Notes on Schemes

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The point of this document is to discuss some basic results and properties of schemes, sheaves, and other related concepts of algebraic geometry. The discussion is rigorous, but we will not define standard concepts like schemes, $\mathcal{O}_X$-modules, etc. A good example of what these notes set out to explain are lemmas like the 'affine communication lemma', 'qcqs lemma', and other similar concepts. Hopefully they will eventually include more advanced concepts like sites, Grothendieck topologies, stacks, descent, sheaf cohomology, etc.

1 Schemes

1.1 Distinguished Opens

Let A be a ring and consider its spectrum $\operatorname{Spec}(A)$. We write $D(f)$ or $D_A(f) \subset \operatorname{Spec}(A)$ for the set

$$D_A(f) \coloneqq \operatorname{Spec}(A) \setminus V_A(f) = \{\mathfrak{p} \in \operatorname{Spec}(A) : f \notin \mathfrak{p}\},$$

which is by definition open. $D_A(f)$ is called the associated 'distinguished open' set of f. Some easy properties are $D_A(f) \cap D_A(g) = D_A(fg)$ and $D_A(f) \cup D_A(g) = D_A(f, g)$ (i.e. we can also define it for ideals). We have a canonical identification

$$D_A(f) = \operatorname{Spec}(A[f^{-1}]) \coloneqq \operatorname{Spec}(A_f).$$

Recall that the sets $D_A(f)$ form a basis of quasicompact, quasiseparated opens for the spectrum of A, since they are the spectrum of a ring. We think of $D_A(f)$ as the set on which f is nonzero, because:

$$\begin{aligned} D_A(f) &= \{\mathfrak{p} \in \operatorname{Spec}(A) : f \neq 0 \text{ in } A/\mathfrak{p}\} && (1)\\ &= \{\mathfrak{p} \in \operatorname{Spec}(A) : f \neq 0 \text{ in } \kappa(\mathfrak{p})\}, && (2) \end{aligned}$$

where $\kappa(\mathfrak{p}) \coloneqq \operatorname{Frac}(A/\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A\mathfrak{p}$ is the residue field at $\mathfrak{p}$.

Let X be a scheme, and suppose $\{U_i = \operatorname{Spec}(A_i)\}$ is an open cover by affine opens. Suppose $U \subset X$ is an arbitrary open subset, then $U \cap \operatorname{Spec}(A_i)$ is open in $\operatorname{Spec}(A_i)$, so we can write $U \cap \operatorname{Spec}(A_i) = \bigcup_j D_{A_i}(f_{ij})$, so

$$U = \bigcup_{i,j} D_{A_i}(f_{ij}) = \bigcup_{i,j} \operatorname{Spec}((A_i)_{f_{ij}}).$$

It follows that X has a basis of affine opens. Inspired by Eq. (2), we can make a similar definition for schemes: If X is a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ is a global section, then for each x we write $f_x \in \mathcal{O}_{X,x}$ for the corresponding germ in the stalk, and also $f_x \in \kappa(x)$ for the corresponding element of the residue field. We will set

$$X_f \coloneqq \{x \in X : f_x \neq 0 \text{ in } \kappa(x)\}.$$

X_f is the 'distinguished open' subset of X, and we interpret it as the set of points where f is nonzero, i.e. f is invertible.

Remark 1.1 If $(X, \mathcal{O}_X)$ is only a ringed space, then for a global section $f \in \Gamma(X, \mathcal{O}_X)$ we define X_f to be $\{x \in X : f_x \in \mathcal{O}_{X,x}^\times\}$, which clearly agrees with the notion for schemes.

In what follows, we write a global section $f \in \Gamma(X, \mathcal{O}_X)$ as 'equal to' its restriction $f|_U$ to an open subscheme. The distinguished open satisfies the following properties:

Proposition 1.2 *Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ a global section.*

1. *If $X = \operatorname{Spec}(A)$ is affine, then $X_f = D_A(f)$.*
2. *If $U \subset X$ is an open subscheme, then $U_f = X_f \cap U$.*
3. *$X_f \subset X$ is an open subset.*
4. *If $g \in \Gamma(X, \mathcal{O}_X)$, then $(X_f)_g = X_{fg}$.*

Proof (1) is trivial by Eq. (2) and the definition. For (2), we have

$$U_f = \{x \in U : f_x \in \mathcal{O}_{U,x}^\times\} = \{x \in X : f_x \in \mathcal{O}_{X,x}^\times\} \cap U = X_f \cap U,$$

so it is because the stalk is unchanged by restricting to a smaller open subset. Next, let $\{U_i\}$ be an affine open cover. By (1), $X_f \cap U_i = U_{i,f}$ is $D(f|_{U_i}) \subset U_i$, which is open. Hence

$$X_f = \bigcup\nolimits_i X_f \cap U_i = \bigcup\nolimits_i D(f|_{U_i})$$

is an open subset. Finally, for (4) notice that $(X_f)_g = X_f \cap X_g$, so it follows from the fact that $f_x g_x \neq 0$ in $\kappa(x)$ if and only if $f_x, g_x \neq 0$. □

Corollary 1.3 (Simultaneous distinguished opens) *Let X be a scheme and $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ two affine opens. Then $U \cap V$ has an open cover of the form*

$$U \cap V = \bigcup\nolimits_i W_i$$

with the following property: For each i, there are $f_i \in A$ and $g_i \in B$ such that

$$W_i = D_A(f_i) = D_B(g_i).$$

In other words, we can cover $U \cap V$ by simultaneous distinguished opens.

Proof Assume U and V meet and pick $x \in U \cap V$. Because distinguished opens form a basis for affine schemes, there is $f \in A$ such that $x \in U_f \subset U \cap V$. Similarly since $U_f \subset V$ is open, there is $g \in B$ such that $x \in V_g \subset U_f$.

First, consider $g \in \Gamma(U_f, \mathcal{O}_X) = A_f$, so we can write $g|_{U_f} = g'/f^n$. We claim:

$$(U_f)_g = \{\mathfrak{p} : g'/f^n \notin \mathfrak{p}\} = \{\mathfrak{p} : g' \notin \mathfrak{p}\} = (U_f)_{g'} = U_{fg'}.$$

Indeed, the second equality is because f^{-n} is a unit in A_f, and then last is by Prop. 1.2(4). On the other hand, by Prop. 1.2(2),

$$V_g = U_f \cap V_g = (U_f)_g = U_{fg'},$$

so we have found a neighborhood which is equal to $D_B(g)$ and $D_A(fg')$. □

Our next goal is to study subsets of the form X_f in general. We will show, eventually, that

$$\Gamma(X, \mathcal{O}_X)_f = \Gamma(X_f, \mathcal{O}_X)$$

for quasicompact and quasiseparated schemes. This is done in a few intermediate steps. In what follows we will use Prop. 1.2 implicitly, especially that $X_f \subset X$ is open.

Proposition 1.4 *Let X be a scheme and $f \in \Gamma(X, \mathcal{O}_X)$ a global section. The image of f under the restriction map*

$$\Gamma(X, \mathcal{O}_X) \to \Gamma(X_f, \mathcal{O}_X)$$

is a unit.

Proof Cover X_f by affine opens $U_i = \mathrm{Spec}(A_i)$. Then

$$U_i = X_f \cap U_i = \mathrm{Spec}(A_i[f^{-1}]),$$

so the restriction of f to U_i is invertible, with inverse $f_i^{-1} \in \Gamma(U_i, \mathcal{O}_X)$. Then both $f_i^{-1}|_{U_i \cap U_j}$ and $f_j^{-1}|_{U_i \cap U_j}$ are the inverse to $f|_{U_i \cap U_j}$, so they coincide. By the sheaf condition there is $f^{-1} \in \Gamma(X_f, \mathcal{O}_X)$ which restricts to the f_i^{-1}. It is an inverse to f since ff^{-1} agrees with the section 1 on the cover. □

Therefore, there is an induced map

$$\Gamma(X, \mathcal{O}_X)_f \to \Gamma(X_f, \mathcal{O}_X) \tag{$*$}$$

factoring through the restriction map.

Lemma 1.5 *Let X be quasicompact. Then the ring map* $(*)$ *is injective.*

Proof Equivalently, we must show that if $s|_{X_f} = 0$, then there is an $n > 0$ for which $f^n s = 0$. Let $\{U_i = \mathrm{Spec}(A_i)\}$ be a finite open cover of X, then $X_f \cap U_i = \mathrm{Spec}(A_i[f^{-1}])$, and the restriction of s to $A_i[f^{-1}]$ is zero. So there is n_i such that $sf^{n_i} = 0$ in $A_i = \Gamma(U_i, \mathcal{O}_X)$. By finiteness, take $n = \max_i n_i$, so $sf^n = 0$ over U_i for each i, hence $sf^n = 0$ over X. □

Theorem 1.6 (qcqs Lemma) *Let X be a quasicompact and quasiseparated scheme. Then the natural map*

$$\Gamma(X, \mathcal{O}_X)_f \to \Gamma(X_f, \mathcal{O}_X)$$

is an isomorphism.

Proof By Lemma 1.5, we need to show the map is onto. Suppose g is in the codomain and take a finite cover $\{U_i = \mathrm{Spec}(A_i)\}$. The restriction of g to $U_i \cap X_f = U_{i,f} = A_i[f^{-1}]$ is of the form s_i/f^{n_i}. So, by finiteness, there are $s_i \in A_i$ and n such that $s_i = f^n g$ in $X_f \cap U_i$ for all i.

Now write $U_{ij} = U_i \cap U_j$, which is quasicompact by assumption, hence

$$\Gamma(U_{ij}, \mathcal{O}_X)_f \to \Gamma(U_{ij,f}, \mathcal{O}_X)$$

is injective by Lemma 1.5. We know that s_i and s_j coincide in $U_{ij,f} = U_i \cap U_j \cap X_f$, so there is N_{ij} such that

$$s_i f^{N_{ij}} = s_j f^{N_{ij}} \quad \text{over } U_{ij}.$$

Letting N be the max, set $t_i = s_i f^N$ so that

$$t_i|_{U_{ij}} = s_i f^{N_{ij}} f^{N-N_{ij}} = s_j f^{N_{ij}} f^{N-N_{ij}} = t_j|_{U_{ij}}.$$

By the sheaf condition there is $t \in \Gamma(X, \mathcal{O}_X)$ such that t restricts to the t_i. Then the image of t/f^{N+n} in $\Gamma(X_f \cap U_i, \mathcal{O}_X)$ is $s_i f^{-n} = g$ for all i, hence the image in $\Gamma(X_f, \mathcal{O}_X)$ is g. $\square$

The somewhat long proof of the qcqs lemma is a good demonstration of using the 'explicit' version of the sheaf condition. We provide the following alternative proof too:

Proof 2 of Thm. 1.6 Let $\{U_i = \mathrm{Spec}(A_i)\}$ be a finite affine open cover of X, and let $\{U_{ijk} = \mathrm{Spec}(A_{ijk})\}$ be a finite affine open cover of $U_i \cap U_j$. We can view the sheaf condition as the following exact sequence of $\Gamma(X, \mathcal{O}_X)$-modules,

$$0 \to \Gamma(X, \mathcal{O}_X) \to \bigoplus_i A_i \to \bigoplus_{i,j} \Gamma(U_i \cap U_j, \mathcal{O}_X) \tag{$*$}$$

where we may use direct sums by finiteness. Also by the sheaf condition we have injections

$$0 \to \Gamma(U_i \cap U_j) \to \bigoplus_k A_{ijk}$$

for each k, so combining these and localizing at f we get

$$0 \to \Gamma(X, \mathcal{O}_X)_f \to \bigoplus_i A_{i,f} \to \bigoplus_{i,j,k} A_{ijk,f}$$

an exact sequence of $\Gamma(X, \mathcal{O}_X)_f$-modules. But $A_{i,f} = \Gamma(U_i \cap X_f, \mathcal{O}_X)$, and $A_{ijk,f} = \Gamma(U_{ijk} \cap X_f, \mathcal{O}_X)$, so the sheaf condition for X_f shows that we have an identification $\Gamma(X, \mathcal{O}_X)_f = \Gamma(X_f, \mathcal{O}_X)$. $\square$

We will end this section with a useful criterion for affine-ness that depends on the qcqs lemma. We will first explicitly construct the 'canonical map'

$$\eta\colon X \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$$

for any scheme X, which is used in the proof. Consider the open cover $\mathcal{U}$ of all affine opens of X, then for each $U = \operatorname{Spec}(A) \in \mathcal{U}$ we set

$$\eta_U = \operatorname{Spec}(\mathrm{res}_U)\colon \operatorname{Spec}(A) \to \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$$

where res_U is the restriction map $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$. We show by Cor. 2.8 that this glues to one morphism η: If $U_i = \operatorname{Spec}(A_i) \in \mathcal{U}$ for $i = 0, 1$, we need to show that

$$\eta_{U_0}|_{U_0 \cap U_1} = \eta_{U_1}|_{U_0 \cap U_1},$$

and by taking an affine open cover of $U_0 \cap U_1$ (any applying Cor. 2.8 again), it suffices to show that if $V = \operatorname{Spec}(A) \subset U_0 \cap U_1$ is affine, then the square

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, hook] & U_1 \arrow[d, hook] \\ U_0 \arrow[r, hook] & \operatorname{Spec}(\Gamma(X, \mathcal{O}_X)) \end{tikzcd}$$

commutes. This follows from functoriality and the fact that

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d, hook] & U_1 \arrow[d, hook] \\ U_0 \arrow[r, hook] & X \end{tikzcd} \qquad \begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r] \arrow[d] & A_1 \arrow[d] \\ A_0 \arrow[r] & A \end{tikzcd}$$

both commute.

Now that we have constructed η, let f be a global section, and consider the open subset $D(f) \subset \operatorname{Spec}(\Gamma(X, \mathcal{O}_X))$. We compute its preimage under η. First, if $U = \operatorname{Spec}(A)$ is affine, then

$$\begin{aligned} \eta_U^{-1}(D(f)) &= \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathrm{res}_U^{-1}(\mathfrak{p}) \in D(f)\} \\ &= \{\mathfrak{p} \in \operatorname{Spec}(A) : \mathrm{res}(f) \notin \mathfrak{p}\} \\ &= D_A(f|_U). \end{aligned}$$

So, if $\{U_i = \operatorname{Spec}(A_i)\}$ is an affine open cover, then

$$\eta^{-1}(D(f)) = \bigcup_i \eta^{-1}(D(f)) \cap U_i = \bigcup_i \eta_{U_i}^{-1}(D(f)) = \bigcup_i D_{A_i}(f|_{U_i}) = X_f.$$

The use of this computation is the following criterion.

Proposition 1.7 *Let X be a scheme. X is affine if and only if there are global sections $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ generating the unit ideal such that X_{f_i} is affine for $i = 1, \ldots, n$.*

Proof One direction is obvious by taking the section 1. On the other hand, we claim that
$$\eta\colon X \to \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$$
is an isomorphism, hence X is affine. It suffices to show this on an open cover of the target, yet $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X)) = \bigcup_i D(f_i)$. Thus, by our computation, we can show
$$X_{f_i} \to D(f_i) = \mathrm{Spec}(\Gamma(X, \mathcal{O}_X)_{f_i})$$
is an isomorphism for each i. Since X_{f_i} is affine, it is qcqs, and since each one is the preimage of $D(f_i)$, which form an open cover of $\mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$, the X_{f_i} form a finite open cover of X. Therefore X is qcqs, so by Thm. 1.6, we need only prove
$$X_{f_i} \xrightarrow{\eta_{X_{f_i}}} \mathrm{Spec}(\Gamma(X_{f_i}, \mathcal{O}_X))$$
is an isomorphism, which is true since X_{f_i} is affine. Hence η is an isomorphism, completing the proof. $\square$

1.2 Properties of Morphisms of Schemes

Now we will do some preparation for studying the properties of morphisms of schemes.

Definition 1.8 Let X be a scheme. A property $\mathcal{P}$ of affine opens of X is *Zariski local* if:

1. If $U = \mathrm{Spec}(A)$ has $\mathcal{P}$, then for each $f \in A$, the subscheme $D(f) = \mathrm{Spec}(A_f)$ has $\mathcal{P}$.
2. If $U = \mathrm{Spec}(A)$ is an open subscheme and there are $f_1, \ldots, f_n \in A$ generating the unit ideal such that $D(f_i) \subset U$ has $\mathcal{P}$, then U has $\mathcal{P}$.

In other words, the property $\mathcal{P}$ is Zariski local if it can be checked on a Zariski open cover. The following result will be useful for studying such properties:

Theorem 1.9 (Affine Communication) *Let X be a scheme and $\mathcal{P}$ be a Zariski local property. If X has an affine open cover $\{U_i\}$ such that each U_i has $\mathcal{P}$, then every affine open has $\mathcal{P}$.*

Proof Write $U_i = \mathrm{Spec}(A_i)$, and fix any affine open $U = \mathrm{Spec}(A)$ in X. Let $I = \{i : U \cap U_i \neq \varnothing\}$, then $U = \bigcap_{i \in I} U \cap U_i$. By Cor. 1.3, write $U \cap U_i = \bigcup_j W_{ij}$ where W_{ij} is distinguished in both U and U_i. The latter implies that W_{ij} has $\mathcal{P}$, while the former means we can write $W_{ij} = D_A(f_{ij})$. Since the W_{ij} cover U and the latter is quasicompact, we have found finitely many f_{ij} which generate the unit ideal and satisfy $\mathcal{P}$. $\square$

Our first application of the affine communication lemma is to affine morphisms. Recall that a morphism $f\colon X \to Y$ is affine if for each affine open $V \subset Y$, the preimage $f^{-1}(V)$ is affine.

Proposition 1.10 *Let $f\colon X \to Y$ be a morphism of schemes. The following are equivalent:*

1. *f is affine.*
2. *Y has an affine open cover $\{U_i\}$ such that $f^{-1}(U_i)$ is affine for each i.*

Proof One implication is trivial. Let $\mathcal{P}$ be the property of affine opens of Y where V has $\mathcal{P}$ if $f^{-1}(V)$ is affine. By Thm. 1.9, we will be done if we can show that $\mathcal{P}$ is Zariski local.

First, suppose $V = \operatorname{Spec}(A) \subset Y$ is so that $f^{-1}(V) = \operatorname{Spec}(B)$ is affine. Then the restriction
$$f\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$$
is $\operatorname{Spec}(\phi)$ for some $\phi\colon A \to B$. Now if $g \in A$, then $f^{-1}(D_A(g)) = D_B(\phi(g)) = \operatorname{Spec}(B_{\phi(g)})$, which is affine. Hence $D_A(g) \subset V$ has $\mathcal{P}$.

On the other hand, suppose we have $(g_1, \ldots, g_n) = A$ where each $D_A(g_i)$ has $\mathcal{P}$. Set $U = f^{-1}(V)$, then we can apply global sections to the map $f\colon U \to V$ to get a ring homomorphism
$$\phi\colon A \to \Gamma(U, \mathcal{O}_X).$$
Clearly $(\phi(g_1), \ldots, \phi(g_n)) = \Gamma(U, \mathcal{O}_X)$. By Prop. 1.7, it suffices to show that $f^{-1}(D_A(g_i)) = U_{\phi(g_i)}$, as this open set is affine by assumption.

Lemma 0.1 *Let $f\colon X \to \operatorname{Spec}(A)$ be a morphism of schemes and write $\phi\colon A \to \Gamma(X, \mathcal{O}_X)$ for the corresponding ring homomorphism. Then for any $g \in A$, we have $f^{-1}(D_A(g)) = X_{\phi(g)}$.*

Proof Let $x \in X$ and consider the associated map on residue fields
$$\kappa_X(f(x)) \to \kappa_A(x),$$
which is necessarily injective and sends $g_{f(x)} \mapsto \phi(g)_{f(x)}$. Thus if $f(x) \in D_A(g)$, then $x \in X_{\phi(g)}$, which means that
$$f^{-1}(D_A(g)) \subset X_{\phi(g)}.$$
On the other hand, suppose $x \in X_{\phi(g)}$, i.e. $\phi(g)_x \in \mathcal{O}_{X,x}^{\times}$, and take $U = \operatorname{Spec}(B) \ni x$. The restriction $f\colon U \to \operatorname{Spec}(A)$ corresponds to $\psi\colon A \to B$, and
$$x \in X_{\phi(g)} \cap U = U_{\phi(g)} = D_B(\phi(g)|_U) = D_B(\psi(g)),$$
since $\mathrm{res}_U \circ \phi = \psi$. But
$$f|_U^{-1}(D_A(g)) = D_B(\psi(g)) = U \cap f^{-1}(D_A(g)),$$
which proves the lemma. ■

Taking in the lemma X to be U and g to be each g_i, we're done. □

This result is very practically useful and has lots of corollaries. We will use shortly these when discussing closed immersions.

Corollary 1.11 *Let Y be an affine scheme and $f\colon X \to Y$ a morphism of schemes. f is affine if and only if X is affine.*

Proof $f^{-1}(Y) = X$. Hence if f is affine, then X is also. If X is affine, then $\{Y\}$ is an affine open cover with affine preimage, we use Prop. 1.10. $\square$

Corollary 1.12 *Let $U, V \subset X$ be affine open subschemes of an affine scheme X. Then $U \cap V$ is affine.*

Proof By Cor. 1.11, the open immersion $j\colon U \hookrightarrow X$ is affine, so $j^{-1}(V) = U \cap V$ is an affine scheme. $\square$

Another corollary which is not a priori obvious is that affineness is a local property of morphisms, in the following sense:

Definition 1.13 Let $\mathcal{P}$ be a property of morphisms of schemes.

1. $\mathcal{P}$ is *target-local* if for each morphism $f\colon X \to Y$ and open cover $\{V_i\}$ of Y, f satisfies $\mathcal{P}$ if and only if $f^{-1}V_i \to V_i$ satisfies $\mathcal{P}$.
2. $\mathcal{P}$ is *source-local* if for each morphism $f\colon X \to Y$ and open cover $\{U_i\}$ of X, f satisfies $\mathcal{P}$ if and only if $f|_{U_i}\colon U_i \to Y$ satisfies $\mathcal{P}$.

Corollary 1.14 *The property of morphisms being affine is target-local.*

Proof If $f\colon X \to Y$ is affine and $U \subset Y$ is any open subset, then $f^{-1}(U) \to U$ is affine: any affine open of U is also one of Y. On the other hand, suppose we have a cover $\{U_i\}$ of Y so that each $f^{-1}(U_i) \to U_i$ is affine. Because we have an affine basis, each $y \in Y$ is contained in $V \subset U_i$ for some i where V is an affine open. Then $f^{-1}(V)$ is affine, so we have found an affine open cover of Y whose preimages are affine. $\square$

Another more interesting use of this result is the following, which we will rely on while studying closed immersions. Recall that a map $f\colon X \to Y$ of topological spaces is a *closed embedding* if it is a homeomorphism $X \xrightarrow{\sim} f(X)$ onto a closed subset $f(X) \subset Y$.

Proposition 1.15 *Let $i\colon Z \to X$ be a morphism of schemes which is a closed embedding of the underlying spaces. Then i is affine.*

Proof We will use the fact that schemes have an affine basis; by identifying Z with $i(Z)$, assume $Z \subset X$ is a closed subset with i the inclusion. Let $x \in X$ and suppose $x \notin Z$. Since $X \setminus Z$ is open, there is an affine open nbhd $U \subset X \setminus Z$ of x, which satisfies $i^{-1}(U) = U \cap Z = \varnothing = \mathrm{Spec}(0)$.

On the other hand, suppose $x \in Z$. Let V be an affine open nbhd of x in X, and let $U \cap Z$ be an affine open nbhd of x in Z contained in $V \cap Z$. Then the composite

$$U \cap Z \hookrightarrow V \cap Z \xrightarrow{i} V$$

is between affine schemes thus is affine. Taking $V' \subset U$ an affine open nbhd of x in X, its preimage under this composite is $V' \cap Z$, which is affine. Hence we have found an affine open cover of X in which each open has affine preimage, completing the proof by Prop. 1.7. $\square$

1.3 Closed Immersions

Closed immersions are morphisms of schemes designed to generalize the morphisms $\operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$, which corresponds to the inclusion of $V(I) \hookrightarrow \operatorname{Spec}(A)$. This will be our basic example.

Definition 1.16

1. A morphism of schemes $i\colon Z \to X$ is a *closed immersion* if it is a closed embedding and the morphism of sheaves

$$i^\sharp\colon \mathcal{O}_X \to i_*\mathcal{O}_Z$$

is surjective.

2. A *closed subscheme* of a scheme X is an isomorphism class of closed immersions.

The nature of the definition means that whenever we have a closed immersion $i\colon Z \to X$, we may always assume that $Z \subset X$ is a closed subset. This definition is somewhat unwieldy due to the second condition - the rest of this subsection is devoted to creating some alternative formulations.

First, recall that the morphism $i^\sharp$ being surjective means that it is surjective on stalks at all $x \in X$. This is equivalent to saying that given any section

$$f \in \Gamma(U, i_*\mathcal{O}_Z) = \Gamma(U \cap Z, \mathcal{O}_Z)$$

over $U \ni x$, there is a smaller open $V \ni x$ such that $f|_V$ is in the image of

$$i^\sharp_V\colon \Gamma(V, \mathcal{O}_X) \to \Gamma(V \cap Z, \mathcal{O}_Z).$$

It is clear that it is the same to assume U and V are elements of some basis for the topology on X. We first need to understand this in the affine case. We note that some of what we do here will become more or less trivial after we study quasicoherent sheaves, but that is delayed until later.

Lemma 1.17 *Let A be a ring and $\varphi\colon M \to N$ a map of A-modules with the following property: For each $y \in N$, there is an open cover $\{D_A(f_i)\}$ of* $\operatorname{Spec}(A)$ *such that $y/1$ is in the image of*

$$\varphi_{f_i}\colon M_{f_i} \to N_{f_i}$$

for each i. Then φ is surjective.

Proof Let $C = \operatorname{cok}(\varphi) = N/\varphi(M)$, so that $M \to N \to C \to 0$ is exact. Choose any class $[y] \in C$; it suffices to show that $[y] = 0$. Take the open cover $\{D(f_i)\}$

associated to y as in the assumption, then localizing at each f_i we see that the image of $[y]$ in C_{f_i} is zero. In other words, $\langle [y] \rangle_{f_i} = 0$ for all i. For any $\mathfrak{p} \in \operatorname{Spec}(A)$, there is some $f_k \notin \mathfrak{p}$; since f_k^n (for all n) is a unit in $A_{\mathfrak{p}}$, it follows that $\langle [y] \rangle_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$. Hence $\langle [y] \rangle = 0$. □

Corollary 1.18 *Let $\phi\colon A \to B$ be a ring homomorphism and write $f = \operatorname{Spec}(\phi)$. The following are equivalent*

1. *ϕ is surjective.*
2. *$f^\sharp\colon \mathcal{O}_{\operatorname{Spec}(A)} \to f_*\mathcal{O}_{\operatorname{Spec}(B)}$ is surjective.*

Proof We know the sets $D_A(g)$ form a basis of $\operatorname{Spec}(A)$, and that

$$f^\sharp_{D_A(g)}\colon A_g \to \Gamma(f^{-1}(D_A(g)), \mathcal{O}_{\operatorname{Spec}(B)}) = \Gamma(D_B(\phi(g)), \mathcal{O}_{\operatorname{Spec}(B)}) = B_{\phi(g)}$$

is just the natural map $A_g \to B_{\phi(g)}$. For (1) $\Rightarrow$ (2), it is clear because in fact each $f^\sharp_{D_A(g)}$ is surjective, so the stalkwise surjectivity is clear too. On the other hand, to prove ϕ is surjective it is enough to show that the A-module homomorphism

$$\phi\colon A \to B_A$$

is onto. Indeed, if $b \in B_A$, then around each point $\mathfrak{p} \in \operatorname{Spec}(A)$ there is a nbhd $D_A(g)$ so that b is in the image of $f^\sharp_{D_A(g)}$. Then by Lemma 1.17, ϕ is surjective. □

Proposition 1.19 *Let A be a ring and $I \subset A$ an ideal. The natural morphism*

$$\operatorname{Spec}(A/I) \to \operatorname{Spec}(A)$$

is a closed immersion.

Proof As topological spaces we know $\operatorname{Spec}(A/I) \cong V(I)$ which is closed in $\operatorname{Spec}(A)$, hence the map is a closed embedding. The second condition follows from Cor. 1.18. □

We can now prove our main result about closed immersions, which is a series of equivalent conditions. Recall that the support $\operatorname{Supp}(\mathcal{F})$ of a sheaf on X is the set $\{x \in X : \mathcal{F}_x \neq 0\}$.

Theorem 1.20 *Let $i\colon Z \to X$ be a morphism of schemes. The following are equivalent:*

1. *i is a closed immersion.*
2. *i is affine and $i^\sharp\colon \mathcal{O}_X \to i_*\mathcal{O}_Z$ is surjective.*
3. *For each affine open $U = \operatorname{Spec}(A) \subset X$, there is an ideal $I \subset A$ and an isomorphism $i^{-1}(U) \cong \operatorname{Spec}(A/I)$ over U.*
4. *There is an affine open cover $\{U_i = \operatorname{Spec}(A_i)\}$ of X such that for each i, there's an $I \subset A_i$ and an isomorphism $i^{-1}(U_i) \cong \operatorname{Spec}(A_i/I)$ over U_i.*

Moreover, if any of these conditions hold, then $i(Z) = \operatorname{Supp}(i_*\mathcal{O}_Z)$.

Proof The obvious implications are (3) $\Rightarrow$ (4) and (1) $\Rightarrow$ (2), by Prop. 1.15. Observe also that (4) $\Rightarrow$ (2), by the locality of affineness, the fact that the condition is stalkwise, and Prop. 1.19. Thus it remains to prove (2) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1). For the former, we know $i^{-1}(U) = \operatorname{Spec}(B)$, so $i^{-1}(U) \to U$ is induced by a ring homomorphism $\phi\colon A \to B$. The sheaf map

$$\mathcal{O}_U \to i_*\mathcal{O}_{i^{-1}(U)}$$

is still surjective because by definition this is a stalkwise condition (and U is open). Hence by Cor. 1.18, ϕ is surjective. Letting $I = \ker \phi$, there is a commutative triangle

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr, "\phi"] & \\ A/I \arrow[rr, "\cong"] & & B \end{tikzcd}$$

where the bottom map is an isomorphism. Taking Spec the result follows.

For (3) $\Rightarrow$ (1), it follows from Prop. 1.19 that each $i^{-1}(U) \to U$ is a closed immersion. Therefore it is already clear that $i^\sharp$ is surjective because its restrictions to an open cover are surjective. Moreover, the condition of being a topological embedding is target local (since the same is true of homeomorphisms), so we just need to show that $i(Z)$ is closed. To do so, we will prove the final statement, that $i(Z) = \operatorname{Supp}(i_*\mathcal{O}_Z)$. This is sufficient because of the following lemma:

Lemma 0.2 *Let X be a space and $\mathcal{F}$ a sheaf of rings on X. Then* $\operatorname{Supp}(\mathcal{F}) \subset X$ *is closed.*

Proof Let $1 \in \Gamma(X, \mathcal{F})$ be the unit section. Then $\mathcal{F}_x = 0$ if and only if $1_x = 0$. So, if $\mathcal{F}_x = 0$, then there is a nbhd $U \ni x$ such that $1|_U = 0$, i.e. $\mathcal{F}_y = 0$ for all $y \in U$. This shows $\{x \in X : \mathcal{F}_x = 0\}$ is open. ■

To finish the proof, let $x \in X$ and suppose $(i_*\mathcal{O}_Z)_x = 0$. This is the same as the unit section being zero at x, so there is an (affine) open nbhd $U = \operatorname{Spec}(A)$ such that $(i_*\mathcal{O}_Z)|_U = 0$. We know

$$A/I = \Gamma(i^{-1}(U), \mathcal{O}_Z) = \Gamma(U, i_*\mathcal{O}_Z) = 0,$$

so $i^{-1}(U) = \operatorname{Spec}(A/I) = \varnothing$, hence $x \notin i(Z)$. On the other hand, suppose $x \notin i(Z)$, choose $U = \operatorname{Spec}(A)$ an affine open nbhd of x, so $i^{-1}(U) = \operatorname{Spec}(A/I)$. Then $x \in D_A(I)$, so

$$(i_*\mathcal{O}_Z)_x = (i_*\mathcal{O}_Z)|_{D_A(I),x} = 0$$

because for $V \subset D_A(I)$,

$$\Gamma(V, i_*\mathcal{O}_Z) = \Gamma(\varnothing, \mathcal{O}_Z) = 0.$$

Thus $i(Z) = \operatorname{Supp}(i_*\mathcal{O}_Z)$. □

There is a little bit more we can say about closed immersions, which we will do after discussing quasicoherent ideal sheaves. Some basic corollaries are as follows:

Corollary 1.21 *The property of being a closed immersion is target-local.*

Proof It is clear since the same is true for affine morphisms so we may use the equivalence of (1) and (2) in Thm. 1.20. □

Definition 1.22 Let $\mathcal{P}$ be a property of morphisms of schemes and $\mathcal{Q}$ be a property of ring homomorphisms. $\mathcal{P}$ is *affine locally* $\mathcal{Q}$ if a morphism $f\colon X \to Y$ satisfies $\mathcal{P}$ if and only if:

1. f is an affine morphism.
2. For each affine open $U = \mathrm{Spec}(A) \subset Y$ so that $f^{-1}(U) = \mathrm{Spec}(B)$, the corresponding morphism $A \to B$ satisfies $\mathcal{Q}$.

Condition (3) of Thm. 1.20 shows that being a closed immersion is affine locally surjective. We will see many other examples of this property which lets us convert properties of ring homomorphisms to properties of morphisms of schemes.

Definition 1.23 A property $\mathcal{Q}$ of ring homomorphisms is *Zariski local* if the following conditions are satisfied:

1. If $\phi\colon A \to B$ has $\mathcal{Q}$, then for each $f \in A$ the map $\phi_f\colon A_f \to B_{\phi(f)}$ has $\mathcal{Q}$.
2. If $\phi\colon A \to B$ and there are $f_1, \ldots, f_n \in A$ generating the unit ideal such that
$$\phi_{f_i}\colon A_{f_i} \to B_{\phi(f_i)}$$
have $\mathcal{Q}$, then ϕ has $\mathcal{Q}$.

Lemma 1.24 *Let $\mathcal{Q}$ be a Zariski local property of ring homomorphisms and suppose $\mathcal{P}$ is affine locally $\mathcal{Q}$. Then the following are equivalent for a morphism $f\colon X \to Y$ of schemes:*

1. *f satisfies $\mathcal{P}$.*
2. *There is an affine open cover $\{V_i\}$ of Y such that $f^{-1}(V_i) \to V_i$ satisfies $\mathcal{P}$ for each i.*

2 Sheaves

2.1 Background

Here we provide some general results on ringed spaces, sheaves, (quasi)coherent sheaves, and more. We will also discuss how they apply to schemes. We shall begin with gluing morphisms of ringed spaces, and then will discuss gluing schemes together.

Lemma 2.1 *Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X such that the induced maps on stalks*

$$f_x \colon \mathcal{F}_x \to \mathcal{G}_x$$

is an injection (isomorphism) for each $x \in X$. Then f is sectionwise an injection (isomorphism).

Proof For the statement about injections, suppose $U \subset X$ and let $s, t \in \mathcal{F}(U)$ such that $f(s) = f(t)$. Then for each $x \in U$, we have

$$f_x(s_x) = f(s)_x = f(t)_x = f_x(t_x),$$

so there is a nbhd $U_x \subset U$ of x, such that $s|_{U_x} = t|_{U_x}$. Since these cover U and $\mathcal{F}$ is a sheaf, $s = t$.

For the isomorphism statement, let $t \in \mathcal{G}(U)$; our assumption means that there is an open cover $\{U_i\}$ of U and $s_i \in \mathcal{F}(U_i)$ and $f(s_i) = t|_{U_i}$. Observe that

$$f(s_i|_{U_i \cap U_j}) = t|_{U_i \cap U_j} = f(s_j|_{U_i \cap U_j}),$$

so by sectionwise injectivity s_i agrees with s_j on overlaps. Therefore there's a section s over U restricting to the s_i, and since

$$f(s)|_{U_i} = f(s_i) = t|_{U_i} \qquad \forall i,$$

we have $f(s) = t$. □

2.2 Sheaves on Bases

In what follows, we will use the following definition for convenience. These are (generally) the categories over which we will consider sheaves.

Definition 2.2 A *concrete category* is either the category of monoids, commutative monoids, groups, abelian groups, or rings. It is always equipped with the canonical forgetful functor $C \to \mathbf{Set}$.

In any concrete category, the forgetful functor creates limits and filtered colimits. This means that we can check the sheaf condition on underlying sets, and describe ‘elements’ of stalks of (pre)sheaves of C-objects.

Concrete categories are also important because they can be exhibited completely categorically from the category of sets (e.g. group objects, ring objects, etc.). In all of these cases, the structure is described using only products and the terminal object, hence finite-product preserving functors preserve C-objects. For a category D with finite products, we write $\mathrm{ObjC}(D)$ for its category of C-objects and C-morphisms.

For now, fix C a category with limits and X a space. Let $\mathcal{V}$ be a basis for the topology on X that is closed under finite intersections. Viewing $\mathcal{V} \subset \mathrm{Open}(X)$

as a full subcategory we write $\mathrm{PSh}(X;\ \mathcal{V})$ for the category of functors $\mathcal{V}^{\mathrm{op}} \to C$. The inclusion induces a functor

$$\mathrm{PSh}(X) \to \mathrm{PSh}(X;\ \mathcal{V}).$$

If $F \in \mathrm{PSh}(X;\ \mathcal{V})$, we say F is a $\mathcal{V}$-sheaf if for any $V \in \mathcal{V}$ and any open cover $V = \bigcup_i V_i$ with $V_i \in \mathcal{V}$, the canonical diagram

$$F(V) \longrightarrow \prod_i F(V_i) \rightrightarrows \prod_{i,j} F(V_i \cap V_j)$$

is an equalizer in C. If F came from a sheaf on X, then it is obviously a $\mathcal{V}$-sheaf; hence there is a natural functor

$$\mathrm{Sh}(X) \to \mathrm{Sh}(X;\ \mathcal{V})$$

to the category of $\mathcal{V}$-sheaves. In fact:

Theorem 2.3 *Let X be space, C be a category with limits, and $\mathcal{V}$ a basis of X closed under finite intersections. The natural functor*

$$\mathrm{Sh}(X) \to \mathrm{Sh}(X;\ \mathcal{V})$$

is an equivalence of categories.

Proof First we show it is fully faithful, i.e. the map

$$\mathrm{Hom}_{\mathrm{Sh}(X)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{Sh}(X;\ \mathcal{V})}(\mathcal{F}_{\mathcal{V}}, \mathcal{G}_{\mathcal{V}})$$

is bijective Let $f\colon \mathcal{F}_{\mathcal{V}} \to \mathcal{G}_{\mathcal{V}}$ be a morphism. □

This will let us define sheaves on a space X by only knowing its values on a basis. We conclude this section with the following technical point: let F be a presheaf on X, U an open set and $\{U_i\}$ an open cover of U. We might ask whether, to know that F satisfies the sheaf condition for this cover, we may check it on a subcover.

Lemma 2.4 *Let F be a C-presheaf on X, $U \subset X$ an open subset, and $\mathcal{U}$ an open cover of U. Suppose that:*

1. *F satisfies the sheaf condition with respect to a subcover $\mathcal{V} \subset \mathcal{U}$ of U.*
2. *For each open $V \subset U$ and open cover $\{V_i\}$ of V, the natural map*

   $$F(V) \to \prod_i F(V_i)$$

   is a monomorphism in C.

Then F satisfies the sheaf condition with respect to $\mathcal{U}$.

Proof □

We remark that the above proof works just as well for the case of a sheaf a on basis, which is our intended application anyway.

Proposition 2.5 *Let $X = \operatorname{Spec}(A)$ and write $\mathcal{O}_X$ for the structure sheaf. Then $\mathcal{O}_X$ is a sheaf on the basis $\{D_A(f) : f \in A\}$ of $\operatorname{Spec}(A)$, and hence extends uniquely up to isomorphism to a sheaf on X.*

Proof □

Proposition 2.6 *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces and $\{U_i : i \in I\}$ be an open cover of X. Suppose we are given morphisms $f_i \colon U_i \to Y$ such that*

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \qquad \forall i, j \in I.$$

Then there is a unique morphism $f \colon X \to Y$ such that $f|_{U_i} = f_i$.

Proof From topology we already know that the statement is true on the underlying spaces, and we get a unique map $|f| \colon |X| \to |Y|$. So it remains to construct the unique sheaf morphism

$$f^\# = f^\#_V \colon \Gamma(V, \mathcal{O}_Y) \to \Gamma(f^{-1}(V), \mathcal{O}_X).$$

Observe that

$$f^{-1}(V) = \bigcup_{i \in I} f^{-1}(V) \cap U_i = \bigcup_{i \in I} f_i^{-1}(V)$$

for any open subset $V \subset X$, and the f_i are equipped with maps

$$f^\#_{i,V} \colon \Gamma(V, \mathcal{O}_Y) \to \Gamma(f_i^{-1}(V), \mathcal{O}_X)$$

making the following diagram commute:

$$\begin{tikzcd}
\Gamma(V, \mathcal{O}_Y) \arrow[r, "f^\#_{i,V}"] \arrow[d, "f^\#_{j,V}"'] & \Gamma(f_i^{-1}(V), \mathcal{O}_X) \arrow[d] \\
\Gamma(f_j^{-1}(V), \mathcal{O}_X) \arrow[r] & \Gamma(f_i^{-1}(V) \cap f_j^{-1}(V), \mathcal{O}_X)
\end{tikzcd}$$

This is because the $\{f_i\}$ agree on overlaps. Therefore by the (equalizer) sheaf condition, this induces the map $f^\#_V$ which is natural in V and restricts to each f_i. Finally, each $f^\#_V$ is unique because of the uniqueness in the equalizer. □

Corollary 2.7 *Let $(X, \mathcal{O}_X)$ be a ringed space and $\{U_i : i \in I\}$ an open cover. Then we have*

$$X = \varinjlim_{(i,j) \in I \times I} \Big(U_i \hookleftarrow U_i \cap U_j \hookrightarrow U_j \Big)$$

in the category of ringed spaces.

Proof This is a rephrasing of Prop. 2.6: A morphism out of X is the same as a morphism out of each U_i which agrees on every pairwise intersection. □

Corollary 2.8 *Let X, Y be schemes, $\{U_i\}$ an open cover of X, and $f_i\colon U_i \to Y$ morphisms of schemes such that*

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j} \qquad \forall i, j \in I.$$

Then there is a unique morphism $f\colon X \to Y$ such that $f|_{U_i} = f_i$ for each i.

Proof We can apply Prop. 2.6, so it just remains to check that the resulting f induces a local map on stalks. But indeed, for any $x \in X$, choose $U_i \ni x$ so that

$$f_x = f_{i,x}\colon \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}.$$

Then f_x is local since f_i is a morphism of schemes. $\square$

We will, for the moment, black-box the following result (until we discuss etale spaces).

Theorem 2.9 *Let C be a locally presentable category (Grothendieck topos), and let X be any topological space. The inclusion*

$$i\colon \mathrm{Sh}(X, C) \hookrightarrow \mathrm{PSh}(X, C)$$

admits a left adjoint $L\colon \mathrm{PSh}(X, C) \to \mathrm{Sh}(X, C)$ called sheafification. L preserves stalks in the sense that the unit map of presheaves $\eta\colon F \to L(F)$ induces stalkwise isomorphisms

$$\eta_x\colon F_x \xrightarrow{\sim} L(F)_x$$

for each $x \in X$.

The most important cases are $C = \mathbf{Set}, \mathbf{Ring}$, and $\mathbf{Ab}$. This tells us how to compute limits and colimits in $\mathrm{Sh}(X)$. The former case is just the same as limits in $\mathrm{PSh}(X)$, which are done sectionwise. In the latter case, we compute the colimit in $\mathrm{PSh}(X)$ and then apply L.

Our adjunction gives sheafification through a universal property: namely, if $\mathcal{G}$ is a sheaf, then any map $F \to \mathcal{G}$ of presheaves extends uniquely to $L(F) \to \mathcal{G}$ along the unit $F \to L(F)$.

Lemma 2.10 *Let F be a presheaf on X and $U \subset X$ an open. There is a canonical isomorphism*

$$L(F|_U) \xrightarrow{\sim} L(F)|_U$$

of sheaves.

Proof The unit $F \to L(F)$ restricts to $F|_U \to L(F)|_U$. This induces a unique map $L(F|_U) \to L(F)|_U$ making the diagram

$$\begin{tikzcd}
F|_U \arrow[d] \arrow[dr] & \\
L(F|_U) \arrow[r] & L(F)|_U
\end{tikzcd}$$

commute. For each $x \in U$, taking the stalk shows that the bottom map is an isomorphism (since the other two are). □

Finally, a restatement of the condition on stalks is phrased as a lemma as follows. We will use this many times.

Lemma 2.11 *Let F be a presheaf on X, $U \subset X$ an open subset, and $t \in \Gamma(U, L(F))$ a section.*

1. *There is an open cover $\{U_i\}$ of U and sections $s_i \in F(U_i)$ such that $\eta(s_i) = s|_{U_i}$.*
2. *Moreover, s_i and s_j agree on an open cover of the intersection $U_i \cap U_j$.*

Proof The second statement follows trivially from the first using the injectivity of η_x for each $x \in U$, and the first follows from its surjectivity. □

2.3 Quasicoherent Sheaves

2.3.1 OX-Modules

Let $(X, \mathcal{O}_X)$ be a ringed space. Recall that an $\mathcal{O}_X$-module $\mathcal{F}$ is a sheaf of abelian groups in $\mathrm{Sh}(X, \mathbf{Ab})$ where each group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module satisfying the following compatibility condition: if $s \in O_X(U)$ and $x \in \mathcal{F}(U)$, then

$$(s \cdot x)|_V = s|_V \cdot x|_V$$

for $V \subset U$. A morphism $\mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of the underlying sheaves such that for each U, $\mathcal{F}(U) \to \mathcal{G}(U)$ is a module homomorphism. The resulting category is written $\mathrm{Mod}(\mathcal{O}_X)$.

Suppose $U \subset X$ and $\mathcal{F}$ is an $\mathcal{O}_X$-module. The restriction map $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ lets us view $\mathcal{F}(U)$ as an $\mathcal{O}_X(X)$-module, and the compatibility condition is the statement that $\mathcal{F}(X) \to \mathcal{F}(U)$ is homomorphism of $\mathcal{O}_X(X)$-modules. Therefore, let $\{U_i\}$ be an open cover of the open U. The sheaf condition says that we have an exact sequence

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

of abelian groups, but the above remark shows that it is an exact sequence of $\mathcal{O}_X(U)$-modules. This will be useful soon.

We will want to construct $\mathcal{O}_X$-modules out of presheaves using sheafification. We will say an $\mathcal{O}_X$*-premodule* is a presheaf $F \in \mathrm{Sh}(X, \mathbf{Ab})$ where $F(U)$ is equipped with an $\mathcal{O}_X(U)$-module structure, such that

$$(s \cdot x)|_V = s|_V \cdot x|_V$$

for all sections. As before, a premodule homomorphism is just a morphism of presheaves which induces sectionwise module homomorphisms. Our main result will be the following.

Theorem 2.12 *Let $(X, \mathcal{O}_X)$ be a ringed space and let F be an $\mathcal{O}_X$-premodule. Then the associated sheaf $L(F)$ has a unique $\mathcal{O}_X$-module structure making*

$$\eta \colon F \to L(F)$$

a premodule homomorphism.

Proof Let $a \in \mathcal{O}_X(U)$ and $y \in \Gamma(U, L(F))$; we want to define $a \cdot y$. By Lemma 2.11, there is an open cover $\{U_i\}$ of U and sections $y_i \in F(U_i)$ such that $\eta(y_i) = y|_{U_i}$. Moreover, y_i and y_j agree on U_{ijk} for $U_i \cap U_j = \bigcup_k U_{ijk}$. Set

$$z_i = \eta(a|_{U_i} \cdot z_i),$$

and observe that

$$z_i|_{U_{ijk}} = \eta(a|_{U_{ijk}} \cdot y_i|U_{ijk}) = \eta(a|_{U_{ijk}} \cdot y_j|_{U_{ijk}}) = z_j|_{U_{ijk}}.$$

Since $L(F)$ is a sheaf, $z_i|_{U_i \cap U_j} = z_j|_{U_i \cap U_j}$. So, there is a unique $z \in \Gamma(U, L(F))$ restricting to the z_i. We put $a \cdot y = z$. Notice that the condition $z_i = \eta(a|_{U_i} \cdot z_i)$ is required if η is a homomorphism, which demonstrates the uniqueness.

This doesn't depend on the choice of open cover; suppose instead we used U_j', y_j', and constructed z' using z_j'. Let $x \in X$, then there is i, j so that $x \in U_i \cap U_j'$. It follows that

$$\eta(y_j')_x = \eta(y_i)_x,$$

and so there is a nbhd $V \subset U_i \cap U_j'$ of x such that y_i and y_j' agree. Hence $z'|_V = z|_V$. Since x was arbitrary $z' = z$.

To show this is a well-defined action, we claim $(ab) \cdot y = a \cdot (b \cdot y)$. For $x \in X$, there is a nbhd V such that

$$((ab) \cdot y)|_V = \eta((ab)|_V \cdot y_0) \quad a \cdot (b \cdot y)|_V = \eta(a|_V \cdot (b|_V y_1))$$

for $\eta(y_0) = y|_V = \eta(y_1)$. By further restricting Y, we may assume $y_0 = y_1$, which proves the claim since x was arbitrary. Our construction also makes $L(F)$ an $\mathcal{O}_X$-module, since any open cover $\{U_i\}$ of U restricts to an open cover $\{U_i \cap V\}$ of $V \subset U$.

It remains to prove that η is a homomorphism. Let $y \in F(U)$, we want $\eta(a \cdot y) = a \cdot \eta(y)$. Since we can compute $a \cdot \eta(y)$ independent of the open cover, use $\{U\}$, then this is by definition. □

There is an alternative perspective on the above theorem, which we briefly discuss. If F is an $\mathcal{O}_X$-premodule, then each section $a \in \mathcal{O}_X(U)$ induces a map $F|_U \to F|_U$. The unit $\eta \colon F \to L(F)$ induces

$$\mathrm{Hom}(F|_U, F|_U) \to \mathrm{Hom}(F|_U, L(F|_U)) \cong \mathrm{Hom}(F|_U, L(F)|_U)$$

by Lemma 2.10. By composing with the adjunction isomorphism, we get a function

$$\mathrm{Hom}_{\mathrm{PSh}(X)}(F|_U, F|_U) \to \mathrm{Hom}_{\mathrm{Sh}(X)}(L(F)|_U, L(F)|_U).$$

This lets us transport the action of a on F to that on $L(F)$. We need to $\mathcal{O}_X$-module homomorphisms out of the sheafification of premodule.

Proposition 2.13 *Let $(X, \mathcal{O}_X)$ be a ringed space, F be an $\mathcal{O}_X$-premodule, and $\mathcal{G}$ be an $\mathcal{O}_X$-module. For any premodule homomorphism $\varphi\colon F \to \mathcal{G}$, there is a unique homomorphism $\widetilde{\varphi}\colon L(F) \to \mathcal{G}$ making the diagram*

$$\begin{tikzcd} F \arrow[d, "\eta"'] \arrow[dr, "\varphi"] & \\ L(F) \arrow[r, "\widetilde{\varphi}"'] & \mathcal{G} \end{tikzcd}$$

commute.

Proof The sheafification adjunction already says this on the level of morphisms of presheaves. It remains to show that $\widetilde{\varphi}$ is a homomorphism, i.e. $\widetilde{\varphi}(a \cdot y) = a \cdot \widetilde{\varphi}(y)$ for $y \in \Gamma(U, L(F))$. Indeed, for $x \in V \subset U$, we may write

$$(a \cdot y)|_V = \eta(a|_V \cdot \widetilde{y})$$

for $\eta(\widetilde{y}) = y|_V$. Hence

$$\begin{aligned} \widetilde{\varphi}(a \cdot y)|_V &= \widetilde{\varphi}(\eta(a|_V \cdot \widetilde{y})) = \varphi(a|_V \cdot \widetilde{y}) \\ &= a \cdot \varphi(\widetilde{y}) = a \cdot \widetilde{\varphi}(y), \end{aligned}$$

so we are done since x was arbitrary. □

Let F be an $\mathcal{O}_X$-premodule. Then F_x is not only an abelian group, but an $\mathcal{O}_{X,x}$ module. Indeed, the multiplication is

$$[a, U] \cdot [y, V] = [a|_{U \cap V} \cdot y|_{U \cap V}, U \cap V],$$

and it is obvious that this is well-defined. Suppose $\varphi\colon F \to G$ is a premodule homomorphism. Then φ_x is a module homomorphism, because

$$\varphi_x([a, U] \cdot [y, V]) = [\varphi(a \cdot y), U \cap V] = [a, U] \cdot [\varphi(y), V].$$

Now the following result is clear from what we have already done.

Corollary 2.14 *Let $(X, \mathcal{O}_X)$ be a ringed space. The inclusion* $\mathrm{Mod}(\mathcal{O}_X) \to \mathrm{PreMod}(\mathcal{O}_X)$ *admits a left adjoint*

$$L\colon \mathrm{PreMod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}_X)$$

with the property that for any F and for each $x \in X$, the induced map

$$\eta_x\colon F_x \to L(F)_x$$

by the unit is an isomorphism of $\mathcal{O}_{X,x}$-modules.

We conclude this section by studying some basic properties of and constructions using $\mathcal{O}_X$-modules. First, 0 is always an $\mathcal{O}_X$-module, and is a zero object in $\mathrm{Mod}(\mathcal{O}_X)$. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_X$-modules. Then the (sectionwise) kernel

$$\ker(\varphi)(U) := \ker(\varphi_U) = \mathrm{Eq}(\mathcal{F} \underset{0}{\overset{\varphi}{\rightrightarrows}} \mathcal{G})$$

is an $\mathcal{O}_X$-module, since limits of $\mathrm{Mod}(\mathcal{O}_X)$ are computed in the premodule category. We may also consider the premodule $\mathrm{cok}(\varphi)$, which in the premodule category is

$$\mathrm{Coeq}(\mathcal{F} \underset{0}{\overset{\varphi}{\rightrightarrows}} \mathcal{G}).$$

Since L preserves limits, the cokernel in $\mathrm{Mod}(\mathcal{O}_X)$ is given by the sheafification

$$\mathrm{cok}(\varphi) := L(U \mapsto \mathrm{cok}(\varphi_U)) = \mathrm{Coeq}_{\mathrm{Mod}(\mathcal{O}_X)}(\mathcal{F} \underset{0}{\overset{\varphi}{\rightrightarrows}} \mathcal{G}),$$

and comes equipped with the natural module homomorphism

$$\mathcal{G} \to \mathrm{cok}(\varphi)$$

Similarly, we can define the image sheaf

$$\mathrm{im}(\varphi) := L(U \mapsto \mathrm{im}(\varphi_U)) = \ker(\mathcal{G} \to \mathrm{cok}(\varphi)).$$

Since the Hom of sheaves is already a sheaf, there is the module Hom

$$\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})(U) := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U),$$

giving another $\mathcal{O}_X$-module. Finally, we define the tensor product as

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} := L\big(U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)\big).$$

In all cases, the unit map η gives a stalkwise isomorphism between the associated sheaf and the corresponding premodule.

Theorem 2.15 *Let $f\colon \mathcal{F} \to \mathcal{G}$ be an $\mathcal{O}_X$-module homomorphism and fix $x \in X$ any point.*

1. *There is an isomorphism $u\colon \ker(f)_x \xrightarrow{\sim} \ker(f_x)$ making the diagram*

$$\begin{tikzcd} \ker(f)_x \arrow[d, "u"] \arrow[dr] & \\ \ker(f_x) \arrow[r] & \mathcal{F}_x \end{tikzcd}$$

commute.

2. *There is an isomorphism $v\colon \mathrm{cok}(f)_x \xrightarrow{\sim} \mathrm{cok}(f_x)$ making the diagram*

$$\begin{tikzcd} \mathcal{G}_x \arrow[r] \arrow[dr] & \mathrm{cok}(f)_x \arrow[d, "v"] \\ & \mathrm{cok}(f_x) \end{tikzcd}$$

commute.

3. *There is a canonical isomorphism* $\mathrm{im}\,(f)_x \cong \mathrm{im}\,(f_x)$.

Proof The last statement follows trivially from the first two. The second is just applying the relevant universal properties and the unit $\eta\colon F \to L(F)$ which induces isomorphisms on stalks. This is because on the level of presheaves, we have

$$\varinjlim_{U \ni x} \mathrm{cok}(f_U) = \mathrm{cok}(\varinjlim_{U \ni x} f_U) = \mathrm{cok}(f_x)$$

since colimits commute with colimits, and

$$\varinjlim_{U \ni x} \ker(f_U) = \ker(\varinjlim_{U \ni x} f_U) = \ker(f_x)$$

since finite limits commute with filtered colimits. □

We could use these operations to define exact sequences in $\mathrm{Mod}(\mathcal{O}_X)$; instead, we will do so stalkwise.

Definition 2.16 Let $(X, \mathcal{O}_X)$ be a ringed space. A sequence

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

of $\mathcal{O}_X$-module homomorphisms is *exact* if for each $x \in X$, the sequence

$$\mathcal{F}'_x \to \mathcal{F}_x \to \mathcal{F}''_x$$

of $\mathcal{O}_{X,x}$-modules is exact.

Theorem 2.17 *Let* $\mathcal{F}' \xrightarrow{f} \mathcal{F} \xrightarrow{g} \mathcal{F}''$ *be a sequence of morphisms of* $\mathcal{O}_X$*-modules.*

1. *Suppose* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ *is exact. Then there is an isomorphism* $\mathcal{F}' \cong \ker(g)$ *making the diagram*

$$\begin{tikzcd} \mathcal{F}' \arrow[d, "\cong"'] \arrow[dr, "f"] & \\ \ker(g) \arrow[r] & \mathcal{F} \end{tikzcd}$$

commute.

2. *Suppose* $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *is exact. Then there is an isomorphism* $\mathrm{cok}(f) \cong \mathcal{F}''$ *making the diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr, "g"'] & \mathrm{cok}(f) \arrow[d, "\cong"] \\ & \mathcal{F}'' \end{tikzcd}$$

commute.

Proof Both proofs are a similar application of Thm. 2.15; we do, e.g., (2). Observe that

$$(g \circ f)_x = g_x \circ f_x = 0 \quad \forall x \in X.$$

This means that $g \circ f$ is zero, since the image of any section is zero on every stalk, hence zero. By the universal property of the coequalizer, there is a unique map $h\colon \operatorname{cok}(f) \to \mathcal{F}$ making

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr, "g"'] & \operatorname{cok}(f) \arrow[d, "h"] \\ & \mathcal{F}'' \end{tikzcd}$$

commute. To show h is an isomorphism, we may show h_x is an isomorphism for each x. Talking stalks and applying Thm. 2.15, we get a commuting diagram:

$$\begin{tikzcd} & \mathcal{F}_x \arrow[dl] \arrow[d] \arrow[dr, "g_x"] & \\ \operatorname{cok}(f_x) \arrow[r, "\cong"] & \operatorname{cok}(f)_x \arrow[r, "h_x"] & \mathcal{F}''_x \end{tikzcd}$$

Our assumption is that the sequence

$$\mathcal{F}'_x \xrightarrow{f_x} \mathcal{F}_x \xrightarrow{g_x} \mathcal{F}''_x \to 0$$

is exact, so it follows that

$$\operatorname{cok}(f_x) \xrightarrow{\sim} \operatorname{cok}(f)_x \xrightarrow{h_x} \mathcal{F}''_x$$

is an isomorphism, hence h_x is. □

We finish this section with an important result about the category $\operatorname{Mod}(\mathcal{O}_X)$:

Theorem 2.18 *Let* $(X, \mathcal{O}_X)$ *be a ringed space.* $\operatorname{Mod}(\mathcal{O}_X)$ *is an abelian category.*

Proof We already know that there is a zero object and that every morphism has a kernel and cokernel. In addition, we have an internal hom whose global sections are the abelian group of morphisms between two $\mathcal{O}_X$-modules; hence $\operatorname{Mod}(\mathcal{O}_X)$ is **Ab**-enriched. It remains to show that $\operatorname{Mod}(\mathcal{O}_X)$ is normal and conormal.

Consider for some $x \in X$ the stalk functor

$$(-)_x\colon \operatorname{Mod}(\mathcal{O}_X) \to \operatorname{Mod}(\mathcal{O}_{X,x}).$$

Since this is defined by a filtered colimit, it commutes with colimits and finite limits, in particular pushouts and pullbacks. Hence $(-)_x$ preserves monomorphisms and epimorphisms. So, if $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a monomorphism, consider the sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \operatorname{cok}(\varphi).$$

The above remark and Thm. 2.15 shows that it is exact at each x, hence exact. Thm. 2.17 shows that φ is the categorical kernel of $\mathcal{G} \to \mathrm{cok}(\varphi)$. We may repeat this same argument instead using the sequence

$$\ker(\varphi) \to \mathcal{F} \to \mathcal{G} \to 0$$

in the case where φ is an epimorphism, showing that φ is a cokernel. □

2.3.2 The Tilde Sheaf

Now we begin to explore $\mathcal{O}_X$-modules in the context of schemes, not arbitrary ringed spaces. Let $X = \mathrm{Spec}(A)$ be an affine scheme and M an A-module. We have a natural $\mathcal{O}_X$-module $\widetilde{M}$ called the *Tilde Sheaf* of M, which is defined on the basis $\{D_A(f)\}$ by

$$\widetilde{M}(D_A(f)) := M_f,$$

equipped with the obvious restriction maps, where M_f is viewed as an $A_f = \Gamma(D_A(f), \mathcal{O}_X)$-module. The following theorem is then proved in the same way as the fact that the structure sheaf is a sheaf.

Theorem 2.19 *Let $X = \mathrm{Spec}(A)$ and M be an A-module. Then $\widetilde{M}$ as defined is an $\mathcal{O}_X$-module.*

Let $\varphi\colon M \to N$ be an A-module homomorphism. Then for each $f \in A$, there are induced homomorphisms

$$\varphi_f\colon M_f \to N_f$$

given as $\varphi \otimes_A A_f$. This makes $\widetilde{(-)}$ into a functor

$$\widetilde{(-)}\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathcal{O}_{\mathrm{Spec}(A)}).$$

On the other hand, we have a functor

$$\Gamma\colon \mathrm{Mod}(\mathcal{O}_{\mathrm{Spec}(A)}) \to \mathrm{Mod}(A)$$

given by taking global sections $\mathcal{F} \mapsto \mathcal{F}(\mathrm{Spec}(A))$. By our definition,

$$\widetilde{M}(\mathrm{Spec}(A)) = \widetilde{M}(D_A(1)) = M_1 = M,$$

so it follows that Γ is a left inverse of the tilde construction. If $\mathcal{F}$ is any $\mathcal{O}_X$-module on an affine scheme $X = \mathrm{Spec}(A)$, there is a natural map

$$\varepsilon\colon \widetilde{\mathcal{F}(X)} \to \mathcal{F}$$

of sheaves, which we define on the basis: Set

$$\varepsilon_{D(f)}\colon \mathcal{F}(X)_f \to \mathcal{F}(D_A(f))$$

to be the localization at f of the restriction map

$$\mathcal{F}(X) \to \mathcal{F}(D_A(f)).$$

Since $\mathcal{F}(D_A(f))$ is already an A_f-module, $\mathcal{F}(D_A(f))_f = \mathcal{F}(D_A(f))$. This glues into a single map $\varepsilon_{D(f)}$ compatible with restriction. We use this construction to prove the following result:

Theorem 2.20 *Let* $X = \mathrm{Spec}(A)$ *be an affine scheme. We have an adjunction*

$$\widetilde{(-)} \dashv \Gamma(X, -)$$

with counit the natural identification $M = \Gamma(X, \widetilde{M})$. *In particular,* $\widetilde{(-)}$ *is fully faithful.*

Proof If $X = \mathrm{Spec}(A)$, we have

$$\mathrm{Hom}(\widetilde{M}, \mathcal{F}) \xrightarrow{\Gamma} \mathrm{Hom}(\widetilde{M}(X) = M, \mathcal{F}(X)),$$

and for $\varphi\colon M \to \mathcal{F}(X)$ any map consider

$$\widetilde{M} \xrightarrow{\tilde{\varphi}} \widetilde{\mathcal{F}(X)} \xrightarrow{\varepsilon} \mathcal{F}.$$

We claim this provides a two-sided inverse to Γ. Indeed, applying Γ to this and using the definition of η gives one direction. On the other hand, let $\varphi\colon \widetilde{M} \to \mathcal{F}$, and we must show the composite

$$\widetilde{M} \xrightarrow{\widetilde{\varphi_X}} \widetilde{\mathcal{F}(X)} \xrightarrow{\varepsilon} \mathcal{F}$$

is equal to φ. Evaluating at $D_A(f)$, we get

$$M_f \xrightarrow{\varphi_X} \mathcal{F}(X)_f \to \mathcal{F}(D_A(f)).$$

This agrees with $\varphi_{D(f)}$ because the diagram of underlying abelian groups

$$\begin{tikzcd} M \arrow[r, "\varphi_x"] \arrow[d] & \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(X)_f \arrow[d, "\varepsilon"] \\ M_f \arrow[r, "\varphi_{D(f)}"] & \mathcal{F}(D(f)) \arrow[r, equal] & \mathcal{F}(D(f))_f \end{tikzcd}$$

commutes, as φ is a sheaf map. ☐

We will use this result many times. For example:

Corollary 2.21 *Let* $X = \mathrm{Spec}(A)$ *be an affine scheme. An* $\mathcal{O}_X$*-module* $\mathcal{F}$ *is isomorphic to* $\widetilde{M}$ *for some* A*-module* M *if and only if the counit* $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$ *is an isomorphism.*

Proof One direction is obvious since $\mathcal{F}(X)$ is an A-module, so for the other direction suppose $\varphi\colon \widetilde{M} \xrightarrow{\sim} \mathcal{F}$ is an isomorphism. By the adjunction, this must make the diagram

$$
\begin{tikzcd}
\widetilde{M} \arrow[r, "\widetilde{\varphi_X}"] \arrow[dr, "\varphi"'] & \widetilde{\mathcal{F}(X)} \arrow[d] \\
& \mathcal{F}
\end{tikzcd}
$$

commute. But φ_X must be an isomorphism, so $\widetilde{\varphi_X}$ is also, which completes the proof. $\square$

We now consider the stalks of $\widetilde{M}$. If $\mathfrak{p} \in \operatorname{Spec}(A)$, then on the underlying abelian groups we have:

$$
\begin{aligned}
\widetilde{M}_{\mathfrak{p}} &= \varinjlim_{D(f)\ni\mathfrak{p}} M(D(f)) = \varinjlim_{D(f)\ni\mathfrak{p}} M \otimes_A A_f \\
&= M \otimes_A \varinjlim_{D(f)\ni\mathfrak{p}} A_f = M \otimes_A A_{\mathfrak{p}} = M_{\mathfrak{p}}.
\end{aligned}
$$

It is clear that the $\mathcal{O}_{\operatorname{Spec}(A),\mathfrak{p}} = A_{\mathfrak{p}}$ structure agrees with the natural one on $M_{\mathfrak{p}}$. Tilde sheaves are a case where we can understand exact sequences completely.

Corollary 2.22 *Let* $X = \operatorname{Spec}(A)$ *be an affine scheme and* M', M, M'' *be* A*-modules. A sequence*

$$\widetilde{M'} \to \widetilde{M} \to \widetilde{M''}$$

is exact in $\operatorname{Mod}(\mathcal{O}_X)$ *if and only if the sequence*

$$M' \to M \to M''$$

obtained by applying $\Gamma(X, -)$ *is exact in* $\operatorname{Mod}(A)$*.*

Proof By full faithfulness, we know the maps are of the form $\widetilde{f}$ for some f. Taking the stalks at each $\mathfrak{p}$ shows that

$$M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}},$$

so $M' \to M \to M''$ is exact. Reversing the same logic gives the opposite direction. $\square$

Since $\widetilde{0}$ is the zero sheaf, it follows that $\widetilde{(-)}$ is an exact functor and preserves finite limits and colimits. But, we already know that all colimits are preserved because it is a left adjoint.

In future sections, it will be important to know when an $\mathcal{O}_X$-module is (isomorphic to) a tilde sheaf. We will develop some criterion for this to conclude this section.

Proposition 2.23 *Let* $X = \operatorname{Spec}(A)$ *be an affine scheme.*

1. *The cokernel, kernel, and image of a morphism between tilde sheaves is a tilde sheaf.*

2. *An $\mathcal{O}_X$-module $\mathcal{F}$ is a tilde sheaf if and only if there is a presentation as an exact sequence of the form*

$$\bigoplus_I \mathcal{O}_X \to \bigoplus_J \mathcal{O}_X \to \mathcal{F} \to 0$$

for some index sets I and J.

Proof (2) follows from (1): As $\mathcal{O}_X$-modules, $\mathcal{O}_X = \widetilde{A}$, and since $\widetilde{(-)}$ preserves colimits, in one direction $\mathcal{F}$ is the cokernel of a morphism of tilde sheaves. On the other hand, if $\mathcal{F} = \widetilde{M}$, then we can make a free resolution by

$$A^{(A^{(M)})} \to A^{(M)} \to M \to 0.$$

For (1), if $\widetilde{\varphi}\colon \widetilde{M} \to \widetilde{N}$ is a morphism of tilde sheaves then we know that

$$\widetilde{M} \to \widetilde{N} \to \widetilde{\operatorname{cok}(\varphi)} \to 0$$

is exact, hence $\operatorname{cok}(\widetilde{\varphi}) \cong \widetilde{\operatorname{cok}(\varphi)}$. The same proof works for the kernel case. Since the image is the kernel of a morphism to the cokernel, we are done. □

We will in the next few results use some terminology from cohomology, which we define now.

Definition 2.24 Let X be a space, $\mathcal{F} \in \operatorname{Sh}(X, \mathbf{Ab})$ be a sheaf of abelian groups, and $\{U_i : i \in I\}$ an open cover. A *Čech cocycle* of $\mathcal{F}$ over $\{U_i\}$ is a collection $(s_{ij})_{i,j}$ of sections $s_{ij} \in \mathcal{F}(U_i \cap U_j)$ satisfying

$$s_{ij} + s_{jk} = s_{ik} \qquad \text{over } U_i \cap U_j \cap U_k$$

for all $i, j, k \in I$.

A cocycle (s_{ij}) is *trivial* if there is a collection $(s_i)_i$ of sections $s_i \in \mathcal{F}(U_i)$ such that

$$s_{ij} = s_j - s_i \qquad \text{over } U_i \cap U_j$$

for all $i, j \in I$.

Proposition 2.25 *Let $X = \operatorname{Spec}(A)$ be an affine scheme with a Zariski open cover $\{D(f_1), \dots, D(f_n)\}$, and M be an A-module. Then every Čech cocycle of $\widetilde{M}$ over $\{D(f_i)\}$ is trivial.*

Proof Let (s_{ij}) be a Čech cocycle over the open cover. Then each $s_{ij} \in M_{f_i f_j}$, so we may write

$$s_{ij} = m_{ij}/(f_i f_j)^N \qquad m_{ij} \in M$$

for N large enough since the cover is finite. The cocycle condition says that $s_{ij} = s_{ik} - s_{jk}$ over $D(f_i f_j f_k)$, hence there is some large R such that

$$f_k^{N+R} \frac{m_{ij}}{(f_i f_j)^N} = f_k^R \left(\frac{m_{ik}}{f_i^N} - \frac{m_{jk}}{f_j^N} \right) \quad \text{in } M_{f_i f_j}.$$

Next, since $D(f_i) = D(f_i^{N+R})$, there are $a_k \in A$ such that $\sum_k a_k f_k^{N+R} = 1$. Finally, set:

$$s_i := \sum_k \frac{-a_k f_k^R m_{ik}}{f_i^N} \quad \text{in } M_{f_i}.$$

Then we have

$$\begin{aligned} s_j - s_i &= \sum_k a_k f_k^R \left(\frac{m_{ik}}{f_i^N} - \frac{m_{jk}}{f_j^N} \right) \\ &= \sum_k a_k f_k^{N+R} \frac{m_{ij}}{(f_i f_j)^N} = 1 \cdot s_{ij}, \end{aligned}$$

which shows that $(s_{ij})_{ij}$ is trivial. $\square$

Theorem 2.26 *Let* $X = \operatorname{Spec}(A)$ *be an affine scheme and*

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

be an exact sequence of $\mathcal{O}_X$*-modules with* $\mathcal{F}_1$ *a tilde sheaf. Then the sequence*

$$0 \to \mathcal{F}_1(X) \to \mathcal{F}_2(X) \to \mathcal{F}_3(X) \to 0$$

of A*-modules is exact.*

Proof First, since $\Gamma(X, -)$ is a right adjoint, it preserves kernels, so we already know that

$$0 \to \mathcal{F}_1(X) \xrightarrow{\psi_X} \mathcal{F}_2(X) \xrightarrow{\varphi_X} \mathcal{F}_3(X)$$

is exact. It remains to show that φ_X is surjective. If $t \in \mathcal{F}_3(X)$ is a global section, stalkwise surjectivity means that there is a Zariski open cover $D(f_1), \ldots, D(f_n)$ of X and sections $t_i \in \mathcal{F}_2(D(f_i))$ such that $\varphi(t_i) = t|_{D(f_i)}$. Restriction preserves exactness so our initial remark also means

$$0 \to \mathcal{F}_1(D(f_i)) \to \mathcal{F}_2(D(f_i)) \to \mathcal{F}_3(D(f_i))$$

is exact. So since $\varphi(t_j - t_i) = 0$, there are $s_{ij} \in \mathcal{F}_1(D(f_i f_j))$ such that $\psi(s_{ij}) = t_j - t_i$. We claim that (s_{ij}) is a Čech cocycle of $\mathcal{F}$ over $\{D(f_i)\}$. Indeed, this is true since

$$\psi(s_{ij} + s_{jk}) = t_j - t_i + t_k - t_j = \psi(s_{ik})$$

and by injectivity of ψ. Since $\mathcal{F}_1$ is a tilde sheaf, (s_{ij}) is trivial by Prop. 2.25; hence there are $s_i \in \mathcal{F}_1(D(f_i))$ such that $s_i - s_j = s_{ji}$, i.e. $\psi(s_i) - \psi(s_j) = t_i - t_j$.

To conclude, set $t_i' := t_i - \psi(s_i)$, so that $\varphi(t_i') = \varphi(t_i) = t|_{D(f_i)}$, yet

$$t_i' - t_j' = t_i - t_j - (\psi(s_i) - \psi(s_j)) = 0.$$

By the sheaf condition, the $\{t_j'\}$ glue into a $t' \in \mathcal{F}_2(X)$ which maps to t under φ, completing the proof. $\square$

Corollary 2.27 *Let* $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ *be an exact sequence of* $\mathcal{O}_X$*-modules. If any two of the* $\mathcal{F}_i$ *are tilde sheaves, then the third is as well.*

Proof In the two cases where $\mathcal{F}_2$ is assumed to be a tilde sheaf, we are already done by Prop. 2.23 using the (co)kernel scenario. Suppose $\mathcal{F}_1$ and $\mathcal{F}_3$ are tilde sheaves. Now consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\mathcal{F}_1(X)} \arrow[r] \arrow[d] & \widetilde{\mathcal{F}_2(X)} \arrow[r] \arrow[d] & \widetilde{\mathcal{F}_3(X)} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F}_3 \arrow[r] & 0 \end{tikzcd}$$

where the vertical maps are components of the counit map. Each square commutes by naturality of the counit. The bottom is exact by assumption, and the top row is exact by Thm. 2.26 and Cor. 2.22. The left and right rows are isomorphisms because F_1 and $\mathcal{F}_3$ are tilde sheaves. Then at each stalk, the five lemma shows that the map

$$\widetilde{\mathcal{F}_2(X)}_x \to \mathcal{F}_{2,x}$$

is an isomorphism, from which it follows the the middle column is an isomorphism, i.e. $\mathcal{F}_2$ is a tilde sheaf. □

2.3.3 Quasicoherent Sheaves

We are finally ready to discuss quasicoherent sheaves on schemes.

Definition 2.28 Let X be a scheme. A quasicoherent sheaf on X is an $\mathcal{O}_X$-module $\mathcal{F}$ for which there is an affine open cover $\{U_i\}$ of X such that for each i, $\mathcal{F}|_{U_i}$ is a tilde sheaf over U_i.

Recall that 'is a tilde sheaf' means that the counit map $\widetilde{\mathcal{F}(U_i)} \to \mathcal{F}|_{U_i}$ is an isomorphism. The crucial result on quasicoherent sheave is the following structure theorem.

Theorem 2.29 *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module over X. The following are equivalent:*

1. *$\mathcal{F}$ is quasicoherent.*
2. *For each affine open $U \subset X$, $\mathcal{F}|_U$ is a tilde sheaf over U.*
3. *X has an open cover $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is quasicoherent over U_i.*
4. *X has an affine open cover $\{U_i\}$ such that for each i there is an exact sequence*
$$\mathcal{O}_{U_i}^{(I)} \to \mathcal{O}_{U_i}^{(J)} \to \mathcal{F}|_{U_i} \to 0$$
for some index sets I, J.
5. *For each affine open $U \subset X$, there is an exact sequence*
$$\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F}|_U \to 0$$

for some index sets I, J.

Proof By Prop. 2.23, (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (5). Obviously (2) $\Rightarrow$ (1), and the equivalence of (1) and (2) will easily prove (3) by taking an affine open cover of each U_i. The same equivalence will also show that (3) $\Rightarrow$ (1). Hence it remains to prove (2) $\Rightarrow$ (1), for which we will use the affine communication lemma (Thm. 1.9): We must show that the property of being a tilde sheaf over U is Zariski local.

Fix $U = \operatorname{Spec}(A)$ an affine open. If $\mathcal{F}|_U$ is a tilde sheaf $\widetilde{M}$, then it is clear that $\mathcal{F}|_{D_A(f)} = \widetilde{M_f}$ because

$$\mathcal{F}|_{D_A(f)}(D_{A_f}(g)) = M_{fg} = (M_f)_g = \widetilde{M_f}(D_{A_f}(g)).$$

On the other hand, suppose that there is a Zariski cover $D_A(f_i)$ such that $\mathcal{F}|_{D_A(f_i)} = \widetilde{M_i}$ is a tilde sheaf. Letting $\mathcal{F}(U) = M$, we show that the counit map

$$\widetilde{M} \to \mathcal{F}|_U$$

is an isomorphism, which we can do by showing that for each g, the map $M_g \to \mathcal{F}(D_A(g))$ is an isomorphism. The sheaf condition gives the exact sequence of A-modules

$$0 \to M \to \bigoplus_i M_i \to \bigoplus_{i,j} M_{i,f_i f_j}$$

where by finiteness we can use sums. But then $\{D_A(f_i g)\}$ gives an open cover of $D_A(g)$, so we have an exact sequence

$$0 \to \mathcal{F}(D_A(g)) \to \bigoplus_i M_{i,g} \to \bigoplus_{i,j} M_{i,f_i f_j g}.$$

Localizing the first sequence at g we get a commutative diagram of A_g-modules

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M_g \arrow[r] \arrow[d] & \bigoplus_i M_{i,g} \arrow[r] \arrow[d] & \bigoplus_{i,j} M_{i,f_i f_j g} \arrow[d] \\
0 \arrow[r] & \mathcal{F}(D_A(g)) \arrow[r] & \bigoplus_i M_{i,g} \arrow[r] & \bigoplus_{i,j} M_{i,f_i f_j g}
\end{tikzcd}$$

with exact rows. It follows that $M_g \to \mathcal{F}(D_A(g))$ is an isomorphism, completing the proof. $\square$

We write $\operatorname{QCoh}(X)$ for the full subcategory of $\operatorname{Mod}(\mathcal{O}_X)$ consisting of quasicoherent schemes. The functor $\widetilde{(-)}$ then clearly restricts to

$$\widetilde{(-)}\colon \operatorname{Mod}(A) \to \operatorname{QCoh}(\operatorname{Spec}(A))$$

in the affine case.

Corollary 2.30 *Every quasicoherent sheaf over an affine scheme X is a tilde sheaf. In particular, the functor*

$$\widetilde{(-)}\colon \operatorname{Mod}(\Gamma(X, \mathcal{O}_X)) \xrightarrow{\sim} \operatorname{QCoh}(X)$$

is an equivalence of categories.

Proof The first statement is clear since X is an affine open of itself, and the second statement follows since we already have full faithfulness. □

We can now use our work in the previous subsection to learn properties of quasicoherent sheaves.

Proposition 2.31 *Let X be a scheme.*

1. *The cokernel, kernel, and image of a morphism between quasicoherent sheaves on X is quasicoherent.*
2. *If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules in which two of the $\mathcal{F}_i$ are quasicoherent, then the third is too.*

Proof Observe that $\operatorname{cok}(\varphi|_U) = \operatorname{cok}(\varphi)|_U$ beacuse sheafification commutes with restriction, and the same holds trivially for ker. Then the cokernel and kernel cases follow from Prop. 2.23, and the image case follows from those. For (2), exact sequences commute with restriction, so we are done by restricting to every affine open using Cor. 2.27. □

We will now prove an analog of the qcqs lemma from earlier. This will be useful in proofs.

Proposition 2.32 *Let X be a quasicompact quasiseparated scheme, $f \in \Gamma(X, \mathcal{O}_X)$ a global section, and $\mathcal{F}$ a quasicoherent sheaf. Then $\Gamma(X_f, \mathcal{F})_f = \Gamma(X_f, \mathcal{F})$, and the localization of the restriction map*

$$\Gamma(X, \mathcal{F})_f \xrightarrow{\sim} \Gamma(X_f, \mathcal{F})$$

is an isomorphism of $\Gamma(X_f, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)_f$-modules.

Proof The first part follows from Prop. 1.4, namely that f is a unit $\Gamma(X_f, \mathcal{O}_X)$, so localizing $\mathcal{F}(X_f)$ at f doesn't affect it. Let $\{U_i\}$ be a finite affine open cover of X with $U_i = \operatorname{Spec}(A_i)$, and we can cover $U_i \cap U_j = \bigcup_k U_{ijk}$ by finitely many affines $U_{ijk} = \operatorname{Spec}(A_{ijk})$. By finiteness, we get an exact sequence of $\Gamma(X, \mathcal{O}_X)_f$-modules

$$0 \to \Gamma(X, \mathcal{F}) \to \bigoplus_i M_i \to \bigoplus_{i,j,k} M_{ijk},$$

and localizing at f we get the sequence

$$0 \to \Gamma(X, \mathcal{F})_f \to \bigoplus_i M_i[f^{-1}] \to \bigoplus_{i,j,k} M_{ijk}[f^{-1}],$$

of $\Gamma(X, \mathcal{O}_X)_f$-modules. On the other hand, $U_i \cap X_f = U_{i,f} = \mathrm{Spec}(A_i[f^{-1}])$ and $U_{ijk,f} = \mathrm{Spec}(A_{ijk}[f^{-1}])$. It follows that $\Gamma(X_f, \mathcal{F})$ fits into the same sequence, and there is a unique isomorphism

$$\Gamma(X, \mathcal{F})_f \xrightarrow{\sim} \Gamma(X_f, \mathcal{F})$$

commuting with the maps into $\bigoplus_i M_i[f^{-1}]$. It is then clear that this isomorphism is the desired one. $\square$