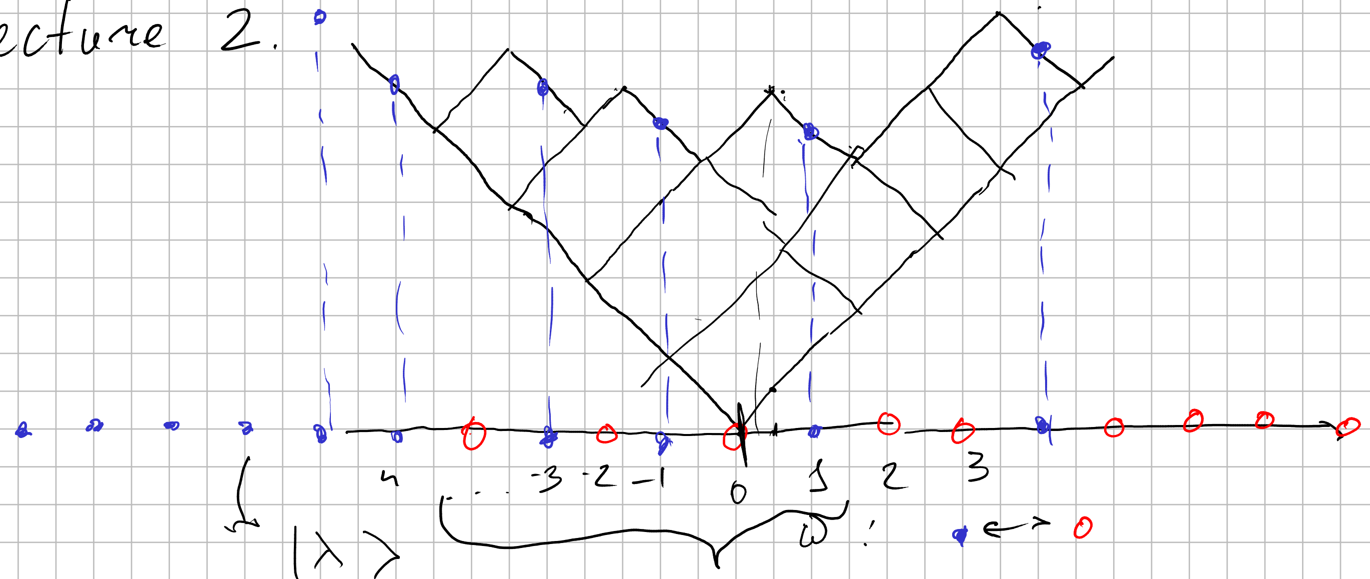
Lecture 2.



$P(\bullet \text{ at } x \text{ in } \lambda) = \langle \lambda | O | \lambda \rangle$

$V \ni v$ vector space

$V \otimes W$ $\quad \{v_i\}$ - basis of V

$\{w_j\}$ basis of W

$\{v_i \otimes w_j\}_{i,j=1}$ $\quad V \otimes V \otimes V \dots \otimes V = T^k V$

$T(V) = TV = \bigoplus_{k=0}^{\infty} T^k V$ $\quad k=0$ $\quad \mathbb{R}, \mathbb{C}$

$I = \{ v \otimes w - w \otimes v \}$

Symmetric algebra $Sym(V) = SV = T(V)/I$

Exterior algebra $\Lambda(V) = \Lambda V = T(V) / \{ v \otimes v \}$

exterior product $v \wedge v = 0$ $\quad (v+w) \wedge (v+w) = 0$

$v \wedge w = - w \wedge v$

$\sigma \in S_k$ $\quad v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(k)} = sgn(\sigma) v_1 \wedge \dots \wedge v_k$

$v_i \in V$ $\quad (-1)^{\#\text{transpositions in } \sigma}$

$\Lambda^k V = \{ v_1 \wedge \dots \wedge v_k \}$ $\quad \dim V = n$ $\quad v_1 \dots v_n$ - basis of V

$\Lambda^n V = \{ v_1 \wedge \dots \wedge v_n \} \cong \mathbb{R}$ or $\mathbb{C}$

$\dim \Lambda^k V = \binom{n}{k}$ $\quad \Lambda^0 V \cong \mathbb{R}$ or $\mathbb{C}$

$\Lambda V = \bigoplus_{k=0}^{n} \Lambda^k V$ $\quad \dim \Lambda V = \sum_{k=0}^{n} \binom{n}{k} = 2^n$

Lie algebra $\mathfrak{g}$ $\quad [\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ bilinear

$[X,Y] = -[Y,X]$

Jacobi identity $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$

$a \cdot b \quad \to \quad [a,b] = a \cdot b - b \cdot a \qquad gl(V)$

Oscillator algebra or Heisenberg algebra $\mathcal{A}$

$\{a_n, n \in \mathbb{Z} ; \mathbb{1}\} \qquad [\mathbb{1}, a_n] = 0$

$$[a_m, a_n] = m\,\delta_{m,-n}\, \mathbb{1}$$

$[a_0, a_n] = 0 \quad a_0$ is a central element

$B = \mathbb{C}[x_1, x_2 \dots]$ Bosonic Fock space

$a_n = \varepsilon_n \frac{\partial}{\partial x_n} \qquad n > 0$

$a_{-n} = \frac{1}{\varepsilon_n} n x_n \qquad a_{-n}$ creation operators

$a_0 = \mu \qquad a_n$ annihilation operators

$\mathbb{1} = id$

$1 = |0\rangle \qquad a_n |0\rangle = 0 \quad n > 0$

$a_0 |0\rangle = \mu |0\rangle$

Prop. If representation V of $\mathcal{A}$ admits a non-zero vector $|0\rangle$ then $a_{-1}^{k_1} \dots a_{-n}^{k_n} |0\rangle$ are indep., if they span V, then $V \cong B$

$\varphi : B \to V \qquad \varphi(P(x_j \dots)) \to P(\ a_j\ )$

Antilinear anti-involution $\omega \qquad + \qquad a_n^{+} = a_{-n}$

$\omega(a_n) = a_{-n}$

$\langle v | w \rangle$

Def Form is contravariant if $\langle a(v) | w \rangle = \langle v | \omega(a) w \rangle$

Prop. V has a unique Hermitean form $\langle \cdot | \cdot \rangle$

$\langle 0 | 0 \rangle = 1 \qquad v$ – vacuum

$$\langle a_{-1}^{k_1} \dots a_{-n}^{k_n}(v) \,|\, a_{-1}^{k_1} \dots a_{-n}^{k_n}(v) \rangle = \prod_{j=1}^{n} k_j!\, j^{k_j}$$

$$\parallel$$

$$\langle 0 | a_n^{k_n} \dots a_1^{k_1} \; a_{-1}^{k_1} \dots a_{-n}^{k_n} | 0 \rangle$$

Corollary : $\langle \ | \ \rangle$ is positive-definite

Def. Vacuum expectation value of $P \in B = \mathbb{C}[x_1 x_2 \dots]$ is its constant term

$\langle a \rangle = \langle 0 | a | 0 \rangle \qquad \langle a^{+} \rangle = \overline{\langle a \rangle} \qquad a \in U(\mathcal{A})$

Def. Degree of $x_1^{j_1} \dots x_k^{j_k}$ is $j_1 + 2j_2 + 3j_3 + \dots + k j_k$

B_j - subspace of B spanned by monomials of degree j

$\dim B_j = p(j)$ - number of partitions of j

$B = \bigoplus_{j\geq 0}^{\infty} B_j$ principal gradation $\qquad p(0) = 1$

$$\dim_q B = \sum_{j\geq 0} (\dim B_j)\, q^j = \frac{1}{\varphi(q)}$$

$$\varphi(q) = \prod_{j>0} (1-q^j) \quad \text{Euler's function}$$

$x_1^{j_1} \dots x_k^{j_k}$ $\qquad q = e^{-\frac{1}{T}} \rightarrow \dim_q = Z$ [free oscillators]

$$\{a, b\} = ab + ba$$

Kac $\quad F \quad \Lambda^{\frac{\infty}{2}} V$ Okounkov $\quad V$ - inf. dim, basis $\{\underline{k}\}$ $\quad k \in \mathbb{Z} + \frac{1}{2}$

$\underline{i_1} \wedge \underline{i_2} \wedge \dots \qquad i_1 > i_2 > i_3 > \dots \qquad \underline{3+\frac{1}{2}} \wedge \underline{1+\frac{1}{2}} \wedge \dots$

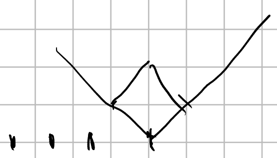
$i_j \in \mathbb{Z} \qquad i_n = i_{n-1} - 1 \qquad i_j \in \mathbb{Z} + \frac{1}{2}$

$\underline{3} \wedge \underline{1} \wedge \underline{-1} \wedge \underline{-2} \wedge \underline{-3} \dots$

Charge decomposition $\qquad |m\rangle = \underline{m} \wedge \underline{m-1} \wedge \underline{m-2} \wedge \dots$

$F^{(m)}$ - spanned by monomials that differ from $|m\rangle$ in finite number of places (do finite number of jumps)

$\varphi = \underline{i_1} \wedge \underline{i_2} \wedge \dots \longrightarrow \lambda_1 = i_1 - m \qquad \lambda_2 = i_2 - (m-1) \dots$



λ^{φ} $\quad$ Energy of $\varphi = |\lambda^{\varphi}| = \sum \lambda_i$

$F_j^{(m)}$ - monomials of charge m and energy j

$$F^{(m)} = \bigoplus_{j\geq 0} F_j^{(m)}$$

$$\dim F_j^{(m)} = p(j)$$

$$\dim_q F^{(m)} = \sum_{j\geq 0} (\dim F_j^{(m)})\, q^j = \frac{1}{\varphi(q)}$$

$$\psi_k\, \underline{i_1} \wedge \underline{i_2} \wedge \dots = \underline{k} \wedge \underline{i_1} \wedge \dots = \begin{cases} 0 & \text{if } k = i_s \\ (-1)^s\, \underline{i_1} \wedge \dots \wedge \underline{i_s} \wedge \underline{k} \wedge \underline{i_{s+1}} \dots & i_s > k > i_{s+1} \end{cases}$$

$$\psi_k^* (\underline{i_1} \wedge \underline{i_2} \wedge \dots) = \begin{cases} 0 & \text{if } k \neq i_s \text{ for all } s \\ (-1)^{s+1} \underline{i_1} \wedge \dots \wedge \underline{i_{s-1}} \wedge \underline{i_{s+1}} \wedge \dots \end{cases}$$

$$\{\psi_i, \psi_j^*\} = \delta_{ij} \qquad \{\psi_i, \psi_j\} = \{\psi_i^*, \psi_j^*\} = 0$$

ψ_i, ψ_j^* generate Clifford algebra Cl $\qquad \psi_j |0\rangle = 0 \quad j \leq 0$

$\psi_j^* |0\rangle = 0 \quad j > 0$

V $\quad (\cdot|\cdot)$ symmetric bilinear

$T(V) / J = \{ x \otimes y - (x|y) \}$

$U \subset V$ $\quad ClV$ has unique F_U (spin module)

with $|0\rangle$ s.t. $U |0\rangle = 0$

$$V = \sum_i \mathbb{C}\psi_i + \sum_i \mathbb{C}\psi_i^* \qquad (\psi_i | \psi_j^*) = \delta_{ij}$$

$$U = \sum_{i \leq 0} \mathbb{C}\psi_i + \sum_{i > 0} \mathbb{C}\psi_i^*$$

$$[\psi_i \psi_j^*, \psi_k] = \delta_{kj} \psi_i \qquad [\psi_i \psi_j^*, \psi_k^*] = -\delta_{ki} \psi_j^*$$

$$a_n = \sum_{j \in \mathbb{Z}} \psi_j \psi_{j+n}^* \qquad n \neq 0$$

$$a_0 = \sum_{j > 0} \psi_j \psi_j^* - \sum_{j \leq 0} \psi_j^* \psi_j$$

$$[a_m, a_n] = m \delta_{m,-n}$$ <u>Exercise</u>