Lecture1

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1 From free fermions to limit shapes and beyond

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1.1 Lecture 1

1.1.1 Teaser pictures

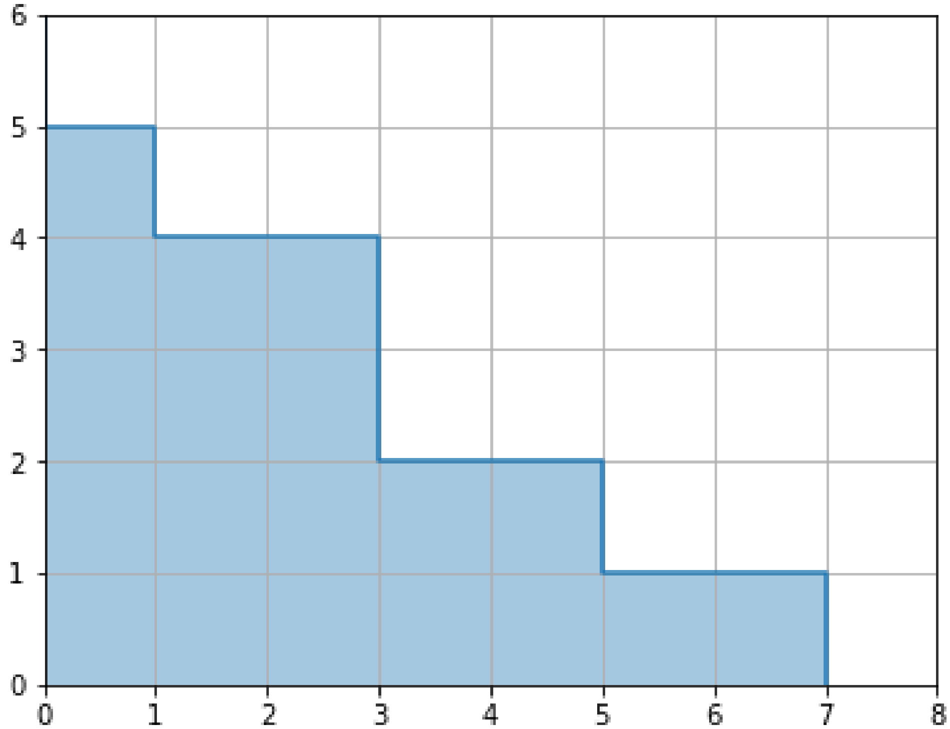
Some imports first:

[11]:
```
%matplotlib inline

import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import matplotlib
```

Young diagram or **partition** - ordered set of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda_i \geq \lambda_{i+1}$.

See an illustration below. We use so called "French" notation here.

[12]:
```
# create a figure and axes
fig = plt.figure(figsize=(6,6))
ax1 = plt.subplot(1,1,1)
ax1.set_xlim(( 0, 8))
ax1.set_ylim((0, 6))

ax1.set_aspect('equal','box')
plt.grid()
rows=[7,5,3,3,1,0]
diagdata=([x for x in rows for _ in range(2)]+[0],
[0]+[y for y in range(1,len(rows)+1) for _ in range(2)])
plt.plot(*diagdata)
plt.fill_between(*diagdata,alpha=0.4)
plt.show()
```



Inverse hook walk Let's generate a random Young diagram by the inverse hook walk algorithm. We start with an empty Young diagram. 1. We create a box at random position (i, j) outside of the diagram. 2. Draw a hook from it and choose next position of the box uniformly from the hook. We proceed to move the box this way until new position is adjacent to the diagram. 3. We then add the box to the diagram and go to step 1.

We add n boxes to sample a random diagram from the Plancherel measure

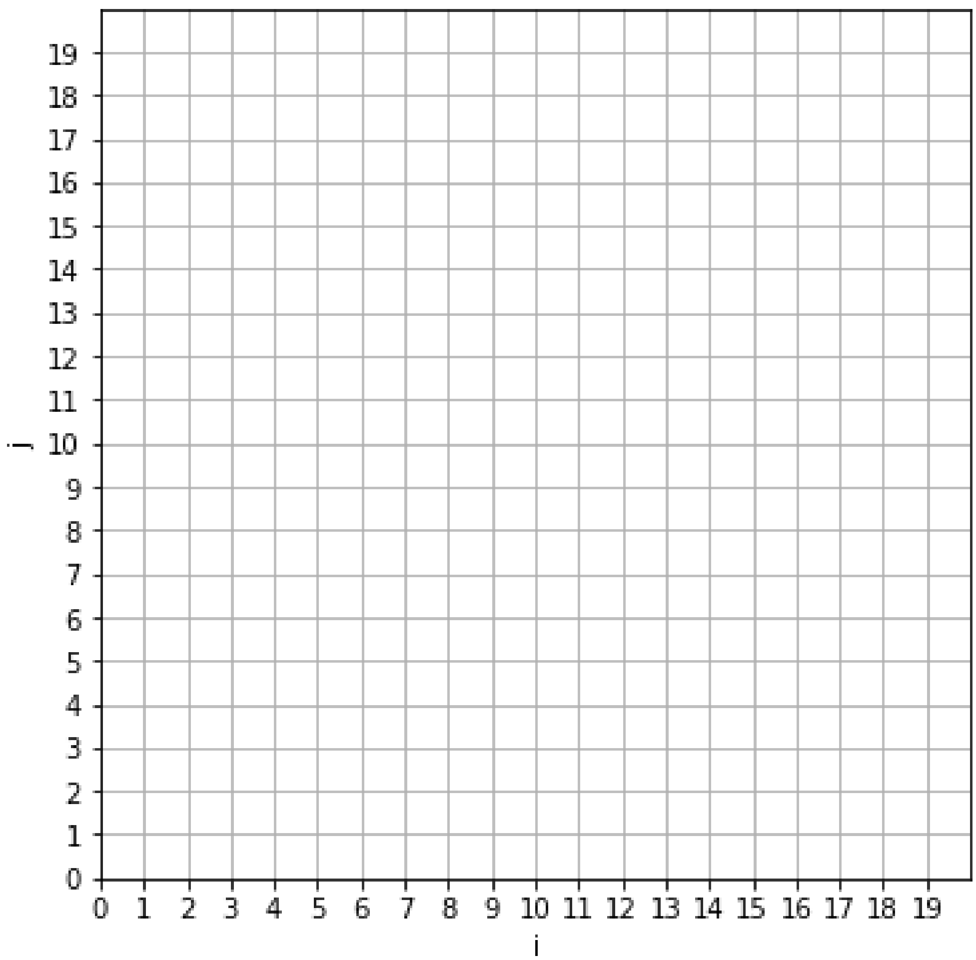
$$\mu_n(\lambda) = \frac{\dim(\lambda)^2}{n!}.$$

Here $\dim(\lambda)$ is the dimension of the permutation group S_n representation, that corresponds to the diagram λ. We will discuss it on next lectures.

Animation of the algorithm Prepare the figure

[13]:
```
%matplotlib inline

import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

n=20 # size of the area
# create a figure and axes
fig = plt.figure(figsize=(6,6))
```

```
ax1 = plt.subplot(1,1,1)

# set up the subplots as needed
ax1.set_xlim(( 0, n))
ax1.set_ylim((0, n))
ax1.set_xlabel('i')
ax1.set_ylabel('j')
ax1.set_title('Young diagram')
ax1.set_aspect('equal','box')
ax1.set_xticks(np.arange(0, n, 1))
ax1.set_yticks(np.arange(0, n, 1))
plt.grid()
# create objects that will change in the animation. These are
# initially empty, and will be given new values for each frame
# in the animation.
txt_title = ax1.set_title('')
box1, = ax1.plot([], [], 'r', lw=2)     # ax.plot returns a list of 2D line␣
 ↪objects
hook1, = ax1.plot([], [], 'y', lw=2)
diagram, = ax1.plot([], [], 'b', lw=2)


from numpy.random import default_rng
rng = default_rng()

ncolumns=0
nrows=0
rows=np.zeros(n,dtype=int)
columns=np.zeros(n,dtype=int)
i=ncolumns+rng.integers(n-ncolumns)
j=nrows+rng.integers(n-nrows)
hooklength=i-rows[j]+j-columns[i]+1
#ax1.legend(['sin','cos']);
```



Animation step

[14]:
```
# animation function. This is called sequentially
def drawframe(framenumber):
    global hooklength, i,j, nrows, ncolumns,rows, columns,n
    #print(i,j,rows,columns)
    if hooklength>1:
        k=rng.integers(hooklength)
        if k<i+1-rows[j]:
            i=rows[j]+k
        else:
            j=j-k+i-rows[j]
        hooklength=i-rows[j]+j-columns[i]+1
    else:
        rows[j]+=1
        columns[i]+=1
        if i>ncolumns:
```

```
            ncolumns=i
        if j>nrows:
            nrows=j
        i=ncolumns+rng.integers(n-ncolumns)
        j=nrows+rng.integers(n-nrows)
        hooklength=i-rows[j]+j-columns[i]+1

    box1.set_data([i,i+1,i+1,i,i],[j,j,j+1,j+1,j])
    hook1.set_data([rows[j],i,i,i+1,i+1,rows[j],rows[j]],
                   [j,j,columns[i],columns[i],j+1,j+1,j])
    diagdata=([0]+[x for x in range(1,n+1) for _ in range(2)],
                     [y for y in columns for _ in range(2)]+[0])

    diagram.set_data(*diagdata)
    txt_title.set_text('Frame = {0:4d}'.format(n))
    return (box1,hook1,diagram)
```

Animate and convert to HTML video

[15]:
```
from matplotlib import animation

# blit=True re-draws only the parts that have changed.
anim = animation.FuncAnimation(fig, drawframe, frames=200, interval=400,␣
 ↪blit=True)
from IPython.display import HTML
HTML(anim.to_html5_video())
```

[15]: <IPython.core.display.HTML object>

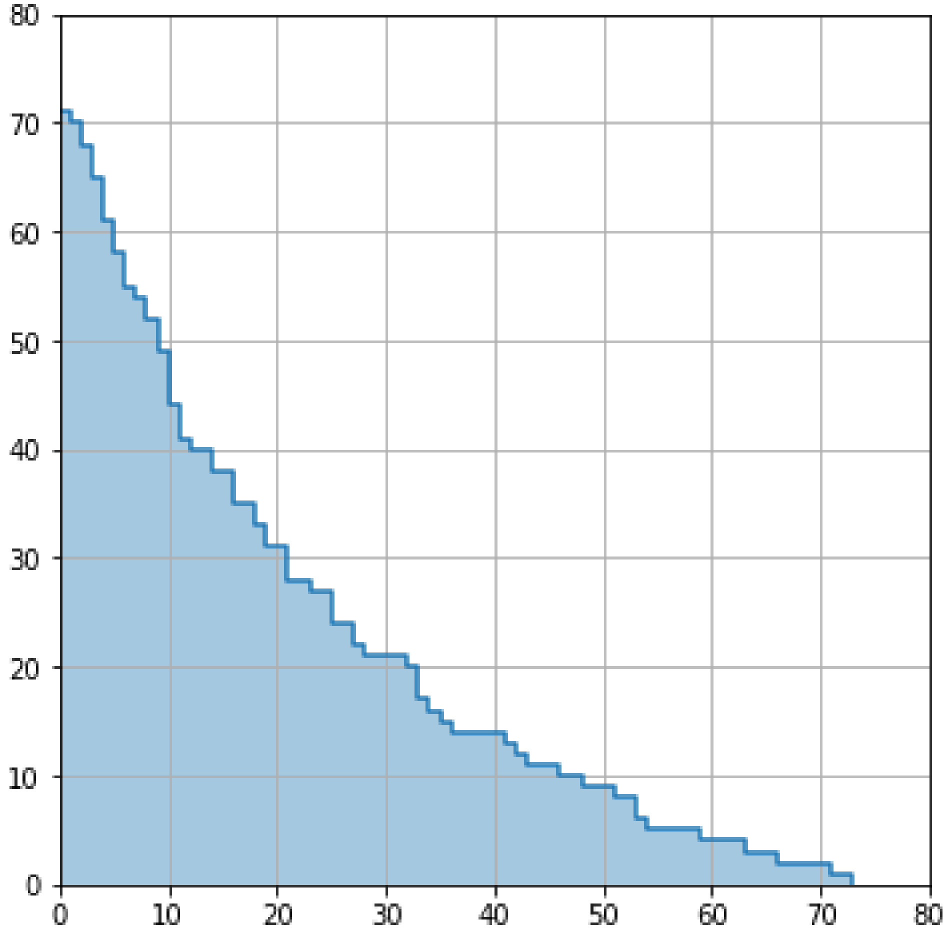
Sampling of large diagram and limit shape Use the algorithm to sample a large diagram of n boxes and draw it. First, introduce the function to sample a diagram.

[16]:
```
def sample_diagram(n):
    ncolumns=0
    nrows=0
    rows=np.zeros(n,dtype=int)
    columns=np.zeros(n,dtype=int)
    for _ in range(n):
        i=ncolumns+rng.integers(n-ncolumns)
        j=nrows+rng.integers(n-nrows)
        hooklength=i-rows[j]+j-columns[i]+1
        while hooklength>1:
            k=rng.integers(hooklength)
            if k<i+1-rows[j]:
                i=rows[j]+k
            else:
                j=j-k+i-rows[j]
```

```
            hooklength=i-rows[j]+j-columns[i]+1
        rows[j]+=1
        columns[i]+=1
        if i>ncolumns:
            ncolumns=i
        if j>nrows:
            nrows=j
        i=ncolumns+rng.integers(n-ncolumns)
        j=nrows+rng.integers(n-nrows)
        hooklength=i-rows[j]+j-columns[i]+1
    return rows[:nrows+1]
```

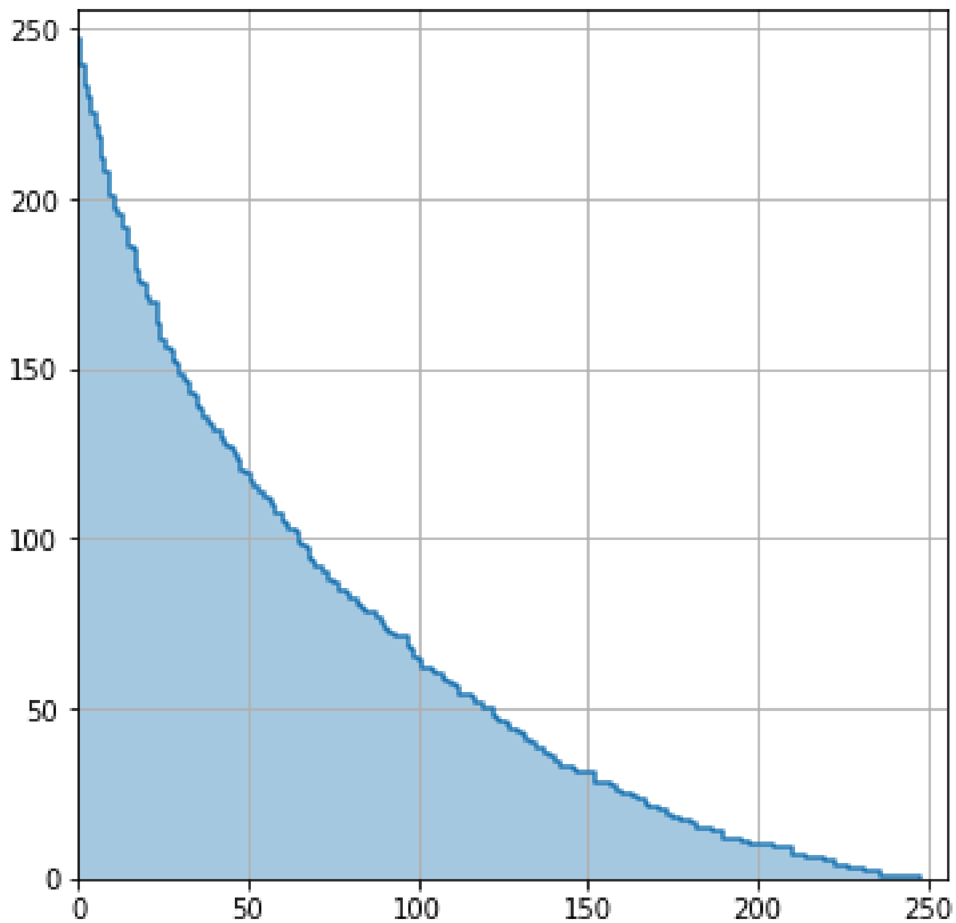
Let’s sample and draw a diagram of 1600 boxes.

[17]:
```
rows=sample_diagram(1600)

# create a figure and axes
fig = plt.figure(figsize=(6,6))
ax1 = plt.subplot(1,1,1)
ax1.set_xlim(( 0, 80))
ax1.set_ylim((0, 80))

ax1.set_aspect('equal','box')
plt.grid()
diagdata=([x for x in rows for _ in range(2)]+[0],
[0]+[y for y in range(1,len(rows)+1) for _ in range(2)])
plt.plot(*diagdata)
plt.fill_between(*diagdata,alpha=0.4)
plt.show()
```



We see that diagram approximates a smooth curve, as we see for larger diagram of 16384 boxes

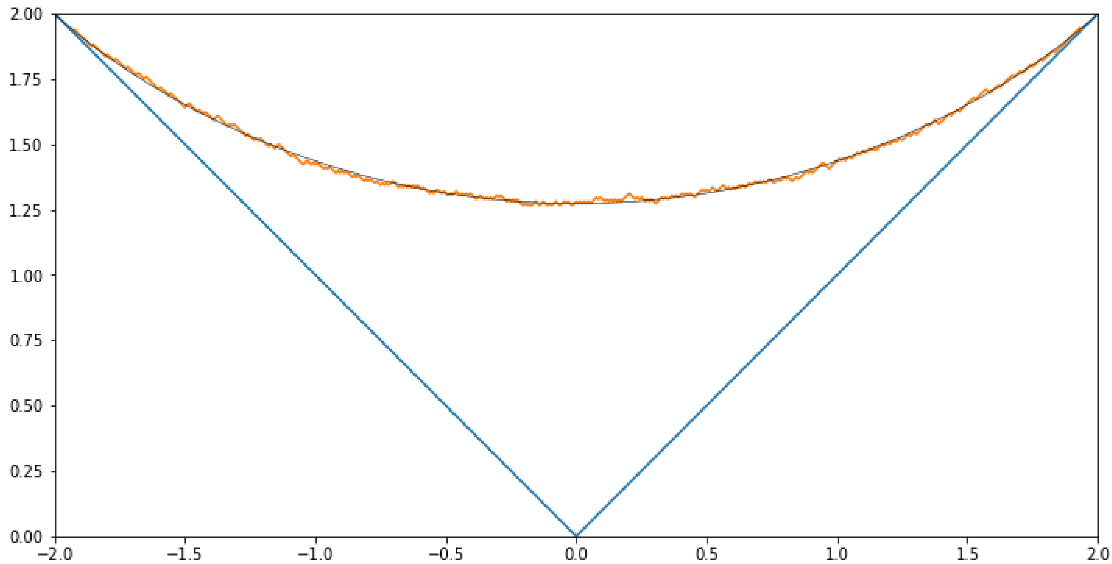
[18]:
```
rows=sample_diagram(16384)

# create a figure and axes
fig = plt.figure(figsize=(6,6))
ax1 = plt.subplot(1,1,1)
ax1.set_xlim(( 0, 256))
ax1.set_ylim((0, 256))

ax1.set_aspect('equal','box')
plt.grid()
diagdata=([x for x in rows for _ in range(2)]+[0],
[0]+[y for y in range(1,len(rows)+1) for _ in range(2)])
plt.plot(*diagdata)
plt.fill_between(*diagdata,alpha=0.4)
plt.show()
```



The diagram not a particularly nice function, it is piecewise-constant. It is often more convenient to rotate it 45° counterclockwise and scale by a factor of $\sqrt{n}$. Then the diagram boundary becomes a piecewise-linear function with derivative ± 1 almost everywhere. Limit theorems can be formulated in terms of Lipshitz functions and Sobolev spaces.

[19]:
```
# create a figure and axes
fig = plt.figure(figsize=(12,12))
ax1 = plt.subplot(1,1,1)
ax1.set_aspect('equal','box')
ax1.set_ylim([0,2])
ax1.set_xlim([-2,2])
c=2
plt.plot([-2,0,2,0,-2],[2,0,2,4,2])

n=np.sqrt(16384)
plt.plot([(x-i-dx)/n for (i,x) in enumerate(rows) for dx in (0,1)],
         [(x+i+dx)/n for (i,x) in enumerate(rows) for dx in (0,1)])
xs=np.arange(-2,2,1/100)
```

```
ys=2/np.pi*(xs*np.arcsin(xs/2)+np.sqrt(4-xs*xs))
with matplotlib.rc_context({'lines.linewidth': 0.5, 'lines.linestyle': '-'}):
    plt.plot(xs,ys,color='k')
plt.show()
```



This is famous Vershik-Kerov-Logan-Shepp curve

$$\Omega(u) = \frac{2}{\pi}\left(u \arcsin\left(\frac{u}{2}\right) + \sqrt{4-u^2}\right).$$

In this course we will learn how to study this limiting behavior. We will connect illustrative examples of random Young diagrams to various advanced mathematical topics, such as representation theory, random matrices, orthogonal polynomials and integrable systems.

1.1.2 Connection to random matrices

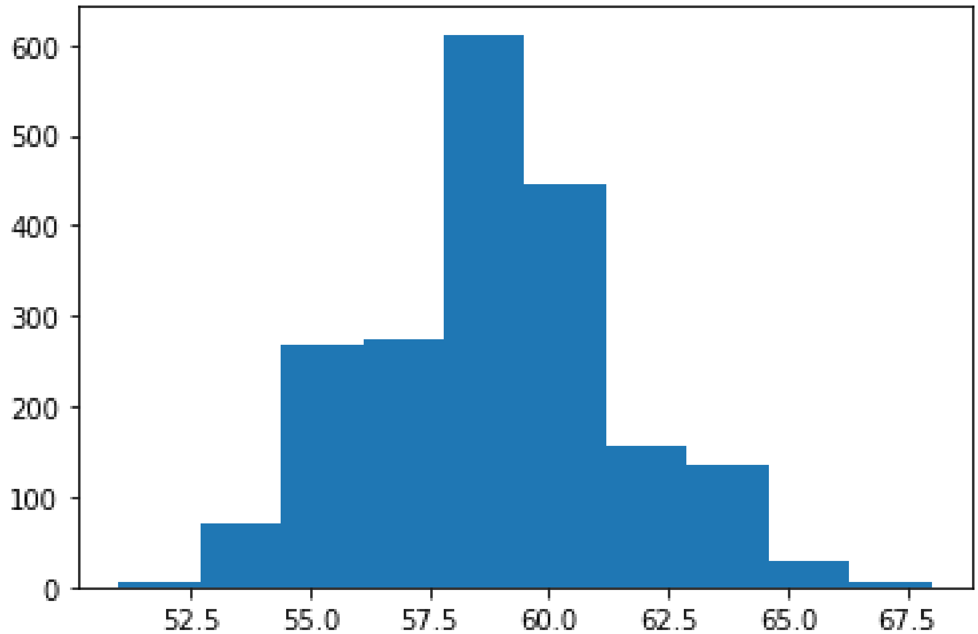
Let us look at the distribution of the length of the first row of our random Young diagram. For that we sample 1000 diagrams of 1024 boxes and plot a histogram of first row lengths. It will take about a minute.

[86]:
```
%%time
n=1024
lmbs=[]
for _ in range(2000):
    d=sample_diagram(n)
    lmbs.append(d[0])
plt.hist(lmbs)
```

```
CPU times: user 1min 21s, sys: 859 µs, total: 1min 21s
Wall time: 1min 21s
```

[86]:

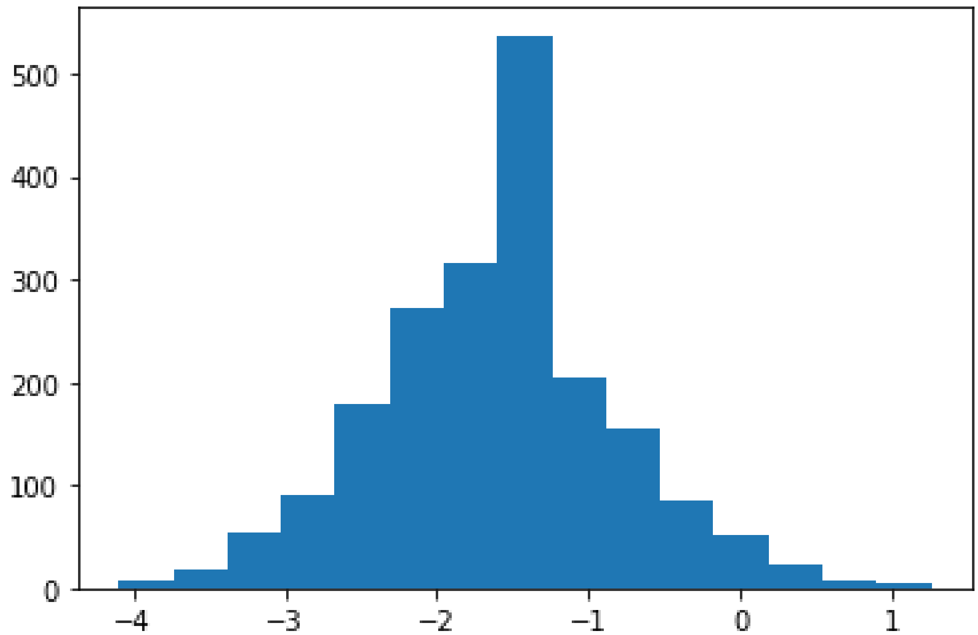
```
(array([  6.,  70., 269., 273., 612., 446., 155., 135.,  30.,   4.]),
 array([51. , 52.7, 54.4, 56.1, 57.8, 59.5, 61.2, 62.9, 64.6, 66.3, 68. ]),
 <BarContainer object of 10 artists>)
```



We see that average length of the first row $\lambda_1 \approx 2\sqrt{n}$, let us shift the histogram and scale by the factor $n^{-\frac{1}{6}}$.

[206]:

```
younghist=plt.hist((np.array(lmbs)-2*np.sqrt(n))/n**(1/6),bins=15)
```



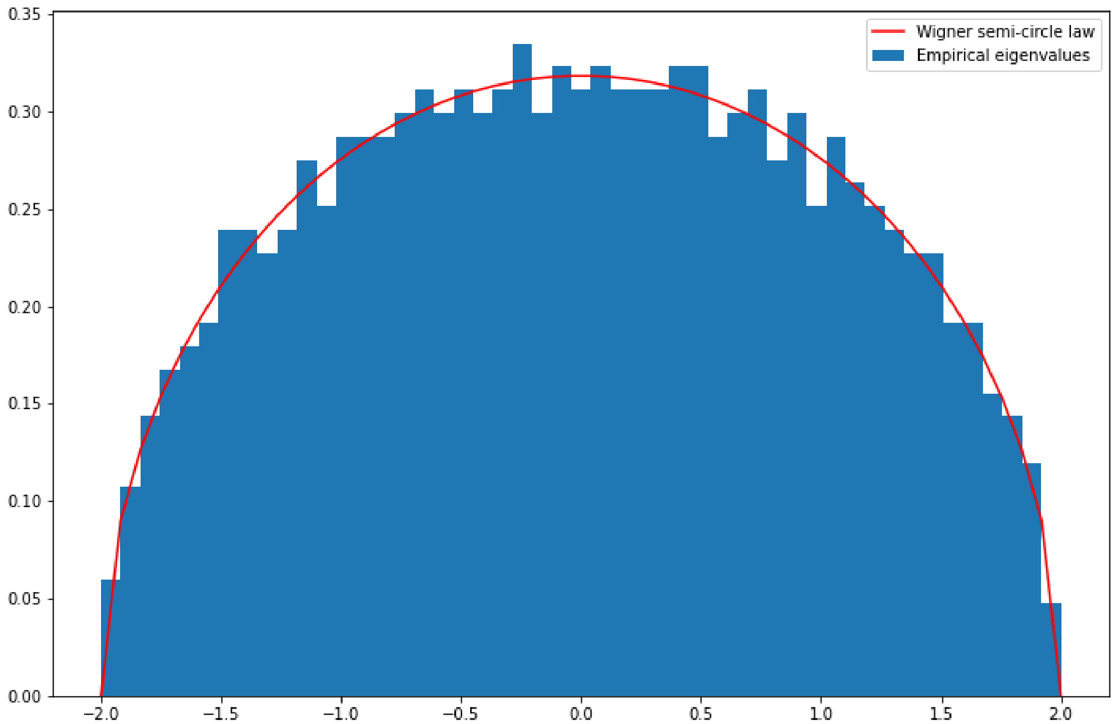
Now compare this to the distribution of the largest eigenvalue of a random complex hermitean gaussian matrix. We sample a complex 1024×1024 matrix with real and imaginary parts of non-diagonal matrix elements from Gaussian standard distribution $\mathcal{N}(0,1)$ and take half a sum of this matric and its conjugate.

[156]:
```
Z=np.array(np.random.randn(n,n))+1j*np.array(np.random.randn(n,n))
X=(Z+Z.T.conjugate())/2
```

Eigenvalues of this matrix satisfy famous **Wigner semicircle law**, that is the scaled density of eigenvalues tends to a semicircle as size of random matrix grows.

[196]:
```
%%time
edges=np.linspace(-2,2,50)

# Limiting measure
mu = np.sqrt( np.maximum.reduce( [4 - edges**2, np.zeros(len(edges))] ) )/2/np.
 ↪pi

# Empirical histogram
fig = plt.figure(figsize=(12,8))
ax1 = plt.subplot(1,1,1)

ax1.hist(np.linalg.eigh(X/np.sqrt(n))[0],bins=edges,weights=1/n/
 ↪(edges[1]-edges[0])*np.ones(n),label='Empirical eigenvalues')
ax1.plot(edges,mu,'r',label='Wigner semi-circle law')
_=plt.legend()
```

```
CPU times: user 10.1 s, sys: 2.41 s, total: 12.5 s
Wall time: 1.73 s
```



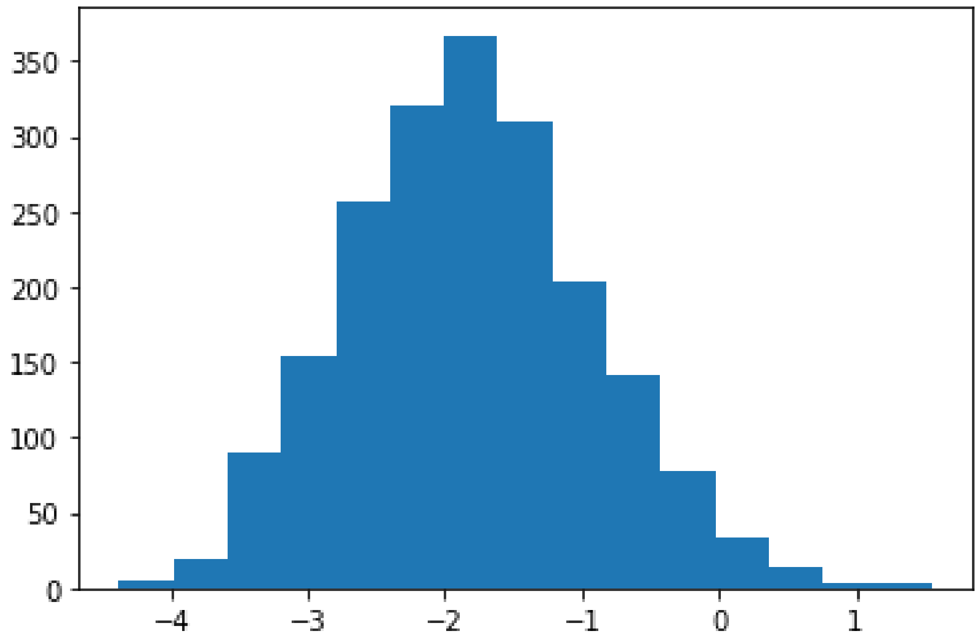
Now let us sample 2000 of such matrices and look at the histogram of the largest eigenvalue.

[169]:
```
%%time
import scipy
eigs=[]
for _ in range(2000):
    Z=np.array(np.random.randn(n,n))+1j*np.array(np.random.randn(n,n))
    X=(Z+Z.T.conjugate())/2
    eigs.append(scipy.sparse.linalg.eigsh(X,1,which='LA')[0][0])
```

```
CPU times: user 1h 22min 6s, sys: 59min 25s, total: 2h 21min 31s
Wall time: 20min 59s
```

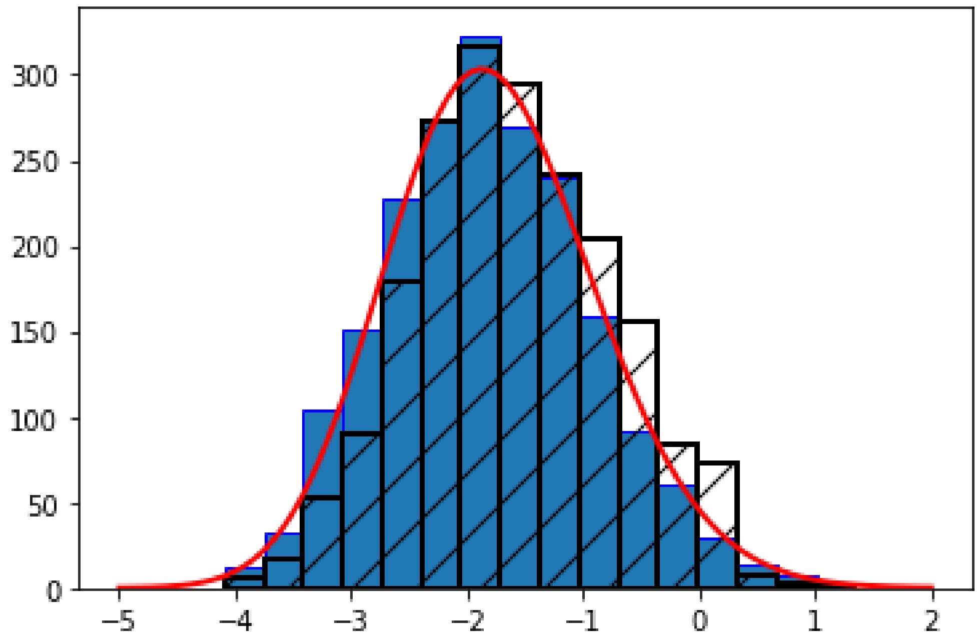
The largest eigenvalue $\lambda_1 \approx 2\sqrt{n}$, and we again shift it and now scale by the factor $n^{\frac{1}{6}}$.

[199]:
```
guehist=plt.hist((np.array(eigs)-2*np.sqrt(n))*n**(1/6),bins=15)
```



Let us compare both distributions on the same picture and plot Tracy-Widom distribution on the same plot.

[205]:
```
from TracyWidom import TracyWidom
tw2 = TracyWidom(beta=2)
xs=np.linspace(-5,2,100)

tw_sample1=(np.array(lmbs)-2*np.sqrt(n))/n**(1/6)
tw_sample2=(np.array(eigs)-2*np.sqrt(n))*n**(1/6)
binwidth=0.34
bins=np.arange(min(tw_sample1), max(tw_sample1) + binwidth, binwidth)
plt.plot(xs,tw2.pdf(xs)*2000*binwidth,color='r',linewidth=2)
plt.hist(tw_sample2,bins=bins,edgecolor='b')
plt.hist(tw_sample1,bins=bins,hatch='/',edgecolor='k',linewidth=2,␣
 ↪facecolor='c',fill=False)

plt.show()
```



We can see the relation of the fluctuations of the first row length for the random Young diagrams and the fluctuations of the largest eigenvalue of the **Gaussian Unitary Ensemble (GUE)**. This is an example of *universality*.

In these lectures I hope to cover limit shape derivation and study fluctuations for certain models that come from representation theory of symmetric group and simple Lie groups.

1.1.3 Course plan

1. Infinite wedge representation
2. Free fermions and bosonization
3. Vertex operators
4. Partitions and Schur polynomials
5. Boson-fermion correspondence
6. Plancherel measure on partitions
7. Correlation kernel from free fermions and limit shape for Plancherel measure
8. Relation of Plancherel measure to RSK algorithm
9. Representations of GL(n), exterior powers and exterior algebra
10. Skew Howe GL(n)-GL(k) duality
11. Probability measure on Young diagrams in the box and dual RSK algorithm
12. Limit shape for skew GL(n)-GL(k) duality from free fermions
13. Skew Howe dualities for other classical series of Lie groups
14. Young diagrams and tableaux for symplectic groups
15. Proctor algorithm for symplectic groups
16. Limit shapes for symplectic groups and scalar Riemann-Hilbert problem
17. Local asymptotics of correlation kernel for skew GL(n)-GL(k) duality, Airy process, local fluctuations around limit shape

18. Global fluctuations around limit shape, Krawtchouk and q-Krawtchouk orthogonal polynomials, Central limit theorem
19. Asymptotics of orthogonal polynomials, Riemann-Hilbert problem and its integrability

1.1.4 Literature

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