Serre Schemes and the Quillen-Suslin Theorem

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1 Introduction

1.1 Serre's Problem

This is an expository paper about the Quillen-Suslin theorem in commutative algebra and algebraic geometry. It is easy to state:

Theorem (Quillen-Suslin) *Let A be a principle ideal domain and fix $n \geq 0$. Every finite projective module over the polynomial ring $A[x_1, \dots, x_n]$ is free.*

The statement of the theorem is deceptively simple. Jean-Pierre Serre first asked in 1957 whether there were any finite projective modules over a polynomial ring. For two decades "Serre's problem" remained open until it was proven independently by Daniel Quillen and Andre Suslin in 1976.

Quillen's proof, only about five pages long, filled in the last claim in a series of reductions completed by Serre, Geoffery Horrocks, and Quillen himself. The proof was algebro-geometric, owing to the following equivalent formulation of Serre's problem:

Theorem *Let A be a principle ideal domain and fix $n \geq 0$. Then every vector bundle over $\mathbf{A}^n_A$ is trivial.*

While Quillen's proof is tractable, it is far from self-contained; fortunately, Vaserstein later greatly simplified the proof into a format which we can present here. This paper, broadly, will give a concise yet rigorous account of Vaserstein's proof of the Quillen-Suslin theorem, yet will in parallel will use a new notion of *Serre scheme* and *Serre morphism* along the way.

1.2 Outline

We will proceed in the rest of the paper as follows: First, we spend some time developing a theory of Serre schemes in Sec. 2, which will serve as an introductory point to the rest of the material.

Next, we prove the Quillen-Suslin theorem as stated, breaking it up into two natural segments. Sec. 3.1 focuses on Serre's crucial early work reducing the problem of finite projective modules to one in stably free modules. We will complete the proof in Sec. 3.2 with Vaserstein's method on rings with the unimodular extension property. Although we give, where insightful, the geometric ideas corresponding to the proof of the Quillen-Suslin theorem, it will be mostly commutative algebra.

In contrast, afterwards we will turn purely to geometry. Our primary goal will be to give an example of a non-affine scheme which admits only trivial vector bundles: a *Serre* scheme. This is done in Sec. 4.1, and will be accompanied by topological analogs of the geometric results we discuss.

Finally, in Sec. 4.2 we conclude this exposition by giving a short account of what else is known in the literature about our notion of Serre schemes; this includes a cohomological characterization of the Serre property, and other peripheral examples and theorems.

2 Serre Schemes

This section will introduce the basic notions considered in the paper. In particular, we define Serre schemes, Serre morphisms, and will begin to build a list of affine examples. The 'proofs' in this section are somewhat terse as the results will be elementary algebra or geometry; in any case standard references will be given.

Definition 2.1 Let $(X, \mathcal{O}_X)$ be a ringed space. A *vector bundle* over X is an $\mathcal{O}_X$-module $\mathcal{F}$ such that for each $x \in X$ there is an open neighborhood U and an $n \geq 0$ satisfying $\mathcal{F}|_U \cong \mathcal{O}_U^{\oplus n}$.

Note that a vector bundle is automatically quasicoherent. We say an $\mathcal{O}_X$-module $\mathcal{F}$ is *trivial* if there is an $n \geq 0$ such that $\mathcal{F} \cong \mathcal{O}_X^{\oplus n}$. Hence a vector bundle is equivalently a locally trivial sheaf. We write $\mathrm{Vect}(X)$ for the full subcategory of $\mathrm{QCoh}(X)$ spanned by the vector bundles. It is clear that this is a local property. Our main definition is the following.

Definition 2.2 A scheme X is *Serre* if every vector bundle over X is trivial. A ring A is *Serre* if every finite projective module over A is free.

The first and most fundamental example of a Serre ring is a local ring, which will help us connect Serre rings to Serre schemes.

Proposition 2.3 ([1, Thm. 2.5]) *Any local ring* $(R, \mathfrak{m})$ *is Serre.*

Proof Let M be a finite projective module and choose a generating set $\{x_1, \ldots, x_n\}$ with n minimal. This defines a surjection from a free module and thus an exact sequence

$$0 \to K \to R^n \to M \to 0,$$

so $R^n = K \oplus M'$ for $M' \cong M$ by projectivity. If $(r_i) \in K$, then $\sum_i r_i x_i = 0$ so by minimality none of the r_i are units, i.e. $K \subset \mathfrak{m}^n$. Then $K \subset \mathfrak{m}K \oplus \mathfrak{m}M'$, which means

$$K/\mathfrak{m}K \subset (\mathfrak{m}M') \cap K = 0.$$

Since K is finitely generated, it is 0 by Nakayama's lemma. $\square$

Remark 2.4 A rather nontrivial result due to Kaplansky is that *any* projective module over a local ring is free.

Corollary 2.5 *Let M be a finite projective A-module. Then $M_\mathfrak{p}$ is free over $A_\mathfrak{p}$ for each prime $\mathfrak{p}$.*

Proof If $M \oplus N = A^n$, then $M_\mathfrak{p} \oplus N_\mathfrak{p} = A_\mathfrak{p}^n$ so we can use Prop. 2.3. $\square$

If M is a finite A-module, then the property of a stalk being 0 can be extended to a neighborhood. Indeed, if $x_1, \ldots, x_n$ generate M and each $x_i \mapsto 0$ in the localization $M_\mathfrak{p}$, there is $f_i \in A \setminus \mathfrak{p}$ such that $f_i x_i = 0$. Thus $f = f_1 \cdots f_n$ annihilates M so $M_f = 0$, and also $M_\mathfrak{q} = 0$ for any $q \in D(f)$. Likewise:

Lemma 2.6 *Let M be a finitely presented A-module and $\mathfrak{p} \subset A$ a prime such that $M_\mathfrak{p}$ is free over $A_\mathfrak{p}$. Then M_f is free over A_f for some $f \in A \setminus \mathfrak{p}$.*

Proof Given a basis for $M_\mathfrak{p}$ over $A_\mathfrak{p}$ we can assume it has no denominators, and so the isomorphism $A_\mathfrak{p}^n \xrightarrow{\sim} M_\mathfrak{p}$ is the localization of a map $A^n \to M$. We have the exact cokernel sequence

$$A^n \to M \to C \to 0,$$

and taking the stalk shows $C_\mathfrak{p} = 0$, Since C is finite, $C_g = 0$ for some g. Hence there is an exact sequence

$$0 \to K \to A_g^n \to M_g \to 0,$$

and M_g is finitely presented so K is finite. Then since $K_\mathfrak{p} = 0$, take h such that $K_h = 0$. Now $A_{gh}^n \to M_{gh}$ has trivial kernel and cokernel. $\square$

Corollary 2.7 *Let M be a finite projective A-module. Then $\widetilde{M}$ is a vector bundle over* $\operatorname{Spec}(A)$.

Proof By Cor. 2.5, Lemma 2.6, and locality of the definition, it is enough to show M is finitely presented. But indeed any presentation

$$0 \to K \to A^n \to M \to 0$$

splits as $A^n \cong K \oplus M$, so K is finite. $\square$

From this it is clear that a quasicoherent sheaf $\mathcal{F}$ over a scheme X is a vector bundle if and only if $\mathcal{F}_x$ a finite free $\mathcal{O}_{X,x}$-module for each $x \in X$. The converse of the corollary holds as well.

Theorem 2.8 ([2, Tag 00NX]) *An A-module M is finite projective if and only if $\widetilde{M}$ is a vector bundle over* $\operatorname{Spec}(A)$. *Therefore A is Serre if and only if* $\operatorname{Spec}(A)$ *is Serre.*

Proof Suppose $\widetilde{M}$ is a vector bundle and choose a Zariski open cover $f_1, \ldots, f_n$ such that M_{f_i} is finite free over A_{f_i}. Since finiteness is Zariski local, M is a finite module, so we can consider the kernel:

$$0 \to K \to A^N \to M \to 0.$$

Localizing at each f_i shows K_{f_i} is a direct summand of $A^N_{f_i}$ hence finite, so K is finite, i.e. M is finitely presented. It is an easy algebra result that in this case $\operatorname{Hom}_A(M,N)_{\mathfrak{p}} = \operatorname{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$, see e.g. [2, Tag 0583].

We may equivalently show $\operatorname{Hom}(M, -)$ is right exact, and we know $M_{\mathfrak{p}}$ is finite free for all $\mathfrak{p}$. If $N' \to N \to N''$ is exact, then localize at all $\mathfrak{p}$ and apply $\operatorname{Hom}(M_{\mathfrak{p}}, -)$ which remains exact by assumption. Hence

$$\operatorname{Hom}(M, N')_{\mathfrak{p}} \to \operatorname{Hom}(M, N)_{\mathfrak{p}} \to \operatorname{Hom}(M, N'')_{\mathfrak{p}}$$

is exact for all $\mathfrak{p}$ completing the proof. □

We next discuss an easy class of Serre schemes, which will eventually be the base case of an induction that proves the Quillen-Suslin theorem:

Proposition 2.9 *Let A be a ring. The following are equivalent:*

i. Any submodule of a finite free module is finite free.

ii. Any nonzero ideal $I \subset A$ is free of rank 1.

iii. A is a PID.

In particular, any PID is Serre.

Proof $(i) \Rightarrow (ii)$ is clear by noting that any free ideal has rank ≤ 1. $(ii) \Rightarrow (iii)$ is clear by considering the image of 1 under an isomorphism $A \xrightarrow{\sim} I$. For the remaining implication, consider a submodule $M \subset A^n$ of a finite free module. Note every ideal is free over A. For $k \leq n$ viewing $A^k \subset A^n$, write $M_k = M \cap A^k$. We show inductively that M_k is free; M_1 is an ideal, so suppose M_k is free and consider the projection $\pi \colon A^{k+1} \to A$ to the last factor. Restricting to $\pi \colon M_{k+1} \to A$, the image is an A-submodule I and the kernel is $M_{k+1} \cap A^k = M_k$. Hence

$$0 \to M_k \to M_{k+1} \xrightarrow{\pi} I \to 0$$

is exact, and since I is free $M_{k+1} \cong M_k \oplus I$ is finite free. □

Remark 2.10 Just as in the local case, *any* projective module over a PID is free. Here, however, it is easy to see using Zorn's lemma [3, Appx. 2].

Remark 2.11 We have relied on the fact that a quasicoherent sheaf over X being locally trivial can be checked on an open cover of X. On the other hand, the property of being Serre is *not* local. For example, if k is a field then $X = \mathbf{P}^1_k$ has an open cover by two copies of $\mathbf{A}^1_k = \mathrm{Spec}(k[x])$, which is Serre by Prop. Prop. 2.9. Yet $\mathbf{P}^1_k$ has nontrivial line bundles.

Let $f\colon X \to Y$ be a morphism of schemes, and consider the associated pullback functor $f^*(\mathcal{F}) = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. In particular, for each $x \in X$ we have $(f^*\mathcal{F})_x = \mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$. So, if $\mathcal{F} \in \mathrm{Vect}(Y)$, then $\mathcal{F}_{f(x)} \cong \mathcal{O}^n_{Y,f(x)}$ shows that $(f^*\mathcal{F})_x$ is free and $f^*\mathcal{F}$ is a vector bundle. Therefore f induces a functor of vector bundles, and using this we may give a better, more general definition of the Serre property:

Definition 2.12 A morphism of schemes $f\colon X \to Y$ is *Serre* if the induced functor

$$f^*\colon \mathrm{Vect}(Y) \to \mathrm{Vect}(X)$$

is essentially surjective.

Remark 2.13 Checking that the class of Serre morphisms contains all isomorphisms and is closed under composition is a special instance of the fact that the pullback defines a *weak 2-functor* or *pseudofunctor*

$$\mathrm{Vect}\colon \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Cat},$$

which follows from the compatible (obvious) natural isomorphisms

$$(g \circ f)^* \cong f^* \circ g^* \qquad (1_X)^* \cong \mathrm{Id}_{\mathrm{Vect}}$$

along with some coherence conditions. Hence Vect forms a weak 2-presheaf of categories over **Sch**. Appropriately ‘topologizing’ the category **Sch** and applying descent for vector bundles shows that Vect is in fact a 2-sheaf or *stack*. In this interpretation, the Serre morphisms are those along which restriction does not produce any new sections of the Vect stack.

Observe that if $f\colon X \to Y$ is Serre and Y is a Serre scheme, then X too is a Serre scheme because f^* preserves trivial bundles. In Sec. 4.1 we will use this to give an example of a non-affine Serre scheme.

Proposition 2.14 *A scheme X is Serre if and only if the unique map $X \to \mathrm{Spec}(\mathbb{Z})$ is Serre.*

Proof One direction follows from the preceding remark and that fact that $\mathrm{Spec}(\mathbb{Z})$ is Serre by Prop. 2.9. On the other hand, if $\mathrm{Vect}(X)$ only consists of trivial bundles then each object therein is isomorphic to the pullback of $\mathcal{O}^{\oplus n}_{\mathrm{Spec}(\mathbb{Z})}$ for some n. □

Example 2.15 Let A be a ring and $0 \le m < n$. The canonical closed immersion $\mathbf{A}^m_A \to \mathbf{A}^n_A$ induced by the map $A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_m]$ sending

$x_{m+1}, \ldots, x_n$ to 0 is a Serre morphism of schemes. Indeed, it suffices to show that if M is finite projective over A, there is a finite projective module N over $A[x]$ so that $N \otimes_{A[x]} A = M$. It is easy to check that $N = \bigoplus_{i \geq 0} Mx^i$ with the obvious action works.

The example implies that $A[x_1, \ldots, x_n]$ cannot be Serre without A being Serre. Our proof is a special instance of a more general fact:

Proposition 2.16 *Any split monomorphism of schemes is Serre. Similarly, any split epimorphism of rings induces a Serre morphism on spectra.*

Proof The second statement follows from the first since all functors preserve split monomorphisms. This same fact applies up to isomorphism to the pseudofunctor Vect: if $i\colon Z \to X$ is a split monomorphism with left inverse p, it follows that $i^* \circ p^* \cong \mathrm{Id}$, hence i^* is full and essentially surjective. $\square$

A split monomorphism $i\colon Z \to X$ is always an immersion. If X and Z are separated (e.g. affine), then i is necessarily a closed immersion. This, however, need not be the case: If k is a field, the open immersion

$$\mathbf{A}^1_k \hookrightarrow \mathbf{A}^1_k \underset{\mathbf{A}^1_k \setminus 0}{\sqcup} \mathbf{A}^1_k$$

of the affine line into the line with two origins is a split monomorphism, and hence Serre, but not a closed immersion.

Another class of Serre morphisms we can derive from Prop. 2.16 is the diagonals: If $f\colon X \to Y$ is any morphism, then the diagonal $\Delta_f\colon X \to X \times_Y X$ is a Serre morphism, since either projection provides a retract. For the absolute case, the map $X \to X^n$, induced in each component by the identity, is Serre for the same reason. Hence if X^n is Serre then X must be, generalizing a conclusion from the first example. Similarly, either of the canonical injections $X \to X \sqcup X$ is Serre, with retract the codiagonal.

We warn that the class of Serre morphisms is not stable under pullbacks, nor is it a local property of morphisms. To see the former, note that the Quillen-Suslin theorem will show that $\mathbf{A}^1 = \mathbf{A}^1_{\mathbb{Z}}$ is Serre, and Prop. 2.3 proved that $\mathrm{Spec}(R)$ is Serre for R a local ring. But in general $\mathbf{A}^1 \times \mathrm{Spec}(R) = \mathbf{A}^1_R$ is not Serre. For locality, we remark that the subclass of Serre morphisms which induce an *equivalence* on categories of vector bundles is local; this is easily checked by descent for bundles.

We conclude this section by stating a different way in which a Serre morphism might arise. This will be a useful criterion to show that an open immersion is Serre which we will use in Sec. 4.1.

Proposition 2.17 *Let $j\colon U \to X$ be an open immersion of schemes such that for each vector bundle $\mathcal{E} \in \mathrm{Vect}(U)$ the pushforward $j_*\mathcal{E}$ is a vector bundle over X. Then j is Serre.*

Proof Notice that for any open $V \subset U$, we have

$$(j^* j_* \mathcal{E})V = j_*(\mathcal{E}|_U(V)) = \mathcal{E}(V),$$

hence j_* provides a right inverse to the pullback defined on bundles. □

3 Proving the Quillen-Suslin Theorem

We are ready to prove the Quillen-Suslin theorem, namely that affine n-space over a PID A is a Serre scheme. The proof is broken into two parts:

(a) [Sec. 3.1] Every finite projective module over $A[x_1, \ldots, x_n]$ is stably free.

(b) [Sec. 3.2] Even stably free module over $A[x_1, \ldots, x_n]$ is free.

As will become clear, the method of proving each part will be quite different: part (a) will consist mostly of manipulating exact sequences and resolutions, while (b) will deal with matrices of elements of rings. Our presentation will correspond structurally to [3, Ch. XXI, Secs. 2, 3], but proofs and statements will vary. We now begin with (a), that is 'Serre's reduction'.

3.1 Serre's Reduction

In this section, all modules without qualification will be over some fixed ring A. As discussed, Serre's result was to reduce 'finite projective' to 'stably free'; we now define the latter:

Definition 3.1 A module M is *stably free* if there is a finite free module F such that $E \oplus F$ is finite free. Two modules M, M' are *stably isomorphic* if there are finite free modules F, F' such that

$$M \oplus F \cong M' \oplus F'.$$

In this case we write $M \simeq_{\mathrm{st}} M'$.

When M is stably free it is finite projective as summand of a finite free module. It is also clear that if M is stably isomorphic to a stably free module, then M is stably free: If $M \oplus F$ is stably free then we can make $M \oplus (F \oplus F')$ finite free.

Throughout this section, we will generally reserve the letter F for finite free modules and E for stably free modules. Much of the theory of stably free modules is about resolutions:

Definition 3.2 Let M be a module. A finite (resp., stably) free resolution of M is a (finite) exact sequence of the form

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to M \to 0$$

where each E_i is finite (stably) free. The finite (stably) free dimension of M, written $\dim_{\mathrm{ff}}(M)$ $(\dim_{\mathrm{sf}}(M))$, is the minimal length of a finite (stably) free resolution in the case that one exists, and otherwise is ∞.

Lemma 3.3 (Schanuel's Lemma) *Suppose that for $i = 1, 2$ the sequences*

$$0 \to K_i \to P_i \to M_i \to 0$$

are exact with P_i projective. Then $K_1 \oplus P_2 \cong K_2 \oplus P_1$.

Proposition 3.4 *Let M_1, M_2 be modules such that $M_1 \simeq_{\text{st}} M_2$, and suppose*

$$0 \to N_i \to E_i \to M_i \to 0 \qquad i = 1, 2$$

are exact with E_i stably free. Then $N_1 \simeq_{\text{st}} N_2$.

Lemma 3.5 *Let M be a module such that $\dim_{\text{sf}}(M) \le n$, and suppose that*

$$E_{n-1} \to \cdots E_0 \to M \to 0$$

is exact with E_i stably free. Then $K = \ker\left(E_{n-1} \to E_{n-2}\right)$ is stably free.

Proof It suffices to show that K is stably isomorphic to a stably free module. By assumption there is a stably free resolution

$$0 \to E'_n \to \cdots \to E'_0 \to M \to 0$$

of length n (extending by zeros if necessary). For convenience let $E'_{-1} = E_{-1} = M$, and write

$$K_i = \ker\left(E_i \to E_{i-1}\right) \qquad K'_i = \ker\left(E'_i \to E_{i-1'}\right)$$

for $0 \le i \le n$. In particular $K_{n-1} = K$ and $K'_{n-1} = E'_n$, so we can show by induction that $K_i \simeq_{\text{st}} K'_i$. Observe that the sequence

$$0 \to K_i \to E_i \to K_{i-1} \to 0$$

is exact, and similarly for K'_i and E'_i: this is since the image of E_i in E_{i-1} is exactly K_{i-1}, and this kernel is still K_i. When $i = 0$ this is clear from Prop. 3.4 because M is stably isomorphic to itself and E_0 is stably free. By induction, the same result implies $K_i \simeq_{\text{st}} K'_i$ from the exact sequence. □

Corollary 3.6 *Assume A is Noetherian. Suppose $0 \to N \to E \to M \to 0$ is an exact sequence with E stably free. Then*

$$\dim_{\text{sf}}(N) \le \dim_{\text{sf}}(M) - 1.$$

Definition 3.7 A module M is *resolvable* if it admits a finite free resolution, i.e. if $\dim_{\text{ff}}(M) < \infty$. A ring A is *pointwise resolvable* if each prime $\mathfrak{p} \in \operatorname{Spec}(A)$ is resolvable as an A-module.

Proposition 3.8 (2-of-3 property) *Assume A is Noetherian. Suppose we have an exact sequence*

$$0 \to M' \to M \to M'' \to 0.$$

If any two of the modules are resolvable then the third is as well.

Proof The proof is a diagram chase; see [3, Ch. XXI, Thm. 2.7]. □

Remark 3.9 In Cor. 3.6 and Prop. 3.8, we assumed A was Noetherian. It is not hard to remove this hypothesis in both cases, but it does simplify the proofs; in any case Noetherianity is crucial for Thm. 3.11 and Cor. 3.13, because it is required in Lemma 3.10. This is to be expected since our results are about finite resolutions, and they work because Serre's problem asks about sheaves which are locally free of *finite* rank.

To prove to prove the following lemma about filtrations, we will need to use associated primes. Let M be an A-module, and write $\mathrm{Ann}_A(M)$ for the ideal $\{a \in A : ax = 0 \ \forall x \in M\}$. Recall that an associated prime of M is a prime $\mathfrak{p}$ of A such that $\mathfrak{p} = \mathrm{Ann}_A(x)$ for some x (i.e. $\langle x \rangle \cong R/\mathfrak{p}$).

If $A \neq \varnothing$ is Noetherian, then M always has an associated prime. Indeed, there is a maximal element of the poset $\{\mathrm{Ann}_A(x) : x \in M\}$ under inclusion, say $I = \mathrm{Ann}_A(x)$. If $a, b \notin I$ and $abx = 0$, then $(a, I) \subset \mathrm{Ann}_A(bx)$, but $bx \neq 0$ and $(a, I) \supsetneq I$ is a contradiction to maximality. Thus I is prime.

Lemma 3.10 *Let A be Noetherian and M a finite A-module. Then there are primes $\{\mathfrak{p}_i\} \in \mathrm{Spec}(A)$ and a finite filtration*

$$M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_0 = 0$$

such that $M_i/M_{i-1} \cong A/\mathfrak{p}_i$ for each i.

Proof By Noetherianity M satisfies the ascending chain condition on submodules. We inductively make the M_i by taking $M_0 = 0$, and once M_i has been constructed let $\mathfrak{p}_{i+1}$ be an associated prime of M/M_i, which corresponds to a submodule $M_{i+1}/M_i \subset M/M_i$ such that $A/\mathfrak{p}_{i+1} \cong M_{i+1}/M_i$. Then indeed $M_i \subset M_{i+1}$ so the sequence must stabilize at some $M_n \subset M$. If $M_n \neq M$ then M/M_n is nonzero and we can find an associated prime which gives a bigger submodule $M_{n+1} \supsetneq M_n$, a contradiction. $\square$

Our next goal is the following theorem:

Theorem 3.11 *Suppose that A is Noetherian and pointwise resolvable. Then $A[x]$ is pointwise resolvable.*

From this Serre's reduction will follow relatively easily, but the proof of Thm. 3.11 requires more than just some exact sequence manipulation as we have been able to get away with so far. This result is at the heart of the Quillen-Suslin theorem; we will first prove the following intermediate step.

Lemma 3.12 *Let A be Noetherian, $P \in \mathrm{Spec}(A[x])$, and write the prime $\mathfrak{p} = P \cap A \in \mathrm{Spec}(A)$. Then there is a finite module N over $A[x]$ with the following properties:*

1. *There exists an exact sequence of $A[x]$-modules*

$$0 \to A[x]/\mathfrak{p}[x] \to P/(\mathfrak{p}[x]) \to N \to 0$$

where $\mathfrak{p}[x] := \mathfrak{p}A[x] \subset A[x]$.

2. *N admits a filtration of the form*

$$N = N_m \supset N_{m-1} \supset \cdots \supset N_1 \supset N_0 = 0$$

such that $N_i/N_{i-1} \cong A[x]/Q_i$ for all i, where each $Q_i \subset A[x]$ is a prime ideal for which $\mathfrak{p} \subsetneq Q_i \cap A$.

Proof Let A_0 be the domain $A/\mathfrak{p}$. $\mathfrak{p}[x] \subset P$, so $P_0 := P/\mathfrak{p}[x]$ is an ideal of the quotient ring $A[x]/\mathfrak{p}[x] \cong A_0[x]$. A_0 is a quotient of A, hence Noetherian so $A_0[x]$ is Noetherian. Thus the ideal P_0 is generated by finitely many polynomials $g_1, \dots, g_n \in A_0[x]$.

Now let $f \in P_0$ be a polynomial of minimal degree, so we can find $q_i, r_i \in k[x]$ such that

$$g_i = q_i f + r_i \qquad i = 1, \dots, n$$

and where $r_i < \deg f$. Let $d_0 \in A_0$ be the (nonzero) product of all denominators, so that

$$d_0 g_i = d_0 q_i f + d_0 r_i \quad \in A_0[x].$$

Looking modulo P_0, shows that $d_0 r_i \in P_0$, but f had minimal degree so $d_0 r_i = 0$. Therefore $d_0 g_i = (d_0 q_i) f \in (f)$, which means $d_0 P_0 \subset (f)$. We can consider the $A_0[x]$-module $P_0/(f)$ as an $A[x]$-module $N := \big(P_0/(f)\big)_{A[x]}$ by restriction of scalars along the quotient map. We have an exact sequence of $A[x]$ modules

$$0 \to (f) \to P_0 \to N \to 0,$$

and we can consider the $A[x]$ map $A_0[x] \to (f)$ defined by $1 \mapsto f$. This is onto, and its kernel is 0 since A_0 (and thus $A_0[x]$) is a domain. Hence we may replace (f) by $A_0[x] = A[x]/\mathfrak{p}[x]$ in the above sequence which proves part (1).

Since $d_0 \neq 0$ in A_0, we get a lift $d \in A$ such that $d \notin \mathfrak{p}$. Now $d_0 \cdot (P_0/(f)) \subset (d_0 P_0)/(f) = 0$ by since $d_0 P_0 \subset (f)$. The definition of restriction of scalars means that $d \cdot N = 0$ and $\mathfrak{p} \cdot N = 0$. So $(d, \mathfrak{p}) \subset \mathrm{Ann}_{A[x]}(N)$. Next, N is clearly finite over $A[x]$ so by Lemma 3.10 we can choose a filtration

$$N = N_m \supset \cdots \supset N_0 = 0$$

such that there are primes $Q_i \subset A[x]$ for which $N_i/N_{i-1} \cong A[x]/Q_i$. But

$$\mathrm{Ann}_{A[x]}(N) \subset \mathrm{Ann}_{A[x]}(N_i) \subset \mathrm{Ann}_{A[x]}(N_i/N_{i-1}) = Q_i$$

for all i because annihilators remain fixed under module isomorphisms. Therefore $(d, \mathfrak{p}) \subset Q_i$, and so $\mathfrak{p} \subset Q_i \cap A$ yet $\mathfrak{p} \neq Q_i \cap A$ because $d \notin \mathfrak{p}$. $\square$

Proof of Thm. 3.11 We have to show that all primes of $A[x]$ are resolvable. Suppose not; by Noetherianity of A, the poset of ideals of A

$$\{P \cap A : P \in \mathrm{Spec}(A) \text{ s.t. } P \text{ is not resolvable}\}$$

has a maximal element, say $P \cap A =: \mathfrak{p}$. Take the corresponding finite module N from Lemma 3.12 which satisfies $N_i/N_{i-1} \cong A[x]/Q_i$ where $\mathfrak{p} \subsetneq Q_i \cap A$. By maximality, Q_i is resolvable for each i. By the exact sequence

$$0 \to Q_i \to A[x] \to N_i/N_{i-1} \to 0$$

of $A[x]$-modules and Prop. 3.8, the quotient N_i/N_{i-1} is resolvable, and by

$$0 \to N_{i-1} \to N_i \to N_i/N_{i-1} \to 0,$$

if N_{i-1} resolvable then N_i is. Since $N_0 = 0$, this means $N_m = N$ is resolvable.

Now we show $\mathfrak{p}[x]$ is a resolvable $A[x]$-module. Indeed, since we assumed A was pointwise resolvable there is an exact sequence of finite free A-modules

$$0 \to F_n \to \cdots F_0 \to \mathfrak{p} \to 0.$$

Recall $A \to A[x]$ is a flat extension; moreover $A^{\oplus n} \otimes_A A[x] = (A[x])^{\oplus n}$, and $\mathfrak{p} \otimes_A A[x] = \mathfrak{p}[x]$, so base changing gives an exact sequence

$$0 \to F_n' \to \cdots \to F_0' \to \mathfrak{p}[x] \to 0$$

of $A[x]$-modules with F_i' finite free, hence $\mathfrak{p}[x]$ is resolvable.

To finish the proof, note that the sequence

$$0 \to \mathfrak{p}[x] \to A[x] \to A[x]/\mathfrak{p}[x] \to 0$$

and Lemma 3.12 implies $A[x]/\mathfrak{p}[x]$ is resolvable. Then the exact sequence of Lemma 3.12(1) along with the resolvability of N implies $P/(\mathfrak{p}[x])$ is resolvable. Finally, the sequence

$$0 \to \mathfrak{p}[x] \to P \to P/(\mathfrak{p}[x]) \to 0$$

implies P is resolvable, which is the desired contradiction. $\square$

Corollary 3.13 (Serre) *Suppose that A is Noetherian and pointwise resolvable. Then all finite $A[x]$-modules are resolvable.*

Proof By Thm. 3.11, all primes of $A[x]$ are resolvable. Let M be a finite $A[x]$-module, and take by Lemma 3.10 a filtration

$$M = M_n \supset \cdots \supset M_0 = 0$$

where $M_i/M_{i-1} \cong A[x]/\mathfrak{p}_i$. Then it follows from the exact sequence

$$0 \to \mathfrak{p}_i \to A[x] \to M_i/M_{i-1} \to 0$$

and Prop. 3.8 that M_i/M_{i-1} is resolvable. Taking $i = 1$ see M_1 is resolvable. Considering the sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0,$$

we can see M_i is resolvable for all i by induction. In particular M is resolvable which completes the proof. $\square$

Lemma 3.14 *Let M a projective module. Then M is stably free if and only if it is resolvable.*

Proof One direction is trivial, so in the other case we induct on the finite free dimension $\dim_{\mathrm{ff}}(M) = n$. Assume $n \geq 1$, and split the resolution into short exact sequences. Projectivity and the induction hypothesis completes the proof. □

Corollary 3.15 *Let A be a PID. Then every finite projective module over $A[x_1, \ldots, x_n]$ is stably free.*

Proof By Lemma 3.14 it suffices to prove that every finite module over $A[x_1, \ldots, x_n]$ is resolvable. In light of Cor. 3.13, an easy induction implies we need only show every prime ideal of A is resolvable. This is trivial since by Prop. 2.9 every ideal is free. □

Notice that the proof can be adapted to any ring whose prime ideals are stably free. A natural question is where in this section the ring A may be replaced by a scheme X and the modules by quasicoherent sheaves. The biggest issue is that finite freeness, which we would like to replace with triviality, is not local. In addition, the fact that vector bundles in $\mathrm{QCoh}(X)$ are not projective objects poses a challenge. For these reasons the best setting for this sequence of results is the affine case.

3.2 Vaserstein's Proof

In this section we complete the proof of the Quillen-Suslin theorem using Vaserstein's simplified proof. The focus is the following definition:

Definition 3.16 Let A be a ring.

(a) A *unimodular vector* $(f_1, \ldots, f_n)$ is an ordered list of elements of A which generate the unit ideal.

(b) A unimodular vector $(f_i) \in A^{(n)}$ has the *unimodular extension property* (UEP) if there is a matrix in $\mathrm{GL}_n(A)$ whose first column is $(f_i)^t$.

(c) The ring A has the *unimodular extension property* (UEP) if every unimodular vector over A has the *unimodular extension property* (UEP).

This is a fundamentally geometric notion: A unimodular vector is just a Zariski open cover of $\mathrm{Spec}(A)$. Heuristically, $\mathrm{GL}_n(A)$ appears here because its elements describe transition functions for a vector bundle on overlapping regions of open covers. The following proposition explains why the definition is useful, in light of the previous section.

Proposition 3.17 *Let A have the UEP. The every stably free module is free.*

Proof By induction, it suffices to show that if $E \oplus A$ is free then E is free. Assume $E \oplus A = A^n$ and let $p\colon A^n = E \oplus A \to A$ be projection to the second factor. Write $u^1 = (a_{11}, \dots, a_{n1})$ for a basis, so by our assumption we can complete this to a matrix $M = (a_{ij})$ of unit determinant. Writing $u^j = Me_j$, we may assume $u^j \in E$. Since M induces an automorphism $A^n \xrightarrow{\sim} A^n$, it maps isomorphically the set $\{e_2, \dots, e_n\} \twoheadrightarrow E$, hence E has a basis. □

The first example of a ring with the UEP will be a PID. There is a bruteforce induction proof, but the following argument is a nice trick.

Proposition 3.18 *Any PID is has the UEP.*

Proof Suppose $\langle f_1, \dots, f_n \rangle = A$, then there are a_i such that $\sum_i a_i f_i = 1$, so

$$0 \to K \to A^n \xrightarrow{\phi} A \to 0$$

is exact. ϕ has a section by $1 \mapsto \sum_i a_i e_i$, so letting $a = \sum a_i e_i$ we have $\langle a \rangle \oplus K = A^n$. Since A^n is a PID, K is free of rank $n-1$ by Prop. 2.9, hence a extends to a basis of R^n, giving the map $R^n \to R^n$ as desired. □

We next need Horrocks' theorem, which we state below. This result is the machinery behind the inductive proof of the Quillen-Suslin theorem. We note that the proof is not difficult, and is found, say, in [3, Ch. XXI, Thm. 3.1].

Theorem 3.19 (Horrocks) *Let $(R, \mathfrak{m})$ be a local ring and write $A = R[x]$. If $f \in A^n$ is a unimodular vector with monic component, then f has the UEP.*

Definition 3.20 For $f, g \in A^n$, we say f and g are A-equivalent and write $f \sim g$ if there is $M \in \mathrm{GL}_n(A)$ such that $f = Mg$.

By Horrock's theorem, if f has a monic component and is unimodular, then it is equivalent to e_1. At this stage, unfortunately the proofs become relatively dense and unmotivated; for brevity we will not prove all the statements. We note, in the proof of the next corollary, the similarity to our original proof that local rings are Serre.

Corollary 3.21 *If R is a local ring and f unimodular with a monic component, then $f \sim f(0)$ over $R[x]$.*

Proof Writing $\sum_i r_i f_i(0) = 1$, if none of the $f_i(0)$ is a unit, they all lie in $\mathfrak{m}$, so $1 \in \mathfrak{m}$ is a contradiction. Hence by Horrock's theorem, f is equivalent to e_i, and the same is true of $f(0)$. □

Lemma 3.22 *Let A be a domain and $S \subset A$ a multiplicative subset. If $f(x) \sim f(0)$ over $(A[S^{-1}])[x]$, then there is $c \in S$ such that $f(x + cy) \sim f(x)$ over $A[x, y]$.*

Proof Let $M(x)$ be a matrix of $A[S^{-1}][x]$ taking $f(0)$ to $f(x)$, then $M^{-1}(x)f(x) = f(0)$ and $M^{-1}(x+y)f(x+y) = M^{-1}(x)f(x)$. Set $G(x,y) = M(x)M^{-1}M^{-1}(x+y)$, then $G(x,y)f(x+y) = f(x)$. In particular, $G(x,0) = I_n$. We may write $G(x,y) = G(x,0) + yH(x,y) = I + yH(x,y)$ where $H(x,y)$ is a polynomial over $A[S^{-1}]$.

Since A is a domain, we can clear denominators by choosing $c \in S$ so that $cH(x,y) \in A[x,y]$, hence $G(x,cy) \in A[x,y]$, and $G(x,cy)f(x+cy) = f(x)$. By definition, $G(x,cy)$ is invertible in $\mathrm{GL}_n(A[S^{-1}][x,y])$; we claim it is invertible in $\mathrm{GL}_n(A[x,y])$. Observe $\det(M(x)) \in (A[S^{-1}][x])^{\times} \subset A[S^{-1}]^{\times}$ since $R[S^{-1}]$ is a domain, so $\det(M(x)) = \det(M(x+cy))$, and therefore indeed $\det(G(x,cy)) = 1$ as desired. □

Proposition 3.23 *Let A be a domain and $f \in (A[x])^n$ a unimodular vector with a monic component. Then $f(x) \sim f(0)$ over $R[x]$.*

Proof Let $J = \{c \in A : f(x+cy) \sim f(x) \text{ over } A[x,y]\}$. If $a \in A$ and $M(x,y)$ is a matrix of $A[x,y]$ for which $M(x,y)f(x) = f(x+cy)$ (i.e. a translation), then $M(x,ay)f(x) = f(x+cay)$. Similarly one can check J is an ideal. Now let $\mathfrak{p} \in \mathrm{Spec}(A)$ so $f(x) \sim f(0)$ by locality over $A_{\mathfrak{p}}[x]$.

We may apply Lemma 3.22 to find $c \notin \mathfrak{p}$ so $f(x+cy) \sim f(x)$ and $c \in J$. It follows that J contains no primes, and thus $J = A$. Hence there is $M(x,y) \in \mathrm{GL}_n(A[x,y])$ such that $M(x,y)f(x) = f(x+y)$ as before. The map $x \mapsto 0$ from $R[x,y] \to R[y]$ extends to

$$\mathrm{GL}_n(R[x,y]) \to \mathrm{GL}_n(R[y]),$$

thus $M(0,y)$ gives (over $A[y]$) that $f(0) \sim f(y)$. □

We can finally state Vaserstein's theorem - we have already set up the main tools of the proof, but we give a reference for the minor details.

Theorem 3.24 (Vaserstein) *Let A be a PID and $f \in A[x_1, \ldots, x_n]^{(m)}$ a unimodular vector. Then f has the UEP.*

Proof The proof is an induction on n (but not m); a slight modification of [3, Ch. XXI, Thm. 3.5] shows that we may start at $n = 0$ and apply the case we already know, namely Prop. 3.18. □

Corollary 3.25 (Quillen-Suslin Theorem) *Let A be a PID. Then every finite projective module over $A[x_1, \ldots, x_n]$ is free. Equivalently, for each PID A and positive integer n, $\mathbf{A}^n_A$ is a Serre scheme.*

Proof We saw in Thm. 3.24 that $A[x_1, \ldots, x_n]$ has the UEP, which by Prop. 3.17 implies that every stably free module over it is free. We are done by Serre's reduction (Cor. 3.15), which showed that every finite projective module over the same ring was stably free. □

4 Applications & Generalizations

4.1 A Non-Affine Serre Scheme

We now aim to give our first example of a Serre scheme which is not affine, which will be the punctured plane $\mathbf{A}^2_k \setminus 0$ over a field. Even equipped with the Quillen-Suslin theorem, the result is not immediate and we resort to a few commutative algebra statements, as well as one geometric theorem (but whose essence is again algebra).

For the proof, we will use the Quillen-Suslin theorem in two nontrivial ways. Nonetheless, the idea of the example is very geometric: Any vector bundle which is 'regular enough' on the punctured plane can be extended to $\mathbf{A}^k_2$, and all vector bundles are regular because of a canonical covering by Serre schemes.

The same result is false in dimensions higher than 2. To build intuition about this, we first consider the topological analog and demonstrate a remarkable similarity between the two cases.

4.1.1 Topological Analogs

We begin with the topological version of the Quillen-Suslin theorem, which is comparatively trivial.

Proposition 4.1 *Let $n \geq 1$. Every real vector bundle over $\mathbb{C}^n$ is trivial.*

Proof Rank d vector bundles over a paracompact Hausdorff space X are classified by homotopy classes of maps $[X, BO(d)]$. Since $X = \mathbb{C}^n$ is homotopy equivalent to a point, we are done. □

Remark 4.2 In fact, there is a geometric formulation of the classification of vector bundles using classifying spaces. In particular, it is a result of motivic homotopy theory that n-dimensional vector bundles over an affine variety X are classified up to isomorphism by $\mathbf{A}^1$-homotopy classes of maps $X \to BGL_n$, where BGL_n is the *simplicial* classifying space of GL_n.

We will attempt to consider the statement of the next proposition in the algebro-geometric case, but the topological formulation provides motivation.

Proposition 4.3 *The space $\mathbb{C}^n \setminus 0$ admits nontrivial vector bundles if and only if $n \neq 2$.*

Proof By connectedness we need only consider the constant rank case. Due to the deformation retract $\mathbb{C}^n \to S^{2n-1}$ given by $z \mapsto z/|z|$, we reduce to computing the homotopy groups $\pi_{2n-1}(BO(d))$. It is well-known that S^k is parallelizeable if and only if $k = 1, 3, 7$, and that S^1 admits the nontrivial Mobius bundle; so it remains to show $n = 2, 4$. The fibration $O(d) \to EO(d) \to BO(d)$ from a weakly contractible space shows that $\pi_{2n-1}(BO(d)) = \pi_{2(n-1)}(O(d))$. Since $O(d)$ is a Lie group, $\pi_2(O(d)) = 0$ which completes $n = 2$. Finally, we claim $\pi_6(O(d))$ is nontrivial for $d = 3$. We may replace $O(3)$ with its identity component $SO(3)$; since $SU(2) \cong S^3$ is the universal cover of $SO(3)$, the fact that $\pi_6(S^3) = \pi_6(S^2) = \mathbb{Z}/12$ completes the classification of vector bundles over punctured complex space. □

Theorem∗ 4.4 (Swan) *Let X be a paracompact Hausdorff space with finite covering dimension. The functor*

$$\mathrm{Vect}_{\mathbb{R}}(X) \longrightarrow \mathrm{FinProj}(C(X, \mathbb{R}))$$

sending a real vector bundle to its module of sections is an equivalence of categories.

It follows that if X is paracompact Hausdorff of finite covering dimension[1] and contractible, then the ring of continuous functions $C(X, \mathbb{R})$ is Serre. However, $C(X, \mathbb{R})$ can be Serre without X being contractible, such as $C(\mathbb{C}^2 \setminus 0, \mathbb{R})$ by Prop. 4.3. The Serre-Swan theorem can be used to give a more interesting example of the non-locality of the Serre property:

Proposition 4.5 *There is a connected affine scheme X with a Zariski cover $\{D(f_i)\}$ such that each $D(f_i)$ is Serre, yet X is not.*

Proof The example is due to Peter Haine. The existence of the Mobius bundle shows that $A := C(S^1, \mathbb{R})$ is not Serre. Moreover, $\mathrm{Spec}(A)$ is connected as there is no element $f \in A$ such that $f^2 = f$ and $f \neq 0, 1$; indeed, such an element would have image $\{0, 1\}$ in $\mathbb{R}$, which cannot be since S^1 is connected.

Let $f \colon S^1 \to \mathbb{R}$ be a function such that for $i = 0, 1$, $U_i := \{z \in S^1 : f(x) \neq i\} \subset S^1$ is connected. Then $\{D(f), D(1 - f)\}$ form an open cover of $\mathrm{Spec}(A)$, and it is easy to see that

$$A_f = C(U_0, \mathbb{R}) \qquad A_{1-f} = C(U_1, \mathbb{R}).$$

Since $U_i \cong (0, 1)$ is paracompact Hausdorff of finite covering dimension and contractible, $D(f)$ and $D(1 - f)$ are Serre. □

4.1.2 A Long Example: The Punctured Plane

Let k be a field, write $\mathbf{A}^2_k := \mathrm{Spec}(k[x, y])$, and set $U := \mathbf{A}^2_k \setminus 0 = \mathbf{A}^2_k \setminus \{\langle x, y\rangle\}$. The goal is to prove the following theorem:

Theorem 4.6 *The open immersion $j \colon U \hookrightarrow \mathbf{A}^2_k$ is a Serre morphism of schemes. In particular, U is a non-affine Serre scheme.*

Remark 4.7 We warn that $\mathbf{A}^n \setminus 0$ is not generally Serre for $n \geq 3$, and just as in Prop. 4.3, $n = 2$ is a very special case. In particular, for any field k there

[1] These assumptions are not that restrictive. In fact, there are examples of non-paracompact and/or non-Hausdorff contractible spaces which admit nontrivial vector bundles. See this MathOverflow post.

is a nontrivial vector bundle over $\mathbf{A}^3_k \setminus 0$, and at least when k is algebraically closed there are nontrivial bundles over $\mathbf{A}^n_k \setminus 0$ for $n \geq 4$.

That j is Serre is more subtle than all of the examples of Serre morphisms we have seen so far, because j is not a split monomorphism. If it were, then since both U and $\mathbf{A}^2_k$ are separated j would be a closed immersion, which contradicts the connectedness of $\mathbf{A}^2_k$. Instead, in order to prove Thm. 4.6, we will use Prop. 2.17. Hence it suffices to show that for each vector bundle $\mathcal{E}$ over U, the pushforward $j_*\mathcal{E}$ is a vector bundle over $\mathbf{A}^2_k$. This is done in three steps: (1) $j_*\mathcal{E}$ is coherent, (2) $j_*\mathcal{E}$ is a reflexive sheaf, (3) $j_*\mathcal{E}$ is a vector bundle. We postpone bringing in external results until as late as possible.

We must study the open subscheme $U \subset \mathbf{A}^2_k$. Observe that $D(x) \cup D(y) = U$, and $D(x) \cap D(y) = D(xy)$. Therefore the sheaf condition shows that

$$\begin{tikzcd} \Gamma(U, \mathcal{O}_U) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & k[x^\pm, y] \arrow[d, hook] \\ k[x, y^\pm] \arrow[r, hook] & k[x^\pm, y^\pm] \end{tikzcd}$$

is a pullback square, which implies $\Gamma(U, \mathcal{O}_U) = k[x^\pm, y] \cap k[x, y^\pm] = k[x, y]$. In particular, the restriction map $\Gamma(\mathbf{A}^2_k, \mathcal{O}_{\mathbf{A}^2_k}) \to \Gamma(U, \mathcal{O}_{\mathbf{A}^2_k})$ is the identity, and we immediately conclude that U is indeed non-affine as otherwise $U \hookrightarrow \mathbf{A}^2_k$ would be onto.

$\mathbf{A}_k$, and U by the open cover, are both Noetherian. This means j is quasicompact and quasiseperated, so j_* preserves quasicoherence. In addition, Noetherianity implies that coherent sheaves are the same as quasicoherent sheaves of finite type. In particular $\mathcal{O}_U$ is coherent over U, and it follows that any vector bundle $\mathcal{E}$ over U is coherent. Then to show $j_*\mathcal{E}$ is coherent, we need only prove it is of finite type.

Since $\mathbf{A}^2_k$ is affine, write $j_*\mathcal{E} = \widetilde{M}$, then by definition $M = \Gamma(\mathbf{A}^2_k, j_*\mathcal{E}) = \Gamma(U, \mathcal{E})$, so it suffices to prove that $\Gamma(U, \mathcal{E})$ is a finitely generated $k[x, y]$-module. Now consider the affine open cover $\{D(x), D(y)\}$ of U from before, the spectra of $k[x^\pm, y]$ and $k[x, y^\pm]$ respectively. We claim $D(x)$ and $D(y)$ are Serre schemes; indeed, write

$$D(x) = \operatorname{Spec}(k[x^\pm][y]) \qquad D(y) = \operatorname{Spec}(k[y^\pm][x]),$$

and observe that $k[z^\pm] = k[z]_z$ is a PID, as it is a localization of a PID. Hence the Quillen-Suslin theorem shows $D(x)$ and $D(y)$ are Serre, so we have isomorphisms

$$\mathcal{E}|_{D(x)} \cong \mathcal{O}_U^{\oplus n}|_{D(x)} \qquad \mathcal{E}|_{D(x)} \cong \mathcal{O}_U^{\oplus m}|_{D(y)}.$$

Finally, the sheaf condition for $\mathcal{E}$ gives the exact sequence of $k[x, y]$-modules

$$0 \to \Gamma(U, \mathcal{E}) \to k[x^\pm, y]^{\oplus n} \oplus k[x, y^\pm]^{\oplus m} \to \Gamma(D(xy), \mathcal{E}),$$

from which it follows that $\Gamma(U, \mathcal{E})$ is finitely generated since $k[x,y]$ is Noetherian. So far we have proven:

Lemma 4.8 *Let $\mathcal{E}$ be a vector bundle over U. Then $j_*\mathcal{E}$ is coherent over $\mathbf{A}^2_k$.*

It is at this point we need some more powerful tools, namely *reflexive sheaves.* If $\mathcal{F}$ is a coherent sheaf on a scheme X, we write

$$\mathcal{F}^{**} := \underline{\mathrm{Hom}}_{\mathcal{O}_X}\big(\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X), \mathcal{O}_X\big)$$

for the weak double dual or reflexive hull of $\mathcal{F}$. We say F is reflexive if the canonical map $F \to \mathcal{F}^{**}$ is an isomorphism. Observe that if X is locally Noetherian, then every vector bundle is coherent and dualizable, so in particular reflexive. Reflexive modules over rings are similarly defined, and one can check that reflexivity can be checked stalkwise. For our first main theorem, we will need the following preliminary results:

Proposition 4.9 *Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent sheaf over X.*

1. *[2, Tag 0E9I] Suppose $j\colon U \to X$ is an open immersion such that for each $z \in X \setminus U$, $\mathrm{depth}(\mathcal{F}_x) \geq 2$. Then $j_*j^*\mathcal{F} \cong \mathcal{F}$.*

Suppose in addition that X is integral.

2. *[2, Tag 0EBH] The hull $\mathcal{F}^{**}$ is reflexive.*
3. *[2, Tag 0EBI] Assume $\mathcal{F}$ is reflexive. For any $x \in X$, if $\mathrm{depth}(\mathcal{O}_{X,x}) \geq 2$, then $\mathrm{depth}(\mathcal{F}_x) \geq 2$.*

The only truly scheme-theoretic result we will need is part (1) of this proposition, as the others follow directly from their commutative algebra counterparts. The following theorem accomplishes most of the work for the example.

Theorem 4.10 *Let $\mathcal{E}$ be a vector bundle over $U \subset \mathbf{A}^2_k$. Then $j_*\mathcal{E}$ is a coherent reflexive sheaf over $\mathbf{A}^2_k$.*

Proof Set $\mathcal{F} := j_*\mathcal{E}$, which is coherent by Lemma 4.8. By Prop. 4.9(2), $\mathcal{F}^{**}$ is reflexive and since j is an open immersion we have $j^*\mathcal{F} = \mathcal{E}$. It follows that

$$j^*(\mathcal{F}^{**}) = (j^*\mathcal{F})^{**} = \mathcal{E}^{**} = \mathcal{E} \tag{$*$}$$

because $\mathcal{E}$ is reflexive. Now $\mathbf{A}^2_k$ is a regular Noetherian local ring of dimension two, and $\dim(\mathcal{O}_{\mathbf{A}^2_k,x}) = \mathrm{depth}(\mathcal{O}_{\mathbf{A}^2_k,x}) = 2$ for each x. Taking $x = 0$, Prop. 4.9(3) shows that $\mathrm{depth}(\mathcal{F}^{**}_0) \geq 2$, and then Prop. 4.9(1) says $j_*j^*\mathcal{F}^{**} \cong \mathcal{F}^{**}$. But according to the computation $(*)$, this means $\mathcal{F}^{**} = j_*\mathcal{E} = \mathcal{F}$, showing that $\mathcal{F}$ is a coherent reflexive sheaf. □

We can finish the proof that U is Serre with one additional commutative algebra lemma which is not very difficult.

Lemma 4.11 ([2, Tag 00NT]) *Let R be a regular local ring. Any maximal Cohen-Macaulay module over R is free.*

Corollary 4.12 *Any coherent reflexive sheaf over $\mathbf{A}^2_k$ is a vector bundle.*

Proof If $\mathcal{F}$ is a coherent reflexive sheaf over $\mathbf{A}^2_k$, we have already seen that for each x we have $\dim(\mathcal{O}_{\mathbf{A}^2_k,x}) = \operatorname{depth}(\mathcal{O}_{\mathbf{A}^2_k,x}) = 2$. But then Prop. 4.9(3) implies $\operatorname{depth}(\mathcal{F}_x) \geq 2$, and since the depth is bounded the Krull dimension of $k[x, y]$, we conclude

$$\operatorname{depth}(\mathcal{F}_x) = \dim(\mathcal{O}_{\mathbf{A}^2_k,x}) = 2 \quad \forall x \in \mathbf{A}^2_k.$$

This means each $\mathcal{F}_x$ is maximal Cohen-Macaulay over the corresponding stalk, and hence (finite) free by Lemma 4.11, i.e. $\mathcal{F}$ is a vector bundle. $\square$

As we have explained, Thm. 4.10 and Cor. 4.12 complete the proof of Thm. 4.6. We have at last shown that $\mathbf{A}^2_k \setminus 0$ is a Serre scheme which is not affine.

4.2 Further Work

In this short section, we will discuss some work on the problem of finding and classifying Serre schemes, but whose full proofs are beyond the scope of this paper. First, there is a mild generalization of the example given in the previous section.

Theorem 4.13 *Let X be a regular integral locally Noetherian scheme of dimension 2. Suppose $U \subset X$ is an open subscheme such that* $\operatorname{depth}(\mathcal{O}_{X,z}) = 2$ *for each $z \in X \setminus U$. Then:*

1. *The open immersion $j : U \hookrightarrow X$ is a Serre morphism.*
2. *The associated pullback and pushforward functors*

$$\operatorname{Vect}(X) \underset{j^*}{\overset{j_*}{\rightleftarrows}} \operatorname{Vect}(Y)$$

form an equivalence of categories.

In particular, if X is Serre then U is Serre.

Proof The second statement implies the first, and [2, Tag 0EBJ] establishes it for the categories of coherent reflexive sheaves. But we may replace these by Vect using [2, Tag 0B3N], which finishes the proof. $\square$

Next, we briefly mention the relationships between sheaf cohomology and the Serre property, of which there are many. One example is an interesting extension of Thm. 2.8 and the Serre-Swan theorem. Recall that if $(X, \mathcal{O}_X)$ is a ringed space, an $\mathcal{O}_X$-module $\mathcal{F}$ is *acyclic* if for each $i > 0$, $H^i(X, \mathcal{F}) = 0$.

Theorem∗ 4.14 ([4, Thm. 2.10]) *Let $(X, \mathcal{O}_X)$ be a ringed space over which vector bundles of bounded rank are acyclic. The global sections functor*

$$\Gamma(X, -)\colon \mathrm{Vect}(X) \to \mathrm{FinProj}(\Gamma(X, \mathcal{O}_X))$$

is an equivalence of categories.

The conditions are satisfied for instance when X is affine by the affine vanishing theorem, but in general would be difficult to show if vector bundles over X are not already well-understood.

If we consider non-abelian cohomology, there is a more interesting description. Let X be a scheme, and consider the presheaf of *groups* $\mathrm{GL}_n(\mathcal{O}_X)$ defined by $U \mapsto \mathrm{GL}_n(\Gamma(U, \mathcal{O}_X))$. One can see that $\mathrm{GL}_n(\mathcal{O}_X)$ is a sheaf of groups, and we have the following result:

Theorem∗ 4.15 ([5, Prop. 11.15]) *For a scheme X, there is a canonical bijection*

$$\left\{\begin{matrix}\textit{vector bundles over } X \\ \textit{of constant rank } n\end{matrix}\right\} / \cong \xrightarrow{\sim} H^1(X, \mathrm{GL}_n(\mathcal{O}_X))$$

of pointed sets, where the left-hand side denotes isomorphism classes. Moreover, if $\{U_i\}$ is an open cover of X, then there is a canonical bijection

$$\left\{\begin{matrix}\textit{rank } n \textit{ vector bundles over } X \\ \textit{trivial on each } U_i\end{matrix}\right\} / \cong \xrightarrow{\sim} \check{H}^1(\{U_i\}, \mathrm{GL}_n(\mathcal{O}_X))$$

of pointed sets.

This gives a method to determine whether a connected scheme is Serre through its non-abelian cohomology; the following corollary is clear.

Corollary 4.16 *Let X be a connected scheme.*

1. *X is Serre if and only if $H^1(X, \mathrm{GL}_n(\mathcal{O}_X))$ is the terminal set for each n.*
2. *Suppose $\{U_i\}$ is an open cover of X such that each U_i is a Serre scheme. Then X is Serre if and only if $\check{H}^1(\{U_i\}, \mathrm{GL}_n(\mathcal{O}_X))$ is the terminal set for each n.*

In practice, the way to apply a cohomological characterization to the problem of showing a scheme is Serre is obstruction theoretic: namely, often vanishing cohomology will imply that a vector bundle can be extended to a larger Serre scheme. A classical theorem of Horrocks which we state now was proven in this way.

Theorem∗ 4.17 (Horrocks [6, Cor. 4.1.1]) *Let $(R, \mathfrak{m})$ be regular local ring of dimension 2 and write $U \coloneqq \mathrm{Spec}(R) \setminus \{\mathfrak{m}\}$. Then U is Serre.*

We note, in fact, that Horrock's result is a special case of Thm. 4.13 because any local ring is Serre. Our proof using reflexive modules substitutes cohomology for some nontrivial algebra. We end this paper by mentioning a result of Swan on the spectra of Laurent polynomial rings:

Theorem∗ 4.18 (Swan [4]) *Let A be a PID and fix $n \geq 0$. Then the scheme* $\mathrm{Spec}(A[x_1^{\pm}, \ldots, x_n^{\pm}])$ *is Serre.*

The proof, according to Swan, is a trivial modification of Quillen's original argument. This example and many others of affine Serre schemes can be found in the book of Lam [7] on this topic.

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