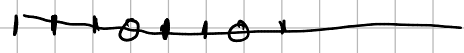
Lecture 4. 31.05.2023

$(s_1, s_2, \dots)$

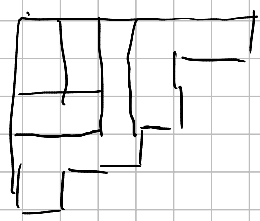
$$\Gamma_\pm(s) = \exp\left(\sum_{n=1}^\infty s_n \alpha_{\pm n}\right)$$

$\lambda_i - i + \frac{1}{2}$



$$\langle\lambda| \;\; \Gamma_-(t)|0\rangle = \sum_\lambda s_\lambda(x_1,\dots)|\lambda\rangle$$

$\langle\lambda|\lambda\rangle = 1$



$(t_1, t_2, \dots)$

$$t_k = \frac{1}{k}\sum_{i=1}^\infty x_i^k$$

$$t'_k = \frac{1}{k}\sum_{i=1}^\infty y_i^k$$

$$\Gamma_-(t)^* = \Gamma_+(t)$$

$$s_\lambda(x_1,\dots) = \langle\lambda|\Gamma_-(t)|0\rangle = \langle 0|\Gamma_+(t)|\lambda\rangle$$

$$s_\lambda(x_1,\dots) = \sum_{T\in SSYT(\lambda)} x^T$$

$$\sum_\lambda s_\lambda(x_1,\dots)\, s_\lambda(y_1,\dots) = \prod_{ij}\frac{1}{1-x_iy_j} = Z$$

Z - partition function

$$\sum_\lambda \langle 0|\Gamma_+(t)|\lambda\rangle\langle\lambda|\Gamma_-(t')|0\rangle = \langle 0|\Gamma_+(t)\Gamma_-(t')|0\rangle$$

$$\sum_\lambda |\lambda\rangle\langle\lambda| = Id \qquad \Gamma_+(t)|0\rangle = |0\rangle$$

$$\Gamma_+(t)\Gamma_-(t') = e^{\sum_{n=1}^\infty n t_n t'_n}\,\Gamma_-(t')\Gamma_+(t)$$

$$e^{\sum_{n=1}^\infty n t_n t'_n} = \exp\left(\sum_{n=1}^\infty \frac{1}{n}\sum_{ij} x_i^n y_j^n\right) = \prod_{ij}\frac{1}{1-x_iy_j} \qquad \text{Ex}$$

$$\frac{1}{1-x_iy_j} = \sum_{n=0}^\infty (x_iy_j)^n$$

$$\underbrace{\Gamma_+(t)\Gamma_-(t') = \prod_{ij}\frac{1}{1-x_iy_j}\,\Gamma_-(t')\Gamma_+(t)}$$

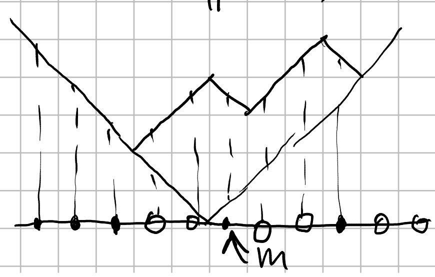
$$= \prod_{ij}\frac{1}{1-x_iy_j}$$

$$\Gamma_\pm(t)\psi(z) = \prod_i \frac{1}{1-x_iz^{\pm1}}\,\psi(z)\Gamma_\pm(t)$$

$$\Gamma_\pm(t)\psi^*(w) = \prod(1-x_iw^{\pm1})\,\psi^*(w)\Gamma_\pm(t)$$

Schur measure $\mu_{Schur}(\lambda | x_1,\dots, y_1,\dots)$

$$\mathbb{P}(\lambda) = \frac{s_\lambda(x_1,\dots)\, s_\lambda(y_1,\dots)}{Z} = s_\lambda(x)\, s_\lambda(y)\prod_{ij}(1-x_iy_j)$$



$$\mathbb{P}(m\in\lambda) = \frac{\langle 0|\Gamma_+(t)\psi_m\psi_m^*\Gamma_-(t')|0\rangle}{\langle 0|\Gamma_+(t)\Gamma_-(t')|0\rangle}$$

$$\psi_m\psi_m^*|\lambda\rangle$$

$|v\rangle \in \mathcal{F}$
$\quad \shortparallel$
$\underline{v_1} \wedge \underline{v_2} \dots$

$$\mu_v(\lambda) = \frac{\langle v|\lambda\rangle^2}{\langle v|v\rangle^2}$$

$$t = (\sqrt{\xi}, 0, 0 \dots)$$

$$t' = (\sqrt{\xi}, 0, 0 \dots)$$

$|\lambda|$ – # of boxes in λ

Poissonized Plancherel measure

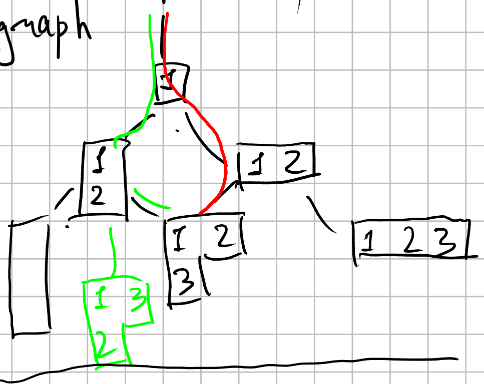
$$\mu_{PP}(\lambda|\xi) = e^{-\xi}\, \xi^{|\lambda|} \left(\frac{\dim \lambda}{|\lambda|!}\right)^2$$

Plancherel measure $\mu_p(\lambda)$
$\shortparallel$
$\left(\frac{\dim \lambda}{|\lambda|!}\right)^2$

$\dim \lambda$ – dimension of $S_{|\lambda|}$ – irreducible representation
$\shortparallel$
$\#SYT(\lambda(|\lambda|))$

Pavel Nikitin

Young graph $= \emptyset$



1	3	5
2	4	
6	9	
7	10	
8		

n boxes $\to$ numbers from 1 to n

$\xi \to \infty \sim |\lambda| \to \infty$
$\qquad\qquad\quad \shortparallel$
$\qquad\qquad\quad N$

Correlation function $\rho(X) = \mathbb{P}(X \subset \lambda) =$

$$= \frac{1}{Z}\sum_{\lambda \supset X} s_\lambda(x_1 \dots)\, s_\lambda(y_1 \dots) = \frac{1}{Z}\langle 0| \Gamma_+(t) \prod_{x\in X} \psi_x \psi_x^* \cdot$$

$$\Gamma_+(t)\Gamma_-(t') = Z\, \Gamma_-(t')\Gamma_+(t) \qquad \cdot\, \Gamma_-(t')|0\rangle$$

$$G = \Gamma_+(t)\Gamma_-(t')^{-1} \qquad \Psi_m = G \psi_m G^{-1}$$

$$\Psi_m^* = G \psi_m^* G^{-1}$$

$$\rho(X) = \langle 0| \prod_{x\in X} \Psi_x \Psi_x^* |0\rangle$$

Thm. (Wick's theorem for free fermions) Schur measure

$$\rho(X) = \det(K(x,y))_{x,y\in X} \qquad K(x,y) = \langle 0| \Psi_x \Psi_y^* |0\rangle$$

propagator or Green's function for

$\hookrightarrow$ correlation kernel

$$\psi_k \psi_j^* = -\psi_j^* \psi_k \qquad j \neq k$$

$$\underline{\psi}_k \underline{\psi}_j^* = -\underline{\psi}_j^* \underline{\psi}_k$$

$$\psi_i \psi_i^* = -\psi_i^* \psi_i + 1$$

$$\rho(X) = \langle 0 | \underline{\psi}_{x_1} \cdots \underline{\psi}_{x_n} \underline{\psi}_{x_n}^* \cdots \underline{\psi}_{x_1}^* | 0 \rangle$$

$$n! \qquad (-1)^b$$

$$\langle 0 | \underline{\psi}_i \underline{\psi}_i^* | 0 \rangle$$

$$K(z,w) = \sum_{i,j \in \mathbb{Z}+\frac{1}{2}} z^i w^{-j} K(i,j) = \langle 0 | G \psi(z) \psi^*(w) G^{-1} | 0 \rangle$$

$$\psi(z) = \sum_i z^i \psi_i$$

$$G \psi(z) \psi^*(w) G^{-1} = \frac{F(z)}{F(w)} \psi(z) \psi^*(w)$$

$$F(z) = \frac{\gamma(z,t)}{\gamma(z^{-1}, t^{-1})} = \exp\Big(\sum_{k \geq 1} t_k z^k - t'_k z^{-k}\Big) =$$

$$= \prod_i \frac{1 - \frac{y_i}{z}}{1 - x_i z}$$

$$\langle 0 | \psi(z) \psi^*(w) | 0 \rangle = \sum_{j \in \mathbb{Z}_{\geq 0} + \frac{1}{2}} \left(\frac{w}{z}\right)^j = \frac{\sqrt{zw}}{z-w} \qquad |w| < |z|$$

Thm Okounkov

$$K(z,w) = \frac{\sqrt{zw}}{z-w} \frac{F(z)}{F(w)}$$

$$K(i,j) = \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{w^j}{z^i} \frac{F(z)}{F(w)} \frac{\sqrt{zw}}{z-w}$$