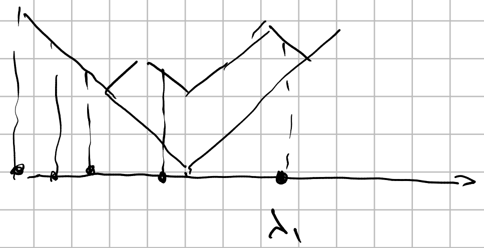
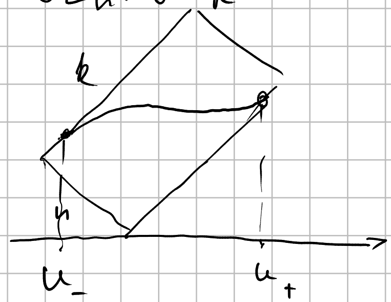
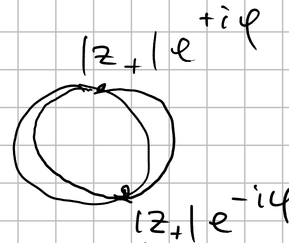
Lecture 12. 2023.06.30 Fluctuations of the first row of λ



$GL_n \times GL_k$ skew Howe duality



$\lambda_1 \approx n u_+ + ?\ldots$



$\rho(u) = K(nu, nu)$ $\cancel{nu, nu}$

$$\lim_{n\to\infty} K_n(nu+m, nu+m') = \iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w} z^{-m} w^{m'} \cdot e^{n(S(z,u)-S(w,u))} \frac{\sqrt{zw}}{z-w} \sim \frac{1}{2\pi i}\int_{|z_+|e^{-i\varphi}}^{|z_+|e^{i\varphi}} \frac{dw}{w^{m-m'+1}} =$$

$$= \frac{\sin(m-m')\varphi}{\pi(m-m')}$$

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda)\dim V_{GL_k}(\lambda')}{2^{nk}}$$

$\lambda_1 \approx n u_+ + n^{\frac{1}{3}}\xi$ $\quad u_+ = \frac{c-1}{2} + \sqrt{c}$ $\quad c = 2$

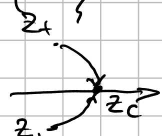
$n = 100$

$m = n u_+ + \frac{\xi}{6} n^{1/3}$

$m' = n u_+ + \eta/6\, n^{1/3}.$ $\quad \lambda_1 = 100 \cdot 1.91 = 192$

$$K_n(m, m') = \iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w} \frac{z^{-\frac{\xi}{6}n^{1/3}} w^{\frac{\eta}{6}n^{1/3}}}{z-w} e^{n(S(z,u_+)-S(w,u_+))}$$

$$= \iint \frac{dz}{2\pi i z}\frac{dw}{2\pi i w} e^{n(S(z,u_+)-S(w,u_+))} e^{\frac{n^{1/3}}{6}(-\xi \ln z + \eta \ln w)} \frac{1}{z-w}$$



$z\partial_z S(z,u) = 0$ $\quad z_\pm$ $\quad z_+, z_- \to z_c \in \mathbb{R}$

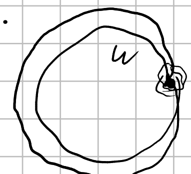
$$S(z_c) = \frac{(\sqrt{c}+1)^2}{2}\ln\frac{\sqrt{c}-1}{\sqrt{c}+1} - c\ln\frac{2\sqrt{c}}{\sqrt{c}+1} - \ln\frac{2}{\sqrt{c}+1}$$

$$S(z,u) = -\ln(1-z) - c\ln(1+z) + (c-u)\ln z$$

$$z_\pm = \frac{c-1 \pm \sqrt{(c-1)^2 - 4(u+1)(c-u)}}{2(u+1)} \qquad u_\pm = \frac{c-1}{2} \pm \sqrt{c}$$

$$z_c = \frac{c-1}{c+2\sqrt{c}+1} = \frac{\sqrt{c}-1}{\sqrt{c}+1}$$

$$z\partial_z S(z,u)\Big|_{\substack{z=z_c\\ u=u_+}} = 0 \qquad \mathrm{Re}\,(S(z)-S(w))<0$$



$$(z\partial_z)^2 S(z,u)\Big|_{\substack{z=z_c\\ u=u_+}} = 0$$

$$S(z) = S(z_c) + \frac{1}{6} S'''(z_c)(z-z_c)^3 + O((z-z_c)^4)$$

$$z - z_c \approx z_c \frac{b\zeta}{n^{1/3}} \qquad z = z_c e^{b\zeta/n^{1/3}} \qquad \zeta \in i\mathbb{R}$$

$$w - z_c \approx z_c \frac{bv}{n^{1/3}} \qquad w = z_c e^{bv/n^{1/3}}$$

$$\frac{n^{1/3}}{b}\left(-\xi \ln z + \eta \ln w\right) = \underline{\frac{n^{1/3}}{b}(\ln z_c)(\eta-\xi)} + \underline{\eta v - \xi\zeta}$$

$$n(S(z)-S(w)) \approx \frac{1}{6} S'''(z_c)\, z_c^3\, b^3 (\zeta^3 - v^3) =$$

$$= \frac{1}{6} b^3 \left((z\partial_z)^3 S(z)\right)\Big|_{z=z_c} (\zeta^3 - v^3) =$$

$$\boxed{b = \frac{2^{1/3}}{(S'''(z_c))^{1/3} z_c}} \qquad = \underline{\frac{1}{3}(\zeta^3 - v^3)}$$

$$z - w = z_c\left(e^{\frac{b\zeta}{n^{1/3}}} - e^{\frac{bv}{n^{1/3}}}\right) \approx z_c\left(\frac{b}{n^{1/3}}(\zeta - v)\right)$$

$$\frac{1}{z-w} \approx \frac{n^{1/3}}{z_c b}\,\frac{1}{\zeta - v} \qquad \frac{dz}{z} = \frac{b}{n^{1/3}} d\zeta \qquad \frac{dw}{w} = \frac{b}{n^{1/3}} dv$$

$$K(\xi,\eta) = \frac{1}{(2\pi i)^2}\int_{i\mathbb{R}+\varepsilon} d\zeta \int_{i\mathbb{R}-\varepsilon} dv\, \frac{\exp\left(\frac{1}{3}(\zeta^3 - v^3) + \eta v - \xi\zeta\right) + \frac{n^{1/3}}{b}\ln z_c(\eta-\xi)}{\zeta - v} \cdot \frac{b}{n^{1/3} z_c}$$

$$K_{Airy}(\xi,\eta) = \frac{1}{(2\pi i)^2}\int_{i\mathbb{R}-\varepsilon} dv \int_{i\mathbb{R}+\varepsilon} d\zeta\, \frac{e^{\frac{\zeta^3 - v^3}{3} + \eta v - \xi\zeta}}{\zeta - v}$$

$$\lim_{n\to\infty} K_n\left(nu_+ + \frac{\xi}{b} n^{1/3}, nu_+ + \frac{\eta}{b} n^{1/3}\right) \cdot \frac{n^{1/3} z_c}{b}\, e^{-\frac{n^{1/3}}{b}(\eta-\xi)\ln z_c} = K_{Airy}(\xi,\eta)$$

Tracy-Widom distribution 1992 TW_2 GUE $n\times n$

$$F_2(x) = \lim_{N\to\infty} \mathbb{P}\left((\lambda_{max} - 2\sqrt{N})\, N^{1/6} < x\right)$$

$$F_2(s) = \det(I - A_s) = 1 + \sum_{e=1}^{\infty} \frac{(-1)^e}{e!} \int_{(s,\infty)^n} \det\left[A_s(x_i, x_j)\right]_{i,j=1}^{n} dx_1 \dots dx_n$$

$$A_s(x,y) = \begin{cases} \dfrac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y} & x \neq y \\ (Ai'(x))^2 - x(Ai(x))^2 & x = y \end{cases}$$

$Ai(x)$ is the solution of ODE

$$y'' - xy = 0 \qquad y \to 0 \text{ as } x \to \infty$$

$$Ai(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)dt = \frac{1}{\pi}\lim_{b\to\infty}\int_0^b \cos\left(\frac{t^3}{3} + xt\right)dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{s^3}{3} + ixs}\, ds$$

Painlevé II $\quad q''(s) = sq(s) + 2q(s)^3$

$$q(s) \sim Ai(s) \quad s \to \infty$$

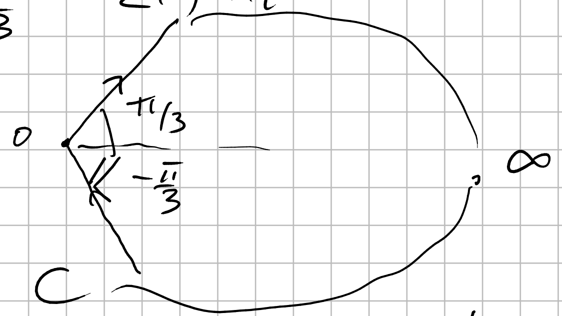
$$F_2(s) = \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right)$$

plancherel measure

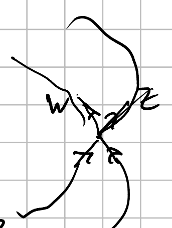
~ 1999 $\quad$ Baik-Deif-Johansson $\quad \lambda_1 \simeq 2\sqrt{N} + \dagger W_2$

$$Ai(0) = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} \qquad Ai'(0) = \frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)}$$

$$Ai(z) = \frac{1}{2\pi i}\int_C \exp\left(\frac{t^3}{3} - zt\right)dt$$



Okounkov
math/0309074
$\zeta, v \to z, w$
$\xi, \eta \to x, y$



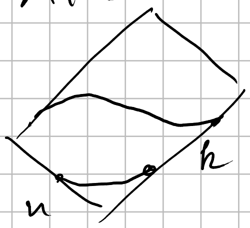
$$\frac{1}{(2\pi)^2}\iint \frac{e^{-i\frac{z^3}{3} - izx + \frac{iw^3}{3} + iwy}}{i(z-w)}\, dz\,dw = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}$$

$$\int e^{-izx + iwy} = \frac{i}{x-y}\left(\partial_z + \partial_w\right)e^{-izx + iwy}$$

$$\frac{i}{(2\pi)^2(x-y)}\iint (z+w)\exp\left(-i\frac{z^3}{3} + i\frac{w^3}{3} - izx + iwy\right)dz\,dw$$

$$Ai'(x) = \frac{1}{2\pi} \int w\, e^{iw^3/3 + iwy}\, dw$$

$$Ai(x) = \frac{1}{2\pi} \int e^{-iz^3/3 - izx}\, dz$$



$$x_i = f\left(\tfrac{i}{n}\right) \qquad y_j = g\left(\tfrac{j}{n}\right)$$

$$(z\partial_z)^3 S(z) = \int_0^1 \left(\frac{2f^2(s) z_c^2}{(1-f(s)z_c)^3} + \frac{2c g(s) z_c^2}{(z_c + g(s))^3} \right) ds$$

$$z_c : \; z\partial_z S(z)\Big|_{z=z_c} = 0$$

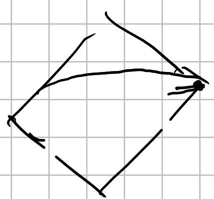
$$(z\partial_z)^2 S(z)\Big|_{z=z_c} = 0$$

$$b = \frac{2^{1/3}}{(S'''(z_c))^{1/3}\, z_c}$$

$$K\left(nu_+ + \frac{\xi}{b} n^{1/3},\; nu_+ + \frac{\eta}{b} n^{1/3}\right) \sim K_{Airy}(\xi, \eta)$$

$z_c = 0$ $\quad u_+ = c$ is root of

$z\partial_z S(z) = 0$, for $z = 0$,



$$(z\partial_z)^2 S(z) = 0 \Rightarrow \int_0^1 f(s)\,ds = c \int_0^1 \frac{ds}{g(s)}$$

$\rho(u_+) = \frac{1}{2}$

$\Omega'(u_+) = 0$ "Critical" in Biane $(c=1)$

Gravner-Tracy-Widom

Borodin-Olshanski 2007

discrete Hermite kernel