Lecture 11 2023.06.28

Skew Howe GL_n-GL_k duality

$$\Lambda(\mathbb{C}^n \otimes \mathbb{C}^k) = \bigoplus_{\lambda \subset n\times k \text{ box}} V_{GL_n}(\lambda) \otimes V_{GL_k}(\lambda')$$

λ' - transposed diagram

Dual Cauchy identity

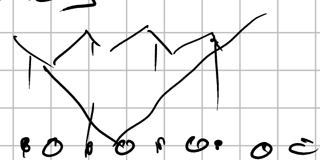
$$\prod_{i=1}^{n} \prod_{j=1}^{k} (1 + x_i y_j) = \sum_{\lambda \subset n\times k \text{ box}} s_\lambda(x_1 \dots x_n)\, s_{\lambda'}(y_1 \dots y_k)$$

$$s_\lambda(x_1 \dots x_n) = \sum_{T \in SSYT(\lambda|n)} x_1^{\#1\text{'s in } T}\, x_2^{\#2\text{'s}} \cdots x_n^{\#n\text{'s}}$$

$$\mu_{n,k}(\lambda | \{x_i\}_{i=1}^{n}, \{y_j\}_{j=1}^{k}) = \frac{s_\lambda(x_1 \dots x_n)\, s_{\lambda'}(y_1 \dots y_k)}{\prod_{i=1}^{n}\prod_{j=1}^{k}(1 + x_i y_j)}$$

if $x_i \geq 0$ $y_j \geq 0$ => probability measure on Young diagrams

$$\mu_{n,k}(\lambda | x, y) = \frac{\langle 0 | \Gamma_+(t) | \lambda \rangle \langle \lambda | \Gamma_-'(t') | 0 \rangle}{\langle 0 | \Gamma_+(t)\, \Gamma_-'(t') | 0 \rangle}$$



$$t = (t_1, t_2 \dots) \qquad t' = (t_1', t_2' \dots)$$

$$t_\ell = \frac{1}{\ell}\sum_{i=1}^{n} x_i^\ell \qquad t_\ell' = \frac{1}{\ell}\sum_{j=1}^{k} y_j^\ell$$

$$\Gamma_-'(t') = \exp\Big(\sum_{\ell=1}^{\infty} (-1)^{\ell+1} \alpha_{-\ell}\Big) = \Gamma_-^{-1}(-y)$$

$$\Gamma_+(t)\, \Gamma_-'(t') = \prod_{i,j}(1 + x_i y_j)\, \Gamma_-'(t')\, \Gamma_+(t)$$

$$\psi(z)\, \Gamma_-'(t') = \prod_{j=1}^{k} \frac{1}{1 + y_j/z}\, \Gamma_-'(t')\, \psi(z)$$

$$\psi^+(w)\, \Gamma_-'(t') = \prod_{j=1}^{k} (1 + y_j/w)\, \Gamma_-'(t')\, \psi^+(w)$$

$$K(m, m') = \frac{\langle 0 | \Gamma_+(t)\, \psi_m \psi_{m'}^+\, \Gamma_-'(t') | 0 \rangle}{\langle 0 | \Gamma_+(t)\, \Gamma_-'(t') | 0 \rangle} =$$

$$= \iint \frac{dz}{2\pi i z} \frac{dw}{2\pi i w}\, z^{-m} w^{m'}\, \frac{\langle 0 | \Gamma_+(t)\, \psi(z) \psi^+(w)\, \Gamma_-'(t') | 0 \rangle}{\langle 0 | \Gamma_+(t)\, \Gamma_-(t) | 0 \rangle}$$

$$K(m,m') = \frac{1}{\prod_{ij}(1+x_i y_j)} \iint_{|w|<|z|} \frac{dz}{2\pi i z}\frac{dw}{2\pi i w}\frac{K(z)}{K(w)} z^{-m} w^{m'} \frac{\sqrt{zw}}{z-w}$$

$$K(z) = \prod_{i=1}^{n} \frac{1}{1-x_i z}\prod_{j=1}^{k} \frac{1}{1+y_j/z}$$

$n\to\infty,\ k\to\infty \qquad \lim \frac{k}{n} = c \qquad k=[cn]$

$x_i=\alpha \ \forall i \qquad y_j=\beta \ \forall j \qquad s_\lambda(\alpha\ldots\alpha)=\alpha^{|\lambda|}\cdot\dim V_{GL_n}(\lambda)$

$$\mu_{n,k}(\lambda|\alpha,\beta) = \frac{(\alpha\beta)^{|\lambda|}\dim V_{GL_n}(\lambda)\dim V_{GL_k}(\lambda')}{(1+\alpha\beta)^{nk}}$$

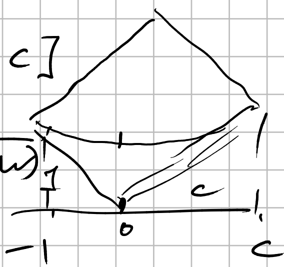
$m,m'\in\mathbb{Z}+\frac{1}{2}$

$] \ \beta = 1$

$$\mu_n(\lambda|c,\alpha) = \frac{\alpha^{|\lambda|}\dim V_{GL_n}(\lambda)\dim V_{GL_{cn}}(\lambda')}{(1+\alpha)^{cn^2}}$$

$m = nu+\frac{1}{2},\ m' = nv+\frac{1}{2} \qquad u,v\in[-1,c]$

$$K_n(u,v) = \iint_{z} \frac{dz}{2\pi i z}\frac{dw}{2\pi i w}\frac{z^{(c-u)n}(1-\alpha w)^n(1+w)^{cn}}{w^{(c-v)n}(1-\alpha z)^n(1+z)^{cn}(z-w)}$$



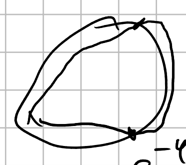
$$\rho(u) = \lim_{n\to\infty} K_n(u,u) = \lim_{n\to\infty}\iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w} e^{n(S(z,u)-S(w,u))}\frac{1}{z-w}$$

$$S(z,u) = -\ln(1-\alpha z)-\ln(1+z)+(c-u)\ln z$$

$\mathrm{Re}\, S(z,u) = 0$

$z\,\partial_z S(z,u)$

$\longrightarrow$



$z_+ = |z_+|e^{i\varphi}$

$z_- = |z_+|e^{-\varphi i}$

$$\varphi = \arccos\frac{\mathrm{Re}\, z_+}{|z_+|} \qquad \int_{z_-}^{z_+}\frac{dz}{2\pi i z} = \frac{\varphi}{\pi}$$

$$z\,\partial_z S(z,u) = 0 \qquad \frac{\alpha z}{1-\alpha z} + \frac{c}{z+1} - u = 0$$

$$z_\pm = \frac{\alpha(c-1)+u(1-\alpha)\mp\sqrt{4\alpha(u+1)(u-c)+[\alpha(c-1)+u(1-\alpha)]^2}}{2\alpha(u+1)}$$

$$u \in [u_-, u_+] \qquad u_\pm = \frac{\alpha(c-1) \pm 2\sqrt{\alpha c}}{\alpha+1}$$

$$\rho(u) = \frac{1}{\pi} \arccos\left(\frac{\alpha(c-1) + u(1-\alpha)}{2\sqrt{\alpha(c-u)(1+u)}} \right)$$

$$\Omega(x) = 1 + \int_{-1}^{x} (1 - 2\rho(u))\, du$$

$\alpha \longleftrightarrow \frac{1}{\alpha}$ mirror images of the curves

"Corner case" $\quad \Omega'(-1) = 0 \quad \rho(-1) = \frac{1}{2}$

$\Omega'(c) = 0 \quad \rho(c) = \frac{1}{2} \quad \alpha(c-1) + c(1-\alpha) = 0$

$\alpha = c$

$$\lim_{u \to -1} \frac{\alpha(c-1) + u(1-\alpha)}{2\sqrt{\alpha(c+1)(u+1)}} = 0$$

$$\alpha(c-1) - (1-\alpha) = 0$$

$$\alpha c = 1 \quad \alpha = \frac{1}{c}$$

$f, g \qquad x_i = f(i/n), \quad y_j = g(j/k)$

$$f, g : [0,1] \to \mathbb{R}_+$$

$$K(z) = \prod_{i=1}^{n} \frac{1}{1 - f(\frac{i}{n}) z} \prod_{j=1}^{k} \frac{1}{1 + g(\frac{j}{k}) \frac{1}{z}}$$

$$K(nu, nv) = \iint \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} e^{n[S(z,u) - S(w,v)]} \frac{1}{z-w}$$

$$S(z,u) = \frac{1}{n} \ln K(z) - u \ln z$$

$$-\frac{1}{n} \sum_{i=1}^{n} \ln(1 - f(\tfrac{i}{n}) z) \approx -\int_0^1 \ln(1 - f(s) z)\, ds$$

$$S(z,u) \approx -\int_0^1 \ln(1 - f(s) z)\, ds - c \int_0^1 \ln(z + g(s))\, ds + (c-u) \ln z$$

$$z \partial_z S(z,u) = \int_0^1 \frac{f(s) z}{1 - f(s) z}\, ds + c \int_0^1 \frac{g(s)}{z + g(s)}\, ds - u = 0$$

For arbitrary f, g solve numerically.

$f(s) = s \qquad g(s) = s \qquad z \ln(1-z)$

$f(s) = e^{-\gamma s} \qquad g(s) = \alpha e^{-c\gamma s}$

$x_i = e^{-\gamma \frac{i}{n}} \qquad y_j = \alpha e^{-c\gamma \frac{j}{k}}$

$x_i = q^{i-1} \qquad y_j = \alpha q^{j-1} \qquad q = e^{-\frac{\gamma}{n}}$

$\|\lambda\| = \sum_{i=1}^{n} (i-1)\lambda_i$

Principal specialization

$$s_\lambda(1, q, q^2, \dots, q^{n-1}) = q^{\|\lambda\|} \dim_q V_{GL_n}(\lambda)$$

$$\dim_q V_{GL_n}(\lambda) = \sum_{\substack{u_1 \dots u_{n-1} \\ \in \\ \mathbb{Z}_{\geq 0}}} q^{\sum u_i} \dim V(\lambda)_{\lambda - \sum u_i \alpha_i}$$

$\alpha_1 \dots \alpha_{n-1}$ simple roots of GL_n

$$e^{\gamma u} = \frac{(1 - e^{-\gamma} z)(z + \alpha)}{(1-z)(z + \alpha e^{-\gamma c})}$$

$$z_\pm = \frac{-\alpha\left(e^{\gamma(u-c)} - e^{-\gamma}\right) + e^{\gamma u} - 1 \pm \sqrt{\left(\alpha\left(e^{\gamma(u-c)} - e^{-\gamma}\right) - e^{\gamma u} + 1\right)^2 - 4\left(e^{\gamma u} - e^{-\gamma}\right) \cdot \alpha\left(1 - e^{\gamma(u-c)}\right)}}{2\left(e^{\gamma u} - e^{-\gamma}\right)}$$

$t_\pm = e^{\gamma u_\pm}$

$$t_\pm = \frac{\alpha^2 e^{\gamma(c-1)} + 2\alpha e^{\gamma(c-1)} - \alpha e^{\gamma c} - \alpha e^{\gamma(2c-1)} + 2\alpha e^{2\gamma c} + e^{2\gamma c}}{(\alpha + e^{\gamma c})^2} \pm$$

$$\pm \frac{2\sqrt{\alpha(1+\alpha)(e^{\gamma} - 1)(e^{\gamma c} - 1)(e^{\gamma(c+1)} + \alpha)}}{(\alpha + e^{\gamma c})^2}$$

$$\rho(u) = \frac{1}{\pi} \arccos\left(\frac{\alpha e^{\gamma c} - e^{\gamma(c+1)} - \alpha e^{\gamma(u+1)} + e^{\gamma(c+u+1)}}{2\sqrt{\alpha\left(e^{\gamma(c+1)} - e^{\gamma(u+1)}\right)\left(e^{\gamma(c+u+1)} - e^{\gamma c}\right)}}\right)$$

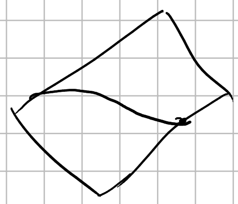
$f(s) = e^{-\gamma s} \qquad g(s) = e^{c\gamma s}$

$$z_\pm = \frac{e^{-\gamma} + e^{\gamma c} \pm \sqrt{(e^{\gamma} - e^{\gamma c})^2 - 4\left(e^{\gamma(c+1)} - e^{\gamma(u+1)}\right)\left(e^{\gamma(u+1)} - 1\right)}}{2\left(e^{\gamma(u+1)} - 1\right)}$$

$$u_\pm = -1 - \frac{1}{\gamma}\ln 2 + \frac{1}{\gamma}\ln\left(1 + e^{\gamma(c+1)} \pm \sqrt{(e^{2\gamma c} - 1)(e^{2\gamma} - 1)}\right)$$

$$\rho(u) = \frac{1}{\pi} \arccos\left(\operatorname{sgn}(-\gamma) \frac{e^{\gamma - \gamma u/2}}{2} \frac{1 - e^{\gamma(c-1)}}{\sqrt{(1-e^{\gamma u})(1-e^{\gamma(c+1-u)})}} \right)$$

$$u \in [u_-, u_+]$$



$$m = u_+ n + \xi n^{\frac{1}{3}} \frac{1}{6}$$

$$z_c : \quad z \partial_z S(z, u) = 0$$

$$S(z) = S(z_c) + \frac{1}{6} S'''(z_c)(z - z_c)^3 + \dots$$

$$z = z_c e^{6 \xi n^{-1/3}} \approx z_c \left(1 + \frac{6\xi}{n^{1/3}} + \dots\right)$$

$$K(\xi, \eta) = \iint \frac{d\zeta \, dv}{(2\pi i)^2} \frac{\exp(\zeta^3/3 - \zeta \xi)}{\exp(v^3/3 - v\eta)} \frac{1}{\zeta - v}$$

$$n(S(z) - S(w)) \approx \frac{1}{6} 6^3 z_c^3 S'''(z_c)(\xi^3 - \eta^3) = \text{Airy kernel}$$

$$= \frac{1}{6} 6^3 \left((z\partial_z)^3 S(z) \Big|_{z = z_c} \right) (\xi^3 - \eta^3)$$