Lecture 7 $\quad R \quad$ R-module $E \quad E^\lambda$

λ - Young diagram $\qquad |\lambda| = n$

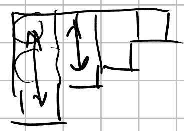
$1 \dots m$

$z_{ij} \qquad i = 1 \dots n$

$j = 1 \dots m$

$$R[\{z_{ij}\}] = R[z]$$

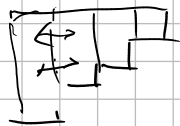
$$D_{i_1 \dots i_p} = \det \begin{pmatrix} z_{1i_1} & z_{1i_2} & \dots & z_{1i_p} \\ z_{2i_1} & & & \vdots \\ \vdots & & & \vdots \\ z_{pi_1} & - & - & z_{pi_p} \end{pmatrix}$$



alternating function $i_1, \dots, i_p$

$\lambda \quad T$ - filling of $\lambda \qquad \mu = \lambda'$

$$D_T = \prod_{j=1}^{\lambda_1} D_{T(1,j)\,T(2,j) \dots T(\mu_j, j)}$$



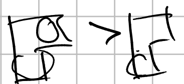
Lemma 3 $\quad e_1 \dots e_m$ - basis of E. $\quad \exists \; E^\lambda \to R[z]$

$e_T \to D_T \qquad$ for all T - fillings of λ

$i_1 \dots i_p \qquad j_1 \dots j_q \qquad q \le p$

$$M = \begin{pmatrix} z_{1i_1} & \dots & z_{1i_p} \\ \vdots & & \\ z_{pi_1} & \dots & z_{pi_p} \end{pmatrix} \qquad N = \begin{pmatrix} z_{1j_1} & \dots & z_{1j_q} & 0 \\ \vdots & & \vdots & \\ z_{pj_1} & \dots & z_{pj_q} & I_{p-q} \end{pmatrix}$$

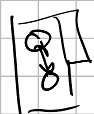
Thm 1. $\quad E^\lambda$ has a basis $\{e_T\}_{T \in SSYT(\lambda|m)}$



$e_T \quad T$ - filling

$T' > T$ if the right-most column which is different in the lowest different box T' has larger value

$$e_T = \sum \alpha_S e_S + Q \qquad S > T$$



$T \to T' \qquad T' > T$

Assume that T is not a YT, in column j there is a position k s.t. $T(k,j) > T(k,j+1)$

$$e_T = \sum e_S \qquad S > T$$

To prove that $\{e_T\}_{T \in SSYT(\lambda|m)}$ is linearly indep.
it is enough to prove independence of $\{D_T\}$

$\{z_{ij}\}$ $\quad z_{ij} < z_{i'j'}$ if $i < i'$
or $i = i'$ and $j < j'$

M - monomial in $\{z\}$ $\quad M_1 < M_2$ if the smallest z_{ij} has smaller power in M_1 than in M_2

$$z_{11} z_{21}^3 < z_{11}^2$$

$M_1 < M_2$ and $N_1 \leq N_2$ $\quad M_1 N_1 < M_2 N_2$

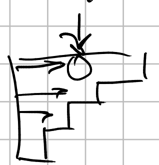
$D_{i_1 \dots i_p}$ $\quad i_1 < i_2 \dots < i_p$

$z_{1 i_1} z_{2 i_2} \dots z_{p i_p}$ - the largest monomial in $D_{i_1 \dots i_p}$

D_T if T has increasing columns

$\prod (z_{ij})^{m_T(i,j)}$ $\quad m_T(ij)$ - # of js in i-th column of T

$T < T'$



$$\begin{matrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & & \end{matrix} < \begin{matrix} 1 & 1 & 3 & 3 & 4 \\ & \dots & & & \end{matrix}$$

smallest i for which $\exists j$ s.t. $m_T(i,j) \neq m_{T'}(i,j)$
$m_T(i,j) > m_{T'}(i,j) \Rightarrow T < T'$

largest in D_T is larger than any monomial in $D_{T'}$

$\sum r_T D_T = 0$ $\quad T$ - minimal s.t. $r_T \neq 0$
then the coeff. of $\prod_{ij} (z_{ij})^{m_T(ij)}$ is r_T
$\Rightarrow$ contradiction $\quad$ TB

Corollary The map from E^λ to $R[Z]$ is injective
its image D^λ is free on D_T $\quad T \in SSYT(\lambda|m)$

$\mathrm{End}_R(E) \to \mathrm{End}(E^\lambda)$ $\quad E$ has a basis $e_1 \dots e_m$
$GL(E)$ $\quad E \cong R^m$

$\mathrm{End}(E) = M_m R$

if $g \in GL(E)$ $(g_{ij})_{ij=1}^m \in M_m R$

if T has entries $j_1 \dots j_n$ in its n boxes

$$g \cdot e_T = \sum_{T'} g_{i_1 j_1} g_{i_2 j_2} \cdots g_{i_n j_n} e_{T'}$$

T' is filling obtained from T $j_1 \dots j_n \to i_1 \dots i_n$

$$g \cdot z_{ij} = \sum_{k=1}^m z_{ik} g_{kj}$$

$R[z]$ $M_m R$ on functions

$$(g \cdot f)(A) = f(A \cdot g) \quad \text{for } g \in M_m R$$

A - $m \times n$ matrix

$$g \cdot D_{j_1 \dots j_p} = \sum g_{i_1 j_1} \cdots g_{i_p j_p} D_{i_1 \dots i_p}$$

$1 \le i_1, \dots, i_p \le m$

$D^\lambda \to D^\lambda$

$E = R^m$ $E^\lambda \to D^\lambda$ isomorphism of $M_m R$-modules

$R = \mathbb{C}$ E^λ - finite-dimensional $GL(E)$-representation

V_ρ of $GL(E)$ is polynomial if the map

$\rho : GL(E) \to GL(V)$ is given by polynomials

N^2 polynomials of m^2 variables

E^λ are polynomial

all holomorphic reps of $GL(E)$ $E^\lambda \otimes D^{\otimes k}$

$D = \Lambda^m E$ $D^{\otimes k}$ - 1-d rep. $GL(E) \to \mathbb{C}$

$$g \to (\det g)^k$$

$e_1 \dots e_m$ $GL(E) = GL(m, \mathbb{C})$

$$\cup$$

$$H = \{ x = \mathrm{diag}(x_1, \dots x_m) \}$$

$v \in V$ is called weight vector with weight $\alpha = (\alpha_1, \dots \alpha_m)$

if $x \cdot v = x_1^{\alpha_1} \dots x_m^{\alpha_m} v$ for all $x \in H$

$$V = \bigoplus_\alpha V_\alpha \qquad V_\alpha = \{ v \in V : x \cdot v = \left(\prod_{j=1}^m x_j^{\alpha_j}\right) v \quad \forall x \in H \}$$

$V = E^\lambda$ $\quad e_T$ is weight vector $\quad \alpha_i = \#\, i\text{s in } T$
$T \in SSYT(\lambda | m)$

$B \subset G$ - upper-triangular

$v \in V$ highest weight vector if $B \cdot v = \mathbb{C} \cdot v$

Lemma 4 Highest weight vector in E^λ is e_T, where

$T = U(\lambda)$
$$\begin{matrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \\ 3 & 3 & & \end{matrix}$$

Proof.

$g \cdot e_T = \sum g_{i_1 j_1} \cdots g_{i_n j_n} e_{T'}$ if $T = U(\lambda)$ and

$g_{ij} = 0$ for $i > j$ $\quad e_{T'} = e_T$

if $T \neq U(\lambda)$ $\quad p$ - first row that contains element $q > p$

$g \in B$ $\quad g_{ij} = 1$ if $i = j$
or $i = p, j = q$

$g_{ij} = 0$ otherwise

$g \cdot e_T = \sum e_{T'}$ $\quad T'$ is obtained from T
by exchang qs to ps

the coeff is 1 which contradicts to h.w. cond. $\boxtimes$

V is irreducible $\iff$ has unique highest weight vector

$V \sim V' \iff$ h.w. vectors have the same weight

$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_m$

Thm 2. (1) If λ has at most m rows E^λ is irreducible representation of $GL(m, \mathbb{C})$ with the highest weight $\lambda = (\alpha_1, \ldots \alpha_m)$

(2) For any $\alpha_1 \geq \ldots \geq \alpha_m$ integers $\exists$ unique irrep of $GL(m, \mathbb{C})$ with h.w. α

$E^\lambda \otimes D^{\otimes k}$ for any $k \in \mathbb{Z}$

$\lambda_i = \alpha_i - k \geq 0$ for all i

$E^\lambda \otimes D^{\otimes k} \cong E^{\lambda'} \otimes D^{\otimes k'} \iff \lambda_i + k = \lambda_i' + k'$ for all i

$SL(E) = SL(m, \mathbb{C})$ $\quad \det = 1$

$H = \{ x = \mathrm{diag}(x_1, \ldots x_m) \mid x_1 \ldots x_m = 1 \}$

$\alpha_1 + \dots + \alpha_m = 0$ $\quad D$ is trivial

$E^\lambda \qquad E^\lambda \simeq E^{\lambda'} \iff \lambda_i - \lambda_i' = \text{const}$ for all i

$\lambda_m = 0$

Character of V $\quad \chi_V(x_1 \dots x_m) =$ Trace over V of $\operatorname{diag}(x)$

$$\chi_V(x) = \sum_\alpha \dim(V_\alpha)\, x_1^{\alpha_1} \dots x_m^{\alpha_m}$$

$E^\lambda \qquad \chi_{E^\lambda}(x) = \sum_{T \in SSYT(\lambda|m)} x^T = S_\lambda(x_1 \dots x_m)$ Schur polynomial

$$\chi_{V \oplus W}(x) = \chi_V(x) + \chi_W(x)$$

$$\chi_{V \otimes W}(x) = \chi_V(x)\, \chi_W(x)$$