Extending the Dold-Kan Correspondence

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1 Introduction

1.1 Motivation

This is an expository paper on the Dold-Kan correspondence, a classic result in simplicial homotopy theory. The question which Dold-Kan answers is quite natural from the perspective of algebraic topology. Namely, if $A_\bullet$ is a simplicial abelian group, we can try to define a singular chain complex (C_*, ∂_*) by setting $C_n = A_n$ and taking the boundary map to be alternating sums of face operators d_i. This defines a functor

$$\mathrm{Moore}\colon \mathbf{sAb} \longrightarrow \mathrm{Ch}_{\geq 0}(\mathbb{Z})$$

from simplicial abelian groups to non-negatively graded chain complexes of abelian groups called the Moore complex. Given the resemblance of the objects (i.e. a sequence of abelian groups) in either category, one might ask whether it is an equivalence.

The answer is no because of the degeneracy operators $s_i\colon A_{n-1} \to A_n$, whose image makes $\mathrm{Moore}(A_\bullet)$ too large. Instead, we will take the quotient of the Moore complex by the subcomplex generated by degenercies, known as the *normalized* Moore complex. The Dold-Kan correspondence says that this construction

$$N_*\colon \mathbf{sAb} \xrightarrow{\sim} \mathrm{Ch}_{\geq 0}(\mathbb{Z})$$

is an equivalence of categories. This will be our first goal. The proof will consist of constructing an inverse functor.

Next, we will attempt to generalize this. It has long been known that we may replace the category of abelian groups by any abelian category and essentially the same proof works with standard modifications [1, Thm. 2.5]. But, following Lurie [2, Ch. 1.2.3], we will eventually prove the equivalence for any additive idempotent complete category.

Along the way, we use the theorem to gain homotopical information about both categories in question. We will also deduce a partial form of the Dold-Kan correspondence in novel generality, for *any* semiadditive category. This is a technical improvement of [2, Thm. 1.2.3.7].

1.2 Roadmap

The outline of this exposition is as follows: We begin by constructing the normalized Moore complex N_* as explained, and then finding a simpler, isomorphic description which will help prove Dold-Kan.

The next step is to define the inverse functor to N_*; in preparation for extending Dold-Kan, we will setup the inverse in as general a setting possible. Then we specialize to a case much closer to **Ab**, namely the category **CMon** of commutative monoids, and prove it is a one-sided inverse.

To finish the Dold-Kan correspondence, we will switch back to **Ab** and swiftly complete the proof. We conclude the section with two important corollaries: (1) Up to chain homotopy, the Moore complex *is* the normalized complex; and (2) the homotopy groups of a simplicial abelian group coincide with the homology of the corresponding chain complex.

Once we have this, we generalize. First we will use our result about **CMon** to deduce that the inverse construction is fully faithful for any semiadditive category. We then need a small categorical detour to discuss idempotent completeness, and we will finish by proving the Dold-Kan correspondence for any additive category which is idempotent complete.

2 The Dold-Kan Correspondence

2.1 The Normalized Moore Complex

Definition 2.1 A *non-negatively graded chain complex* over $\mathbb{Z}$ is a sequence $A = A_\bullet$ of abelian groups

$$\cdots \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots \to A_1 \xrightarrow{\partial_1} A_0$$

such that $\partial_n \partial_{n+1} = 0$ for all $n \in \mathbb{N}$. A morphism of such complexes $f\colon (A, \partial) \to (B, \partial)$ is a sequence of maps $f_n\colon A_n \to B_n$ making

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "\partial_n"] & B_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

commute for all n. We write $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ for the corresponding category.

On the other hand, let $\mathbf{sAb} = [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Ab}]$ be the category of simplicial abelian groups. The goal is to construct an equivalence of categories $\mathbf{sAb} \xrightarrow{\sim} \mathrm{Ch}_{\geq 0}(\mathbb{Z})$. We will first set up three intimately related functors $\mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$ and learn their properties.

1. *Moore Complex.* Let A be a simplicial abelian group. We associate to it a chain complex (A_*, ∂) given by

$$\cdots \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

where $\partial_n = \sum_{i=0}^{n}(-1)^i d_i$. This both a chain complex and is functorial in A:

Proposition 2.2 *Let A be a simplicial abelian group. Then A_* is a chain complex, and the assignment $A \mapsto A_*$ defines a functor* $\mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$.

Proof We first need $\partial_{n-1}\partial_n = 0$. From the definition

$$\begin{aligned}
\partial_{n-1}\partial_n &= \partial_{n-1}\sum_i (-1)^i d_i = \sum_{i,j}(-1)^{i+j} d_j d_i \\
&= \sum_{i\leq j}(-1)^{i+j} d_j d_i + \sum_{i>j}(-1)^{i+j} d_{i-1} d_j \\
&= \sum_{i\leq j}(-1)^{i+j} d_j d_i - \sum_{l\leq k}(-1)^{k+l} d_k d_l = 0,
\end{aligned}$$

where in the second step we used the simplicial identities, and in the third we set $k = i-1$ and $l = j$. For functoriality, note that a simplicial map $f\colon A \to B$ is a natural transformation, so

$$\begin{array}{ccc}
A_n & \xrightarrow{d_i} & A_{n-1} \\
{\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n-1}} \\
B_n & \xrightarrow{d_i} & B_{n-1}
\end{array}$$

must commute. But since ∂_n is defined as a linear combination of face maps, so the same square obtained by replacing d_i with ∂_n still commutes. □

Definition 2.3 The chain complex A_* is the *Moore complex* associated to a simplicial abelian group.

Example 2.4 Let X be a topological space and $\mathrm{Sing}(X)$ the associated singular simplicial set. Then the chain complex $\mathrm{Sing}(X)_*$ is the usual singular complex $C_*(X)$ with $\mathbb{Z}$ coefficients from topology, whose homology is that of X.

2. Reduced Moore Complex Recall that an n-simplex $x \in A_n$ of a simplicial abelian group is *degenerate* if there is $y \in A_{n-1}$ such that $s_i(y) = x$ for some $i \leq n-1$. We claim there is a subcomplex $DA_* \subset A_*$ such that DA_n is generated by the degenerate n-simplices of A_n. We need to show the restriction of ∂_n to DA_n takes values in DA_{n-1}; i.e. $\partial(s_i x) \in DA_{n-1}$ for $x \in A_{n-1}$. Since $d_i s_i = d_{i+1}s_i = 1$, we have:

$$\partial(s_i x) = \sum_j (-1)^j d_j s_i(x) = \sum_{j\neq i,i+1}(-1)^j d_j s_i(x)$$

$$= \sum_{j>i+1} (-1)^j d_j s_i(x) + \sum_{j<i} (-1)^j d_j s_i(x)$$
$$= \sum_{j>i+1} (-1)^j s_i d_{j-1}(x) + \sum_{j<i} (-1)^j s_{i-1} d_j(x)$$

by the simplicial identities, so we have represented $\partial(s_i x)$ as a sum of degenerate simplices.

Definition 2.5 The *reduced complex* associated to a simplicial abelian group is the quotient complex $A_*/DA_* =: (A/DA)_*$. We will write

$$q \colon A_* \twoheadrightarrow (A/DA)_*$$

for the canonical chain quotient map.

This is clearly functorial because the construction of $DA_* \subset A_*$ is, giving our second functor $\mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$; then the quotient map q is a natural transformation.

3. Normalized Complex. Let A be a simplicial abelian group, and set

$$NA_n = \bigcap_{i=1}^{n} \ker\left(d_i \colon A_n \to A_{n-1}\right) \subset A_n.$$

We take boundary maps $NA_n \to NA_{n-1}$ to be d_0, which is well-defined because if $x \in NA_n$ and $0 < i \leq n-1$ then

$$d_i d_0(x) = d_0 d_{i+1}(x) = 0.$$

Moreover, $d_0 d_0(x) = d_0 d_1(x) = 0$ implies that (NA_*, d_0) is a chain complex. While it is not a subcomplex, the 'inclusion' $i \colon NA_* \to A_*$ is a chain map. Indeed,

$$\begin{tikzcd} NA_n \arrow[r, "d_0"] \arrow[d, hook] & NA_{n-1} \arrow[d, hook] \\ A_n \arrow[r, "\partial_n"] & A_{n-1} \end{tikzcd}$$

commutes since if $x \in NA_n$ then $\partial_n(x) = d_0 x + \sum_{i\geq 1} d_i(x) = d_0(x)$.

Proposition 2.6 *The assignment $A \mapsto NA_*$ is functorial, and i is a natural transformation.*

Proof If $f \colon A \to B$ is a simplicial map, then for all i

$$\begin{tikzcd} A_n \arrow[r, "f_n"] \arrow[d, "d_i"] & B_n \arrow[d, "d_i"] \\ A_{n-1} \arrow[r, "f_{n-1}"] & B_{n-1} \end{tikzcd}$$

commutes. Taking $i = 0$ shows that f induces a chain map as long as f_n carries NA_n to NB_n. But indeed, taking $i \geq 1$ we have

$$d_i(f_n(x)) = f_{n-1}(d_i(x)) = f_{n-1}(0) = 0.$$

The statement about i is clear. $\square$

Definition 2.7 The *normalized complex* $N_* \colon \mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$ is the functor $A \mapsto NA_*$ defined above.

The Dold-Kan theorem is that N_* is an equivalence of categories. The first step in the proof will be that in fact the reduced complex is isomorphic to the normalized complex. We will need a lemma first.

Fix n; for $1 \leq k \leq n$, define

$$N_k A_n = \bigcap_{i=1}^{k} \ker d_i \subset A_n \qquad D_k A_n = \sum_{i=0}^{k-1} \operatorname{im} s_i \subset A_n,$$

so that when $k = n$ we recover NA_n and DA_n.

Lemma 2.8 *Let A be a simplicial abelian group and $n \geq 1$.*[1]

1. *The sequence*

$$0 \to A_{n-1}/D_{k-1}A_{n-1} \xrightarrow{s_k} A_n/D_{k-1}A_n \to A_n/D_k A_n \to 0$$

is well-defined and exact.

2. *The sequence*

$$0 \to N_{k-1}A_{n-1} \xrightarrow{s_k} N_{k-1}A_n \xrightarrow{\phi} N_k A_n \to 0$$

is well-defined and exact, where $\phi(x) = x - s_k d_k x$.

Proof (1) s_k here is indeed well-defined: if $x = \sum_{i<k} s_i(x_i) \in D_{k-1}A_{n-1}$, then

$$s_k(x) = \sum_{i<k} s_i s_{k-1} x_i \in D_{k-1}A_n$$

by the simplicial identities. To show exactness, it is clear the composite of the nonzero arrows is 0 since $D_k A_n \supset \operatorname{im} s_k$. The second map is clearly surjective since it is a quotient map. s_k is injective on the quotients since if $s_k(x) \in D_{k-1}A_n$, then $s_k(x) = \sum_{i<k} s_i(x_i)$, so by the simplicial identities

$$x = d_k s_k(x) = \sum_{i<k} s_i d_{k-1}(x_i) \in D_{k-1}A_{n-1}.$$

Lastly, for exactness in the middle, if $[x] \in A_n/D_{k-1}A_n$ has $x \in D_k A_n$, then

$$x = \sum_{i \leq k} s_i(x_i) \equiv s_k(x_k) \pmod{D_{k-1}A_n},$$

[1] This lemma needs to be fixed to reflect that I've changed from using $(-1)^n d_n$ for N_* to d_0.

completing the proof of exactness.

(2) To see that ϕ actually maps into $N_k A_n$, observe that if $i < k$ then

$$d_i\phi(x) = 0 - d_i s_k d_k(x) = -s_{k-1}d_{k-1}d_i(x) = 0,$$

and $d_k\phi(x) = d_k x - d_k x = 0$. That s_k maps into $N_{k-1}A_n$ is clear since $d_i s_k(x) = s_{k-1}d_i(x) = 0$ for $i < k$.

For exactness, observe $\phi(s_k(x)) = s_k(x) - s_k(x) = 0$, and we know s_k is injective (via d_k). If $x \in N_k A_n$, then it is also in $N_{k-1}A_n$ and $\phi(x) = x - s_k d_k(x) = x$, so ϕ is onto. Finally, for exactness in the middle, suppose $\phi(x) = 0$, i.e. $x = s_k d_k(x)$. So we just need to show $d_k(x) \in N_{k-1}A_{n-1}$, but for $i < k$

$$d_i d_k(x) = d_{k-1}d_i(x) = 0$$

since $x \in N_{k-1}A_n$, completing the proof. □

Theorem 2.9 *Let A be a simplicial abelian group. Then the map*

$$NA_* \oplus DA_* \to A_*$$

induced by i and the inclusion $DA_* \to A_*$ is an isomorphism. In particular,

$$NA_* \xrightarrow{i} A_* \xrightarrow{p} (A/DA)_*$$

is an isomorphism.

Proof It suffices to show the second, a priori weaker, statement; indeed, given that there's an exact sequence of chain complexes

$$0 \to DA_* \hookrightarrow A_* \xrightarrow{(pi)^{-1}p} NA_* \to 0,$$

but then clearly $i\colon NA_* \to A_*$ is a splitting. To prove it we show that for all $n \geq 0$, and for all $1 \leq k \leq n$, the composite

$$N_k A_n \to A_n \to A_n/D_k A_n$$

is an isomorphism (then $k = n$ gives the theorem). We do this by induction on n; if $n = 0$ then there are no $k \geq 1$, so suppose we have the isomorphism $N_j A_{n-1} \cong A_{n-1}/D_j A_{n-1}$ for every $j < n$. To prove the statement for n, we do a second induction on $k \geq 1$.

If $k = 1$, $N_1 A_n = \ker d_1$ and $D_1 A_n = \operatorname{im} s_0$. But the sequence

$$0 \to N_1 A_n \to A_n \xrightarrow{d_1} A_{n-1} \to 0$$

is exact and s_0 is a retract of d_1 (by the identity $d_1 s_0 = 1$), so $A_n = \ker d_1 \oplus \operatorname{im} s_0$, hence $A_n/D_1 A_n \cong N_1 A_n$. Now suppose we know $N_{k-1}A_n \to A_n/D_{k-1}A_n$ is an isomorphism. Put the two exact sequences of Lemma 2.8

into a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & N_{k-1}A_{n-1} \arrow[r, "s_k"] \arrow[d, "pi"'] & N_{k-1}A_n \arrow[r, "\phi"] \arrow[d, "pi"'] & N_kA_n \arrow[r] \arrow[d, "pi"'] & 0 \\
0 \arrow[r] & A_{n-1}/D_{k-1}A_{n-1} \arrow[r, "s_k"] & A_n/D_{k-1}A_n \arrow[r] & A_n/D_kA_n \arrow[r] & 0
\end{tikzcd}$$

The left square commutes by definition of how s_k passes to the quotients. The right square commutes since

$$pi\phi(x) = p(x - s_kd_kx) = pi(x)$$

since $s_k(d_kx) \in D_kA_n$. The first vertical is an isomorphism by our induction on n, and the second is one by induction on k. Hence $pi\colon N_kA_n \to A_n/D_kA_n$ is an isomorphism by a trivial case of the five lemma. $\square$

The isomorphism of Theorem 2.9 will be important in the proof of Dold-Kan as well as its corollaries. In particular, we can use it to construct chain homotopy inverses to i and q. Before that, however, we need to construct an inverse to the functor N_*.

2.2 The Eilenberg-MacLane Functor

To prove Dold-Kan, will explicitly describe a functor

$$\mathrm{K}\colon \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \longrightarrow \mathbf{sAb}$$

inverse to the normalized complex construction N_*. But, since this functor is a special case of what will be the focus of Section 3, we define K in full generality here, namely for semiadditive categories:

Definition 2.10 A category $\mathcal{A}$ is *semiadditive* if it has all finite biproducts and a zero object.

Example 2.11 Clearly any abelian category, in particular **Ab**, is semiadditive. A less trivial example is **Rel**, the category of sets and relations. Here the product and coproduct are the disjoint union.

The most important (and in some sense canonical in light of Prop. 3.1) example of a semiadditive category is **CMon**, the category of commutative monoids. Indeed, since monoid maps must preserve the unit, the trivial monoid $\{e\}$ is a zero object. We can also see that $A \times B$ represents homomorphisms *out of* A and B individually: if $f\colon A \to C$ and $g\colon B \to C$, the corresponding unique map is $(a, b) \mapsto f(a)g(b)$ (which uses that C is commutative). As done here, in this paper we will write commutative monoids multiplicatively to distinguish them from abelian groups.

Let $\mathcal{A}$ be a semiadditive category. Since it has 0, and thus zero morphisms, $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is well-defined. Of course, $\mathbf{s}\mathcal{A}$ is always well-defined and means

$[\mathbf{\Delta}^{\mathrm{op}}, \mathcal{A}]$. The construction of K factors through the category of *semisimplicial* objects:

Definition 2.12 Write $\mathbf{\Delta}_{\mathrm{inj}} \subset \mathbf{\Delta}$ for the wide subcategory consisting of monomorphisms, i.e. injective order preserving maps. Let C be any category. Then we define

$$s\mathbf{s}A := [\mathbf{\Delta}_{\mathrm{inj}}^{\mathrm{op}}, C]$$

to be the category of *semisimplicial* objects of C.

Now suppose $\mathcal{A}$ be semiadditive and let (A, ∂) be an chain complex in $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. There is an associated semisimplicial object $K^s(A)$ defined as follows:

1. $[n] \mapsto K^s(A)_n = A_n$.
2. For $j\colon [m] \hookrightarrow [n]$ an injection, $K^s(j)\colon A_n \to A_m$ is the zero morphism unless $m = n - 1$ and $j = d^0$, in which case $K^s(d_0) = \partial_n\colon A_n \to A_{n-1}$.

To see that $K^s(A)$ is a presheaf, consider a composite

$$[l] \xrightarrow{j} [m] \xrightarrow{i} [n].$$

$K^sA(ij) = K^sA(i) \circ K^sA(j)$ is trivial in the case that neither i nor j are of the form d^n or d^m, or that one of them is and the other is 0 or the identity. The only other case $j = d^0$ and $i = d^0$, but this case is true too because $\partial^2 = 0$.

It is also clear that this construction is functorial because the assignment induced by a chain map f is always a natural transformation: indeed the required composites are 0, unless we are checking the d_0 case where it follows from f being a chain map.

Definition 2.13 Let $\mathcal{A}$ be a semiadditive category. The functor

$$K^s_{\mathcal{A}}\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to s\mathbf{s}\mathcal{A}$$

defined by $A \mapsto K^s(A)$ is the *semisimplicial Eilenberg-MacLane functor.*

It remains to provide the second half of the composite $s\mathbf{s}\mathcal{A} \to \mathbf{s}\mathcal{A}$. Recall the following elementary result about $\mathbf{\Delta}$:

Lemma 2.14 (epimonic factorization) *Every map $u\colon [m] \to [n]$ in $\mathbf{\Delta}$ has a unique factorization of the form*

$$[m] \xrightarrow{\pi} [k] \xrightarrow{i} [n]$$

into a split epi π and a mono i.

Proof This is clear: factor u into the inclusion of the image after the restriction to the image; the image is uniquely isomorphic to $[k]$ for some k. The image of this composite always has cardinality $k + 1$, so $[k]$ is the only object which works. No other map factoring through $[k]$ except for the aforesaid one can be order-preserving. $\square$

Proposition 2.15 *Let $\mathcal{A}$ be a semiadditive category. There is a canonical functor $E\colon s\mathbf{s}\mathcal{A} \to \mathbf{s}\mathcal{A}$ defined by*

$$E(F)_n \coloneqq \bigoplus_{[n]\twoheadrightarrow[k]} F_k$$

for each object F of $s\mathbf{s}\mathcal{A}$, where the biproduct is taken over surjections $[n] \twoheadrightarrow [k]$ in the simplex category $\mathbf{\Delta}$.

Proof Let $u\colon [n] \to [n']$ be a morphism of $\mathbf{\Delta}$. To define the corresponding morphism $u^*\colon E(F)_{n'} \to E(F)_n$, we define the map

$$u^*_{\phi'}\colon F_{k',\phi'} \to \bigoplus_{[n]\twoheadrightarrow[k]} F_k$$

for each surjection $\phi'\colon [n'] \twoheadrightarrow [k']$. Form by Lemma 2.14 the epimonic splitting of the composite $\phi \circ u$ to get a unique commutative square:

$$\begin{tikzcd}
{[n]} \arrow[r, "u"] \arrow[d, dashed, two heads, "\phi"] & {[n']} \arrow[d, two heads, "\phi'"] \\
{[k]} \arrow[r, dashed, hook, "v"] & {[k']}
\end{tikzcd}$$

Then define $u^*_{\phi'}$ on components of the target by taking $F_{k',\phi'} \to F_{k,\phi}$ to be $F(v)$, and all other components the zero map. It remains to check that this respects composition in $\mathbf{\Delta}$ and that it is functorial in F.

For the former it suffices to show the claim component-wise; hence consider $[m] \to [n] \to [p]$ in $\mathbf{\Delta}$ and a surjection out of $[p]$, forming the diagram:

$$\begin{tikzcd}
m \arrow[r] \arrow[d, dashed, two heads] & {[n]} \arrow[r] \arrow[d, dashed, two heads] & {[p]} \arrow[d] \\
{[j]} \arrow[r, dashed, hook] & {[k]} \arrow[r, dashed, hook] & {[l]}
\end{tikzcd}$$

where the left square is formed using epimonic factorization from the first square. Now, by uniqueness, the outer square gives a factorization the composite map, so we can see $E(F)$ respects composition using the fact that F does.

Finally, that E is functorial is clear: given a semisimplicial map $\eta\colon F \to G$, the associated simplicial map is defined by

$$E(\eta)_n \coloneqq \bigoplus_{[n]\twoheadrightarrow[k]} \eta_k\colon \bigoplus_{[n]\twoheadrightarrow[k]} F_k \to \bigoplus_{[n]\twoheadrightarrow[k]} G_k$$

and it is easy to check that this respects composition. □

Definition 2.16 Let $\mathcal{A}$ be a semiadditive category. The composite functor

$$\mathrm{K}_{\mathcal{A}}\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \xrightarrow{K^s} s\mathbf{s}\mathcal{A} \xrightarrow{E} \mathbf{s}\mathcal{A}$$

from Def. 2.13 and Prop. 2.15 is the *Eilenberg-MacLane* functor.

In the special case of $\mathcal{A} = \mathbf{Ab}$, the Dold-Kan correspondence is that $\mathrm{K}_{\mathbf{Ab}}$ is the left and right inverse to N_*. We will prove the first part in the next section; the second, however, can be done in more generality, for $\mathcal{A} = \mathbf{CMon}$.

Observe that Def. 2.7 and the first part of Prop. 2.6 works for any commutative monoid. Indeed, given $f\colon B \to B'$ a morphism of commutative monoids, we can still set $\ker f = f^{-1}(1)$ for the identity $1 \in B'$. Hence for a simplicial commutative monoid A put

$$NA_n = \bigcap_{i=1}^{n} \ker d_i \subset A_n$$

with differential $d_0\colon NA_n \to NA_{n-1}$, where the simplicial identities again imply this is well-defined and that $d^2 = 1$. Then the same proof as before shows that NA_* is a functor $\mathbf{sCMon} \to \mathrm{Ch}_{\geq 0}(\mathbf{CMon})$. Note, however, that there is no Moore complex A_*, which we needed '-1' to define.

Theorem 2.17 *Let A be a simplicial commutative monoid. There is a canonical identification*

$$(N_* \circ \mathrm{K}_{\mathbf{CMon}} A)_n = A_n$$

natural in A. In particular, we have natural isomorphisms

$$N_* \circ \mathrm{K}_{\mathbf{CMon}} \cong \mathrm{Id}_{\mathrm{Ch}_{\geq 0}(\mathbf{CMon})} \qquad N_* \circ \mathrm{K}_{\mathbf{Ab}} \cong \mathrm{Id}_{\mathrm{Ch}_{\geq 0}(\mathbf{Ab})}.$$

Proof Every abelian group is a commutative monoid, and group homomorphisms are the same as morphisms of the underlying monoids; so it is clear the natural isomorphism for **CMon** gives one for **Ab**, and proving the first part suffices. From now on write $\mathrm{K}_{\mathbf{CMon}} = K$.

The unique surjection $[n] \to [n]$ is the identity, thus the definition of $K(A)$ has

$$A_n \subset K(A)_n = \bigoplus_{[n] \twoheadrightarrow [k]} A_k$$

as a direct summand. First we show $A_n \subset N(KA)_n$, for which we must prove that for $i \geq 1$, the map

$$d_i\colon \bigoplus_{[n] \twoheadrightarrow [k]} A_k \to \bigoplus_{[n-1] \twoheadrightarrow [k']} A_{k'}$$

sends A_n to 1. Indeed, A_n is associated to the identity, so the associated splitting is just:

$$\begin{tikzcd} {[n-1]} \arrow[r, "d^i"] \arrow[d, dashed, two heads, "1"] & {[n]} \arrow[d, two heads, "1"] \\ {[n-1]} \arrow[r, dashed, hook, "d^i"] & {[n]} \end{tikzcd}$$

Thus the only non-constant component is

$$d_i \colon A_n^{\mathrm{id}} \to A_{n-1}^{\mathrm{id}},$$

given by $K^s(d^i)$. But by definition of K^s this is only non-constant if $i = 0$.

Next, to see $N(KA)_n \subset A_n$, suppose $\phi \colon [n] \twoheadrightarrow [k]$ is a surjection for some $k \neq n$. Then there is $i \geq 0 \in [n]$ such that $\phi(i) = \phi(i-1)$, therefore $\phi \circ d^i$ is still a surjection $[n-1] \to [k]$. The factorization is:

$$\begin{tikzcd}
{[n-1]} \arrow[r, "d^i"] \arrow[d, dashed, two heads, "\phi d^i"] & {[n]} \arrow[d, two heads, "\phi"] \\
{[k]} \arrow[r, hook, dashed, "1"] & {[k]}
\end{tikzcd}$$

Therefore the corresponding component of d_i given by

$$d_i|_{A_k} \colon A_k^{\phi d^i} \to A_k^{\phi}$$

is the identity map. If $x = (x_\phi)_\phi \in N(KA)_n$, then $x_\phi = d_i(x)_\phi = 1$ for an i, so the only non-constant component of x is that associated to $\mathrm{id} \colon [n] \to [n]$. $\square$

2.3 Finishing Dold-Kan

In this section we complete the proof of the Dold-Kan correspondence:

Theorem 2.18 (Dold-Kan correspondence) *The functor*

$$N_* \colon \mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$$

is an equivalence of categories, with inverse given by $\mathrm{K}_{\mathbf{Ab}}$.

For the remainder of this section fix $K = \mathrm{K}_{\mathbf{Ab}}$ for the Eilenberg-MacLane functor. In Thm. 2.17 we already saw one direction of the theorem. To finish we construct an isomorphism

$$\mu \colon K \circ N_* \to \mathrm{Id}_{\mathbf{sAb}}.$$

For A a simplicial abelian group and $n \geq 0$, we set

$$\mu(A)_n \colon K(NA_*)_n = \bigoplus_{[n] \twoheadrightarrow [k]} NA_k \to A_n$$

to be defined on the component associated to $\phi \colon [n] \twoheadrightarrow [k]$ by the composite

$$NA_k^{\phi} \hookrightarrow A_k \xrightarrow{\phi^*} A_n.$$

Proposition 2.19 *μ is a well-defined natural transformation $KN_* \to \mathrm{Id}$.*

Proof It is obvious from the definition that μ is natural as long as $\mu(A)$ is a valid simplicial map $K(NA_*) \to A$. Consider $u \colon [m] \to [n]$; to check commutativity of the relevant square, it suffices to show that

$$\begin{tikzcd} NA_k^\phi \arrow[r] \arrow[d, "u^*"] & A_n \arrow[d, "u^*"] \\ \bigoplus_{[m]\twoheadrightarrow[j]} NA_j \arrow[r] & A_m \end{tikzcd}$$

commutes for every surjection $\phi\colon [n] \twoheadrightarrow [k]$. This follows from the fact that

$$\begin{tikzcd} NA_k \arrow[r, hook] \arrow[d, dashed] & A_k \arrow[r, "\phi^*"] \arrow[d, dashed] & A_n \arrow[d, "u^*"] \\ NA_j \arrow[r, hook] & A_j \arrow[r, dashed] & A_m \end{tikzcd}$$

commutes, where the right-hand square is the application of A to the epimonic factorization of the composite $\phi \circ u$. $\square$

Proving Thm. 2.18 consists of showing μ is an isomorphism, for which it suffices to show it is levelwise bijective, i.e. for each simplicial abelian group A and $n \geq 0$, $\mu(A)_n$ is bijective. Levelwise surjectivity is much easier and is done by a simple induction; $n = 0$ is just $A_0 \cong A_0$. Now by Thm. 2.9, recall $A_n \cong NA_n \oplus DA_n$, so we can show $\mu(A)_n$ surjects onto both NA_n and DA_n. The former is clear since NA_n is a direct summand of the domain. For the latter, it suffices to prove $s_i(y)$ is in the image for $y \in A_{n-1}$ and any i, so by induction let $x = (x_\phi)$ such that $\mu(A)_{n-1}(x) = y$, i.e. $\sum_\phi \phi^*(x_\phi) = y$. Then

$$s_i(y) = \sum_\phi s_i\phi^*(x_\phi) = \sum_\phi (\phi s^i)^*(x_\phi),$$

where ϕs^i is a surjection $[n] \to [n-1] \to [k]$, hence $s_i(y)$ is in the image. Now levelwise injectivity remains:

Proof (of Theorem 2.18) It suffices to show that the map

$$\mu(A)_n\colon \bigoplus_{[n]\twoheadrightarrow[k]} NA_k \to A_n$$

is injective. Once again we do this inductively, noting that when $n = 0$, $\mu(A)_0$ is an equality. So let $x = (x_\phi)_\phi$ where $x_\phi \in A_k^\phi$ for which

$$\mu(A)_n(x) = \sum_{\phi\colon [n]\twoheadrightarrow[k]} \phi^*(x_\phi) = 0.$$

We must prove $x_\phi = 0$ for all surjections $\phi\colon [n] \twoheadrightarrow [k]$. First observe that $x_{\mathrm{id}} \in NA_n$ is 0. Indeed, for $x_\phi \in A_k$ and $k < n$, we have $\phi^*(x_\phi) \in DA_n$ in the splitting $A_n = NA_n \oplus DA_n$. This is because there must be some i such that

$\phi(i-1) = \phi(i)$, so ϕ factors through s^i. But then

$$\sum\nolimits_\phi \phi^*(x_\phi) \equiv x_{\mathrm{id}} \pmod{DA_n},$$

so if the sum is zero then x_{id} is 0 in the quotient $A_n/DA_n = NA_n$, i.e. it is zero.

Now suppose for contradiction that $\phi_0 \colon [n] \twoheadrightarrow [k]$ is so that $x_{\phi_0} \neq 0$. We have just shown $k < n$. Now choose a section ψ of ϕ_0 which is not a section of any other surjection $[n] \twoheadrightarrow [k]$. Since $\phi_0(0) = 0$, we may modify ψ appropriately to assume $\psi(0) = 0$. Then by assumption

$$\psi^* \sum\nolimits_\phi \phi^*(x_\phi) = \sum\nolimits_\phi (\phi\psi)^*(x_\phi) = 0 \tag{$*$}$$

First consider the ϕ for which $\phi\psi$ is not a surjection. Then since $\phi(\psi(0)) = 0$, $\phi\psi$ factors through d^i for some $i \geq 1$ and hence $(\phi\psi)^* x_\phi = 0$ because we are using the normalized complex. Therefore the equation $(*)$ is of the form $\mu(A)_k((y)) = 0$ for $k < n$, and by induction $(y) = 0$. But the identity component here is x_{ϕ_0} since $\phi_0\psi = \mathrm{id}$, and no other ϕ had ψ as a section. Therefore $x_{\phi_0} = 0$, which is what we wanted.

We have shown that $\mu(A)_n$ is levelwise bijective, and hence μ is a natural isomorphism $KN_* \cong \mathrm{Id}$. Alongside Thm. 2.17, this proves the result. $\square$

We will briefly discuss some applications of the Dold-Kan correspondence to topology. Recall that every simplicial abelian group is naturally a pointed Kan complex, and we have a forgetful functor:

$$\mathbf{sAb} \to \mathbf{Kan}_*$$

Hence we may consider homotopy groups. We will first state, but not prove, two additional results which follow readily from what we have done:

Proposition 2.20 ([1, Cor. 2.7]) *Let A be a simplicial abelian group, and write $\pi_n(A) := \pi_n(A, 0)$ for its homotopy groups. Then there is a natural isomorphism*

$$\pi_n(A) \cong H_n(NA_*)$$

for each $n \geq 1$.

Lemma 2.21 ([1, Thm. 2.4]) *Let A be a simplicial abelian group and write $pi \colon NA_* \to (A/DA)_*$ for the isomorphism of Thm. 2.9. Then the composite*

$$A_* \xrightarrow{p} (A/DA)_* \xrightarrow{(pi)^{-1}} NA_* \xrightarrow{i} A_*$$

is chain homotopic to the identity map $A_* \to A_*$.

Corollary 2.22 *Let A be a simplicial abelian group. Then $i \colon NA_* \to A_*$ is a homotopy equivalence.*

Proof We claim that

$$A_* \xrightarrow{p} (A/DA)_* \xrightarrow{(pi)^{-1}} NA_*$$

is a homotopy inverse. Indeed, Lemma 2.21 gives one direction and the other is obvious since $((pi)^{-1}p)i = 1$. □

In particular, i is a quasi-isomorphism so $\pi_n(A) = H_n(A_*)$, the Moore complex. Now we have two interesting corollaries. First:

Proposition 2.23 *There is a functor*

$$K \colon \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{Top}_*$$

such that $\pi_n(K(B)) = H_n(B)$ for every chain complex B.

Proof Consider the composite:

$$\mathrm{Ch}_{\geq 0}(\mathbb{Z}) \xrightarrow{\mathrm{K}_{\mathbf{Ab}}} \mathbf{sAb} \to \mathbf{Kan}_* \xrightarrow{|\cdot|} \mathbf{Top}_*$$

Since the homotopy groups of a Kan complex coincide with those of its realization, the result is immediate since $N_*\mathrm{K}_{\mathbf{Ab}} \cong \mathrm{Id}$. □

Corollary 2.24 *There is, for each $n \geq 1$, a functor*

$$K(-, n) \colon \mathbf{Ab} \to \mathbf{Top}_*$$

such that for any G, $\pi_k(K(G, n)) = 0$ unless $k = n$, in which case it is G.

Proof Let $(-, n) \colon \mathbf{Ab} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$ be the functor sending an abelian group to the chain complex concentrated in degree n. It is obvious what the homology is, so precomposing with the functor of Prop. 2.23 gives the result. □

This explains the name 'Eilenberg-MacLane' functor for $\mathrm{K}_{\mathcal{A}}$. Our Dold-Kan construction is superior to the classical topological construction of Eilenberg-MacLane spaces because it is functorial. Lastly:

Proposition 2.25 *There is a functor*

$$F \colon \mathbf{Top} \to \mathbf{Top}_*$$

such that for each space X, we have

$$H_n(X) = \pi_n(F(X))$$

for all $n \geq 1$, where the left-hand side refers to the singular homology.

Proof Consider the composite:

$$\mathbf{Top} \xrightarrow{\mathrm{Sing}} \mathbf{Set} \xrightarrow{\mathbb{Z}[-]} \mathbf{sAb} \xrightarrow{(-)_*} \mathrm{Ch}_{\geq 0}(\mathbb{Z})$$

Here $\mathbb{Z}[-]$ is the free abelian group construction. By the definition of Sing, this is just the singular complex $C_*(X)$ of a space X, and the homology of this complex is just the homology of X. Now compose with the functor K of Prop. 2.23 to get the result. $\square$

3 Extending Dold-Kan

In Thm. 2.18, we proved the Dold-Kan correspondence for chain complexes and simplicial objects in the category **Ab**. We did this by defining N_* and then showing that $\mathrm{K}_{\mathbf{Ab}}$ was its inverse. Although we needed the structure of **Ab** to construct N_*, the Eilenberg-MacLane functor $\mathrm{K}_{\mathcal{A}}$ is defined for any semiadditive category (Def. 2.16). So, we may ask when it is an equivalence of categories, or at least fully faithful. The generalized Dold-Kan correspondence, which we aim to prove in this section, is the answer:

Theorem (Generalized Dold-Kan Correspondence) *Let $\mathcal{A}$ be a semiadditive category. The Eilenberg-MacLane functor*

$$\mathrm{K}_{\mathcal{A}} \colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A}$$

is fully faithful. If in addition $\mathcal{A}$ is additive and idempotent complete, then $\mathrm{K}_{\mathcal{A}}$ *is an equivalence of categories.*

The slogan of this section will be deducing properties of semiadditive (resp., additive) categories by proving them for the category in which they are canonically enriched, namely **CMon** (resp., **Ab**). We will need to develop some general theory about semiadditive and additive categories to deduce the first part of the theorem. Next, we will study the notion of idempotent completeness in abstract, and will then be able to deduce the general Dold-Kan correspondence without too much difficulty.

3.1 Semiadditive Categories

Recall (Def. 2.10) that a semiadditive category is one with zero and biproducts. In what follows $\mathcal{A}$ without qualification will be a semiadditive category, with biproduct written $\oplus$. We will write i_k, p_k for the canonical injections and projections associated with the product.

For any $a \in \mathcal{A}$ there is always a diagonal and codiagonal map:

$$\Delta_a \colon a \to a \oplus a \qquad \nabla_a \colon a \oplus a \to a$$

These are induced by the product and coproduct using the identities. In any semiadditive category, there is a canonical addition on the hom-sets $\mathrm{Hom}_{\mathcal{A}}(a, b)$. For $f, g \colon a \to b$, write

$$f + g \coloneqq a \xrightarrow{\langle f,g \rangle} b \oplus b \xrightarrow{\nabla_b} b \tag{1}$$

where the first map is induced by the product. Notice $\nabla_b \circ \langle f, 0\rangle = \nabla_b \circ \langle 0, f\rangle = f$, because always $p_1 \circ i_1 = 1_b$, but $p_1 \circ i_1 \circ \nabla_b \circ \langle f, 0\rangle = p_1 \circ \langle f, 0\rangle = f$. In fact we have:

Proposition 3.1 *The addition in Eq. 1 defines an enrichment of $\mathcal{A}$ over* **CMon***; we write the corresponding hom-monoids $\mathcal{A}(a, b)$. Moreover, the enrichment is unique.*

Proof For the first part one shows that $+$ is commutative and associative, and that it is bilinear with respect to compositions. This is very standard (see [3]), so we don't check everything here; e.g. for commutativity we use the natural braiding $\gamma_{a,b} \colon a \oplus b \cong b \oplus a$, and then observe that

$$\begin{tikzcd} a \arrow[r, "\langle f,g\rangle"] \arrow[d, equal] & b \oplus b \arrow[r, "\nabla_b"] \arrow[d, "\gamma_{b,b}"'] & b \arrow[d, equal] \\ a \arrow[r, "\langle g,f\rangle"'] & b \oplus b \arrow[r, "\nabla_b"'] & b \end{tikzcd}$$

commutes. This works because ∇_b is invariant after the braiding because its induced by the pair of maps $(1_b, 1_b)$ which is symmetric. We can do the same thing with the natural associator $\alpha_{a,b,c}$. To conclude that $(f+g)\circ h = f\circ h+g\circ h$ and the other way around, one can use naturality of $\nabla = \nabla_a \colon a \to a \oplus a$.

For uniqueness of the enrichment, suppose $f \cdot g$ is another **CMon**-enrichment with unit 1. By (1) and the fact that $\nabla_b \circ \langle f, 0\rangle = f$, it suffices to show that

$$\nabla_b \circ \langle f \cdot g, 0\rangle = \nabla_b \circ \langle f, g\rangle \,. \tag{$*$}$$

First we claim that in all hom-monoids $1 = 0$. Indeed by definition $0 = a \to 0 \to b$, thus $1 \cdot 0 = (1 \cdot 0_b) \circ (1 \cdot {}_a0) = 1 \circ 1 = 1$, where the second equality is because there is only one map in $\mathrm{Hom}(a, 0)$ and $\mathrm{Hom}(0, b)$. But also $1 \cdot 0 = 0$, so $0 = 1$. Now we'll show $\langle f, 0\rangle \cdot \langle 0, g\rangle = \langle f, g\rangle$. By universal property:

$$p_1 \circ (\langle f, 0\rangle \cdot \langle 0, g\rangle) = (p_1 \circ \langle f, 0\rangle) \cdot (p_2 \circ \langle 0, g\rangle) = f \cdot 0 = f$$

since $0 = 1$. We do the same for p_2 and g, showing it is indeed $\langle f, g\rangle$. Finally we have:

$$\begin{aligned} \nabla_b \circ \langle f \cdot g, 0\rangle &= f \cdot g = (\nabla_b \circ \langle f, 0\rangle) \cdot (\nabla_b \circ \langle 0, g\rangle) \\ &= \nabla_b \circ (\langle f, 0\rangle \cdot \langle g, 0\rangle) \\ &= \nabla_b \circ \langle f, g\rangle \,, \end{aligned}$$

which is what we wanted. $\square$

Now let $よ \colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] =: \mathrm{PSh}(\mathcal{A})$ be the Yoneda embedding and $U \colon \mathbf{CMon} \to \mathbf{Set}$ be the forgetful functor. We have an induced functor

$$U_* \colon [\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}] \to \mathrm{PSh}(\mathcal{A}),$$

which is faithful because U is, and presheaf maps are checked for equality pointwise. Prop. 3.1 says that the Yoneda embedding factors through $[\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]$ so that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\text{よ}^e"] \arrow[dr, "\text{よ}"'] & {[\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]} \arrow[d, "U_*"] \\
& \mathrm{PSh}(A)
\end{tikzcd}$$

defining the functor よ^e, which we call the *enriched* Yoneda embedding.

Lemma 3.2 *Let A, B, C be categories and suppose*

$$\begin{tikzcd}
A \arrow[r, "G"] \arrow[dr, "F"'] & B \arrow[d, "H"] \\
& C
\end{tikzcd}$$

commutes where F is fully faithful and H is faithful. Then G is fully faithful.

Proof Suppose $f, g\colon a \to b$; if $Gf = Gg$ then applying H we get $Ff = Fg$ so $f = g$, i.e. it is faithful. On the other hand, if $f\colon Ga \to Gb$, then $Hf\colon Fa \to Fb$, so there is $g\colon a \to b$ such that $Fg = Hf$. But then $HGg = Hf$, so $Gg = f$ by faithfulness, thus we have shown that G is full. $\square$

Now the following result is clear in light of the lemma and the commutative diagram above it:

Corollary 3.3 *For a semiadditive category $\mathcal{A}$, $\text{よ}^e\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]$ is fully faithful.*

The forgetful functors U and U_* have a much stronger property than just faithfulness; this is partly responsible for the last step in the proof of generalized Dold-Kan. We will need this result for both $\mathbf{CMon}$ and $\mathbf{Ab}$, so we prove it generally:

Proposition 3.4 *Let C be a category. The forgetful functors*

$$U\colon \mathbf{CMon} \to \mathbf{Set} \qquad U_*\colon [C^{\mathrm{op}}, \mathbf{CMon}] \to \mathrm{PSh}(C)$$

are faithful, conservative, and limit creating. Similarly the functors

$$U\colon \mathbf{Ab} \to \mathbf{Set} \qquad U_*\colon [C^{\mathrm{op}}, \mathbf{Ab}] \to \mathrm{PSh}(C)$$

are also faithful, conservative, and limit creating.

Proof It is obvious that all four functors are faithful by definition of the respective homomorphisms. Similarly, since any bijective homomorphism of monoids or groups is an isomorphism, U is conservative, and the same holds for U_* since isomorphisms of presheaves are checked pointwise. Finally, U is right adjoint to the free abelian group or commutative monoid construction, and thus U_* is right adjoint to the pointwise free construction. This completes the proof since conservative functors reflect any (co)limits which they preserve. $\square$

Recall that a functor $F\colon \mathcal{A} \to \mathcal{B}$ between semiadditive categories is *additive* if it preserves finite (and nullary) biproducts. It is always sufficient to show that F preserves 0 and binary products or coproducts. In this case it is clear that $F(f+g) = Ff + Fg$ because the addition in Eq. 1 is defined solely with canonical biproduct data.

If C is a category and $\mathcal{A}$ is semiadditive, then $[C, \mathcal{A}]$ is semiadditive, which is immediate: (co)products are computed pointwise, so it has biproducts and 0. In particular, $[\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]$ is semiadditive.

Proposition 3.5 *The enriched Yoneda embedding* $\text{よ}^e\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]$ *is additive.*

Proof It suffices to show よ^e preserves limits, and since U_* creates limits we can show $U_* \circ \text{よ}^e$ preserves limits, and this composite is just $\text{よ}\colon \mathcal{A} \to \mathrm{PSh}(\mathcal{A})$. But よ preserves limits because hom functors do in each variable and limits are computed pointwise. $\square$

Definition 3.6 A semiadditive category $\mathcal{A}$ is *additive* if for any objects a and b, the monoid $\mathcal{A}(a, b)$ is an abelian group.

If $\mathcal{A}$ is an additive category, then the enriched Yoneda factors through presheaves of abelian groups; observe from what we have done that the corresponding functor

$$\text{よ}^e\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}]$$

is still fully faithful and additive. This follows from Lemma 3.2 and the fact that the forgetful functor $\mathbf{Ab} \to \mathbf{CMon}$ (and hence the induced $[\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}] \to [\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]$) is fully faithful and additive. We will use the notation $\widehat{\mathcal{A}} \coloneqq [\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}]$ for the abelian presheaves into which $\mathcal{A}$ embeds.

For both semiadditive and additive categories, there is an 'enriched' Yoneda lemma where よ^e takes the place of よ, and additive presheaves (where the functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ is additive) take the place of presheaves. We will need to use a slightly weaker statement for the additive case:

Lemma 3.7 (enriched Yoneda lemma) *Suppose $\mathcal{A}$ is an additive category and X is an additive presheaf in $\widehat{\mathcal{A}}$. The natural map induced by the forgetful functor $U_*\colon \widehat{\mathcal{A}} \to \mathrm{PSh}(\mathcal{A})$*

$$\mathrm{Hom}_{\mathcal{A}}(\text{よ}^e(x), X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{A})}(\text{よ}(x), U_* X)$$

is a bijection, natural in $x \in \mathcal{A}$.

Proof Naturality is clear since this is induced by U_*, and we already know the map is injective since U_* is faithful. For surjectivity, let $f\colon \text{よ}(x) \to U_*X$ be any morphism. This corresponds under the Yoneda lemma to some section $s \in U(X(x)) = X(x)$. So f must be of the following form

$$\begin{aligned} f = f_y \colon \operatorname{Hom}(y, x) &\to U(X(y)) \\ g &\mapsto Xg(s), \end{aligned}$$

and we may let $\widetilde{f}\colon \text{よ}^e(x) \to X$ be

$$\begin{aligned} \widetilde{f} = \widetilde{f}_y \colon \mathcal{A}(y, x) &\to X(y) \\ g &\mapsto Xg(s). \end{aligned}$$

If this is a valid morphism, then clearly $U_*(\widetilde{f}) = f$ so we will be done. Since X is an additive functor, it defines a homomorphism of abelian groups

$$\mathcal{A}(y, x) \to \mathbf{Ab}(X(y), X(x)).$$

Therefore, because of the group structure given to $\mathbf{Ab}(X(y), X(x))$, we have

$$\begin{aligned} \widetilde{f}_y(g + h) &= X(g + h)(s) = (X(g(s) + X(h)(s)) \\ &= X(g)s + X(h)s = \widetilde{f}_y(g) + \widetilde{f}_y(h) \end{aligned}$$

so $\widetilde{f}_y$ is a group homomorphism. To to check naturality, let $u\colon y \to z$ be a map of $\mathcal{A}$. Since f was natural we have

$$U(Xu) \circ U(\widetilde{f}_z) = U(\widetilde{f}_y) \circ U(\text{よ}^e(y)),$$

so $\widetilde{f}$ is natural by functoriality and faithfulness of U, hence a morphism. □

3.2 Part 1: Full Faithfulness

In the previous section we developed some theory about semiadditive and additive categories. Here we begin to apply it to Dold-Kan. Recall that in Thm. 2.17 we showed

$$N_* \circ \mathrm{K}_{\mathbf{CMon}} \cong \mathrm{Id}.$$

While the Dold-Kan correspondence shows that $\mathrm{K}_{\mathbf{Ab}}$ is an equivalence and so fully faithful, we do not yet know if $\mathrm{K}_{\mathbf{CMon}}$ is fully faithful. The first step will be to prove this, and the following proposition suffices by fully faithful adjointness:

Proposition 3.8 *The functor* $N_*\colon \mathbf{sCMon} \to \mathrm{Ch}_{\geq 0}(\mathbf{CMon})$ *is right adjoint to* $\mathrm{K}_{\mathbf{CMon}}$.

Proof We will give an explicit adjunction formula using the natural isomorphism above. Write $K = \mathrm{K}_{\mathbf{CMon}}$, let A be a chain complex, and B a simplicial commutative monoid; considering the composite

$$\operatorname{Hom}(KA, B) \xrightarrow{N_*} \operatorname{Hom}(NK(A), NB) \xrightarrow{\sim} \operatorname{Hom}(A, NB), \qquad (*)$$

we need to show this is bijective. Explicitly, given

$$f = f_n \colon \bigoplus_{[n]\twoheadrightarrow[k]} A_k \to B_n,$$

the corresponding map $A \to NB$ is given by $Nf_n = f|_{A_n} \colon A_n \to NB_n$ (where the domain uses the identification $NK = \mathrm{Id}$). We construct an inverse as follows: for $g = g_k \colon A_k \to NB_k$, let $\Phi_n \colon KA_n \to B_n$ be defined on the component $\phi \colon [n] \twoheadrightarrow [k]$ by $\phi^* \circ g_k$. In other words, Φ is defined by the diagram:

$$\begin{tikzcd}
A_k^\phi \arrow[r, "g_k"'] \arrow[d] & NB_k \arrow[r, hook] & B_k \arrow[d, "\phi^*"'] \\
\bigoplus_{[n]\twoheadrightarrow[k]} A_k \arrow[rr, dashed, "\Phi_n"] & & B_n
\end{tikzcd}$$

We have to show this is a two-sided inverse to $(*)$:

1. Let f as above be given, so that the corresponding map is $f|_{A_n}$. We want $\Phi_n = f_n$, and by definition

$$\begin{tikzcd}
A_k^\phi \arrow[r, "f_k|_{A_k^\phi}"] \arrow[d] & B_k \arrow[d, "\phi^*"'] \\
\bigoplus A_k \arrow[r, "\Phi_n"] & B_n
\end{tikzcd}$$

   commutes. By the universal property of the biproduct it suffices to show that $f_n|_{A_k^\phi} = \phi^* \circ f_k|_{A_k}$. In other words, we want to show that the outer square of

$$\begin{tikzcd}
 & \bigoplus_{[n]\twoheadrightarrow[k]} A_k \arrow[r, "f_n"] \arrow[d, "\phi^*"'] & B_n \arrow[d, "\phi^*"'] \\
A_k^{\mathrm{id}} \arrow[ur] \arrow[r] & \bigoplus_{[k]\twoheadrightarrow[l]} A_l \arrow[r, "f_k"] & B_k
\end{tikzcd}$$

   commutes. The right square commutes since f is a morphism, and the left triangle commutes because

$$\begin{tikzcd}
{[n]} \arrow[r, hook] \arrow[d, two heads, "\phi^*"] & {[n]} \arrow[d, "\phi^*"'] \\
{[k]} \arrow[r, two heads, "\mathrm{id}"] & {[k]}
\end{tikzcd}$$

   is the relevant epimonic factorization for the definition of K.

2. For the other direction let $g = g_n \colon A_n \to NB_n$ be given, and consider the corresponding morphism Φ. We want to show $N\Phi_n = \Phi_n|_{A_n} = g_n$.

By construction of Φ, the commutative diagram describing the identity component of Φ_n is:

$$\begin{tikzcd} A_n^{\mathrm{id}} \arrow[r, "g_n"] \arrow[d] & B_n \arrow[d, "\mathrm{Id}"] \\ \bigoplus A_k \arrow[r, "\Phi_n"] & B_n \end{tikzcd}$$

This says precisely that $g_n = \Phi_n|_{A_n}$.

Therefore we have shown $(*)$ is bijective, and naturality is clear because we defined the map functorially. Hence $K \dashv N_*$. $\square$

Corollary 3.9 *The Eilenberg-MacLane functor*

$$\mathrm{K}_{\mathbf{CMon}} \colon \mathrm{Ch}_{\geq 0}(\mathbf{CMon}) \to \mathbf{sCMon}$$

is fully faithful. Moreover it exhibits the essential image of $\mathrm{Ch}_{\geq 0}(\mathbf{CMon})$ *as coreflective subcategory of* $\mathbf{sCMon}$.

Now we will work on the general case of $\mathrm{K}_{\mathcal{A}}$ for any semiadditive category $\mathcal{A}$. Let $f \colon \mathcal{A} \to \mathcal{B}$ be an additive functor between semiadditive categories. We will write the canonically induced functors as

$$f_* \colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{B}) \qquad f_* \colon \mathbf{s}\mathcal{A} \to \mathbf{s}\mathcal{B}$$

which on the left is object-wise application and on the right is post-composition $- \circ f$. It defines a functor of chain complexes because it takes zero maps to zero maps (so $\partial^2 = 0$ is satisfied).

Lemma 3.10 *Let* $f \colon \mathcal{A} \to \mathcal{B}$ *an additive functor between semiadditive categories. Then the square*

$$\begin{tikzcd} \mathrm{Ch}_{\geq 0}(\mathcal{A}) \arrow[r, "\mathrm{K}_{\mathcal{A}}"] \arrow[d, "f_*"'] & \mathbf{s}\mathcal{A} \arrow[d, "f_*"] \\ \mathrm{Ch}_{\geq 0}(\mathcal{B}) \arrow[r, "\mathrm{K}_{\mathcal{B}}"] & \mathbf{s}\mathcal{B} \end{tikzcd}$$

commutes up to natural isomorphism.

Proof For any chain complex $A = (A_*, \partial)$, we want $f_*(\mathrm{K}_{\mathcal{A}}(A) \cong \mathrm{K}_{\mathcal{B}}(f_*(A))$ naturally in A. On the LHS, evaluate at n to get

$$\begin{aligned} f_*(\mathrm{K}_{\mathcal{A}}(A))_n = f(\mathrm{K}_{\mathcal{A}}(A)_n) = f\Big(\bigoplus_{[n]\twoheadrightarrow[k]} A_k\Big) \\ \cong \bigoplus_{[n]\twoheadrightarrow[k]} f(A_k) = \mathrm{K}_{\mathcal{B}}(f_*(A))_n \end{aligned}$$

because f is additive. For naturality with respect to a chain map $\phi\colon A \to B$, unwinding the definitions gives the square

$$\begin{tikzcd}
f\left(\bigoplus A_k\right) \arrow[r, "\simeq"] \arrow[d, "f(\bigoplus \phi_k)"'] & \bigoplus f(A_k) \arrow[d, "\bigoplus f(\phi_k)"] \\
f\left(\bigoplus B_k\right) \arrow[r, "\simeq"] & \bigoplus f(B_k)
\end{tikzcd}$$

This commutes since f must preserve the maps the biproduct is equipped with, completing the proof. $\square$

Remark 3.11 If we write **SemiAdd** for the 2-category of semiadditive categories with additive functors, then there are two 2-functors

$$\mathrm{Ch}_{\geq 0}, [\Delta^{\mathrm{op}}, -]\colon \mathbf{SemiAdd} \to \mathbf{Cat}$$

defined in the obvious way. Lemma 3.10 shows (after some checks) that

$$\mathrm{K}\colon \mathrm{Ch}_{\geq 0} \to [\Delta^{\mathrm{op}}, -]$$

is a *lax natural transformation*; this is the same as a 2-morphism in the 3-category **2Cat** of 2-categories.

Let C, D, and E be categories. The cartesian closed structure in the 2-category of categories gives canonical isomorphisms of categories

$$[C,[D,E]] \cong [C \times D, E] \cong [D \times C, E] \cong [D,[C,E]] \tag{$*$}$$

Given a functor $f\colon C \to [D,E]$ the associated functor $\widetilde{f}\colon D \to [C,E]$ is just $\widetilde{f}(d)(c) = f(c)(d)$. The definition on natural transformations is clear; we will denote the isomorphism $f \mapsto \widetilde{f}$ by μ.

Now fix a category J and an additive category $\mathcal{B}$. From now on we will write $\mathrm{PSh}_C(B) = [B^{\mathrm{op}}, C]$. The first special case of $(*)$ is by taking $C = \boldsymbol{\Delta}^{\mathrm{op}}$, $D = J^{\mathrm{op}}$, $E = \mathcal{B}$, giving

$$\mu\colon \mathbf{s}\mathrm{PSh}_{\mathcal{B}}(J) \cong \mathrm{PSh}_{\mathbf{s}\mathcal{B}}(J). \tag{2}$$

Next, let $\mathbb{Z}_{\geq 0}$ be its namesake poset category, and consider a functor $\mathbb{Z}_{\geq 0}^{\mathrm{op}} \to \mathcal{A}$ for a semiadditive category $\mathcal{A}$. This is just a sequence of arrows

$$\cdots \to A_3 \to A_2 \to A_1 \to A_0$$

in $\mathcal{A}$. It is a chain complex exactly when $A(n \to n-1) = 0$. Observe that a map of chain complexes is just any natural transformation of the functors; thus we can equivalently say:

Proposition 3.12 *Let $\mathcal{A}$ be a semiadditive category. Then the chain complexes*

$$\mathrm{Ch}_{\geq 0}(\mathcal{A}) \hookrightarrow [\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathcal{A}]$$

are the full subcategory spanned by functors A such that $A(n \to k) = 0$ whenever $k \leq n-2$.

The second special case of the isomorphism $(*)$ is

$$\mu\colon [\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathrm{PSh}_{\mathcal{B}}(J)] \cong \mathrm{PSh}_{[\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathcal{B}]}(J). \tag{3}$$

We know $\mathrm{PSh}_{\mathcal{B}}(J) = [J^{\mathrm{op}}, \mathcal{B}]$ is semiadditive as $\mathcal{B}$ is, so we may ask whether this isomorphism restricts to the categories of chain complexes. Indeed:

Lemma 3.13 *The isomorphism of categories (3) descends to*

$$\mu\colon \mathrm{Ch}_{\geq 0}(\mathrm{PSh}_{\mathcal{B}}(J)) \cong \mathrm{PSh}_{\mathrm{Ch}_{\geq 0}(\mathcal{B})}(J),$$

the restriction of μ.

Proof We have the correspondence

$$A\colon \mathbb{Z}_{\geq 0}^{\mathrm{op}} \to [J^{\mathrm{op}}, \mathcal{B}] \quad \longleftrightarrow \quad \widetilde{A}\colon J^{\mathrm{op}} \to [\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathcal{B}],$$

and we have to check that A is a chain complex if and only if $\widetilde{A}(j)$ is a chain complex for all j. By Prop. 3.12, it suffices to show that $A(n \to k) = 0$ iff $\widetilde{A}(j)(n \to k) = 0$ for all j. But this is a tautology: by definition,

$$\widetilde{A}(j)(n \to k) = A_n(j) \xrightarrow{A(n \to k)_j} A_k(j),$$

and a natural transformation is zero exactly when its components are. $\square$

In other words, chain complexes of presheaves are presheaves with values in chain complexes. The two isomorphisms of (2) and Lemma 3.13 fit into a canonical commutative square relating to the Eilenberg-MacLane functor.

Proposition 3.14 *Let J be a category and $\mathcal{B}$ be a semiadditive category. The diagram*

$$\begin{tikzcd}
\mathrm{Ch}_{\geq 0}(\mathrm{PSh}_{\mathcal{B}}(J)) \arrow[r, "\mathrm{K}_{\mathrm{PSh}_{\mathcal{B}}(J)}"] \arrow[d, "\mu"', "\simeq"] & \mathbf{s}\mathrm{PSh}_{\mathcal{B}}(J) \arrow[d, "\simeq"', "\mu"] \\
\mathrm{PSh}_{\mathrm{Ch}_{\geq 0}(\mathcal{B})}(J) \arrow[r, "\mathrm{K}_{\mathcal{B}} \circ -"'] & \mathrm{PSh}_{\mathbf{s}\mathcal{B}}(J)
\end{tikzcd}$$

commutes up to natural isomorphism.

Proof Let F be a chain complex with terms in $\mathrm{PSh}_{\mathcal{B}}(J)$. Either composite of the square is a presheaf $J^{\mathrm{op}} \to \mathbf{s}\mathcal{B}$, so we evaluate at any k, and then look at the n-simplices. On one hand

$$\mu(\mathrm{K}F)_j(n) = (\mathrm{K}F)_n(j) = \Big(\bigoplus_{[n] \twoheadrightarrow [k]} F_n\Big)(j) \cong \bigoplus_{[n] \twoheadrightarrow [k]} F_n(j)$$

where we have used the fact that (co)limits are computed pointwise in the presheaf category (in particular evaluation functors preserve limits). So far this is clearly natural. in F. On the other hand,

$$(\mathrm{K} \circ \mu(F))_j(n) = \mathrm{K}(\mu(F)_j(n)) = \mathrm{K}(F_n(j)) = \bigoplus_{[n] \twoheadrightarrow [k]} F_n(j)$$

proving the commutativity up to natural isomorphism. $\square$

We need a categorical lemma, and then the full faithfulness of $\mathrm{K}_{\mathcal{A}}$ will follow easily. If $f\colon D \to E$ is a functor, then we denote for post-composition $f_* = f \circ -\colon [C, E] \to [C, D]$.

Lemma 3.15 *Let $f\colon D \to E$ be a functor and consider $f_*\colon [C, D] \to [C, E]$. Then the assignment $f \mapsto f_*$ preserves the following properties:*

(a) Being faithful.

(b) Being fully faithful.

(c) Being an equivalence.

Proof For (a), suppose $\alpha, \beta\colon F \to G$ such that $f_*\alpha = f_*\beta\colon f \circ F \to f \circ G$. Then

$$(f_*\alpha)_c = (f_*\beta)_c \quad \forall c \implies f(\alpha_c) = f(\beta_c) \quad \forall c$$

and $\alpha_c = \beta_c$ since f is faithful. For (b), suppose in addition that f is full. Let $\beta\colon f \circ F \to f \circ G$ be a map, then for each c we have $\beta_c\colon f(Fc) \to f(Gc)$, so since f is full there is $\alpha_c\colon Fc \to Gc$ such that $f(\alpha_c) = \beta_c$. Since β is natural, for any $u\colon c \to d$ the square

$$\begin{tikzcd} f(Fc) \arrow[r, "f(Fu)"] \arrow[d, "f(\alpha_c)"'] & f(Fd) \arrow[d, "f(\alpha_d)"] \\ f(Gc) \arrow[r, "f(Gu)"] & f(Gd) \end{tikzcd} \implies \begin{tikzcd} Fc \arrow[r, "Fu"] \arrow[d, "\alpha_c"] & Fd \arrow[d, "\alpha_d"] \\ Gc \arrow[r, "Gu"] & Gd \end{tikzcd}$$

commutes. Then the square on the right commutes by functoriality and since f is faithful. For (c), suppose that f is an equivalence of categories, then let $g\colon E \to D$ be the inverse. We claim g_* is an inverse to f. Indeed,

$$(g_* \circ f_*(F))c = gf(Fc) \cong Fc \implies (g_* \circ f_*)F \cong F,$$

naturally in F, where the implication uses naturality of $gf \cong \mathrm{id}$, and the other direction is the same. $\square$

Recall that for the special case $\mathcal{B} = \mathbf{Ab}$, we write $\widehat{J} := \mathrm{PSh}_{\mathcal{B}}(J)$.

Corollary 3.16 *Let J be a category.*

(a) *The functor*

$$\mathrm{K}_{[J^{\mathrm{op}},\mathbf{CMon}]}\colon \mathrm{Ch}_{\geq 0}([J^{\mathrm{op}},\mathbf{CMon}]) \to \mathbf{s}[J^{\mathrm{op}},\mathbf{CMon}]$$

is fully faithful.

(b) *The functor*

$$\mathrm{K}_{\widehat{J}}\colon \mathrm{Ch}_{\geq 0}(\widehat{J}) \to \mathbf{s}\widehat{J}$$

is an equivalence of categories.

Proof By Prop. 3.14, if $\mathcal{B}$ is semiadditive then $\mathrm{K}_{\mathrm{PSh}_{\mathcal{B}}(J)}$ is naturally isomorphic to the composite

$$\mu^{-1} \circ (\mathrm{K}_{\mathcal{B}})_* \circ \mu$$

where the μ are isomorphisms. Thus if $\mathrm{K}_{\mathcal{B}}$ is fully faithful (an equivalence), $\mathrm{K}_{\mathrm{PSh}_{\mathcal{B}}(J)}$ is also fully faithful (an equivalence) by Lemma 3.15. Taking $\mathcal{B} = \mathbf{CMon}$, Cor. 3.9 proves (a). Taking $\mathcal{B} = \mathbf{Ab}$, the Dold-Kan correspondence (Thm. 2.18) proves (b). □

Theorem 3.17 *Let $\mathcal{A}$ be a semiadditive category. The Eilenberg-MacLane functor*

$$\mathrm{K}_{\mathcal{A}}\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A}$$

is fully faithful.

Proof By Prop. 3.5, the enriched Yoneda embedding $よ^e\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}},\mathbf{CMon}]$ is additive, and it is fully faithful by Cor. 3.3. Consider the diagram

$$\begin{tikzcd}
\mathrm{Ch}_{\geq 0}(\mathcal{A}) \arrow[r, "よ^e_*"'] \arrow[d, hook] & \mathrm{Ch}_{\geq 0}([\mathcal{A}^{\mathrm{op}},\mathbf{CMon}]) \arrow[d, hook] \\
{[\mathbb{Z}^{\mathrm{op}},\mathcal{A}]} \arrow[r, "よ^e_*"] & {[\mathbb{Z}^{\mathrm{op}},[\mathcal{A}^{\mathrm{op}},\mathbf{CMon}]]}
\end{tikzcd}$$

which commutes by our definition of the induced functor on chain complex categories. The vertical functors are fully faithful by Prop. 3.12, and the bottom is fully faithful by Lemma 3.15. Hence the top is fully faithful by Lemma 3.2, since the left-hand composite is fully faithful.

Now by additivity apply Lemma 3.10 to get the diagram

$$\begin{tikzcd}
\mathrm{Ch}_{\geq 0}(\mathcal{A}) \arrow[r, "\mathrm{K}_{\mathcal{A}}"] \arrow[d, "よ^e_*"] & \mathbf{s}\mathcal{A} \arrow[d, "よ^e_*"] \\
\mathrm{Ch}_{\geq 0}([\mathcal{A}^{\mathrm{op}},\mathbf{CMon}]) \arrow[r, "\mathrm{K}_{[\mathcal{A}^{\mathrm{op}},\mathbf{CMon}]}"] & \mathbf{s}[\mathcal{A}^{\mathrm{op}},\mathbf{CMon}]
\end{tikzcd}$$

which commutes up to isomorphism. We just proved the left vertical map is fully faithful, and the right side is fully faithful by Lemma 3.15. The bottom is fully faithful by Cor. 3.16, and so another application of Lemma 3.2 shows $\mathrm{K}_{\mathcal{A}}$ is naturally isomorphic to a fully faithful functor, proving the result. □

3.3 Idempotent Completeness

In Cor. 3.16 we showed that the Eilenberg-MacLane functor induced an equivalence of categories

$$\mathrm{Ch}_{\geq 0}(\widehat{\mathcal{A}}) \simeq \mathbf{s}\widehat{\mathcal{A}}, \tag{$*$}$$

and the goal is to deduce the equivalence for $\mathcal{A}$. When $\mathcal{A}$ is additive, we know $\mathcal{A}$ embeds into $\widehat{\mathcal{A}}$ by the enriched Yoneda embedding.

But it is not immediate that the equivalence restricts to $\mathcal{A}$ because it is not a priori clear how to recover $\mathcal{A}$ from $\widehat{\mathcal{A}}$; the remainder of this paper focuses on this. First, we show that if a category J is *idempotent complete*, then it can be recovered categorically as the full subcategory of atomic objects of $\mathrm{PSh}(J)$. In that case we can at least say that $J \simeq J'$ iff $\mathrm{PSh}(J) \simeq \mathrm{PSh}(J')$.

While this doesn't directly apply to the enriched case, in the next section we will use the properties of the forgetful functor $U_* \colon \widehat{\mathcal{A}} \to \mathrm{PSh}(\mathcal{A})$ that we already showed in Prop. 3.4. This will let us prove the Dold-Kan theorem in full generality.

Fix an ordinary category J and recall we write $\mathrm{PSh}(J) := [J^{\mathrm{op}}, \mathbf{Set}]$. The main concepts underlying this section are *retracts* of objects and morphisms:

Definition 3.18

1. Let x, y be objects of J. We say y is a retract of x if there are maps $s \colon y \to x$ and $r \colon x \to y$ such that $rs = 1_y$.
2. Let $f \colon x \to y$, $f' \colon x' \to y'$ be morphisms in J. Then we say f' is a retract of f if it is so in the arrow category $\mathrm{Arr}(J)$.
3. Given these, we can ask if a full subcategory $J_0 \subset J$ is closed under retracts, or if a subset of morphisms $S \subset \mathrm{mor}(J)$ is stable under retracts.

For example, it is easy to see that the class of all isomorphisms in J is closed under retracts. For the definition we need to show that given a commutative diagram

$$\begin{tikzcd} y \arrow[r, "s"] \arrow[d, "f"] & x \arrow[r, "r"] \arrow[d, "g"] & y \arrow[d, "f"] \\ y' \arrow[r, "s'"] & x' \arrow[r, "r"] & y' \end{tikzcd}$$

where the rows compose to 1_y and $1_{y'}$, if g is an isomorphism then f is too. Indeed, let $h = rg^{-1}s' \colon y' \to y$. Then $hf = rg^{-1}s'f = rgg^{-1}s = 1$ and $fh = frg^{-1}s' = r'gg^{-1}s' = 1$. Observe that if we have a retract diagram $y \xrightarrow{s} x \xrightarrow{r} y$, then $(sr)sr = sr$, so in particular we get an idempotent from every retraction/section pair. On the other hand, if we start with an idempotent $e \colon X \to X$ (so that $ee = e$), we say it *splits* if it comes from such a pair. Notice that these retraction pairs are 'absolute' in the sense that they are preserved by any functor.

Definition 3.19 An object x of J is *atomic* if the functor $\mathrm{Hom}_J(x,-)\colon J \to \mathbf{Set}$ preserves colimits. Let $J^{\mathrm{at}} \subset J^{\mathrm{at}}$ be the full subcategory spanned by the atomic objects of J.

For any J, the subcategory J^{at} will be our first example of a full subcategory closed under retracts. We need a quick categorical lemma first:

Lemma 3.20 *Let G, F be functors in $[J, \mathbf{Set}]$ such that F is a retract of G. Then the natural map*

$$\varinjlim_{i\in I} G(z_i) \to G(\varinjlim_{i\in I} z_i)$$

is a retract of the natural map $\varinjlim_i F(z_i) \to F(\varinjlim_i z_i)$, provided that the colimits exist.

Proof We have to show that

$$\begin{tikzcd} \varinjlim_i F(z_i) \arrow[r] \arrow[d] & \varinjlim_i G(z_i) \arrow[r] \arrow[d] & \varinjlim_i F(z_i) \arrow[d] \\ F(\varinjlim_i z_i) \arrow[r] & G(\varinjlim_i z_i) \arrow[r] & F(\varinjlim_i z_i) \end{tikzcd}$$

commutes and the rows compose to the identity. The latter part is immediate using the functoriality of $\varinjlim$ and the fact that F is a retract of G. Commutativity is from the more general fact that given $f\colon F \to G$, the diagram

$$\begin{tikzcd} \varinjlim_i F(z_i) \arrow[r, "f"] \arrow[d] & \varinjlim_i G(z_i) \arrow[d] \\ F(\varinjlim_i z_i) \arrow[r, "f"] & G(\varinjlim_i z_i) \end{tikzcd}$$

commutes. This is the case because there are two ways to get from $F(z_i)$ to $G(\varinjlim_i z_i)$ which respect the colimit maps (and are thus equal): we can either apply f before or after taking the colimit, and the relevant diagrams commute by naturality of f. $\square$

Proposition 3.21 $J^{\mathrm{at}} \subset J$ *is closed under retracts.*

Proof Let y be a retract of x where x is atomic. Applying the contravariant Yoneda functor $J^{\mathrm{op}} \to [J, \mathbf{Set}]$, we deduce that $\mathrm{Hom}(y,-)$ is a retract of $\mathrm{Hom}(x-)$ in the functor category, so the natural map

$$\varinjlim_{i\in I} \mathrm{Hom}(y, z_i) \to \mathrm{Hom}(y, \varinjlim_{i\in I} z_i)$$

is a retract of

$$\varinjlim_{i\in I} \mathrm{Hom}(x, z_i) \to \mathrm{Hom}(x, \varinjlim_{i\in I} z_i)$$

by the lemma. But since x is atomic, the bottom map is an isomorphism, and retracts of isomorphisms are isomorphisms, so y is atomic. $\square$

The canonical example of atomic objects are representable presheaves in $\mathrm{PSh}(J)$. Indeed:

$$\mathrm{Hom}(\text{よ}(x), \varinjlim_{i\in I} F_i) = \left(\varinjlim_{i\in I} F_i\right)(x) = \varinjlim_{i\in I} F_i(x) = \varinjlim_{i\in I} \mathrm{Hom}(\text{よ}(x), F_i)$$

by the Yoneda lemma and since (co)limits are evaluated pointwise. Retracts are related to the notion of idempotent completeness, which will be crucial for generalized Dold-Kan.

Definition 3.22 A category J is *idempotent complete* if all its idempotents split.

We can characterize this in a more categorical manner:

Proposition 3.23 *Let $e\colon x \to x$ be an idempotent in J. The following are equivalent:*

(i) *e splits.*

(ii) *The equalizer* $\mathrm{Eq}(e, 1_x)$ *exists.*

(iii) *The coequalizer* $\mathrm{Coeq}(e, 1_x)$ *exists.*

Proof Observe that (ii) and (iii) are equivalent since if e is idempotent in J then it is also one in J^{op}. For $(i) \Rightarrow (ii)$, suppose $e = x \xrightarrow{r} y \xrightarrow{s} x$ where $rs = 1$. The claim is that

$$\begin{tikzcd}
z \arrow[d, dashed, "rg"'] \arrow[dr, "g"] & & \\
y \arrow[r, "s"] & x \arrow[r, shift left, "e"] \arrow[r, shift right, "1"'] & x
\end{tikzcd}$$

is an equalizer diagram. Indeed, $es = srs = s$, and if $eg = g$ as above, then $rg\colon z \to y$ such that $g = s(rg) = eg = g$. It's unique since if $sh = g$, then $rsh = rg = h$.

On the other hand, if we already have the equalizer, then consider

$$\begin{tikzcd}
x \arrow[d, dashed, "r"'] \arrow[dr, "e"] & & \\
\mathrm{Eq}(e,1) \arrow[r, "s"] & x \arrow[r, shift left, "e"] \arrow[r, shift right, "1"'] & x
\end{tikzcd}$$

so that $sr = e$. Then $srs = es = s \circ 1_x$ implies that $rs = 1$, since equalizer maps (s) are always monic. □

Corollary 3.24 *If J has equalizers or coequalizers then it is idempotent complete.*

Thus, in particular, presheaf categories, (most) concrete categories, abelian categories, and topoi, are all idempotent-complete.

Lemma 3.25 *If J is idempotent-complete, then the essential image of the Yoneda embedding $\text{よ}\colon J \to \mathrm{PSh}(J)$, denoted $\text{よ}(J)$, is closed under retracts.*

Proof Suppose we have a retract diagram

$$1 = Y \xrightarrow{s} \text{よ}(x) \xrightarrow{r} Y$$

in $\mathrm{PSh}(J)$. Now sr is an idempotent of $\text{よ}(J)$, and since J is idempotent-complete and $J \simeq \text{よ}(J)$, it splits so that $s'r' = sr$ and $r's' = 1$, i.e.

$$\begin{tikzcd} \text{よ}(x) \arrow[r, "r'"] \arrow[d, "r"] & \text{よ}(y) \arrow[d, "s'"] \\ Y \arrow[r, "s"] & \text{よ}(x) \end{tikzcd}$$

commutes for some y in J. But then

$$Y \cong \mathrm{Eq}(sr, 1) = \mathrm{Eq}(s'r', 1) \cong \text{よ}(y)$$

by Prop. 3.23, so we're done. $\square$

In the case that the category J is *not* idempotent-complete, we can complete it into a universal idempotent complete category into which J embeds.

Definition 3.26 The idempotent completion of J is an idempotent complete category J^{idem} equipped with a fully faithful functor $y\colon J \to J^{\text{idem}}$ such that for any idempotent-complete C, the induced map

$$- \circ y\colon [J^{\text{idem}}, C] \to [J, C]$$

is an equivalence of categories.

We first have to show that the completion exists. To do so we will embed J into $\mathrm{PSh}(J)$ and take the 'retract-closure':

Lemma 3.27 *Let $J_0 \subset J$ be a full subcategory. Then:*

1. *There is a smallest full subcategory of J containing J_0 which is closed under retracts, written $\overline{J_0}$.*
2. *Each object of $\overline{J_0}$ is a retract of an object of J_0.*

Proof Let $\overline{J_0}$ be the full subcategory spanned by all retracts of objects of J_0. Then $J_0 \subset \overline{J_0}$ since all objects are retracts of themselves. Moreover clearly $\overline{J_0}$ is contained in any full subcategory containing J_0 which is closed under retracts. So showing $\overline{J_0}$ itself is closed will prove (1) and (2). Suppose y is a retract of $x \in \overline{J_0}$ via $rs = 1$; then x is a retract of some $w \in J_0$ via $r's' = 1$, so the outer square in

$$\begin{tikzcd} y \arrow[r, "s"] \arrow[d, equal] & x \arrow[r, "s'"] \arrow[d, equal] & w \arrow[d, equal] \\ y & x \arrow[l, "r"'] & w \arrow[l, "r'"'] \end{tikzcd}$$

gives y as a retract of w. $\square$

Proposition 3.28 *Let $\overline{よ(J)} \subset \mathrm{PSh}(J)$ be the retract-closure of the essential image of the Yoneda embedding. Then:*

1. *$\overline{よ(J)}$ is idempotent-complete.*
2. *The Yoneda embedding $よ\colon J \to \overline{よ(J)}$ exhibits $\overline{よ(J)}$ as the idempotent completion of J.*

Proof For (1), let $e\colon X \to X$ be an idempotent in $\overline{よ(J)}$. Since $\mathrm{PSh}(X)$ is idempotent-complete, this splits in $\mathrm{PSh}(X)$ as a retract of X onto some $Y \in \mathrm{PSh}(J)$, but then $Y \in \overline{よ(J)}$ since $\overline{よ(J)}$ is closed under retracts.

For (2), fix an idempotent-complete category C; we will construct an inverse

$$[J, C] \to [\overline{よ(J)}, C]$$

to $- \circ よ$. Let $F\colon J \to C$ be any functor, and define $\widetilde{F}\colon \overline{よ(J)} \to C$ as follows: On representable presheaves $よ(x)$ take $\widetilde{F}(よ(x)) = F(x)$, and any other object is a retract of representables:

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[dr, equal] & よ(x) \arrow[d, "r"] \\ & Y \end{tikzcd}$$

so define $\widetilde{F}(Y) = \mathrm{Eq}(\widetilde{F}(sr), 1)$, where $\widetilde{F}$ is defined on sr by the Yoneda lemma (it corresponds to an idempotent $y \to y$). It is easy to see that $F \mapsto \widetilde{F}$ is a (natural) inverse to $- \circ よ$ *if* it is well-defined; so we just have to show that.

In particular, suppose Y is a retract of $よ(x), よ(x')$:

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "s'"'] \arrow[dr, equal] & よ(x) \arrow[d, "r"] \\ よ(y) \arrow[r, "r'"'] & Y \end{tikzcd}$$

we must check $\mathrm{Eq}(\widetilde{F}(sr), 1) = \mathrm{Eq}(\widetilde{F}(s'r'), 1)$. It suffices to give an isomorphism $よ(x) \cong よ(y)$ making the diagram

$$\begin{tikzcd} よ(x) \arrow[r, "\simeq"] \arrow[d, "sr"] & よ(y) \arrow[d, "s'r'"] \\ よ(x) \arrow[r, "\simeq"] & よ(y) \end{tikzcd}$$

commute. But indeed $s'r$ is such an isomorphism, and its inverse is $r's$. $\square$

Therefore the idempotent completion J^{idem} always exists and is unique up to a specified isomorphism. We now make a few useful observations. First, if J is idempotent-complete, then is is clear that $\mathrm{Id}_J \colon J \to J$ exhibits J as its own idempotent completion. But, more interestingly, passing to the presheaf category doesn't distinguish between J and its idempotent completion:

Proposition 3.29 *The inclusion $J \hookrightarrow J^{\mathrm{idem}}$ induces an equivalence of categories* $\mathrm{PSh}(J^{\mathrm{idem}}) \simeq \mathrm{PSh}(J)$*.*

Proof Let $y \colon J \to J^{\mathrm{idem}}$ be the idempotent-completion. Then we have the opposite functor $y^{\mathrm{op}} \colon J^{\mathrm{op}} \to (J^{\mathrm{idem}})^{\mathrm{op}}$, and observe that

$$- \circ y^{\mathrm{op}} \colon [(J^{\mathrm{op}})^{\mathrm{idem}}, C] \to [J^{op}, C]$$

is still an equivalence of categories since $- \circ y$ was. Therefore $(J^{\mathrm{op}})^{\mathrm{idem}} = (J^{\mathrm{idem}})^{\mathrm{op}}$ by the universal property. Since **Set** is idempotent-complete, a special case of the equivalence

$$[(J^{\mathrm{idem}})^{\mathrm{op}}, C] \simeq [J^{op}, C]$$

is $\mathrm{PSh}(J^{\mathrm{idem}}) \simeq \mathrm{PSh}(J)$. □

This statement shows that we definitely cannot recover a category which isn't idempotent-complete from its presheaf category. The next few results show, however, that this is the only thing that can fail. We recall a useful fact about colimits in **Set** first.

Lemma 3.30 *Suppose $\varinjlim_i Z_i = Z$ is a colimit in* **Set** *with limiting cocone $\eta = \eta_i \colon Z_i \to Z$. Then for each $z \in Z$, there is an i and an $x \in Z_i$ such that $\eta_i(x) = z$.*

Proof Just as in any category, we can express the colimit as a coequalizer of coproducts,

$$\bigsqcup_{i \to j} Z_i \rightrightarrows \bigsqcup_i Z_i \longrightarrow \varinjlim_{i \in I} Z_i$$

where the last map is induced by the family $\{\eta_i \colon Z_i \to \varinjlim_i Z_i\}$. But coequalizers are always epi, and thus surjective in **Set**, and that this map is a surjection from the disjoint union is a rephrasing of what we wanted to show. □

Proposition 3.31 *A presheaf F in* $\mathrm{PSh}(J)$ *is atomic if and only if it is a retract of a representable presheaf.*

Proof We know that representables are atomic and that the atomic subcategory is closed under retracts (Prop. 3.21), which proves one direction. For the other, write an atomic presheaf F as the colimit of representables $F = \varinjlim_{i \in I} \text{よ}(x_i)$ by the density theorem. Then

$$\mathrm{Hom}(F, F) = \mathrm{Hom}(F, \varinjlim_{i \in I} \text{よ}(x_i)) = \varinjlim_{i \in I} \mathrm{Hom}(F, \text{よ}(x_i)),$$

and the colimit maps $\mathrm{Hom}(F, よ(x_i)) \to \mathrm{Hom}(F, F)$ are composition with the maps $f_i \colon よ(x_i) \to F$ from the density theorem. Applying Lem. 3.30 to the identity $1_F \colon F \to F$, there is some $i \in I$ and a map $g \colon F \to よ(x_i)$ such that $g \circ f_i = 1_F$. Thus F is a retract of $よ(x_i)$. □

Corollary 3.32 *The Yoneda embedding* $よ\colon J \to \mathrm{PSh}(J)^{\mathrm{at}}$ *exhibits the atomic presheaves* $\mathrm{PSh}(J)^{\mathrm{at}}$ *as the idempotent completion of* J*.*

Proof Apply Prop. 3.28, and recall by Lem. 3.27 that $\overline{よ(J)}$ is the subcategory of $\mathrm{PSh}(J)$ consisting of retracts of representable presheaves, which we now know is just $\mathrm{PSh}(J)^{\mathrm{at}}$. □

So, if J is idempotent-complete then $よ\colon J \simeq \mathrm{PSh}(J)^{\mathrm{at}}$ is an equivalence of categories. This is useful because of the following result:

Corollary 3.33 *(a) The induced functor* $f^{\mathrm{idem}} \colon J^{\mathrm{idem}} \to C^{\mathrm{idem}}$ *is an equivalence.*

(b) The functor $f^* \colon \mathrm{PSh}(C) \to \mathrm{PSh}(C)$ *is an equivalence.*

Alternatively, $J^{\mathrm{idem}} \simeq C^{\mathrm{idem}}$ *if and only if* $\mathrm{PSh}(C) \simeq \mathrm{PSh}(J)$*.*

Proof Without maps, $\mathrm{PSh}(J) \simeq \mathrm{PSh}(J^{\mathrm{idem}}) \simeq \mathrm{PSh}(C^{\mathrm{idem}}) \simeq \mathrm{PSh}(C)$, and on the other hand if $\mathrm{PSh}(J) \simeq \mathrm{PSh}(C)$, then since the equivalence preserves colimits we have $\mathrm{PSh}(J)^{\mathrm{at}} \simeq \mathrm{PSh}(C)^{\mathrm{at}}$, so $J^{\mathrm{idem}} \simeq C^{\mathrm{idem}}$.

With maps, for the forward direction note

$$\begin{tikzcd} J \arrow[r, "f"] \arrow[d, hook] \arrow[dr, dashed] & C \arrow[d, hook] \\ J^{\mathrm{idem}} \arrow[r, "f^{\mathrm{idem}}"] & C^{\mathrm{idem}} \end{tikzcd}$$

commutes, and by taking precompositions and applying the universal property we see

$$\begin{tikzcd} \mathrm{PSh}(J) & \mathrm{PSh}(C) \arrow[l, "f^*"'] \\ \mathrm{PSh}(J^{\mathrm{idem}}) \arrow[u] & \mathrm{PSh}(C^{\mathrm{idem}}) \arrow[l, "(f^{\mathrm{idem}})^*"'] \arrow[u] \end{tikzcd}$$

commutes. But then the bottom map and both vertical ones are equivalences (Prop. 3.29), so f^* is one too. Conversely, we have a commutative diagram

$$\begin{tikzcd} \mathrm{PSh}(J)^{\mathrm{at}} \arrow[d, hook] & \mathrm{PSh}(C)^{\mathrm{at}} \arrow[l, "f^*"'] \arrow[d, hook] \\ \mathrm{PSh}(J) & \mathrm{PSh}(C) \arrow[l, "f^*"'] \end{tikzcd}$$

as the bottom equivalence restricts to one on the top. Then we can take (the restriction of) f^* to be f^{idem}. □

Now we have resolved the issue at the beginning of this section: While we can't generally recover a category from the presheaves over it, we can recover its idempotent completion; hence if it is idempotent complete, we can get the category itself.

3.4 Part 2: Essential Surjectivity

In this section, we finish the proof of the generalized Dold-Kan correspondence. Let $\mathcal{A}$ be an additive category, and consider the commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "\text{よ}^e"] \arrow[d, "\text{よ}"] & & \widehat{A} \arrow[d, "U_*"'] \\ \mathrm{PSh}(\mathcal{A})^{\mathrm{at}} \arrow[rr, hook, "j"] & & \mathrm{PSh}(\mathcal{A}) \end{tikzcd} \tag{4}$$

Here よ^e is additive and fully faithful, j is fully faithful. よ is fully faithful, and an equivalence if $\mathcal{A}$ is idempotent-complete. Recall from Prop. 3.4 that U_* is faithful, conservative, and creates limits. In the idempotent complete case, U_* will let us test whether, given an arbitrary presheaf of abelian groups $X \in \widehat{A}$, it is in fact in the essential image of よ^e.

Lemma 3.34 *Let $\mathcal{A}$ be an additive idempotent complete category. For $X \in \widehat{A}$, there is an $x \in \mathcal{A}$ such that $\text{よ}^e(x) \cong X$ if and only if $U^*(X)$ is an atomic presheaf.*

Proof If $X \cong \text{よ}^e(x)$, then $U_*X \cong j\text{よ}(x)$, i.e. U_*X is isomorphic to an atomic object and hence is atomic. On the other hand, suppose U_*X is atomic, then since $\text{よ}\colon \mathcal{A} \to \mathrm{PSh}(\mathcal{A})^{\mathrm{at}}$ is an equivalence there is x such that $f\colon j\text{よ}(x) \cong U_*X$ is an isomorphism. By commutativity of diagram (4), we have $f\colon U_*\text{よ}^e(x) \cong U_*X$. By conservativity of U_*, it will suffice to show f is of the form $U_*\widetilde{f}$, since then $\widetilde{f}\colon \text{よ}^e(x) \cong X$.

But, recall that by Lemma 3.7, if X is an additive presheaf then any map $\text{よ}(x) \to X$ lifts under U_* to one of the form $\text{よ}^e(x) \to X$. Hence it we may prove X is additive, for which it suffices to show X preserves limits. But U_* creates limits, so we just need that $U_*X \cong \text{よ}(x)$ preserves limits. Indeed,

$$\text{よ}(x)(\varinjlim_{i\in I} a_i) \cong \mathrm{Hom}(\varinjlim_{i\in I} a_i, x) \cong \varprojlim_{i\in I} \mathrm{Hom}(a_i, x) \cong \varprojlim_{i\in I} \text{よ}(x)(a_i)$$

noting the contravariance). $\square$

Dold-Kan will now follow from a simple observation about the Eilenberg-MacLane functor: If $A_* \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is a chain complex, then for each n, A_n is a retract of $\mathrm{K}_{\mathcal{A}}(A_*)_n$. Indeed, $\mathrm{K}_{\mathcal{A}}(A_*)_n$ is a finite biproduct with A_n a summand, hence the canonical injection and projection provides the retract diagram. Finally:

Theorem 3.35 (generalized Dold-Kan correspondence) *Let $\mathcal{A}$ be a semiadditive category. The Eilenberg-MacLane functor*

$$\mathrm{K}_{\mathcal{A}} \colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A}$$

is fully faithful. If $\mathcal{A}$ is additive and idempotent complete, then $\mathrm{K}_{\mathcal{A}}$ is an equivalence of categories.

Proof The first part was proven as Thm. 3.17. For the second, since よ^e is additive we have by Lemma 3.10 a diagram

$$\begin{tikzcd} \mathrm{Ch}_{\geq 0}(\mathcal{A}) \arrow[r, "\mathrm{K}_{\mathcal{A}}"] \arrow[d, "\text{よ}^e_*"'] & \mathbf{s}\mathcal{A} \arrow[d, "\text{よ}^e_*"] \\ \mathrm{Ch}_{\geq 0}(\widehat{\mathcal{A}}) \arrow[r, "\mathrm{K}_{\widehat{\mathcal{A}}}"] & \mathbf{s}\widehat{\mathcal{A}} \end{tikzcd}$$

which commutes up to natural isomorphism. We know from the first part that $\mathrm{K}_{\mathcal{A}}$ is fully faithful since any additive category is semiadditive, and the bottom map $\mathrm{K}_{\widehat{\mathcal{A}}}$ is an equivalence by Cor. 3.9(b). By Lemma 3.15 the right hand よ^e_* is fully faithful. To show $\mathrm{K}_{\mathcal{A}}$ is essentially surjective, let $X \in \mathbf{s}\mathcal{A}$, and since the bottom is an equivalence choose a complex $(\widetilde{B}_*, \widetilde{\partial})$ so that $\mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{B}_*) \cong \text{よ}^e_*(X)$. It suffices to show that each $\widetilde{A}_k$ is of the form $\text{よ}^e(A_k)$ for $A_k \in \mathcal{A}$: indeed, by full faithfulness

$$\widetilde{\partial}_k = \text{よ}(\partial_k) \colon \text{よ}^e(A_k) \to \text{よ}(A_{k-1})$$

and $\partial^2 = 0$ since よ^e reflects limits. So in that case we'll have $(A_*, \partial) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ such that $\text{よ}^e_*(A_*) = \widetilde{A}_*$, and

$$\text{よ}^e_*(X) \cong \mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{A}_*) = \mathrm{K}_{\widehat{\mathcal{A}}}\text{よ}^e_*(\widetilde{A}_*) \cong \text{よ}^e_*\mathrm{K}_{\mathcal{A}}(A_*),$$

hence $X \cong \mathrm{K}_{\mathcal{A}}(A_*)$ by full faithfullness. Thus it remains to show that $\widetilde{A}_k$ is in the (essential) image of よ^e.

By Lem. 3.34, we only need to prove that $U_*(\widetilde{A}_k)$ is an atomic presheaf. But by assumption $\mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{A}_*)_n \cong \text{よ}^e(A_n)$, so $U_*\mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{A}_*)_n$ is an atomic presheaf. Since $\widetilde{A}_n$ is a retract of $\mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{A}_*)_n$, $U_*\widetilde{A}_n$ is the retract of an atomic presheaf, so is still atomic by Prop. 3.21. □

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