Lecture 21 2023-08-23

Skew $GL_n \times GL_k$ Howe duality

$$\Lambda(\mathbb{C}^n \otimes \mathbb{C}^k) = \bigoplus_{\lambda \subset k^n} V_n(\lambda) \otimes V_k(\lambda')$$

$$\mu_{n,k}(\lambda \mid \{x\}, \{y\}) = \frac{s_\lambda(x_1 \dots x_n)\, s_{\lambda'}(y_1 \dots y_k)}{\prod_i \prod_j (1 + x_i y_j)}$$

$m, l \in \mathbb{Z} + \frac{1}{2}$

$$K(m,l) = \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{w^l}{z^m} \frac{F(z)}{F(w)} \frac{\sqrt{zw}}{z-w}$$

$$\frac{F(z)}{F(w)} = e^{n(S(z)-S(w))}$$

$$F(z) = \prod_i \frac{1}{1 - x_i z} \prod_j \frac{1}{1 + y_j/z}$$

$$S(z) = \frac{1}{n} \ln F(z) \qquad \frac{df}{dt} = \frac{df}{dz}\frac{dz}{dt}$$

$z = t\tilde{z} \quad w = t\tilde{w}$

$$K_n(m,l) = \iint_{|\tilde{w}|<|\tilde{z}|} \frac{d\tilde{z}}{2\pi i \tilde{z}} \frac{d\tilde{w}}{2\pi i \tilde{w}} \frac{\tilde{w}^l}{\tilde{z}^m} t^{l-m} \frac{F(t\tilde{z})}{F(t\tilde{w})} \frac{\sqrt{\tilde{z}\tilde{w}}}{\tilde{z}-\tilde{w}}$$

$$\left(\frac{d}{dt} K(m,l)\right)\Big|_{t=1} = 0 = \underbrace{\iint_{|\tilde{w}|<|\tilde{z}|} \frac{d\tilde{z}}{2\pi i \tilde{z}} \frac{d\tilde{w}}{2\pi i \tilde{w}} \frac{\tilde{w}^l}{\tilde{z}^m} (l-m) \frac{F(\tilde{z})}{F(\tilde{w})} \frac{\sqrt{\tilde{z}\tilde{w}}}{\tilde{z}-\tilde{w}}}_{= K(m,l)(l-m)}$$

$$+ \iint_{|\tilde{w}|<|\tilde{z}|} \frac{d\tilde{z}}{2\pi i \tilde{w}} \frac{d\tilde{w}}{2\pi i \tilde{w}} \frac{\tilde{w}^l}{\tilde{z}^m} \frac{\sqrt{\tilde{z}\tilde{w}}}{\tilde{z}-\tilde{w}} n\left(\tilde{z}\frac{d}{dz} S(z) - \tilde{w}\frac{d}{dw} S(w)\right) \frac{F(\tilde{z})}{F(\tilde{w})}$$

$\tilde{z} = z \quad \tilde{w} = w$ since $t=1$

$$(m-l) K_n(m,l) = \iint \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{w^l}{z^m} \frac{\sqrt{zw}\, n\left(z\frac{d}{dz}S(z) - w\frac{d}{dw}S(w)\right)}{z-w} \cdot e^{n(S(z)-S(w))}$$

$x_i = 1 \quad y_j = 1 \qquad S(z) = \frac{1}{n}\ln F(z) = -\ln(1-z) - \frac{k}{n}\ln\left(1 + \frac{1}{z}\right)$

$n, k \to \infty \quad c = \frac{k}{n}$

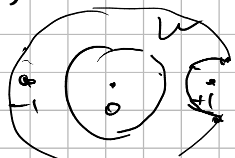
$$z\frac{d}{dz} S(z) = \frac{z}{1-z} - \frac{\frac{k}{n} z}{1+z} + \frac{k}{n} = \frac{1}{1-z} + \frac{k/n}{1+z} - 1$$

$$z\frac{d}{dz} S(z) - w\frac{d}{dw} S(w) = \frac{1}{1-z} - \frac{1}{1-w} + \frac{k}{n}\left(\frac{1}{1+z} - \frac{1}{1+w}\right) =$$

$$= \frac{(z-w)}{(1-z)(1+w)} - \frac{k(z-w)}{n(1+z)(1+w)}$$

$$K_n(m,l) = \frac{n}{m-l} \iint_{|w|<|z|} \frac{dz}{2\pi i z}\,\frac{dw}{2\pi i w}\, \frac{z^{k-m+\frac12}}{(1-z)^n(1+z)^k}\, \frac{(1-w)^n(1+w)^k}{w^{k-l-\frac12}} \cdot$$

$$\cdot \left(\frac{1}{(1-z)(1-w)} - \frac{k}{n}\,\frac{1}{(1+z)(1+w)}\right)$$



$l' = l + \frac12 \in \mathbb{Z}$

$$\int_{C_\varepsilon} \frac{dw}{2\pi i w}\, \frac{(1-w)^{n-1}(1+w)^k}{w^{k-l'}}$$ — a polynomial in $(k-l')$ or n

$z = \frac{1+\zeta}{1-\zeta}$ $\quad K(m',l') \sim \frac{p_n(m')p_{n-1}(l') - p_{n-1}(m')p_n(l')}{m'-l}$ Krawtchouk polynomials

Kravchuk

classical discrete orthogonal polynomials

R. Koekoek, P. Lesky, R. Swarttouw "Hypergeometric orthogonal polynomials and their q-analogues" 2010

Krawtchouk polynomials

$$K_m(x;p,N) = {}_2F_1\left(-m,-x,-N;\frac{1}{p}\right) \qquad m=0\ldots N$$

$$(a)_k = \prod_{i=1}^{k}(a+i-1) = a(a+1)\ldots(a+k-1)$$

$$(-N)_m = (-N)(-N+1)\ldots(-N+m-1) = (-1)^m \frac{N!}{(N-m)!}$$

$$ {}_2F_1(a,b,c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\,\frac{z^k}{k!}$$

$\downarrow$ satisfies 2 order ODE

$$z(1-z)\varphi''(z) + (c-(a+b+1)z)\varphi'(z) - ab\,\varphi(z) = 0$$

Classical orthogonal polynomials

$Q(x)$ $L(x)$ polynomials in x

$$Q(x) f_m''(x) + L(x) f_m'(x) + \lambda_m f_m(x) = 0$$

Look for polynomial solutions of degree m

$$D f_m = \lambda_m f_m \qquad \lambda_m \in \mathbb{R}$$

There are such solutions if $Q(x) = ex^2 + 2fx + g$

$L(x) = 2\varepsilon x + \delta$

Hermite, Laguerre, Jacobi

$\lambda_n \neq \lambda_m$ for all $m \neq n$

Monic polynomial $f_m(x) = x^m + \dots$ Three-term rec. rel.

$$f_{m+1}(x) = (x-\alpha_m) f_m(x) - \beta_m f_{m-1}(x)$$
$$x f_m(x) = f_{m+1}(x) + \alpha_m f_m(x) + \beta_m f_{m-1}(x)$$

$[a, b]$ $W(x)$
$$\int_a^b W(x) f_n(x) f_m(x)\, dx = 0 \quad n \neq m$$
$$\frac{d}{dx}\big(W(x) Q(x)\big) = W(x) L(x)$$
$$W(a) Q(a) = 0 \quad W(b) Q(b) = 0$$

$$(ex^2 + 2fx + g) y_n''(x) + (2\varepsilon x + \delta) y_n'(x) = n(e(n-1) + 2\varepsilon) y_n(x)$$

1. $e = f = 0$ $g = 1$ $W(x) = e^{\varepsilon x^2 + \delta x}$ Hermite polynomials on $(-\infty, \infty)$
2. $e = 0$ $f = \frac{1}{2}$ $\varepsilon < 0$ $\alpha > -1$ $W(x) = (x-a)^\alpha e^{2\varepsilon x}$ Laguerre polynomials on (a, ∞)
3. $e = 1$ $f^2 > g$ $\varepsilon > 0$ $\alpha > -1$ $\beta > -1$ $W(x) = (x-a)^\alpha (b-x)^\beta$ Jacobi polynomials on (a, b)

Krawtchok $W(x) = \binom{N}{x} p^x (1-p)^{N-x}$
$x \in \mathbb{Z}$

Discrete orthogonal polynomials $x \in \mathbb{Z}$

$$\Delta f(x) = f(x+1) - f(x)$$
$$(ex^2 + 2fx + g)(\Delta^2 y_n)(x) + (2\varepsilon x + \delta)(\Delta y_n)(x) =$$
$$= n(e(n-1) + 2\varepsilon) y_n(x+1)$$

$n = 0, 1, 2 \dots$ $e, f, g, \varepsilon, \delta \in \mathbb{R}$

$e = f = 0$ $g = 1$ $2\varepsilon < 0$ $\delta = 2\varepsilon + 1$

Charlier discrete orthogonal polynomials

$$W(x) = \frac{1}{(-2\varepsilon)^x} \frac{1}{x!}$$

Krawtchouk polynomials $e=0$ $f=\frac{1}{2}$ $2\varepsilon>1$

$g=\gamma-2\varepsilon+1$ $\gamma-2\varepsilon=N\in\mathbb{Z}_{>0}$

$$W(x)=\frac{1}{(2\varepsilon-1)^x}\frac{1}{x!}\frac{1}{(N+1-x)!}$$

q-analogues! consider polynomials on a lattice q^x $x\in\mathbb{Z}$

Askey scheme of orthogonal polynomial

Sakai classification for Painleve equations

$$K_m(x;p,N)={}_2F_1\left(-m,-x,-N;\frac{1}{p}\right)$$

Orthogonality
$$\sum_{x=0}^{N}\binom{N}{x}p^x(1-p)^{N-x}K_m(x;p,N)K_n(x;p,N)=\frac{(-1)^m m!}{(-N)_m}\left(\frac{1-p}{p}\right)^n\cdot\delta_{mn}$$

Three term rec. relation

$$-xK_n(x;p,N)=p(N-n)K_{n+1}(x;p,N)-(p(N-n)+n(1-p))K_n(x;p,N)+n(1-p)K_{n-1}(x;p,N)$$

Monic $P_n(x)=(-N)_n p^n K_n(x;p,N)$

$$xP_n(x)=P_{n+1}(x)+[p(N-n)+n(1-p)]P_n(x)+np(1-p)(N+1-n)P_{n-1}(x)$$

Difference equation

$$-nK_n(x)=p(N-x)K_n(x+1)-(p(N-x)+x(1-p))K_n(x)+x(1-p)K_n(x-1)$$

Self-duality $K_n(x;p,N)=K_x(n;p,N)$

Dual orthogonality

$$\sum_{n=0}^{N}\binom{N}{n}p^n(1-p)^{N-n}K_n(x;p,N)K_n(y;p,N)=\left(\frac{1-p}{p}\right)^x\binom{N}{x}^{-1}\delta_{xy}$$

Orthonormal polynomials $p=\frac{1}{2}$ $N=n+b-1$

$$\frac{(-1)^m m!}{(-N)_m}\left(\frac{1-p}{p}\right)^m=\binom{N}{m}^{-1}$$

$$(-N)_m=(-1)^m\frac{N!}{(N-m)!}$$

$$p_m(x) = K_m(x; \tfrac{1}{2}, n+k-1)\sqrt{\binom{n+k-1}{m}}$$

Leading coeff. $k_m = \frac{1}{m!}\frac{2^m}{\sqrt{\binom{n+k-1}{m}}}$ $\qquad p_m(x) = \frac{2^m}{\sqrt{\binom{n+k-1}{m}}}\int_{C_\varepsilon}\frac{dw}{2\pi i w}\frac{(1+w)^x(1-w)^{n+k-1-x}}{w^m}$

$${}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$$

$$K_m(x;p,N) = K_x(m;p,N)$$

$$\frac{p_m(x)}{\sqrt{\binom{n+k-1}{m}}} = \frac{p_x(m)}{\sqrt{\binom{n+k-1}{x}}}$$

$$p_{k-e'}^{(n-1)} \cdot 2^{e'-k}\sqrt{\binom{n+k-1}{k-e'}}$$

$$\int\frac{dw}{2\pi i w}\frac{(1-w)^{n-1}(1+w)^k}{w^{k-e'}} = \frac{p_{n-1}(k-e')\, 2^{e'-k}\binom{n+k-1}{k-e'}}{\sqrt{\binom{n+k-1}{n-1}}}$$

$$\int\frac{dw}{2\pi i w}\frac{(1-w)^n(1+w)^{k-1}}{w^{k-e'}} = \frac{p_n(k-e')\, 2^{e'-k}\binom{n+k-1}{k-e'}}{\sqrt{\binom{n+k-1}{n-1}}}$$

$$\int\frac{dz}{2\pi i z}\frac{z^{k-m'+1}}{(1-z)^{n+1}(1+z)^k} = \frac{(-1)^{n+1}}{2^{n+k}}\int_{C_\varepsilon}\frac{d\zeta}{2\pi i\zeta}\frac{(1+\zeta)^{k-m'}(1-\zeta)^{n+m'-1}}{\zeta^n} \sim p_n(k-m')$$

$$z = \frac{1+\zeta}{1-\zeta} \qquad \zeta = \frac{z-1}{z+1} \qquad 1-z = \frac{-2\zeta}{1-\zeta} \qquad \frac{dz}{z} = \frac{2d\zeta}{1-\zeta^2}$$

$$1+z = \frac{2}{1-\zeta}$$

$$\int\frac{dz}{2\pi i z}\frac{z^{k-m'+1}}{(1-z)^n(1+z)^{k+1}} = \frac{(-1)^n}{2^{n+k}}\int\frac{d\zeta}{2\pi i\zeta}\frac{(1+\zeta)^{k-m'}(1-\zeta)^{n+m'-1}}{\zeta^{n-1}} \sim p_{n-1}(k-m')$$

Ex:

$$K_n(m', e') \sim \frac{p_n(k-m')\,p_{n-1}(k-e') - \dots p_{n-1}(k-m')\,p_n(k-e')}{m'-e'}$$

Central limit theorem for global fluctuations

2d Riemann-Hilbert problem for asymptotics of ~~non~~ orthogonal polynomials

Asymptotics of recurrence coefs $\longleftrightarrow$ Painleve eqn.