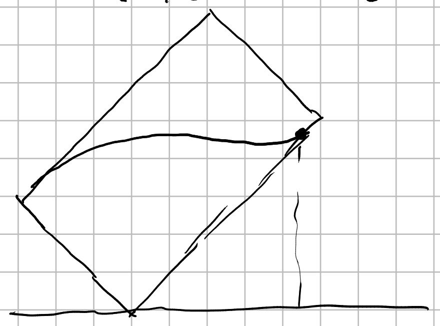
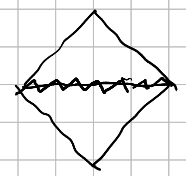
Lecture 13 2023-07-05



$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda)\,\dim V_{GL_k}(\lambda')}{2^{nk}}$$

$$c = \lim_{n,k\to\infty} \frac{k}{n}$$

$c = 1$



$\lambda_1 = n + \dots$

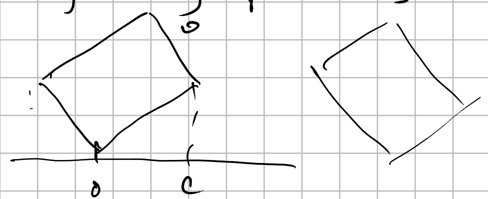
Some distribution that does not depend on n.

$$\mu_{n,k}(\lambda \mid \alpha) = \frac{\alpha^{|\lambda|}\dim V_{GL_n}(\lambda)\dim V_{GL_k}(\lambda')}{(1+\alpha)^{nk}}$$

Universality $\alpha = c \to$ the distribution in the corner

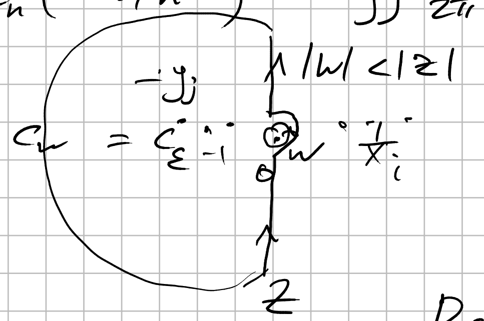
$$\mu_{n,k}(\lambda \mid f,g) = \frac{S_\lambda(x_1 \dots x_n)\, S_{\lambda'}(y_1 \dots y_k)}{\prod_{i=1}^{n}\prod_{j=1}^{k}(1+x_i y_j)} \qquad x_i = f\left(\tfrac{i}{n}\right) \quad y_j = g\left(\tfrac{j}{k}\right)$$

If $\int_0^1 f(s)\,ds = c\int_0^1 \frac{ds}{g(s)}$ then the diagram hits the corner and we have the same discrete distribution of the first row



$K(m, m')$ $\quad m = cn - h$, $\quad m' = cn - h'$, $\quad h, h' \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$

$$K_n(h,h') = \iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w}\, e^{n[S(z)-S(w)]}\, z^{h} w^{-h'} \frac{\sqrt{zw}}{z-w}$$



C_z encircles all $\{-y_j\}_{j=1}^{k}$ and does not contain any of $\{\frac{1}{x_i}\}_{i=1}^{n}$

$\operatorname{Re} z = 0$

$u_+ = c \iff z = 0$ critical pt. of S $\partial_z S(z)\big|_{z=0} = 0$

$$S(z) = -\ln(1-z) - c\ln(1+z) + (c - u_+)\ln z$$

$\operatorname{Re} S(z) \leq 0$

$S(0) = 0$

$$S(z,\alpha) = -\ln(1-\alpha z) - c\ln(1+z) \qquad \alpha = c$$

$$S'(0) = 0 = \frac{1}{1-z} - \frac{1}{1+z} = 0$$

$$S''(0) = \left(\frac{1}{(1-z)^2} + \frac{1}{(1+z)^2}\right)\Bigg|_{z=0} = 2 \qquad \frac{c}{1-cz} - \frac{c}{1+z} = 0$$

$$-nS(w) \approx -nS(0) - \frac{n}{2}S''(0)\,w^2$$

$$\oint_{C_\varepsilon} e^{-\frac{n}{2}S''(0)w^2}\, w^{-h'} \frac{\sqrt{zw}}{z-w}\, \frac{dw}{2\pi i w} =$$

$|w| < |z|$

$$= \oint_{C_\varepsilon} \left(\sum_{l=0}^{\infty} \frac{(-1)^l n^l (S''(0))^l}{2^l \cdot l!} w^{2l}\right) w^{-h'+\frac{1}{2}} \left(\sum_{j=0}^{\infty} \frac{w^j}{z^{j+\frac{1}{2}}}\right) \frac{dw}{2\pi i w}$$

$$= \sum_{l=0}^{N(h')} \frac{(-1)^l n^l (S''(0))^l}{2^l \cdot l!} z^{-h'+2l} \qquad 2l + j - h' + \tfrac{1}{2} \overset{!}{=} 0$$

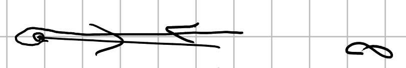
$$N(h') = \left\lfloor \frac{h' - \frac{1}{2}}{2} \right\rfloor$$

$$\int_{-i\infty}^{i\infty} e^{nS(z)} z^h \left(\sum_{l=0}^{N(h')} \frac{(-1)^l n^l (S''(0))^l}{2^l \cdot l!} z^{-h'+2l}\right) \frac{dz}{2\pi i z} =$$

$\mathrm{Re}\, S(z) < 0 \qquad nS(z) \approx nS(0) + \frac{n}{2} S''(0)\, z^2$

$$= \int_{-i\infty}^{i\infty} \frac{dz}{2\pi i z}\, e^{\frac{n}{2}S''(0)z^2} z^h \left(\sum_{l=0}^{\lfloor h'/2 - \frac{1}{4}\rfloor} \frac{(-1)^l n^l (S''(0))^l}{2^l \cdot l!} z^{-h'+2l}\right)$$

$$z = \frac{-i\sqrt{2t}}{\sqrt{nS''(0)}} \qquad t$$



$$K_n(h, h') = \frac{1}{2\pi} \sum_{l=0}^{N(h')} \int_{\infty}^{0^+} dt\, \frac{(-1)^l n^l (S''(0))^l}{2^l \cdot l!} (-1)^{\frac{h-h'+1}{2}+l} \cdot 2^{\frac{h-h'}{2}+l-1} (nS''(0))^{-\frac{h-h'}{2}-l}\, t^{\frac{h-h'}{2}+l-1} \cdot e^{-t}$$

Hankel's representation for the Γ-function:

$$\Gamma(z) = -\frac{1}{2i\sin \pi z} \int_{\infty}^{0^+} dt\, (-t)^{z-1} e^{-t}$$

if $\frac{h-h'}{2} + l \geq 0$ then

else by residue we obtain $\dfrac{(-1)^{\frac{h-h'}{2}+l} \cdot 2\pi i}{\left(\frac{h'-h}{2} - l\right)!}$

$$\mathbb{P}\left(\lambda_1 - nc \le -\Delta\right) = \det\left(\delta_{ij} - K\left(i+\tfrac{1}{2}, j+\tfrac{1}{2}\right)\right)_{ij=0}^{\Delta-1}$$

a^{i-j}

$$\lim_{n\to\infty} \mathbb{P}\left(\lambda_1 - nc \le -\Delta\right) = \det\left[\delta_{ij} - K_{crit}(i,j)\right]_{ij=0}^{\Delta-1}$$

$$K_{crit}(i,j) = \sum_{\ell=0}^{\Delta-j-1} \begin{cases} \frac{1}{2\pi}\frac{1}{\ell!}\sin\frac{\pi(j-i)}{2}\,\Gamma\left(\ell+\frac{j-i}{2}\right) & \text{if } \ell+\frac{j-i}{2} > 0 \\ \frac{1}{2}\frac{(-1)^{\ell}}{\ell!\left(\frac{i-j}{2}-\ell\right)!} & \ell+\frac{j-i}{2} \le 0 \end{cases}$$

$$S(z,f,g) = -\int_0^1 \ln(1-f(s)z)\,ds - c\int_0^1 \ln(g(s)+z)\,ds + (c-u)\ln z \qquad u \in [u_-, u_+]$$

$z\partial_z S(z) = 0$ double root at $z=0$ $\qquad z_\pm : z\partial_z S(z)\big|_{z=z_\pm} = 0$

$u = u_+ \quad z_c = z_+ = z_-$

$z_c = 0$

For Tracy-Widom we had $S'(z_c) = 0$, $S''(z_c) = 0$

$S''(z_c) \neq 0$

$z_c = 0 \quad (z\partial_z)^2 S(z)\big|_{z=z_c} = 0 \not\Rightarrow S''(z_c) = 0$ for $z_c = 0$

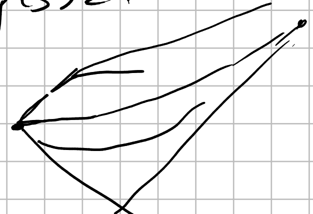
$$\int_0^1 \frac{f(s)}{1-f(s)z}\,ds - c\int_0^1 \frac{ds}{g(s)+z} = 0$$

$$z = 0 \qquad \int_0^1 f(s)\,ds = c\int_0^1 \frac{ds}{g(s)}$$

$$f(s) = e^s \qquad \int_0^1 f(s)\,ds = e-1 = c\int_0^1 \frac{ds}{g(s)}$$

N., Scrimshaw, Betea 2022

$c = e-1 \qquad g(s) = 1$



Schur-Weyl duality

$GL_n \times S_k \qquad c = \frac{\sqrt{k}}{n} \qquad c = 1$

Fluctuations of the first column are in the same universality class

Ex. Do this simulation.

Borodin - Olshanski

$$k = cn^2 + s\, n^{3/2}$$

Discrete Hermite kernel

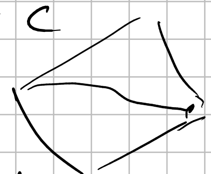
$$H_\ell(x) = \frac{\ell!}{2\pi i}\oint_0 \frac{e^{2tx-t^2}}{t^{\ell+1}} \quad \text{Physics}$$

$$K^H_s(\ell,\ell') = \frac{1}{\sqrt{2\pi \ell! \ell'!}} e^{-\frac{s^2}{2}} \left(\frac{He_{\ell+1}(s) He_{\ell'}(s) - He_\ell(s) He_{\ell'+1}(s)}{\ell-\ell'}\right)$$

$$He_\ell(x) = \frac{\ell!}{2\pi i}\oint_0 \frac{e^{tx-t^2/2}}{t^{\ell+1}} \quad \text{Probability}$$

$$He_\ell(x) = 2^{-\frac{\ell}{2}} H_\ell\left(\frac{x}{\sqrt{2}}\right)$$

$GL_n \times GL_k$ $\qquad k = cn + \frac{s}{\sigma}\sqrt{n} \qquad u_+ \approx c$



$$m = n u_+ + \frac{\sqrt{n}}{\sigma} s + \ell \qquad s=0$$

$$m' = n u_+ + \frac{\sqrt{n}}{\sigma} s + \ell'$$

$\searrow$ previous discrete distribution (of Gravner-Tracy-Widom)

$$u_+ = c$$

$$K(m,m') = \iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w} \frac{z^{-m} w^{m'} \sqrt{zw}}{z-w} \frac{K(z)}{K(w)} =$$

$$K(z) = \frac{1}{(1-z)^n} \frac{1}{\left(1+\frac{1}{z}\right)^k} \qquad k = cn + \frac{s}{\sigma}\sqrt{n}$$

$$= \iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w} e^{n\left[-\ln(1-z) - c\ln(1+z) + \cancel{c\ln z} - \cancel{u_+ \ln z} - \frac{s}{\sigma}\frac{1}{\sqrt{n}} \cdot \ln(1+z) + \cancel{\frac{s}{\sigma}\frac{1}{\sqrt{n}}\ln z} - \cancel{\frac{s}{\sigma}\frac{1}{\sqrt{n}}\ln z} + \ln(1-w) + c\ln(1+w) + \frac{s}{\sigma}\frac{1}{\sqrt{n}}\ln(1+w)\right]} \frac{z^{-\ell+\frac{1}{2}} w^{\ell'+\frac{1}{2}}}{z-w} =$$

$$z = z_c + \frac{\zeta}{\sqrt{n}}\sigma \qquad z_c = 0 \qquad dz = \frac{\sigma}{\sqrt{n}} d\zeta$$

$$w = z_c + \frac{\upsilon}{\sqrt{n}}\sigma$$

$$= \iint \frac{d\zeta\, d\upsilon}{(2\pi i)^2} \frac{\zeta^{-\ell-\frac{1}{2}} \upsilon^{\ell'-\frac{1}{2}}}{\zeta-\upsilon} \left(\frac{\sigma}{\sqrt{n}}\right)^{\ell'-\ell+1} e^{\frac{S''(z_c)\sigma^2}{2}(\zeta^2-\upsilon^2) - \zeta s + \upsilon s}$$

$$S(z_c) = 0$$
$$S'(z_c) = 0$$

$$z = \frac{\zeta}{\sqrt{n}}\sigma \qquad \ln(1+z) \approx \frac{\zeta}{\sqrt{n}}\sigma$$