Lecture 23 2023-08-30

Orthogonal polynomials and Riemann-Hilbert problem

$w(x)$ $\quad \int_{-\infty}^{\infty} |x|^k w(x)\,dx < \infty$

$P_m(x) = x^m + \dots \qquad \int P_m(x)\, x^j\, w(x)\,dx = 0$

$j = 0, 1 \dots m-1$

$p_m(x) = \varkappa_m P_m(x)$

$$\varkappa_m = \frac{1}{\|P_m\|_2} = \left(\int P_m^2(x)\, w(x)\,dx\right)^{-\frac{1}{2}}$$

$$x P_m(x) = P_{m+1}(x) + b_m P_m(x) + a_m^2 P_{n-1}(x)$$

$$K_n(x,y) = \sum_{j=0}^{n-1} p_j(x)\, p_j(y) \qquad \int K_n(x,y)\, q(y)\, w(y)\,dy = q(x)$$

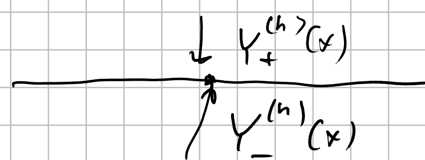
$\deg q < n$

$$K_n(x,y) = \varkappa_{n-1}^2 \frac{P_n(x) P_{n-1}(y) - P_n(y) P_{n-1}(x)}{x-y}$$

Riemann-Hilbert problem: find an analytic function with jump conditions along some curve and given asymptotics

$Y^{(n)}(z) : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$

- $Y^{(n)}(z)$ is analytic in $\mathbb{C}\setminus\mathbb{R}$
- $Y_+^{(n)}(x) = Y_-^{(n)}(x) \begin{pmatrix} 1 & w(x) \\ 0 & 1 \end{pmatrix} \quad x \in \mathbb{R}$



- $Y^{(n)}(z) = (I + o(1)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad z \to \infty$

Thm

$$Y^{(n)}(z) = \begin{pmatrix} P_n(z) & \frac{1}{2\pi i} \int \frac{P_n(x)\, w(x)\,dx}{x-z} \\ -2\pi i \varkappa_{n-1}^2 P_{n-1}(z) & -\varkappa_{n-1}^2 \int \frac{P_{n-1}(x)\, w(x)\,dx}{x-z} \end{pmatrix}$$

$(Y^{(n)}(z))_{11} \qquad (Y_+^{(n)}(x))_{11} = (Y_-^{(n)}(x))_{11} \quad x \in \mathbb{R}$

$(Y^{(n)}(z))_{11} = z^n \left(1 + O\left(\frac{1}{z}\right)\right) \quad z \to \infty$

$(Y^{(n)}(z))_{11}$ is holomorphic on $\mathbb{C}$ (entire function)

$\Rightarrow \left(Y^{(n)}(z)\right)_{11} = q_n(z)$ monic polynomial

Look at $\left(Y^{(n)}(z)\right)_{12}$
$$\begin{cases} \left(Y_+^{(n)}(x)\right)_{12} = \left(Y_-^{(n)}(x)\right)_{12} + q_n(x)\,w(x) & x\in\mathbb{R} \\ \left(Y^{(n)}(z)\right)_{12} = O\left(\frac{1}{z^{n+1}}\right) & z\to\infty \end{cases}$$

$$\left(Y^{(n)}(z)\right)_{12} = \frac{1}{2\pi i}\int \frac{q_n(y)\,w(y)\,dy}{y-z}$$

$$\frac{1}{y-z} = -\sum_j \frac{y^j}{z^{j+1}}$$

$$\left(Y^{(n)}(z)\right)_{12} = -\frac{1}{2\pi i}\sum_{j=0}^{n-1}\frac{1}{z^{j+1}}\int q_n(y)\,y^j\,w(y)\,dy + O\left(\frac{1}{z^{n+1}}\right)$$

$$\int q_n(y)\,y^j\,w(y)\,dy = 0 \qquad j=0,1\dots n-1$$

$$q_n(y) = P_n(y)$$

$$K_n(x,y) = \begin{pmatrix} 0 & 1\end{pmatrix}\frac{\left(Y_+^{(n)}(y)\right)^{-1} Y_+(x)}{2\pi i(x-y)}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}$$

$$\det Y^{(n)} = 1$$

$$\left(Y^{(n)}\right)^{-1} = \begin{pmatrix} * & * \\ 2\pi i\gamma_{n-1}^2 P_{n-1} & P_n\end{pmatrix}\cdot\begin{pmatrix} P_n(x)\\ -2\pi i\gamma_{n-1}^2 P_{n-1}(x)\end{pmatrix}$$

$$= \gamma_{n-1}^2\,\frac{P_{n-1}(y)\,P_n(x) - P_{n-1}(x)\,P_n(y)}{x-y}$$

$$Y^{(n)}(z) = \left(I + Y^{(n,1)}/z + Y^{(n,2)}/z^2 + \dots\right)\begin{pmatrix} z^n & 0\\ 0 & z^{-n}\end{pmatrix}$$

$$\begin{cases} a_n^2 = \left(Y^{(n,1)}\right)_{12}\left(Y^{(n,1)}\right)_{21} \\ b_n = \dfrac{\left(Y^{(n,2)}\right)_{12}}{\left(Y^{(n,1)}\right)_{12}} - \left(Y^{(n,1)}\right)_{22}\end{cases}$$

$$\left(Y^{(n,1)}\right)_{21} = -2\pi i\,\gamma_{n-1}^2$$

since the jump does not depend on n

$Y^{(n+1)}\left(Y^{(n)}\right)^{-1}$ has no jump, so it is entire function

$$Y^{(n+1)}_{(z)}\left(Y^{(n)}_{(z)}\right)^{-1} = \left(I + O\left(\frac{1}{z}\right)\right)\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix}\left(I + O\left(\frac{1}{z}\right)\right)$$

$$= \begin{pmatrix} z + c_1 & c_2 \\ c_3 & 0 \end{pmatrix} \qquad Y^{(n+1)}_{(z)} = \begin{pmatrix} z + c_1 & c_2 \\ c_3 & 0 \end{pmatrix} Y^{(n)}(z)$$

Look at 21 : $-2\pi i \kappa_n^2 P_n \qquad -2\pi i \kappa_{n-1}^2 P_{n-1}$

!! $$Y^{(n+1)}(z) = \begin{pmatrix} z - b_n & \frac{a_n^2}{2\pi i \kappa_{n-1}^2} \\ -2\pi i \kappa_n^2 & 0 \end{pmatrix} Y^{(n)}(z)$$

$$\cdot \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \text{ and } z \to \infty$$

$$\left(I + O\left(\frac{1}{z}\right)\right)\begin{pmatrix} z & 0 \\ 0 & \frac{1}{z} \end{pmatrix} = \begin{pmatrix} z - b_n & \frac{a_n^2}{2\pi i \kappa_{n-1}^2} \\ -2\pi i \kappa_{n-1}^2 & 0 \end{pmatrix}\left(I + \frac{Y^{(n,1)}}{z} + \frac{Y^{(n,2)}}{z^2} + \dots\right)$$

$$O\left(\frac{1}{z^2}\right) = \left(Y^{(n,1)}\right)_{12} + \frac{a_n^2}{2\pi i \kappa_{n-1}^2} + \frac{1}{z}\left(\left(Y^{(n,2)}\right)_{12} - b_n\left(Y^{(n,1)}\right)_{12} + \frac{a_n^2\left(Y^{(n,1)}\right)_{22}}{2\pi i \kappa_{n-1}^2}\right) + O\left(\frac{1}{z^2}\right)$$

$$\left(Y^{(n,1)}\right)_{12} = -\frac{a_n^2}{2\pi i \kappa_{n-1}^2} \qquad z \to \infty$$

$$\left(Y^{(n,1)}\right)_{21} = -2\pi i \kappa_{n-1}^2$$

$$a_n^2 = \left(Y^{(n,1)}\right)_{12}\left(Y^{(n,1)}\right)_{21}$$

$$b_n = \frac{\left(Y^{(n,2)}\right)_{12}}{\left(Y^{(n,1)}\right)_{12}} + \frac{1}{\left(Y^{(n,1)}\right)_{12}}\frac{a_n^2\left(Y^{(n,1)}\right)_{22}}{2\pi i \kappa_{n-1}^2} =$$

$$= \frac{\left(Y^{(n,2)}\right)_{12}}{\left(Y^{(n,1)}\right)_{12}} - \left(Y^{(n,1)}\right)_{22}$$

$$w(a) = \binom{n+k}{a} \qquad a = nx \qquad \frac{k}{n} \to c$$

$$w(x) \approx e^{n\left(x \ln x + (c+1-x)\ln(c+1-x)\right)}$$

$$w(x) = e^{-N V(x)}$$ $V(x)$ is a polynomial of even degree

$N \to \infty \quad N \sim n$

$Y^{(n)}(z)$

$$\Psi_n(z) = Y^{(n)}(z) \begin{pmatrix} e^{-NV(z)/2} & 0 \\ 0 & e^{NV(z)/2} \end{pmatrix}$$

$\Psi_n(z)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$

$$\begin{cases} \Psi_{n,+} = \Psi_{n,-} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{on } \mathbb{R} \\ \Psi_n(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n e^{-NV(z)/2} & 0 \\ 0 & z^{-n} e^{NV(z)/2} \end{pmatrix} \end{cases}$$

$$\frac{\partial \Psi_n}{\partial z} (\Psi_n)^{-1} = A_n(z) \quad \text{entire function}$$

$$\deg A_n(z) = \deg V - 1$$

$$\Psi_{n+1}(z) = \underbrace{\begin{pmatrix} z - b_n & 2\pi i\, a_n^2 \kappa_{n-1} \\ -2\pi i\, \kappa_n^2 & 0 \end{pmatrix}}_{B_n(z)} \Psi_n(z) = B_n(z) \Psi_n(z)$$

$$\begin{cases} \frac{\partial \Psi_n}{\partial z} = A_n(z) \Psi_n(z) \\ \Psi_{n+1}(z) = B_n(z) \Psi_n(z) \end{cases} \quad \text{Lax pair}$$

$$A_{n+1}(z) B_n(z) = B_n'(z) + B_n(z) A_n(z)$$

$$V(x) = \frac{x^4}{4} + t\frac{x^2}{2} \qquad a_n^2 (t + a_{n+1}^2 + a_n^2 + a_{n-1}^2) = \frac{n}{N}$$

Freud equation, string equ.

discrete Painlevé equation

$$n \|P_{n-1}\|_2^2 = \int P_n'(x) P_{n-1}(x) e^{-NV(x)} dx =$$

$$= -\int P_n(x) P_{n-1}'(x) e^{-NV(x)} dx + N \int P_n(x) P_{n-1}(x) V'(x) e^{-NV(x)} dx$$

$$= N \int V'(x) P_n(x) P_{n-1}(x) e^{-NV(x)} dx$$

Asymptotics of orthogonal polynomials

$W(x) = e^{-nV(x)}$ $\qquad Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ s.t.

- Y is analytic in $\mathbb{C} \setminus \mathbb{R}$
- $Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-n\underline{\underline{V(x)}}} \\ 0 & 1 \end{pmatrix}$
- $Y(z) = (I + o(1)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \quad z \to \infty$

$$\Psi \to T \to S \to R$$ T, S, R satisfy RHPs

$$\begin{cases} R_+ = R_- J_R & J_R \to I \text{ as } n \to \infty \\ R(z) = I + O(\tfrac{1}{z}) & z \to \infty \end{cases}$$

$$J[\rho] = \iint \rho(x)\rho(y) \ln |x-y|^{-1} dx\,dy + \int \underline{V(x)}\rho(x)\,dx$$

$$2\int \ln|x-y|\rho(y)\,dy - V(x) = \ell \qquad x \in \operatorname{supp}\rho$$

$$2\int \ln|x-y|\rho(y)\,dy - V(x) \le \ell \qquad x \notin \operatorname{supp}\rho$$

$$\int \frac{\rho(y)\,dy}{y-x} + V'(x) = 0$$

$$g(z) = \int_{\operatorname{supp}\rho} \ln(z-x)\rho(x)\,dx \qquad \int \rho(x)\,dx = 1$$

AS $z \to \infty$ $\quad g(z) = \ln z - \sum_{j=1}^{\infty} \frac{g_j}{z^j}$

$$g_j = \int \frac{x^j}{j}\rho(x)\,dx$$

$$T(z) = \begin{pmatrix} e^{-n\ell/2} & 0 \\ 0 & e^{n\ell/2} \end{pmatrix} \Psi(z) \begin{pmatrix} e^{-n(g(z)-\frac{\ell}{2})} & 0 \\ 0 & e^{n(g(z)-\frac{\ell}{2})} \end{pmatrix}$$

$z \to \infty$

$$T(z) = I + O(\tfrac{1}{z})$$

$$J_T = T_-^{-1} T_+ = \begin{pmatrix} e^{n(g_- -\frac{\ell}{2})} & 0 \\ 0 & e^{-n(g_- -\frac{\ell}{2})} \end{pmatrix} \Psi_-^{-1}\Psi_+ \cdot$$

$$\cdot \begin{pmatrix} e^{-n(g_+-\frac{\ell}{2})} & 0 \\ 0 & e^{n(g_+-\frac{\ell}{2})} \end{pmatrix}$$

T solves Riemann-Hilbert problem

- T is analytic in $\mathbb{C}\setminus\mathbb{R}$
- $T_+(x) = T_-(x) \begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & e^{n(g_+(x)+g_-(x)-V(x)-\ell)} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix}$

– $T(z) = I + O\left(\frac{1}{z}\right) \qquad z \to \infty$

$$\operatorname{supp}\rho = [a,b]$$

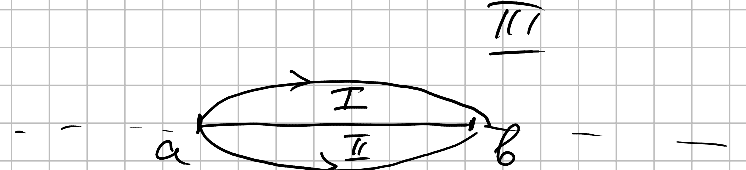
$$\begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & e^{n(g_+(x)+g_-(x)-V(x)-\ell)} \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix} =$$

$$= \begin{cases} \begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & 1 \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix} & x \in \operatorname{supp}\rho \\ \begin{pmatrix} 1 & e^{n\left(\overbrace{g_+(x)+g_-(x)-V(x)-\ell}^{<0}\right)} \\ 0 & 1 \end{pmatrix} & x \notin \operatorname{supp}\rho \end{cases}$$

$$\begin{pmatrix} e^{-n(g_+(x)-g_-(x))} & 1 \\ 0 & e^{n(g_+(x)-g_-(x))} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{n(g_+(x)-g_-(x))} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & 0 \\ e^{-n(g_+(x)-g_-(x))} & 1 \end{pmatrix}$$

$T \to S$ "opening of lenses"



$$S = \begin{cases} T \begin{pmatrix} 1 & 0 \\ e^{-n(g_+(x)-g_-(x))} & 1 \end{pmatrix} & \text{in } \underline{I} \\ T \begin{pmatrix} 1 & 0 \\ e^{n(g_+(x)-g_-(x))} & 0 \end{pmatrix} & \text{in } \underline{II} \\ T & \text{in region } \underline{III} \end{cases}$$

Now jumps for S are exponentially small in n except for the jump at $\operatorname{supp}\rho$ which does not depend on n

Assume that P_∞ is the solution of the

the model RHP $\quad S = R P_\infty$

- P_∞ is analytic in $\mathbb{C} \setminus \operatorname{supp}\rho$
- $P_{\infty,+}(x) = P_{\infty,-}(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\qquad$ P_∞ parametrix
- $P_\infty(z) = I + o(1) \quad z \to \infty$

$\operatorname{supp}\rho = [a,b]$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$\tilde{P}_\infty(z) = \frac{1}{2} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} P_\infty(z) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$\begin{cases} \tilde{P}_\infty(z) \text{ is analytic in } \mathbb{C} \setminus \operatorname{supp}\rho \\ \tilde{P}_{\infty,+}(x) = \tilde{P}_{\infty,-}(x) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad x \in [a,b] \\ \tilde{P}_\infty(z) = I + o(1) \quad z \to \infty \end{cases}$$

$$\tilde{P}_\infty(z) = \begin{pmatrix} \exp\left(\frac{1}{2\pi i} \int_a^b \frac{\ln i}{s-z}\, ds\right) & 0 \\ 0 & \exp\left(\frac{1}{2\pi i} \int_a^b \frac{\ln(-i)}{s-z}\, ds\right) \end{pmatrix} =$$

$$= \begin{pmatrix} \gamma^{-1} & 0 \\ 0 & \gamma \end{pmatrix} \qquad \gamma = \left(\frac{z-a}{z-b}\right)^{1/4}$$ with a cut $[a,b]$

$\gamma(\infty) = 1$

$$P_\infty(z) = \begin{pmatrix} \frac{\gamma(z) + \gamma^{-1}(z)}{2} & \frac{\gamma(z) - \gamma^{-1}(z)}{(-2i)} \\ \frac{\gamma(z) - \gamma^{-1}(z)}{2i} & \frac{\gamma(z) + \gamma^{-1}(z)}{2} \end{pmatrix}$$

Look for S in the form

$S = \tilde{R} P_\infty$ with $S \sim P_\infty \quad n \to \infty$

$\tilde{R} = S P_\infty^{-1}$ should solve RHP!

- $\tilde{R}$ is analytic in $\mathbb{C} \setminus [a,b]$

- $\tilde{R}_+(x) = \tilde{R}_-(x) J_{\tilde{R}}(x) \quad x \in [a, b]$
- $\tilde{R}(z) = I + O(1) \quad z \to \infty$

$J_{\tilde{R}} = P_\infty J_S P_\infty^{-1} \qquad J_{\tilde{R}} \to I$ as $n \to \infty$

pointwise at every $x \in [a, b]$

Next time: local analysis near a, b
then final asymptotic result for Y