Lecture 6. 07.06.2023

Robinson-Schensted-Knuth

$$n! = \sum_{\lambda \vdash n} (\dim \lambda)^2$$

perm $(1 \dots n) \longleftrightarrow P, Q \in SYT(\lambda | n)$

$\underbrace{3\,7\,1\,2\,6\,5\,4}$

$\boxed{3} \to \boxed{3|7} \to \begin{matrix}1&7\\3\end{matrix} \to \begin{matrix}1&2\\3&7\end{matrix} \to \begin{matrix}1&2&6\\3&7\end{matrix} \to \begin{matrix}1&2&5\\3&6\\7\end{matrix} \to \begin{matrix}1&2&4\\3&5\\6\\7\end{matrix}$

$\boxed{1} \to \boxed{1|2} \quad \begin{matrix}1&2\\3\end{matrix} \quad \begin{matrix}1&2\\3&4\end{matrix} \to \begin{matrix}1&2&5\\3&4\end{matrix} \to \begin{matrix}1&2&5\\3&4\\6\end{matrix} \to \begin{matrix}1&2&5\\3&4\\6\\7\end{matrix}$

$i_1\ i_2 \dots i_k$

$1 \le i_j \le n$

$3\,1\,2\,2\,1\,2 \to 3 \to \begin{matrix}1\\3\end{matrix} \to \begin{matrix}1&2&2\\3\end{matrix} \to \begin{matrix}1&1&2\\2\\3\end{matrix} \to \begin{matrix}1&1&2&2\\2\\3\end{matrix}$

Semistandard Young tableau

$\{\ \} \longleftrightarrow P, Q \quad \begin{matrix}P \in SSYT(\lambda | n)\\ Q \in SYT(\lambda | k)\end{matrix}$

Schur-Weyl duality $\quad S_k \quad GL(n)$

$V = \mathbb{C}^n$

$$\overbrace{V \otimes \dots \otimes V}^{k \text{ times}} = \bigoplus V_{GL(n)}(\lambda) \otimes V_{S_k}(\lambda)$$

$\ell(\lambda) \le n$

$V = \langle e_1 \dots e_n \rangle \qquad e_{i_1} \otimes \dots \otimes e_{i_k} \longleftrightarrow e_P \otimes e_Q$

$\boxed{i_j} \to e_{i_j}$

Even weaker conditions:

$i_1 \dots i_k$ are allowed to be the same

$\ell_1 \dots \ell_k \qquad \ell_1 \le \ell_2 \le \dots \le \ell_k$

(?)

$3\,1\,2\,2\,1\,2 \qquad 3 \to \begin{matrix}1\\3\end{matrix} \to \begin{matrix}1&2\\3\end{matrix} \to \begin{matrix}1&2&2\\3\end{matrix} \to \begin{matrix}1&1&2\\2\\3\end{matrix} \to \begin{matrix}1&1&2&2\\2\\3\end{matrix}$

$1\,1\,2\,2\,3\,4 \qquad 1 \to \begin{matrix}1\\1\end{matrix} \to \begin{matrix}1&2\\1\end{matrix} \quad \begin{matrix}1&2&2\\1\end{matrix} \to \begin{matrix}1&2&2\\1\\3\end{matrix} \to \begin{matrix}1&2&2&4\\1\\3\end{matrix}$

$\begin{matrix}3\,1\,2\,2\,1\,2\\1\,1\,2\,2\,3\,4\end{matrix} \to$

$3 \to \begin{matrix}1\\3\end{matrix} \to \begin{matrix}1&2\\3\end{matrix} \to \begin{matrix}1&2\\2\\3\end{matrix} \to \begin{matrix}1&2\\1\\2\\3\end{matrix} \to \begin{matrix}1&2\\1&2\\2\\3\end{matrix} \to \begin{matrix}1&1&2&3\\2&2\end{matrix}$

$1 \to \begin{matrix}1\\1\end{matrix} \to \begin{matrix}1&2\\1\end{matrix} \to \begin{matrix}1&2\\1\\2\end{matrix} \to \begin{matrix}1&2\\1\\2\\3\end{matrix} \to \begin{matrix}1&2\\1&4\\2\\3\end{matrix}$

dual RSK

$P' \in SSYT(\lambda | n)$

$Q \in SSYT(\lambda | k)$

$$\Lambda(\mathbb{C}^n \otimes \mathbb{C}^k) = \bigoplus_{\lambda} V_{GL(n)}(\lambda) \otimes V_{GL(k)}(\lambda')$$

$\downarrow$ $\quad$ $\downarrow$

$e_1 \ldots e_n \quad \tilde{e}_1 \ldots \tilde{e}_k$

$$(e_{i_1} \otimes \tilde{e}_{j_1}) \wedge (e_{i_2} \otimes \tilde{e}_{j_2}) \ldots$$

$$\overset{\shortparallel}{e_{i_1 j_1}}$$

n rows
k columns

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{matrix} 1 & 3 & 1 & 2 & 4 & 1 & 4 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 \end{matrix}$$

SSYT$(\lambda' | k)$

ψ

$$\begin{matrix} 1 & 3 & 1 & 2 & 4 & 1 & 4 \\ 1 & 1 & 2 & 2 & 2 & 3 & 3 \end{matrix}$$

$$1 \quad\; \begin{matrix}1&3\end{matrix} \quad\; \begin{matrix}1&3\\1&\end{matrix} \quad\; \begin{matrix}1&2\\1&3\end{matrix} \quad\; \begin{matrix}1&2&4\\1&3&\end{matrix} \quad\; \begin{matrix}1&2&4\\1&3&\\1&&\end{matrix} \quad\; \begin{matrix}1&2&4\\1&3&4\\1&&\end{matrix}$$

$$1 \quad\; \begin{matrix}1&1\end{matrix} \quad\; \begin{matrix}1&1\\2&\end{matrix} \quad\; \begin{matrix}1&1\\2&2\end{matrix} \quad\; \begin{matrix}1&1&2\\2&2&\end{matrix} \quad\; \begin{matrix}1&1&2\\2&2&\\3&&\end{matrix} \quad\; \begin{matrix}1&1&2\\2&2&3\\3&&\end{matrix}$$

$\nearrow$

SSYT(λ / n)

Representation theory of $GL(n)$

$V \qquad M \in GL(V) \qquad V = \mathbb{C}^n \qquad GL(n, \mathbb{C})$

$\dim V = n \qquad M$ - $n \times n$ ~~mat~~ matrix $\qquad \det M \neq 0$

Lie group is group that is smooth manifold

$g_1 \cdot g_2$

$gl(V) \ni a, b \quad [a, b] = ab - ba$

Representation ρ on $V \qquad \rho : G \to GL(V)$

$\rho(g_1 \cdot g_2) = \rho(g_1) \cdot \rho(g_2)$

$\rho_1 \; \rho_2$ on $V_1, V_2 \qquad \rho_1 \cong \rho_2 \qquad T : V_1 \to V_2$

$T \rho_1(g) T^{-1} = \rho_2(g)$

$\forall g \in G$

R - ring $(+, \cdot)$

Left R-module $M \qquad (M, +)$

$\mathbb{C}[G]$ - module $V \qquad (\alpha g_1 + \beta g_2) v = g_1(\alpha v) + g_2(\beta v)$

R - commutative ring $\qquad R$-module E, λ - Young diagram we will construct R-module E^λ

if $\lambda = \underbrace{\boxed{}}_{}$ (n boxes) $\qquad E^\lambda = \mathrm{Sym}^n E$

if $\lambda = \boxed{}$ (column) $\qquad E^\lambda = \Lambda^n E$

$n = |\lambda|$

$$\varphi : E^{\times\lambda} \to F \qquad F \text{ - } R\text{-module}$$

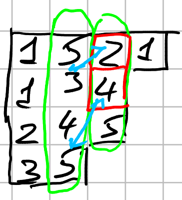
1. φ is R-multilinear
2. φ is alternating in the entries of any column of λ

$$\varphi(v) = -\varphi(v') \quad \text{if } v' \text{ is obtained from } v \text{ by exchanging 2 values in one column}$$

3. $\forall\, v \in E^{\times\lambda} \qquad \varphi(v) = \sum_w \varphi(w)$

where w are obtained from v by exchanges in $\overset{\text{2 given}}{\vee}$ columns with a given subset of boxes in the right column

exchange



$$\longrightarrow \quad \begin{matrix} 1 & 5 & 3 & 1 \\ 1 & 2 & 5 & \\ 2 & 4 & 5 & \\ 3 & 4 & & \end{matrix}$$

$\binom{c}{k}$ elements in the sum

$$\varphi\begin{pmatrix} x & \boxed{u} \\ y & v \\ z & w \end{pmatrix} = \varphi\begin{pmatrix} u & x \\ y & v \\ z & w \end{pmatrix} + \varphi\begin{pmatrix} x & y \\ u & v \\ z & w \end{pmatrix} + \varphi\begin{pmatrix} x & z \\ y & v \\ u & w \end{pmatrix}$$

$$\varphi\begin{pmatrix} x & \boxed{\begin{matrix} u \\ v \end{matrix}} \\ y & \\ z & w \end{pmatrix} = \varphi\begin{pmatrix} u & x \\ v & y \\ z & w \end{pmatrix} + \varphi\begin{pmatrix} u & x \\ y & z \\ v & w \end{pmatrix} + \varphi\begin{pmatrix} x & y \\ u & z \\ v & w \end{pmatrix}$$

$$\varphi\begin{pmatrix} x & \boxed{\begin{matrix} u \\ v \\ w \end{matrix}} \\ y & \\ z & \end{pmatrix} = \varphi\begin{pmatrix} u & x \\ v & y \\ w & z \end{pmatrix}$$

Define Schur module E^λ to be an R-module s.t.

map $E^{\times\lambda} \to E^\lambda \quad v \to v^\lambda$ satisfies (1), (2), (3)

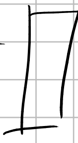
$\forall\ \varphi : E^{\times\lambda} \to F$ satisfying (1-3) $\quad \exists\ \tilde{\varphi} : E^\lambda \to F$

s.t. $\varphi(v) = \tilde{\varphi}(v^\lambda)$ for all v in $E^{\times\lambda}$

$\lambda = \boxed{} \qquad E^\lambda$ is $\mathrm{Sym}^n(E)$

$$E^{\otimes n} / \{ v_1 \otimes \ldots \otimes v_n - v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \}$$

for all σ in S_n

$\lambda =$  E^{λ} is $\Lambda^n(E)$

$E^{\otimes n} / \{v_1 \otimes \dots \otimes v_n \quad v_i = v_j\}$

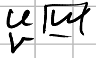
$E^{\times\lambda}$

(1): $E^{\otimes|\lambda|}$ $\quad \lambda = (\lambda_1 \dots \lambda_{\ell(\lambda)})$ $\quad \mu = \lambda'$

(2): $\Lambda^{\mu_1} E \otimes_R \dots \otimes_R \Lambda^{\mu_\ell} E$ $\quad \mu_1 \dots \mu_\ell$ column length of λ

$$E^{\times\lambda} \ni \begin{array}{|c|c|} \hline x & u \\ \hline y & v \\ \hline z & w \\ \hline \end{array} \to (x \wedge y \wedge z) \otimes (u \wedge v \wedge w)$$

(3): $E^{\lambda} = \Lambda^{\mu_1} E \otimes \dots \otimes \Lambda^{\mu_\ell} E / Q^{\lambda}(E)$

  $\qquad \wedge v - \sum_w \wedge w$

$E^{(2,1)} = \Lambda^2 E \otimes E / \{u \wedge v \otimes w - w \wedge v \otimes u - u \wedge w \otimes v\}$

$E \to F \qquad E^{\lambda} \to F^{\lambda}$

$R \to R' \qquad (E \otimes_R R')^{\lambda} \cong E^{\lambda} \otimes_R R'$

$e_1 \dots e_m \in E \qquad T$ - filling of λ with $(1 \dots m)$

$E^{\times\lambda} \qquad \boxed{i} \leftrightarrow e_i \qquad E^{\lambda}$ is e_T

Lemma 1. If $e_1 \dots e_m$ is basis of E then $E^{\lambda} = \{e_T\}/Q$

Q: (i) e_T if T has two equal entries in any column

(ii) $e_T + e_{T'}$ $\quad T'$ is obtained from T by interchanging 2 values in a column

(iii) $e_T - \sum_S e_S$ $\quad S$ is obtained from T by an exchange

Lemma 2. $\det M \cdot \det N = \sum \det M' \cdot \det N'$

M, N are $p \times p$ matrices

M', N' are obtained by interchanging k columns of N with any k columns of M.

$$|v_1 \dots v_p| \cdot |w_1 \dots w_p| = \sum_{i_1 < \dots < i_k} |v_1 \dots w_1 \dots w_k \dots v_p| \cdot |v_{i_1} \dots v_{i_k} w_{k+1} \dots w_p|$$

$p+1$ vectors $v_1 \dots v_p, w_1 \qquad \Lambda^{p+1} \mathbb{C}^p = 0$

$v_i = v_{i+1}$ ✓

$v_p = w_1$ $i_1 < \dots < i_{k-1}$