Lecture 24 2023-09-01

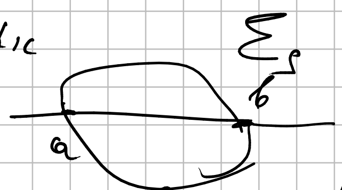
$$\underbrace{Y^{(n)} \to T \to S}\to R$$

P_∞ - asymptotic solution for model RHP for S

$\tilde{R} = S P_\infty^{-1}$ $\quad$ $\tilde{R}$ satisfies RHP

- $\tilde{R} : \mathbb{C} \setminus \Sigma_S \to \mathbb{C}^{2\times 2}$ analytic

$\mathbb{C} \setminus [a,b]$



- $\tilde{R}_+(x) = \tilde{R}_-(x) J_{\tilde{R}}(x)$
- $\tilde{R}(z) = I + o(1)$, $z \to \infty$

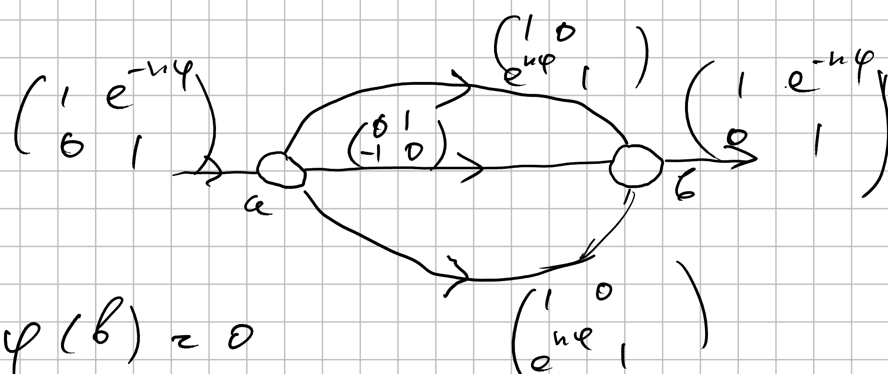
$V(x)$ is polynomial of even power

$\operatorname{supp} \rho = [a,b]$

$J_{\tilde{R}} = P_\infty J_S P_\infty^{-1}$ $\quad$ J_S

$\varphi = g_+(z) - g_-(z)$

$J_S \to I$ as $n \to \infty$ pointwise



$\tilde{R} \not\to I$ as $n \to \infty$

At point a, b $\quad \varphi(b) = 0$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

We need another parametrix near a, b

$z_- = a$ $\quad \odot\ \mathcal{U}_- \ni z_-$ $\quad a = -b$

$z_+ = b$ $\quad \odot\ \mathcal{U}_+ \ni z_+$ $\quad \mathcal{U}_- = -\mathcal{U}_+$

$P_\pm$ s.t. $P_\pm$ have the same jumps as S in $\mathcal{U}_\pm$

$P_\pm$ satisfy matching condition

$$P_\pm(z) = P_\infty(z)\left(I + O\left(\tfrac{1}{n}\right)\right) \quad n \to \infty$$
$$z \in \partial \mathcal{U}_\pm$$

$$2\int \ln|x-y|\,\rho(y)\,dy - V(x) = \ell$$

$$g_+(x) + g_-(x) = V(x) + \ell \quad x \in [a,b] \qquad g(z) = \int \rho(x)\ln(x-z)\,dx$$

$$2g_+(x) = V(x) + \ell - 2\pi i \int_b^x \rho(y)\,dy$$

$$(2g_+(x) - V(x)) - (2g_+(b) - V(b)) = -2\pi i \int_b^x \rho(y)\,dy$$

$$\varphi(x) = -2\pi i \int_b^x \rho(s)\,ds \qquad x\in[a,b]$$

If $V(x)$ is a polynomial

$$\rho(x) = \frac{1}{2\pi i}\, h(x) \sqrt{R_+(x)} \qquad R(x) = (x-a)(x-b)$$

$h(x)$ is polynomial of degree $\deg V - 2$

$a \qquad b$

$$\frac{V'(x)}{\sqrt{R(x)}} = h(x) + O\left(\frac{1}{x}\right) \quad \text{as } x\to\infty$$

$$V(x) = x^2 \qquad h(x) = 1 \qquad \rho(x) = \frac{\sqrt{2-x^2}}{\pi}$$

Wigner semicircle law

$$V(x) = t x^{2m}$$

$$\rho(x) = -\frac{mit}{\pi}\left(\sqrt{x^2-a^2}\right)_+ h_1(x)$$

$$a = \left(mt\prod_{l=1}^{m}\frac{2l-1}{2l}\right)^{-\frac{1}{2m}}$$

$$h_1(x) = x^{2m-2} + \sum_{j=1}^{m-1} x^{2m-2-2j} a^{2j} \prod_{l=1}^{j} \frac{2l-1}{2l}$$

$g(z)$ and $\sqrt{R(z)}$ analytic for $\operatorname{Im} z > 0$

$$(2g_+(b) - V(b)) - (2g(z) - V(z)) = \int_b^z h(s)\sqrt{R(s)}\,ds$$

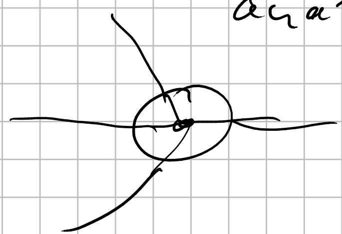
$$\varphi(z) = \int_b^z h(s)\sqrt{R(s)}\,ds = \underset{\neq 0}{\text{const}}\,(z-b)^{3/2} + O((z-b)^{5/2}) \qquad z\to b$$

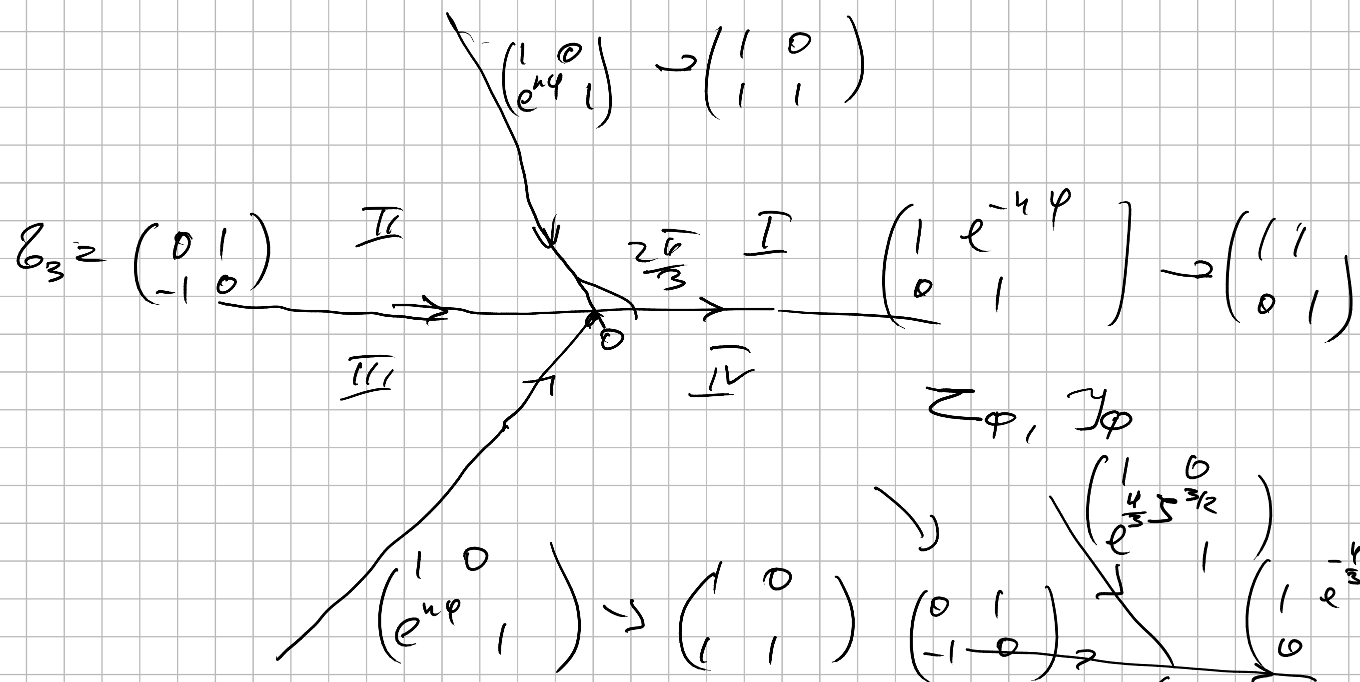
$2 \quad \sqrt{z-b}$

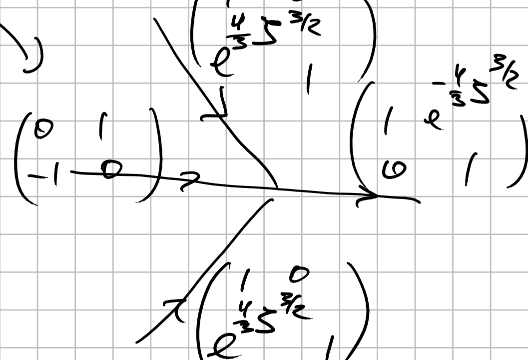
$$= \frac{4}{3}\zeta^{3/2}$$

$$\zeta = \beta(z) = \left(\frac{3}{4}\left[(2g_+(b) - V(b)) - (2g(z) - V(z))\right]\right)^{2/3}$$

analytic near $z = b$ $\qquad \beta'(b) > 0$







$\Phi : \mathbb{C} \setminus \Sigma_\Phi \to \mathbb{C}^{2\times 2}$ analytic

$\Phi_+(\zeta) = \Phi_-(\zeta)\, J_\Phi \quad \zeta \in \Sigma_\Phi$

$$\Phi(\zeta) = \frac{1}{2}\begin{pmatrix}1&-1\\-i&-i\end{pmatrix}\begin{pmatrix}\zeta^{-\frac{1}{4}}&0\\0&\zeta^{\frac{1}{4}}\end{pmatrix}\begin{pmatrix}1&i\\-1&i\end{pmatrix}\left(I + O(\zeta^{-\frac{3}{2}})\right)$$
as $\zeta \to \infty$

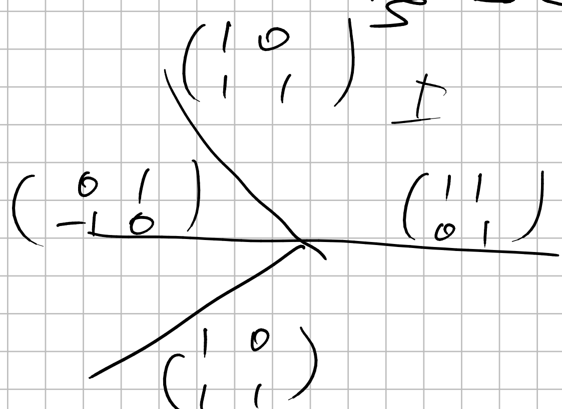
$$\Psi(\zeta) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix}1&i\\-1&i\end{pmatrix}\Phi(\zeta)\begin{pmatrix}e^{-\frac{2}{3}\zeta^{3/2}}&0\\0&e^{\frac{2}{3}\zeta^{3/2}}\end{pmatrix}$$

Ψ is analytic in $\mathbb{C}\setminus\Sigma_\Psi$

$\Psi_+(\zeta) = \Psi_-(\zeta)\, J_\Psi \quad \zeta \in \Sigma_\Psi$

$$\Psi(\zeta) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix}\zeta^{-\frac{1}{4}}&0\\0&\zeta^{\frac{1}{4}}\end{pmatrix}\begin{pmatrix}1&i\\-1&i\end{pmatrix}\left(I + O(\zeta^{-\frac{3}{2}})\right)\cdot\begin{pmatrix}e^{-\frac{2}{3}\zeta^{3/2}}&0\\0&e^{\frac{2}{3}\zeta^{3/2}}\end{pmatrix}$$
$\zeta \to \infty$

Ψ bounded at $\zeta = 0$



$Ai(\zeta)$ solves the ODE $y'' = \zeta y$

for any $\varepsilon > 0$ $\quad \pi - \varepsilon \le \arg\zeta \le \pi + \varepsilon$

$$Ai(\zeta) = \frac{1}{2\sqrt{\pi}\,\zeta^{1/4}} e^{-\frac{2}{3}\zeta^{3/2}} \left(1 + O(\zeta^{-\frac{3}{2}})\right)$$

$$\omega = e^{\frac{2\pi i}{3}} \qquad Ai(\omega\zeta) \quad Ai(\omega^2\zeta)$$

$$Ai(\zeta) + \omega Ai(\omega\zeta) + \omega^2 Ai(\omega^2\zeta) = 0$$

$$y_0(\zeta) = Ai(\zeta) \quad y_1(\zeta) = \omega Ai(\omega\zeta)$$

$$y_2(\zeta) = \omega^2 Ai(\omega^2\zeta)$$

$$\Psi(\zeta) = \begin{cases} \begin{pmatrix} y_0(\zeta) & -y_2(\zeta) \\ y_0'(\zeta) & -y_2'(\zeta) \end{pmatrix} & 0 < \arg\zeta < \frac{2\pi}{3} \quad \underline{+} \\ \begin{pmatrix} -y_1(\zeta) & -y_2(\zeta) \\ -y_1'(\zeta) & -y_2'(\zeta) \end{pmatrix} & \frac{2\pi}{3} < \arg\zeta < \pi \quad \underline{\Pi} \\ \begin{pmatrix} -y_2(\zeta) & y_1(\zeta) \\ -y_2'(\zeta) & y_1'(\zeta) \end{pmatrix} & -\pi < \arg < \zeta < -\frac{2\pi}{3} \\ \begin{pmatrix} y_0(\zeta) & y_1(\zeta) \\ y_0'(\zeta) & y_1'(\zeta) \end{pmatrix} & -\frac{2\pi}{3} < \arg\zeta < 0 \end{cases}$$

$$\Phi(\zeta) = \frac{\sqrt{\pi}}{2} \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \Psi(\zeta) \begin{pmatrix} e^{\frac{2}{3}\zeta^{3/2}} & 0 \\ 0 & e^{-\frac{2}{3}\zeta^{3/2}} \end{pmatrix}$$

$$P_+(z) = E_n(z)\,\Phi(n^{2/3}\beta(z))$$

$$E_n(z) = \frac{1}{2} P_\infty(z) \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} \begin{pmatrix} n^{\frac{1}{6}}(\beta(z))^{\frac{1}{4}} & 0 \\ 0 & n^{-\frac{1}{6}}\beta(z)^{-\frac{1}{4}} \end{pmatrix} \begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix}$$

$$P_+(z) = E_n(z)\,\Phi(n^{2/3}\beta(z)) = P_\infty(z)\left(I + O(\tfrac{1}{n})\right)$$

$a = -b \quad \operatorname{supp}\rho = [-b, b]$ $\quad n \to \infty$ uniformly for $z \in \partial U_+$

$$P_-(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P_+(-z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

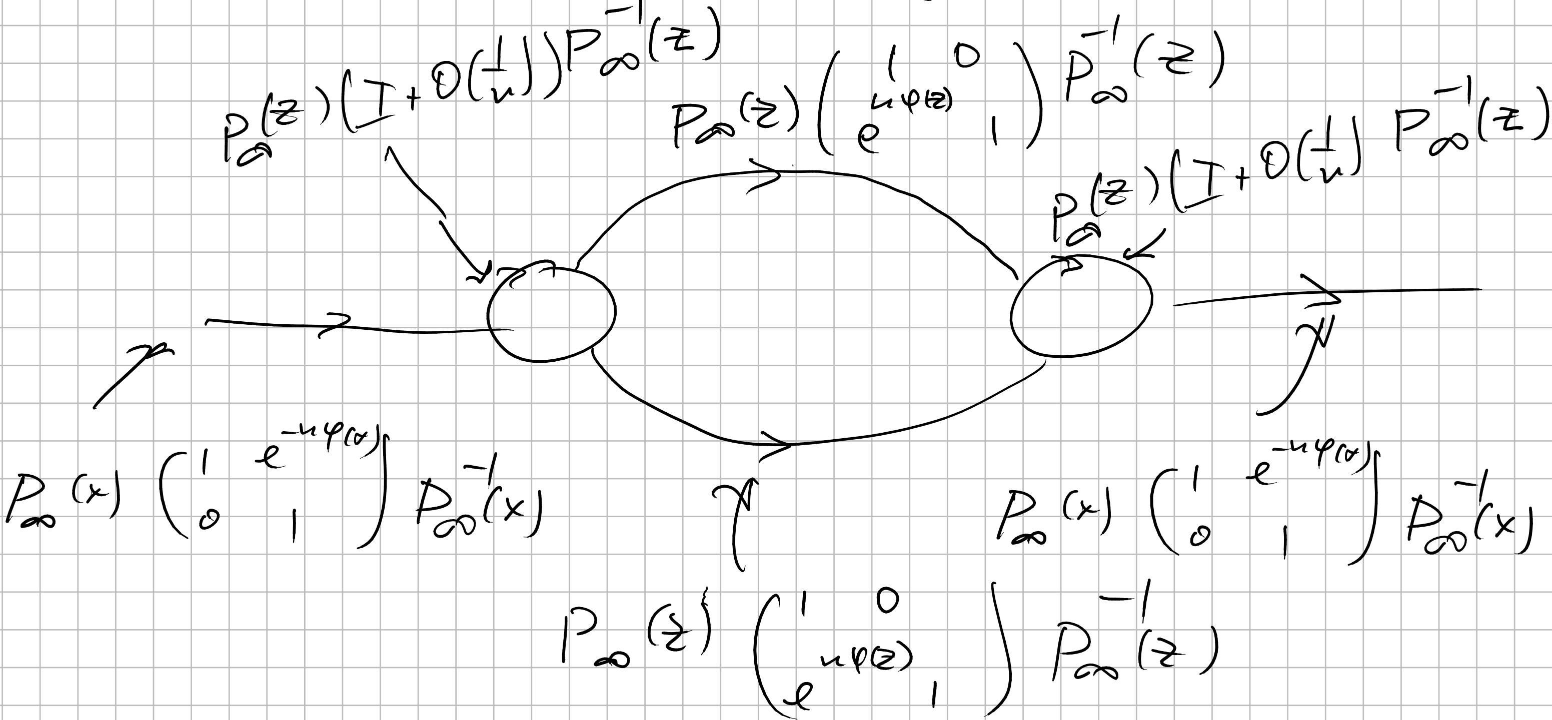
$S \to R$

$$R(z) = \begin{cases} S(z)\,P_\infty(z)^{-1} & z \in \mathbb{C} \setminus (U_+ \cup U_- \cup \Sigma_S) \\ S(z)\,P_+(z)^{-1} & z \in U_+ \setminus \Sigma_S \\ S(z)\,P_-(z)^{-1} & z \in U_- \setminus \Sigma_S \end{cases}$$

$R : \mathbb{C}\setminus[a,b] \to \mathbb{C}^{2\times 2}$ analytic

$R_+(x) = R_-(x)\, J_R(x) \qquad x \in \Sigma_R$

$R(z) = I + o(1)$ as $z \to \infty$



$\|J_R - I\|_{\infty, z} = O(\frac{1}{n})$ as $n \to \infty$ on $\partial U_\pm$

$R(z) = I + O(\frac{1}{n})$ as $n \to \infty$

uniformly in z in compact subsets of $\mathbb{C}\setminus\Sigma_R$

$z \to \infty \quad R(z) \cong I + \sum_{j=1}^{\infty} \frac{R_j}{z^j}$

$J_R^0(z) := J_R(z) - I \qquad J_R^0(z) = O(\frac{1}{n})$ $n \to \infty$

thm. Assume that $v(z)$ for $z \in \Sigma_R$ solves the equation

$J_R^0(z) = O(e^{-c(z)n})$

$c(z) > \ln z$

$z \to \infty$

$$v(z) = I - \frac{1}{2\pi i}\int_{\Sigma_R}\left(\frac{v(u)\, J_R^0(u)}{z - u}\right)du$$

$z \in \mathbb{C}\setminus\Sigma_R$

$$R(z) = I - \frac{1}{2\pi i}\int_{\Sigma_R}\frac{v(u)\, J_R^0(u)}{z-u}\,du$$

solves RHP

proof

$R_-(z) = v(z)$ for $z \in \Sigma_R \qquad R_+(z) - R_-(z) = v(z)\, J_R^0(z) =$

$= R_-(z)\, J_R^0(z)$

$$R_+(z) = R_-(z)\, J_R(z)$$

$$R(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty$$

$$v(z) = I + \sum_{k=1}^{\infty} v_k(z)$$

For $k \geq 1$

$$v_k(z) = -\frac{1}{2\pi i} \int_{\Sigma_R} \left(\frac{v_{k-1}(u)\, J_R^0(u)}{z-u} \right) du \qquad z \in \Sigma_R$$

$$v_0(z) = I$$

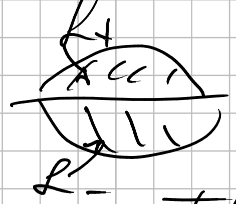
$$|v_k(z)| < \frac{C^k}{n^k(1+|z|)} \qquad J_R^0(z) = O\left(e^{-C(z)n}\right)$$

$$R_k(z) = -\frac{1}{2\pi i} \int_{\Sigma_R} \frac{v_{k-1}(u)\, J_R^0(u)}{z-u}\, du$$

$$R_1(z) = -\frac{1}{2\pi i} \int \frac{J_R^0(u)}{z-u}\, du$$

$$|R_k(z)| \leq \frac{C_0\, C^k}{n^k(1+|z|)}$$

$$R(z) = I + O\left(\frac{1}{n(1+|z|)}\right)$$ as $n \to \infty$ uniformly for $z \in \mathbb{C} \setminus \Sigma_R$



$$S(z) = \begin{cases} \left(I + O\left(\frac{1}{n(1+|z|)}\right)\right) P_\infty(z) & z \in \mathbb{C} \setminus \mathcal{U}_\pm \\ \left(I + O\left(\frac{1}{n(1+|z|)}\right)\right) P_\pm(z) & z \in \mathcal{U}_\pm \end{cases}$$

$$T(z) = \begin{cases} S(z) \begin{pmatrix} 1 & 0 \\ e^{-n\varphi(z)} & 1 \end{pmatrix} & z \in \mathcal{L}_+ \\ S(z) \begin{pmatrix} 1 & 0 \\ e^{n\varphi(z)} & 1 \end{pmatrix} & z \in \mathcal{L}_- \\ S(z) & z \in \mathbb{C} \setminus (\mathcal{L}_+ \cup \mathcal{L}_-) \end{cases}$$

$$e^{\frac{n\ell}{2}\sigma_3}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Y(z) = \begin{cases} e^{\frac{n\ell}{2}\sigma_3} \left(I + O\left(\frac{1}{n(1+|z|)}\right)\right) P_\infty(z) \begin{pmatrix} 1 & 0 \\ \pm e^{\mp n\varphi(z)} & 1 \end{pmatrix} \cdot e^{n\left(g(z) - \frac{\ell}{2}\right)\sigma_3} & z \in \mathcal{L}_\pm \setminus \mathcal{U}_\pm \\ e^{\frac{n\ell}{2}\sigma_3} \left(I + O\left(\frac{1}{n(1+|z|)}\right)\right) P_\pm(z)\, e^{n\left(g(z) - \frac{\ell}{2}\right)\sigma_3} & z \in \mathcal{U}_\pm \end{cases}$$

$$\Big(e^{\frac{n\ell}{2}\sigma_3}\Big(I+O\Big(\frac{1}{n(1+|z|)}\Big)\Big)P_\infty(z)\, e^{n(g(z)-\frac{\ell}{2})\sigma_3}$$

$$z\in\mathbb{C}\setminus(\mathcal{U}_\pm\cup\mathcal{L}_\pm)$$

a_n^2

$$I+\frac{Y^{(1)}}{z}+\frac{Y^{(2)}}{z^2}+\dots = Y^{(n)}(z)\, e^{-n\sigma_3} =$$

$$= e^{\frac{n\ell}{2}\sigma_3} T(z)\, e^{n(g(z)-\frac{\ell}{2}-\ell_n z)\sigma_3} = e^{\frac{n\ell}{2}\sigma_3}\cdot$$

$$\cdot\Big(I+\frac{T^{(1)}}{z}+\frac{T^{(2)}}{z^3}+\dots\Big)e^{n(g(z)-\frac{\ell}{2}-\ell_n z)\sigma_3}$$

$$(Y^{(1)})_{12} = e^{n\ell}(T^{(1)})_{12}$$

$$(Y^{(1)})_{21} = e^{-n\ell}(T^{(1)})_{21}$$

$$a_n^2 = (Y^{(1)})_{12}(Y^{(1)})_{21} = (T^{(1)})_{12}(T^{(1)})_{21}$$

$$a_n^2 = (P_\infty^{(1)})_{12}(P_\infty^{(1)})_{21} + O\Big(\frac{1}{n}\Big)$$

$$P_\infty(z) = \begin{pmatrix} \frac{\gamma(z)+\gamma^{-1}(z)}{2} & \frac{\gamma(z)-\gamma^{-1}(z)}{-2i} \\ \frac{\gamma(z)-\gamma^{-1}(z)}{2i} & \frac{\gamma(z)+\gamma^{-1}(z)}{2} \end{pmatrix}$$

$$\gamma(z) = \Big(\frac{z-a}{z-b}\Big)^{1/4} \qquad \gamma(z) = 1+\frac{b-a}{4}\frac{1}{z}+\frac{(-3a^2-2ab+5b^2)}{32z^2}+\dots$$

$$(P_\infty^{(2)})_{21} \approx \frac{-8a^2+8b^2}{32} = \frac{(b-a)(b+a)}{4} \qquad \gamma(z)^{-1} = 1+\frac{a-b}{4}\frac{1}{z}+\frac{5a^2-2ab-3b^2}{32z^2}+\dots$$

$$a_n^2 = \Big(\frac{b-a}{2}\Big)\frac{1}{(-2i)}\Big(\frac{b-a}{2}\Big)\frac{1}{2i} = \frac{(b-a)^2}{16}+O\Big(\frac{1}{n}\Big)$$

$$a_n \to \frac{b-a}{4}$$

for b_n

$$(Y^{(1)})_{11} = (T^{(1)})_{11} + ng_1$$

$$(Y^{(2)})_{11} = e^{-n\ell}\big((T^{(2)})_{11}+(T^{(1)})_{21}ng_1\big)$$

$$b_n = \frac{(Y^{(2)})_{21}}{(Y^{(1)})_{21}} - (Y^{(1)})_{11} = \frac{(T^{(2)})_{21}}{(T^{(1)})_{21}} - (T^{(1)})_{11}$$

$$= \frac{(P_\infty^{(2)})_{21}}{(P_\infty^{(1)})_{21}} - \underbrace{(P_\infty^{(1)})_{11}}_{0} + O\left(\frac{1}{n}\right)$$

$$\frac{(b-a)(b+a)}{4} \Big/ \frac{b-a}{2} = \frac{b+a}{2}$$

$$b_n \to \frac{b+a}{2} + O\left(\frac{1}{n}\right)$$

Lectures: Bleher 0801.1858

Baik, Deift, Its, Zhou

Riemann-Hilbert approach to asymptotics of orthogonal polynomials