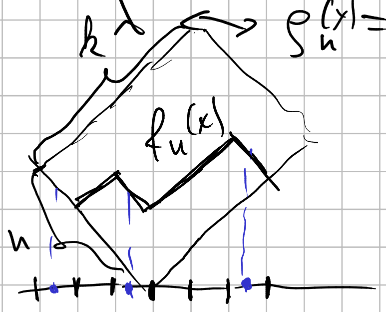
Lecture 20   2023-08-18

skew $Sp(2n,\mathbb{C})\times Sp(2k,\mathbb{C})$ Howe duality

$$\Lambda(\mathbb{C}^{2n}\otimes\mathbb{C}^{k}) = \bigoplus_{\lambda\in k^n} V_{Sp_{2n}}(\lambda)\otimes V_{Sp_{2k}}(\tilde{\lambda}')$$

$$\mu_{n,k}(\lambda) = \frac{\dim V_{Sp_{2n}}(\lambda)\,\dim V_{Sp_{2k}}(\tilde{\lambda}')}{2^{2nk}} = \frac{1}{Z_{n,k}}e^{-n^2 J[\rho_n]}$$



$$\rho_n(x) = \begin{cases} 1 & \text{if } x\in\left[\frac{i}{2n},\frac{i+1}{2n}\right] \text{ that contains a particle}\\ 0 & \text{otherwise}\end{cases}$$

$c = \frac{k}{n}$

$\rho_n(x)$ is defined on $\left[0,\frac{c+1}{2}\right]$

$\rho_n(-x) = \rho_n(x)$

$$J[\rho_n] = \iint_{-\frac{c+1}{2}}^{\frac{c+1}{2}} \rho_n(x)\rho_n(y)\ln|x-y|^{-1}\,dx\,dy \;+$$

$$+\int_{-\frac{c+1}{2}}^{\frac{c+1}{2}} \rho_n(x)\underbrace{\left[\left(\frac{c+1}{2}-x\right)\ln\left(\frac{c+1}{2}-x\right)+\left(\frac{c+1}{2}+x\right)\ln\left(\frac{c+1}{2}+x\right)\right]}_{V(x)}dx$$

$$\int_{-\frac{c+1}{2}}^{\frac{c+1}{2}} \rho_n(x)\,dx = 1 \qquad 2n \text{ particles}$$

$\operatorname{supp}\rho = [-a,a]$

$\rho_n(x)$ "$\to$" $\rho(x)$

Euler-Lagrange eqn:

$$\int_{-a}^{a}\ln|x-y|^{-1}\rho(y)\,dy + V(x) = \text{const}$$

$$-\int_{-a}^{a}\frac{\rho(y)\,dy}{y-x} + V'(x) = 0 \qquad \text{Equilibrium condition}$$

$x,y\in\mathbb{R}$

$f_n'(x) = 1-2\rho_n(x)$

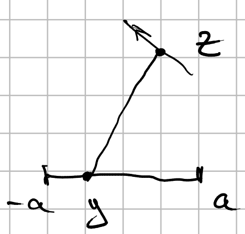
$\Omega'(x) = 1-2\rho(x)$ $f_n \to \Omega$ uniformly

$\sim \ln|z|$

$\Delta\varphi = 0$

$\frac{1}{z}$

$$G(z) := -i\int_{-a}^{a}\frac{\rho(y)}{y-z}\,dy$$



$z \to x\in[-a,a]$

$G(z)$ is analytic for $z\in\mathbb{C}\setminus[-a,a]$

$$G_\pm(x) = \lim_{\varepsilon\to0}\frac{1}{i}\int_{-a}^{a}\frac{\rho(y)\,dy}{y-(x\pm i\varepsilon)} = \lim_{\varepsilon\to0}\frac{1}{i}\int_{-a}^{a}\rho(y)\frac{y-x\pm i\varepsilon}{(y-x)^2+\varepsilon^2}\,dy$$

$z = x\pm i\varepsilon$

$$\frac{\varepsilon}{\pi(x^2+\varepsilon^2)} \xrightarrow[\varepsilon\to0]{} \delta(x)$$

$$G_\pm(x) = -i\ \text{p.v.}\int_{-a}^{a}\frac{\rho(y)\,dy}{y-x} \pm \pi\rho(x) = i V'(x) \pm \pi\rho(x)$$

$-a$ a $2\pi\rho(x)$

$G(z)$ is analytic in $\mathbb{C}\setminus[-a,a]$

scalar Riemann-Hilbert problem

$$G_+(x) + G_-(x) = 2iV'(x) \qquad x\in[-a,a]$$
$$G_+(x) - G_-(x) = 0 \qquad x\notin[-a,a] \quad x\in\mathbb{R}$$
$$G(z)\to 0 \qquad z\to\infty$$

$$G_+(x) - G_-(x) = \dots$$

$$\tilde G(z) = \frac{G(z)}{\sqrt{z^2-a^2}} \qquad \left(\sqrt{x^2-a^2}\right)_+ = -\left(\sqrt{x^2-a^2}\right)_-$$

$-a$ $+$ $-$ a

$$\tilde G_+(x) - \tilde G_-(x) = \frac{G_+(z)}{\left(\sqrt{x^2-a^2}\right)_+} - \frac{G_-(z)}{\left(\sqrt{x^2-a^2}\right)_-} = \frac{G_+(z)+G_-(z)}{\left(\sqrt{x^2-a^2}\right)_+} =$$

$$\tilde G_+(x) - \tilde G_-(x) = \frac{2iV'(x)}{\left(\sqrt{x^2-a^2}\right)_+} \qquad x\in[-a,a]$$
$$\tilde G_+(x) - \tilde G_-(x) = 0 \qquad x\notin[-a,a]$$
$$\tilde G(z)\to 0 \qquad z\to\infty$$

Plemelj formula:

$$\tilde G(z) = \frac{1}{2\pi i}\int_{-a}^{a}\frac{2iV'(s)\,ds}{\left(\sqrt{s^2-a^2}\right)_+(s-z)}$$

$$G(z) = \frac{\sqrt{z^2-a^2}}{\pi}\int_{-a}^{a}\frac{V'(s)\,ds}{\left(\sqrt{s^2-a^2}\right)_+(s-z)}$$

$$G(z) = \frac{z+\dots}{\pi}\left(-\frac{1}{z}\right)\int_{-a}^{a}\frac{V'(s)}{\left(\sqrt{s^2-a^2}\right)_+}\left(1+\frac{s}{z}+O\left(\frac{1}{z^2}\right)\right)ds$$

$$G(z)\to 0 \qquad z\to\infty \qquad \int_{-a}^{a}\frac{V'(s)}{\left(\sqrt{s^2-a^2}\right)_+}ds = 0$$

$V(s)$ even
$V'(s)$ odd

$$G(z) = -i\int_{-a}^{a}\frac{\rho(y)\,dy}{y-z} \approx \frac{i}{z}\underbrace{\int_{-a}^{a}\rho(y)\,dy}_{1} + O\left(\frac{1}{z^2}\right)$$

$$-\frac{1}{\pi}\int_{-a}^{a}\frac{V'(s)\,s\,ds}{\left(\sqrt{s^2-a^2}\right)_+} = i$$

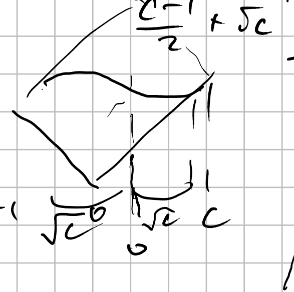
$$\frac{1}{2}\int_{-a}^{a}\frac{s}{\sqrt{s^2-a^2}}\;\frac{(-1)}{\pi}\ln\left|\frac{s+\frac{c+1}{2}}{s-\frac{c+1}{2}}\right|ds=i$$

$$\int\frac{s}{\sqrt{s^2-a^2}}\ln\left|\frac{s+\frac{c+1}{2}}{s-\frac{c+1}{2}}\right|ds=\frac{1}{2}\Big(\big(2\sqrt{s^2-a^2}-\sqrt{(c+1)^2-4a^2}\big)\cdot$$
$$\cdot\ln(c+1-2s)+\big(\sqrt{(c+1)^2-4a^2}-2\sqrt{s^2-a^2}\big)\ln(c+1+2s)$$
$$-\sqrt{(c+1)^2-4a^2}\,\ln\big(\sqrt{(c+1)^2-4a^2}\sqrt{s^2-a^2}-2a^2-(c+1)s\big)$$
$$+\sqrt{(c+1)^2-4a^2}\,\ln\big(\sqrt{(c+1)^2-4a^2}\sqrt{s^2-a^2}-2a^2+(c+1)s\big)$$
$$-2(c+1)\ln\big(\sqrt{s^2-a^2}+s\big)\Big)+\text{const}$$

$$\frac{c+1}{2}\left(1-\sqrt{1-\left(\frac{2a}{c+1}\right)^2}\right)=1$$

$c\geq 1 \qquad a=\sqrt{c}$

GL: $\operatorname{supp}\rho=\left[\frac{c-1}{2}-\sqrt{c};\ \frac{c-1}{2}+\sqrt{c}\right]$



$$\rho(x)=\frac{1}{\pi}\operatorname{Re}\left[G_+(x)\right]=$$
$$=\frac{1}{\pi^2}\operatorname{Re}\left[\sqrt{x^2-c}\int_{-\sqrt{c}}^{\sqrt{c}}\frac{\frac{1}{2}\left[\ln\left(\frac{c+1}{2}+s\right)-\ln\left(\frac{c+1}{2}-s\right)\right]ds}{\left(\sqrt{s^2-c}\right)_+(s-x)}\right]-1$$

$$\frac{1}{\pi}\int_{-\sqrt{c}}^{\sqrt{c}}\underbrace{\frac{1}{\sqrt{c-s^2}\,(s-x)}}_{=f(s)}\;\underbrace{\frac{1}{\pi}\ln\left|\frac{s-\frac{c+1}{2}}{s+\frac{c+1}{2}}\right|}_{\text{Hilbert transform of } 1\!\!1_{[-\frac{c+1}{2},\frac{c+1}{2}]}}ds$$

$$\int_{-\frac{c+1}{2}}^{\frac{c+1}{2}}\frac{dy}{s-y}$$

$$\rho(x)=\frac{1}{\pi}\arccos\frac{c-1}{\sqrt{(c-x)(x+1)}}$$

f, g

$f\in L^p(\mathbb{R})$

$g\in L^q(\mathbb{R})$ $\qquad \frac{1}{p}+\frac{1}{q}=1 \qquad \tilde{f}(s)$ - Hilbert transform of f

$$\int_{-\infty}^{\infty}f(s)\,\tilde{g}(s)\,ds=-\int_{-\infty}^{\infty}\tilde{f}(s)\,g(s)\,ds$$

$$g=1\!\!1_{[-\frac{c+1}{2},\frac{c+1}{2}]}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}f(s)\ln\left|\frac{s-\frac{c+1}{2}}{s+\frac{c+1}{2}}\right|ds=-\int_{-\frac{c+1}{2}}^{\frac{c+1}{2}}\tilde{f}(s)\,ds$$

$$f(y)=\begin{cases}\frac{1}{\pi}\frac{1}{\sqrt{c-y^2}\,(y-x)} & y\in[-\sqrt{c},\sqrt{c}]\\ 0 & \text{otherwise}\end{cases}$$

$$y = \sqrt{c}\,\frac{c-t^2}{c+\frac{t^2}{\sqrt{c}}} \qquad \frac{dy}{\sqrt{c-y^2}} = \frac{-2\sqrt{c}\,dt}{c+t^2}$$

$$\tilde{f}(z) = \frac{1}{\pi^2}\int \frac{ds}{\sqrt{c-s^2}(s-x)(s-z)} = \frac{1}{\pi}\frac{1}{x-z}\left(\frac{1}{\sqrt{z^2-c}} - \frac{1}{\sqrt{x^2-c}}\right)$$

$$\rho(x) = \frac{1}{\pi}\,\mathrm{Re}\left[\sqrt{c-x^2}\int_{-\frac{c+1}{2}}^{\frac{c+1}{2}} \frac{1}{2}\left(\frac{1}{(x-z)\sqrt{z^2-c}} - \frac{1}{(x-z)\sqrt{x^2-c}}\right)dz\right]$$

$$\rho(x) = -\frac{1}{2\pi}\Big[\mathrm{Im}\Big[\ln\left(\sqrt{(c-1)^2}\sqrt{x^2-c} - (c+1)x + 2c\right) + $$
$$+ \ln\left(\sqrt{(c-1)^2}\sqrt{x^2-c} + (c+1)x + 2c\right)\Big] - \pi\Big]$$

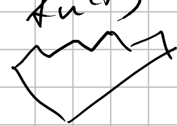
$$c \geq 1 \qquad |x| < \sqrt{c}$$

$$\rho(x) = \frac{1}{2\pi}\left[\arctan\left(\frac{-(c+1)x+2c}{(c-1)\sqrt{c-x^2}}\right) + \arctan\left(\frac{(c+1)x+2c}{(c-1)\sqrt{c-x^2}}\right)\right]$$

$$\Rightarrow \rho(x) = \frac{1}{\pi}\arccos\frac{c-1}{2\sqrt{\left(\frac{c+1}{2}\right)^2 - x^2}}$$

$$\Omega(u) = 1 + \int_0^u (1-2\rho(x))\,dx$$

$f_n(x)$



For all $\varepsilon > 0$

$$\mathbb{P}\left(\sup_{x\in\mathbb{R}} |f_n(x) - \Omega(x)| > \varepsilon\right) \xrightarrow[\substack{n\to\infty \\ k\to\infty}]{} 0 \qquad c = \frac{k}{n}$$

$f_n \to \Omega$ uniformly in the supremum norm

$f_n'(x) = 1 - 2\rho_n(x)$ $\qquad \|\cdot\|_\infty = \sup|\ \ |$

Dan Romik
Surprising mathematics of longest increasing subsequences

$J[\rho_n]$

$$J[f_n] = Q[f_n] + C$$

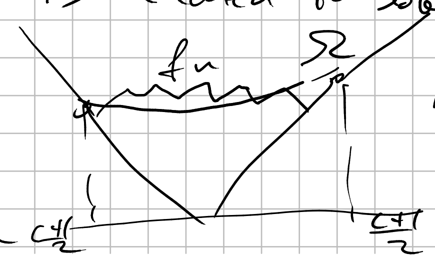
$$Q[f_n] = \frac{1}{2}\iint_{-\frac{c+1}{2}}^{\frac{c+1}{2}} f_n'(x) f_n'(y) \ln|x-y|^{-1}\,dx\,dy$$

is positive definite proof in Romik

on compactly supported Lipschitz functions

/it is related to Sobolev norm/ $H^{\frac{1}{2}}$

$$\|f\|_Q = \sqrt{Q[f]}$$



$$\left.\begin{aligned}\Omega(x) &= |x| \\ f_n(x) &= |x|\end{aligned}\right\} \quad |x| > \frac{c+1}{2}$$

$$d_Q(f_1, f_2) = \|f_1 - f_2\|_Q$$

$$\|f\|_\infty = \sup_x |f(x)| \le C\,(Q[f])^{1/4}$$

$$\mu_{n,k}(f_n) = \exp\Big(-n^2\big(\underbrace{Q[f_n] + C}_{d_Q(f_n,\Omega)}\big)\Big)$$

$$Q[\Omega] + C = 0$$

$$d_Q(f_n, \Omega) = \varepsilon$$

$$\mu_{n,k}(f_n) \le C e^{-n^2\varepsilon^2 + O(n\ln n)}$$

Hardi-Ramanujan formula N - number of boxes in Young diagram

$$p(N) \sim \frac{1}{4N\sqrt{3}} e^{\frac{2\pi}{\sqrt{6}}\sqrt{N}}$$

cn^2 total number of diagrams $\sim C e^{cn}$

$$\mathbb{P}\left(\sup_x |f_n(x) - \Omega(x)| > \varepsilon\right) \to 0$$

$$\sim e^{-n^2\varepsilon^2} \cdot e^{cn} \cdot e^{n\ln n}$$

Recall that in $GL_n \times GL_k$

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda)\dim V_{GL_k}(\lambda')}{2^{nk}}$$

$$\dim V_{GL_k}(\lambda') = \prod_{i<j}(a_i - a_j)\prod_\ell \binom{n+k}{a_\ell}$$

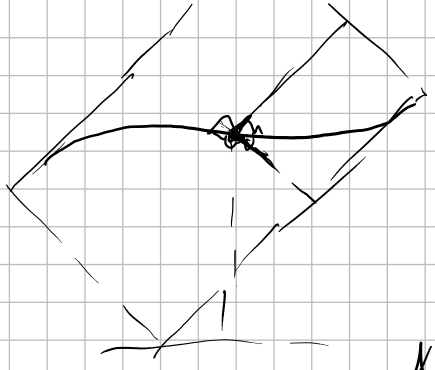
$$a_i = \lambda_i + n - i \qquad \frac{n+k}{n} = c+1$$

$$\dim V_{GL_n}(\lambda) \sim \prod_{i<j}\frac{a_i - a_j}{(i-j)}$$

$$\mu_{n,k}(\lambda) = \prod_{i<j}(a_i - a_j)^2 \prod_\ell e^{-nV\left(\frac{a_\ell}{n}\right)}$$

$$V(u) = u\ln u + ((c+1) - u)\ln((c+1) - u)$$

$$x = u - \frac{c+1}{2}$$



We have demonstrated that Sp is "half" of GL

Conjecture: Sp fluctuations are different

Next time: Orthogonal polynomials

asymptotics of orthogonal polynomials is connected to 2d Riemann-Hilbert problem