On singular elements in conformal field theory

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Talk outline

We describe the structure of affine Lie algebra modules and its role in coset construction of conformal field theory models.

- WZNW and coset models of CFT
- Weyl-Kac formula and singular elements of algebra modules
- Singular element decomposition and branching
- Splints, theta functions and branching functions

WZNW-action

$$S = S_0 + k\Gamma, \quad k \in \mathbb{Z} \tag{1}$$

Here S_0 is the action of non-linear sigma model:

$$S_0 = -\frac{k}{8\pi} \int_{S^2} d^2x \; Tr(\partial^\mu g^{-1} \partial_\mu g), \quad g(x) : \mathbb{C} \cup \{\infty\} \sim S^2 \to G \tag{2}$$

We need to add topological Wess-Zumino term:

$$\Gamma = -\frac{i}{24\pi} \int_B \epsilon_{ijk} Tr \left(\tilde{g}^{-1} \frac{\partial \tilde{g}}{\partial y^i} \tilde{g}^{-1} \frac{\partial \tilde{g}}{\partial y^j} \tilde{g}^{-1} \frac{\partial \tilde{g}}{\partial y^k} \right) d^3y \tag{3}$$

Here Γ is defined on 3d manifold B, such that $\partial B = S^2$. $\tilde{g}$ is a continuation of g to B.
$\pi_3(G) = \mathbb{Z} \Rightarrow k \in \mathbb{Z}, \; e^{-S[g]}$ is single-valued.

Affine Lie algebra

- The currents are $J(z) = -k\partial_z g g^{-1} \quad \bar{J}(\bar{z}) = k g^{-1}\partial_{\bar{z}} g$
- We have gauge invariance $g(z,\bar{z}) \to \Omega(z) g(z,\bar{z}) \bar{\Omega}^{-1}(\bar{z})$, where $\Omega,\ \bar{\Omega} \in G$
- Ward identities for $\Omega = 1 + \omega$:

$$\delta_{\omega,\bar{\omega}} \langle X \rangle = -\frac{1}{2\pi i} \oint dz \sum \omega^a \langle J^a X \rangle + \frac{1}{2\pi i} \oint d\bar{z} \sum \bar{\omega}^a \langle \bar{J}^a X \rangle$$

- $J(z) = \sum_a J^a(z) t^a = \sum_a \sum_n J^a_n t^a z^{n-1} \Rightarrow$ commutation relations of affine Lie algebra $\hat{\mathfrak{g}}$:

$$\left[J^a_n, J^b_m \right] = \sum_c i f^{abc} J^c_{n+m} + kn\delta^{ab}\delta_{n+m,0}$$

- Virasoro generators are given by Sugawara construction $L_n = \frac{1}{2(k+h^v)} \sum_a \sum_m : J^a_m J^a_{n-m} : \Leftrightarrow Vir \subset U(\hat{\mathfrak{g}})$.

Primary fields

- Full chiral algebra of the model is $\hat{\mathfrak{g}} \ltimes Vir$:

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \\ [L_n, J^a_m] &= -mJ^a_{n+m} \end{aligned} \tag{4}$$

- $[L_0, L_m] = -mL_m, \quad [L_0, J^a_m] = -mJ^a_m \quad$ – grading
- Primary fields are defined by operator product expansion $J^a_{\mathfrak{g}}(z)\phi_i(w) \sim \frac{-t^a_i\phi_i(w)}{z-w}$.
- Primary fields ϕ_λ correspond to highest weights of representations. Field-state correspondence: $|\lambda\rangle = \lim_{z\to 0}\phi_\lambda(z)\,|\Omega\rangle$:

$$J^a_0\,|\phi_\lambda\rangle = -t^a_\lambda\,|\phi_\lambda\rangle \quad J^a_n\,|\phi_\lambda\rangle = 0 \quad \text{for } n>0$$

$$L_0\,|\phi_\lambda\rangle = \frac{1}{2(k+h^v)}\sum_a J^a_0 J^a_0\,|\phi_\lambda\rangle = \frac{(\lambda,\lambda+2\rho)}{2(k+h^v)}\,|\phi_\lambda\rangle = h_\lambda\,|\phi_\lambda\rangle$$

- Singular vectors $\begin{aligned} &J^a_n\,|v\rangle = 0 \quad \text{for } n>0 \\ &J^+_0\,|v\rangle = 0 \end{aligned}$

Weyl-Kac character formula and singular elements

Verma module

$$M^{\mu} = U(\mathfrak{g}) \underset{U(\mathfrak{b}_+)}{\otimes} D^{\mu}(\mathfrak{b}_+) \quad \text{where} \quad \mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-, \mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$$

$D^{\mu}(\mathfrak{b}_+) : D(E^{\alpha}) = 0,\ D(H) = \mu(H) \quad \forall \alpha > 0.$

$$\mathrm{ch} M^{\mu} = \frac{e^{\mu}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)}} = \frac{e^{\mu}}{\sum_{w \in W} \epsilon(w) e^{w\rho - \rho}}, \quad \epsilon(w) := \det(w)$$

M^{μ} has unique maximal submodule and unique non-trivial factormodule L^{μ} – *irreducible highest weight module*. $L^{\mu} \sim U(\mathfrak{n}_-)/ < \Psi^{\mu} >$.

$$\mathrm{ch} L^{\mu} = \frac{\Psi^{\mu}}{R} = \frac{\sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho}}{\sum_{w \in W} \epsilon(w) e^{w\rho-\rho}} = \sum_{w \in W} \epsilon(w)\, \mathrm{ch} M^{w(\mu+\rho)-\rho} (\mathrm{BGG})$$

Coset-construction and gauged WZNW-model

Let's add pure gauge fields $A, \bar{A}$ with the values in subalgebra $\mathfrak{a} \subset \mathfrak{g}$ to the action:

$$S(g, A) = S_{WZNW}(g) +$$
$$\frac{k}{4\pi} \int d^2 z \left(Tr(Ag^{-1}\bar{\partial} g) - Tr(\bar{A}(\partial g) g^{-1}) + Tr(Ag^{-1}\bar{A}g) - Tr(A\bar{A}) \right)$$

The currents are

$$J_{(\mathfrak{g},\mathfrak{a})} = -k\partial g g^{-1} - kgAg^{-1}$$

Using Ward identities we obtain

$$\left\langle A^b(z)\phi_1 \dots \phi_N \right\rangle = \frac{2}{k + 2h^v_{\mathfrak{a}}} \sum_k \frac{\tilde{t}^b_k}{z - z_k} \left\langle \phi_1 \dots \phi_N \right\rangle$$

Algebraic structure is connected with $\hat{\mathfrak{g}}, \hat{\mathfrak{a}} : \hat{\mathfrak{a}} \subset \hat{\mathfrak{g}}$.
Virasoro generators are given by the difference of Sugawara expressions:

$$L_n = L^{\mathfrak{g}}_n - L^{\mathfrak{a}}_n$$

Primary fields and singular elements

Primary fields are labeled with pairs of weights $(\mu,\nu)\in\mathfrak{h}^*_{\hat{\mathfrak{g}}}\oplus\mathfrak{h}^*_{\hat{\mathfrak{a}}}$ such that $b^{\mu}_{\nu}(q)\neq 0$. Some pairs are equivalent. The equivalence is given by the action of simple currents $(J,\tilde{J})$ such that $h_J-h_{\tilde{J}}=0$.
Conformal weight of primary field is

$$L_0\left|\phi_{(\mu,\nu)}\right\rangle=\left(\frac{1}{2(k+h^v)}\sum_a J^a_0 J^a_0-\frac{1}{2(k+h^v_{\mathfrak{a}})}\sum_b \tilde{J}^b_0\tilde{J}^b_0\right)\left|\phi_\lambda\right\rangle= \left(\frac{(\mu,\mu+2\rho)}{2(k+h^v)}-\frac{(\nu,\nu+2\rho_{\mathfrak{a}})}{2(k+h^v)}\right)\left|\phi_{(\mu,\nu)}\right\rangle \quad (5)$$

We can decompose $\hat{\mathfrak{g}}$ modules

$$L^{\mu}_{\hat{\mathfrak{g}}}=\bigoplus_{\nu} L^{\nu}_{\hat{\mathfrak{a}}}\otimes V^{(\mu,\nu)}$$

Singular element decomposition

We can rewrite the decomposition with characters

$$\pi_{\mathfrak{a}}\left(\frac{\sum_{\omega\in W}\epsilon(\omega)e^{\omega(\mu+\rho)-\rho}}{\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}\right)=\sum_{\nu\in P^+_{\mathfrak{a}}} b^{(\mu)}_{\nu}\frac{\sum_{\omega\in W_{\mathfrak{a}}}\epsilon(\omega)e^{\omega(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta\in\Delta^+_{\mathfrak{a}}}(1-e^{-\beta})^{\mathrm{mult}_{\mathfrak{a}}(\beta)}}.$$

We need to compute branching coefficients. Let us multiply by denominator and rearrange sum as recurrent relation.

Consider roots orthogonal to $\Delta_{\mathfrak{a}}$.

Let $\Delta^+_{\mathfrak{b}}=\left\{\alpha\in\Delta^+_{\mathfrak{g}}:\forall\beta\in\Delta_{\mathfrak{a}};\alpha\perp\beta\right\}$ – subset of positive roots of $\mathfrak{g}$, orthogonal to root system of $\mathfrak{a}$.

Denote by $W_{\mathfrak{b}}$ subgroup of Weyl group W, generated by reflections ω_β, corresponding to roots $\beta\in\Delta^+_{\mathfrak{b}}$.

Subsystem $\Delta_{\mathfrak{b}}$ determines subalgebra $\mathfrak{b}=\mathfrak{a}_{\perp}\subset\mathfrak{g}$.

$\mathfrak{a}, \mathfrak{b}$ – "orthogonal pair" of subalgebras $\mathfrak{g}$, $\mathfrak{b}$ is regular.
Cartan subalgebra is decomposed as $\mathfrak{h} = \mathfrak{h}_{\mathfrak{a}} + \mathfrak{h}_{\perp} + \mathfrak{h}_{\mathfrak{b}}$.
Introduce

$$\mathcal{D}_{\mathfrak{a}} := \rho_{\mathfrak{a}} - \pi_{\mathfrak{a}}\rho.$$
$$\mathcal{D}_{\mathfrak{b}} := \rho_{\mathfrak{b}} - \pi_{\mathfrak{b}}\rho.$$

Lemma

Let $\widetilde{\mathfrak{a}_{\perp}} = \mathfrak{a}_{\perp} \oplus \mathfrak{h}_{\perp}$, $\widetilde{\mathfrak{a}} = \mathfrak{a} \oplus \mathfrak{h}_{\perp}$,
L^{μ} *– irreducible module with singular element* $\Psi^{(\mu)}$,
$R_{\mathfrak{a}_{\perp}}$ *– Weyl denominator for subalgebra* $\mathfrak{a}_{\perp}$. $U \sim W/W_{\mathfrak{a}_{\perp}}$.
Then $\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_{\perp})} = \pi_{\mathfrak{a}}\left(\frac{\Psi^{\mu}_{\mathfrak{g}}}{R_{\mathfrak{a}_{\perp}}}\right)$ *can be present as the sum over* $u \in U$*:*

$$\Psi^{(\mu)}_{(\mathfrak{a},\mathfrak{a}_{\perp})} = \quad \pi_{\mathfrak{a}}\left(\frac{\Psi^{\mu}}{R_{\mathfrak{a}_{\perp}}}\right) = \sum_{u \in U} \epsilon(u) \dim\left(L^{\mu_{\widetilde{\mathfrak{a}_{\perp}}}(u)}_{\widetilde{\mathfrak{a}_{\perp}}}\right) e^{\mu_{\mathfrak{a}}(u)}.$$

Recurrent relations on branching coefficients

$$k_{\xi}^{(\mu)} = -\frac{1}{s\left(\gamma_0\right)}\left(\sum_{u\in W/W_{\mathfrak{b}}} \epsilon(u)\,\dim\left(L_{\mathfrak{b}}^{\pi_{(\mathfrak{b})}[u(\mu+\rho)-\rho]-\mathcal{D}_{\mathfrak{b}}}\right)\right.$$
$$\left.\delta_{\xi-\gamma_0,\pi_{(\mathfrak{a}\oplus\mathfrak{h}_{\perp})}[u(\mu+\rho)-\rho]+\mathcal{D}_{\mathfrak{b}}} + \sum_{\gamma\in\Gamma_{\hat{\mathfrak{a}}\subset\hat{\mathfrak{g}}}} s\left(\gamma+\gamma_0\right) k_{\xi+\gamma}^{(\mu)}\right).$$

The recursion is governed by the set $\Gamma_{\hat{\mathfrak{a}}\subset\hat{\mathfrak{g}}}$ of weights $\{\xi\}$ in expansion

$$\prod_{\alpha\in\Delta^+\setminus\Delta^+_{\mathfrak{b}}}\left(1-e^{-\pi_{\hat{\mathfrak{a}}}\alpha}\right)^{\mathrm{mult}(\alpha)-\mathrm{mult}_{\hat{\mathfrak{a}}}(\pi_{\hat{\mathfrak{a}}}\alpha)} = -\sum_{\gamma\in P_{\hat{\mathfrak{a}}}} s(\gamma)e^{-\gamma}$$

We need to shift weights on γ_0 – minimal weight in $\{\xi\}$, and exclude zero element:

$$\Gamma_{\hat{\mathfrak{a}}\subset\hat{\mathfrak{g}}} = \{\xi-\gamma_0\}\setminus\{0\}.$$

Simple example: $A_1 \subset B_2$

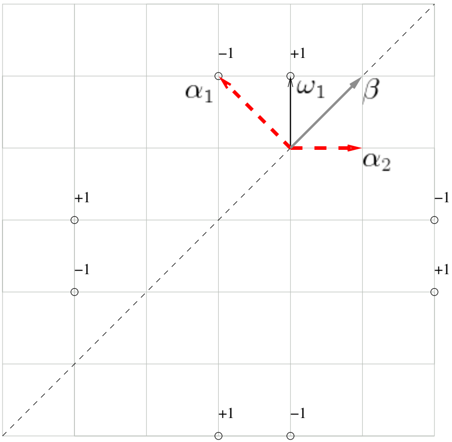


Figure: Roots of B_2, A_1 and Ψ^{ω_1}

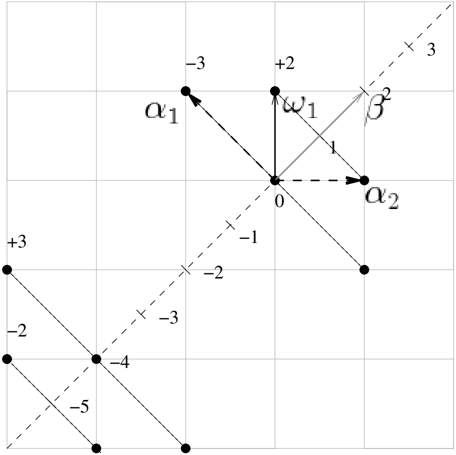
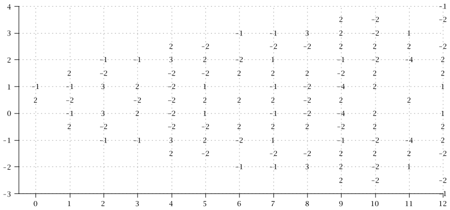


Figure: Orthogonal subalgebra $\mathfrak{b}$ and dimensions of $\mathfrak{b}$-modules



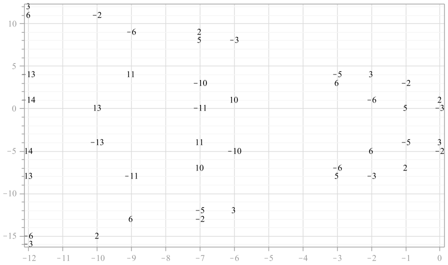
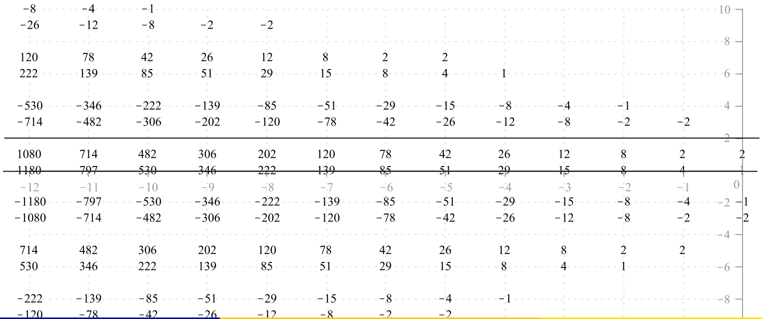


Figure: $\Gamma_{\hat{A}_1 \subset \hat{B}_2}$

Figure: $\pi_{\hat{A}_1}\left(\Psi^{\omega_1}_{\hat{B}_2}\right)$



Splints

Definition

ϕ – "embedding" $\Delta_0 \hookrightarrow \Delta$:
$\phi(\gamma) = \phi(\alpha) + \phi(\beta) \quad \forall \alpha, \beta, \gamma \in P_0 : \gamma = \alpha + \beta.$

ϕ induces embedding of formal algebras: $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ and for $\mathcal{E}_i = \mathrm{Im}_\phi(\mathcal{E}_0)$ and $\phi^{-1} : \mathcal{E}_i \longrightarrow \mathcal{E}_0$.

Definition

Root system Δ "splints" to (Δ_1, Δ_2) if there exist embeddings $\phi_1 : \Delta_1 \hookrightarrow \Delta$ and $\phi_2 : \Delta_2 \hookrightarrow \Delta$ where (a) Δ – disjoint union of images of ϕ_1 and ϕ_2, (b) rank of Δ_1 and rank Δ_2 is less or equal to rank of Δ.

Let $\Delta_1 = \Delta_{\mathfrak{a}}$. $\Delta_{\mathfrak{s}} := \Delta_2 = \Delta \setminus \Delta_{\mathfrak{a}}$ determines injection fan $\Gamma_{\mathfrak{a} \hookrightarrow \mathfrak{g}}$.
$\prod_{\beta \in \Delta_{\mathfrak{s}}^+} \left(1 - e^{-\beta}\right) = -\sum_{\gamma \in P} s(\gamma) e^{-\gamma}$
$\Psi_{\mathfrak{g}}^{(\mu)} = e^{-\rho} \sum_{w \in W_{\mathfrak{a}}} w \circ \left(e^{\rho_{\mathfrak{a}}} \phi_2 \left(\Psi^{\widetilde{\mu} + \rho_{\mathfrak{s}}}\right)\right) \qquad \mu = \sum m_k \omega^k, \; \widetilde{\mu} = \sum m_k \omega_{\mathfrak{s}}^k$

Example

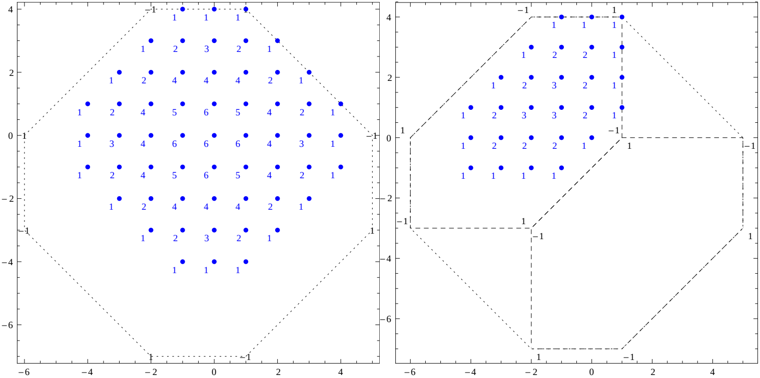


Figure: Weights and multiplicities of $L_{B_2}^{[3,2]}$ on the left. Short dash is a singular element contour. On the right is a decomposition of singular element $\Psi_{B_2}(L_{B_2}^{[3,2]})$ into sum of images of singular elements $\Psi_{A_2}(L^{[3,2]})$ (long dash). $L_{A_2}^{[3,2]}$ multiplicities = branching coefficients for $L_{B_2 \downarrow A_1 \oplus u(1)}^{[3,2]}$.

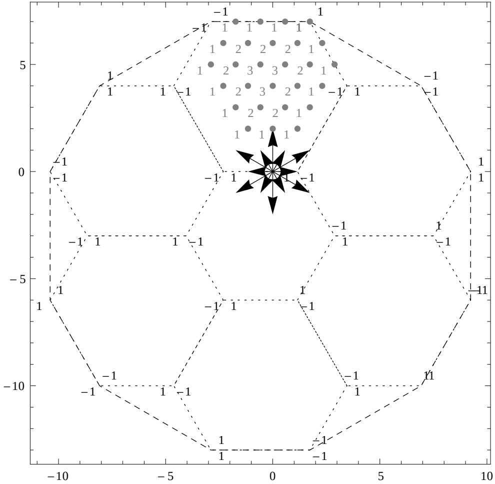


Figure: Weyl group orbit (long dash) for $\Psi_{G_2}(L^{[3,2]})$ and its decomposition into sum of images of singular elements of A_2 modules (short dash). Weight multiplicities of $L^{[3,2]}_{A_2}$ coincide with branching coefficients $L^{[3,2]}_{G_2\downarrow A_2}$.

Affine extension $\hat{\mathfrak{a}} \subset \hat{\mathfrak{g}}$. Since $\mathrm{rank}\mathfrak{g} \leq \mathrm{rank}\mathfrak{a} + \mathrm{rank}\mathfrak{s}$ for Weyl denominators we have

$$\prod_{\alpha \in \hat{\Delta}_1^+} (1 - e^{-\alpha})^{\mathrm{mult}(\alpha)} \prod_{\beta \in \hat{\Delta}_2^+} (1 - e^{\phi \circ \beta})^{\mathrm{mult}(\beta)} =$$

$$\prod_{\gamma \in \hat{\Delta}^+} (1 - e^{-\gamma})^{\mathrm{mult}(\gamma)} \prod_{n=0}^{\infty} (1 - e^{-n\delta})^{\mathrm{rank}\mathfrak{a} + \mathrm{rank}\mathfrak{s} - \mathrm{rank}\mathfrak{g}}$$

$$\Theta^{(\hat{\mathfrak{g}})}_{\widehat{\lambda}=(\lambda,k,0)}(\tau, z) = \sum_{\xi \in Q_{\mathfrak{g}} + \frac{\lambda}{k}} e^{2\pi i k \left(\frac{1}{2}(\xi,\xi)\tau + (\xi,z) \right)}$$

Then we obtain relation on theta-functions of algebras $\hat{\mathfrak{g}}, \hat{\mathfrak{s}}, \hat{\mathfrak{a}}$:

$$\left(\sum_{v \in W_{\mathfrak{a}}} \epsilon(v) \Theta^{(\hat{\mathfrak{a}})}_{v\rho_{\mathfrak{a}}}(\tau, z) \right) \cdot \left(\sum_{u \in W_{\mathfrak{s}}} \epsilon(u) \Theta^{(\hat{\mathfrak{s}})}_{\phi \circ (u\rho_{\mathfrak{s}})}(\tau, z) \right) =$$

$$\left(\sum_{w \in W} \epsilon(w) \Theta^{(\hat{\mathfrak{g}})}_{w\rho_{\mathfrak{g}}}(\tau, z) \right)$$

Branching to finite-dimensional subalgebras

Consider branching of $\hat{\mathfrak{g}}$-module to $\mathfrak{g}$-modules

$$\mathrm{ch} L_{\hat{\mathfrak{g}}}^{\hat{\mu}} = \sum_{n=0}^{\infty} e^{-n\delta} \sum_{\nu \in P} b_{\nu}^{(\hat{\mu})}(n) \mathrm{ch} L_{\mathfrak{g}}^{\nu} \quad m_{\hat{\nu}=(\nu,k,n)}^{(\hat{\mu})} = \sum_{\xi \in P} b_{\xi}^{(\hat{\mu})}(n) m_{\nu}^{(\xi)}$$

Introduce functions $b_{\nu}^{(\hat{\mu})}(q) := \sum_{n=0}^{\infty} b_{\nu}^{(\hat{\mu})}(n) q^n$, which are connected with q-dimension
$\dim_q L_{\hat{\mathfrak{g}}}^{\hat{\mu}} = \sum_{n=0}^{\infty} q^n \sum_{\nu \in P} b_{\nu}^{(\hat{\mu})}(n) \dim L_{\mathfrak{g}}^{\nu} = \sum_{\nu \in P} b_{\nu}^{(\hat{\mu})}(q) \dim L_{\mathfrak{g}}^{\nu}$.
$\sigma_{\nu}^{(\hat{\mu})}(q) = \sum_{\xi \in P} m_{\nu}^{(\xi)} b_{\xi}^{(\hat{\mu})}(q)$.
Introduce the order on the set of the roots ξ: assign to the weight ξ the value (ρ, ξ) and order weights using this values. Then

$$\sigma(q) = M b(q) \quad b(q) = M^{-1} \sigma(q)$$

$\sigma(q)$ and $b(q)$ are infinite vectors of string functions and branching functions. The matrix M contains weight multiplicities of $\mathfrak{g}$-modules. Inverse matrix M^{-1} consists of recurrent relations on weight multiplicities.

Matrix relations for splints

Consider branching of $\hat{\mathfrak{g}}$-modules into $\mathfrak{a}$-modules when there exists a splint $\Delta^+_{\mathfrak{g}} = \Delta^+_{\mathfrak{a}} \cup \phi(\Delta^+_{\mathfrak{s}})$. Decompose $\mathfrak{g}$-modules into $\mathfrak{a}$-modules using splint properties:

$$\mathrm{ch} L^{\hat{\mu}}_{\hat{\mathfrak{g}}} = \sum_{n=0}^{\infty} e^{-n\delta} \sum_{\nu \in P_{\mathfrak{a}}} b^{(\hat{\mu})}_{(\hat{\mathfrak{g}}\downarrow\mathfrak{a})\nu}(n) \mathrm{ch} L^{\nu}_{\mathfrak{a}} =$$

$$\sum_{n=0}^{\infty} e^{-n\delta} \sum_{\nu \in P} b^{(\hat{\mu})}_{(\hat{\mathfrak{g}}\downarrow\mathfrak{g})\nu}(n) \sum_{\xi \in P_{\mathfrak{a}}} b^{(\nu)}_{(\mathfrak{g}\downarrow\mathfrak{a})\xi} \mathrm{ch} L^{\xi}_{\mathfrak{a}} =$$

$$= \sum_{n=0}^{\infty} e^{-n\delta} \sum_{\nu \in P} b^{(\hat{\mu})}_{(\hat{\mathfrak{g}}\downarrow\mathfrak{g})\nu}(n) \sum_{\xi \in P_{\mathfrak{a}}} M^{\widetilde{\nu}}_{\widetilde{\nu}-\phi^{-1}(\nu-\xi)} \mathrm{ch} L^{\xi}_{\mathfrak{a}} \quad (6)$$

Matrix relation holds for branching functions $b_{(\hat{\mathfrak{g}}\downarrow\mathfrak{a})}(q) = M_{\mathfrak{s}}\ b_{(\hat{\mathfrak{g}}\downarrow\mathfrak{g})}(q)$ and $\sigma(q) = M_{\mathfrak{a}}\ b_{(\hat{\mathfrak{g}}\downarrow\mathfrak{a})}(q)$. If we know branching coefficients for the embedding $\mathfrak{g} \subset \hat{\mathfrak{g}}$ we immediately obtain (graded) branching functions for the embedding $\mathfrak{a} \subset \hat{\mathfrak{g}}$.

Conclusion

- Singular elements determine structure of modules
- Decomposition of singular element leads to relations on branching coefficients
- We can get different decompositions and new relations if we consider deformed root subsystems

Thank you!