Lecture 17 2023-08-09

Last lecture: symplectic group $Sp(2n, \mathbb{C})$

$SP(\lambda|n)$ symplectic tableaux $1, \bar{1} \ldots n, \bar{n}$

semi-standard in this order

$\ell, \bar{\ell}$ cannot appear below row ℓ

symplectic characters

$$sp_\lambda(x_1, x_1^{-1} \ldots x_n, x_n^{-1}) = \sum_{T \in SP(\lambda|n)} \prod x_i^{\#i \text{ in } T}$$

Dual Cauchy-like identity

$$\sum_{\substack{\lambda \subset n \times k \\ \text{rectangle}}} sp_\lambda(x_1^{\pm 1}, \ldots x_n^{\pm 1}) \, sp_{\bar{\lambda}'}(y_1^{\pm 1} \ldots y_k^{\pm 1}) = \prod_{i=1}^{n} \prod_{j=1}^{k} (x_i + x_i^{-1} + y_j + y_j^{-1})$$

$$\bigoplus_\lambda V_{sp_{2n}}(\lambda) \otimes V_{sp_{2k}}(\bar{\lambda}') = \Lambda(\mathbb{C}^{2n} \otimes \mathbb{C}^k)$$

Cauchy-like identity

$$\sum_\lambda sp_\lambda(x_1^{\pm 1} \ldots x_n^{\pm 1}) \, s_\lambda(y_1 \ldots y_n) = \prod_{i<j} (1 - y_i y_j) \prod_{i,j=1}^{n} \frac{1}{(1 - y_i x_j)(1 - y_i x_j^{-1})}$$

$$\mu_{sp}(\lambda) \sim sp_\lambda(X) \, s_\lambda(Y)$$

Baker "Vertex operator realization of symplectic and orthogonal s-functions" 1996

Dan Betea "Correlators for the symplectic and orthogonal Schur measures" 1804.08495

$$H(Y; t) = \prod_i \frac{1}{1 - t y_i} = \sum_{k \geq 0} h_k(Y) t^k \qquad h_k(Y) = \sum_{i_1 \leq i_2 \leq \ldots \leq i_k} y_{i_1} \cdots y_{i_k}$$

homogeneous symm. functions

$$E(Y; t) = \prod_i (1 + t y_i) = \sum_{k \geq 0} e_k(Y) t^k$$

$$e_k(Y) = \sum_{i_1 < \ldots < i_k} y_{i_1} \cdots y_{i_k}$$

elementary symmetric functions

$$H(Y; t) E(Y; -t) = 1$$

$$p_k(Y) = \sum_i y_i^k$$

$$\exp\left(\sum_{k \geq 1} \frac{p_k(Y) t^k}{k}\right) = H(Y; t)$$

$$H(Y, Z) = \prod_{i,j} \frac{1}{1 - y_i z_j} = \exp\left(\sum_{k \geq 1} \frac{p_k(Y) p_k(Z)}{k}\right)$$

$$E(Y, Z) = \prod_{i,j} (1 + y_i z_j) = \exp\left(\sum_{k \geq 1} (-1)^{k+1} \frac{p_k(Y) p_k(Z)}{k}\right)$$

$h_{sp}(Y) = \prod_{i<j}(1-y_iy_j)$ | $h_o(Y) = \prod_{i\le j}(1-y_iy_j)$

$$s_\lambda(Y) = \det\left[h_{\lambda_i-i+j}(Y)\right]_{i,j=1}^{\ell(\lambda)} = \det\left[e_{\lambda'_i-i+j}(Y)\right]_{i,j=1}^{\lambda_1}$$

$$s_{\lambda/\mu}(Y) = \det\left[h_{\lambda_i-i-\mu_j+j}(Y)\right]_{i,j=1}^{\ell(\lambda)}$$

$\mu < \lambda$

$$sp_\lambda(X) = \tfrac{1}{2}\det\left(h_{\lambda_i-i+j}(x) + h_{\lambda_i-i-j+2}(x)\right)_{i,j=1}^{\ell(\lambda)} =$$

$$= \det\left(e_{\lambda'_i-i+j}(X) - e_{\lambda'_i-i-j}(X)\right)_{i,j=1}^{\lambda_1}$$

$\lambda = (\lambda_1 \dots \lambda_n)$ Frobenius coordinates $(a_1\, a_2 \dots \mid b_1, b_2 \dots)$

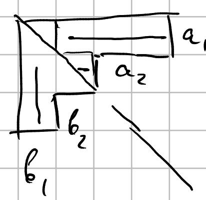
$a_i = \max(\lambda_i - i, 0)$

$b_i = \max(\lambda'_i - i, 0)$

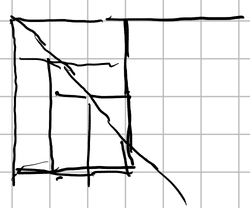
$\lambda = (4,2,1)$

$\downarrow$

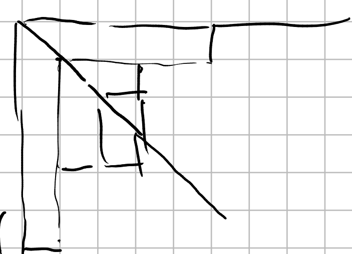
$(3,0,0 \mid 2,0,0,0)$



$\alpha = (2,1,0 \mid 3,2,1)$



$\alpha = (4,1,0 \mid 5,2,1)$



symplectic Schur polynomial

$$sp_\lambda(z_1 \dots z_{2n}) = \sum_{T \in SP(\lambda \mid n)} \prod_i z_i^{\#i \text{ in } T}$$

$z_1 \to \# 1$

$z_2 \to \# \bar{1}$

$z_{2i-1} = x_i$

$z_{2i} = \frac{1}{x_i}$ $\to$ symplectic character

$$sp_\lambda(X) = \sum_\alpha (-1)^{\frac{|\alpha|}{2}} s_{\lambda/\alpha}(X)$$

$$\alpha = (a_1, a_2 \dots \mid a_1+1, a_2+1 \dots)$$

Specialization ρ homomorphism of algebra of symm. functions to $\mathbb{C}$ $\rho(h_k) \in \mathbb{C}$

$$h_k''(\rho)$$

$$H(\rho,t) = \sum_{k\geq 0} h_k(\rho)\, t^k = \exp\left(\sum_{k\geq 1} \frac{p_k(\rho)\, t^k}{k}\right)$$

$$H(\rho,\rho') = \exp\left(\sum_{k\geq 1} \frac{p_k(\rho)p_k(\rho')}{k}\right)$$

$$h_{sp}(\rho) = \exp\left(\sum_{k\geq 1} \frac{p_{2k}(\rho)}{2k} - \frac{p_k^2(\rho)}{2k}\right)$$

Ex: demonstrate that this is an equivalent definition

$$\sum_\lambda sp_\lambda(\rho)\, s_\lambda(\rho') = h_{sp}(\rho)\, H(\rho,\rho') = \exp\left(\sum_{k\geq 1} \frac{p_k(\rho)p_k(\rho')}{k} + \frac{p_{2k}(\rho)}{2k} - \frac{p_k^2(\rho)}{2k}\right)$$

Free fermions

$\mathcal{F}$ $\qquad |S\rangle = \underline{s_1} \wedge \underline{s_2} \wedge \dots \qquad s_1 > s_2 > \dots \qquad s_i \in \mathbb{Z} + \frac{1}{2}$

$$S \longleftrightarrow (\lambda, c) \qquad |\lambda, 0\rangle = |\lambda\rangle$$

$$\mathcal{F} = \bigoplus_{c\in\mathbb{Z}} \mathcal{F}_c \qquad S(\lambda,c) = \{\lambda_i - i + \tfrac{1}{2} + c\}$$

$|0\rangle$

$$\psi_k |S\rangle = \underline{k} \wedge |S\rangle$$

$$\psi_k^* |S\rangle = \begin{cases} (-1)^{i-1}\, \underline{s_1} \wedge \dots \wedge \underline{s_{i-1}} \wedge \underline{s_{i+1}} \wedge \dots & k = s_i \\ 0 & \text{otherwise} \end{cases}$$

$$\{\psi_k, \psi_\ell^*\} = \delta_{k,\ell} \qquad \psi(z) = \sum_k \psi_k z^k$$

$$\psi_k|0\rangle = \psi_{-k}^*|0\rangle = 0 \quad k<0 \qquad \psi^*(w) = \sum_k \psi_k^* w^{-k}$$

Charge operator
$$C = \sum_{k>0} \psi_k \psi_k^* - \sum_{k<0} \psi_k^* \psi_k$$

$$C|\lambda, c\rangle = c|\lambda, c\rangle \qquad \mathcal{F}_0 : C|\lambda\rangle = 0$$

$$\mathcal{F}_0 = \ker C$$

Shift operator $\quad R: \mathcal{F}_c \to \mathcal{F}_{c+1} \qquad R|\lambda, c\rangle = |\lambda, c+1\rangle$

$$\alpha_n = \sum_k \psi_{k-n} \psi_k^* \qquad n = \pm 1, \pm 2 \dots$$

$$\alpha_0 = C$$

$$\alpha_n^* = \alpha_{-n} \qquad \alpha_n|0\rangle = 0 \quad n>0$$

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}$$

$$[\alpha_n, \psi(z)] = z^n \psi(z) \qquad [\alpha_n, \psi^*(w)] = -w^n \psi^*(w)$$

Half-vertex operators

$$\Gamma_\pm(Y) = \exp\left(\sum_{k\geq 1} \frac{p_k(Y)\alpha_{\pm k}}{k}\right)$$

$$\Gamma_\pm(Y) = \prod_{y\in Y} \Gamma_\pm(y) \qquad \Gamma_+(A)\Gamma_+(B) = \Gamma_+(B)\Gamma_+(A)$$

$$\Gamma_+(A)|0\rangle = |0\rangle$$

$$\langle 0|\Gamma_-(A) = \langle 0|$$

$$\Gamma_+(A)\Gamma_-(B) = H(A,B)\Gamma_-(B)\Gamma_+(A)$$

$$\Gamma_\pm(A)\psi(z) = H(A, z^{\pm 1})\psi(z)\Gamma_\pm(A)$$

$$\Gamma_\pm(A)\psi^*(w) = H(A, w^{\pm 1})^{-1}\psi^*(w)\Gamma_\pm(A)$$

$$\langle \lambda, c|\Gamma_+(A)|\mu, d\rangle = \langle \mu, d|\Gamma_-(A)|\lambda, c\rangle = s_{\mu/\lambda}(A)\delta_{c,d}$$

$$\psi(z) = z^{C-\frac{1}{2}} R\, \Gamma_-(z)\Gamma_+^{-1}(z^{-1})$$

$$\psi^*(w) = R^{-1} w^{-C+\frac{1}{2}} \Gamma_-^{-1}(w)\Gamma_+(w^{-1})$$

$$\Gamma_{sp\pm}(X) = \exp\left(\sum_{k\geq 1} \frac{1}{k}\left(p_k(X)\alpha_{\pm k} + \frac{\alpha_{\pm 2k}}{2} - \frac{\alpha_{\pm k}^2}{2}\right)\right)$$

$$\Gamma_{sp+}(X)|0\rangle = |0\rangle \qquad \langle 0|\Gamma_{sp-}(X) = \langle 0|$$

Commutation relations:

1. $\Gamma_{sp+}(X)\Gamma_+(Y) = \Gamma_+(Y)\Gamma_{sp+}(X)$
2. $\Gamma_{sp+}(X)\Gamma_-(Y) = H(X,Y)\,h_{sp}(Y)\,\Gamma_-(Y)\Gamma_+(Y)^{-1}\Gamma_{sp+}(X)$
3. $\Gamma_{sp+}(X)\psi(z)\Gamma_{sp+}^{-1}(X) = H(X,z)\psi(z)\Gamma_+^{-1}(z)$
4. $\Gamma_{sp+}(X)\psi^*(w)\Gamma_{sp+}^{-1}(X) = (1-w^2)H(X,w)^{-1}\psi^*(w)\Gamma_+(w)$

$\psi(z) = z^{C-\frac{1}{2}} R\, \Gamma_-(z)\Gamma_+^{-1}(\frac{1}{z})$ subst. into (3) and get

$$\Gamma_{sp+}(X)\Gamma_-(z)\Gamma_{sp+}^{-1}(X) = H(X,z)\Gamma_-(z)\Gamma_+^{-1}(z)$$

$$\Gamma_{sp+}(X)\Gamma_-(Y)\Gamma_{sp+}^{-1}(X) = \prod_{i=1}^n \Gamma_{sp+}(X)\Gamma_-(y_i)\Gamma_{sp+}^{-1}(X)$$

$$= \prod_{i=1}^n H(X,y_i)\Gamma_-(y_i)\Gamma_+^{-1}(y_i) = H(X,Y)\,h_{sp}(Y)\Gamma_-(Y)\Gamma_+^{-1}(X)$$

this gives (2) from (3)

$$\Gamma_{sp+}(x) = \exp(M_1 + M_2 + M_3) \qquad M_1 = \sum_{k\geq 1} \frac{p_k(x)}{k}\alpha_k$$

$$M_2 = \sum_{k \geq 1} \frac{\alpha_{2k}}{2k}$$

$$e^{M_1}\psi(z)e^{-M_1} = H(x,z)\psi(z) \qquad M_3 = -\sum_{k\geq 1}\frac{\alpha_k^2}{2k}$$

Baker-Campbell-Hausdorff formula

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]]$$

$$[\alpha_n, \psi(z)] = z^n \psi(z)$$

$$e^{M_2}\psi(z)e^{-M_2} = \frac{1}{\sqrt{1-z^2}}\psi(z)$$

$$[\alpha_n^2, \psi(z)] = z^n\alpha_n\psi(z) + z^n\psi(z)\alpha_n = z^{2n}\psi(z) + 2z^n\psi(z)\alpha_n$$

$$[M_3, \psi(z)] = \psi(z)\left(a(z) - M_+(z)\right)$$

$$a(z) = \ln\sqrt{1-z^2} \qquad M_+(z) = \sum_{k\geq 1}\frac{z^k}{k}\alpha_k = \ln\Gamma_+(z)$$

$$\underbrace{[M_3[M_3 \ldots [M_3}_{k}, \psi(z)] \ldots]] = \psi(z)\left(a(z) - M_+(z)\right)^k$$

$$e^{M_3}\psi(z)e^{-M_3} = \psi(z) + [M_3, \psi(z)] + \ldots = \psi(z)e^{a(z) - M_+(z)} =$$

$$= \sqrt{1-z^2}\,\psi(z)\Gamma_+^{-1}(z)$$

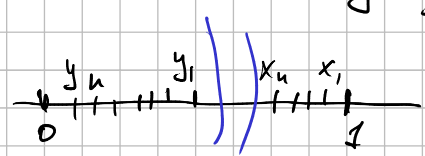
(4) is derived in the same way $\sqrt{1-w^2}$

$$[\alpha_n, \psi^*(w)] = -w^n\psi^*(w)$$

$$[M_3, \psi^*(w)] = \psi^*(w)\left(a(w) + M_+(z)\right)$$

$$\mu_{sp}(\lambda) = \frac{sp_\lambda(x)s_\lambda(y)}{H(x,y)h_{sp}(y)} \qquad 1 \geq x_1 \geq x_2 \ldots \geq x_n > y_1 \geq y_2 \geq \ldots \geq y_n \geq 0$$

$= Z$



$$\rho_m^{sp}(k_1 \ldots k_m) = \mu_{sp}\left(\{k_1 \ldots k_m\} \subset \{\lambda_i - i + \tfrac{1}{2}\}\right)$$

Thm $\rho_m^{sp}(k_1 \ldots k_m) = \det\left[K_{sp}(k_i, k_j)\right]_{i,j=1}^m$

$$K_{sp}(a,b) = \iint\limits_{y_1 < |w| < |z| < x_n} \frac{F_{sp}(z)}{F_{sp}(w)} \frac{(1-w^2)}{(1-wz)(1-wz^{-1})} \frac{dz\,dw}{(2\pi i)^2 z^{a+\frac{3}{2}} w^{-b+\frac{1}{2}}}$$

$$F_{sp}(z) = \frac{M(X,z)}{M(Y,z)\,M(Y,z^{-1})}$$