Tannakian Reconstruction for Tensor Categories

Vihaan Dheer

1 Introduction

This is a short exposition on P. Deligne's 'Tannakian reconstruction' theorem regarding certain tensor categories. The definitions needed to state the theorem, although relatively simple, are numerous; hence the goal of this paper is to state Deligne's theorem and discuss examples, particularly in representation theory and algebraic geometry.

The main source are [1], and [2]; geometric content comes form [3]. An introductory and expository source was [4]. Most proofs can be found in those papers and will therefore be omitted here for brevity.

1.1 Tannakian Duality

The goal of Tannakian reconstruction or duality is to recover an object from its associated category of 'representations', appropriately construed. In most scenarios this is not possible so one determines under what conditions this can be done. 'Tannakian' usually connotes that the category should be viewed with a symmetric monoidal structure.

There are two classes of reconstruction theorems. One ponders whether a representation category *determines* the underlying object, and the other asks when we can, given a category satisfying some properties, produce an object whose representations are the category. Deligne's theorem is of the latter form, which is interesting when the former question is trivial; we begin with some examples of such reconstruction results.

If R is a ring, the category of its modules $\mathrm{Mod}(R)$ are its representations. We might ask when an additive equivalence $\mathrm{Mod}(R) \cong \mathrm{Mod}(S)$ implies $R \cong S$. It is true for instance if R and S are commutative; in general this is the theory of Morita equivalence.[1]

We may ask the same thing for representations of a group G in set, so $\mathrm{Rep}(G) \coloneqq \mathbf{Set}(G)$. Indeed, in this case we always have:

Theorem *Let G, G' be groups such that $\mathbf{Set}(G) \cong \mathbf{Set}(G')$. Then $G \cong G'$.*

Proof The Yoneda lemma gives a canonical isomorphism

$$\mathrm{Aut}\big(\mathbf{Set}(G) \to \mathbf{Set}\big) \xrightarrow{\sim} G$$

[1]Note, however, that if we regard the categories as monoidal and ask for a monoidal equivalence, it is completely trivial.

where $\mathbf{Set}(G) \to \mathbf{Set}$ is the forgetful functor regarded as an object of the functor category $\mathrm{Fun}(\mathbf{Set}(G), \mathbf{Set})$. $\square$

If we think of G as a one-object groupoid with morphisms the elements of G, then $\mathbf{Set}(G) = \mathrm{Fun}(G, \mathbf{Set})$. Hence it is natural to think of $\mathrm{PSh}(C) := \mathrm{Fun}(C^{\mathrm{op}}, \mathbf{Set})$ as a category of representations of an arbitrary small category C. So, when does $\mathrm{PSh}(C) \cong \mathrm{PSh}(D)$ imply $C \cong D$? It is another (easy) reconstruction theorem that it holds if and only if C and D are idempotent complete, i.e. every idempotent splits as a section/retract pair.

Our example of commutative rings shows that if X and Y are affine schemes with the same quasicoherent sheaves, namely an additive equivalence

$$\mathrm{QCoh}(X) \xrightarrow{\sim} \mathrm{QCoh}(Y),$$

then $X \cong Y$. The case is more interesting when X and Y are not affine; in some cases the derived category of only coherent sheaves is enough to conclude $X \cong Y$ (see the Bondal-Orlov theorem).

All of these examples are of the first class. Deligne's theorem, however, is about constructing an object from a potential representation category. We state it now:

Theorem (Deligne) *Let k be an algebraically closed field of characteristic zero, and $\mathcal{A}$ a regular k-tensor category. There is an affine algebraic supergroup G and an inner parity σ such that there is an equivalence*

$$\mathcal{A} \xrightarrow{\sim} \mathrm{Rep}_k(G, \sigma)$$

of tensor categories.

This is a theorem fundamentally about 'superalgebra', which we will discuss in the next section. There is a similar algebro-geometric Deligne's theorem as follows:

Theorem (Deligne 2) *Let k be an algebraically closed field of characteristic zero, and $\mathcal{A}$ a regular k-tensor category. There is an affine group scheme G and an equivalence*

$$\mathcal{A} \xrightarrow{\sim} \mathrm{Rep}_k(G)$$

of tensor categories.

The rest of this paper will be devoted to explaining the statement of super-Deligne's theorem. We will briefly discuss the algebraic geometry statement as well.

2 Super-representation Theory

In this section we will describe the superalgebra involved in stating Deligne's theorem. By the end we will be able to define a supergroup and its k-tensor category of super-representations. This discussion will *not* be entirely self-contained as we will need the following result from the representation theory of finite groups. By *partition* of a number n we mean a sequence $n_1 \geq n_2 \geq \cdots \geq n_k$ such that $\sum_i n_i = n$.

Theorem 2.1 *Let k be a field of characteristic 0. For each $n \geq 0$, the irreducible representations of the symmetric group S_n are classified by the partitions of n.*

For each partition $\lambda = (n_1, \ldots, n_k)$ of n, we write V_λ for the corresponding irreducible representation $S^n \to \mathrm{GL}(V_\lambda)$. This will appear as part of the definition of the Schur functor as we will see. We begin the general theory now.

2.1 Hopf algebras

Let $C = (C, \otimes, I)$ be a symmetric monoidal category. In general, we will write the associator, unitors, and braiding as identities; this can either be reconciled by inserting them everywhere and applying MacLane's coherence theorem, or by replacing C with a monoidally equivalent strict monoidal category.

Recall that a monoid in C is an object c equipped with morphisms $m\colon c \otimes c \to c$ and $\eta\colon I \to c$ such that the diagrams

$$\begin{tikzcd} c \otimes c \otimes c \arrow[r, "1\otimes m"] \arrow[d, "m\otimes 1"'] & c \otimes c \arrow[d, "m"] \\ c \otimes c \arrow[r, "m"] & c \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "1\otimes\eta"] \arrow[dr, equal] & c \otimes c \arrow[d] & c \arrow[l, "\eta\otimes 1"'] \arrow[dl, equal] \\ & c & \end{tikzcd}$$

commute. A comonoid in C is a monoid in C^{op}. A (co)monoid is (co)commutative if the diagram

$$\begin{tikzcd} c \otimes c \arrow[rr, "\mathrm{flip}"] \arrow[dr, "m"'] & & c \otimes c \arrow[dl, "m"] \\ & c & \end{tikzcd}$$

commutes in C (C^{op}). A morphism of monoids in C is a morphism $f\colon c \to d$ so that

$$\begin{tikzcd} c \otimes c \arrow[r, "f\otimes f"] \arrow[d, "m"'] & d \otimes d \arrow[d, "m"] \\ c \arrow[r, "f"] & d \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, "\eta"'] \arrow[dr, "\eta"] & \\ c \arrow[r, "f"] & d \end{tikzcd}$$

commute, and likewise for comonoids. Then $\mathrm{Mon}(C, \otimes)$ and $\mathrm{Comon}(C, \otimes)$ form categories. Some examples are:

$$\mathrm{Mon}(\mathbf{Set}, \times) = \{\text{monoids}\}, \quad \mathrm{Mon}(\mathbf{Cat}, \times) = \{\text{strict monoidal categories}\}$$
$$\mathrm{Mon}(\mathbf{Grp}, \times) = \mathbf{Ab}, \quad \mathrm{Mon}(\mathrm{Vect}(k), \otimes) = \{\text{associative } k\text{-algebras}\}$$
$$\mathrm{Mon}(C, \sqcup) = \mathrm{Comon}(C, \times) = C$$

In the last example C is any category with finite products or coproducts. It is clear from the definition that $\mathrm{Mon}(C)$ admits a faithful functor to C. A *bialgebra* in C is an object c which is both a monoid and comonoid such that the compatibility diagram

$$\begin{tikzcd} c \otimes c \arrow[r, "m"] \arrow[d, "w\otimes w"] & c \arrow[r, "w"] & c \otimes c \\ c \otimes c \otimes c \otimes c \arrow[rr, "1\otimes\mathrm{flip}\otimes 1"] & & c \otimes c \otimes c \otimes c \arrow[u, "m\otimes m"'] \end{tikzcd}$$

commutes, as do the identity diagrams:

$$\begin{tikzcd} c \otimes c \arrow[r, "m"] \arrow[dr, "\nu\otimes\nu"'] & c \arrow[d, "w"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\eta"] \arrow[dr, equal] & c \arrow[d, "\nu"] \\ & I \end{tikzcd} \qquad \begin{tikzcd} I \arrow[r, "\eta"] \arrow[dr, "\eta\otimes\eta"'] & c \arrow[d, "w"] \\ & c \otimes c \end{tikzcd}$$

A general example is any semiadditive (i.e. pointed with finite biproducts) category: Indeed, each object is uniquely a bimonoid of $(C, \oplus, 0)$, and $\mathrm{Bimon}(C, \oplus) = C$. More interesting is the case of $C = \mathrm{Vect}(k)$ with $\otimes_k$, in which case a bimonoid is a biassociative bialgebra. Let $A = k[x_1, \ldots, x_n]$ be the standard commutative polynomial algebra. A also has a coalgebra structure where

$$w(f) = f(0, \ldots, 0) \qquad \nu(x_i) = x_i \otimes 1 + 1 \otimes x_i,$$

and it is easy to A becomes a bialgebra.

Finally, we define a *Hopf algebra* in C as a bialgebra c equipped with a map $S\colon c \to c$ making

$$\begin{tikzcd} c \otimes c \arrow[d, "1\otimes S"'] & c \arrow[l, "w"'] \arrow[r, "w"] \arrow[d, "\eta\circ\nu"] & c \arrow[d, "S\otimes 1"] \\ c \otimes c \arrow[r, "m"] & c & c \otimes c \arrow[l, "m"'] \end{tikzcd}$$

commute. As before, these form a category $\mathrm{Hopf}(C)$. For example, $\mathrm{Hopf}(\mathbf{Set}, \times) = \mathbf{Grp}$, and taking $C = \mathrm{Vect}(k)$ with $\otimes$ recovers a classical Hopf algebra. A nice example is the cohomology of Lie groups: For k a field, there is a functor

$$H^\bullet(-, k)\colon \mathrm{LieGrp} \to \mathrm{Hopf}(\mathrm{Vect}(k))$$

taking a Lie group to its graded cohomology with cup product, with comultiplication induced by the group operation.[2] For fun, we may write $\mathrm{LieGrp} = \mathrm{Hopf}(\mathrm{SmMan}, \times)$. For a final example, we can take our polynomial bialgebra before and define $S(x_i) = -x_i$ to get a Hopf algebra.

2.2 Superalgebra

We will now briefly discuss 'superobjects', which means objects with a $\mathbb{Z}_2$ grading. For example, a *super vector space* over k is a vector space V with a decomposition $V = V_0 \oplus V_1$. We think of V_0 as the 'even' subspace and V_1 as the odd subspace. A homogeneous element is one contained in $V_0 \cup V_1 \setminus 0$, and its parity $p(v) \in \mathbb{Z}_2$ is the summand it is in.

It is clear how to define $\oplus$ of two super vector spaces. For $\otimes$, we can set

$$V \otimes W = \big((V_0 \otimes W_0) \oplus (V_1 \otimes W_1)\big)_0 \oplus \big((V_0 \otimes W_1) \oplus (V_1 \otimes W_0)\big)_1$$

so that a homogeneous element is a sum of simple tensors $v \otimes w$ where either all v and w have the same parity, or they all have opposite parity. This shows $\mathrm{Hom}_k(V, W)$ is a super vector space: $\mathrm{Hom}_k(V, W)_0$ consists of *even maps* which preserve the grading, and $\mathrm{Hom}_k(V, W)_1$ has odd maps which reverse it.

Based on this, we define the category $\mathrm{SVect}(k)$ of super vector spaces with hom sets $\mathrm{Hom}_k(V, W)_0$ (even maps). In particular, it is enriched in $\mathrm{Vect}(k)$, but not $\mathrm{SVect}(k)$. Our definition of the tensor product shows that $\mathrm{SVect}(k)$ is a symmetric monoidal category after a quick verification that all maps involved are even. Hence there is a natural faithful monoidal inclusion $F\colon \mathrm{SVect}(k) \to \mathrm{Vect}(k)$.

There is also a canonical map back to $\mathrm{SVect}(k)$ defined by sending $V \mapsto (V)_0 \oplus (0)_1$. All linear maps between such super vector spaces are even, hence this is a fully faithful embedding $i\colon \mathrm{Vect}(k) \hookrightarrow \mathrm{SVect}(k)$. Observe that an even map $f\colon V \to i(W)$ is the same as a map $V \to W$ whose kernel contains V_1. In other words, there is

[2] Note that, indeed, we must have k a field to get an isomorphism $H^\bullet(G, k) \otimes H^\bullet(G, k) \cong H^\bullet(G \times G, k)$ by Kunneth's formula.

another functor $p\colon \mathrm{SVect}(k) \to \mathrm{Vect}(k)$ defined by collapsing the odd summand, and the identification

$$\mathrm{Hom}_{\mathrm{Vect}(k)}(p(V), W) \cong \mathrm{Hom}_{\mathrm{SVect}(k)}(V, i(W))$$

exhibits a fully faithful adjunction $p \dashv i$. On the other hand, for $W = W_0 \oplus W_1$ a linear map $V \to F(W)$ is the same as a pair of maps $V \to W_0$ and $V \to W_1$, which is in turn the same as a graded map $(V)_0 \oplus (V)_1 \to W$. Hence there is yet another functor $d\colon \mathrm{Vect}(k) \to \mathrm{SVect}(k)$ so that $d(V) = V \oplus V$ with the obvious grading, and the identification

$$\mathrm{Hom}_{\mathrm{SVect}(k)}(d(V), W) \cong \mathrm{Hom}_{\mathrm{Vect}(k)}(V, F(W))$$

as an adjunction $d \dashv F$. Notice that $p \circ d \cong \mathrm{Id}$. This is actually a *biadjoint*: flipping the same logic also shows

$$\mathrm{Hom}_{\mathrm{Vect}(k)}(F(V), W) \cong \mathrm{Hom}_{\mathrm{SVect}(k)}(V, d(W)),$$

and hence $F \dashv d$. If the pattern is not already clear, the same works for our first adjunction too: $i \dashv p$. This pair of biadjoint pairs is remarkably strong.

Because of our generality in the previous section, we can apply everything to $\mathrm{SVect}(k)$ to get the notions of superalgebra, superbialgebra, and super Hopf algebra. We will, however, add our own notion of 'supercommutativity': A superalgebra will be supercommutative if $ab = (-1)^{p(a)p(b)}ba$. In other words, if A is a superalgebra then $A_0 \subset A$ is a commutative subsuperalgebra, but A_1 is neither commutative nor a subalgebra. Given two superalgebras A, B, we make $A \otimes B$ a superalgebra be declaring

$$(a \otimes b)(a' \otimes b') = (-1)^{p(a')p(b)}(aa' \otimes bb')$$

for all a, b' and homogeneous a', b.

We finish this part with an important example of a Hopf algebra on $\mathrm{Vect}(k)$ and $\mathrm{SVect}(k)$. Let G be a group and consider the group algebra $k[G]$. Define $w(g) = g \otimes g$, and set $\nu(g) = 0$ unless $g = e \in G$, in which case $\nu(e) = 1$. Finally, setting $S(g) = g^{-1}$ gives a Hopf algebra structure on $k[G]$. Given a Hopf algebra H, its dual $H*$ is canonically a Hopf algebra. We will write

$$\mathcal{O}(G) \coloneqq k[G]^*$$

for the dual Hopf algebra of the group algebra. Unwinding the definitions shows that, for $f \in \mathcal{O}(G)$, $S^*(f)(x) = f(x^{-1})$, and $\eta^*(f) = f(e)$. For (co)multiplication, we have

$$m^*(f) = f \circ m \qquad w^*(f \otimes g) = (f \otimes g) \circ w.$$

By $\mathbb{Z}_2$-grading $k[G]$, we get the corresponding definition $\mathcal{O}(G)$ of the associated Hopf superalgebra.

2.3 Tensor Categories

In this section we define the notion of a k-tensor category; we will provide the definition at the start and work backwards.

Definition 2.2 Let k be a field. A k-tensor category $\mathcal{A}$ is a k-linear, rigid, symmetric monoidal category.

That $\mathcal{A}$ is k-linear means it is enriched in $\mathrm{Vect}(k)$; for example, $\mathrm{Vect}(k)$, or $\mathrm{SVect}(k)$ by our earlier remark. An object $A \in \mathcal{A}$ is left rigid if there is $B \in \mathcal{A}$ and morphisms

$$\mathrm{coev}\colon I \to A \otimes B \qquad \mathrm{ev}\colon B \otimes A \to I$$

such that the composites

$$A \xrightarrow{\mathrm{coev}\otimes 1} A \otimes B \otimes A \xrightarrow{1\otimes \mathrm{ev}} A$$
$$B \xrightarrow{1\otimes \mathrm{coev}} B \otimes A \otimes B \xrightarrow{\mathrm{ev}\otimes 1} B$$

are the identities. We say $\mathcal{A}$ is rigid if each object is both left and right rigid. Once again, both $\mathrm{Vect}(k)$ and $\mathrm{SVect}(k)$ are easily seen to be rigid by taking duals, and hence k-tensor categories.

Not all symmetric monoidal categories are rigid. For example, if X is a scheme then $\mathrm{QCoh}(X)$ is a symmetric monoidal category which is not generally rigid. A sheaf $\mathcal{F}$ is rigid if and only if it is a vector bundle, i.e. it is locally trivial. If X is affine, the following are even equivalent:

(a) $\mathrm{QCoh}(X)$ is a rigid symmetric monoidal category.

(b) X is the spectrum of a semisimple Artinian ring.

(c) X is a discrete finite set.

Unfortunately, not all k-tensor categories satisfy Deligne's theorem; we will need a regular property. Recall from Thm. 2.1 that V_λ denotes the irreducible representation of S_n corresponding to a partition λ of n.

Remark 2.3 Let $\mathcal{A}$ be a k-tensor category. For the sake of succinctness, in the following definitions we will assume that objects of $\mathcal{A}$ look like k-vector spaces. This is not totally unfounded. If $X \in \mathcal{A}$, the enriched Yoneda lemma says that the assignment $X \mapsto \mathrm{Hom}_k(-, X)$ defines a fully faithful embedding

$$\mathcal{A} \to \mathrm{Fun}(\mathcal{A}^{\mathrm{op}}, \mathrm{Vect}(k)).$$

This latter category behaves very much like $\mathrm{Vect}(k)$: it is abelian, $\mathrm{Vect}(k)$-enriched, and limits and colimits are computed pointwise in $\mathrm{Vect}(k)$. In any case, however, the real condition we want on $\mathcal{A}$ is at least that it has some abelian-ity.

Now let X be an object of a tensor category $\mathcal{A}$, and consider the natural action of S^n on

$$X^{\otimes n} = X \otimes \cdots \otimes X.$$

Now V_λ and $X^{\otimes n}$ are representations of S_n, so we define the *Schur functor*

$$S_\lambda(X) \coloneqq (V_\lambda \otimes X^{\otimes n})^{S_n}$$

to be the subspace of invariants of the tensor product. If $\mathcal{A}$ is a k-tensor category, we say it is finitely generated by $\otimes$ if there is an object X such that any object is a subquotient of a direct sum of objects $X^{\otimes n}$. For instance, the *odd* space $k = (0)_0 \oplus (k)_1$ finitely generates $\mathrm{SVect}(k)_{\mathrm{fin}}$. We conclude with a central definition:

Definition 2.4 A k-tensor category $\mathcal{A}$ is *regular* if $\mathcal{A}$ is finitely generated and its Schur functor $S_\bullet$ is non-singular: For each $X \in \mathcal{A}$ there is n, λ such that $S_\lambda(X) = 0$.

In other words, the action of some S^n on the nth tensor power is faithful up to tensoring by an irreducible representation of S_n.

3 The Deligne Theorems

In this section we state Deligne's theorem on Tannakian reconstruction. Let H be a Hopf superalgebra over k and write $\mathcal{O}(H) = H^*$ for its dual. An H-module is a super vector space V equipped with an even action of the algebra structure on H, i.e. a superalgebra map $\rho\colon H \to \mathrm{Hom}_k(V, V)$. Dually, an H-comodule is a super vector space W with a coaction $\rho\colon V \to V \otimes A$. One can see that an H-comodule (resp. module) is the same as an $\mathcal{O}(H)$-module (comodule). We write the category $\mathrm{Rep}(H)$ and $\mathrm{Rep}(\mathcal{O}(H))$ of H-modules and comodules; if H is finite dimensional, each is a regular k-tensor category.

Definition 3.1 Let k be an algebraically closed field of characteristic zero. An affine algebraic supergroup is a Hopf superalgebra G which is dual to a supercommutative Hopf superalgebra $\mathcal{O}(G)$.

For any superalgebra A, there is a canonical involution $i\colon A \to A$ such that for each homogeneous element $h \in A$, $i(h) = (-1)^{p(h)}h$. Now let $g \in G$ be an element of supergroup. We write $g^2 = 1$ if $(g \otimes g) \circ w = \nu$. Each such g induces an automorphism of $\mathcal{O}(G)$ in a canonical way.

If G is a supergroup, an inner parity of G is an element $g \in G$ for which $g^2 = 1$ and g induces the canonical involution i. Fixing an inner parity g, we set define a representation of (G, g) as a representation of G which respects the parity. These assemble into a k-tensor category $\mathrm{Rep}(G, g)$. If the Hopf algebra $\mathcal{O}(G)$ is finite dimensional, then $\mathrm{Rep}(G, g)$ is a regular k-tensor category. At last, Deligne's theorem can be stated:

Theorem 3.2 (Deligne) *Let k be an algebraically closed field of characteristic zero, and $\mathcal{A}$ a regular k-tensor category. There is a supergroup G and an inner parity $g \in G$ such that there is an equivalence*

$$\mathcal{A} \xrightarrow{\sim} \mathrm{Rep}_k(G, g)$$

of tensor categories.

Hence, the supergroup G and the parity $g \in G$ may be reconstructed from the data of the tensor category. Moreover, this is unique: considering the forgetful functor

$$F\colon \ \mathrm{Rep}(G, \sigma) \to \mathrm{SVect}(k)_{\mathrm{fin}}$$

called a 'fiber functor', we may recover the supergroup as automorphisms of F, suitably construed. Finally, we give the geometric statement. Recall that an affine group scheme is an affine k-scheme which equipped with a group object structure in $(\mathbf{Sch}_k, \times_k, 1)$. It is not hard to see that Spec induces an equivalence between affine group schemes and commutative Hopf algebras over k. A simple example is $\mathbb{G}_m \coloneqq \mathrm{Spec}(k[x^{\pm}])$. Given such a group scheme, we can define its category of representations $\mathrm{Rep}(G)$; geometric Deligne's theorem then says:

Theorem 3.3 *Let k be an algebraically closed field of characteristic zero, and $\mathcal{A}$ a regular k-tensor category. There is an affine group scheme G and an equivalence*

$$\mathcal{A} \xrightarrow{\sim} \mathrm{Rep}_k(G)$$

of tensor categories.

We remark that similarly to the supergroup case, the affine group scheme G is determined up to isomorphism by considering the enriched automorphism group of the associated fiber functor.

References

[1] Deligne, P.: Catégories Tensorielles. Moscow Mathematical Journal (2002)

[2] Deligne, P.: Catégories Tannakiennes. In: The Grothendieck Festschrift, Vol. II. Progr. Math., vol. 87, pp. 111–195. Birkhäuser Boston, Boston, MA (1990)

[3] Deligne, P., Milne, J.: Tannakian Categories. From Milne's website. (2018)

[4] Lepine, D.: Deligne's Theorem on Tensor Categories. Available online.