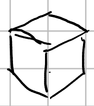
Lecture 15 2023.07.12



SSYT $\longleftrightarrow$ lozenge tilings

1	1	2	3
2	3		
3			

$\longleftrightarrow x_1^2 x_2^2 x_3^3$

$$s_\lambda(x_1 \ldots x_n) = \sum_{T \in SSYT(\lambda | n)} \prod_{i=1}^{n} x_i^{\#i\text{'s in } T}$$

$T \in SSYT(\lambda|n) \longleftrightarrow$ Gelfand-Tsetlin pattern

$b_i^{(j)} = \#$ of boxes $\leq j$ in the row i of T

$\tilde{b}_i^{(j)} = b_i^{(j)} + j - i$

$b_1^{(n)} \geq b_2^{(n)} \ldots b_n^{(n)} \to (\lambda_1, \lambda_2 \ldots \lambda_n)$ $\quad b_1^{(n)} = \lambda_1$, $\quad b_2^{(n)} = \lambda_2$

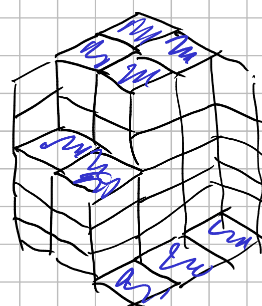
$b_1^{(n-1)} \quad b_2^{(n-1)} \ldots b_{n-1}^{(n-1)}$

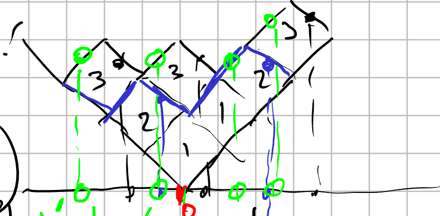
$b_1^{(n-2)} \ldots b_{n-2}^{(n-2)}$

$b_1^{(1)}$

$\tilde{b}_1^{(n)} > \tilde{b}_2^{(n)} > \ldots \tilde{b}_n^{(n)} \to (\lambda_1, \lambda_2 - 1, \lambda_3 - 2 \ldots)$

$\tilde{b}_1^{(n-1)} > \ldots$





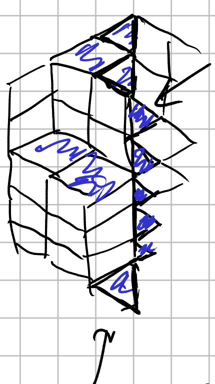
$-\lambda_1' + \frac{1}{2}$ $\quad -\lambda_2' + \frac{3}{2}$ $\quad \lambda_2 - 2 + \frac{1}{2}$ $\quad \lambda_1 - 1 + \frac{1}{2}$

$$s_\lambda(x)\, s_\lambda(y) = \Big(\sum_{T \in SSYT(\lambda|n)} x^T\Big)\Big(\sum_{T' \in SSYT(\lambda|k)} y^{T'}\Big)$$

$x^T y^{T'} \longleftrightarrow$ lozenge tiling of a hexagon ("cubes in the corner")

skew howe duality

$$s_\lambda(x)\, s_{\lambda'}(y) = \Big(\sum_{T \in SSYT(\lambda|n)} x^T\Big)\Big(\sum_{T' \in SSYT(\lambda'|n)} y^{T'}\Big)$$



$-\lambda_i' + i$ (or $\lambda_i' - i$ from the top)

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \dim V_{GL_k}(\lambda')}{2^{nk}}$$

$\lambda_i - i$ from the bottom

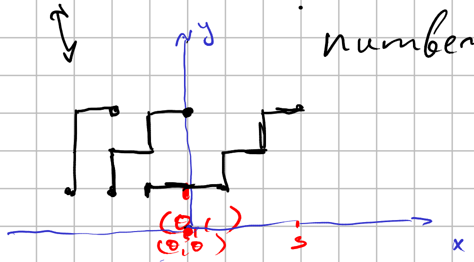
SSYT $\longleftrightarrow$ Non-intersecting lattice paths (NILP)

Row-reading

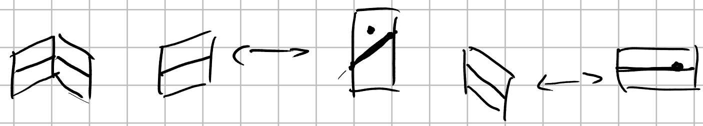
1 1 2 3
2 3
3

row $i \longleftrightarrow$ path P_i — [$(0,1)$ or $(1,0)$

start at $(-i, 0)$, end at $(\lambda_i - i, n)$



number of horizontal steps on level j = #j's in row i



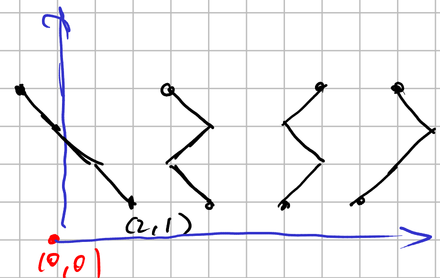
Column reading

column $\longleftrightarrow$ P_i' \ / $(-1,1)$ or $(1,1)$

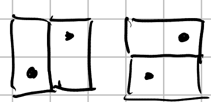
start at $(2i, 1)$ end at $(2i-2\lambda_i'+n, n)$

j-th step is $(-1,1)$ if j is in column i,
otherwise $(1,1)$

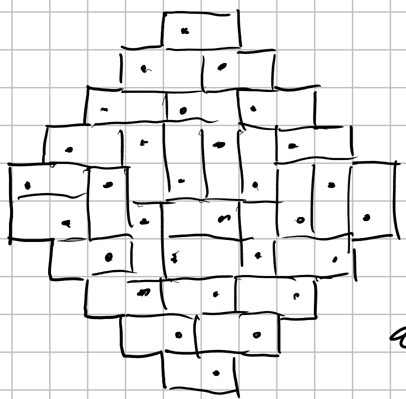
1 1 2 3
2 3
3



Aztec diamond, domino tilings

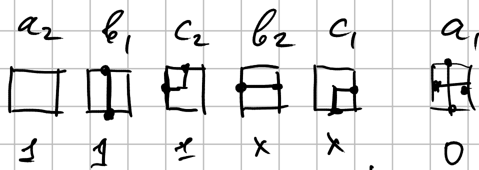


4 kinds of dominos

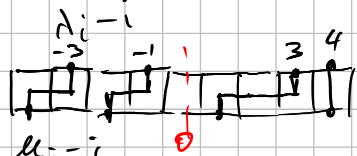


$a_1a_2+b_1b_2-c_1c_2=0$ free fermionic condition

Five vertex model



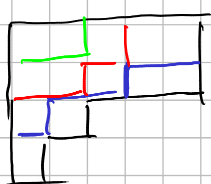
$2^4=16$



$\lambda_1=5$ $\mu_1=5$
$\lambda_2=5$ $\mu_2=3$
$\lambda_3=2$ $\mu_3=1$
$\lambda_4=1$ $\mu_4=0$

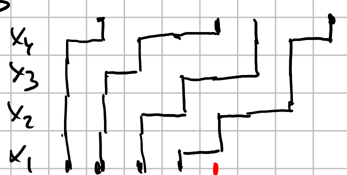
$\lambda \succcurlyeq \mu$ $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \dots$

$wt = \prod_{\square} wt(\square) = x^4$



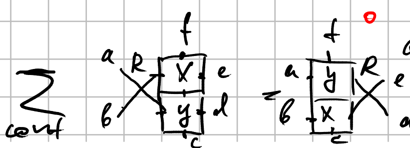
SSYT

1 2 2 4
2 3 3
3 4 4
4



$x_1x_2^3x_3^3x_4^4$

$s_\lambda(x_1\dots x_n)=\sum_{conf} wt(conf)$
$=Z_\lambda(x_1\dots x_n)$

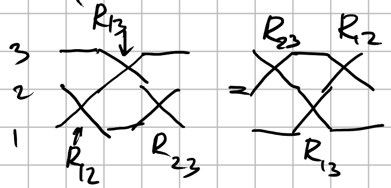


$a,b,c,d,f = \{\cdot, \bullet\}$

$R_{12}L_1L_2 = L_2L_1R_{12}$

$R:$ 

$1 \qquad y/x \qquad 1 \qquad y/x \qquad 1-\frac{y}{x}$

Yang-Baxter equation 

$$R_{23}R_{13}R_{12} = R_{12}R_{13}R_{23}$$

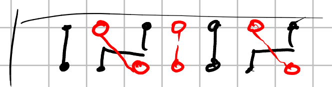
$$e^{\sum_{\ell=1}^{\infty} t_\ell \alpha_\ell} \qquad t_\ell = \frac{1}{\ell}\sum_{i=1}^{n} x_i^\ell$$

$$= e^{\sum_{i=1}^{n}\sum_{\ell=1}^{\infty} \frac{x_i^\ell}{\ell}\alpha_\ell} = \prod_{i=1}^{n} \underbrace{e^{\sum_{\ell=1} \frac{x_i^\ell}{\ell}\alpha_\ell}}_{\substack{\shortparallel \\ T(x_i)}}$$

$$T(x) = e^{\sum_{\ell\geq 1} \frac{x^\ell}{\ell}\alpha_\ell}$$

$$S_\lambda(x\ldots) = \sum \prod_{i=1}^{n} \langle \lambda^{(i-1)} | T(x_i) | \lambda^{(i)} \rangle$$

$$\emptyset = \lambda^{(0)}, \lambda^{(1)} \ldots \lambda^{(n)} = \lambda$$

$T(x) \longrightarrow$ 

$$L \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & x & 1 & 0 \\ 0 & x & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T(x) = \ldots L_{a,-1}(x)\, L_{a,0}(x)\, L_{a,1}(x) \ldots$$

$$R_{ab}(x,y) = L_{a,k}(x)\, L_{b,k}(y) = L_{b,k}(y)\, L_{a,k}(x) \cdot$$

$$\cdot R_{a,b}(x,y)$$