Question 3

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1 Answer

1.1 Naive Bayes as linear classifier

A multinomial naive bayes is a linear classifier under certain assumptions:

$$P(\mathbf{x}|C_k) = \frac{(\sum_i x_i)!}{\prod_i x_i!} \prod_i p_{k_i}^{x_i}$$

The multinomial naive Bayes classifier becomes a linear classifier when expressed in log-space:

$$\begin{aligned} P(C_k|\mathbf{x}) \propto log\left(p(C_k)\prod_{i=1}^{n} p(k_i)^{x_i}\right) \\ = logp(C_k) + \sum_{i=1}^{n} x_i.log(p(k_i)) \\ = b + \mathbf{w_k^T x} \end{aligned}$$

where $b = logp(C_k)$ and $w_{ki} = logp(k_i)$

If we look at this much more carefully with a binary class classification problem of gaussian naive bayes.

$$
\begin{aligned}
P(Y=1|X) &= \frac{P(Y=1)P(X|Y=1)}{P(Y=1)P(X|Y=1)+P(Y=0)P(X|Y=0)} \\
&= \frac{1}{1+\frac{P(Y=0)P(X|Y=0}{P(Y=1)P(X|Y=1}} \\
&= \frac{1}{1+\exp(\log\frac{P(Y=0)P(X|Y=0}{P(Y=1)P(X|Y=1})} \\
P(Y=1) &= \theta \\
P(X_j|Y=1) &= \frac{1}{\sigma_j\sqrt{2\pi}}\exp\left\{\frac{-(X_j-\mu_{j1})^2}{2\sigma_j^2}\right\} \\
P(X_j|Y=0) &= \frac{1}{\sigma_j\sqrt{2\pi}}\exp\left\{\frac{-(X_j-\mu_{j0})^2}{2\sigma_j^2}\right\} \\
&\log\frac{P(Y=0)P(X|Y=0)}{P(Y=1)P(X|Y=1)} \\
&= \log P(Y=0)+\log P(X|Y=0)-\log P(Y=1)-\log P(X|Y=1) \\
&= \log\frac{P(Y=0)}{P(Y=1)}+\sum_j\log\frac{P(X_j|Y=0)}{P(X_j|Y=1)} \\
&= \log\frac{1-\theta}{\theta}+\sum_j\frac{(\mu_{j0}-\mu_{j1})X_j}{\sigma_j^2}+\frac{\mu_{j1}^2-\mu_{j0}^2}{2\sigma_j^2} \\
&= b+\mathbf{w}^\top X
\end{aligned}
$$

The boundary is linear only when the covariance is shared between the class. When the covariance is not shared the decision boundary is quadratic function. It looks like an ellipse.

1.2 logistic regression as linear classifier

$$P(Y=0|X) = \frac{1}{1+\exp(w_0+\sum_i w_i X_i)}$$

$$P(Y=1|X) = 1-P(Y=0|X) = \frac{\exp(w_0+\sum_i w_i X_i)}{1+\exp(w_0+\sum_i w_i X_i)}$$

Prediction is 1 if:

$$P(Y=1|X) > P(Y=0|X)$$

Taking log and writing this

$$log(\frac{P(Y=1|X)}{P(Y=0|X)}) > 1$$

If we expand this using the first equation

$$log(\exp(w_0 + \sum_i w_i X_i)) - log(1 + \exp(w_0 + \sum_i w_i X_i)) - log(1) + log(1 + \exp(w_0 + \sum_i w_i X_i)) > 0$$

$$w_0 + \sum_i w_i X_i > 0$$

We see that this is equivalent to the linear boundary.