Lecture 18 $\qquad X = x_1, x_1^{-1}, x_2, x_2^{-1} \dots x_n, x_n^{-1}$

$$\mu_{sp}(\lambda \mid X, Y) = \frac{sp_\lambda(X)\, S_\lambda(Y)}{H(X,Y)\, h_{sp}(Y)} \qquad H(X,Y)\, h_{sp}(Y) = Z$$

m-point corr. function $\rho_{sp}(k_1 \dots k_m) = \mu_{sp}(\{k_1 \dots k_m\} \subset \{\lambda_i - i + \tfrac12\}_{i=1}^{\infty})$

Thm (Be-ka '18)

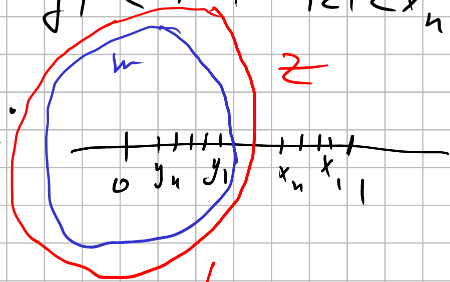
$$\rho_{sp}(k_1 \dots k_m) = \det\left[K_{sp}(k_i, k_j)\right]_{i,j=1}^{m}$$

$$K_{sp}(a,b) = \oint_z \oint_w \frac{F_{sp}(z)}{F_{sp}(w)} \frac{(1-w^2)}{(1-wz)(1-\frac{w}{z})} \frac{dw\, dz}{(2\pi i)^2 z^{a+\frac32} w^{-b+\frac12}}$$

$$F_{sp}(z) = \frac{H(X,z)}{H(Y,z)\, H(Y,z^{-1})}$$

$0 < y_n \le y_{n-1} \le \dots \le y_1 < x_n \le \dots \le x_1 < 1$

$y_1 < |w| < |z| < x_n$



Proof:

$$\rho_{sp}(k_1 \dots k_m) = \frac{1}{Z} \langle 0 | \Gamma_{sp+}(X) \prod_{i=1}^{m} \psi_{k_i} \psi^*_{k_i} \cdot \Gamma_-(Y) | 0 \rangle =$$

$$= \frac{1}{Z} \left[\frac{z_1^{k_1} \dots z_m^{k_m}}{w_1^{k_1} \dots w_m^{k_m}} \right] \langle 0 | \Gamma_{sp+}(X) \prod_{i=1}^{m} \psi(z) \psi^*(w) \cdot \Gamma_-(Y) | 0 \rangle$$

$[z^k] f(z)$ – coeff of z^k in the function $f(z)$

1-pt correlator $\qquad \Gamma_{sp+}^{-1}(X)\Gamma_{sp+}(X)$

$$\frac{1}{Z} \langle 0 | \Gamma_{sp+}(X) \psi(z) \psi^*(w) \Gamma_-(Y) | 0 \rangle =$$

$$\Gamma_{sp+}(X)\psi(z)\Gamma_{sp+}^{-1}(X) = \dots \psi(z)\Gamma_+^{-1}(z)$$

$$= (1-w^2) \frac{H(X,z)}{H(X,w)} \langle 0 | \psi(z) \Gamma_+^{-1}(z) \psi^*(w) \Gamma_+(w) \Gamma_-(Y) \cdot | 0 \rangle =$$

$$= (1-w^2) \frac{H(X,z)\, H(Y,w)\, H(Y,w^{-1})}{H(X,w)\, H(Y,z)\, H(Y,z^{-1})} \langle 0 | \psi(z) \Gamma_+^{-1}(z) \psi^*(w) | 0 \rangle$$

$F_{sp}(z)$ $\qquad z^{C-\frac12} R\, \Gamma_-(z)\, \Gamma_+^{-1}(z^{-1})$

$$= \frac{F_{sp}(z)}{F_{sp}(w)} (1-w^2) \sqrt{\frac{w}{z}} \frac{1}{(1-wz)(1-wz^{-1})}$$

$$\mathrm{Ad}_G A = G A G^{-1}$$

$$\frac{1}{Z}\langle 0|\Gamma_{sp+}(x)\prod_i \psi(z_i)\psi^*(w_i)\Gamma_-(y)|0\rangle = \frac{1}{Z}\langle 0|\Big(\prod_i \mathrm{Ad}_{\Gamma_{sp+}(x)}\psi(z_i)\cdot \mathrm{Ad}_{\Gamma_{sp+}(x)}\psi^*(w_i)\Big)\Gamma_{sp+}(x)\Gamma_-(y)|0\rangle =$$

$$= \prod_i (1-w_i^2)\frac{H(x,z_i)}{H(x,w_i)}\langle 0|\Big(\prod_i \psi(z_i)\Gamma_+^{-1}(z_i)\psi^*(w_i)\Gamma_+(w_i)\Big)\cdot\Gamma_-(y)|0\rangle =$$

$$= \prod_i(1-w_i^2)\frac{H(x,z_i)}{H(x,w_i)}\frac{H(y,w_j)}{H(y,z_i)}\prod_{i<j}\frac{H(w_i,z_j)}{H(z_i,z_j)H(w_i,w_j)}\prod_{i\le j}H(z_i,w_j)$$

$$\langle 0|\prod_i\psi(z_i)\psi^*(w_i)\Gamma_-(y)|0\rangle = \prod_{i=1}^m(1-w_i^2)\frac{H(x,z_i)H(y,w_i)}{H(x,w_i)H(y,z_i)}\cdot\frac{H(y,w_i^{-1})}{H(y,z_i^{-1})}\prod_{i<j}(1-z_iz_j)(1-w_iw_j)\prod_{i,j=1}^m\frac{1}{1-z_iw_j}\langle 0|\prod_i\psi(z_i)\psi^*(w_i)|0\rangle$$

$$= \prod_{i=1}^m(1-w_i^2)\underbrace{\frac{H(x,z_i)H(y,w_i)H(y,w_i^{-1})}{H(x,w_i)H(y,z_i)H(y,z_i^{-1})}}_{F_{sp}(z)}\prod_{i<j}(1-z_iz_j)(1-w_iw_j)\prod_{i,j}\frac{1}{1-z_iw_j}\prod_i\sqrt{\frac{w_i}{z_i}}\prod_{i<j}(z_i^{-1}-z_j^{-1})(w_i-w_j)\prod_{i,j}\frac{1}{1-w_iz_j^{-1}} = (=)$$

BC-Cauchy determinant

$$\det\left[\frac{1}{(1-w_iz_j)(1-w_iz_j^{-1})}\right]_{i,j=1}^m = \prod_{i<j}(z_i-z_j)(w_i-w_j)(1-w_iw_j)\cdot\Big(1-\frac{1}{z_iz_j}\Big)\prod_{i,j=1}^m\frac{1}{(1-w_iz_j)(1-w_iz_j^{-1})}$$

$$(=)\ \det\left[(1-w_i^2)\sqrt{\frac{w_i}{z_i}}\frac{F_{sp}(z_i)}{F_{sp}(w_i)}\frac{1}{(1-w_iz_j)(1-w_iz_j^{-1})}\right]_{i,j=1}^m$$

$$\rho_{sp}(k_1\dots k_m) = \oint_{z_1}\oint_{w_1}\cdots\oint_{z_m}\oint_{w_m}\prod_{i=1}^m\frac{w_i^{k_i-1}}{z_i^{k_i+1}}\frac{dz_i\,dw_i}{(2\pi i)^2}\det\left[\frac{F_{sp}(z_i)}{F_{sp}(w_i)}\cdot\frac{(1-w_i^2)w_i^{1/2}z_i^{-1/2}}{(1-w_iz_j)(1-w_iz_j^{-1})}\right]_{i,j=1}^m = \det\left[\oint\oint_{z,w}\frac{F_{sp}(z)}{F_{sp}(w)}\frac{1-w^2}{(1-wz)(1-wz^{-1})}\frac{dz\,dw}{z^{k_i+3/2}}\frac{1}{w^{-k_j+1/2}}\frac{1}{(2\pi i)^2}\right]_{i,j=1}^m$$

$$\sum_\lambda sp_\lambda(X)\, s_\lambda(Y) = \prod_{i<j}(1-y_iy_j)\prod_{ij}\frac{1}{(1-y_ix_j^{-1})(1-y_ix_j)}$$

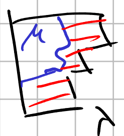
θ (Pois.) Plancherel specialization $= Pl_\theta$ $\qquad p_1(Pl_\theta)=\theta$

$$h_k(Pl_\theta) = \frac{\theta^k}{k!} \qquad p_k(Pl_\theta)=0 \quad k\geq 2$$

$$P_{sp}(\lambda) = \exp\left(-\frac{3\theta}{2}\right) sp_\lambda(Pl_{2\theta})\, s_\lambda(Pl_\theta)$$

$$\theta\to\infty \qquad |\lambda|\to\infty \qquad sp_\lambda(Pl_{2\theta}) \sim \theta^{|\lambda|}$$

$$s_{\lambda/\mu}(Pl_\theta) = \theta^{|\lambda/\mu|}\frac{\dim(\lambda/\mu)}{|\lambda/\mu|!} \qquad |\lambda/\mu| = |\lambda|-|\mu|$$



$$\dim(\lambda/\mu) = \#SSYT(\lambda/\mu)$$

$$P_{sp}(\lambda) \sim \theta^{|\lambda|}\frac{\dim\lambda}{|\lambda|!}\sum_\alpha (-1)^{\frac{|\alpha|}{2}}(2\theta)^{|\lambda/\alpha|}\frac{\dim(\lambda/\alpha)}{|\lambda/\alpha|!}$$

$$\alpha\in\{(a_1,a_2\dots \mid a_1+1, a_2+1\dots)\}$$

$$F(z) = \exp\left(\theta(z-z^{-1})\right) = \sum_{k\in\mathbb{Z}} z^k J_k(2\theta)$$

J_k – Bessel function of the first kind

$$x^2f''(x) + xf'(x) + (x^2-k^2)f(x) = 0$$

finite at $x=0$

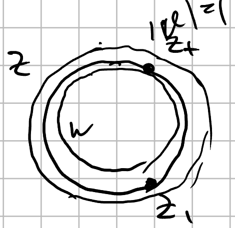
$$J_k(z) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,(m+k)!}\left(\frac{z}{2}\right)^{2m+k}$$

$$\theta\to\infty \qquad K_{sp}(a,b) \qquad a=\alpha\theta+m \quad b=\alpha\theta+l$$

$$K_{sp}(a,b) = K^{\alpha}_{sp}(m,l) = \iint\frac{dz}{2\pi i}\frac{dw}{2\pi i}\, e^{\theta(S(z)-S(w))}\frac{1-w^2}{(1-wz)(1-wz^{-1})}\frac{w^{l-1}}{z^m}$$

$$S(z) = z - z^{-1} - \alpha\ln z$$

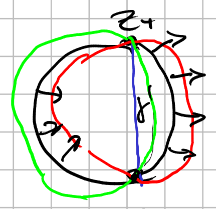
$$\frac{1-w^2}{(1-wz)(1-wz^{-1})} = \frac{w}{z}\frac{1}{1-wz^{-1}} + \underbrace{\frac{1}{1-wz}}$$



Critical points $z\partial_z S(z)=0$

$$z_\pm = \frac{\alpha \pm \sqrt{\alpha^2-4}}{2}$$

$|\alpha|<2$



$\mathrm{Re}\,(S(z)-S(z_+))=0$ on the circle

$\mathrm{Re}(S(z)-S(w))<0$

First residue gives

$$z_\pm = |z_\pm| e^{\pm i\varphi} \qquad \frac{1}{2\pi i}\int_{e^{-i\varphi}}^{e^{i\varphi}} w^{\ell-m-1}\,dw = \frac{\sin\varphi(\ell-m)}{\pi(\ell-m)}$$

Second residue at $wz=1$ $\quad \frac{1}{2\pi i}\int_{\gamma'} e^{2\theta(z-z^{-1}-\alpha\ln z)} z^{-\ell-m-1}\,dz \to 0$

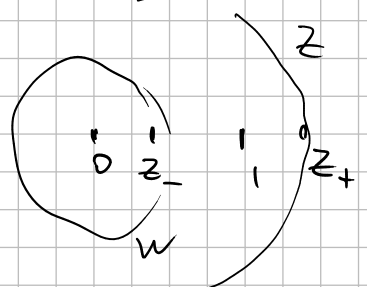
$|z|<1 \quad \mathrm{Re}\,S(z)<0$

$$\varphi = \arg\frac{\alpha+\sqrt{\alpha^2-4}}{2} = \arccos\frac{\alpha}{2} \qquad \Omega'(\alpha)=1-2\rho(\alpha)$$

$$K^\alpha_{sp}(m,\ell) \to \frac{\sin\varphi(m-\ell)}{\pi(m-\ell)} \qquad \rho(\alpha)=\frac{1}{\pi}\arccos\frac{\alpha}{2}$$

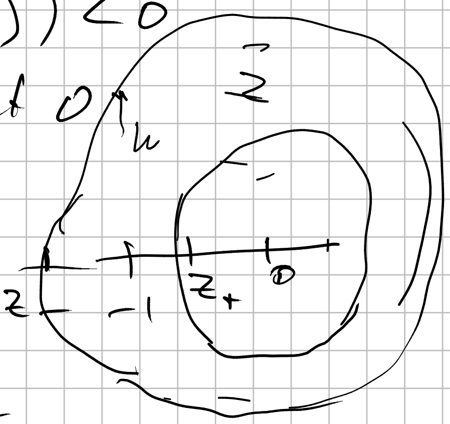
$$\Omega(\alpha) = \begin{cases} \frac{2}{\pi}\left(\alpha\arcsin\frac{\alpha}{2}+\sqrt{4-\alpha^2}\right) & |\alpha|\le 2 \\ |\alpha| & |\alpha|>2 \end{cases}$$

$\alpha>2 \quad z_\pm\in\mathbb{R} \quad z_+>1 \quad 0<z_-<1$



$\mathrm{Re}(S(z)-S(w))<0$

$\Rightarrow$ we get 0



$\alpha<-2 \qquad z_-<-1<z_+<0$

$w=z \qquad \ell\neq m \Rightarrow 0$

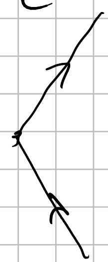
$\ell=m \Rightarrow 1 \qquad \varphi=\pi$

Fluctuations of the first row

$z_0=1$ Double critical point of $S(z)$

$$A^+_{2\to1}(x,y)=\int_0^\infty Ai(x+s)Ai(y+s)\,ds+\int_0^\infty Ai(x-s)Ai(y+s)\,ds =$$

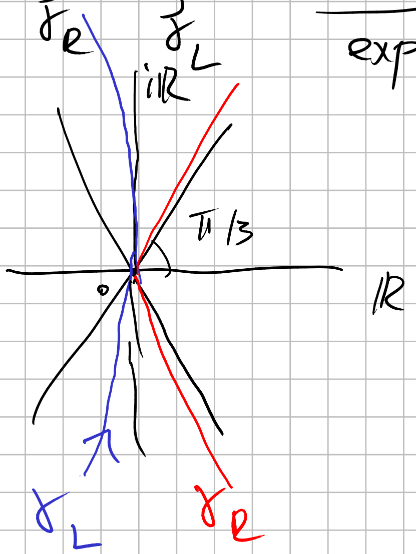
$$A_i(x) = \int_C \exp\left(\frac{\zeta^3}{3} - x\zeta\right)\frac{d\zeta}{2\pi i}$$



$$\frac{1}{u \pm v} = \int_0^\infty e^{-s(u\pm v)}\,ds \quad \text{if } \operatorname{Re}(u\pm v)>0$$

$$= \frac{1}{(2\pi i)^2}\left(\int_{\gamma_R} du \int_{\gamma_L} dv \, \frac{\exp\left(\frac{u^3}{3} - yu\right)}{\exp\left(\frac{v^3}{3} - xv\right)} \frac{1}{u-v} + \right.$$

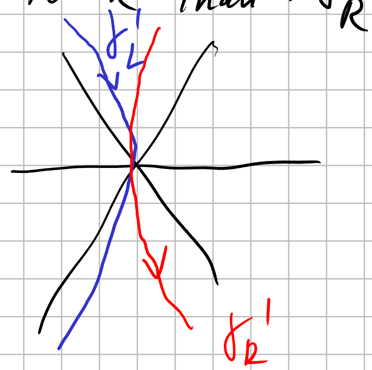
$$\left. + \int_{\gamma_R} du \int_{\gamma_L} dv \, \frac{\exp\left(\frac{u^3}{3} - yu\right)}{\exp\left(\frac{v^3}{3} - xv\right)} \frac{1}{u+v} \right) =$$



γ_L is closer to $i\mathbb{R}$ than $-\gamma_R$

$\gamma_L' = -\gamma_R$

$\gamma_R' = -\gamma_L$



$$= \frac{1}{(2\pi i)^2}\left(\int_{\gamma_L'} dw \int_{\gamma_R'} d\zeta \; e^{\frac{\zeta^3}{3} - x\zeta - \frac{w^3}{3} + yw} \frac{1}{\zeta - w} + \right.$$

$$\left. + \int_{\gamma_L'} dw \int_{\gamma_R'} d\zeta \; e^{\zeta^3/3 - x\zeta - \frac{w^3}{3} + yw} \frac{-1}{w+\zeta} \right)$$

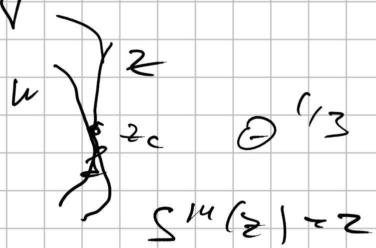
$$a = 2\Theta + x\,\Theta^{1/3}$$

$$b = 2\Theta + y\,\Theta^{1/3} \quad \text{as } \Theta \to \infty$$

$$\Theta^{1/3} K_{sp}(a,b) \longrightarrow A^{*}_{2\to 1}(x,y)$$

$|z| = 1+\delta \qquad \delta > 0$

$|w| = 1-\delta$



$$\frac{z^{-a} F_{sp}(z)}{w^{-b} F_{sp}(w)} \approx \frac{\exp\left(\theta(z - z^{-1} - 2\ln z) - x\theta^{1/3}\ln z\right)}{\exp\left(\theta(w - w^{-1} - 2\ln w) - y\,\theta^{1/3}\ln w\right)}$$

$$z = \exp(\zeta\,\theta^{-1/3}) \qquad w = \exp(\omega\,\theta^{-1/3})$$

$$\frac{1-w^2}{(1-wz^{-1})(1-wz)} \longrightarrow \frac{2\omega}{(\zeta-\omega)(\zeta+\omega)} = \frac{1}{\zeta-\omega} + \frac{-1}{\zeta+\omega}$$

$$\frac{\exp(\zeta^3/3 - x\zeta)}{\exp(\omega^3/3 - y\omega)} \qquad \frac{dz\,dw}{zw} = \theta^{-\frac{2}{3}}\,d\zeta\,d\omega$$