Lecture 10   23. 06. 2023

Limit shapes for skew Howe GL_n-GL_k duality

$$\Lambda(\mathbb{C}^n \otimes \mathbb{C}^k) = \bigoplus_{\lambda \subset n \times k} V_{GL_n}(\lambda) \otimes V_{GL_k}(\lambda')$$

Dual Cauchy identity $\prod_{i=1}^{n} \prod_{j=1}^{k} (1 + x_i y_j) = \sum_\lambda s_\lambda(x_1 \dots x_n) s_{\lambda'}(y_1 \dots y_k)$

$$\mu_{n,k}(\lambda | x_1 \dots x_n; y_1 \dots y_k) = \frac{s_\lambda(x_1 \dots x_n) s_{\lambda'}(y_1 \dots y_k)}{\prod_{i=1}^{n} \prod_{j=1}^{k} (1 + x_i y_j)}$$

$n, k \to \infty$ $\quad \lim \frac{k}{n} = c$ $\qquad s_\lambda(x_1 \dots x_n) = \sum_{T \in SSYT(\lambda|n)} x_1^{\#1 \text{ in } T} \cdots x_n^{\#n \text{ in } T}$

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \dim V_{GL_k}(\lambda')}{2^{nk}}$$

$t_\ell = \frac{1}{\ell} \sum_i x_i^\ell \quad t'_\ell = \frac{1}{\ell} \sum_j y_j^\ell$

$$\mu_{n,k}(\lambda | x, y) = \frac{\langle 0 | \Gamma_+(t) | \lambda \rangle \langle \lambda | \Gamma'_-(t') | 0 \rangle}{\langle 0 | \Gamma_+(t) \Gamma'_-(t') | 0 \rangle}$$

$y_j \to -y_j$, $t'_\ell = (-1)^{\ell-1} \dots$

$$\Gamma'_-(t') = \exp\left(\sum_{\ell=1}^{\infty} (-1)^{\ell-1} t'_\ell \alpha_{-\ell}\right) = \Gamma_-^{-1}(-Y)$$

$\psi(z)$

$\psi^*(w)$

$$\langle 0 | \Gamma_+(t) \Gamma_-(t') | 0 \rangle = \frac{1}{\prod_{ij} (1 - x_i y_j)}$$

$$\langle 0 | \Gamma_+(t) \Gamma_-^{-1}(-Y) | 0 \rangle = \prod_{ij} (1 + x_i y_j)$$

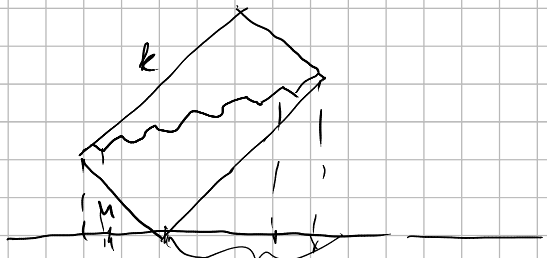
$\psi(z) \Gamma_-^{-1}$

$$K(m, m') = \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{K(z)}{K(w)} z^{-m} w^{m'} \frac{\sqrt{zw}}{z - w}$$

$$F(z) = \prod_{i=1}^{n} \prod_{j=1}^{k} \frac{1 - y_j/z}{1 - x_i z} \qquad K(z) = \prod_{i=1}^{n} \frac{1}{1 - x_i z} \prod_{j=1}^{k} \frac{1}{1 + y_j/z}$$

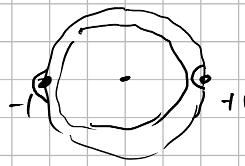
$x_i, y_j = 1 \qquad K(z) = \frac{1}{(1-z)^n} \frac{1}{(1 + 1/z)^k}$

$$K(m, m') = \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{z^{k-m} (1-w)^n (1+w)^k}{(1-z)^n (1+z)^k w^{k-m'}} \frac{\sqrt{zw}}{z - w}$$



$k - m \geq 0$

$k - n' \geq 0$



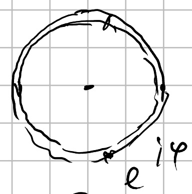
Poissonized Plancherel measure

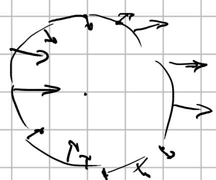
$$e^{\sqrt{\xi}\,(S(z)-S(w)-x\ln z+y\ln w)}\,\frac{\sqrt{zw}}{z-w}$$

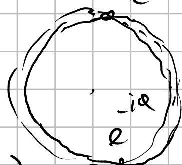
on $|z|=1$

$S(z)=z-\frac{1}{z}$

$\mathrm{Re}\, S(z)=0$







(z,w) $\quad \mathrm{Re}(S(z)-S(w))<0$

$\xi\to\infty$

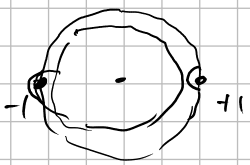
$\int_{e^{-i\varphi}}^{e^{i\varphi}}\cdots \to$ Sine kernel

$n,k\to\infty \quad k\approx cn \qquad x,y\in[-1,c]$

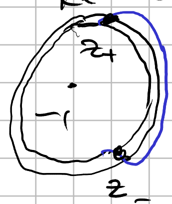
$m=nx \quad m'=ny$

$$K(x,y)=\iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w}\, e^{n[-\ln(1-z)-c\ln(1+z)+(c-x)\ln z+\ln(1-w)+c\ln(1+w)-(c-y)\ln w]}\,\frac{\sqrt{zw}}{z-w}$$

$S(z,x)=-\ln(1-z)-c\ln(1+z)+(c-x)\ln z$



$\mathrm{Re}\, S(z,x)=0$



Critical points

$z\partial_z S(z,x)=0$

$\mathrm{Re}[S(z,x)-S(w,y)]\leq 0$

$n\to\infty \quad e^{\cdots}\to 0$

$$z\partial_z S(z,x)=\frac{z}{1-z}-\frac{cz}{1+z}-x+c=0$$

$$z^2(x+1)+z(1-c)+c-x=0$$

$$z_\pm=\frac{c-1\pm\sqrt{(c-1)^2-4(x+1)(c-x)}}{2(x+1)}$$

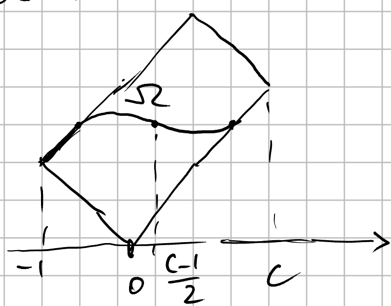
$x\in \mathrm{supp}\,\rho = [x_-,x_+]$

$z_\pm=|z_\pm|e^{\pm i\varphi}$

$|z_\pm|=\cancel{\frac{\sqrt{(c-1)^2-(c-1)^2+4(x+1)(c-x)}}{2(x+1)}}$

$$4x^2+4(1-c)x+c^2-6c+1=0$$

$$\varphi=\arccos\left(\frac{c-1}{2\sqrt{(x+1)(c-x)}}\right) \qquad x_\pm=\frac{c-1}{2}\pm\sqrt{c}$$



$$\rho(x)=\frac{1}{\pi}\mathrm{Re}\left[\arccos\frac{c-1}{2\sqrt{(x+1)(c-x)}}\right]$$

$$\Omega(u)=1+\int_{-1}^{u}(1-2\rho(x))\,dx$$

$$\begin{array}{cc} & k \\ n & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

$$P(0) = \frac{1}{1+x_i y_j} \qquad \mu(\lambda|x,y) = \frac{s_\lambda(x) s_{\lambda'}(y)}{\prod(1+x_i y_j)}$$

$$P(1) = \frac{x_i y_j}{1+x_i y_j}$$

$$\mu(\lambda|\alpha,\beta) = \frac{s_\lambda(\alpha,\dots,\alpha)\, s_{\lambda'}(\beta\dots\beta)}{(1+\alpha\beta)^{nk}} = \frac{\alpha^{|\lambda|}\beta^{|\lambda|} \dim V_{GL_n}(\lambda) \dim V_{GL_k}(\lambda')}{(1+\alpha\beta)^{nk}}$$

$$\mu(\lambda|\alpha) = \frac{s_\lambda(\alpha\dots\alpha)\, s_{\lambda'}(1\dots1)}{(1+\alpha)^{nk}} = \frac{\alpha^{|\lambda|} \dim V_{GL_n}(\lambda) \dim V_{GL_k}(\lambda')}{(1+\alpha)^{nk}}$$

$$K(z) = \prod_i \frac{1}{1-x_i z} \prod_j \frac{1}{1+y_j/z}$$

$$S(z,x) = -\ln(1-\alpha z) - c\ln(1+1/z) - x\ln z$$

$$z\partial_z S(z,x) = \frac{\alpha z}{1-\alpha z} + \frac{c}{z+1} - x = 0$$

$$z_\pm = \frac{\alpha(c-1)+x(1-\alpha) \mp \sqrt{4\alpha(x+1)(x-c)+(\alpha(c-1)+x(1-\alpha))^2}}{2\alpha(x+1)}$$

$$x_\pm = \frac{\alpha(c-1) \pm 2\sqrt{\alpha c}}{\alpha+1} \qquad \Big| \ \alpha = 1 \quad x_\pm = \frac{c-1}{2} \pm \sqrt{c}$$

$$\rho(x) = \mathrm{Re}\left[\frac{1}{\pi} \arccos \frac{\alpha(c-1)+x(1-\alpha)}{\sqrt{\alpha(c-x)(x+1)}}\right] \quad \rightarrow \alpha(c-x-1)+x$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad P(0) = \frac{1}{1+\alpha} \qquad P(1) = \frac{\alpha}{1+\alpha}$$

$\alpha = c$ is critical value