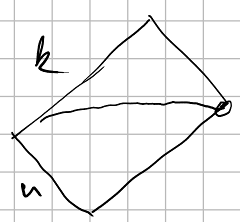
Lecture 14. 2023.07.07 Discrete Hermite kernel in $GL_n \times GL_k$ skew Howe duality



$n, k \to \infty$ $\quad \lim_{n,k\to\infty} \frac{k}{n} = c$

$k = \underbrace{cn}_{} + \underline{\frac{s}{\sigma}\sqrt{n}}$

$[u_-, u_+]$ $\quad u_+ = c$ $\qquad s = 0 \Rightarrow$ GTW discrete "critical" distribution

$m = nc + \frac{\sqrt{n}}{\sigma} s + \ell \qquad m \le k$

$m' = nc + \frac{s}{\sigma}\sqrt{n} + \ell' \qquad \ell', \ell \le 0$

$$K(m, m') = \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{z^{-m} w^{m'} \sqrt{zw}}{z - w} \frac{K(z)}{K(w)} \;(=)$$

$k = cn + \frac{s}{\sigma}\sqrt{n}$

$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \dim V_{GL_k}(\lambda')}{2^{nk}}$ $\qquad c = 1$

$K(z) = \frac{1}{(1-z)^n} \frac{z^k}{\left(\frac{1+z}{2}\right)^k}$

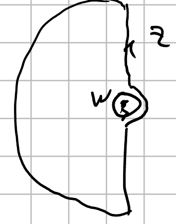
$\mu_{n,k}(\lambda | f, g) \longrightarrow K(z) = \prod_{i=1}^{n} \frac{1}{1 - z f(\frac{i}{n})} \cdot \prod_{j=1}^{k} \frac{z}{z + g(j/k)},$

$$(=) \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} e^{n\left[-\ln(1-z) - c\ln(1+z) + \cancel{(c-u_+)\ln z} - \frac{s}{\sigma}\frac{1}{\sqrt{n}}\ln(1+z) + \cancel{\frac{s}{\sigma}\frac{1}{\sqrt{n}}\ln z} - \cancel{\frac{s}{\sigma}\frac{1}{\sqrt{n}}\ln z} + \ln(1-w) + c\ln(1+w) + \frac{s}{\sigma}\frac{1}{\sqrt{n}}\ln(1+w)\right]}$$

$$\cdot \frac{z^{-\ell+\frac{1}{2}} w^{\ell'+\frac{1}{2}}}{z - w}$$

w-contour encircle $0 \Rightarrow$ can be made C_ε

z-contour encircle $-1 \Rightarrow$



$S(z) = -\ln(1-z) - \ln(1+z)$

$z \in i\mathbb{R}$ $\qquad \operatorname{Re} S(z) < 0$

$w = \frac{\sigma}{\sqrt{n}} v \qquad dw = \frac{\sigma}{\sqrt{n}} dv$

near 0: $S(z) \approx \frac{S''(0)}{2} z^2$ $\qquad S(0) = 0$

$S'(0) = 0$

$z = \frac{\sigma}{\sqrt{n}} \zeta$ $\qquad S''(0) = 2$

$$K(\ell, \ell') = \int_{-i\infty}^{i\infty} \frac{d\zeta}{2\pi i} \int_{C_\varepsilon} \frac{dv}{2\pi i} \left(\frac{\sigma}{\sqrt{n}}\right)^{\ell'-\ell} \frac{\zeta^{-\ell-\frac{1}{2}} v^{\ell'-\frac{1}{2}}}{\zeta - v} \cdot \exp\left(\frac{S''(0)\sigma^2}{2}(\zeta^2 - v^2) - \zeta s + v s\right)$$

$\left(\frac{1}{2n}\right)^{\frac{\ell'-\ell}{2}}$ $\qquad \frac{S''(0)\sigma^2}{2} \to 1/2$

$\sigma^2 = \frac{1}{S''(0)} = \frac{1}{2} \qquad (c = 1)$

Physics $$He_\ell(x) = \frac{\ell!}{2\pi i}\oint_0 e^{2tx-t^2}\cdot t^{-\ell-1}\,dt$$

$$\int_{-\infty}^{\infty} He_\ell(x)\, H_m(x)\, e^{-t^2}\,dt \sim \delta_{\ell m}$$

probability $$He_\ell(x) = \frac{\ell!}{2\pi i}\oint e^{tx-\frac{t^2}{2}}\,t^{-\ell-1}\,dt$$

$$H_\ell\left(\frac{s}{\sqrt{2}}\right) = \frac{\ell!}{2\pi i}\int e^{\sqrt{2}ts-t^2}\,t^{-\ell-1}\,dt = \frac{\ell!}{2\pi i}\,2^{\frac{\ell}{2}}\oint e^{t's-\frac{t'^2}{2}}\,t'^{-\ell-1}\,dt'$$

$t'=\sqrt{2}t$ $\qquad$ $He_\ell''(s)$

$$H_\ell\left(\frac{s}{\sqrt{2}}\right) = 2^{\frac{\ell}{2}}\,He_\ell(s)$$

$\ell \neq \ell'$ $$s\,He_\ell(s) = He_{\ell+1}(s) + \ell\, He_{\ell-1}(s)$$

$$\underbrace{\frac{1}{(\ell'-\ell)}\left(\zeta\frac{\partial}{\partial\zeta} + v\frac{\partial}{\partial v} + 1\right)\zeta^{-\ell-\frac{1}{2}}\,v^{\ell'-\frac{1}{2}}}\; = \zeta^{-\ell-\frac{1}{2}}\,v^{\ell'-\frac{1}{2}}$$

$$\left(\zeta\frac{\partial}{\partial\zeta} + v\frac{\partial}{\partial v} + 1\right)\frac{e^{\frac{\zeta^2}{2}-\zeta s}\,e^{vs-\frac{v^2}{2}}}{\zeta - v} = \left(\frac{\zeta(\zeta-s)}{\zeta-v} + \frac{+v(s-v)}{\zeta-v} + \cancel{-\frac{\zeta}{(\zeta-v)^2} + \frac{v}{(\zeta-v)^2} + \frac{1}{\zeta-v}}\right)e^{\frac{\zeta^2}{2}-\zeta s}\,e^{vs-\frac{v^2}{2}}$$

$$\frac{\zeta^2 - \zeta s + vs - v^2 +}{\zeta - v} = (\zeta + v - s)\,e^{\frac{\zeta^2}{2}-\zeta s}\,e^{vs-\frac{v^2}{2}}$$

$$K(\ell,\ell') = \left(\frac{1}{2n}\right)^{\frac{\ell'-\ell}{2}}\int_{-i\infty}^{i\infty}\frac{d\zeta}{2\pi i}\int_{C_\varepsilon}\frac{dv}{2\pi i}\,\frac{\zeta+v-s}{\ell'-\ell}\,\zeta^{-\ell-\frac{1}{2}}\,e^{\frac{\zeta^2}{2}-\zeta s}\,\underbrace{e^{vs-\frac{v^2}{2}}\,v^{\ell'-\frac{1}{2}}}$$

$$\zeta\int\frac{dv}{2\pi i}\,v^{\ell'-\frac{1}{2}}\,e^{vs-\frac{v^2}{2}} = \frac{1}{\left(-\ell'-\frac{1}{2}\right)!}\;He_{-\ell'-\frac{1}{2}}(s)$$

$$\int\frac{dv}{2\pi i}\,v^{\ell'+\frac{1}{2}}\,e^{vs-\frac{v^2}{2}} = \frac{1}{\left(-\ell'-\frac{3}{2}\right)!}\;\cancel{He_{-\ell'-\frac{3}{2}}}(s)$$

$$-s\int\frac{dv}{2\pi i}\,v^{\ell'-\frac{1}{2}}\,e^{vs-\frac{v^2}{2}} = \frac{1}{\left(-\ell'-\frac{1}{2}\right)!}\left(-He_{-\ell'+\frac{1}{2}}(s) + \left(\ell'-\tfrac{1}{2}\right)\cdot\cancel{He_{-\ell'-\frac{3}{2}}}(s)\right)$$

$$\overset{\shortparallel}{\frac{1}{\left(-\ell'-\frac{1}{2}\right)!}}\,He_{-\ell'-\frac{1}{2}}(s)$$

$$K(\ell,\ell') = \left(\frac{1}{2n}\right)^{\frac{\ell'-\ell}{2}} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\, \frac{s^{-\ell-\frac12}}{\ell'-\ell}\, e^{\frac{s^2}{2}-\varsigma s}\left(\frac{s\, He_{-\ell'-\frac12}(s) - He_{-\ell'+\frac12}(s)}{(-\ell'-\frac12)!}\right)$$

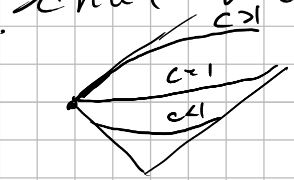
$$He_\ell(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x+iy)^\ell e^{-\frac{y^2}{2}}\,dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(i\tilde y)^\ell e^{-\frac{\tilde y^2}{2}-i\tilde y x+\frac{x^2}{2}}\,d\tilde y$$

$$i\tilde y = x+iy \qquad -y^2 = -\tilde y^2 - 2i\tilde y x + x^2$$

$$i\eta = s \qquad e^{-\frac{s^2}{2}}\int_{-\infty}^{\infty} d\eta\,(i\eta)^{-\ell-\frac12}\, e^{-\frac{\eta^2}{2}-i\eta s+\frac{s^2}{2}}$$

$$K_s(\ell,\ell') = \left(\frac{1}{2n}\right)^{\frac{\ell'-\ell}{2}}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{s^2}{2}}\,\frac{He_{-\ell+\frac12}(s)\,He_{-\ell'-\frac12}(s) - He_{-\ell-\frac12}(s)\,He_{-\ell'-\frac12}(s)}{(\ell'-\ell)\,(-\ell'-\frac12)!}$$

Borodin-Olshanski 2007 Schur-Weyl duality $c=1$



$GL_n \times S_k$

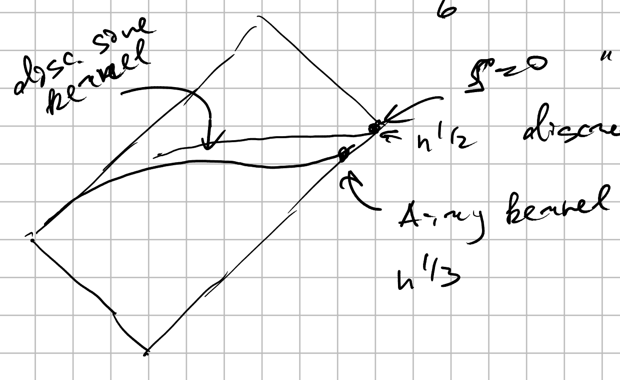
$k = n^2 + s n^{3/2}$

$$K_s^{He}(\ell,\ell') = \frac{1}{\sqrt{2\pi\,\ell!\,\ell'!}}\, e^{-\frac{s^2}{2}}\, \frac{He_{\ell+1}(s)\,He_{\ell'}(s) - He_\ell(s)\,He_{\ell'+1}(s)}{\ell-\ell'}$$

$$K_s(\ell,\ell') = \left(\frac{1}{2n}\right)^{\frac{\ell'-\ell}{2}}\sqrt{\frac{(-\ell'-\frac12)!}{(-\ell-\frac12)!}}\; K_s^{He}\left(-\ell-\frac12, -\ell'-\frac12\right)$$

$\ell=\ell'$ $$K_s^{He}(\ell,\ell) = \frac{1}{\sqrt{\pi}\,\ell!}\int_s^\infty e^{-t^2/2}\, He_\ell^2(t)\,dt$$

$$\mathbb{P}\left(\lambda < cn + \frac{s}{2}\sqrt{n} - \Delta\right) = \det\left(\delta_{ij} - K_s\left(-i-\tfrac12, -j-\tfrac12\right)\right)_{i,j=0}^{\Delta-1}$$



$s=0$ "critical" kernel $d\,$Herm$_{s=0}$

discrete Hermite kernel → Borodin-Olshanski 2014

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