Lecture 19 2023-08-16

Recall: Proctor's algorithm for $Sp_{2n}\times Sp_{2k}$ skew Howe duality

How to describe limit shapes?

It is difficult to use free fermions in this case

Different approach based on the explicit formula

Last time $\mu(\lambda) \sim sp_\lambda(X)\, s_\lambda(Y)$ $\Gamma_{sp+}(X)$

$Sp_{2n}\times Sp_{2k}$ skew Howe duality

$$\Lambda(\mathbb{C}^{2n}\otimes\mathbb{C}^k) = \bigoplus_{\lambda\subset k^n} V_{Sp_{2n}}(\lambda)\otimes V_{Sp_{2k}}(\bar\lambda')$$

Dual Cauchy identity for symplectic characters

$$\sum_{\lambda\subset k^n} sp_\lambda(X)\, sp_{\bar\lambda'}(Y) = \prod_{i=1}^{n}\prod_{j=1}^{k}\left(x_i+\frac{1}{x_i}+y_j+\frac{1}{y_j}\right) =$$

$$= \prod_{i,j}(x_i+y_j)\left(1+\frac{1}{x_iy_j}\right) = \left(\prod_{i=1}^{n}\prod_{j=1}^{k}(1+x_iy_j)\left(1+\frac{x_i}{y_j}\right)\right)\prod_i\frac{1}{x_i^k}$$

$$\mu(\lambda\mid x,y) = \frac{sp_\lambda(x)\, sp_{\bar\lambda'}(y)}{\prod_{i,j}\left(x_i+\frac{1}{x_i}+y_j+\frac{1}{y_j}\right)}$$

$$\Gamma_{sp-}(x) = \exp\left(\sum_{k\geq 1}\frac{p_k(X)}{k}\alpha_{-k} + \frac{\alpha_{-2k}}{2} - \frac{\alpha_{-k}^2}{2}\right)$$

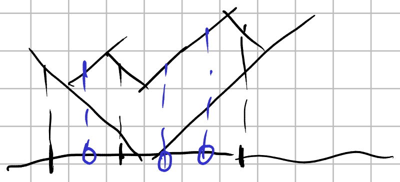
$$p_k(X) = \sum_{i=1}^{n} x_i^k$$

$$\Gamma_{sp-}(x)|0\rangle = \sum_\lambda sp_\lambda(x)|\lambda\rangle$$

In $GL_{2n}\times GL_{2k}$ case $\sum_\lambda s_\lambda(x)\, s_{\lambda'}(Y) = \prod(1+x_iy_j)$

$\langle 0|\Gamma_+(x)\,\Gamma'_-(y)|0\rangle$

ω: exchange particles and holes



$\omega s_\lambda(x) = s_{\lambda'}(x)$ | $\omega sp_\lambda(x) \neq sp_{\lambda'}(x)$

$\omega p_\ell(x) = (-1)^{\ell-1} p_\ell(x)$

$\omega \Gamma_-(x) = \Gamma_-'(x) = \Gamma_-^{-1}(-x)$

$$E(X,Y) = \prod_{ij} (1 + x_i y_j)$$

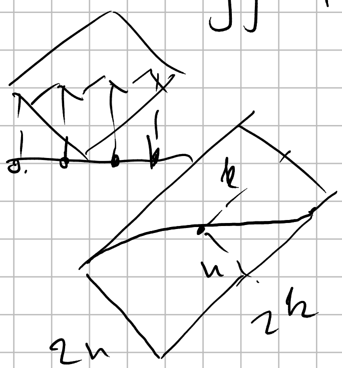
$$E(X,Y^{-1}) = \prod_{ij} \left(1 + \frac{x_i}{y_j}\right)$$

$2n$ $\{x_i, x_i^{-1}\}_{i=1}^n$ $E(X,Y)\,E(X,Y^{-1})\,E(X^{-1},Y)\,E(X^{-1},Y^{-1})$

Q: Vertex operator description of the dual Cauchy identity?

$x_i = 1$
$y_j = 1$

$sp_\lambda(1\ldots 1) = \dim V_{Sp_{2n}}(\lambda)$



$$\mu_{n,k}(\lambda) = \frac{\dim V_{Sp_{2n}}(\lambda)\,\dim V_{Sp_{2k}}(\bar\lambda')}{2^{2nk}}$$

Write in this form:

$\sim \exp(-n^2\, J[\rho])$

Solve variational problem: minimize $J[\rho]$

$\dim V_{Sp_{2k}}(\bar\lambda') = \#\, SP(\bar\lambda', 1,\bar 1 \ldots k, \bar k)$

symplectic or King tableaux

$k = 2$

$\begin{array}{|c|c|c|}\hline 1 & 1 & \bar 1 \\ \hline 2 & \bar 2 & \\ \hline \end{array} \to \begin{array}{ccc} 3 & & 2 \\ & 3 & & 1 \\ & & 3 \\ & & & 2 \end{array}$ Proctor patterns (halves of GT-patterns)

$\begin{array}{cccccccccc} 3 & & 2 & & 0 & & -2 & & -3 \\ & 3 & & 1 & & -1 & & -3 \\ & & 3 & & 0 & & -3 \\ & & & 2 & & -2 \\ & & & & 0 \end{array}$ $\xrightarrow[k+1]{\text{add}}$ $\begin{array}{ccccccccc} 6 & & 5 & & 3 & & 1 & & 0 \\ & 6 & & 4 & & 2 & & 0 \\ & & 6 & & 3 & & 0 \\ & & & 5 & & 1 \\ & & & & 3 \end{array}$

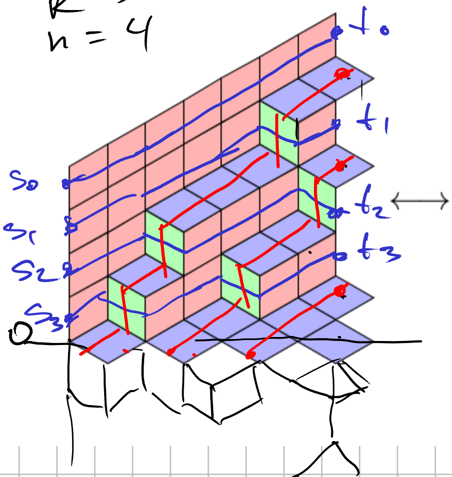
symmetric lozenge tilings of half-hexagon

$GL_{2k+1} \supset Sp_{2k}$

NILP (n paths from $(1\ldots n) \to (\lambda_1 \ldots \lambda_n)$)

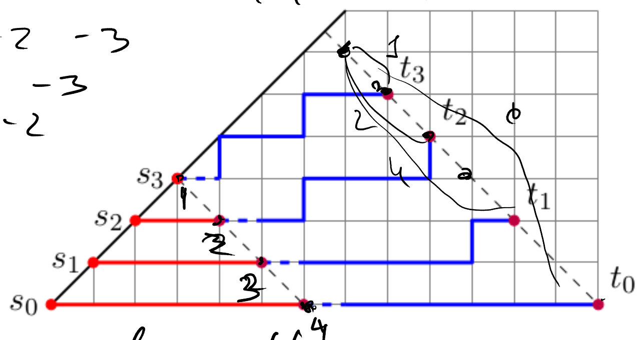
$\simeq$ NILP (k paths from $(1\ldots k) \to (\bar\lambda_1' \ldots \bar\lambda_k')$)

$k=3$
$n=4$

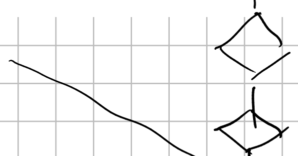


$$\begin{array}{cccccc} 3 & 2 & 0 & 0 & -2 & -3 \\ 3 & 1 & 0 & -1 & -3 & \\ 2 & 1 & -1 & -2 & & \\ 2 & 0 & -2 & & & \\ 2 & -2 & & & & \\ 1 & & & & & \end{array}$$

$6,4,2,1=$
$=\lambda_1+n-1, \lambda_2+n-2 \ldots$



Dyck paths



Symmetric tiling of $2k+1 \times n$ half hexagon

triangle Catalan numbers

$$C_{\tilde n,\tilde k} = \frac{\tilde n - \tilde k + 1}{\tilde n + 1}\binom{\tilde n + \tilde k}{\tilde k} = \binom{\tilde n + \tilde k}{\tilde k} - \binom{\tilde n + \tilde k}{\tilde k - 1}$$

count the number of paths of $\tilde n$ horizontal steps and $\tilde k$ of vertical steps that do not go above the diagonal

Lindström-Gessel-Viennot lemma

NILP from $s_0, s_1 \ldots s_{n-1}$ to $t_0, \ldots t_{n-1}$

$= \det\left[\#\text{paths from } s_i \to t_j\right]_{i,j=0}^{n-1}$

start at s_j

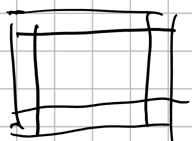
$\tilde n = k + \lambda_i - j \qquad \tilde k = k - \lambda_i + i - j$

$$\dim V_{sp_{2k}}(\bar\lambda') = \det\left(\underbrace{\frac{2\lambda_i + 2j + i}{k+\lambda_i+j+1}\binom{2k+i}{k-\lambda_i+i-j}}\right)_{i,j=1}^{n}$$

product formula use Desnanot-Jacobi identity

M - $n\times n$ matrix

M_A^B - submatrix obtained from M by removing columns A and rows B

$M_1^1 =$  $\qquad M_{1,n}^{1,n} =$ 

$$\det M \cdot \det M_{1,n}^{1,n} = \det M_1^1 \cdot \det M_n^n - \det M_1^n \cdot \det M_n^1$$

$$\dim V_{Sp_{2k}}(\bar\lambda') = \prod_{i=1}^{n} \frac{(2k+2i-1)!}{(k-\lambda_i+i-1)!\,(k+2n+\lambda_i+i+1)!} \cdot (2\lambda_i+2n-2i+2) \cdot \prod_{i<j}(\lambda_i-\lambda_j+j-i)(\lambda_i+\lambda_j+2n-i-j+2)$$

Ex!
Exact derivation
N. Scrimshaw, Podnova

$$a_i = \lambda_i + n - i + 1$$

$$\dim V_{Sp_{2k}}(\bar\lambda') = \prod_{i=1}^{n} \frac{(2k-1+2i)!}{(k+n+a_i)!\,(k+n-a_i)!} (2a_i) \prod_{i<j}(a_i^2-a_j^2)$$

$$\dim V_{Sp_{2n}}(\lambda) = \prod_{i<j} \frac{(a_i^2-a_j^2)}{(j-i)(2n+2-i-j)} \cdot \prod_{\ell}(2a_\ell)$$

Weyl dimension formula

Ex. prove it using NILPs and LGV lemma

$$\mu^{sp}_{n,k}(\{a_i\}_{i=1}^n) = \underbrace{2^{2n-2nk}}_{Z_{n,k}} \prod_{i<j} \frac{(a_i^2-a_j^2)^2}{(i-j)(2n+2-i-j)} \prod_{i=1}^{n} \frac{(2k-1+2i)!}{(k+n+a_i)!}$$

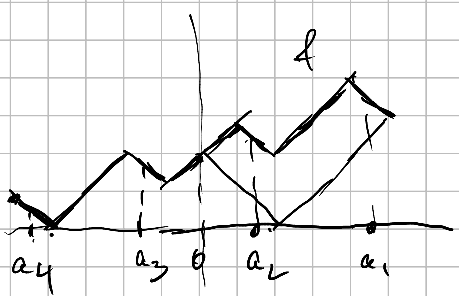
$$\frac{1}{(k+n-a_i)!} \quad a_i^2$$

Introduce mirror images $\qquad u_i = \frac{a_i}{2n}, \quad \frac{k+n}{2n} = \frac{c+1}{2}$

$$a_{2n+1-i} := -a_i$$

GL case:

$$\mu_{n,k}(a_i) = \prod_{i<j}(a_i-a_j)^2 \prod_i \binom{k+n}{a_i}$$



$n=2 \qquad (a_i^2-a_j^2) = (a_i-a_j)(a_i+a_j)$

$$\mu^{sp}_{n,k}(\{a_i\}) = \frac{1}{Z_{n,k}} \prod_{\substack{i<j\\ i,j=1}}^{2n} |a_i-a_j| \prod_{\ell=1}^{2n} \exp\left(-2n \cdot \left(V\left(\frac{a_\ell}{2n}\right) + e_n(a_\ell)\right)\right)$$

$$\ln n! \approx n\ln n - \tfrac{1}{2}\ln n$$

$$V(u) = \frac{1}{4}\left(\left(\frac{c+1}{2}+u\right)\ln\left(\frac{c+1}{2}+u\right) + \left(\frac{c+1}{2}-u\right)\ln\left(\frac{c+1}{2}-u\right)\right)$$

$\frac{k}{n} = c$

$$e_n(u) = \frac{1}{4}\ln\left(((c+2)n)^2-u^2\right) + \frac{1}{2}\ln|u| + O\left(\frac{1}{n}\right)$$

In the limit $n \to \infty$ $\quad \rho_n(u)$ instead of particle positions u_i

$u_i \in \left[-\frac{c+1}{2}, \frac{c+1}{2}\right]$

$$\mu_{n,k}(\{u_i\}) \approx \frac{1}{Z_{n,k}} \exp\left(-n^2 J[\rho_n(u)]\right)$$

$$J[\rho_n] = \iint_{-\frac{c+1}{2}}^{\frac{c+1}{2}} \rho_n(x)\rho_n(y) \ln|x-y|^{-1}\, dx\, dy + \int_{-\frac{c+1}{2}}^{\frac{c+1}{2}} \rho_n(x) V(x)\, dx$$

$\rho_n(x) = 0$ if no particle at interval $(\lfloor nx \rfloor, \lfloor nx \rfloor + 1)$

$\ 1$ if there is a particle

$$\rho_n(x) \to \rho(x)$$

Minimize $J[\rho]$ with a constraint

$$\int_{-\frac{c+1}{2}}^{\frac{c+1}{2}} \rho(x)\, dx = 1 \quad \text{and} \quad \rho(-x) = \rho(x)$$

$$\text{supp}\, \rho = [-a, a]$$

Euler-Lagrange eqn: $\displaystyle\int_{-a}^{a} \ln|x-y|^{-1} \rho(y)\, dy + V(x) = \text{const}$

$\frac{d}{dx} \to \displaystyle -\int_{-a}^{a} \frac{1}{y-x} \rho(y)\, dy + V'(x) = 0$ external field strength

field strength from distributions

equilibrium condition for a charged particle at position x

Next time: write it as scalar Riemann-Hilbert problem, solve by Plemelj formula.

potential theory