

Fan, splint and branching rules.

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Abstract

Splint of root system of simple Lie algebra appears naturally in the study of (regular) embeddings of reductive subalgebras. It can be used to derive branching rules. We show that application of splint properties drastically simplifies calculations of branching coefficients.

1 Introduction

Splint ϕ of a root system Δ_1 into a root system Δ is a bijective map of roots of Δ_1 to (proper) subset of Δ which commutes with vector composition law in Δ_1 and Δ .

$$\phi : \Delta_1 \longrightarrow \Delta$$

$$\phi \circ (\alpha + \beta) = \phi \circ \alpha + \phi \circ \beta, \quad \alpha, \beta \in \Delta_1$$

Note that the image $Im(\phi)$ must not inherit the root system properties except the addition rules equivalent to the addition rules in Δ_1 (for pre-images). The term *splint* was introduced by D. Richter in [1] where the classification of splints for simple Lie algebras was obtained. There was also

mentioned that the notion of splint must have tight connections with the injection fan approach. The fan $\Gamma \subset \Delta$ was introduced in [2] as the subset of root system describing recurrent properties of branching coefficients for maximal embeddings. Injection fan is an efficient tool to study branching rules. Later this construction was generalized to non-maximal embeddings and infinite-dimensional Lie algebras in [3, 4].

In the present paper we study the connection of splint with injection fan for regular embeddings of reductive subalgebras $\mathfrak{a} \longrightarrow \mathfrak{g}$. We show that (under certain conditions described in section 3) splint is a natural tool to study reduction properties of simple Lie algebra \mathfrak{g} -modules with respect to a subalgebra $\mathfrak{a} \longrightarrow \mathfrak{g}$. Using this tool we obtain the main result of the present paper – the one-to-one correspondence between weight multiplicities in irreducible modules of splint and branching coefficients for a reduced module $L_{\mathfrak{g} \downarrow \mathfrak{a}}^\mu$.

2 Injections and splints

Consider a simple Lie algebra \mathfrak{g} and its regular subalgebra $\mathfrak{a} \longrightarrow \mathfrak{g}$ such that \mathfrak{a} is a reductive subalgebra $\mathfrak{a} \subset \mathfrak{g}$ with correlated root spaces: $\mathfrak{h}_{\mathfrak{a}}^* \subset \mathfrak{h}_{\mathfrak{g}}^*$. Let \mathfrak{a}^S be a semisimple summand of \mathfrak{a} , this means that $\mathfrak{a} = \mathfrak{a}^S \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \dots$. We shall consider \mathfrak{a}^S to be a proper regular subalgebra and \mathfrak{a} to be the maximal subalgebra with \mathfrak{a}^S fixed that is the rank r of \mathfrak{a} is equal to that of \mathfrak{g} .

- r , $(r_{\mathfrak{a}^S})$ — the rank of \mathfrak{g} (resp. \mathfrak{a}^S);
- Δ ($\Delta_{\mathfrak{a}}$) — the root system; Δ^+ (resp. $\Delta_{\mathfrak{a}}^+$) — the positive root system (of \mathfrak{g} and \mathfrak{a} respectively);
- S ($S_{\mathfrak{a}}$) — the system of simple roots (of \mathfrak{g} and \mathfrak{a} respectively);
- α_i , $(\alpha_{(\mathfrak{a})j})$ — the i -th (resp. j -th) simple root for \mathfrak{g} (resp. \mathfrak{a}); $i = 0, \dots, r$, $(j = 0, \dots, r_{\mathfrak{a}^S})$;
- W , $(W_{\mathfrak{a}})$ — the corresponding Weyl group;
- C , $(C_{\mathfrak{a}})$ — the fundamental Weyl chamber;
- \bar{C} , $(\bar{C}_{\mathfrak{a}})$ — the closure of the fundamental Weyl chamber;
- $\epsilon(w) := (-1)^{\text{length}(w)}$;
- ρ , $(\rho_{\mathfrak{a}})$ — the Weyl vector;
- L^μ ($L_{\mathfrak{a}}^\nu$) — the integrable module of \mathfrak{g} with the highest weight μ ; (resp. integrable \mathfrak{a} -module with the highest weight ν);
- \mathcal{N}^μ , $(\mathcal{N}_{\mathfrak{a}}^\nu)$ — the weight diagram of L^μ (resp. $L_{\mathfrak{a}}^\nu$);
- P (resp. $P_{\mathfrak{a}}$) — the weight lattice;
- P^+ (resp. $P_{\mathfrak{a}}^+$) — the dominant weight lattice;

\mathcal{E} (resp. $\mathcal{E}_{\mathfrak{a}}$) — the formal algebra;
 $m_{\xi}^{(\mu)}$, $(m_{\xi}^{(\nu)})$ — the multiplicity of the weight $\xi \in P$ (resp. $\in P_{\mathfrak{a}}$) in the module L^{μ} , (resp. $\xi \in L_{\mathfrak{a}}^{\nu}$);
 $ch(L^{\mu})$ (resp. $ch(L_{\mathfrak{a}}^{\nu})$) — the formal character of L^{μ} (resp. $L_{\mathfrak{a}}^{\nu}$);
 $ch(L^{\mu}) = \frac{\sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1-e^{-\alpha})}$ — the Weyl formula;
 $R := \prod_{\alpha \in \Delta^+} (1-e^{-\alpha})$ (resp. $R_{\mathfrak{a}} := \prod_{\alpha \in \Delta_{\mathfrak{a}}^+} (1-e^{-\alpha})$) — the Weyl denominator.

Let L^{μ} be completely reducible with respect to \mathfrak{a} ,

$$\begin{aligned}
 L_{\mathfrak{g} \downarrow \mathfrak{a}}^{\mu} &= \bigoplus_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} L_{\mathfrak{a}}^{\nu}. \\
 \pi_{\mathfrak{a}} ch(L^{\mu}) &= \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} ch(L_{\mathfrak{a}}^{\nu}). \tag{1}
 \end{aligned}$$

For the modules we are interested in the Weyl formula for $ch(L^{\mu})$ can be written in terms of singular elements [5]

$$\Psi^{(\mu)} := \sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho},$$

namely,

$$ch(L^{\mu}) = \frac{\Psi^{(\mu)}}{\Psi^{(0)}} = \frac{\Psi^{(\mu)}}{R}. \tag{2}$$

The same is true for submodules $ch(L_{\mathfrak{a}}^{\nu})$ in (1)

$$ch(L_{\mathfrak{a}}^{\nu}) = \frac{\Psi_{\mathfrak{a}}^{(\nu)}}{\Psi_{\mathfrak{a}}^{(0)}} = \frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}},$$

with

$$\Psi_{\mathfrak{a}}^{(\nu)} := \sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}.$$

Applying formula (2) to the branching rule (1) we get the relation connecting the singular elements $\Psi^{(\mu)}$ and $\Psi_{\mathfrak{a}}^{(\nu)}$:

$$\begin{aligned}
 \frac{\sum_{w \in W} \epsilon(w) e^{w(\mu+\rho)-\rho}}{\prod_{\alpha \in \Delta^+} (1-e^{-\alpha})} &= \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\sum_{w \in W_{\mathfrak{a}}} \epsilon(w) e^{w(\nu+\rho_{\mathfrak{a}})-\rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{a}}^+} (1-e^{-\beta})}, \\
 \frac{\Psi^{(\mu)}}{R} &= \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \frac{\Psi_{\mathfrak{a}}^{(\nu)}}{R_{\mathfrak{a}}}. \tag{3}
 \end{aligned}$$

In [3] it was proven that branching coefficients $b_\xi^{(\mu)}$ corresponding to the injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ are subject to the set of recurrent relations:

$$b_\xi^{(\mu)} = -\frac{1}{s(\gamma_0)} \left(\sum_{u \in U} \epsilon(u) \dim \left(L_{\mathfrak{a}_\perp}^{\mu_{\mathfrak{a}_\perp}(u)} \right) \delta_{\xi - \gamma_0, \pi_{\mathfrak{a}}(u(\mu + \rho) - \rho)} + \sum_{\gamma \in \Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}}} s(\gamma + \gamma_0) b_{\xi + \gamma}^{(\mu)} \right). \quad (4)$$

where \mathfrak{a}_\perp is the subalgebra determined by the roots of \mathfrak{g} orthogonal to roots of \mathfrak{a}

$$\Delta_{\mathfrak{a}_\perp} : = \{ \beta \in \Delta_{\mathfrak{g}} | \forall h \in \mathfrak{h}_{\mathfrak{a}}; \beta(h) = 0 \}, \quad (5)$$

$$\widetilde{\mathfrak{a}_\perp} := \mathfrak{a}_\perp \oplus \mathfrak{h}_\perp \quad \widetilde{\mathfrak{a}} := \mathfrak{a} \oplus \mathfrak{h}_\perp \quad (6)$$

and π is the projection operator. When the injection is maximal the projection becomes trivial and the relation (4) is simplified:

$$b_\xi^{(\mu)} = -\frac{1}{s(\gamma_0)} \left(\sum_{u \in W} \epsilon(u) \delta_{\xi - \gamma_0, u(\mu + \rho) - \rho} + \sum_{\gamma \in \Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}}} s(\gamma + \gamma_0) b_{\xi + \gamma}^{(\mu)} \right). \quad (7)$$

The recursion is governed by the set $\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}}$ called the injection fan. The latter is defined by the carrier set $\{\xi\}_{\mathfrak{a} \rightarrow \mathfrak{g}}$ for the coefficient function $s(\xi)$

$$\{\xi\}_{\mathfrak{a} \rightarrow \mathfrak{g}} := \{\xi \in P_{\mathfrak{a}} | s(\xi) \neq 0\}$$

appearing in the expansion

$$\prod_{\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{a}}^+} (1 - e^{-\alpha}) = - \sum_{\gamma \in P_{\mathfrak{a}}} s(\gamma) e^{-\gamma}; \quad (8)$$

Now we remind two definitions introduced in [1]

Definition 2.1. Suppose Δ_0 and Δ are root systems with the corresponding weight lattices P_0 and P . Then ϕ is an “embedding”,

$$\phi : \begin{cases} \Delta_0 \hookrightarrow \Delta, \\ P_0 \hookrightarrow P, \end{cases} \quad (9)$$

if

(a) it injects Δ_0 in Δ , and

(b) acts homomorphically with respect to the vector groups in P_0 and P :

$$\phi(\gamma) = \phi(\alpha) + \phi(\beta)$$

for any triple $\alpha, \beta, \gamma \in P_0$ such that $\gamma = \alpha + \beta$.

ϕ induces an injection of formal algebras : $\mathcal{E}_0 \longrightarrow \mathcal{E}$ and for the image $\mathcal{E}_i = Im_\phi(\mathcal{E}_0)$ one can consider its inverse $\phi^{-1} : \mathcal{E}_i \longrightarrow \mathcal{E}_0$.

Notice that one must distinguish two classes of embeddings: when the scalar product (defined by the Killing form) in the root space P_0 is invariant with respect to ϕ and when it is not ϕ -invariant. The first embedding is called "metric" , the second – "nonmetric".

Definition 2.2. A root system Δ "splinters" as (Δ_1, Δ_2) if there are two embeddings $\phi_1 : \Delta_1 \hookrightarrow \Delta$ and $\phi_2 : \Delta_2 \hookrightarrow \Delta$ where (a) Δ is the disjoint union of the images of ϕ_1 and ϕ_2 and (b) neither the rank of Δ_1 nor the rank of Δ_2 exceeds the rank of Δ .

It is equivalent to say that (Δ_1, Δ_2) is a "splint" of Δ and we shall denote this by $\Delta \approx (\Delta_1, \Delta_2)$. Each component Δ_1 and Δ_2 is a "stem" of the splint (Δ_1, Δ_2) .

In [1] it was indicated that the injection fan technique used in [2] is close to that of splints.

To demonstrate the relations between these instruments consider the case where one of the stems $\Delta_1 = \Delta_{\mathfrak{a}}$ is a root subsystem in Δ .

Splint $\Delta \approx (\Delta_1, \Delta_2)$ is called "injective" if one of its stems, say $\Delta_1 = \Delta_{\mathfrak{a}}$, is a root subsystem in Δ corresponding to a regular reductive subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$.

In case of injective splint the second stem $\Delta_s := \Delta_2 = \Delta \setminus \Delta_{\mathfrak{a}}$ can be translated into a product (8) and defines an injection fan $\Gamma_{\mathfrak{a} \hookrightarrow \mathfrak{g}}$.

Thus we see that

Conjecture 2.3. Each injective splint $\Delta \approx (\Delta_{\mathfrak{a}}, \Delta_s)$ defines an injection fan with the carrier $\{\xi\}_{\mathfrak{a} \hookrightarrow \mathfrak{g}}$ fixed by the product

$$\prod_{\beta \in \Delta_s^+} (1 - e^{-\beta}) = - \sum_{\gamma \in P} s(\gamma) e^{-\gamma} \quad (10)$$

In this case we say that the subalgebra $\mathfrak{a} \hookrightarrow \mathfrak{g}$ splinters Δ (and call \mathfrak{a} the "splinting subalgebra" of \mathfrak{g}). In [1] splints are classified (see Appendix there) and the first three types of them are injective.

3 How stems define multiplicity functions

In this Section we consider the situation where the splinting subalgebra is "larger" than the Lie algebra \mathfrak{s} defined by the root system $\Delta_{\mathfrak{s}0} = CoIm_\phi(\Delta_{\mathfrak{s}})$

(the details will be given below). In such a case to find branching coefficients for a splinting injection $\mathfrak{a} \hookrightarrow \mathfrak{g}$ means to find weight multiplicities for an irreducible \mathfrak{s} -module $L_{\mathfrak{s}}^{\nu}$ with fixed highest weight ν . Notice that \mathfrak{s} must not be a subalgebra of \mathfrak{g} .

Let us return to relation (3) and multiply both sides by $R_{\mathfrak{a}}$:

$$\frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}}^+} (1 - e^{-\beta})} \Psi_{\mathfrak{g}}^{(\mu)} = \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \Psi_{\mathfrak{a}}^{(\nu)}. \quad (11)$$

Here the first factor in the l.h.s. is the inverse of the fan $\Gamma_{\mathfrak{a} \rightarrow \mathfrak{g}}$. Consider the highest weight module $L_{\mathfrak{s}}^{\nu}$. An embedding $\phi : \Delta_{\mathfrak{s}0} \rightarrow \Delta_{\mathfrak{g}}$ sends the singular element $\Psi_{\mathfrak{s}}^{(\nu)}$ into $\Psi_{\mathfrak{g}}^{(\mu)}$. Applying the inverse morphism ϕ^{-1} to the product $\left(\prod_{\beta \in \Delta_{\mathfrak{s}}^+} (1 - e^{-\beta}) \right)^{-1} \phi \left(\Psi_{\mathfrak{s}}^{(\nu)} \right)$ one gets the character of the module $L_{\mathfrak{s}}^{\nu}$,

$$\phi^{-1} \left(\frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}}^+} (1 - e^{-\beta})} \phi \left(\Psi_{\mathfrak{s}}^{(\nu)} \right) \right) = \frac{1}{\prod_{\beta \in \Delta_{\mathfrak{s}(0)}^+} (1 - e^{-\beta})} \Psi_{\mathfrak{s}}^{(\nu)} = \text{ch}(L_{\mathfrak{s}}^{\nu}). \quad (12)$$

Our task is to prove that the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ contains the element $\Psi_{\mathfrak{s}}^{(\xi)}$ for a module $L_{\mathfrak{s}}^{\xi}$ uniquely defined by $L_{\mathfrak{g}}^{\mu}$ and that the branching coefficients $b_{\nu}^{(\mu)}$ in the r.h.s. of (11) coincide with multiplicities $m_{\zeta}^{(\xi)}$ of the corresponding weights in $\mathcal{N}_{\mathfrak{s}}^{\xi}$.

For a highest weight irreducible module $L_{\mathfrak{g}}^{\mu}$ the singular element $\Psi_{\mathfrak{g}}^{(\mu)}$ is an element of \mathcal{E} corresponding to the shifted Weyl-orbit of the weight $(\mu + \rho) \in P^+$ with the sign function $\epsilon(w)$. It is convenient to use also unshifted singular elements

$$\Phi^{(\mu)} := \Psi^{(\mu)} e^{\rho}. \quad (13)$$

In these terms the relation (11) looks like

$$\frac{e^{\rho_{\mathfrak{g}} - \rho_{\mathfrak{a}}}}{\prod_{\beta \in \Delta_{\mathfrak{s}}^+} (1 - e^{-\beta})} \Phi_{\mathfrak{g}}^{(\mu)} = \sum_{\nu \in P_{\mathfrak{a}}^+} b_{\nu}^{(\mu)} \Phi_{\mathfrak{a}}^{(\nu)}. \quad (14)$$

The orbit related to $\Phi_{\mathfrak{g}}^{(\mu)}$ is completely defined by the set of edges $\{\lambda_i\}_{i=1, \dots, r}$ adjusted to the end of the highest weight vector $\mu + \rho$. Let $\mu = \sum m_i \omega_i$ then these edges are

$$\lambda_i = -(m_i + 1) \alpha_i, \quad i = 1, \dots, r \quad (15)$$

Each formal exponent $e^{\mu+\rho+\lambda_i}$ in $\Phi_{\mathfrak{g}}^{(\mu)}$ bears the sign coefficient $\epsilon = (-)$. The defining property of $\Phi_{\mathfrak{g}}^{(\mu)}$ is as follows. Consider any pair of edges λ_i, λ_j and the corresponding weights $\mu+\rho, \mu+\rho+\lambda_i$ and $\mu+\rho+\lambda_j$. Apply the reflection s_{α_i} (or s_{α_j}),

$$s_{\alpha_i} \circ \begin{cases} (\mu + \rho) \\ (\mu + \rho + \lambda_i) \\ (\mu + \rho + \lambda_j) \end{cases} = \begin{cases} (\mu + \rho + \lambda_i) \\ (\mu + \rho) \\ (\mu + \rho + \lambda_i - (m_j + 1)s_{\alpha_i} \circ \alpha_j) \end{cases} \quad (16)$$

Property 3.1. *The edge $\lambda_{i,j}$ of $\Phi_{\mathfrak{g}}^{(\mu)}$ starting at the weight $(\mu + \rho + \lambda_i)$ along the root $-s_{\alpha_i} \circ \alpha_j$ has the same length in $(s_{\alpha_i} \circ \alpha_j)$ as λ_j has in α_j . (The same is true for the edge $\lambda_{j,i}$, its length in $(s_{\alpha_j} \circ \alpha_i)$ is equal to the length of λ_i in α_i .)*

In $\Phi_{\mathfrak{g}}^{(\mu)}$ the elements $e^{(\mu+\rho+\lambda_i-(m_j+1)s_{\alpha_i}\alpha_j)}$ and $e^{(\mu+\rho+\lambda_j-(m_i+1)s_{\alpha_j}\alpha_i)}$ have the sign coefficient $\epsilon = (+)$.

Remember that only the first three types in Richter classification are injective splints and thus are naturally connected with branching. Below we reproduce the part of the splints table from [1] corresponding to injective splints:

type	Δ	$\Delta_{\mathfrak{a}}$	$\Delta_{\mathfrak{s}}$
(i)	G_2	A_2	A_2
	F_4	D_4	D_4
(ii)	$B_r(r \geq 2)$	D_r	$\oplus^r A_1$
	$C_r(r \geq 3)$	D_r	$\oplus^r A_1$
(iii)	$A_r(r \geq 2)$	$A_{r-1} \oplus u(1)$	$\oplus^r A_1$
	B_2	$A_1 \oplus u(1)$	A_2

Each row in the table gives a splint $(\Delta_{\mathfrak{a}}, \Delta_{\mathfrak{s}})$ of the simple root system Δ . In the first two types both $\Delta_{\mathfrak{a}}$ and $\Delta_{\mathfrak{s}}$ are embedded metrically. Stems in the first type splints are equivalent and in the second are not. In the third type splints only $\Delta_{\mathfrak{a}}$ is embedded metrically. the summands $u(1)$ are added to keep $r_{\mathfrak{a}} = r$. This does not change the principle properties of branching but makes it possible to use the multiplicities of \mathfrak{s} -modules without further projecting their weights.

Splints induce a decomposition of the set $S = S_{\mathfrak{c}} \cup S_{\mathfrak{d}}$ with $S_{\mathfrak{c}} = S \cap S_{\mathfrak{a}}$ and $S_{\mathfrak{d}} = S \cap S_{\mathfrak{s}}$. It is easy to check that for all the injective splints the subset $S_{\mathfrak{d}}$ is nonempty. It follows that in the set $\{\lambda_i\}_{i=1, \dots, r}$ one can always

find simple roots $\beta_k \in \Delta_{\mathfrak{s}}$ and that the orbit corresponding to $\Phi_{\mathfrak{g}}^{(\mu)}$ contains the edges

$$\lambda_k = -(m_k + 1) \beta_k \quad (17)$$

attached to the weight $\mu + \rho$. As far as $\Delta_{\mathfrak{a}}$ is a root system and for any pair of simple roots from $S_{\mathfrak{c}}$ the property 3.1 is fulfilled, the element $\Phi_{\mathfrak{g}}^{(\mu)}$ being a singular element for a set of \mathfrak{a} -modules. Consider $\beta_l \in \Delta_{\mathfrak{s}}$ whose coimage in $\Delta_{\mathfrak{s}0}$ is simple. In Appendix it is shown that for any such β_l there corresponds a root $\alpha_l \in S_{\mathfrak{c}}$ such that $\beta_l = \alpha_l + \beta_k$. It is easily seen that the corresponding edge intersects the boundary plane of the fundamental chamber $\bar{C}_{\mathfrak{a}}$ orthogonal to the root α_l ,

$$s_{\alpha_l}(\mu + \rho - p\beta_l) = s_{\alpha_l}(\mu + \rho) - ps_{\alpha_l}\beta_l = \mu + \rho - p\beta_l \quad (18)$$

$$\mu + \rho - s_{\alpha_l}(\mu + \rho) = (m_l + 1)\alpha_l = (m_l + 1)\beta_l - (m_l + 1)\beta_k = p\beta_l - ps_{\alpha_l}\beta_l \quad (19)$$

It follows that $p = (m_l + 1)$ and $s_{\alpha_l}\beta_l = \beta_k$. Now apply the operator s_{β_k} and find that the edge along the root $s_{\beta_k}\alpha_l$ attached at the weight $s_{\beta_k}(\mu + \rho)$ is also equal to $-ps_{\beta_k}\alpha_l$. This means that for the triple of roots β_k, β_l and $s_{\beta_k}\alpha_l$ in $\Delta_{\mathfrak{s}}$ the edges $\lambda_k = -(m_k + 1)\beta_k$, $\lambda_l = -(m_l + 1)\beta_l$ and $\lambda_{kl} = -(m_l + 1)s_{\beta_k}\alpha_l$ demonstrate the property 3.1. One can continue this procedure further in the 2-dimensional subspace fixed by the roots β_k and β_l and find the set of formal exponents that being supplied with the corresponding sign factors compose the coimage of the singular element of a module for the subalgebra in \mathfrak{s} (this subalgebra has rank $r = 2$).

The same can be proven for any positive root $\beta_l \in \Delta$ that is simple in $\Delta_{\mathfrak{s}0}$ and correspondingly for any $r = 2$ subalgebra in \mathfrak{s} . The latter means that to "find" a singular element of \mathfrak{s} -module in $\Phi_{\mathfrak{g}}^{(\mu)}$ it is necessary to incorporate in it additional formal elements $\{-e^{\mu+\rho-(m_l+1)\beta_l}|\beta_l \in S_{\mathfrak{c}}\}$. This fixes the starting edges of the diagram $\phi^{-1}(\Phi_{\mathfrak{s}}^{\tilde{\mu}})$. As it follows from the reconstruction procedure the highest weight $\tilde{\mu}$ is totally defined by the weight μ , they have the same Dynkin numbers:

$$\mu = \sum m_k \omega_k \quad \implies \quad \tilde{\mu} = \sum m_k \tilde{\omega}_k \quad (20)$$

The next step is to construct the full $W_{\mathfrak{s}}$ -orbit $\Phi_{\mathfrak{s}}^{(\tilde{\mu})}$ in $P_{\mathfrak{s}}$. It is easily seen that its coimage belongs to $\bar{C}_{\mathfrak{a}}$. and that the set $\phi^{-1}(\Phi_{\mathfrak{s}}^{\tilde{\mu}}) \setminus \Phi_{\mathfrak{g}}^{(\mu)}|_{\bar{C}_{\mathfrak{a}}}$ corresponds to the weights belonging to the boundary $\bar{C}_{\mathfrak{a}}$ (including the subset $\{-e^{\mu+\rho-(m_l+1)\beta_l}|\beta_l \in S_{\mathfrak{c}}\}$). Thus we have constructed all the formal elements

with the appropriate sing factors that after being added to $\Phi_{\mathfrak{g}}^{(\mu)}|_{\bar{C}_{\mathfrak{a}}}$ form the diagram $\phi^{-1}(\Phi_{\mathfrak{s}}^{\tilde{\mu}})$ in $\bar{C}_{\mathfrak{a}}$.

Now let us return to the relation (14). One can add to $\Phi_{\mathfrak{g}}^{(\mu)}$ pairs of additional formal elements constructed above with the opposite signs: $\epsilon(w)|_{w \in W_{\mathfrak{s}}}$ and $-\epsilon(w)|_{w \in W_{\mathfrak{s}}}$. Attribute the signs $\epsilon(w)|_{w \in W_{\mathfrak{s}}}$ to the elements whose weights we shall consider belonging to $\bar{C}_{\mathfrak{a}}$. The same elements with the opposite signs are to be referred to the neighbor Weyl chambers of $C_{\mathfrak{a}}^{(l)}$ (the latter are connected with the main one via simple reflections s_{α_l} so the signes $-\epsilon(w)|_{w \in W_{\mathfrak{s}}}$ are natural for them). In fact one can repeat the procedure of finding additional singular weights in any Weyl chamber $C_{\mathfrak{a}}^{(\bar{m})}$ and in them additional singular weights always have the signs opposite to that in their nearest neighbors. Thus without changing in fact the element $\Phi_{\mathfrak{g}}^{(\mu)}$ one can present it as a sum

$$\Phi_{\mathfrak{g}}^{(\mu)} = \sum_{w \in W_{\mathfrak{a}}} w \circ (e^{\rho_{\mathfrak{a}}} \Psi^{\tilde{\mu} + \rho_{\mathfrak{s}}}) \quad (21)$$

where the weight $\tilde{\mu} = \sum m_k \omega_{\mathfrak{s}}^k$ was defined above. The decomposition (21) provides the possibility to apply the factor $\left(\prod_{\beta \in \Delta_{\mathfrak{s}}^+} (1 - e^{-\beta})\right)^{-1}$ to each summand of the singular element $\Phi_{\mathfrak{g}}^{(\mu)}$ separately because the sets of weights from different Weyl summands do not intersect. Taking into account the isomorphism ϕ one can see that in the main Weyl chamber $\bar{C}_{\mathfrak{a}}$ the set of weights generated by the factor $\left(\prod_{\beta \in \Delta_{\mathfrak{s}}^+} (1 - e^{-\beta})\right)^{-1}$ is isomorphic to the weight diagram $\mathcal{N}_{\mathfrak{s}}^{\tilde{\mu}}$ of the \mathfrak{s} -module $L_{\mathfrak{s}}^{\tilde{\mu}}$. Finally one must restrict relation (14) to $\bar{C}_{\mathfrak{a}}$ and obtain the main result:

Property 3.2.

$$\frac{e^{\rho_{\mathfrak{g}}}}{\prod_{\beta \in \Delta_{\mathfrak{s}}^+} (1 - e^{-\beta})} (\Psi^{\tilde{\mu} + \rho_{\mathfrak{s}}}) = \sum_{\tilde{\nu} \in \mathcal{N}_{\mathfrak{s}}^{\tilde{\mu}}} M_{(\mathfrak{s})\tilde{\nu}}^{\tilde{\mu}} e^{(\mu - \phi(\tilde{\mu} - \tilde{\nu}))} = \sum_{\nu \in P_{\mathfrak{a}}^{++}} b_{\nu}^{(\mu)} e^{\nu}. \quad (22)$$

Any weight with nonzero multiplicity in the r. h. s. is equal to one of the highest weights in the decomposition. The multiplicity $M_{(\mathfrak{s})\tilde{\nu}}^{\tilde{\mu}}$ of the weight $\tilde{\nu} \in \mathcal{N}_{\mathfrak{s}}^{\tilde{\mu}}$ defines the branching coefficient $b_{\nu}^{(\mu)}$ for the highest weight $\nu = (\mu - \phi(\tilde{\mu} - \tilde{\nu}))$:

$$b_{(\mu - \phi(\tilde{\mu} - \tilde{\nu}))}^{(\mu)} = M_{(\mathfrak{s})\tilde{\nu}}^{\tilde{\mu}}.$$

4 Examples

Example 4.1. Consider Lie algebra $A_2(\mathfrak{sl}(3))$ and branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra $A_1 \oplus u(1)$ with root system spanned by first simple root of A_2 . Singular element of $L^{(3,2)}$ is decomposed into the sum of splint images of singular elements of $A_1 \oplus A_1$ -modules and branching coefficients coincide with weight multiplicities of $A_1 \oplus A_1$ -module (see Fig. 1).

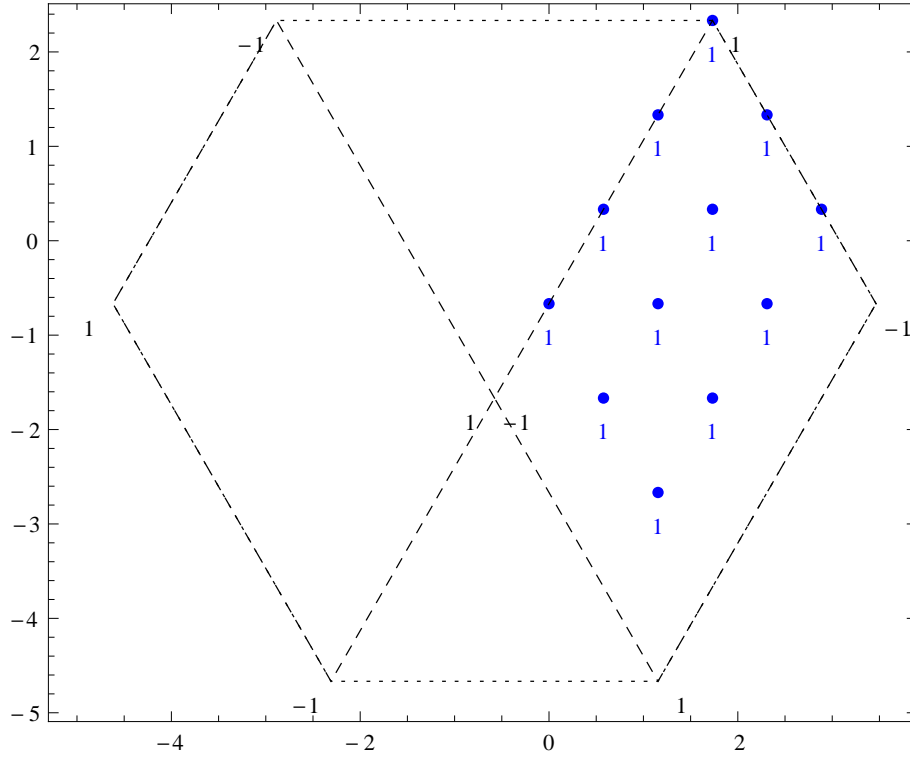


Figure 1: Weyl group orbit (dotted) producing singular element of A_2 -module $L^{(3,2)}$ and its decomposition into the sum of splint images of singular elements of $A_1 \oplus A_1$ -modules (dashed). Weight multiplicities of $A_1 \oplus A_1$ -module coincide with branching coefficients for the reduction $L^{(3,2)}_{A_2 \downarrow A_1 \oplus u(1)}$.

Example 4.2. Now consider Lie algebra $B_2(\mathfrak{so}(5))$ and branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra $A_1 \oplus u(1)$ with root system spanned by first simple root of B_2 . Singular element of

$L^{(3,2)}$ is decomposed into the sum of splint images of singular elements of A_2 -modules and branching coefficients coincide with weight multiplicities of A_2 -module (see Fig. 2).

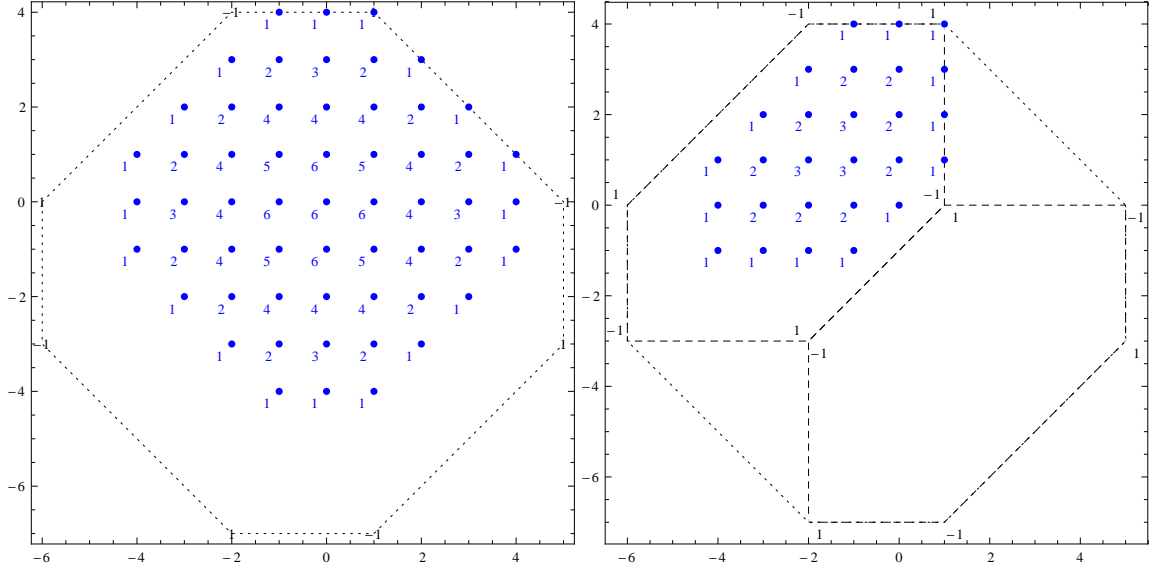


Figure 2: B_2 -module $L^{(3,2)}$ is shown on the left. Weight multiplicities are indicated. Contour of singular element is shown by dotted line. Right figure represents the decomposition of $L^{(3,2)}$ -singular element into the sum of splint images of singular elements of A_2 -modules (dashed). Weight multiplicities of A_2 -module coincide with branching coefficients for the reduction $L^{(3,2)}_{B_2 \downarrow A_1 \oplus u(1)}$.

Example 4.3. Lie algebra G_2 has regular subalgebra A_2 with root system built on long roots of G_2 . Consider branching of its irreducible module $L^{(3,2)}$ into the modules of reductive subalgebra A_2 . Singular element of $L^{(3,2)}$ is decomposed into the sum of splint images of singular elements of A_2 -modules and branching coefficients coincide with weight multiplicities of A_2 -module (see Fig. 3).

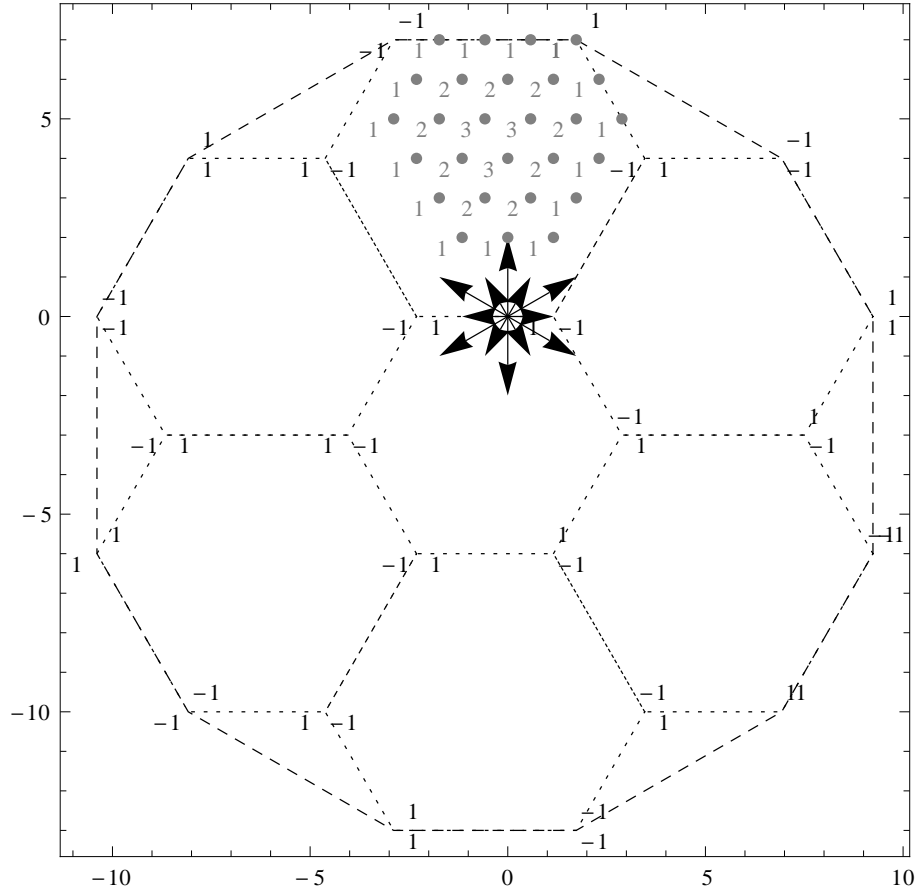


Figure 3: Weyl group orbit (dotted) producing singular element of G_2 -module $L^{(3,2)}$ and its decomposition into the sum of splint images of singular elements of A_2 -modules (dashed). Weight multiplicities of A_2 -module coincide with branching coefficients for the reduction $L_{G_2 \downarrow A_2}^{(3,2)}$.

5 Conclusions

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6 Appendix

Let us demonstrate that the injection splints

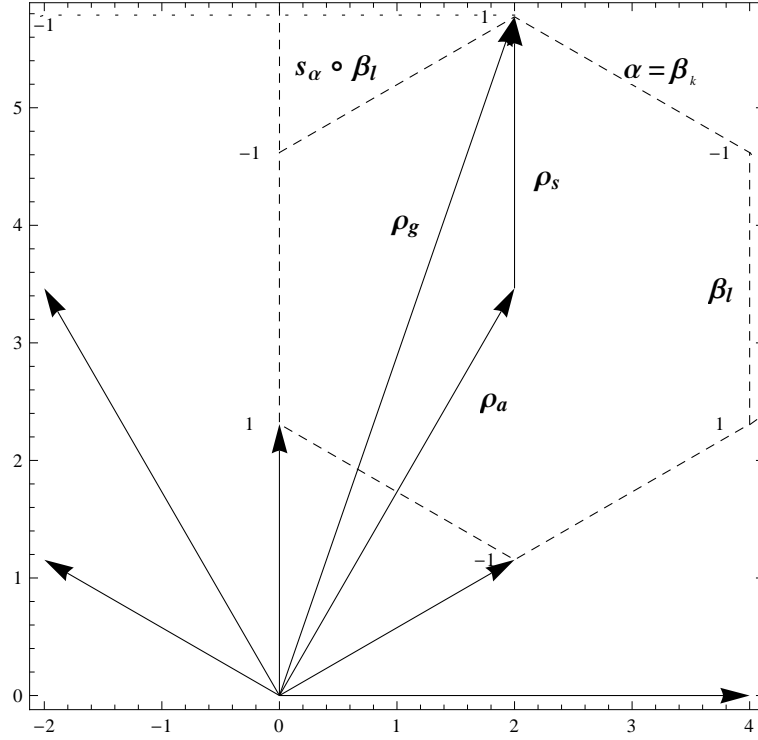


Figure 4: Weyl group orbit (dotted) producing singular element of A_2 -module $L^{(3,2)}$ and its decomposition into the sum of splint images of singular elements of $A_1 \oplus A_1$ -modules (dashed). Weight multiplicities of $A_1 \oplus A_1$ -module coincide with branching coefficients for the reduction $L_{A_2 \downarrow A_1 \oplus u(1)}^{(3,2)}$.