DIGITAL BIOSIGNAL PROCESSING

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Lecture 2

- Discrete-time signals
- Discrete-time systems
- Linear, time-invariant (LTI) systems
- Discrete-time convolution
- Frequency response of LTI systems

Discrete-time signals

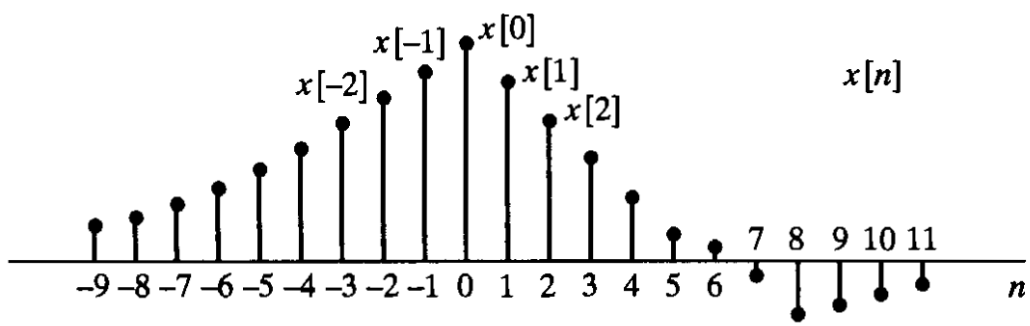
A discrete-time signal is mathematically represented as a sequence of numbers/data (data stream). The sequence x contains the numbers $x[n]$ indexed by the integer n (which is referred to as discrete time):

$$x = \{x[n]\}, \qquad -\infty < n < \infty$$

As we have seen in the previous lecture, the sequence can be obtained from sampling a continuous-time signal:

$$x[n] = x_c(nT) \qquad -\infty < n < \infty$$

However, the notation $x[n]$ is more general



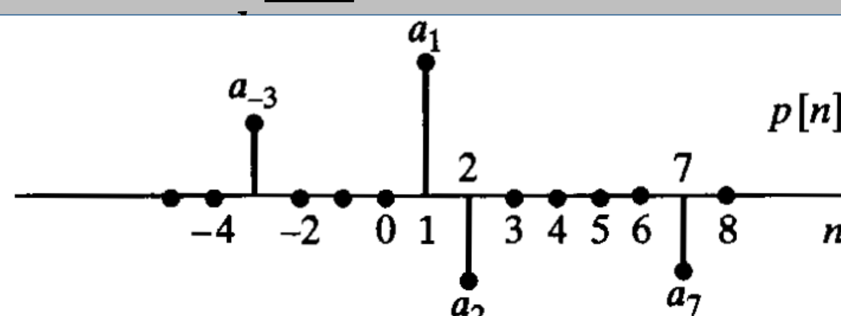
The unit sample sequence

The unit sample sequence is defined as (Kronecker delta):

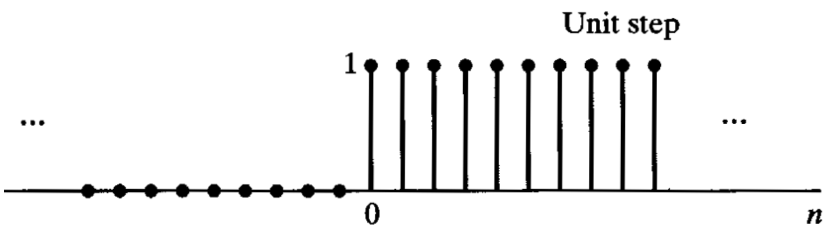
$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$

Any arbitrary sequence can be expressed as the sum of scaled and delayed impulses:

$$x[n] = \sum^{+\infty} x[k]\delta[n-k]$$



Other relevant sequences



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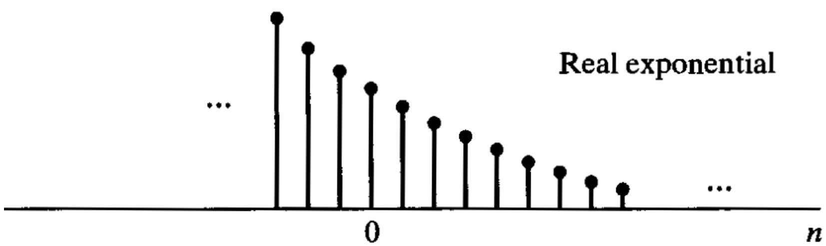
Unit step:

$$u[n] = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases} = \sum_{k=0}^{+\infty} \delta[n-k]$$



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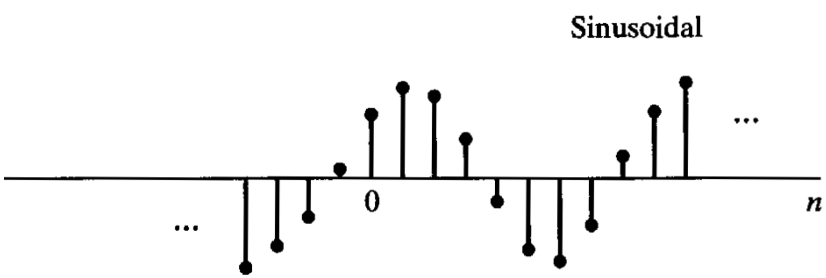
Exponential:
$$x[n] = A\alpha^n$$



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Sinusoidal:
$$x[n] = A\cos(\omega_0 n + \phi)$$

Sinusoidal sequences

Given the sinusoidal sequence

$$x[n] = A\cos(\omega_0 n + \phi)$$

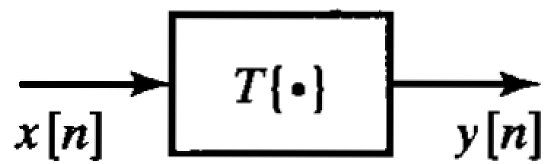
the frequencies $\omega_0 = 0$ and $\omega_0 = 2\pi$ are indistinguishable, and the same for any multiple of 2π, $\omega_0 = k2\pi$

This is an important difference with respect to continuous-time sinusoidal signals

Discrete-time systems

A discrete-time system maps an input sequence $x[n]$ in an output sequence $y[n]$:

$y[n] = T\{x[n]\}$



Example:

$$y[n] = x[n - n_d]$$

It shifts the sequence in time by the delay n_d

Type of discrete-time systems

A system is **memoryless** if its output at time n depends exclusively on the input at time n

Example: $y[n] = (x[n])^{\alpha}$

A system is **linear** if given $y_1[n] = T\{x_1[n]\}$ and $y_2[n] = T\{x_2[n]\}$, then:

$$T\{ax_1[n] + bx_2[n]\}=ay_1[n]+by_2[n]$$

The memoryless system $y[n] = (x[n])^{\alpha}$ is non-linear for $\alpha \neq 1$

Time-invariant systems

A system is **time-invariant** if a shift in its input sequence causes a corresponding shift in its output sequence. Thus, if $y[n] = T\{x[n]\}$, then for all n_0, the input sequence $x_1[n] = x[n - n_0]$ produces at output the sequence $y_1[n] = y[n - n_0]$

Examples:
Show that the accumulator system

$y[n] = \sum_{k=-\infty}^{n} x[k]$ is time-invariant,

while the down-sampler $y[n] = x[Mn]$ with M integer is not time-invariant

Causality and stability

A system is **causal** if for all indexes n_0 its output depends only on input values $n \leq n_0$

At each time, causal systems determine their output using previous or current values of the input

A system is **stable** if every bounded input always corresponds to bounded outputs (bounded input, bounded output stability; BIBO):

For all bounded sequences

$|x[n]| \leq B_x < \infty$ (B_x finite positive),

the output is bounded

$|y[n]| \leq B_y < \infty$ (B_y finite positive)

Summary of type of discrete-time systems

- **Memoryless:** Output at n depends only on input at n
- **Linear:** $T\{ax_1[n] + bx_2[n]\}$=$ay_1[n]$+$by_2[n]$
- **Time-invariant:** if $y[n] = T\{x[n]\}$ and $x_1[n] = x[n - n_0]$, then $y_1[n] = y[n - n_0]$
- **Causal:** Output at n depends only on inputs at $n \leq n_0$
- **Stable (BIBO):** if input $|x[n]| \leq B_x < \infty$ then output $|y[n]| \leq B_y < \infty$

A particularly important class of systems is that of Linear and Time-Invariant (LTI) systems

Linear and time-invariant (LTI) systems

Remember we can write any sequence as the linear combination of impulses:

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Let's consider a **linear** system, then its output can be expressed by linear superposition:

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\right\} =$$

$$= \sum_{k=-\infty}^{+\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]$$

Where $h_k[n]$ is the response of the system to the impulse, which depends on the delay of the impulse in general

Linear and time-invariant (LTI) systems

Now let's assume that, in addition to linear, the system is also **time-invariant** (i.e., it is an LTI system). Then, by definition, if $h[n]$ is the response of the system to the impulse $\delta[n]$, $h[n-k]$ is the response of the system to $\delta[n-k]$, so that:

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]\right\} =$$

$$= \sum_{k=-\infty}^{+\infty} x[k]T\{\delta[n-k]\} = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

The output of a LTI system is fully determined by the response of the system to the impulse

Discrete-time convolution

The relation between input and output of a LTI system is expressed by the convolution operator:

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n]$$

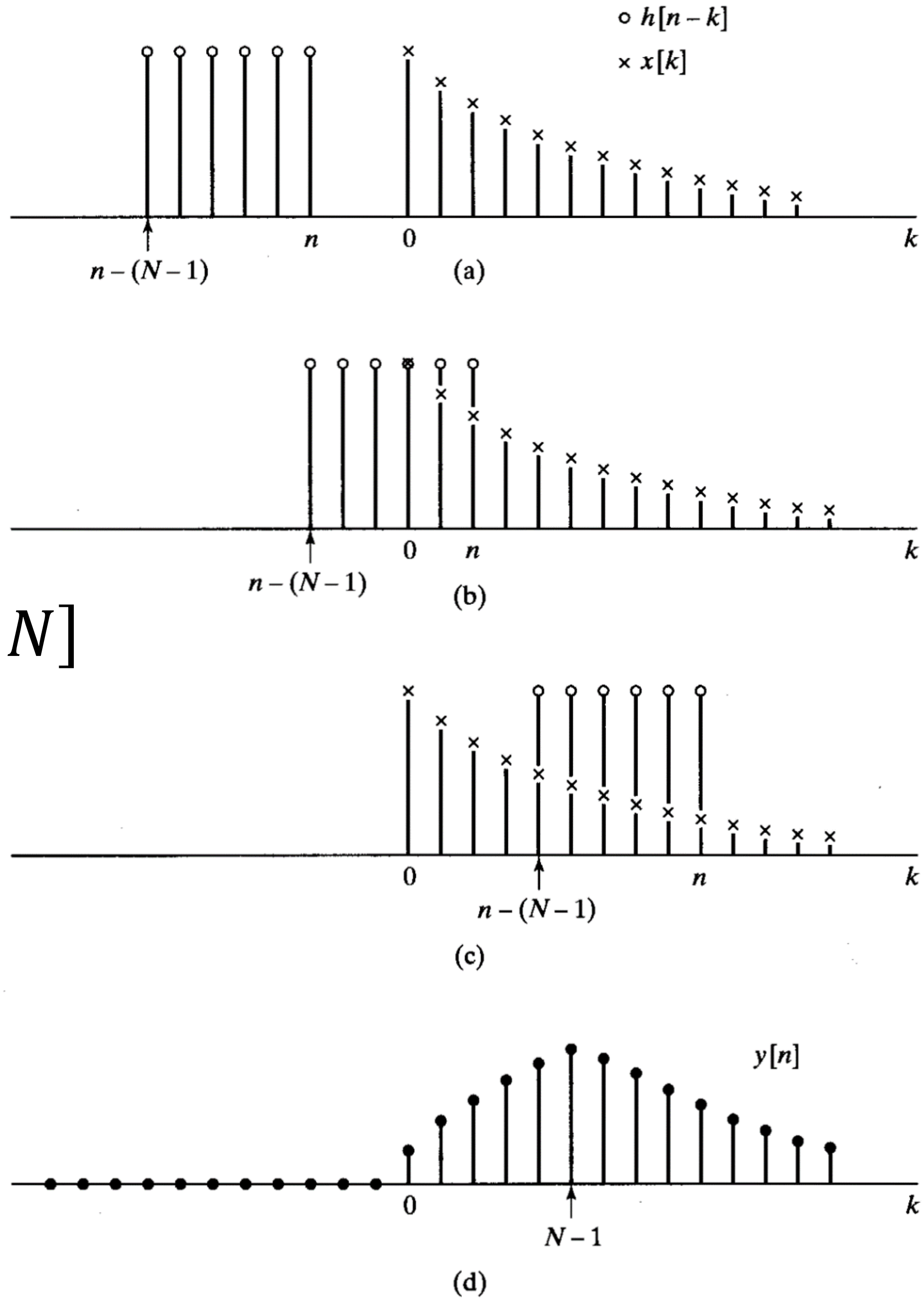
Each sample of the output sequence is obtained as the multiplication of the input sequence with the delayed impulse response

Convolution

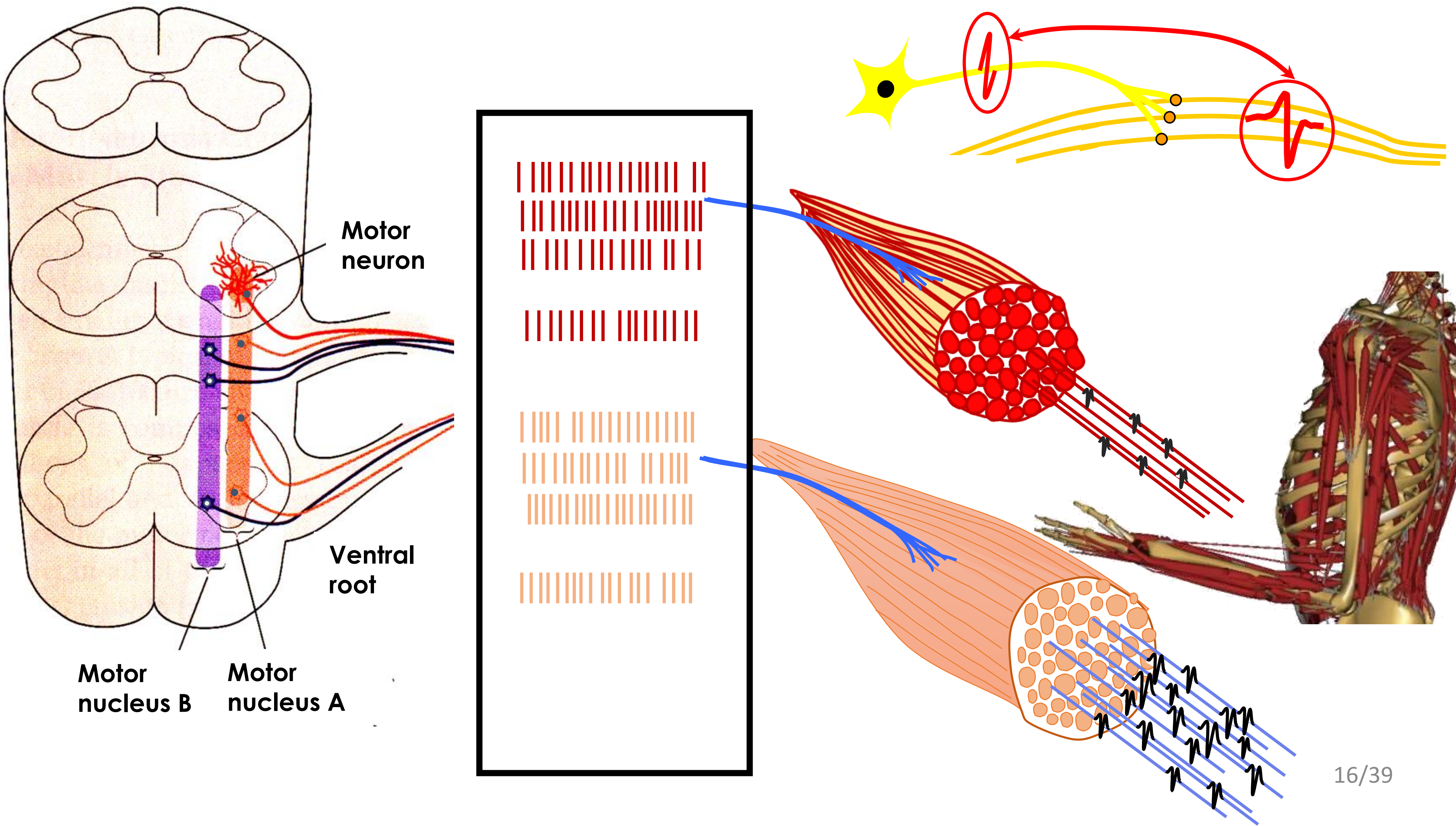
Example:

$$h[n] = u[n] - u[n-N]$$
$$= \begin{cases} 1, 0 \le n \le N-1 \\ 0, \quad\quad \text{otherwise} \end{cases}$$

$$x[n] = a^n u[n]$$



Example: EMG signal generation model

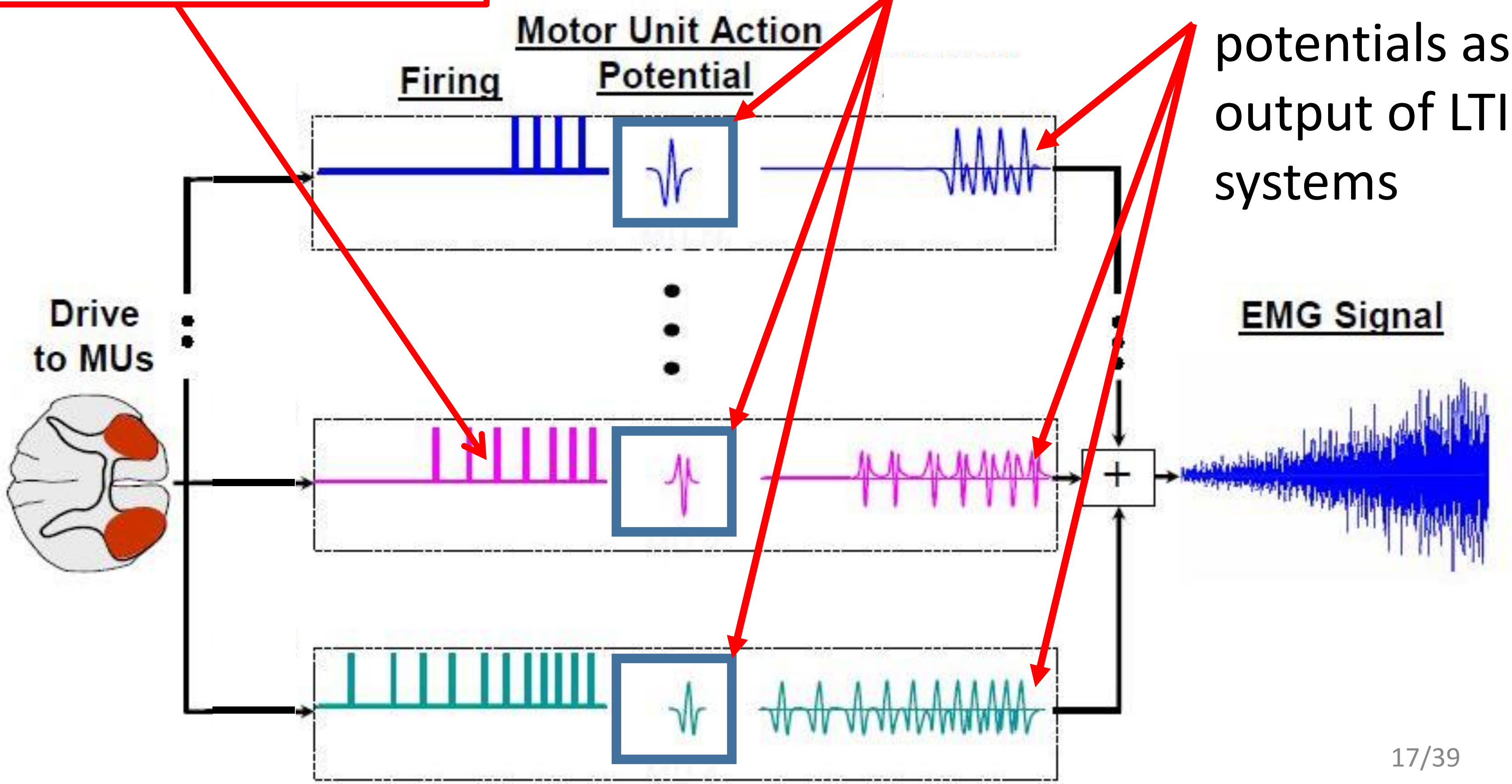


Generation model of the EMG signal with LTI systems

$$p_i[k] = \sum_{r=-\infty}^{\infty} \delta[k - rT_i]$$

LTI systems

Trains of motor unit action potentials as output of LTI systems



Generation model of the EMG signal with LTI systems

The model can be described with discrete-time convolutions. For a single motor neuron activity and trains of muscle fibres action potentials:

$$g_i[n] = \sum_{k=-\infty}^{\infty} p_i[k] h_i[n-k]$$

The input $p_i[k]$ is the sequence representing the impulses of activation of the motor neuron i. Assuming perfectly periodic discharges by the motor neuron:

$$p_i[k] = \sum_{r=-\infty}^{\infty} \delta[k - rT_i]$$

Thus:

$$g_i[n] = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \delta[k - rT_i]\, h_i[n-k]$$

$\boldsymbol{g_i[n]}$ **is a single motor unit action potential train**. It corresponds to the muscle electrical activity elicited by a single motor neuron in the spinal cord

Generation model of the EMG signal with LTI systems

From the single motor unit action potential trains:

$$g_i[n] = \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \delta[k - rT_i]\, h_i[n-k]$$

The EMG signal $s[n]$ is obtained as the sum of the activity of M muscle units driven by M motor neurons:

$$s[n] = \sum_{i=1}^{M} g_i[n] =$$

$$= \sum_{i=1}^{M} \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \delta[k - rT_i]\, h_i[n-k]$$

Use of the model of EMG generation

From the model we just built:

$$s[n] = \sum_{i=1}^{M} \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \delta[k - rT_i]\, h_i[n-k]$$

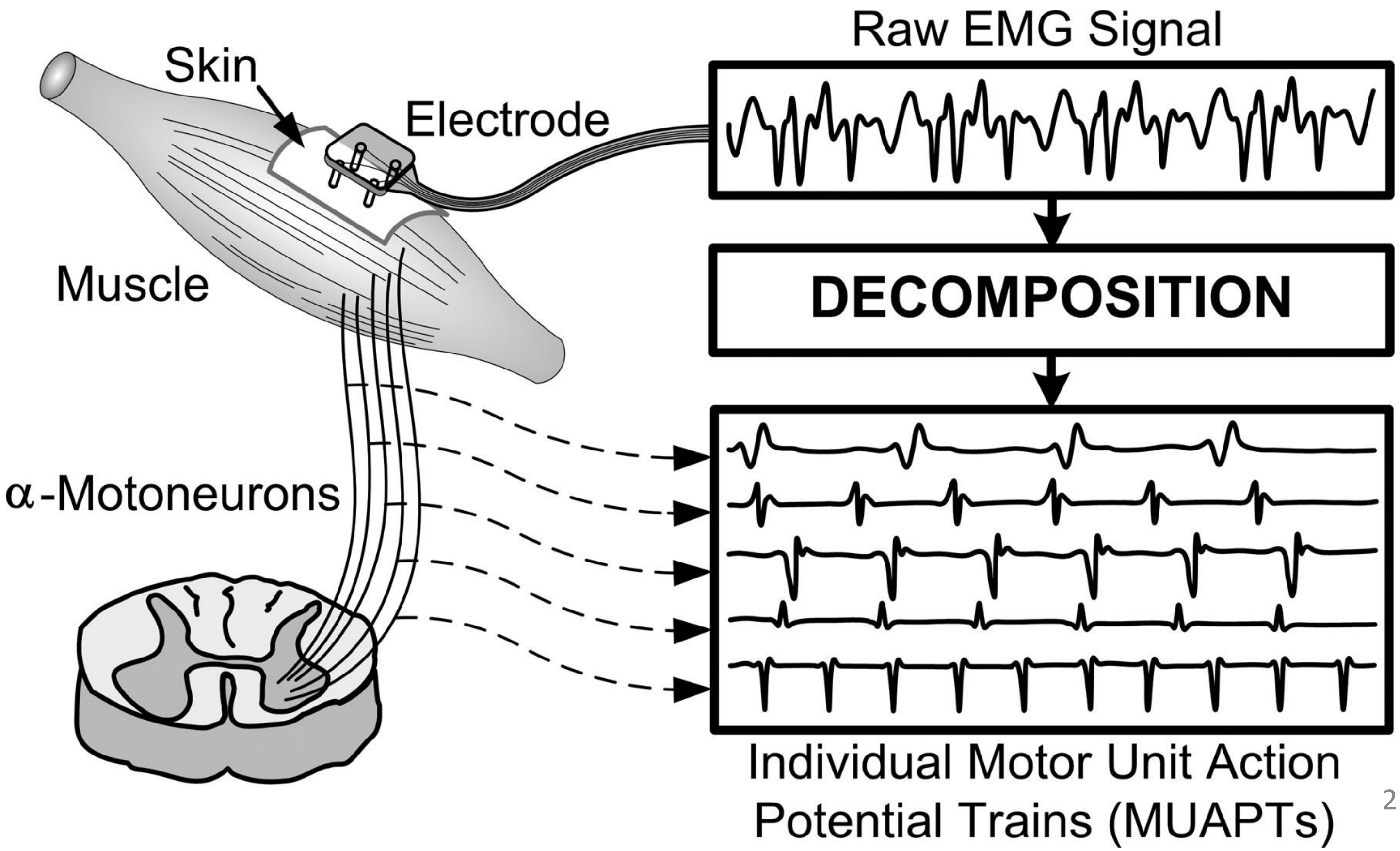
- We can develop processing methods for extracting physiologically-relevant information (see envelope estimate)
- We can determine the characteristics of the signal, given the underlying physiological mechanisms
- We can also develop methods to directly invert the signal generation model and estimate the neural activation from the EMG

Inverse modeling

Inverting the model:

$$s[n] = \sum_{i=1}^{M} \sum_{k=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} \delta[k - rT_i]\, h_i[n-k]$$

EMG decomposition



Properties of LTI systems

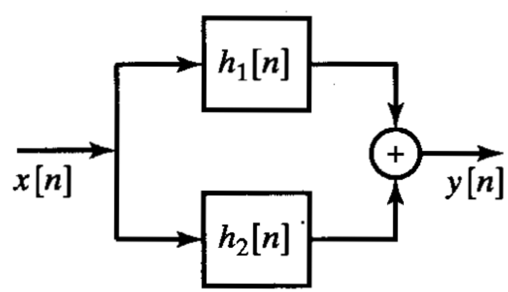
The LTI systems are fully characterized by the convolution operator, thus their properties are those of this operator

Commutative property:

$$x[n] * h[n] = h[n] * x[n]$$

Distributive property:

$$x[n] * (h_1[n] + h_2[n]) =$$
$$= x[n] * h_1[n] + x[n] * h_2[n]$$



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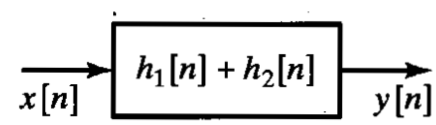

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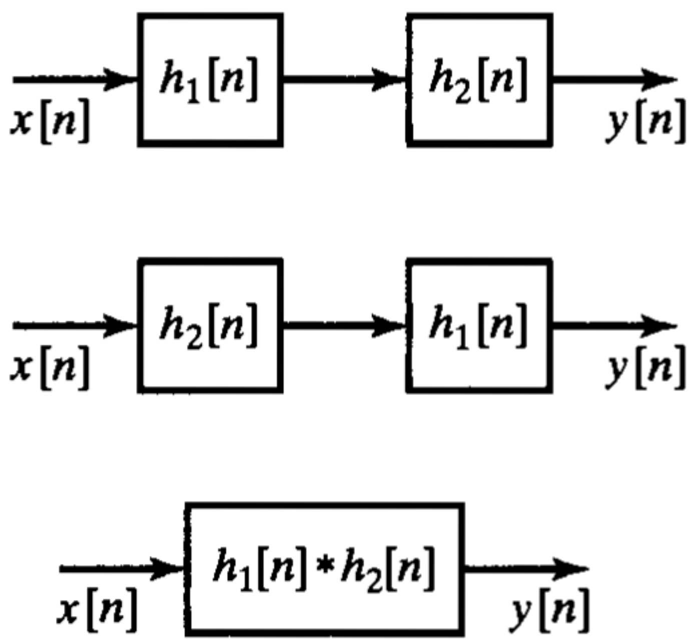
Properties of LTI systems

Associative property:

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]) =$$
$$= (x[n] * h_2[n]) * h_1[n]$$



Properties of LTI systems

Stability of LTI systems:

An LTI system is stable if and only if its impulse response is absolutely summable:

$$B_y = \sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

Causality of LTI systems:

An LTI system is causal if and only if:

$$h[n] = 0, \qquad n < 0$$

Properties of LTI systems

Example (moving average filter):

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]$$

Stability ($B_y = \sum_{k=-\infty}^{+\infty} |h[n]| < \infty$): $\sum_{k=-\infty}^{+\infty} |h[n]|$=1$< \infty$ (stable system)

The moving average filter has finite-duration impulse response (FIR). ***Any FIR system is stable***.

Causality ($h[n] = 0, n < 0$): The moving average filter is causal if $M_2 \geq 0$ and $-M_1 \geq 0$

Frequency response of LTI systems

Let's consider the following input sequence:

$$x[n] = e^{j\omega n}, \qquad -\infty < n < \infty$$

The output of an LTI system with this input is:

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} e^{j\omega k} h[n-k] = e^{j\omega n} \sum_{k=-\infty}^{+\infty} h[k]e^{-j\omega k}$$

$$= x[n]H(e^{j\omega})$$

Thus, **the sequences $e^{j\omega n}$ are the eigenfunctions of LTI systems**

$H(e^{j\omega})$ is the frequency response of the system

Frequency response of LTI systems

$H(e^{j\omega})$ is periodic in ω:

$$H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{+\infty} h[k]e^{-j\omega k}$$

$$H\left(e^{j(\omega+r2\pi}\right) = \sum_{r=-\infty}^{+\infty} h[k]e^{-j(\omega+r2\pi)k} =$$
$$= \sum_{r=-\infty}^{+\infty} h[k]e^{-j\omega k}e^{-jr2\pi k} = H\left(e^{j\omega}\right)$$

All frequency responses of discrete-time LTI systems are periodic

Frequency response of LTI systems

If we now consider a sinusoidal input at frequency ω_0 :

$$x[n] = A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi}\ e^{-j\omega_0 n}$$

$$y[n] = T\{x[n]\} = \frac{A}{2} e^{j\phi} T\left\{e^{j\omega_0 n}\right\} + \frac{A}{2} e^{-j\phi} T\left\{e^{-j\omega_0 n}\right\} =$$

$$= \frac{A}{2} e^{j\phi} e^{j\omega_0 n} H\left(e^{j\omega_0}\right) + \frac{A}{2} e^{-j\phi} e^{j\omega_0 n} H\left(e^{-j\omega_0}\right)$$

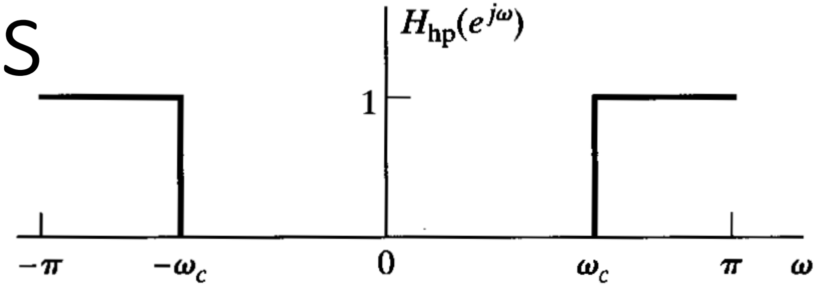
If $h[n]$ is real, then $H\left(e^{-j\omega_0}\right) = H^*\left(e^{j\omega_0}\right)$ and:

$$y[n] = A\left|H\left(e^{j\omega_0}\right)\right| \cos\left(\omega_0 n + \phi + \angle H\left(e^{-j\omega_0}\right)\right)$$

Sinusoidal sequences are modified only in amplitude and phase when processed by an LTI system

Frequency response of LTI systems

Example of frequency responses of ideal low-pass, high-pass, and band-pass LTI systems



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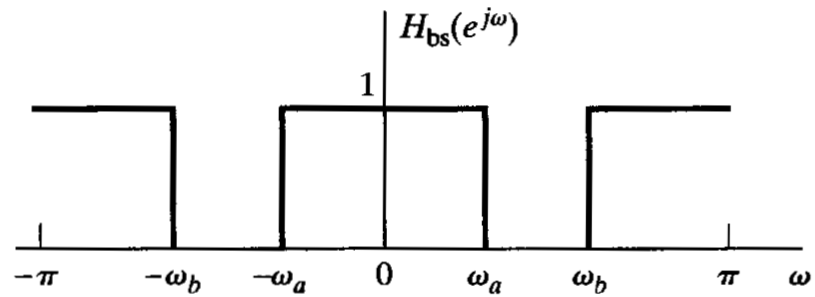

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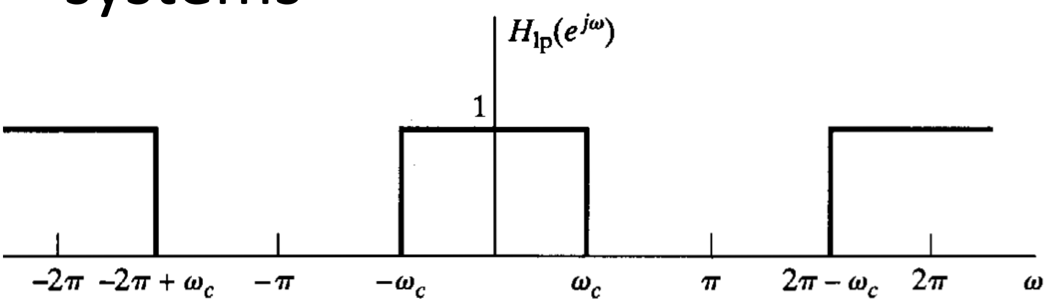

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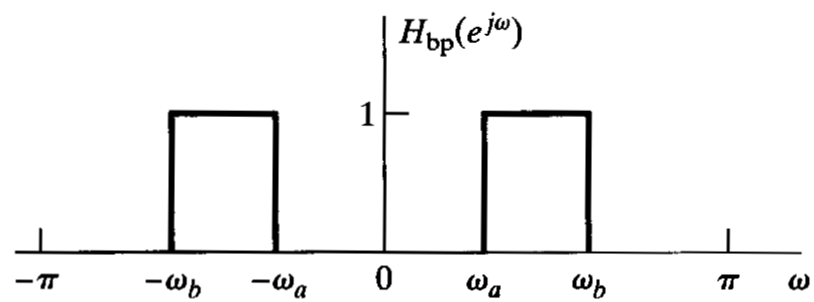
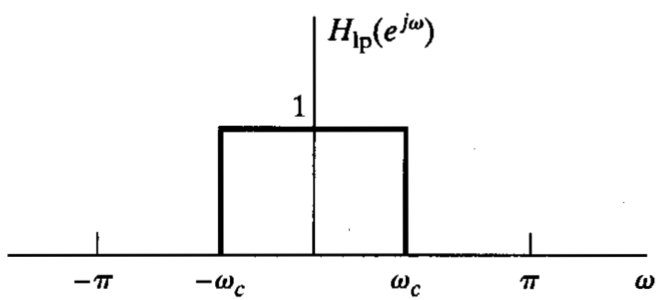




Frequency response of the moving average system

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k] =$$

$$= \begin{cases} \frac{1}{M_1 + M_2 + 1}, & -M_1 \le n \le M_2 \\ 0, & \text{otherwise} \end{cases}$$

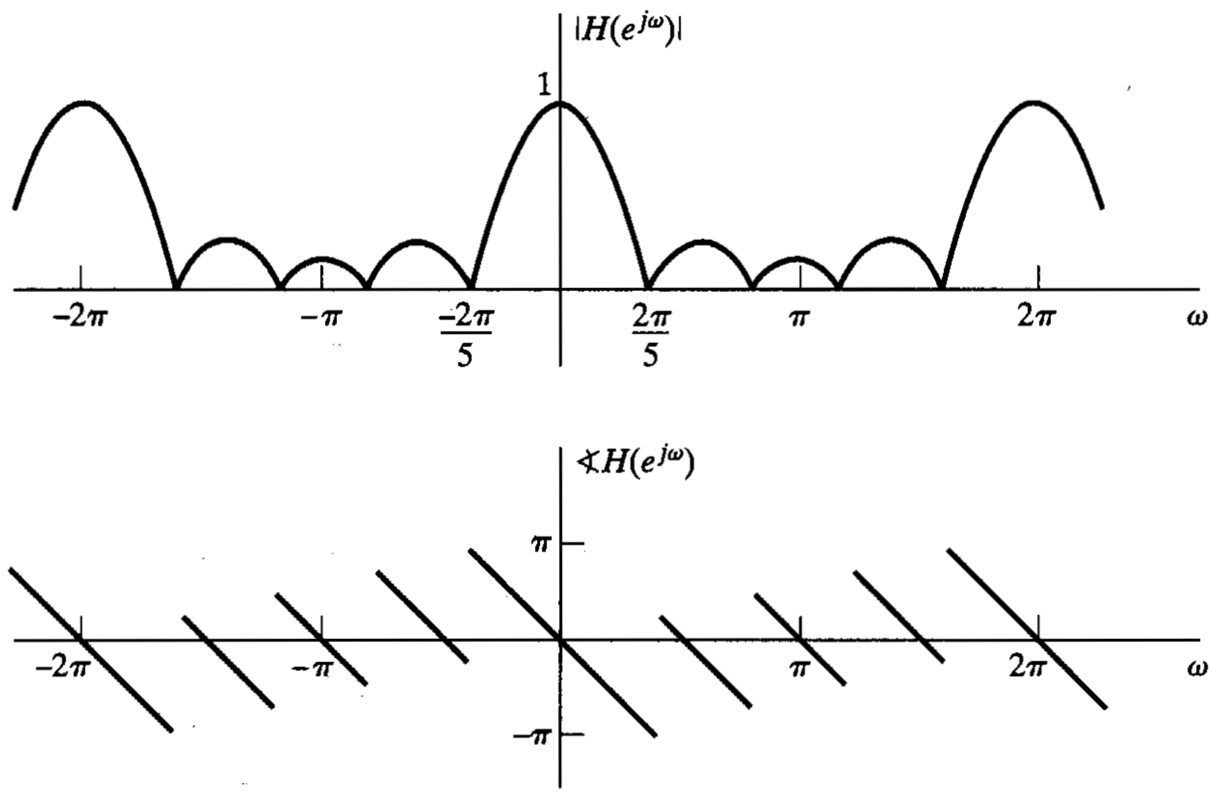
Let's consider the causal case with $M_1 = 0$:

$$H\left(e^{j\omega}\right) = \frac{1}{M_2+1} \sum_{k=0}^{M_2} e^{-j\omega k} = \frac{1}{M_2+1} \left(\frac{1-e^{-j\omega(M_2+1)}}{1-e^{-j\omega}} \right) =$$

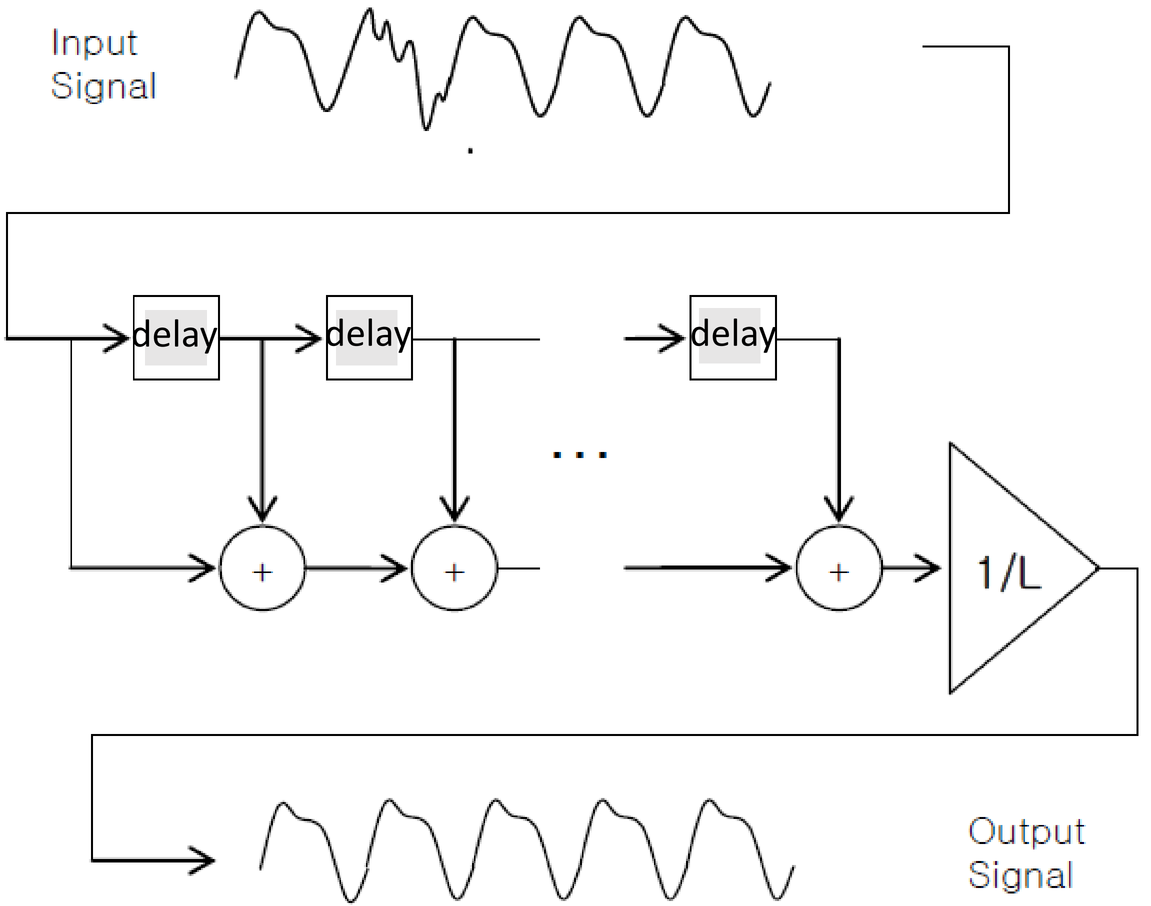
$$= \frac{1}{M_2 + 1} \frac{\sin[\omega(M_2+1)/2]}{\sin(\omega/2)} e^{-j\omega M_2/2}$$

Frequency response of the moving average system

$$H\left(e^{j\omega}\right) = \frac{1}{M_2+1}\frac{\sin[\omega(M_2+1)/2]}{\sin(\omega/2)}e^{-j\omega M_2/2}$$

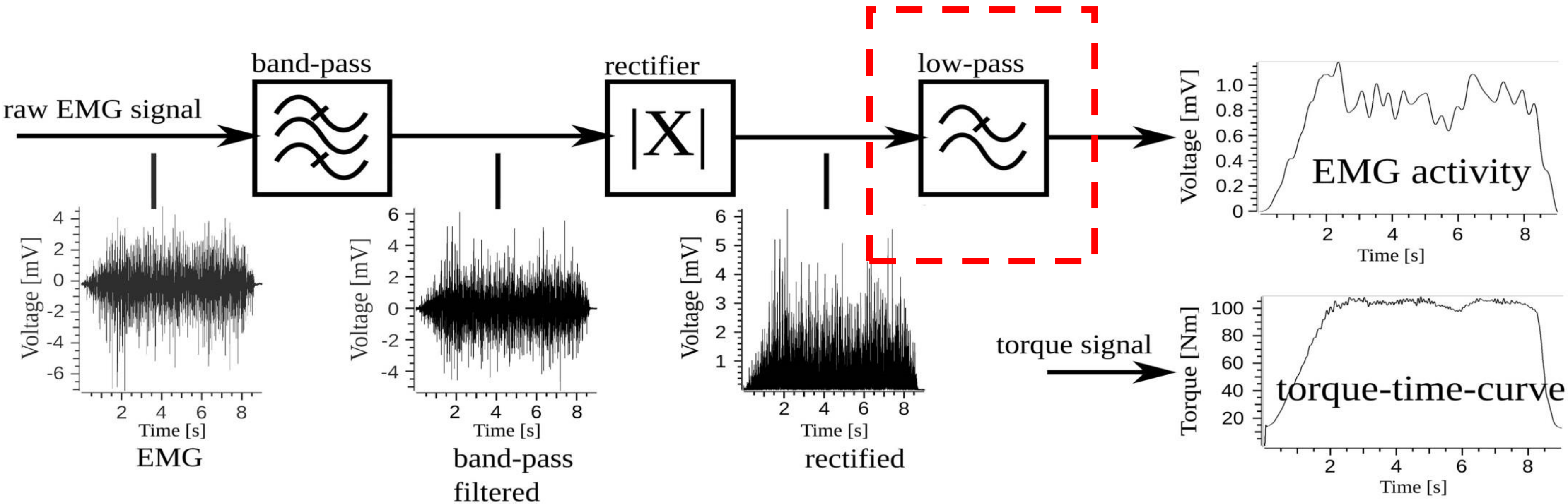


Example: Simple filtering for Photoplethysmography

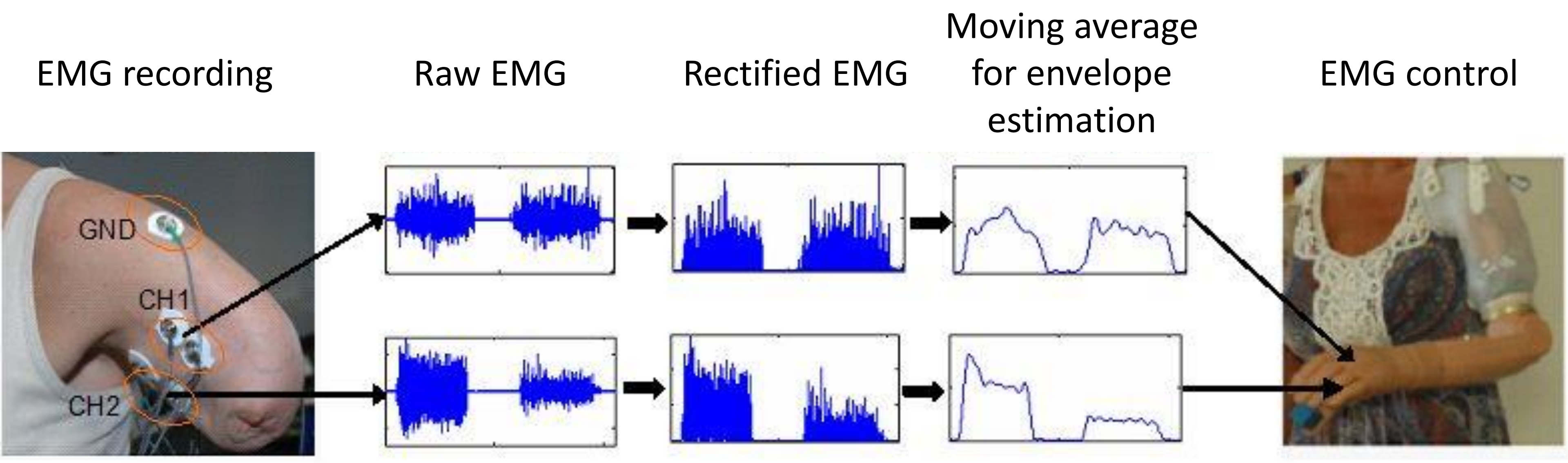


Extraction of EMG envelope

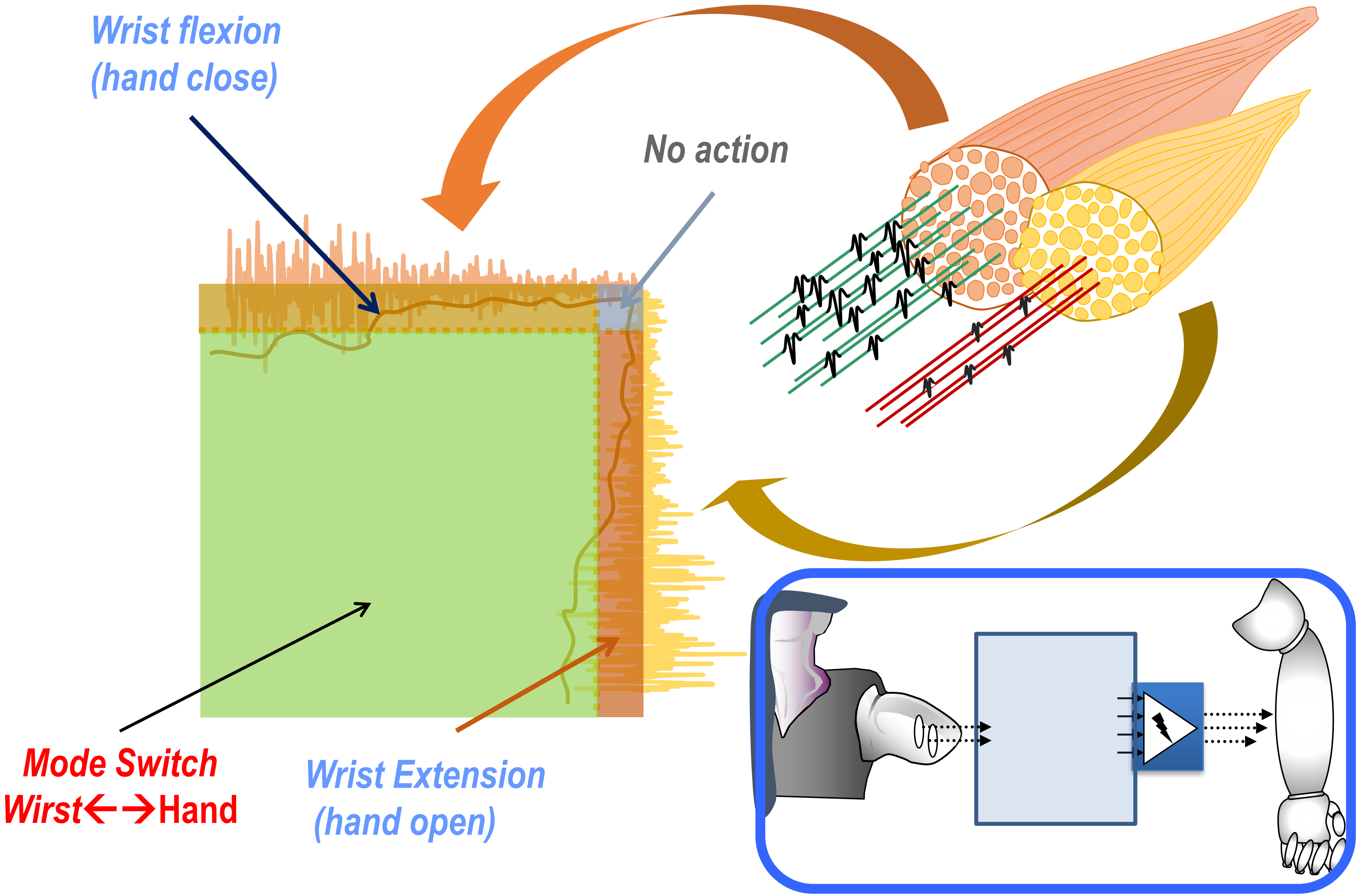
Smoothing rectified EMG by moving average filtering



Extraction of EMG envelope for prosthesis control

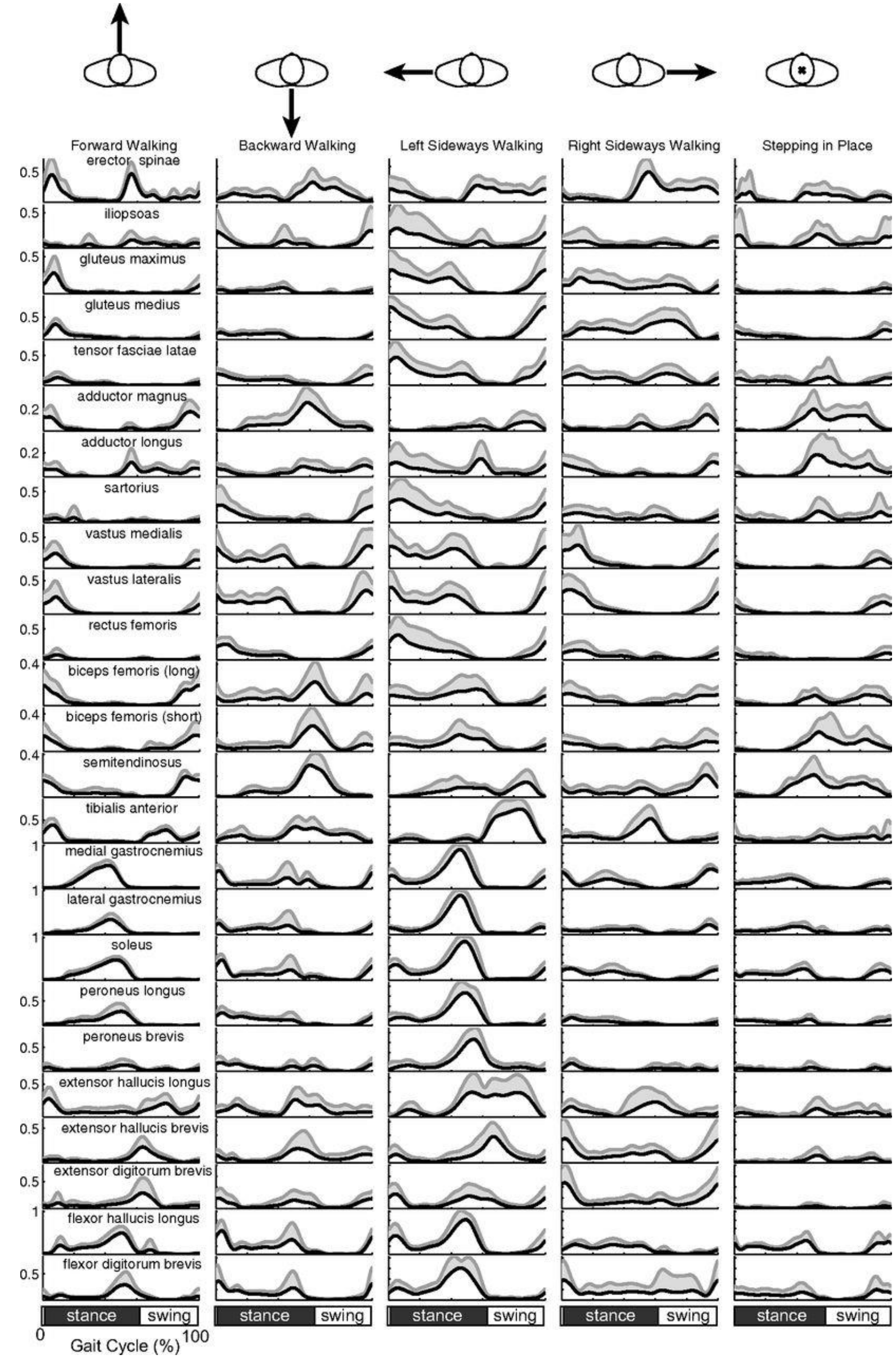


Myoelectric control of prostheses



EMG envelope in gait analysis

Activity of muscles during walking estimated as the envelope of the EMG



Summary

- Discrete-time signals are sequences of numbers (data streams) indexed by an integer representing the discrete time
- Discrete-time signals may or may not derive from sampling of continuous-time signals
- Discrete-time systems are operators defined on input discrete-time sequences
- Special properties of discrete-time systems are linearity and time-invariance (LTI systems) which, together, allow to describe the effect of the system on any input as a convolution by the impulse response
- The complex exponential are the eigenfunctions of LTI systems

Selected book exercises

Selected exercises Lecture 2:

2.1, 2.2, 2.6, 2.7, 2.11, 2.13, 2.18, 2.19, 2.20 (pp. 70-74)

Next week

We will introduce the discrete-time Fourier transform (DTFT) as the Fourier transform of sequences

We will introduce the discrete Fourier transform (DFT) as a Fourier transform that is itself a sequence

We will introduce the concept of multi-rate processing