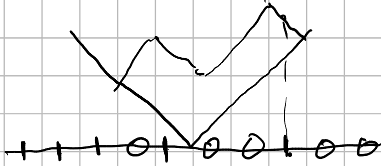
Lecture 5. 02.06.2023



$$K(i,j) = \langle 0 | \psi_i \psi^*_j | 0 \rangle$$

$i,j \in \mathbb{Z}+\frac{1}{2}$

$\{x_i\} \to \{t=(t_1,t_2\dots)\}$

$\{y_j\} \to \{t'=(t'_1,t'_2\dots)\}$

$z, w \in \mathbb{C}$

$$K(z,w) = \sum_{i,j} z^i w^{-j} K(i,j)$$

$$K(z,w) = \frac{\sqrt{zw}}{z-w} \frac{F(z)}{F(w)}$$

$|w| < |z|$

$$F(z) = \prod_{i=1}^{\infty} \frac{1-y_i/z}{1-x_i z} = e^{\sum_{k=1}^{\infty} t_k z^k - t'_k z^{-k}}$$

$t = (\sqrt{\xi}, 0, 0 \dots)$

$t' = (\sqrt{\xi}, 0 \dots)$

$f: \mathbb{C} \to \mathbb{C}$ holomorphic in $U \subset \mathbb{C}$

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$c \subset U$ $\quad \oint_c f(z)\, dz = 0$

$z_0 \in U$ $\quad f(z_0) = \frac{1}{2\pi i} \oint_c \frac{f(z)}{z - z_0} dz$

$$f(z) = a_0 + a_1 z + \dots$$

$$a_k = \oint_0 f(z) z^{-k} \frac{dz}{2\pi i z}$$

$$K(i,j) = \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{w^j}{z^i} \frac{F(z)}{F(w)} \frac{\sqrt{zw}}{z-w}$$

$$\mu(\lambda | x, y) = s_\lambda(x) s_\lambda(y) \prod_{i,j}(1 - x_i y_j)$$

$t = (\sqrt{\xi}, 0, \dots)$

$t' = (\sqrt{\xi}, 0 \dots)$

$$K(x,y) = \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{\exp(\sqrt{\xi}(z - z^{-1}) - x \ln z)\sqrt{zw}}{\exp(\sqrt{\xi}(w - w^{-1}) - y \ln w)(z-w)}$$

$\xi \to \infty$ $\quad x, y \in \mathbb{Z} + \frac{1}{2}$ $\quad \frac{x}{\sqrt{\xi}}, \frac{y}{\sqrt{\xi}} \to u$

$$\exp\Big(\underset{\downarrow\,\infty}{\sqrt{\xi}}\Big[\underbrace{z - z^{-1}}_{S(z)} - u\ln z\Big]\Big) \qquad x-y \text{ is fixed}$$

$$e^{\sqrt{\xi}\left(S(z) - u\ln z - S(w) + u\ln w\right)}$$

$$\int e^{MS(x)}dx \qquad M\to\infty$$

x_0

$$MS(x) = MS(x_0) + \frac{MS''(x_0)}{2}(x-x_0)^2 + \dots$$

$$x - x_0 = \frac{t}{\sqrt{-MS''(x_0)}}$$

$$\int e^{MS(x)}dx \simeq e^{MS(x_0)}\int_{-\infty}^{\infty} e^{-t^2/2}dt = \sqrt{2\pi}\,\frac{e^{MS(x_0)}}{\sqrt{-MS''(x_0)}}$$

$$\mathrm{Re}\left(S'(z) - \frac{u}{z}\right) = 0$$

$$zS'(z) - u = 0$$

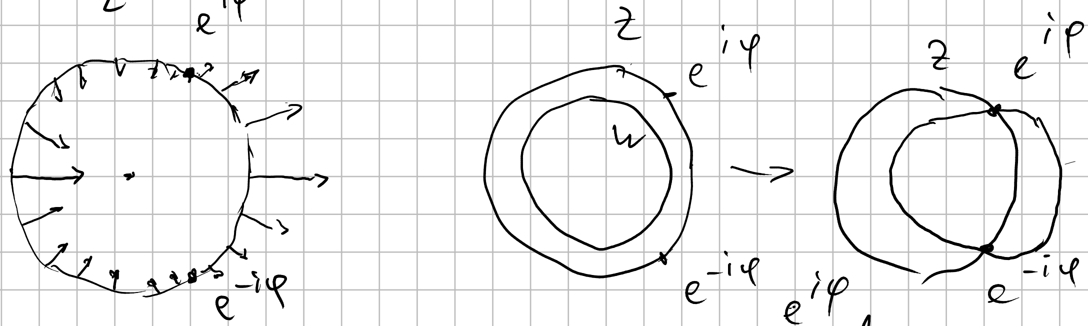
$$z + \frac{1}{z} - u = 0 \qquad u\in(-2,2)$$

$$z = e^{\pm i\varphi}$$

$$\varphi = \arccos\frac{u}{2}$$

$$\mathrm{Re}\left(S(z) - u\ln z\right) =$$

$$= \mathrm{Re}\left(z - \frac{1}{z} - u\ln z\right) = 0 \quad |z| = 1$$

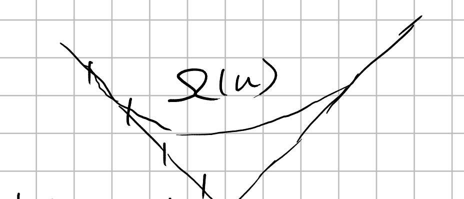


There is a pole at $z = w$ $\quad \frac{1}{2\pi i}\int_{e^{-i\varphi}}^{e^{i\varphi}}\frac{dw}{w^{x-y+1}}$

$$K(x,y\,|\,\xi) \xrightarrow[\xi\to\infty]{} \frac{1}{2\pi i}\int_{e^{-i\varphi}}^{e^{i\varphi}}\frac{dw}{w^{x-y+1}} = \frac{\sin(x-y)\varphi}{\pi\,\varphi}$$

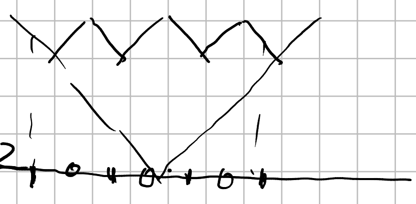
$$\overset{x=y}{\rho(u)} = K\left(\frac{x}{\sqrt{\xi}},\frac{x}{\sqrt{\xi}}\right) = \frac{1}{2\pi i}\int_{e^{-i\varphi}}^{e^{i\varphi}}\frac{dw}{w} = \frac{\varphi}{\pi}$$

$$\rho(u) = \frac{1}{\pi}\arccos\frac{u}{2}$$



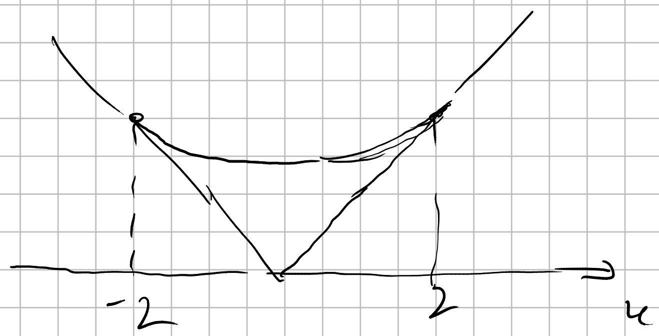
$$\rho(u)\big| = 1 \qquad \rho(u) = 0$$

$\Omega'(u)=-1$ $\quad \Omega'(u)=1$ $\quad \rho(u)=\frac{1}{2}$ $\quad \Omega'(u)=0$



$\Omega'(u)=1-2\rho(u)$

$$\Omega(u)=\begin{cases}\frac{2}{\pi}\left(u\arcsin\frac{u}{2}+\sqrt{4-u^2}\right) & |u|\le 2\\ |u| & |u|\ge 2\end{cases}$$



Schur measure

$$\mu(\lambda|x_1\dots,y_1\dots)=s_\lambda(x)s_\lambda(y)\prod_{i,j}(1-x_iy_j)=$$

$$=\frac{\langle 0|\Gamma_+(t)|\lambda\rangle\langle\lambda|\Gamma_-(t')|0\rangle}{\langle 0|\Gamma_+(t)\Gamma_-(t')|0\rangle}$$

$$e^{\sum_{k\ge 1}kt_kt'_k}\to e^{-\xi}$$

take specialization $t=(\sqrt{\xi},0\dots)$ $\quad t'=(\sqrt{\xi},0\dots)$

$$\mu(\lambda|\xi)=\xi^{|\lambda|}\left(\frac{\dim\lambda}{|\lambda|!}\right)^2 e^{-\xi}\qquad s_\lambda(x)\to\underline{(\sqrt{\xi})^{|\lambda|}\frac{\dim\lambda}{|\lambda|!}}$$

$s_\lambda(x_1,x_2\dots x_n)$ $\qquad x_i=a$

$$s_\lambda(x_1\dots x_n)=\frac{\det\left(x_j^{\lambda_i+n-i}\right)_{i,j=1}^n}{\prod_{i<j}(x_i-x_j)}\qquad \prod_{i<j}(x_i-x_j)=\det\left(x_j^{n-i}\right)_{i,j=1}^n$$

$x_1=aq^{n-1}$ $\qquad q\to 1$

$x_2=aq^{n-2}$

$\vdots$

$x_n=a$

$$\sim \begin{pmatrix} a^{\lambda_1+n-1} q^{(n-1)(\lambda_1+n-1)}, & a^{\lambda_1+n-1} q^{(n-2)(\lambda_1+n-1)} & \dots & a^{\lambda_1+n-1} \\ a^{\lambda_2+n-2} q^{(n-1)(\lambda_2+n-2)} & a^{\lambda_2+n-2} q^{(n-2)(\lambda_2+n-2)} & \dots & a^{\lambda_2+n-2} \\ - & - & - & - \\ a^{\lambda_n} q^{(n-1)\lambda_n} & \dots & - & a^{\lambda_n} \end{pmatrix}$$

$$\downarrow \quad a^{|\lambda| + \frac{n(n-1)}{2}} \Big/ a^{\frac{n(n-1)}{2}} \longrightarrow a^{|\lambda|}$$

$$\det\left[\left(q^{\lambda_i+n-i}\right)^{n-j}\right]_{i,j=1}^{n} = V_n\left(q^{\lambda_1+n-1}, \dots, q^{\lambda_n}\right)$$

$$\shortparallel$$

$$V_n(u_1 \dots u_n) = \prod_{i<j} (u_i - u_j) \qquad \prod_{i<j} q^{\lambda_i+n-i} - q^{\lambda_j+n-j}$$

$$S_\lambda(q^{n-1} \dots 1) = \prod_{i<j} \frac{q^{\lambda_i+n-i} - q^{\lambda_j+n-j}}{q^{n-i} - q^{n-j}} = \prod_{i<j} \frac{q^{\lambda_i-i} - q^{\lambda_j-j}}{q - 1} \cdot$$

$$\cdot \frac{q-1}{q^{-i} - q^{-j}} \qquad \frac{q^a - q^b}{q-1} \xrightarrow[q \to 1]{} a-b$$

$$S_\lambda(a \dots a) = a^{|\lambda|} \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j-i} \qquad \prod_{i<j} (j-i) = 0!\, 1! \dots (n-1)!$$

$\dim \lambda$ – dim of group of $S_{|\lambda|}$ corresp. to λ

$$= SYT(\lambda \,|\, |\lambda|)$$

$$\dim \lambda = \frac{|\lambda|! \prod_{i<j} (\lambda_i - \lambda_j + j - i)}{\prod_{i=1}^{n} (\lambda_i + n - i)!}$$

$$t = (\sqrt{\xi}, 0 \dots) \qquad p_1 = \sqrt{\xi} \qquad p_2 = p_3 = \dots = 0$$

$$x_i = \frac{\sqrt{\xi}}{n} \qquad p_k = \sum_{i=1}^{n} x_i^k = n\left(\frac{\sqrt{\xi}}{n}\right)^k \xrightarrow[n\to\infty]{} 0 \quad k \geq 1$$

$$s_\lambda\left(\frac{\sqrt{\xi}}{n} \dots \frac{\sqrt{\xi}}{n}\right) = \left(\frac{\sqrt{\xi}}{n}\right)^{|\lambda|} \frac{\dim\lambda}{|\lambda|!} \frac{\prod_i (\lambda_i + n - i)!}{0!\,1!\dots(n-1)!} \ (\ast)$$

$$\frac{(\lambda_i + n - i)!}{(n-i)!} = \underbrace{(n-i+1)(n-i+2)\dots(\lambda_i + n - i)}_{\lambda_i \text{ terms}} =$$

$$= \prod_{j=1}^{\lambda_i} (n+j-i)$$

$$(\ast) \left(\sqrt{\xi}\right)^{|\lambda|} \frac{\dim\lambda}{|\lambda|!} \prod_{i=1}^{n} \prod_{j=1}^{\lambda_i} \left(1 + \frac{j-i}{n}\right) \xrightarrow[n\to\infty]{} \left(\sqrt{\xi}\right)^{|\lambda|} \frac{\dim\lambda}{|\lambda|!}$$

$$t = (\sqrt{\xi}, 0 \dots) \qquad t' = (\sqrt{\xi}, 0 \dots)$$

$$\mu_{PP}(\lambda \mid \xi) = \xi^{|\lambda|} e^{-\xi} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2$$ Poissonized plancherel measure

$$\mu_{Pl,|\lambda|=n}(\lambda) = \frac{(\dim\lambda)^2}{|\lambda|!}$$ Plancherel measure

$$\mathrm{Poisson}_\xi(n) = \frac{e^{-\xi}\xi^n}{n!}$$

$$\mu_{PP}(\lambda \mid \xi) = \sum_{n=0}^{\infty} \mu_{Pl,n}(\lambda)\Big|_{\lambda \vdash n} \frac{e^{-\xi}\xi^n}{n!} = \sum_{n=0}^{\infty} \frac{(\dim\lambda)^2}{(n!)^2} e^{-\xi}\xi^n$$

Ensemble equivalence $\quad \lim\limits_{n\to\infty} \mu_{Pl,n}(\lambda) = \lim\limits_{\xi\to\infty} \mu_{PP}(\lambda \mid \xi)$

$$X \sim \mathrm{Poisson}_\xi \qquad \mathbb{P}(X = n) = \frac{e^{-\xi}\xi^n}{n!}$$

$$\mathbb{E}\, X^{\downarrow k} = \mathbb{E}\, X(X-1)\dots(X-k+1) = \sum_{n=0}^{\infty} \frac{e^{-\xi}\xi^n}{n!} n(n-1)\dots(n-k+1)$$

$$= \xi^k e^{-\xi} \sum_n \frac{\xi^{n-k}}{(n-k)!} = \xi^k$$

$$\mathbb{E}\, X = \xi \qquad \mathbb{E}[X^2 - X] = \xi^2 \qquad \mathbb{D} X = \mathbb{E}\, X^2 - (\mathbb{E} X)^2 = \xi$$

$$X \simeq \xi + O\left(\frac{1}{\sqrt{\xi}}\right)$$

Robinson - Schensted - Knuth algorithm

perm. $1 \dots n \longleftrightarrow P, Q \in SYT(n)$

$$\text{shape} P = \text{shape}\, Q = \lambda$$

$$n! = \sum_{\lambda \vdash n} (\dim \lambda)^2 \qquad \mu_{pe}(\lambda) = \frac{(\dim \lambda)^2}{n!}$$

$$3\ 7\ 1\ 2\ 6\ 5\ 4$$

$$P = \boxed{3} \xrightarrow{7} \boxed{3\,|\,7} \xrightarrow{1} \begin{array}{ll} 1 & 7 \\ 3 & \end{array} \xrightarrow{2} \begin{array}{ll} 1 & 2 \\ 3 & 7 \end{array} \xrightarrow{6} \begin{array}{lll} 1 & 2 & 6 \\ 3 & 7 & \end{array} \xrightarrow{5} \begin{array}{lll} 1 & 2 & 5 \\ 3 & 6 & \\ 7 & & \end{array} \xrightarrow{4} \begin{array}{lll} 1 & 2 & 4 \\ 3 & 5 & \\ 6 & & \\ 7 & & \end{array}$$

$$Q = \boxed{1} \to \boxed{1\,|\,2} \to \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \to \begin{array}{ll} 1 & 2 \\ 3 & 4 \end{array} \to \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & \end{array} \to \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & & \end{array} \to \begin{array}{lll} 1 & 2 & 5 \\ 3 & 4 & \\ 6 & & \\ 7 & & \end{array}$$