Lecture 22 2023-08-25

$$K_{n,k}(m',l') = \frac{n}{n-l} \iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w}\left(\frac{z^{k-m'+1}}{(1-z)^n(1+z)^k}\,\frac{(1-w)^n(1+w)^k}{w^{k-l'}}\right)\left(\frac{1}{(1-z)(1-w)} - \frac{k/n}{(1+z)(1+w)}\right)$$

$$\int \frac{dw}{2\pi i w}\,\frac{(1-w)^n(1+w)^{k-1}}{w^{k-l'}} = (-1)^{k-l'}\int \frac{dw}{2\pi i w}\,\frac{(1+w)^n(1-w)^{k-1}}{w^{k-l'}} =$$

$w \to -w$

$$\int \frac{dw}{2\pi i w}\,\frac{(1-w)^{n-1}(1+w)^k}{w^{k-l'}} = \quad = (-1)^{k-l'}\,2^{-(k-l')}\sqrt{\binom{n+k-1}{k-l'}}\,P^{(n)}_{k-l'} =$$

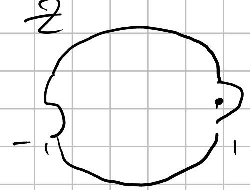
$$\int \frac{dz}{2\pi i z}\,\frac{z^{k-m'+1}}{(1-z)^{n+1}(1+z)^k} = . \quad = (-1)^{k-l'}\,2^{-(k-l')}\,\frac{\binom{n+k-1}{k-l'}}{\sqrt{\binom{n+k-1}{n}}}\,P_n(k-l')$$

$$\int \frac{dz}{2\pi i z}\,\frac{z^{k-m'+1}}{(1-z)^n(1+z)^{k+1}} =$$

$$K_m \sim {}_2F_1$$

$$p_m(x) = \frac{2^m}{\sqrt{\binom{n+k-1}{m}}}\int_{c_\varepsilon} \frac{dw}{2\pi i w}\,\frac{(1+w)^x(1-w)^{n+k-1-x}}{w^m}$$

$$K_m(x) = K_x(m)$$



$$\frac{p_m(x)}{\sqrt{\binom{n+k-1}{m}}} = \frac{p_x(m)}{\sqrt{\binom{n+k-1}{x}}}$$

$$\int \frac{dw}{2\pi i w}\,\frac{(1-w)^{n-1}(1+w)^k}{w^{k-l'}} = (-1)^{k-l'}\,2^{-(k-l')}\,\frac{\binom{n+k-1}{k-l'}}{\sqrt{\binom{n+k-1}{n-1}}}\,P_{n-1}(k-l')$$

$$\zeta = \frac{z-1}{z+1}$$

$$\int \frac{dz}{2\pi i z}\,\frac{z^{k-m'+1}}{(1-z)^{n+1}(1+z)^k} \quad z = \frac{1+\zeta}{1-\zeta} \quad \frac{(-1)^{n+1}}{2^{n-k}}\int \frac{d\zeta}{2\pi i \zeta}\,\frac{(1+\zeta)^{k-m'}(1-\zeta)^{n+m'-1}}{\zeta^n} =$$

$$\int \frac{dz}{2\pi i z}\,\frac{z^{k-m'+1}}{(1-z)^n(1+z)^{k+1}} = \quad = (-1)^{n+1}\sqrt{\binom{n+k-1}{n}}\,2^{-2n-k}\,P_n(k-m')$$

$$= \frac{(-1)^n}{2^{n+k}}\int \frac{d\zeta}{2\pi i \zeta}\,\frac{(1+\zeta)^{k-m'}(1-\zeta)^{n+m'-1}}{\zeta^{n-1}} = (-1)^n\,2^{-2n-k}\sqrt{\binom{n+k-1}{n-1}}\,P_{n-1}^{(k-m')}$$

$$K_{n,k}(m',l') = \frac{(-1)^{n+k-l'}}{2^{2n+2k}}\, 2^{l'} \left(\frac{n \binom{n+k-1}{k-l'} \sqrt{\binom{n+k-1}{n}}}{\sqrt{\frac{k}{n}}\sqrt{\binom{n+k-1}{n-1}}} p_{n-1}(k-l')\, p_n(k-m') \right.$$

$$= \sqrt{\frac{n}{k}}$$

$$\left. - k \frac{\binom{n+k-1}{k-l'} \sqrt{\binom{n+k-1}{n-1}}}{\sqrt{\binom{n+k-1}{n}}} p_n(k-l')\, p_{n-1}(k-m') \right) \frac{1}{m'-l'} =$$

$$\frac{\binom{n+k-1}{n}}{\binom{n+k-1}{n-1}} = \frac{(n+k-1)!}{n!\,(k-1)!} \frac{(n-1)!\,k!}{(n+k-1)!} = \frac{k}{n}$$

$$= \frac{(-1)^{n+k-l'}\, 2^{l'}}{2^{2n+2k}} \sqrt{nk} \binom{n+k-1}{k-l'} \frac{p_{n-1}(k-l')\, p_n(k-m') - p_n(k-l')\, p_{n-1}(k-m')}{m'-l'}$$

$$w(k-l') = \binom{n+k-1}{k-l'} 2^{-n-k+1}$$

$$w(k-m') = \binom{n+k-1}{k-m'} 2^{-n-k+1}$$

$$k_m = \frac{1}{m!} \frac{2^m}{\sqrt{\binom{n+k-1}{m}}} \qquad \frac{k_{n-1}}{k_n} = \frac{n}{2}\sqrt{\frac{k}{n}} = \frac{\sqrt{kn}}{2}$$

$$K(m',l') = \frac{(-1)^{n+k-l'}\, 2^{l'}}{2^{n+k}} \frac{k_{n-1}}{k_n} \frac{p_{n-1}(k-l')\, p_n(k-m') - p_n(k-l')\, p_{n-1}(k-m')}{m'-l'} \cdot \sqrt{w(k-l')\, w(k-m')} \left(\sqrt{\binom{n+k-1}{k-l'}} \Big/ \sqrt{\binom{n+k-1}{k-m'}} \right)$$

$f_j(x)$ – polynomial of degree j in x

$$h_j = \sum_x f_j(x)\, f_j(x)\, w(x)$$

$$\sum_{j=0}^{n-1} \frac{f_j(x) f_j(y)}{h_j} = \frac{k_{n-1}}{k_n} \frac{1}{h_{n-1}} \frac{f_n(x) f_{n-1}(y) - f_{n-1}(x) f_n(y)}{x-y}$$

Christoffel–Darboux formula

$$K(x,y) = \sum_{l=0}^{n-1} p_l(x)\, p_l(y) \sqrt{w(x)\, w(y)}$$

$$\rho(x_1, \dots x_m) =: \mu(\{\lambda : x_1, \dots x_m \in \lambda\})$$

$$= \det\left[K(x_i, x_j)\right]_{i,j=1}^{m}$$

orthogonal polynomial ensemble

$$\mu_n(\{x_1 \dots x_n\}) = \det\left(\sqrt{w(x_i)w(x_j)} \sum_{l=0}^{n-1} p_l(x_i)\, p_l(x_j)\right)_{i,j=1}^{m}$$

$$x\, p_m(x) = a_{m+1} p_{m+1}(x) + b_{m+1} p_m(x) + a_m p_{m-1}(x)$$

$\{p_m(x)\}_{m=0}^{\infty}$ $\qquad \sum_x p_m(x)\, p_l(x)\, w(x) = \delta_{lm}$

$$\int_\alpha^\beta p_m(x)\, p_l(x) \underbrace{w(x)\, dx}_{d\nu} = \delta_{lm}$$

$L^2([\alpha,\beta], d\nu)$ $\qquad \int_\alpha^\beta |f(x)|^2\, d\nu < \infty$

$$\langle f, g\rangle = \int \bar{f}(x)\, g(x)\, d\nu$$

$\ell^2(\mathbb{Z}_+) = \{(a_1, a_2 \dots)\}$ $\qquad a_i \in \mathbb{R}$

$$\left(\sum_i |a_i|^2\right)^{1/2} = \|a\|_2 < \infty$$

$$\langle a, b\rangle = \sum_i \bar{a}_i b_i$$

$\{p_m(x)\}_{m=0}^{n}$ $\qquad \ell^2(\{0, 1 \dots L\}) \xrightarrow[L\to\infty]{n\to\infty} \ell^2(\mathbb{Z}_+)$

$x = 0, 1 \dots L$ $\qquad \{f(0), f(1) \dots\}$

$\{0, 1 \dots L\}$ $\qquad \ell^2(\{0, 1 \dots L\}) \ni f(x)$

$$\sqrt{w(x)}\, p_l(x) = \{\sqrt{w(0)}\, p_l(0), \sqrt{w(1)}\, p_l(1) \dots \sqrt{w(L)}\, p_l(L)\}$$

$$\sum_{x=0}^{L} K_n(x,y) f(x) = \sum_{x=0}^{L} \sum_{l=0}^{n-1} \sqrt{w(x)w(y)}\, p_l(x)\, p_l(y)\, f(x) =$$

$$= \sum_{l=0}^{n-1} \left(\underbrace{\sum_{x=0}^{L} \sqrt{w(x)}\, p_l(x)\, f(x)}_{\langle f, \sqrt{w(\cdot)}\, p_l(\cdot)\rangle}\right) p_l(y) \sqrt{w(y)}$$

Projection to the first n basis elements in ℓ^2.

$$J\, p_m(x)\sqrt{w(x)} = x\, p_m(x) \sqrt{w(x)}$$

$$J = \begin{pmatrix} b_1 & a_1 & & 0 \\ a_1 & b_2 & a_2 & \\ 0 & a_2 & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

Jacobi matrix

$$A(m)\, p_m^{(n,L)}(x) = \underbrace{B(x)\, p_m(x+1) - (B(x)+C(x))\, p_m(x) + C(x)\, p_m(x-1)}$$

$$A(m)\, p_m(x) = D^{(n,L)} p_m(x)$$

$$n, L \to \infty \qquad D^{(n,L)} \to D$$

bounded self adjoint

$$\ell^2(\mathbb{Z}_+) \overset{\sim}{\longleftrightarrow} L^2([\alpha,\beta], d\nu)$$

with spectrum $[\alpha,\beta]$ $d\nu$

$$k_m(x) = \sqrt{w(x)}\, p_m(x)$$

$$\frac{A(m)}{\sqrt{w(x)}} k_m(x) = \frac{B(x)}{\sqrt{w(x+1)}} k_m(x+1) - \frac{(B(x)+C(x))}{\sqrt{w(x)}} k_m(x) + \frac{C(x)}{\sqrt{w(x-1)}} k_m(x-1)$$

$$\underbrace{B(x)\sqrt{\frac{w(x)}{w(x+1)}}}\, k_m(x+1) + C(x)\underbrace{\sqrt{\frac{w(x)}{w(x-1)}}}\, k_m(x-1) = \left(A(m) + B(x) + C(x)\right) k_m(x)$$

$$D^{(n,L)}(x,y) = \begin{pmatrix} 0 & C(x)\sqrt{\frac{w(x)}{w(x+1)}} & & 0 \\ B(x)\sqrt{\frac{w(x)}{w(x-1)}} & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & 0 \end{pmatrix} \qquad x = 0 \ldots L$$

$$\ell^2(\mathbb{Z}_+)$$

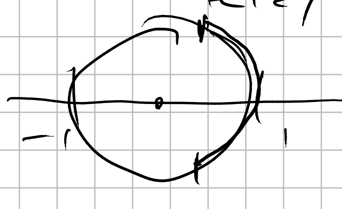
$$D^{(n,L)} \to D$$

$$\ell^2(\mathbb{Z}_+) \sim L^2([\alpha,\beta], d\nu)$$

$$\tilde{D} f(x) = f(x+1) + f(x-1)$$

$$\ell^2(\mathbb{Z}_+) \to L^2(|z|=1) \qquad \hat{f}(\underset{e^{i\varphi}}{z}) = \sum_{x=0}^{L} f(x)\, e^{ix\varphi}$$

$|z|=1$



$$z + \bar{z} = 2\,\mathrm{Re}\, z \qquad [-2, 2]$$

$$\frac{\sin\varphi(x-y)}{\pi(x-y)} \longleftarrow 1\!\!1_{[e^{-i\varphi}, e^{i\varphi}]}$$

$$p(L-x)\, k_m(x+1; p, L) + x(2p-1)\, k_m(x; p, L) + x(1-p)\, k_m(x-1; p, L) =$$
$$= (pL - m)\, k_m(x; p, L)$$

k_m !

$$D = \frac{\sqrt{(L-x)(x+1)}}{L} K_m(x+1) + \frac{x(2p-1)}{L\sqrt{p(1-p)}} K_m(x) + \frac{\sqrt{(L-x+1)x}}{L} K_m(x-1)$$

$$= \frac{pL-m}{L\sqrt{p(1-p)}} K_m(x)$$

$m = 0 \dots L$

Corr. kernel projects to

$$\left\{ \frac{pL-m}{L\sqrt{p(1-p)}},\ m = 0 \dots n-1 \right\}$$

$$L = n+k-1 \approx (c+1)n \qquad x = nt \qquad t \in [0, c+1]$$

$$\left[\frac{p(c+1)-1}{(c+1)\sqrt{p(1-p)}},\ \frac{p}{\sqrt{p(1-p)}} \right]$$

$$D: \quad \frac{\sqrt{t(c+1-t)}}{c+1} \qquad \frac{(2p-1)t}{(c+1)\sqrt{p(1-p)}} \qquad \frac{\sqrt{(c+1-t)t}}{c+1}$$

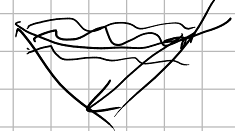
subtract diagonal and divide by $\frac{\sqrt{(c+1-t)t}}{c+1}$

$$f \to \frac{(2p-1)t}{(c+1)\sqrt{p(1-p)}} f \qquad D \to \tilde{D} f(x) = f(x+1) + f(x-1)$$

$$\left[\frac{p(c+1)-1-(2p-1)t}{\sqrt{t(c+1-t)}\sqrt{p(1-p)}},\ \frac{p(c+1)-(2p-1)t}{\sqrt{p(1-p)}\sqrt{t(c+1-t)}} \right]$$

$$p = \frac{1}{2}: \left[\frac{c-1}{\sqrt{t(c+1-t)}},\ \frac{c+1}{\sqrt{t(c+1-t)}} \right]$$

$$\varphi = \arccos\left(\frac{c-1}{2\sqrt{t(c+1-t)}} \right)$$



We see limit shapes in orthogonal polynomials

Fluctuations:

$$\{x_1 \dots x_n\} = \left\{ \frac{\lambda_1+n-1}{n} \dots \frac{\lambda_n+n-1}{n} \right\}$$

$$f(x) \in C^1(\mathbb{R}) \qquad X_f = \sum_{i=1}^{n} f(x_i)$$

Thm (Breuer, Duits 2016)

$$\{p_m^{(n)}(x)\}_{m=0}^{n-1} \qquad x p_m^{(n)}(x) = a_{m+1}^{(n)} p_{m+1}^{(n)}(x) + b_{m+1}^{(n)} p_m^{(n)}(x) + a_m^{(n)} p_{m-1}^{(n)}(x)$$

$$\{n_j\}_{j=1}^{\infty} \qquad a > 0 \qquad b \in \mathbb{R} \qquad b_{n_j+k}^{(n_j)} \to b \qquad a_{n_j+k}^{(n_j)} \to a \text{ for all } k$$

$$X_f^{(n_j)} - \mathbb{E} X_f^{(n_j)} \to \mathcal{N}\left(0, \sum_{\ell \geq 1} \ell |\hat{f}_\ell|^2\right)$$

$$\hat{f}_\ell = \frac{1}{2\pi} \int_0^{2\pi} f(2a\cos\theta + b) e^{-i\ell\theta} \, d\theta$$

<u>if $n_j = j$</u> $\quad a_n^{(n)} \to a \quad b_n^{(n)} \to b$

monic $\quad x P_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$

$$x P_n(x) = P_{n+1}(x) + [p(L-n) + n(1-p)] P_n(x) + $$
$$+ np(1-p)(L+1-n) P_{n-1}(x)$$

orthonormal

$$a_n = \sqrt{\beta_n} \qquad b_{n+1} = \alpha_n \qquad b_n = \frac{p(\overset{c+1}{\cancel{L}}-n) + n(1-p)}{n}$$

$$b = 1 + p(c-1)$$

$$a = \sqrt{c}\sqrt{p(1-p)} \qquad a_n = \sqrt{\frac{p(1-p)(L+1-n)}{n}}$$

$$[b-2a, b+2a] \overset{p=\frac{1}{2}}{=} \left[\frac{c+1}{2} - \sqrt{c}, \frac{c+1}{2} + \sqrt{c}\right]$$

$f(x) = (x+1)^2 \qquad X_f - EX_f \sim \mathcal{N}(0, \sum_{\ell \geq 1} \ell |\hat{f}_\ell|^2)$