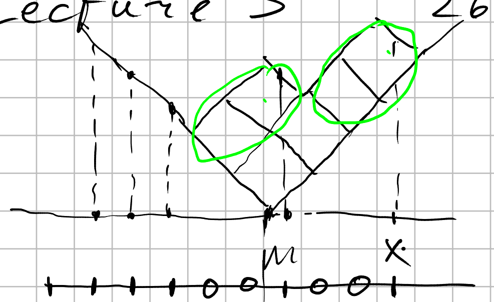
Lecture 3 26.05.2023



$x \in \mathbb{Z} + \frac{1}{2}$ Okounkov

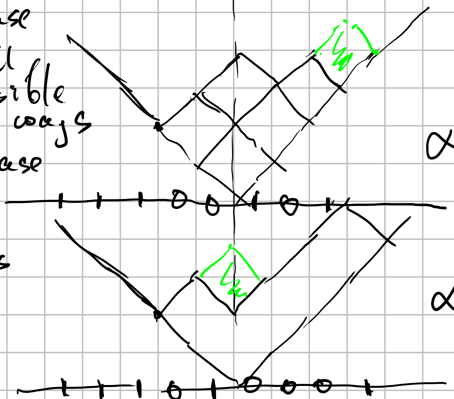
$x \in \mathbb{Z}$ Kac

$\downarrow F^{(m)}$

$$|\lambda\rangle = \underline{\lambda_1 + m - 1 \left(+\tfrac{1}{2}\right)} \wedge \underline{\lambda_2 + m - 2 \left(+\tfrac{1}{2}\right)} \wedge$$

α_1 – erase 1 box all possible ways

α_2 – erase a strip of 2 boxes



$$\alpha_n = \sum_{j \in \mathbb{Z}\left(+\frac{1}{2}\right)}' \psi_j \psi^*_{j+n}$$

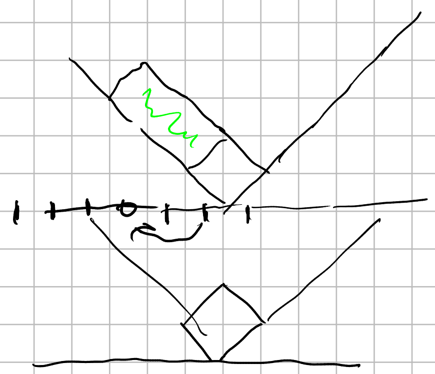
$$\alpha_0 = \sum_{j > 0 \left(\frac{1}{2}\right)} \psi_j \psi_j^* - \sum_{j < 0 \left(\frac{1}{2}\right)} \psi_j^* \psi_j$$

$$[\alpha_m, \alpha_n] = m\,\delta_{m,-n}$$

$\alpha_n \quad n > 0$

erases a continuous strip of n boxes

α_2



Strip

α_{-n} – adjoining strip of n boxes in all possible ways

$$[\alpha_m, \alpha_n] = \Big[\sum_j \psi_j \psi^*_{j+m}, \sum_\ell \psi_\ell \psi^*_{\ell+n}\Big] =$$

$$= \sum_j \sum_\ell \left(\psi_j \psi^*_{j+m} \psi_\ell \psi^*_{\ell+n} - \psi_\ell \psi^*_{\ell+n} \psi_j \psi^*_{j+m}\right) =$$

$$= \sum_j \sum_\ell \Big(\cancel{-\psi_j \psi_\ell \psi^*_{j+m} \psi^*_{\ell+n}} + \delta_{\ell, j+m} \psi_j \psi^*_{\ell+n} +$$

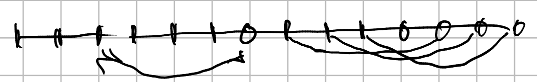
$$+ \cancel{\psi_\ell \psi_j \psi^*_{\ell+n} \psi^*_{j+m}} - \delta_{j, \ell+n} \psi_\ell \psi^*_{j+m}\Big) =$$

$$= \sum_\ell \left(\psi_{\ell-m} \psi^*_{\ell+n} - \psi_\ell \psi^*_{\ell+n+m}\right) = m\,\delta_{m,-n}$$

if $m \neq -n$ $\quad 0$

$m = -n \qquad \sum_{\ell} \left(\psi_{\ell-m}\psi^*_{\ell-m} - \psi_\ell \psi^*_\ell \right)$

m



$1 + m-1 = m$

$\alpha_n^* = \alpha_{-n} \qquad :\psi_k \psi_k^*: = \begin{cases} \psi_k \psi_k^* & k > 0 \left(\frac{1}{2}\right) \\ -\psi_k^* \psi_k & k \le 0 \left(\frac{1}{2}\right) \end{cases}$

$C = \alpha_0 = \sum_k :\psi_k \psi_k^*:$

$H = \sum_k k :\psi_k \psi_k^*:$

translation

$R\ \underline{s_1} \wedge \underline{s_2} \wedge \dots = \underline{s_1+1} \wedge \underline{s_2+1} \wedge \dots$

$R \psi_k R^{-1} = \psi_{k+1}$

$R \psi_k^* R^{-1} = \psi_{k+1}^* \qquad \alpha_n \leftrightarrow a_n$

$R^{-k} \alpha_0 R^k = \alpha_0 + k$

$R^{-k} H R^k = H + k\alpha_0 + \frac{k^2}{2}$

We would like: $B \overset{?}{\cong} F$ | $\psi(z) = \sum_{i \in \mathbb{Z}+\frac{1}{2}} z^i \psi_i$

$\psi^*(w) = \sum_{j \in \mathbb{Z}+\frac{1}{2}} w^{-j} \psi_j^*$

$[\alpha_n, \psi(z)] = z^n \psi(z)$

$[\alpha_n, \psi^*(w)] = -w^n \psi^*(w)$

$s = (s_1, s_2, s_3 \dots)$ infinite sequence

$\Gamma_{\pm}(s) = \exp\left(\sum_{n=1}^{\infty} s_n \alpha_{\pm n} \right) = 1 + \sum_{n=1}^{\infty} s_n \alpha_n + \frac{1}{2}\left(\sum_{n=1}^{\infty} s_n \alpha_n \right)^2 + \dots$

$\Gamma_+(s)|m\rangle = |m\rangle \qquad \Gamma_+^* = \Gamma_-$

$\Gamma_-^* = \Gamma_+$

$$\Gamma_+(s)\,\Gamma_-(s') = e^{\sum_{n\geq 1}^{\infty} n s_n s_n'}\,\Gamma_-(s')\,\Gamma_+(s)$$

$$\Gamma_{\pm}(s)\,\psi(z) = \gamma(z^{\pm 1}, s)\,\psi(z)\,\Gamma_{\pm}(s)$$

$$\Gamma_{\pm}(s)\,\psi^*(w) = \gamma(w^{\pm 1}, s)^{-1}\,\psi^*(w)\,\Gamma_{\pm}(s) \quad \text{Ex.}$$

$$\gamma(z,t) = \exp\Big(\sum_{n\geq 1} t_n z^n\Big) = \prod_i \frac{1}{1-x_i z} = \sum_{n\geq 0} z^n h_n(x)$$

complete homogeneous symmetric functions

$t_1, t_2 \dots \qquad t_k = \frac{1}{k}\sum_i x_i^k$

$$h_k(x) = \sum_{i_1\leq i_2\leq\dots\leq i_k} x_{i_1}\cdot x_{i_2}\dots x_{i_k} = \sum_{\lambda\vdash k} m_\lambda(x)$$

$$m_\lambda(x) = \sum_{a\sim\lambda} \underbrace{x^a}_{= x_1^{a_1}\dots x_n^{a_n}}$$

monomial symmetric function

a - rearrangement of $\lambda_1 \dots \lambda_n$

symmetricfunctions.com

$$\psi(z) = z^{\alpha_0} R\, \Gamma_-\Big(\underbrace{\{z\}}_{=(z, \frac{z^2}{2}, \frac{z^3}{3}\dots)}\Big)\, \Gamma_+\Big(\underbrace{-\{z^{-1}\}}_{= -\frac{1}{z}, -\frac{1}{2}z^{-2}\dots}\Big)$$

$$\psi^*(w) = R^{-1} z^{-\alpha_0}\, \Gamma_-(-\{z\})\, \Gamma_+(\{z^{-1}\})$$

$$\Gamma_-(\{z\}) = \exp\Big(\sum_{n\geq 1}\frac{z^n}{n} a_{-n}\Big) = \exp\Big(\sum_{n\geq 1} z^n x_n\Big)$$

$$\Gamma_+(-\{z^{-1}\}) = \exp\Big(\sum_{n\geq 1} -\frac{z^{-n}}{n} a_n\Big) = \exp\Big(\sum_{n\geq 1}\frac{z^{-n}}{n}\frac{\partial}{\partial x_n}\Big)^{-1}$$

Schur polynomials

elementary Schur polynomials $\quad S_m(x)$

$$\sum_{m\in\mathbb{Z}} S_m(x) z^m = \exp\Big(\sum_{n\geq 1} z^n x_n\Big)$$

$S_m(x) = 0 \quad m<0$

$S_0(x) = 1$

$$S_m(x) = \sum_{\substack{k_1, k_2 \dots \\ k_1+2k_2+3k_3+\dots = m}} \frac{x_1^{k_1}}{k_1!}\cdot\frac{x_2^{k_2}}{k_2!}\dots$$

For $\lambda = (\lambda_1, \lambda_2 \dots)$ $\qquad \ell(\lambda) \qquad \lambda_0$

$$s_\lambda(x) = \det\left(s_{\lambda_i + j - i}(x)\right)_{i,j=1}^{\ell(\lambda)} \qquad F = \bigoplus_m F^{(m)}$$

Bosonic Fock space:

$a_n = \frac{\partial}{\partial x_n}$ $\qquad \mathbb{C}[x_1, x_2 \dots]$

$a_{-n} = n x_n$ $\qquad \mathbb{C}[x_1, x_2 \dots ; q, q^{-1}]$ $\qquad a_0 = q \cdot \frac{\partial}{\partial q}$

$a_0 = \mu \, Id$ $\qquad B^{(m)} = q^m \, \mathbb{C}[x_1 \dots]$

$$\exists! \ F \xrightarrow{\sim} B \qquad \sigma(F^{(m)}) = B^{(m)}$$

$$\langle \lambda | \mu \rangle = H(\underline{\lambda_1 - 1 \wedge \dots}, \underline{\mu_1 - 1 \wedge \dots})$$

$$H_B\left(q^m P(x), q^n Q(x)\right) = \delta_{mn} \, P\left(\frac{\partial}{\partial x_1}, \frac{1}{2}\frac{\partial}{\partial x_2} \dots\right) Q(x)\Big|_{x=0}$$

$$\sigma(R) = q$$

$$\varphi \leftrightarrow \lambda^\varphi \qquad \sigma(\varphi) = q^m s_{\lambda^\varphi}(x)$$

Thm σ is an isomorphism $\qquad$ Kac Thm 14.10

Symmetric functions: $\qquad$ Generating function

Elementary symmetric functions

$$e_k(x) = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \qquad E(z) = \sum_{k=0}^{\infty} e_k(x) z^k$$

Complete homogeneous symmetric functions

$$h_k(x) = \sum_{i_1 \le i_2 \le \dots \le i_k} x_{i_1} \cdot x_{i_2} \cdots x_{i_k} \qquad H(z) = \sum_{k=0}^{\infty} h_k(x) z^k$$

Power sums

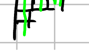
$$p_k(x) = \sum_i x_i^k \qquad P(z) = \sum_{k=1}^{\infty} p_k(x) z^{k-1}$$

$$H(z) = \prod_i \frac{1}{1 - x_i z} \qquad s_\lambda(x_1 \dots x_N) = \frac{\det\left(x_i^{\lambda_j + N - j}\right)_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}$$

$$E(z) = \prod_i (1 + x_i z)$$

$$P(z) = \frac{d}{dz} \sum_i \ln \frac{1}{1 - x_i z}$$

Transposed diagram λ'

λ   λ'

Jacobi-Trudi identity

$$s_\lambda(x) = \det\left(h_{\lambda_i - i + j}\right)_{i,j=1}^{\ell(\lambda)} = \det\left(e_{\lambda'_i - i + j}\right)_{i,j=1}^{\ell(\lambda')}$$

Cauchy identity

$$\sum_\lambda s_\lambda(x)\, s_\lambda(y) = \prod_{ij} \frac{1}{1 - x_i y_j}$$

$$\mu(\lambda) = \frac{s_\lambda(x)\, s_\lambda(y)}{\sum_\nu s_\nu(x)\, s_\nu(y)} \rightarrow \text{Schur measure}$$

Miwa variables $\quad t_k = \frac{1}{k} \sum_i x_i^k \qquad t'_k = \frac{1}{k} \sum_i y_i^k$

$$t = t' = (\sqrt{\xi}, 0, 0 \dots)$$

$$\mu(\lambda) = e^{-\xi}\, \xi^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!}\right)^2 \quad \text{Poissonized Plancherel measure}$$

$$\Gamma_-(t)\,|0\rangle = \sum_\lambda s_\lambda(x)\,|\lambda\rangle \rightsquigarrow \text{proof next time}$$

Skew Schur functions $\qquad x = (x_1 \dots)$
$\qquad\qquad\qquad\qquad\qquad\quad y = (y_1 \dots)$

$$s_\lambda(x, y) = \sum_\mu{}' \underbrace{s_{\lambda/\mu}(x)}_{\text{skew Schur functions}}\, s_\mu(y)$$

skew Cauchy identity

$$\sum_\mu s_{\mu/\lambda}(x)\, s_{\mu/\nu}(y) = \prod_{ij} \frac{1}{1 - x_i y_j} \sum_\varkappa s_{\lambda/\varkappa}(y)\, s_{\nu/\varkappa}(x)$$

$$s_{\lambda/\mu}(x) = \det\left(h_{\lambda_i - \mu_j - i + j}\right)_{i,j=1}^{\max(\ell(\lambda), \ell(\mu))} =$$

$$= \sum_{T \in SSYT(\lambda/\mu)}{}' x^T$$

$$S_\lambda(x) = \sum_{T \in SSYT(\lambda)} x^T$$