Lecture 16 2023.07.14

Orthogonal groups

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$ 2x2 matrix $\varphi \in [0, 2\pi)$

$SO(2)$

$\mathbb{C}^n$ $\quad SO(n,\mathbb{C}) \subset GL(n,\mathbb{C})$

$$SO(n,\mathbb{C}) = \{ g \in GL(n,\mathbb{C}),\ (g v_1, g \cdot v_2) = (v_1, v_2) \}$$

$n \to \infty$ $\quad (\mathbb{C}^n)^{\otimes k} = \bigoplus M^\lambda V_{SO(n)}(\lambda)$

$$\mu_n^{SO}(\lambda) = \frac{M^\lambda \dim V_{SO(n)}(\lambda)}{n^k}$$

M^λ — dim of Brauer algebra representations

Symplectic Lie groups

$\mathbb{C}^{2n}$ $(\mathbb{R}^{2n})$

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$(v_1, \Omega v_2)$

$$Sp(2n,\mathbb{C}) = \{ M \in GL(2n,\mathbb{C}) : (M v_1, \Omega M v_2) = (v_1, \Omega v_2) \quad M^T \Omega M = \Omega \}$$

rank $sp(2n,\mathbb{C}) = n$

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$n \to \infty$

$V_{Sp(2n,\mathbb{C})}(\lambda)$ $\quad \lambda = (\lambda_1 \dots \lambda_n)$

King's tableaux

GL_n λ $\quad SP(\lambda, n)$

$SSYT(\lambda, n)$

$1, \bar{1}, 2, \bar{2} \dots n, \bar{n}$

$1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n}$

$\bar{1}$	2	$\bar{3}$
2	3	
$\bar{3}$		

$\to x_1^{-1} x_2^2 x_3^{\cdot} x_3^{-2}$

- weak increase in rows
- strict increase in columns
- entries in row i are $\geq i$

(even) Symplectic Schur polynomials = characters of symplectic group irreps

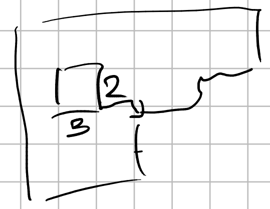
$$sp_\lambda(x_1, x_1^{-1}, x_2, x_2^{-1} \dots x_n, x_n^{-1}) = \sum_{T \in SP(\lambda, n)} \prod_{i=1}^{n} x_i^{\#i\text{'s in } T}$$

$$sp_\lambda(x_1^{\pm1}, x_2^{\pm1} \dots x_n^{\pm1}) = \frac{\det\left(x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)}\right)_{ij=1}^n}{\det\left(x_i^{n-j+1} - x_i^{-(n-j+1)}\right)_{ij=1}^n}$$

Schensted insertion → Berele insertion

row i $\boxed{\bar{i}}$ ← i

I need erase



$\downarrow 1$

$$\begin{matrix} \bar1 & 2 & \bar2 \\ 2 & 3 & 3 \\ \bar3 & \bar3 & \end{matrix} \to \begin{matrix} \square & 2 & \bar2 \\ 2 & 3 & 3 \\ \bar3 & \bar3 & \end{matrix} \to \begin{matrix} 2 & 2 & \bar2 \\ \square & 3 & 3 \\ \bar3 & \bar3 & \end{matrix} \to \begin{matrix} 2 & 2 & \bar2 \\ 3 & \square & 3 \\ \bar3 & \bar3 & \end{matrix} \to \begin{matrix} 2 & 2 & \bar2 \\ 3 & 3 & \\ \bar3 & 3 & \end{matrix}$$

Proctor algorithm (symplectic analogue of dual RSK)

$$\Lambda(\mathbb{C}^n \otimes \mathbb{C}^{2k}) = \bigoplus_{\lambda \subset k^n} V_{Sp(2n,\mathbb{C})}(\lambda) \otimes V_{Sp(2k,\mathbb{C})}(\bar\lambda')$$

$$\Lambda(\mathbb{C}^n \otimes \mathbb{C}^k) = \bigoplus_{\lambda \subset k^n} V_{GL_n}(\lambda) \otimes V_{GL_k}(\lambda')$$

$n \times 2k$ 0-1 matrix

$\updownarrow$

$P \in SP(\lambda | n)$

$Q \in SP(\bar\lambda', k)$

$n \times n$ 0-1 matrices $\xleftrightarrow[\text{dual RSK}]{\text{bijection}}$ $P \in SSYT(\lambda | n)$, $Q \in SSYT(\lambda' | k)$

skew $GL_n \times GL_k$ Howe duality

$$n\overbrace{\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ 1 & 1 & & & \end{pmatrix}}^{2k}$$

$$n=4 \begin{cases} & 0 \ 0 & \\ & 0 \ 1 & i \\ n-1 & 1 \ 1 & \\ n & 0 \ 1 & \end{cases}$$

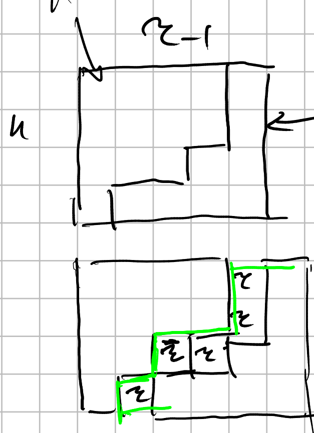
$\bar3, 1$

if in row i we have

0 0	→ i
0 1	→ nothing
1 0	→ $\bar{i}, i$
1 1	→ $\bar{i}$

On step $r-1$ P, Q

$P \in SP(\lambda, n)$



$\leftarrow Q \in SP(\bar\lambda'_{n \times r-1}, r-1)$

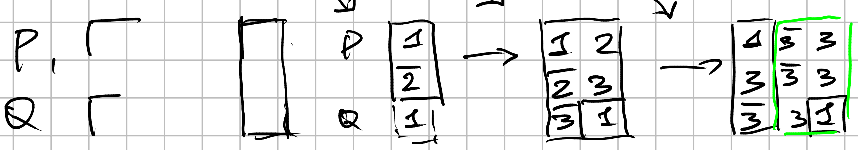
n empty boxes do Berele insertion

$k=3$

$$n=3 \quad \begin{matrix} 0 & 0 & | & 0 & 1 & | & 1 & 0 \\ 1 & 1 & | & 0 & 0 & | & 0 & 1 \\ 0 & 1 & | & 1 & 0 & | & 0 & 1 \end{matrix}$$

$\bar2, 1 \mid \bar3, 3, 2 \mid \bar1, 1$

$$\begin{matrix} 1 \\ 2 \end{matrix} \to \begin{matrix} 1 & \bar3 \\ \bar2 & \end{matrix} \to \begin{matrix} 1 & 3 \\ \bar2 & \bar3 \end{matrix} \to \begin{matrix} 1 & 2 \\ \bar2 & 3 \\ \bar3 & \end{matrix}$$



$P = \begin{matrix} 1 \\ \bar3 \\ 3 \end{matrix}$

$$\begin{array}{ll} 1 & 2 \\ \bar{2} & 3 \\ \bar{3} & \end{array} \to \begin{array}{ll} 1 & \bar{1} \\ \square & 3 \\ \bar{3} & \end{array} \to \begin{array}{ll} 1 & \bar{1} \\ 3 & \square \\ \bar{3} & \end{array} \to \begin{array}{ll} 1 & \square \\ 3 & \\ \bar{3} & \end{array} \qquad Q = \begin{vmatrix} 1 & 3 & 3 \\ 3 & \bar{3} & \bar{3} \end{vmatrix}$$

$$\sum_{\lambda \subset k^n} sp_\lambda(x_1^{\pm 1} \dots x_n^{\pm 1})\, sp_{\bar{\lambda}'}(y_1^{\pm 1} \dots y_k^{\pm 1}) =$$

$$= \prod_{i=1}^n \prod_{j=1}^k \left(x_i^{1/2} y_j^{1/2} + x_i^{-\frac{1}{2}} y_j^{-\frac{1}{2}}\right)\left(x_i^{\frac{1}{2}} y_j^{-\frac{1}{2}} + x_i^{-\frac{1}{2}} y_j^{\frac{1}{2}}\right) =$$

$$= \prod_{i,j} (1 + x_i y_j)\left(\frac{1}{x_i} + \frac{1}{y_j}\right) = \prod_{ij} (x_i + y_j)\left(1 + \frac{1}{x_i y_j}\right) =$$

$$= \prod_{ij} \left(x_i + y_j + \frac{1}{x_i} + \frac{1}{y_j}\right)$$

$$\mu^{sp}_{n,k}(\lambda \mid \{x_i\}, \{y_j\}) = \frac{sp_\lambda(x)\, sp_{\bar{\lambda}'}(y)}{\prod_{ij}(x_i + y_j + x_i^{-1} + y_j^{-1})}$$

Put 1 at position $(i, 2j-1)$ with $\mathbb{P} = \frac{1}{1 + x_i y_j}$

$$\mathbb{P}(0 \text{ at } (i, 2j-1)) = \frac{x_i y_j}{1 + x_i y_j}$$

$$\mathbb{P}(1 \text{ at } (i, 2j)) = \frac{1}{1 + x_i/y_j}$$

$$\mathbb{P}(0 \text{ at } (i, 2j)) = \frac{1}{1 + y_j/x_i}$$

$$x_i = f\left(\frac{i}{n}\right)$$

$$y_j = g\left(\frac{j}{k}\right)$$

$$f, g \in C([0,1])$$

$$n, k \to \infty$$

$\mu_{n,k}(\lambda \mid f, g)$ – not so easy

$$\mu_{n,k}(\lambda) = \frac{\dim V_{sp(2n,\mathbb{C})}(\lambda)\, \dim V_{sp(2k,\mathbb{C})}(\bar{\lambda}')}{2^{2nk}}$$

The limit shape is "half" of limit shape for $GL_{2n} \times GL_{2k}$