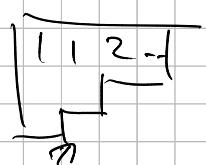
Lecture 9. 21.06.2023

$\mathbb{C}^n \circlearrowleft GL_n$

$$\underbrace{\mathbb{C}^n \otimes \mathbb{C}^n \otimes \dots \otimes \mathbb{C}^n}_{k} \;(\dim = n^k) = \bigoplus_{\lambda \vdash k} V_{GL_n}(\lambda) \otimes V_{S_k}(\lambda)$$

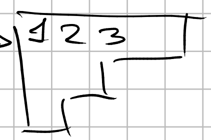
$\circlearrowleft S_k$ Schur-Weyl duality

$\boxed{i_1} \otimes \boxed{i_2} \dots \longleftrightarrow$



$1 \quad 2 \quad 3$

$SSYT(\lambda|n)$



$\in SYT(\lambda|k)$

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \dim \lambda}{n^k}$$

$n, k \to \infty$

$\sqrt{k} \sim n \qquad \lim \frac{\sqrt{k}}{n} = c$

Poissonized measure $\mu(\lambda|\xi) = \frac{\xi^{|\lambda|} e^{-\xi}}{|\lambda|!} \frac{\dim \lambda \dim V_{GL_n}(\lambda)}{n^{|\lambda|}}$

$s_\lambda(y_1 \dots y_k) \quad y_1 \dots y_k = \frac{\sqrt{\xi}}{k} \qquad k \to \infty$

$$\lim_{k\to\infty} s_\lambda\left(\frac{\sqrt{\xi}}{k}, \dots \frac{\sqrt{\xi}}{k}\right) = (\sqrt{\xi})^{|\lambda|} \frac{\dim \lambda}{|\lambda|!}$$

$s_\lambda(x_1 \dots x_n) \qquad x_i = \frac{\sqrt{\xi}}{n}$

$$s_\lambda\left(\frac{\sqrt{\xi}}{n}, \dots \frac{\sqrt{\xi}}{n}\right) = \frac{(\sqrt{\xi})^{|\lambda|}}{n^{|\lambda|}} \dim V_{GL_n}(\lambda)$$

$$\prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j-i}$$

$$\mu(\lambda | x, y) = \prod_{i=1}^{n}\prod_{j=1}^{k} (1 - x_i y_j) \cdot s_\lambda(x_1 \dots) s_\lambda(y_1 \dots)$$

$\downarrow$

$$\lim_{k\to\infty} \left(1 - \frac{\xi}{kn}\right)^{kn} = e^{-\xi}$$

$$\lim_{k\to\infty} \mu\left(\lambda \,\Big|\, \left\{\frac{\sqrt{\xi}}{n}\right\}_{i=1}^{n}, \left\{\frac{\sqrt{\xi}}{k}\right\}_{j=1}^{k}\right) = \frac{e^{-\xi}}{|\lambda|!} \frac{\xi^{|\lambda|}}{n^{|\lambda|}} \dim \lambda \dim V_{GL_n}(\lambda)$$

$$\frac{\langle 0|\Gamma_+(t)|\lambda\rangle\langle\lambda|\Gamma_-(t')|0\rangle}{\langle 0|\Gamma_+(t)\Gamma_-(t')|0\rangle}$$

$$K(i,j) = \oint\!\!\oint_{|w|<|z|} \frac{dz}{2\pi i z}\frac{dw}{2\pi i w}\frac{w^j}{z^i}\frac{F(z)}{F(w)}\frac{\sqrt{zw}}{z-w}$$

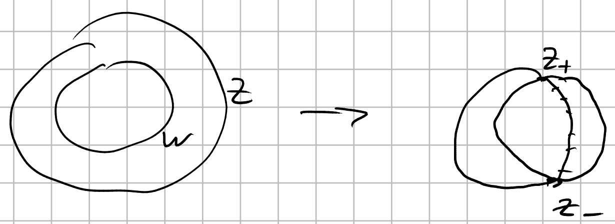
$$F(z) = \frac{\prod_{j=1}^{k}(1-y_j/z)}{\prod_{i=1}^{n}(1-x_i z)} = \frac{\left(1-\frac{\sqrt{\xi}}{kz}\right)^k}{\left(1-\frac{\sqrt{\xi}\,z}{n}\right)^n} \xrightarrow[k\to\infty]{} \frac{e^{-\sqrt{\xi}/z}}{\left(1-\frac{\sqrt{\xi}}{n}z\right)^n} = F(z)$$

$$\sqrt{\xi} = cn \qquad \ln F(z) = n\left(-\frac{c}{z} - \ln(1-cz)\right)$$

$$i = nx \qquad S(z,x) = \frac{1}{n}\ln F(z) - x\ln z$$

$$j = ny$$

$$K(x,y) = \iint_{|w|<|z|} \frac{dz}{2\pi i z}\frac{dw}{2\pi i w}\, e^{n(S(z,x)-S(w,y))}\,\frac{\sqrt{zw}}{z-w}$$



$$S(z,x) = -\frac{c}{z} - \ln(1-cz) - x\ln z$$

$$z\partial_z S(z,x) = \frac{c}{z} + \frac{cz}{1-cz} - x = 0$$

$$\to (x+1)z^2 - \left(c+\frac{x}{c}\right)z + 1 = 0$$

$$z_\pm = \frac{c + x/c \pm \sqrt{(c+x/c)^2 - 4x - 4}}{2(x+1)}$$

$$x \in [x_-, x_+] \qquad z_- = \bar{z}_+$$

$$\left(c+\frac{x}{c}\right)^2 - 4x - 4 = 0 \qquad \left(\frac{x}{c} - c - 2\right)\left(\frac{x}{c} - c + 2\right) = 0$$

$$x \in [c(c-2),\ c(c+2)]$$

$$\varphi = \arg z \qquad z = |z|e^{i\varphi} = |z|(\cos\varphi + i\sin\varphi)$$

$$|z|\cos\varphi = \frac{c + x/c}{2(x+1)}$$

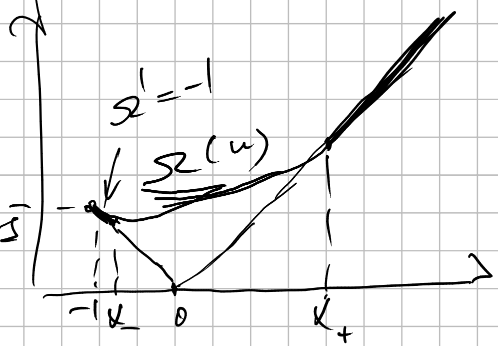
$$|z| = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2} = \sqrt{\frac{\cancel{(c+x/c)^2} - \cancel{(c+\frac{x}{c})^2} + 4x + 4}{4(x+1)^2}} =$$

$$= \frac{1}{\sqrt{x+1}} \qquad \cos\varphi = \frac{c + x/c}{2\sqrt{x+1}}$$

$$\varphi = \arccos \frac{c + x/c}{2\sqrt{x+1}}$$

$$K(x,y) \xrightarrow[n\to\infty]{} \frac{\sin(x-y)\varphi}{\pi(x-y)} \qquad \operatorname{supp}\rho(x) = [x_-, x_+]$$

$$K(x,x) = \rho(x) = \operatorname{Re}\left[\frac{1}{\pi} \arccos \frac{c + x/c}{2\sqrt{x+1}}\right]$$



$$\Omega'(x) = 1 - 2\rho(x)$$

$$\Omega(u) = 1 + \int_{-1}^{u} (1 - 2\rho(x))\,dx$$

We have obtained result of P. Biane '00

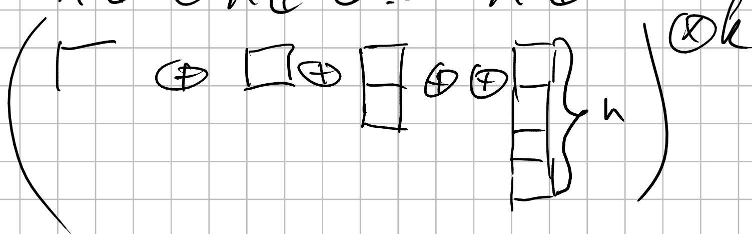
Skew Howe duality

dual RSK

$$\Lambda(\mathbb{C}^n \otimes \mathbb{C}^k) = (\Lambda\mathbb{C}^n)^{\otimes k} = (\Lambda\mathbb{C}^k)^{\otimes n} = \bigoplus_{\lambda \text{ in } n\times k \text{ box}} V_{GL_n}(\lambda) \otimes V_{GL_k}(\lambda')$$

$GL_n \quad GL_k$

$$\Lambda^0\mathbb{C}^n \oplus \Lambda^1\mathbb{C}^n \oplus \dots \oplus \Lambda^n\mathbb{C}^n$$



$e_1 \dots e_n \quad e_1 \dots e_k$

$$e_i \otimes e_j = e_{ij}$$

$$\underbrace{e_{i_1 j_1} \wedge e_{i_2 j_2} \wedge \dots}_{S_k}$$

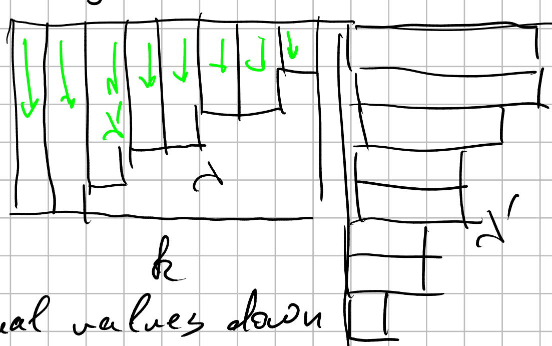
e_{ij}

$$\begin{array}{cc} & j \\ i & \begin{pmatrix} 0 & \boxed{1} & 0 \\ & & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array} \to \begin{pmatrix} i_1 & i_2 & \dots \\ j_1 & j_2 & \dots \end{pmatrix}$$

2^{nk}

λ' — transposed λ

↳ diagram in a box $n \times k$



dual Schensted insertion   bump equal values down

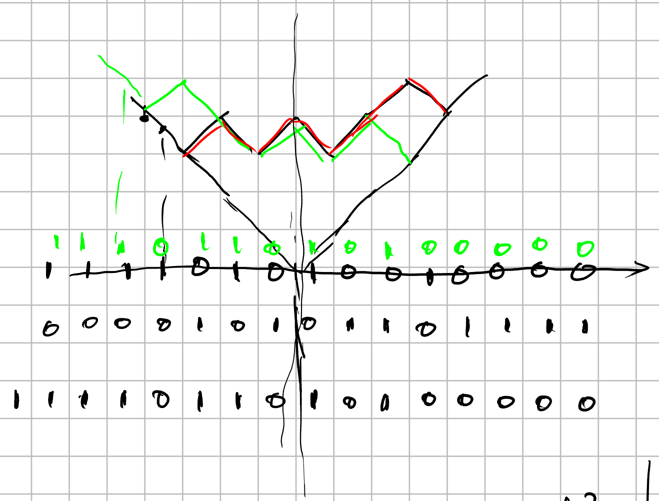
$P' \in SSYT(\lambda' \mid k)$ $\quad Q \in SSYT(\lambda \mid n)$

$$\begin{array}{c|c} & k=4 \\ \hline n=3 & \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \end{array} \to \begin{matrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 2 & 4 & 1 & 4 \end{matrix} \xrightarrow[\text{RSK}]{\text{dual}} \quad P: \begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 4 & & \\ 2 & & & \end{matrix} \qquad Q: \begin{matrix} 1 & 1 & 1 & 2 \\ 2 & 3 & & \\ 3 & & & \end{matrix}$$

$$P' = \begin{matrix} 1 & 1 & 2 \\ 2 & 4 & \\ 3 & & \\ 4 & & \end{matrix}$$

dual Cauchy identity

$$\prod_{i=1}^{n}\prod_{j=1}^{k}(1+x_i y_j) = \sum_{\lambda \text{ in } \boxed{n\times k}} s_\lambda(x_1 \ldots x_n)\, s_{\lambda'}(y_1 \ldots y_k)$$



ω = exchange particles and holes and take a mirror image

$$\omega|\lambda\rangle = |\lambda'\rangle$$

$$\omega \alpha_n \omega^{-1} = (-1)^{n-1} \alpha_n \qquad \text{Ex}$$

$$\omega \,\Big|\; \cdot\; p_\ell(y_1 \ldots) = \frac{1}{\ell}\sum_{i=1}^{\infty} y_i^\ell$$

$$\omega p_\ell(y_1 \ldots) = (-1)^{\ell-1} p_\ell(y_1 \ldots)$$

$$\omega s_\lambda(\underbrace{y_1 \ldots}_{Y}) = s_{\lambda'}(y_1 \ldots)$$

$$\Gamma'_\pm(Y) = \exp\Big(\sum_{\ell=1}^{\infty} (-1)^{\ell-1} p_\ell(Y)\alpha_{\pm\ell}\Big) = \Gamma_\pm^{-1}(-Y)$$

$$\langle 0|\Gamma'_+(Y)|\lambda\rangle = s_{\lambda'}(Y)$$

$$\Gamma'_+(Y)|\lambda\rangle = \Gamma_+(Y)|\lambda'\rangle$$

$$\sum_\lambda s_\lambda(X) s_{\lambda'}(Y) = \sum_\lambda \langle 0|\Gamma_+(X)|\lambda\rangle\langle\lambda|\Gamma'_-(Y)|0\rangle$$

$$= \langle 0|\Gamma_+(X)\Gamma_-^{-1}(-Y)|0\rangle = H(X,-Y)^{-1}\langle 0|\Gamma_-^{-1}(-Y)$$

$$\Gamma_+(X)|0\rangle =$$
$$= \prod_{ij}(1+x_i y_j)$$

$$H(X,Y) = \prod_{ij}\frac{1}{1-x_i y_j}$$

$$H^{-1}(X,-Y) = \prod_{ij}(1+x_i y_j)$$

$$\mu_{n,k}(\lambda) = \frac{\dim V_{GL_n}(\lambda) \dim V_{GL_k}(\lambda')}{2^{nk}}$$