcal{N}^{\nu}_{\mathfrak{a}})$ — the weight diagram of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$);

P (resp. $P_{\mathfrak{a}}$) — the weight lattice;

P^{+} (resp. $P^{+}_{\mathfrak{a}}$) — the dominant weight lattice;

$\mathcal{E}$ (resp. $\mathcal{E}_{\mathfrak{a}}$) — the formal algebra;

$m^{(\mu)}_{\xi}$, $\left(m^{(\nu)}_{\xi}\right)$ — the multiplicity of the weight $\xi \in P$ (resp. $\in \mathrm{P}_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L^{\nu}_{\mathfrak{a}}$);

$ch\left(L^{\mu}\right)$ (resp. $\mathrm{ch}\left(L^{\nu}_{\mathfrak{a}}\right)$)— the formal character of L^{μ} (resp. $L^{\nu}_{\mathfrak{a}}$);

$ch\left(L^{\mu}\right) = \frac{\sum_{w \in W} \epsilon(w) e^{w \circ (\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^{+}}(1-e^{-\alpha})}$ — the Weyl formula;

$R := \prod_{\alpha \in \Delta^{+}} \left(1 - e^{-\alpha}\right)$ (resp. $R_{\mathfrak{a}} := \prod_{\alpha \in \Delta^{+}_{\mathfrak{a}}} \left(1 - e^{-\alpha}\right)$) — the Weyl denominator.

Let L^{μ} be completely reducible with respect to $\mathfrak{a}$,

$$L^{\mu}_{\mathfrak{g} \downarrow \mathfrak{a}} = \bigoplus_{\nu \in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} L^{\nu}_{\mathfrak{a}}.$$

$$\pi_{\mathfrak{a}} ch\left(L^{\mu}\right) = \sum_{\nu \in P^{+}_{\mathfrak{a}}} b^{(\mu)}_{\nu} ch\left(L^{\nu}_{\mathfrak{a}}\right). \tag{1}$$

For the modules we are interested in the Weyl formula for $\mathrm{ch}\left(L^{\mu}\right)$ can be written in terms of singular elements [5]

$$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho},$$

namely,

$$\mathrm{ch}\left(L^{\mu}\right) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}. \tag{2}$$

The same is true for submodules $\mathrm{ch}\left(L^{\nu}_{\mathfrak{a}}\right)$ in (1)

$$\mathrm{ch}\left(L^{\nu}_{\mathfrak{a}}\right) = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{\Psi^{(0)}_{\mathfrak{a}}} = \frac{\Psi^{(\nu)}_{\mathfrak{a}}}{R_{\mathfrak{a}}},$$

with

$$\Psi^{(\nu)}_{\mathfrak{a}} := \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}.$$

Applying formula (2) to the branching rule (1) we get a relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{\mathfrak{a}}^{(\nu)}$:

$$\begin{aligned} \frac{\sum_{w\in W}\epsilon(w)e^{w(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})} &= \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\sum_{w\in W_{\mathfrak{a}}}\epsilon(w)e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta_{\mathfrak{a}}^+}(1-e^{-\beta})}, \\ \frac{\Psi^{(\mu)}}{R} &= \sum_{\nu\in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}}. \end{aligned} \tag{3}$$

In [3] it was proven that branching coefficients $b_{\xi}^{(\mu)}$ corresponding to the injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ are subject to the set of recurrent relations:

$$\begin{aligned} b_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)} \Big(&\sum_{u\in W/W_{\perp}} \epsilon(u)\ \dim\left(L_{\mathfrak{a}_{\perp}}^{\mu_{\mathfrak{a}_{\perp}}(u)}\right) \delta_{\xi-\gamma_0,\pi_{\widetilde{\mathfrak{a}}}(u(\mu+\rho)-\rho)} + \\ &+ \sum_{\gamma\in\Gamma_{\widetilde{\mathfrak{a}}\to\mathfrak{g}}} s(\gamma+\gamma_0)\, b_{\xi+\gamma}^{(\mu)} \Big). \end{aligned} \tag{4}$$

where $\mathfrak{a}_{\perp}$ is the subalgebra determined by the roots of $\mathfrak{g}$ orthogonal to roots of $\mathfrak{a}$ and $W_{\perp}$ is a Weyl group of $\mathfrak{a}_{\perp}$

$$\Delta_{\mathfrak{a}_{\perp}} \ :\ = \{\beta\in\Delta_{\mathfrak{g}} | \forall h\in\mathfrak{h}_{\mathfrak{a}};\beta(h)=0\}, \tag{5}$$

$$\widetilde{\mathfrak{a}_{\perp}} := \mathfrak{a}_{\perp}\oplus\mathfrak{h}_{\perp} \qquad \widetilde{\mathfrak{a}} := \mathfrak{a}\oplus\mathfrak{h}_{\perp} \tag{6}$$

and π is the projection operator. When an injection is maximal the projection becomes trivial and the relation (4) is simplified:

$$\begin{aligned} b_{\xi}^{(\mu)} = -\frac{1}{s(\gamma_0)} \Big(&\sum_{u\in W} \epsilon(u)\delta_{\xi-\gamma_0,u(\mu+\rho)-\rho} + \\ &+ \sum_{\gamma\in\Gamma_{\mathfrak{a}\to\mathfrak{g}}} s(\gamma+\gamma_0)\, b_{\xi+\gamma}^{(\mu)} \Big). \end{aligned} \tag{7}$$

The recursion is goverened by the set $\Gamma_{\mathfrak{a}\to\mathfrak{g}}$ called the injection fan. The latter is defined by the carrier set $\{\xi\}_{\mathfrak{a}\to\mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\mathfrak{a}\to\mathfrak{g}} := \{\xi\in P_{\mathfrak{a}} | s(\xi)\neq 0\}$$

appearing in the expansion

$$\prod_{\alpha\in\Delta^+\setminus\Delta_{\mathfrak{a}}^+} \left(1-e^{-\alpha}\right) = -\sum_{\gamma\in P_{\mathfrak{a}}} s(\gamma)e^{-\gamma}; \tag{8}$$

Now we remind two definitions introduced in [1]

Definition 2.1. *Suppose Δ_0 and Δ are root systems with corresponding weight lattices P_0 and P. Then ϕ is an "embedding",*

$$\phi : \begin{cases} \Delta_0 \hookrightarrow \Delta, \\ P_0 \hookrightarrow P, \end{cases} \tag{9}$$

if
(a) it injects Δ_0 in Δ, and
(b) acts homomorphically with respect to the vector groups in P_0 and P:

$$\phi(\gamma) = \phi(\alpha) + \phi(\beta)$$

for any triple $\alpha, \beta, \gamma \in P_0$ such that $\gamma = \alpha + \beta$.

ϕ induces an injection of formal algebras : $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ and for the image $\mathcal{E}_i = Im_\phi(\mathcal{E}_0)$ one can consider its inverse $\phi^{-1} : \mathcal{E}_i \longrightarrow \mathcal{E}_0$.

Notice that one must distinguish two classes of embeddings: when the scalar product (defined by the Killing form) in the root space P_0 is invariant with respect to ϕ and when it is not ϕ-invariant. The first embedding is called "metric" , the second – "nonmetric".

Definition 2.2. *A root system Δ "splinters" as (Δ_1, Δ_2) if there are two embeddings $\phi_1 : \Delta_1 \hookrightarrow \Delta$ and $\phi_2 : \Delta_2 \hookrightarrow \Delta$ where (a) Δ is the disjoint union of the images of ϕ_1 and ϕ_2 and (b) neither the rank of Δ_1 nor the rank of Δ_2 exceeds the rank of Δ.*

It is equivalent to say that (Δ_1, Δ_2) is a "splint" of Δ and we shall denote this by $\Delta \approx (\Delta_1, \Delta_2)$. Each component Δ_1 and Δ_2 is a "stem" of the splint (Δ_1, Δ_2).

To study relations between injection fan technique and splint let us consider the case when one of the stems $\Delta_1 = \Delta_{\mathfrak{a}}$ is a root subsystem.

Splint $\Delta \approx (\Delta_1, \Delta_2)$ is called "injective" if $\Delta_1 = \Delta_{\mathfrak{a}}$, is a root subsystem in Δ corresponding to a regular reductive subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$.

In case of injective splint the second stem $\Delta_{\mathfrak{s}} := \Delta_2 = \Delta \setminus \Delta_{\mathfrak{a}}$ can be translated into a product (8) and it defines an injection fan $\Gamma_{\mathfrak{a} \hookrightarrow \mathfrak{g}}$. Denote by $\Delta_{\mathfrak{s}0}$ the coimage of the second embedding $\phi : \Delta_{\mathfrak{s}0} \to \Delta_{\mathfrak{g}}$. The following conjecture follows.

Conjecture 2.3. *Each injective splint $\Delta \approx (\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$ defines an injection fan with the carrier $\{\xi\}_{\mathfrak{a} \to \mathfrak{g}}$ fixed by the product*

$$\prod_{\beta \in \Delta_{\mathfrak{s}}^+} \left(1 - e^{-\beta}\right) = -\sum_{\gamma \in P} s(\gamma) e^{-\gamma} \tag{10}$$

In case of injective splint we say that subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ splinters Δ (and call $\mathfrak{a}$ the "splinting subalgebra" of $\mathfrak{g}$). In [1] splints are classified (see Appendix there) and the first three types of them are injective.

3 How stems define multiplicity functions

In this Section we study properties of injective splints $\Delta \approx (\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$. It will be demonstrated that in this case to find branching coefficients for a splinting injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ means to find weight multiplicities of an irreducible $\mathfrak{s}$-module $L_{\mathfrak{s}}^{\nu}$ with fixed highest weight ν. Notice that $\mathfrak{s}$ must not be a subalgebra of $\mathfrak{g}$.

Let us return to relation (3) and multiply both sides by $R_{\mathfrak{a}}$:

$$\frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}}^{+}} (1 - e^{-\beta})} \Psi_{\mathfrak{g}}^{(\mu)} = \sum_{\nu \in P_{\mathfrak{a}}^{+}} b_{\nu}^{(\mu)} \Psi_{\mathfrak{a}}^{(\nu)}. \tag{11}$$

Here the first factor in the l.h.s. is the inverse of the fan $\Gamma_{\mathfrak{a} \to \mathfrak{g}}$. Consider the highest weight module $L_{\mathfrak{s}}^{\nu}$. The embedding $\phi : \Delta_{\mathfrak{s}\,0} \longrightarrow \Delta_{\mathfrak{g}}$ sends the singular element $\Psi_{\mathfrak{s}}^{(\nu)}$ into $\Psi_{\mathfrak{g}}^{(\mu)}$. Applying the inverse morphism ϕ^{-1} to the product $\left(\prod_{\beta \in \Delta_{\mathfrak{s}}^{+}} (1 - e^{-\beta})\right)^{-1} \phi\left(\Psi_{\mathfrak{s}}^{(\nu)}\right)$ one gets the character of the module $L_{\mathfrak{s}}^{\nu}$,

$$\phi^{-1}\left(\frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}}^{+}} (1 - e^{-\beta})} \phi\left(\Psi_{\mathfrak{s}}^{(\nu)}\right)\right) = \frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}0}^{+}} (1 - e^{-\beta})} \Psi_{\mathfrak{s}}^{(\nu)} = \mathrm{ch}\left(L_{\mathfrak{s}}^{\nu}\right). \tag{12}$$

Our task is to prove that the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ contains the element $\Psi_{\mathfrak{s}}^{(\xi)}$ for a module $L_{\mathfrak{s}}^{\xi}$ uniquely defined by $L_{\mathfrak{g}}^{\mu}$ and that the branching coefficients $b_{\nu}^{(\mu)}$ in the r.h.s. of (11) coincide with multiplicities $m_{\zeta}^{(\xi)}$ of the corresponding weights in $\mathcal{N}_{\mathfrak{s}}^{\xi}$.

For a highest weight irreducible module $L_{\mathfrak{g}}^{\mu}$ the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ is an element of $\mathcal{E}$ corresponding to the shifted Weyl-orbit of the weight $(\mu + \rho) \in P^{+}$ with the sign function $\epsilon(w)$. It is convenient to use also unshifted singular elements

$$\Phi^{(\mu)} := \Psi^{(\mu)} e^{\rho}. \tag{13}$$

In these terms the relation (11) looks like

$$\frac{e^{\rho_{\mathfrak{g}}-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta_{\mathfrak{s}}^{+}}(1-e^{-\beta})}\Phi_{\mathfrak{g}}^{(\mu)}=\sum_{\nu\in P_{\mathfrak{a}}^{+}}b_{\nu}^{(\mu)}\Phi_{\mathfrak{a}}^{(\nu)}. \tag{14}$$

The orbit related to $\Phi_{\mathfrak{g}}^{(\mu)}$ is completely defined by the set of edges $\{\lambda_i\}_{i=1,\dots,r}$ adjusted to the end of the highest weight vector $\mu+\rho$. For $\mu=\sum m_i\omega_i$ these edges are

$$\lambda_i=-\left(m_i+1\right)\alpha_i,\quad i=1,\dots,r. \tag{15}$$

Each formal exponent $e^{\mu+\rho+\lambda_i}$ in $\Phi_{\mathfrak{g}}^{(\mu)}$ bears the sign coefficient $\epsilon=(-)$. The defining property of $\Phi_{\mathfrak{g}}^{(\mu)}$ is as follows. Consider any pair of edges λ_i,λ_j and the corresponding weights $\mu+\rho$, $\mu+\rho+\lambda_i$ and $\mu+\rho+\lambda_j$. Apply the reflection s_{α_i} (or s_{α_j}),

$$s_{\alpha_i}\circ\left\{\begin{array}{l}(\mu+\rho)\\(\mu+\rho+\lambda_i)\\(\mu+\rho+\lambda_j)\end{array}\right.=\left\{\begin{array}{l}(\mu+\rho+\lambda_i)\\(\mu+\rho)\\(\mu+\rho+\lambda_i-(m_j+1)s_{\alpha_i}\circ\alpha_j)\end{array}\right. \tag{16}$$

Property 3.1. *The edge $\lambda_{i,j}$ of $\Phi_{\mathfrak{g}}^{(\mu)}$ starting at the weight $(\mu+\rho+\lambda_i)$ along the root $-s_{\alpha_i}\circ\alpha_j$ has the same length in $(s_{\alpha_i}\circ\alpha_j)$ as λ_j has in α_j. (The same is true for the edge $\lambda_{j,i}$, its length in $(s_{\alpha_j}\circ\alpha_i)$ is equal to the length of λ_i in α_i.)*

In $\Phi_{\mathfrak{g}}^{(\mu)}$ the elements $e^{\left(\mu+\rho+\lambda_i-(m_j+1)s_{\alpha_i}\circ\alpha_j\right)}$ and $e^{\left(\mu+\rho+\lambda_j-(m_i+1)s_{\alpha_j}\circ\alpha_i\right)}$ have the sign coefficient $\epsilon=(+)$.

Remember that only three types of splints are injective and thus are naturally connected with branching. Below we reproduce the part of the splints table from [1] corresponding to injective splints:

type	Δ	$\Delta_{\mathfrak{a}}$	$\Delta_{\mathfrak{s}}$
(i)	G_2	A_2	A_2
	F_4	D_4	D_4
(ii)	$B_r(r\geq 2)$	D_r	$\oplus^r A_1$
	$C_r(r\geq 3)$	D_r	$\oplus^r A_1$
(iii)	$A_r(r\geq 2)$	$A_{r-1}\oplus u(1)$	$\oplus^r A_1$
	B_2	$A_1\oplus u(1)$	A_2

Each row in the table gives a splint $(\Delta_{\mathfrak{a}},\Delta_{\mathfrak{s}})$ of the simple root system Δ. In the first two types both $\Delta_{\mathfrak{a}}$ and $\Delta_{\mathfrak{s}}$ are embedded metrically. Stems in the first type splints are equivalent and in the second are not. In the third type splints only $\Delta_{\mathfrak{a}}$ is embedded metrically. The summands $u(1)$ are added to keep $r_{\mathfrak{a}} = r$. This does not change the principle properties of branching but makes it possible to use the multiplicities of $\mathfrak{s}$ -modules without further projecting their weights.

Splints induce a decomposition of the set $S = S_{\mathfrak{c}} \cup S_{\mathfrak{d}}$ with $S_{\mathfrak{c}} = S \cap S_{\mathfrak{a}}$ and $S_{\mathfrak{d}} = S \cap S_{\mathfrak{s}}$. It is easy to check that for any injective splint the subset $S_{\mathfrak{d}}$ is nonempty. It follows that in the set $\{\lambda_i\}_{i=1,\ldots,r}$ one can always find simple roots $\beta_k \in \Delta_{\mathfrak{s}}$ and that the orbit corresponding to $\Phi_{\mathfrak{g}}^{(\mu)}$ contains the edges

$$\lambda_k = -(m_k + 1)\,\beta_k \tag{17}$$

attached to the weight $\mu + \rho$. As far as $\Delta_{\mathfrak{a}}$ is a root system and for any pair of simple roots from $S_{\mathfrak{c}}$ the property 3.1 is fulfilled, the element $\Phi_{\mathfrak{g}}^{(\mu)}$ being a singular element for a set of $\mathfrak{a}$-modules. Consider $\beta_l \in \Delta_{\mathfrak{s}}$ whose coimage in $\Delta_{\mathfrak{s}0}$ is simple. In Appendix it is shown that for any such β_l there exists a root $\alpha_l \in S_{\mathfrak{c}}$ such that $\beta_l = \alpha_l + \beta_k$. It is easily seen that the corresponding edge intersects the boundary plane of the fundamental chamber $\bar{C}_{\mathfrak{a}}$ orthogonal to the root α_l,

$$s_{\alpha_l}(\mu + \rho - p\beta_l) = s_{\alpha_l}(\mu + \rho) - p s_{\alpha_l}\beta_l = \mu + \rho - p\beta_l, \tag{18}$$

$$\mu + \rho - s_{\alpha_l}(\mu + \rho) = (m_l + 1)\,\alpha_l = (m_l + 1)\,\beta_l - (m_l + 1)\,\beta_k = p\beta_l - p s_{\alpha_l}\beta_l. \tag{19}$$

It follows that $p = (m_l + 1)$ and $s_{\alpha_l}\beta_l = \beta_k$. Now apply the operator s_{β_k} and find that the edge along the root $s_{\beta_k}\alpha_l$ attached at the weight $s_{\beta_k}(\mu+\rho)$ is also equal to $-p s_{\beta_k}\alpha_l$. This means that for the triple of roots β_k, β_l and $s_{\beta_k}\alpha_l$ in $\Delta_{\mathfrak{s}}$ the edges $\lambda_k = -(m_k + 1)\,\beta_k$, $\lambda_l = -(m_l + 1)\,\beta_l$ and $\lambda_{kl} = -(m_l + 1)\,s_{\beta_k}\alpha_l$ demonstrate the property 3.1. One can continue this procedure further in the 2-dimensional subspace fixed by the roots β_k and β_l and find the set of formal exponents that being supplied with the corresponding sign factors compose the coimage of the singular element of a module for the subalgebra in $\mathfrak{s}$ (this subalgebra has rank $r = 2$).

The same can be proven for any positive root $\beta_l \in \Delta$ that is simple in $\Delta_{\mathfrak{s}0}$ and correspondingly for any $r = 2$ subalgebra in $\mathfrak{s}$. The latter means that to "find" a singular element of $\mathfrak{s}$-module in $\Phi_{\mathfrak{g}}^{(\mu)}$ it is necessary to incorporate in it additional formal elements $\left\{-e^{\mu+\rho-(m_l+1)\beta_l} | \beta_l \in S_{\mathfrak{c}}\right\}$. This fixes the starting edges of the diagram $\phi^{-1}\left(\Phi_{\mathfrak{s}}^{\widetilde{\mu}}\right)$. As it follows from the reconstruction procedure the highest weight $\widetilde{\mu}$ is totally defined by the weight μ, they have the same Dynkin numbers:

$$\mu=\sum m_k\omega_k \qquad \Longrightarrow \qquad \widetilde{\mu}=\sum m_k\widetilde{\omega}_k. \tag{20}$$

The next step is to construct the full $W_{\mathfrak{s}}$-orbit $\Phi_{\mathfrak{s}}^{(\widetilde{\mu})}$ in $P_{\mathfrak{s}}$. It is easily seen that its coimage belongs to $\bar{C}_{\mathfrak{a}}$ and that the set $\phi^{-1}\left(\Phi_{\mathfrak{s}}^{\widetilde{\mu}}\right)\setminus\Phi_{\mathfrak{g}}^{(\mu)}|_{\bar{C}_{\mathfrak{a}}}$ corresponds to the weights belonging to the boundary $\bar{C}_{\mathfrak{a}}$ (including the subset $\left\{-e^{\mu+\rho-(m_l+1)\beta_l}|\beta_l\in S_{\mathfrak{c}}\right\}$). Thus we have constructed all the formal elements with the appropriate sign factors that after being added to $\Phi_{\mathfrak{g}}^{(\mu)}|_{\bar{C}_{\mathfrak{a}}}$ form the diagram $\phi^{-1}\left(\Phi_{\mathfrak{s}}^{\widetilde{\mu}}\right)$ in $\bar{C}_{\mathfrak{a}}$.

Now let us return to relation (14). One can add to $\Phi_{\mathfrak{g}}^{(\mu)}$ pairs of formal elements constructed above with the opposite signs: $\epsilon\left(w\right)|_{w\in W_{\mathfrak{s}}}$ and $-\epsilon\left(w\right)|_{w\in W_{\mathfrak{s}}}$. Attribute the signs $\epsilon\left(w\right)|_{w\in W_{\mathfrak{s}}}$ to the elements whose weights we shall attribute to $\bar{C}_{\mathfrak{a}}$. The same elements with the opposite signs are to be referred to the neighboring Weyl chambers of $\bar{C}_{\mathfrak{a}}^{(l)}$ (the latter are connected with the main one via simple reflections s_{α_l} so the signes $-\epsilon\left(w\right)|_{w\in W_{\mathfrak{s}}}$ are natural for them). In fact one can repeat the procedure and find additional singular weights in any Weyl chamber $\bar{C}_{\mathfrak{a}}^{(m)}$ and in them additional singular weights always have the signs opposite to that in their nearest neighbors. Thus without changing the element $\Phi_{\mathfrak{g}}^{(\mu)}$ one can present it as a sum

$$\Phi_{\mathfrak{g}}^{(\mu)}=\sum_{w\in W_{\mathfrak{a}}} w\circ\left(e^{\rho_{\mathfrak{a}}}\Psi^{\widetilde{\mu}+\rho_{\mathfrak{s}}}\right) \tag{21}$$

where the weight $\widetilde{\mu}=\sum m_k\omega_{\mathfrak{s}}^k$ was defined above. The decomposition (21) provides the possibility to apply the factor $\left(\prod_{\beta\in\Delta_{\mathfrak{s}}^+}(1-e^{-\beta})\right)^{-1}$ to each summand of the singular element $\Phi_{\mathfrak{g}}^{(\mu)}$ separately because the sets of weights from different Weyl summands do not intersect. Taking into account the isomorphism ϕ one can see that in the main Weyl chamber $\bar{C}_{\mathfrak{a}}$ the set of weights generated by the factor $\left(\prod_{\beta\in\Delta_{\mathfrak{s}}^+}(1-e^{-\beta})\right)^{-1}$ is isomorphic to the weight diagram $\mathcal{N}_{\mathfrak{s}}^{\widetilde{\mu}}$ of the $\mathfrak{s}$-module $L_{\mathfrak{s}}^{\widetilde{\mu}}$. Now one can restrict relation (14) to $\bar{C}_{\mathfrak{a}}$ and obtain the main result:

Property 3.2.

$$\frac{e^{\rho_{\mathfrak{g}}}}{\prod_{\beta\in\Delta_{\mathfrak{s}}^+}(1-e^{-\beta})}\left(\Psi^{\widetilde{\mu}+\rho_{\mathfrak{s}}}\right)=\sum_{\widetilde{\nu}\in\mathcal{N}_{\mathfrak{s}}^{\widetilde{\mu}}} M_{(\mathfrak{s})\widetilde{\nu}}^{\widetilde{\mu}} e^{(\mu-\phi(\widetilde{\mu}-\widetilde{\nu}))}=\sum_{\nu\in P_{\mathfrak{a}}^{++}} b_{\nu}^{(\mu)}e^{\nu}. \tag{22}$$

Any weight with nonzero multiplicity in the r. h. s. is equal to one of the highest weights in the decomposition. The multiplicity $M^{\widetilde{\mu}}_{(\mathfrak{s})\widetilde{\nu}}$ of the weight $\widetilde{\nu} \in \mathcal{N}^{\widetilde{\mu}}_{\mathfrak{s}}$ defines the branching coefficient $b^{(\mu)}_{\nu}$ for the highest weight $\nu = (\mu - \phi(\widetilde{\mu} - \widetilde{\nu}))$:

$$b^{(\mu)}_{(\mu-\phi(\widetilde{\mu}-\widetilde{\nu}))} = M^{\widetilde{\mu}}_{(\mathfrak{s})\widetilde{\nu}}.$$

4 Examples

Example 4.1. *Consider the Lie algebra $A_2 = \mathbf{sl(3)}$ and branching of its irreducible module $L^{[3,2]}_{A_2}$ with respect to the reductive subalgebra $A_1 \oplus u(1)$. The root system $\Delta_{\mathfrak{a}} = \Delta_{A_1 \oplus u(1)}$ contains the simple root of $\alpha_1 = e_1 - e_2$ of A_2. The singular element $\Psi^{[3,2]}_{\mathfrak{a}}$ is decomposed into a sum of splint images of singular elements of $A_1 \oplus A_1$-modules. Branching coefficients $b^{[3,2]}_{\nu}$ coincide with weight multiplicities of $L^{[3,2]}_{A_1 \oplus A_1}$-module (see Fig. 1).*

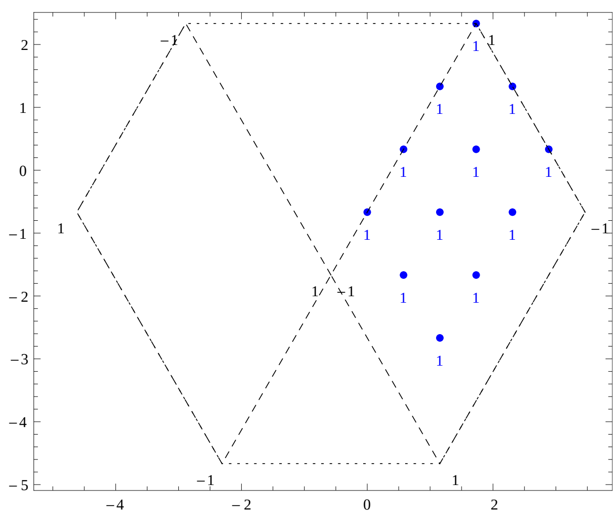

Figure 1: Weyl group orbit (dotted) producing singular element of $L^{[3,2]}_{A_2}$ and its decomposition into the sum of splint images of singular elements of modules $L^{[3,2]}_{A_1 \oplus A_1}$ (dashed). Weight multiplicities of $L^{[3,2]}_{A_1 \oplus A_1}$-module coincide with branching coefficients for the reduction $L^{[3,2]}_{A_2 \downarrow A_1 \oplus u(1)}$.

Example 4.2. *For the Lie algebra $B_2 = \mathbf{so(5)}$ branching of its irreducible module $L^{[3,2]}$ into modules of a reductive subalgebra $A_1 \oplus u(1)$ with the root* *system spanned by the first simple root $\alpha_1 = e_1 - e_2$ of B_2. Singular element of $\Psi_{B_2}^{[3,2]}$ is decomposed into the sum of splint images of singular elements of A_2-modules and branching coefficients coincide with weight multiplicities of A_2-module (see Fig. 2).*

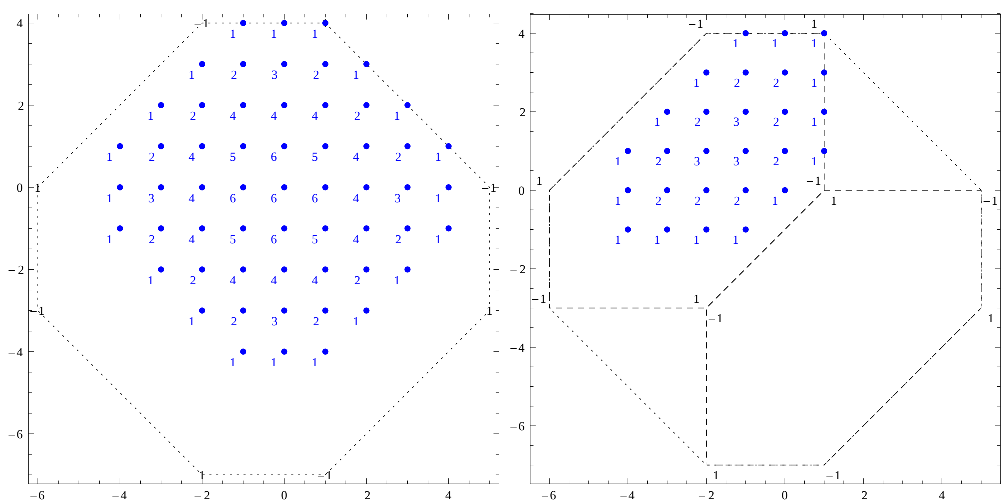

Figure 2: Weights of the B_2-module $L^{[3,2]}$ are indicated by dots in the left picture (their multiplicities are also indicated). Contour of the singular element is shown by dotted line. The right picture presents the decomposition of $\Psi_{B_2}(L_{B_2}^{[3,2]})$-singular element into the sum of splint images of singular elements $\Psi_{A_2}(L^{[3,2]})$ (dashed). Weight multiplicities of $L_{A_2}^{[3,2]}$-module coincide with branching coefficients for the reduction $L_{B_2 \downarrow A_1 \oplus u(1)}^{[3,2]}$.

Example 4.3. *Lie algebra G_2 has a regular subalgebra A_2 with root system $\Delta_{\mathfrak{a}} = \Delta_{A_2}$ containing the G_2 long roots. Consider branching of an irreducible module $L_{G_2}^{(3,2)}$ into the A_2-modules. Singular element $\Psi_{G_2}(L^{[3,2]})$ is decomposed into the sum of splint images of singular elements $\Psi_{A_2}(L^{[3,2]})$ and the corresponding branching coefficients coincide with weight multiplicities of $L_{A_2}^{[3,2]}$-module (see Fig. 3).*

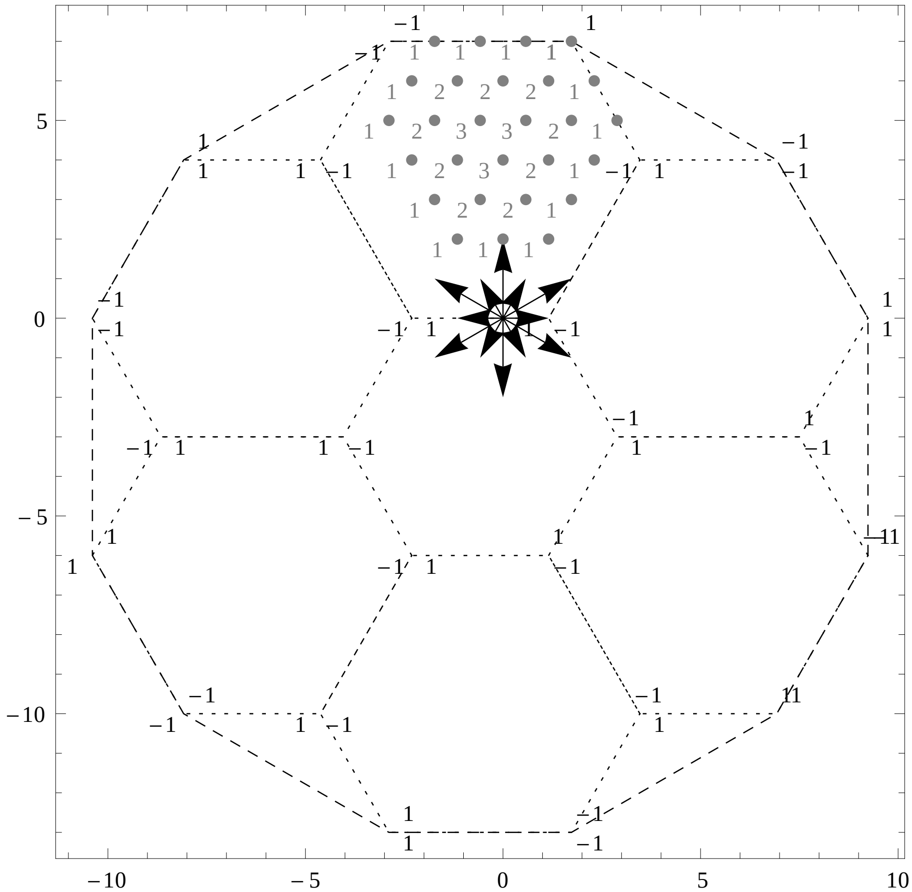

Figure 3: Weyl group orbit (dotted) for the singular element $\Psi_{G_2}(L^{[3,2]})$ and its decomposition into the sum of splint images of singular elements of A_2-modules (dashed). Weight multiplicities of $L^{[3,2]}_{A_2}$-module coincide with branching coefficients for the reduction $L^{[3,2]}_{G_2\downarrow A_2}$.

5 Conclusions

It is explicitly demonstrated that splint presents a very effective tool to find branching coefficients. In particular injective splints provide a possibility to reduce branching rules calculations for highest weight modules to a determination of weight multiplicities for a module with the same Dynkin labels referred to the Lie algebra $\mathfrak{s}$. This algebra $\mathfrak{s}$ must not be a subalgebra in the initial $\mathfrak{g}$, it has the same rank $r_{\mathfrak{s}} = r$, but obviously less "complicated" than

$\mathfrak{g}$ – only a subset of the initial root system is involved in the second stem $\Delta_{\mathfrak{s}}$.

It is significant that for the injections $D_r \hookrightarrow B_r$, $D_r \hookrightarrow C_r$ and $A_{r-1} \oplus u(1) \hookrightarrow A_r$ splint technique shows transparently Gelfand-Tzeytlin rules for branching: the reduction is multiplicity free (all nonzero branching coefficients are equal to 1). Here it is an immediate consequence of the structure of the second stem being a direct sum of A_1 algebras and the fact that the corresponding module $L_{\mathfrak{s}}^{\mu}$ is irreducible.

Acknowledgements

The work was supported in part by the RFFI grant N 09-01-00504. A.A.N. thanks the Chebyshev Laboratory (Department of Mathematics and Mechanics, Saint-Petersburg State University) for support under the grant 11.G34.31.0026 of the Government of Russian Federation.

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Appendix

Let us demonstrate that for injective splints of classical Lie algebras the following property is valid:

Property 5.1. *Let $\Delta \approx (\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$ be an injective splint with the decomposition of simple roots $S = S_{\mathfrak{c}} \cup S_{\mathfrak{d}}$ with $S_{\mathfrak{c}} = S \cap S_{\mathfrak{a}}$ and $S_{\mathfrak{d}} = S \cap S_{\mathfrak{s}}$.*

Thus for any simple root $\beta \in S_{\mathfrak{s}}$ there exists the pair of roots (α ,β') with $\alpha \in S_{\mathfrak{c}}, \beta' \in S_{\mathfrak{s}}$ such that $\alpha = \beta - \beta'$

- Type 1. $\Delta_{G_2} \approx (\Delta_{A_2}, \Delta_{A_2})$.

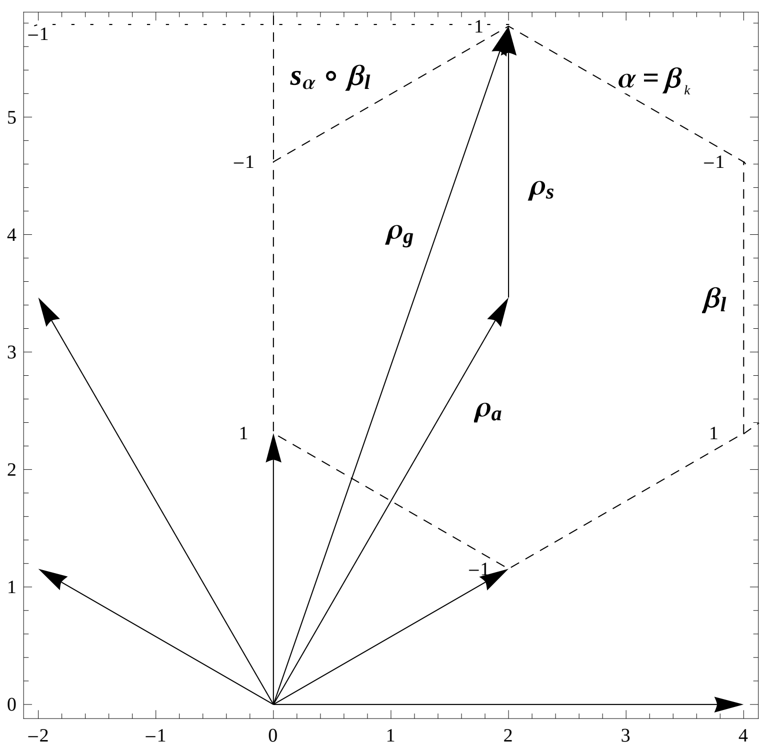

Figure 4: Positive roots of G_2 and formation of singular element $\Phi_{\mathfrak{s}}^{(0)}$ in the main Weyl chamber of $\mathfrak{a} = A_2$.

Here both stems are metric and the corresponding root systems are equivalent. In Figure 4 a part of the singular element $\Phi_{G_2}^{(0)}$ is presented. The boundaries of $\bar{C}_{\mathfrak{a}}$ are the dashed lines starting at the center of the singular element. It contains the edge $\lambda_2 = -\alpha_2 = -\beta_2$ and the roots $-\beta_1 = -s_{\alpha_2} \circ \beta_3$ and $-\beta_3$ (β_3 is indicated as β_l). For the root β_1 the necessary pair is (α_1, β_2): $\alpha_1 = \beta_1 - \beta_2$. The $\lambda^{\mathfrak{s}}_{2,3} = \beta_3$ edge is equal to $\lambda^{\mathfrak{s}}_1 = \beta_1 = s_{\alpha_2} \circ \beta_3$ and m_1 index is aquired by the $\mathfrak{s}$-module that also inherit the second index m_2. In this particular case they are $m_1 = m_2 = 0$. The general case with the initial module L^{μ} and $\mu = m_1\omega_1 + m_2\omega_2$ can be treated in the same way: one finds an edge $\lambda_2 = -(m_2 + 1)\beta_2$ and put $\lambda^{\mathfrak{s}}_1 = -(m_1 + 1)\beta_1$, its end belongs to the boundary $\bar{C}_{\mathfrak{a}}$. The reflection s_{β_2} sends β_1 to β_3 and the corresponding edge $\lambda^{\mathfrak{s}}_{2,3} = -(m_1 + 1)\beta_3$ has the length $(m_1 + 1)$. Now consider $\lambda^{\mathfrak{s}}_1$ (or $\lambda^{\mathfrak{s}}_{2,3}$) and $\lambda^{\mathfrak{s}}_{1,3}$ (or $\lambda^{\mathfrak{s}}_{2,3,1}$) edges to find that they belong to the boundary $\bar{C}_{\mathfrak{a}}$ and the Weyl symmetry predicts that $\lambda^{\mathfrak{s}}_{1,3} = -(m_2 + 1)\beta_3$ ($\lambda^{\mathfrak{s}}_{2,3,1} = -(m_2 + 1)\beta_1$) . Finally the edge $\lambda^{\mathfrak{s}}_{1,3,2} = -(m_1 + 1)\beta_2$ closes the polytope. Its vertices correspond to weights of the singular element $\Phi^{(\widetilde{\mu})}_{\mathfrak{s}} = \sum_{w \in W_{\mathfrak{s}}} \varepsilon(w)\, e^{w \circ (\widetilde{\mu} + \rho_{\mathfrak{s}})}$ of the module $L^{(\widetilde{\mu})}_{\mathfrak{s}}$ with $\widetilde{\mu} = m_1\widetilde{\omega}_1 + m_2\widetilde{\omega}_2$. Notice that in this case the sign factors can be obtained directly in the initial weight system as far as the stem is metric.

- Type 1. $\Delta_{F_4} \approx (\Delta_{D_4}, \Delta_{D_4})$.

  Both stems are metric here and the corresponding root systems are equivalent. The system Δ_{D_4} of the subalgebra $\mathfrak{a} = D_4$ is formed by the set $\{\pm e_i \pm e_j\}_{|i,j=1,\ldots 4,\ i \neq j}$. The simple roots $S_{\mathfrak{c}}$ are $\{e_2 - e_3, e_3 - e_4\}$ and $S_{\mathfrak{d}} = \left\{e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\right\}$. For a module L^{μ} with $\mu = \sum m_k\omega_k$ consider the edge $\lambda_3 = -(m_3 + 1)e_4 = -(m_3 + 1)\beta_3$. Compose an edge $\lambda^{\mathfrak{s}}_2 = -(\widetilde{m}_2 + 1)\beta_2$. The necessary pair of roots is $(\alpha_2 = e_3 - e_4, \beta_3)$. The intersection of $\lambda^{\mathfrak{s}}_2$ with the α_2 -boundary of $\bar{C}_{\mathfrak{a}}$ fixes its length to be $\lambda^{\mathfrak{s}}_2 = -(m_2 + 1)\beta_2$ and the length of the edge $\lambda^{\mathfrak{s}}_{3,2}$ is equal to that of $\lambda^{\mathfrak{s}}_2$. Next consider the edge $\lambda^{\mathfrak{s}}_2 = -(m_2 + 1)\beta_2$ and the pair $(\alpha_1 = e_2 - e_3, \beta_1 = e_2)$. The length of $\lambda^{\mathfrak{s}}_1$ becomes equal to $\lambda^{\mathfrak{s}}_1 = -(m_1 + 1)\beta_1$. Proceed further till the closure of the polytope. The edges looking along the roots of the α_4-type, $\alpha_4 = \beta_4 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$, are treated similarly and finally the singular element $\Phi^{(\widetilde{\mu})}_{\mathfrak{s}} = \sum_{w \in W_{\mathfrak{s}}} \varepsilon(w)\, e^{w \circ (\widetilde{\mu} + \rho_{\mathfrak{s}})}$ for the module $L^{(\widetilde{\mu})}_{\mathfrak{s}}$ with $\widetilde{\mu} = \sum m_k\widetilde{\omega}_k$ is formed in $\bar{C}_{\mathfrak{a}}$.

- Type 2. $\Delta_{B_r} \approx (\Delta_{D_r}, \Delta_{\oplus^r A_1})$.

  Both stems are metric. An injection is fixed by the stem Δ_{D_r} simple roots $S_{\mathfrak{a}} = \{e_1 - e_2, e_2 - e_3, \ldots, e_{r-1} - e_r, e_{r-1} + e_r\}$. The second stem corresponds to a direct sum of algebras A_1 with the simple roots $S_{\mathfrak{s}} = \{e_1, e_2, \ldots, e_{r-1}, e_r\}$. Consider the edge $\lambda_r = -(m_r + 1)\beta_r$ (here $\beta_r = e_r$) and $\lambda_{r-1} = -(\widetilde{m}_{r-1} + 1)\beta_{r-1}$ attached to it (here $\beta_{r-1} = e_{r-1}$). The corresponding pair is $(\alpha_{r-1} = e_{r-1} - e_r, \beta_{r-1} = e_{r-1})$. The intersection condition fixes the second edge to be $\lambda_{r-1} = -(m_{r-1} + 1)\beta_{r-1}$, it is orthogonal to β_r so the opposite edge has the same length. The Dynkin index m_{r-1} now refers also to the simple root β_{r-1} . Next consider the obtained edge $\lambda_{r-1} = -(m_{r-1} + 1)\beta_{r-1}$ and $\lambda_{r-2} = -(\widetilde{m}_{r-2} + 1)\beta_{r-2}$ to fix the index $\widetilde{m}_{r-2} = m_{r-2}$ and the edge $\lambda_{r-2} = -(m_{r-2} + 1)\beta_{r-2}$ and so on till all the pairs of edges are properly fixed. Finally in $\bar{C}_{D_r}$ the element $\Phi^{(\widetilde{\mu})}_{\oplus^r A_1} = \sum_{w \in W_{\oplus^r A_1}} \varepsilon(w)\, e^{w \circ \left(\widetilde{\mu} + \frac{1}{2}\sum e_k\right)}$ can be constructed for the module $L^{(\widetilde{\mu})}_{\oplus^r A_1}$ with $\widetilde{\mu} = \sum m_k \frac{1}{2} e_k$.

- Type 2. $\Delta_{C_r} \approx (\Delta_{D_r}, \Delta_{\oplus^r A_1})$.

  The situation in this case is eqivalent to the previous one and the additional edges are constructed similarly.

- Type 3 $\Delta_{A_r} \approx (\Delta_{A_{r-1} \oplus u_1}, \Delta_{\oplus^r A_1})$.

  Here only the first stem is metric and it fixes the injection with simple roots $S_{\mathfrak{a}} = \{e_1 - e_2, e_2 - e_3, \ldots, e_{r-1} - e_r\}$. The second stem corresponding to a direct sum of r copies of A_1 has the simple roots $S_{\mathfrak{s}} = \{e_1 - e_{r+1}, e_2 - e_{r+1}, \ldots, e_r - e_{r+1}\}$. Consider the edge $\lambda_r = -(m_r + 1)\beta_r$ with $\beta_r = e_r - e_{r+1}$ and $\lambda_{r-1} = -(\widetilde{m}_{r-1} + 1)\beta_{r-1}$ with $\beta_{r-1} = e_{r-1} - e_{r+1}$ attached to it. Then the corresponding pair is $(\alpha_{r-1} = e_{r-1} - e_r, \beta_{r-1} = e_{r-1} - e_{r+1})$. The intersection with the boundary of $\bar{C}_{A_{r-1}}$ orthogonal to α_{r-1} fixes the second edge to be $\lambda_{r-1} = -(m_{r-1} + 1)\beta_{r-1}$. The Dynkin index m_{r-1} is to be used for the fundamental weight ω_{r-1}. The reflection s_{β_r} sends $\lambda_{r-1} = -(m_{r-1} + 1)\beta_{r-1}$ to $\lambda_{r,r-1} = -(m_{r-1} + 1)\beta_{r-1}$. Next consider the obtained edge $\lambda_{r-1} = -(m_{r-1} + 1)\beta_{r-1}$ and $\lambda_{r-2} = -(\widetilde{m}_{r-2} + 1)\beta_{r-2}$ with $\beta_{r-2} = e_{r-2} - e_{r+1}$ to obtain the index $\widetilde{m}_{r-2} = m_{r-2}$ and the edge $\lambda_{r-2} = -(m_{r-2} + 1)\beta_{r-2}$ and so on till all the pairs of edges are properly fixed. Finally in $\bar{C}_{D_r}$ the element $\Phi^{(\widetilde{\mu})}_{\oplus^r A_1} = \sum_{w \in W_{\oplus^r A_1}} \varepsilon(w)\, e^{w \circ (\widetilde{\mu} + \widetilde{\rho})}$ can be constructed for the module $L^{(\widetilde{\mu})}_{\oplus^r A_1}$ with $\widetilde{\mu} = \sum m_k \beta_k$. The simplest case $\Delta_{A_2} \approx (\Delta_{A_1 \oplus u_1}, \Delta_{A_1 \oplus A_1})$ is presented in Example 4.1 and Figure 1.

- Type 3 $\Delta_{B_2} \approx (\Delta_{A_1}, \Delta_{A_2})$.

  This splint is illustrated in Example 4.1 and Figure 1, $S_{A\_1} = \{e_1 - e_2\}$, $S_{A\_2} = \{e_1, e_2\}$. The edge $\lambda_{\alpha_2} = \lambda_{\beta_2} = -(m_2+1)\beta_2$ is followed by $\lambda_{\beta_1} = -(\widetilde{m}_1+1)\beta_1$. Consider the pair $(\alpha_1 = e_1 - e_2, \beta_1 = e_1)$. The end of the edge λ_{β_1} must indicate a weight invariant under the reflexion s_{α_1}. Its length is thus fixed: $\lambda_{\beta_1} = -(m_1+1)\beta_1$. In the coimage of the second stem, that is in the root system Δ_{A_2}, the reflection s_{β_2} sends $\lambda_{\beta_1} = -(m_1+1)\beta_1$ to $\lambda_{2,3}$, thus the latter edge has the same length in $\beta_3 = e_1 + e_3$, we have $\lambda_{2,3} = -(m_1+1)\beta_3$ with $\beta_3 = e_1 + e_3$. The irreducible $\mathfrak{s}$-module has the highest weight $\widetilde{\mu} = m_1\widetilde{\omega}_1 + m_2\widetilde{\omega}_2$. In Figure 1 we see the details of these relations in a particular case where $L_{B_2}^{[3,2]}$ is reduced to a subalgebra $A_1 \oplus u(1)$ and the corresponding highest weights (with their multiplicities) form the diagram $\mathcal{N}_{A_2}^{[3,2]}$.