Lecture 8 14.06.2023

GL_n . E^λ $V_n(\lambda)$

$$x^T \to x_1^{\#1 \text{ in } T}\, x_2^{\#2 \text{ in } T} \cdots$$

$$\chi_{E^\lambda}(x_1 \dots x_n) = \sum_{T \in SSYT(\lambda|n)} x^T = s_\lambda(x_1 \dots x_n)$$

$$\mu(\lambda | \{x_i\}_{i=1}^n, \{y_j\}_{j=1}^k) = \left(\prod_{i=1}^n \prod_{j=1}^k (1 - x_i y_j) \right) s_\lambda(x_1 \dots x_n) s_\lambda(y_1 \dots y_k)$$

$\downarrow$

$$t_\ell = \frac{1}{\ell} \sum_i x_i^\ell \qquad t = (\sqrt{\xi}, 0, 0 \dots)$$

$$t'_\ell = \frac{1}{\ell} \sum_j y_j^\ell \qquad t' = (\sqrt{\xi}, 0, 0 \dots)$$

$$\sqrt{\xi} \to \infty$$

$$s_\lambda(1 \dots 1) = \chi_{E^\lambda}(1, \dots, 1) = \sum_\alpha \dim V_\alpha \, x_1^{\alpha_1} \cdots x_n^{\alpha_n} \Big|_{x_i = 1}$$
$$= \dim E^\lambda = \dim V_n(\lambda)$$
$$= \# \, SSYT(\lambda | n)$$

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$c_1, c_2 \dots c_7 = 3\ 2\ 3\ 2\ 1\ 3\ 1$

$r_1 \dots r_6 = 1\ 2\ 5\ 2\ 4\ 1$

$P, Q \in SSYT(\lambda | n)$

i r_i times in Q

j c_j times in P

$$\downarrow \begin{pmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 6 \\ 3 & 6 & 6 & 1 & 2 & 3 & 4 & 6 & 3 & 5 & 1 & 1 & 2 & 4 & 7 \end{pmatrix}$$

$$\boxed{3} \to \boxed{3\,6} \to 366 \to \begin{array}{l} 166 \\ 3 \end{array} \to \begin{array}{l} 126 \\ 36 \end{array} \to \begin{array}{l} 123 \\ 366 \end{array} \to \begin{array}{l} 1234 \\ 366 \end{array} \to \begin{array}{l} 12346 \\ 366 \end{array} \to$$

$$\boxed{1} \to \boxed{1\,2} \to 122 \quad \begin{array}{l} 122 \\ 3 \end{array} \to \begin{array}{l} 122 \\ 33 \end{array} \to \begin{array}{l} 122 \\ 333 \end{array} \to \begin{array}{l} 1223 \\ 333 \end{array} \quad \begin{array}{l} 12233 \\ 333 \end{array} \to$$

$$\to \begin{array}{l} 12336 \\ 346 \\ 6 \end{array} \to \begin{array}{l} 12335 \\ 3466 \\ 6 \end{array} \to \begin{array}{l} 11335 \\ 2466 \\ 3 \\ 6 \end{array} \to \begin{array}{l} 11135 \\ 2366 \\ 34 \\ 6 \end{array} \to \begin{array}{l} 11125 \\ 2336 \\ 346 \\ 6 \end{array} \to$$

$$\to \begin{array}{l} 12233 \\ 333 \\ 4 \end{array} \quad \begin{array}{l} 12233 \\ 3334 \\ 4 \end{array} \to \begin{array}{l} 12233 \\ 3334 \\ 4 \\ 5 \end{array} \quad \begin{array}{l} 12233 \\ 3334 \\ 45 \\ 5 \end{array} \to \begin{array}{l} 12233 \\ 3334 \\ 455 \\ 5 \end{array}$$

$$\begin{array}{ccccc} 1&1&1&2&4\\ 2&3&3&5&\\ 3&4&6&6&\\ 6&&&& \end{array} \to \begin{array}{|cccccc} 1&1&1&2&4&7\\ 2&3&3&5&&\\ 3&4&6&6&&\\ 6&&&&& \end{array} \longleftrightarrow \begin{array}{ccccccccccccc} 6&&4&&4&&1&&0&&0&&0\\ &5&&4&&4&&1&&0&&0&\\ &&5&&4&&2&&0&&0&&\\ &&&5&&3&&2&&0&&&\\ &&&&4&&3&&1&&&&\\ &&&&&4&&1&&&&&\\ &&&&&&3&&&&&& \end{array}$$

$$\begin{array}{ccccc} 1&2&2&3&3\\ 3&3&3&4&\\ 4&5&5&5&\\ 5&&&& \end{array} \to \begin{array}{|cccccc} 1&2&2&3&3&6\\ 3&3&3&4&&\\ 4&5&5&5&&\\ 5&&&&& \end{array}$$

Gelfand-Tsetlin patterns

$$\begin{array}{ccccc} b_1^{(n)} \geq b_2^{(n)} & & \cdots & & b_n^{(n)} \\ \rotatebox{90}{$\leq$}\; b_1^{(n-1)} & b_2^{(n-1)} & \cdots & b_{n-1}^{(n-1)} & \\ & \ddots & & \vdots & \\ & & b_1^{(1)} & & \end{array}$$

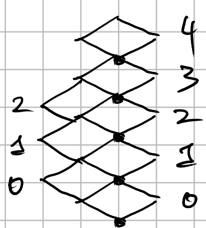
$b_i^{(j)}$ = # of boxes $\leq j$ in the row i

$$\begin{array}{|cccccc} 1&2&2&3&3&6\\ 3&3&3&4&&\\ 4&5&5&5&&\\ 5&&&&& \end{array} \qquad \begin{array}{ccccccccccccc} 6&&4&&4&&1&&0&&0&&0\\ &5&&4&&4&&1&&0&&0&\\ &&5&&4&&1&&0&&0&&\\ &&&5&&3&&0&&0&&&\\ &&&&3&&0&&0&&&&\\ &&&&&1&&0&&&&&\\ &&&&&&0&&&&&& \end{array} \qquad \begin{array}{ccccccccccccc} 12&&9&&8&&4&&2&&1&&0\\ &10&&8&&7&&3&&1&&0&\\ &&9&&7&&3&&1&&0&&\\ &&&8&&5&&1&&0&&&\\ &&&&5&&1&&0&&&&\\ &&&&&2&&0&&&&&\\ &&&&&&0&&&&&& \end{array}$$

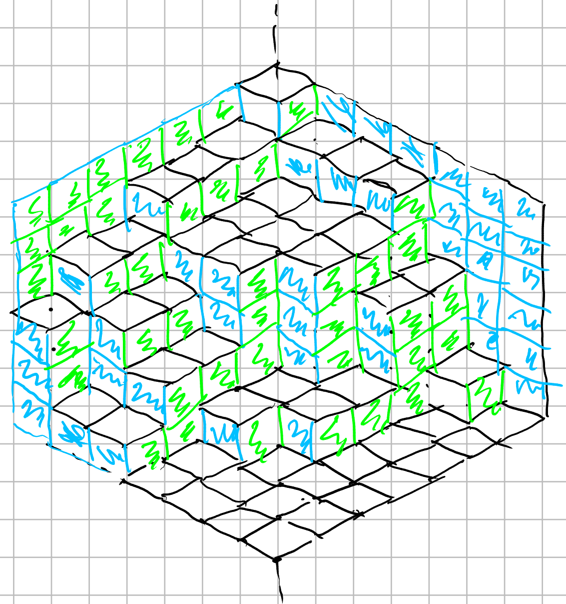
$$\tilde{b}_i^{(j)} = b_i^{(j)} + j - i$$

$$\begin{array}{ccccccccccccc} 6&&4&&4&&1&&0&&0&&0\\ &5&&4&&4&&1&&0&&0&\\ &&5&&4&&2&&0&&0&&\\ &&&5&&3&&2&&0&&&\\ &&&&4&&3&&1&&&&\\ &&&&&4&&1&&&&&\\ &&&&&&3&&&&&& \end{array} \qquad \begin{array}{ccccccccccccc} 12>&&9>&&8>&&4&&2&&1&&0\\ &10&&8&&7&&3&&1&&0&\\ &&9&&7&&4&&1&&0&&\\ &&&8&&5&&3&&0&&&\\ &&&&6&&4&&1&&&&\\ &&&&&5&&1&&&&&\\ &&&&&&3&&&&&& \end{array}$$

$$\begin{array}{ccccccccccccc} 12>&&9>&&8>&&4&&2&&1&&0\\ &10&&8&&7&&3&&1&&0&\\ &&9&&7&&4&&1&&0&&\\ &&&8&&5&&3&&0&&&\\ &&&&6&&4&&1&&&&\\ &&&&&5&&1&&&&&\\ &&&&&&3&&&&&& \end{array}$$



12 9 8 4 2 1 0
10 8 7 3 1 0
9 7 3 1 0
8 5 1 0
5 1 0
2 0
0



$P, Q \in SSYT(\lambda, n)$

$\updownarrow$

lozenge tilings of a hexagon

$\parallel$

pile of cubes in the corner

$$\mathrm{Sym}\, V = \bigoplus_{\lambda} V_n(\lambda) \otimes V_k(\lambda)$$

$$\mathbb{C}^n = V \qquad GL_n \times S_k$$

$$(\mathbb{C}^n)^{\otimes k} = \underset{\{e_1 \dots e_n\}}{V} \otimes V \otimes \dots \otimes V = \bigoplus_{\lambda \vdash k} V_{GL_n}(\lambda) \otimes V_{S_k}(\lambda)$$

Schur-Weyl duality

$\boxed{i_1} \otimes \boxed{i_2} \otimes \dots \otimes \boxed{i_k}$ RSK $\to SSYT(\lambda|n)$, $\to SYT(\lambda|k)$

$$\dim \quad n^k = \sum_{\lambda \vdash k} \dim V_{GL_n}(\lambda) \ \dim \lambda$$

$\dim V_{GL_n}(\lambda) = s_\lambda(1, \dots, 1)$ $\qquad s_\lambda(y_1 \dots y_k)$

$t' = (\sqrt{\xi}, 0 \dots 0)$ $\to$ $\sqrt{\xi}^{|\lambda|} \dfrac{\dim \lambda}{|\lambda|!}$

$$\mu_{n,k}(\lambda) = \frac{\dim V_n(\lambda) \dim \lambda}{n^k}$$

$n, k \to \infty$ $\qquad \dim V_n(\lambda) \to \dim \lambda$

$n \sim k$ $\qquad$ Plancherel case $\qquad$ Kerov 1985

$n \sim \sqrt{k}$ $\qquad c = \dfrac{\sqrt{k}}{n}$

$$\mu_n(\lambda | \xi) = \frac{\dim V_n(\lambda)}{n^{|\lambda|}} \sqrt{\xi}^{|\lambda|} \frac{\dim \lambda}{|\lambda|!} e^{-\sqrt{\xi}}$$

$$K(z, w) = \frac{\sqrt{zw}}{z - w} \frac{F(z)}{F(w)} \qquad F(z) = \frac{\prod_{j=1}^{k} (1 - y_j/z)}{\prod_{i=1}^{n} (1 - x_i z)} = e^{\sum_{\ell \geq 1} t_\ell z^\ell - t'_\ell z^{-\ell}}$$

$$F(z) = \frac{e^{-\sum_\ell t'_\ell z^{-\ell}}}{\prod (1 - x_i z)} = \frac{e^{-\frac{\sqrt{\xi}}{z}}}{(1 - z)^n}$$

$$K(m,m') = \iint_{|w|<|z|} \frac{dz}{2\pi i z} \frac{dw}{2\pi i w} \frac{\sqrt{zw}}{z-w} z^{-m} w^{m'} \frac{F(z)}{F(w)}$$

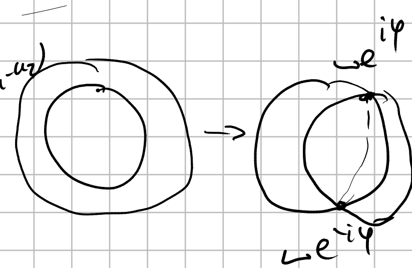
$$F(z) = \exp\left(-\sqrt{\xi}\,\frac{1}{z} - n \ln(1-z)\right)$$

m, m', n, ξ

$$\frac{\sqrt{\xi}}{n} \to c$$

$m = nu + u_1 \to \frac{\sin \varphi(u_1-u_2)}{\pi}$

$m' = nu + u_2$



$$e^{-n \underbrace{\left(\frac{c}{z} + \ln(1-z) + u \ln z\right)}_{S(z;u)}}$$

$$z\partial_z S(z;u) = -\frac{c}{z} + \frac{z}{z-1} + u = 0$$

$$u\,z(z-1) - c(z-1) + z^2 = 0$$

Ex: solve for $z_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$z = |z| e^{\pm i\varphi}$

$b^2 - 4ac = 0 \mid u \to \operatorname{supp} \rho(u)$

$$\rho(u) = \frac{1}{\pi} \arccos \varphi$$