The Dold-Kan Correspondence

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1 Kan Complexes

We begin with two (unrelated) results about Kan complexes. Recall that the singular complex functor $\mathrm{Sing}\colon \mathbf{Top} \to \mathbf{sSet}$ is right adjoint to the realization functor $|-|$. We first show that $\mathrm{Sing}(T)$ is always a Kan complex.

Proposition 1.1 *For any space T, $\mathrm{Sing}(T)$ is a Kan complex.*

Proof We need to show that any morphism $f\colon \Lambda^n_k \to \mathrm{Sing}(T)$ extends to Δ^n along the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$. By the adjunction, this is equivalent to extending a continuous map $f\colon |\Lambda^n_k| \to T$ to all of $|\Delta^n|$. In particular, it suffices to show $|\Delta^n|$ retracts onto $|\Lambda^n_k|$. By construction $|\Delta^n|$ is the standard n-simplex in $\mathbb{R}^{n+1}$, and thus is homeomorphic to the disk D^n. We claim $|\Lambda^n_k| \cong S^{n-1} \setminus \{p\}$ for any point p. Observe that this will complete the proof, since there's an easy retract $D^n \to S^{n-1} \setminus \{p\}$: just send any x to the endpoint (on the boundary) of the unique line segment starting at p, passing through x, and hitting the boundary.

For $|\Lambda^n_k| \cong S^{n-1} \setminus \{p\}$, recall the presentation

$$\coprod_{0\le i\le j\le n} \Delta^{n-2} \rightrightarrows \coprod_{i=0,\, i\neq k}^{n} \Delta^{n-1} \longrightarrow \Lambda^n_k \tag{1}$$

of the horn as a coequalizer in $\mathbf{sSet}$ (e.g. [1, Lem. 3.1]). $|-|$ preserves colimits, thus $|\Lambda^n_k|$ is obtained from topological standard $(n-1)$-simplices by gluing along $(n-2)$-simplices. So, $|\Lambda^n_k|$ is exactly what we exact: the subspace of $\partial|\Delta^n|$ where we remove one face. This is homeomorphic to $S^{n-1} \setminus \{p\}$. $\square$

We next prove the same but for simplicial groups, i.e. objects of $\mathbf{sGrp} :=$ $[\mathbf{\Delta}^{op}, \mathbf{Grp}]$. A simplicial group is equivalently a group object in $\mathbf{sSet}$.

Proposition 1.2 ([2, Tag 00MG]) *If G is a simplicial group, then it (i.e. its underlying set) is a Kan complex.*

Proof We must show, given $0 \le k \le n$, then any map $\Lambda^n_k \to G$ extends to Δ^n along the inclusion. Because of the coequalizer (1), the condition is the same as saying: for each tuple $(x_0, \dots, \hat{x_k}, \dots, x_n)$ of simplices $x_i \in G_{n-1}$ satisfying $d_i(x_j) = d_{j-1}(x_i)$ for $i < j$ not equal to k, there is a simplex $x \in G_n$ so that $d_i x = x_i$ for each $i \neq k$.

The proof is by a double induction. First assume the n-tuple is of the form $(x_0, \dots, x_{k-1}, e, \dots, e)$. If the first k components are all e, then take $x = e$ and we're done. If not, let x_j be the left-most non-e component, for $j \leq k-1$. Suppose the statement has already been proved when the leftmost non-e component is at $j+1$. Write $x_i' = x_i \cdot (d_i s_j x_j)^{-1} = x_i \cdot d_i(s_j x_j)^{-1}$ for $i \neq k$. Then, on one hand, when $l < i$ not k we have

$$\begin{aligned} d_l x_i' &= d_l(x_i) \cdot d_l d_i (s_j x_j)^{-1} \\ &= d_{i-1}(x_l) \cdot d_{i-1} d_l (s_j x_j)^{-1} \\ &= d_{i-1}(x_l \cdot d_l (s_j x_j)^{-1}) = d_{i-1} x_l' \end{aligned}$$

by the fact that (x_i) is a map $\Lambda^n_k \to G$ and the simplicial identities. In particular (x_i') is such a map too. On the other hand, note $x_i = e$ for $i \geq k+1$, and $d_i s_j x_j = e$, so $x_i' = e$ for $i \geq k+1$. Similarly, since $x_i = e$ when $i \leq j-1$, we will have $x_i' = e$. But, since $d_j s_j (x_j) = x_j$, $x_j' = e$ (as opposed to $x_j \neq e$). Thus by supposition there is some $x' \in G_n$ such that $d_i x' = x_i'$. Now write $x = x' s_j x_j$, and consider $d_i x = x_i' \cdot d_i s_j (x_j)$: when $i \leq j-1$, and when $i \geq k+1$, this is just e. But when $i = j$, we get x_j, and when $j \leq i \leq k-1$, we get x_i. So we have found an extension of the original map. In the above we assumed that the components after $k+1$ were all e; but, now just apply the same argument - this time taking $s_{j-1} x_j$ for x_j the rightmost non-e component, and we get the same result. $\square$

2 The Dold-Kan Correspondence

Recall that a *non-negatively graded chain complex* over $\mathbb{Z}$ is a sequence $A = A_\bullet$ of abelian groups

$$\cdots \xrightarrow{\partial_{n+1}} A_n \xrightarrow{\partial_n} A_{n-1} \to \cdots \to A_1 \xrightarrow{\partial_1} A_0$$

such that $\partial_n \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. A morphism of such complexes $f\colon (A, \partial) \to (B, \partial)$ is a sequence of maps $f_n\colon A_n \to B_n$ making

$$\begin{tikzcd} \cdots \arrow[r] & A_n \arrow[r, "\partial_n"] \arrow[d, "f_n"'] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & B_n \arrow[r, "\partial_n"] & B_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

commute for all n. We write $\mathrm{Ch}_{\geq 0}(\mathbb{Z})$ for the corresponding category. On the other hand, let $\mathbf{sAb} = [\mathbf{\Delta}^{op}, \mathbf{Ab}]$ be the category of simplicial abelian groups. The goal is to prove an equivalence of categories $\mathbf{sAb} \simeq \mathrm{Ch}_{\geq 0}(\mathbb{Z})$. First, we construct three related functors $\mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$.

Let A be a simplicial abelian group. We have an associated *Moore complex* A_* given by

$$\cdots \xrightarrow{\partial_3} A_2 \xrightarrow{\partial_2} A_1 \xrightarrow{\partial_1} A_0$$

where $\partial_n = \sum_{i=1}^{0} (-1)^i d_i$.

Proposition 2.1 *The Moore complex A_* is a chain complex, and the assignment $A \mapsto A_*$ defines a functor* $\mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$.

Proof We first need $\partial_{n-1}\partial_n = 0$. From the definition

$$\begin{aligned}\partial_{n-1}\partial_n = \partial_{n-1}\sum_i(-1)^i d_i &= \sum_{i,j}(-1)^{i+j}d_jd_i\\ &= \sum_{i\le j}(-1)^{i+j}d_jd_i + \sum_{i>j}(-1)^{i+j}d_{i-1}d_j\\ &= \sum_{i\le j}(-1)^{i+j}d_jd_i - \sum_{l\le k}(-1)^{k+l}d_kd_l = 0,\end{aligned}$$

where in the second step we used the simplicial identities, and in the third we set $k = i-1$ and $l = j$. For functoriality, observe that a simplicial map $f\colon A \to B$ is a natural transformation, so

$$\begin{tikzcd} A_n \arrow[r, "d_i"] \arrow[d, "f_n"'] & A_{n-1} \arrow[d, "f_{n-1}"] \\ B_n \arrow[r, "d_i"] & B_{n-1} \end{tikzcd}$$

must commute. But since ∂_n is defined as a linear combination of face maps, it too will fit into the same square. $\square$

An n-simplex $x \in A_n$ in a simplicial abelian group is *degenerate* if there is $y \in A_{n-1}$ such that $s_i(y) = x$ for some $i \le n-1$. We claim there is a subcomplex $DA_* \subset A_*$ such that DA_n is generated by the degenerate n-simplices. By definition DA_n is a subgroup of A_n. We need to show the restriction of ∂_n to DA_n takes values in DA_{n-1}; i.e. $\partial(s_ix) \in DA_{n-1}$ for $x \in A_{n-1}$. Remember that $d_is_i = d_{i+1}s_i = \mathrm{id}$, so we compute:

$$\begin{aligned}\partial(s_ix) = \sum_j(-1)^j d_js_i(x) &= \sum_{j\ne i,i+1}(-1)^j d_js_i(x)\\ &= \sum_{j>i+1}(-1)^j d_js_i(x) + \sum_{j<i}(-1)^j d_js_i(x)\\ &= \sum_{j>i+1}(-1)^j s_id_{j-1}(x) + \sum_{j<i}(-1)^j s_{i-1}d_j(x)\end{aligned}$$

by the simplicial identities. But now we have represented $\partial(s_ix)$ as a sum of degenerate simplices. Therefore we can write the quotient

$$q\colon A_* \to (A/DA)_*$$

by degenerate simplices. This defines a second functor $\mathbf{sAb} \to \mathrm{Ch}_{\ge 0}(\mathbb{Z})$, and the quotient map is a natural transformation.

Finally, we define the *normalized complex* NA_*, where we set

$$(NA)_n := \bigcap_{i=0}^{n-1} \ker d_i$$

and take $NA_n \to NA_{n-1}$ to be $(-1)^n d_n$. This is indeed well-defined since if $x \in NA_n$, then $d_id_n(x) = d_{n-1}d_i(x) = 0$ for $i < n-1$ by the simplicial identities since $i < n$. But also taking $i = n-1$ shows that NA_* is a chain complex. Finally, the inclusion map

$$i\colon NA_* \to A_*$$

defines a map of complexes since

$$\begin{tikzcd} \cdots \arrow[r] & NA_n \arrow[r, "(-1)^n d_n"] \arrow[d] & NA_{n-1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & A_n \arrow[r, "\partial_n"] & A_n \arrow[r] & \cdots \end{tikzcd}$$

commutes: if $x \in NA_n$, then $\partial_n(x) = \sum_{i<n}(-1)^i d_i(x) + (-1)^n d_n(x) = (-1)^n d_n(x)$, which explains the factor of $(-1)^n$. We can now state the Dold-Kan correspondence.

Theorem 2.2 (Dold-Kan) *With the categories and complexes defined above:*

(a) The functor $\mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$ *defined by*

$$A \mapsto NA_*$$

is an equivalence of categories.

(b) There is a canonical isomorphism $NA_* \cong (A/DA)_*$*, and a homotopy equivalence*

$$NA_* \cong (A/DA) \simeq A_*$$

arising naturally.

To prove part (a), we will have to define an inverse

$$K\colon \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to \mathbf{sAb}$$

called the *Eilenberg-MacLane* functor. This will be defined as a composite

$$\mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to s\mathbf{sAb} \to \mathbf{sAb},$$

where $s\mathbf{sAb}$ is the category of *semisimplicial* abelian groups, i.e. functors $\mathbf{\Delta}_{\mathrm{inj}} \to \mathbf{Ab}$ from the wide subcategory of injections of $\mathbf{\Delta}$. To show K is a homotopy inverse, the direction

$$\mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to s\mathbf{sAb} \to \mathbf{sAb} \to \mathrm{Ch}_{\geq 0}(\mathbb{Z})$$

will be easy. For the other direction, we will need to first prove part (b), or at least the canonical isomorphism. Hence we begin there.

We will need a lemma for this; momentarily fix n. for $k < n$, set

$$N_k A_n = \bigcap_{i=0}^{k} \ker d_i \subset A_n \qquad D_k A_n = \sum_{i=0}^{k} \mathrm{im}\, s_i \subset A_n,$$

so that when $k = n-1$ we recover NA_n and DA_n.

Lemma 2.3 *Let A be a simplicial abelian group and $n \geq 1$.*

1. The sequence

$$0 \to A_{n-1}/D_{k-1}A_{n-1} \xrightarrow{s_k} A_n/D_{k-1}A_n \to A_n/D_k A_n \to 0$$

is well-defined and exact.

2. *The sequence*

$$0 \to N_{k-1}A_{n-1} \xrightarrow{s_k} N_{k-1}A_n \xrightarrow{\phi} N_kA_n \to 0$$

is well-defined and exact, where $\phi(x) = x - s_kd_kx$.

Proof (1) s_k here is indeed well-defined: if $x = \sum_{i<k} s_i(x_i) \in D_{k-1}A_{n-1}$, then

$$s_k(x) = \sum_{i<k} s_is_{k-1}x_i \in D_{k-1}A_n$$

by the simplicial identities. To show exactness, it is clear the composite of the nonzero arrows is 0 since $D_kA_n \supset \operatorname{im} s_k$. The second map is clearly surjective since it is a quotient map. s_k is injective on the quotients since if $s_k(x) \in D_{k-1}A_n$, then $s_k(x) = \sum_{i<k} s_i(x_i)$, so by the simplicial identities

$$x = d_ks_k(x) = \sum_{i<k} s_id_{k-1}(x_i) \in D_{k-1}A_{n-1}.$$

Lastly, for exactness in the middle, if $[x] \in A_n/D_{k-1}A_n$ has $x \in D_kA_n$, then

$$x = \sum_{i\leq k} s_i(x_i) \equiv s_k(x_k) \pmod{D_{k-1}A_n},$$

completing the proof of exactness.

(2) To see that ϕ actually maps into N_kA_n, observe that if $i < k$ then

$$d_i\phi(x) = 0 - d_is_kd_k(x) = -s_{k-1}d_{k-1}d_i(x) = 0,$$

and $d_k\phi(x) = d_kx - d_kx = 0$. That s_k maps into $N_{k-1}A_n$ is clear since $d_is_k(x) = s_{k-1}d_i(x) = 0$ for $i < k$.

For exactness, observe $\phi(s_k(x)) = s_k(x) - s_k(x) = 0$, and we know s_k is injective (via d_k). If $x \in N_kA_n$, then it is also in $N_{k-1}A_n$ and $\phi(x) = x - s_kd_k(x) = x$, so ϕ is onto. Finally, for exactness in the middle, suppose $\phi(x) = 0$, i.e. $x = s_kd_k(x)$. So we just need to show $d_k(x) \in N_{k-1}A_{n-1}$, but for $i < k$

$$d_id_k(x) = d_{k-1}d_i(x) = 0$$

since $x \in N_{k-1}A_n$, completing the proof. □

Using this we prove the theorem:

Theorem 2.4 *Let A be a simplicial abelian group. Then the map*

$$NA_* \oplus DA_* \to A_*$$

induced by i and the inclusion $DA_* \to A_*$ is an isomorphism. In particular,

$$NA_* \xrightarrow{i} A_* \xrightarrow{p} (A/DA)_*$$

is an isomorphism.

Proof It suffices to show the second, a priori weaker, statement; indeed, if that is true then there's an exact sequence of chain complexes

$$0 \to DA_* \hookrightarrow A_* \xrightarrow{(pi)^{-1}p} NA_* \to 0,$$

but then clearly $i\colon NA_* \to A_*$ is a splitting. To prove it we show that for all $n \geq 0$, and for all $k < n$, the composite

$$N_k A_n \to A_n \to A_n / D_k A_n$$

is an isomorphism (then $k = n - 1$ gives the theorem). We do this by induction on n; if $n = 0$ then there are no $k < n$, so suppose we have the isomorphism $N_j A_{n-1} \cong A_{n-1}/D_j A_{n-1}$ for every $j < n - 1$. To prove the statement for n, we do a second induction on $k < n$.

If $k = 0$, $N_0 A_n = \ker d_0$ and $D_0 A_n = \operatorname{im} s_0$. But the sequence

$$0 \to N_0 A_n \to A_n \xrightarrow{d_0} A_{n-1} \to 0$$

is exact and s_0 is a retract of d_0, so $A_n = \ker d_0 \oplus \operatorname{im} s_0$, $A_n/D_0 A_n \cong N_0 A_n$. Now suppose we know $N_{k-1} A_n \to A_n / D_{k-1} A_n$ is an isomorphism. Put the two exact sequences of Lemma 2.3 into a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & N_{k-1}A_{n-1} \arrow[r, "s_k"] \arrow[d, "pi"'] & N_{k-1}A_n \arrow[r, "\phi"] \arrow[d, "pi"'] & N_k A_n \arrow[r] \arrow[d, "pi"'] & 0 \\
0 \arrow[r] & A_{n-1}/D_{k-1}A_{n-1} \arrow[r, "s_k"] & A_n/D_{k-1}A_n \arrow[r] & A_n/D_k A_n \arrow[r] & 0
\end{tikzcd}$$

The left square commutes by definition of how s_k passes to the quotients. The right square commutes since

$$pi\phi(x) = p(x - s_k d_k x) = pi(x)$$

since $s_k(d_k x) \in D_k A_n$. The first vertical is an isomorphism by our induction on n, and the second is one by induction on $k < n$. Hence $pi\colon N_k A_n \to A_n/D_k A_n$ is an isomorphism by a trivial case of the five lemma. □

Next we have the task of defining the Eilenberg-MacLane functor K into simplicial abelian groups. As mentioned, we first define it into semisimplicial abelian groups.

Definition 2.1 The *semisimplicilal Eilenberg-MacLane functor*

$$K_s\colon \mathrm{Ch}_{\geq 0}(\mathbb{Z}) \to s\mathbf{sAb}$$

is the functorial assignment $(C, \partial) \mapsto K_s(C)$ where $K_s(C)_n = C_n$, and given an injection $i\colon [m] \hookrightarrow [n]$ in $\mathbf{\Delta}_{\mathrm{inj}}$ we define:

1. If $m = n$, hence $i = 1_{[n]}$, then $K_s(C)(i) = 1$
2. If $m = n - 1$ and $i = d^n$, then $K_s(C)(i) = \partial_n$.
3. Otherwise $K_s(C)(i) = 0$.

The functoriality of K_s is clear because of the commutative diagram any chain map must satisfy. To $K_s C$ is a presheaf, consider the composite

$$[l] \xrightarrow{j} [m] \xrightarrow{i} [n].$$

$K_s C(ij) = K_s C(i) \circ K_s C(j)$ is trivial in the case that neither i nor j are of the form d^n or d^m, or that one of them is and the other is 0 or the identity. The only other case $j = d^{n-1}$ and $i = d^n$, but this case is true too because $\partial^2 = 0$ as C is a complex.

Lemma 2.5 (epimonic factorization) *Every map $u\colon [m] \to [n]$ in $\mathbf{\Delta}$ has a unique factorization of the form*

$$[m] \xrightarrow{\pi} [k] \xrightarrow{i} [n]$$

into a split epi π and a mono i.

Proof Clear: Factor it into the inclusion of the image after the restriction to the image; the image is uniquely isomorphic to $[k]$ for some k. The image of this composite always has cardinality $k+1$, so $[k]$ is the only object which works. No other map factoring through $[k]$ except for the aforesaid one can be order-preserving. $\square$

3 Half of generalized Dold-Kan: Full faithfulness

We aim to prove most of Dold-Kan in any additive category $\mathcal{A}$, and all of Dold-Kan for $\mathcal{A}$ if the category is idempotent-complete. We start by reviewing basic results of semi-additive categories.

Definition 3.1 A *semiadditive category* $\mathcal{A}$ is a category with a zero object and biproducts.

In such a category, let $a \oplus b$ denote the biproduct; recall this is equipped with projections p_1, p_2 and 'injections' i_1, i_2 satisfying standard conditions which are exactly enough so that canonical map

$$a \sqcup b \cong a \times b$$

is an isomorphism (induced by $(1_a, 0)$ and $(0, 1_b)$). In $\mathcal{A}$ there is a canonical addition on $\mathrm{Hom}_{\mathcal{A}}(a,b)$: given $f, g\colon a \to b$, let

$$f + g = a \xrightarrow{\langle f,g\rangle} b \oplus b \xrightarrow{\nabla_b} b \tag{2}$$

where the first map is induced by product and the second is the codiagonal. Notice $\nabla_b \circ \langle f, 0\rangle = \nabla_b \circ \langle 0, f\rangle = f$, because always $p_1 \circ i_1 = 1_b$, but $p_1 \circ i_1 \circ \nabla_b \circ \langle f, 0\rangle = p_1 \circ \langle f, 0\rangle = f$. In fact we have:

Proposition 3.1 *The addition in Eq. 2 defines an enrichment of $\mathcal{A}$ over* **CMon***; we write the corresponding hom-monoids $\mathcal{A}(a,b)$. Moreover, the enrichment is unique.*

Proof For the first part one shows that $+$ is commutative and associative, and that it is bilinear with respect to compositions. This is very standard so we don't check everything here, but e.g. for commutativity we use the natural braiding $\gamma_{a,b}\colon a \oplus b \cong b \oplus a$, and then observe that

$$\begin{tikzcd}
a \arrow[r, "\langle f,g\rangle"] \arrow[d, equal] & b \oplus b \arrow[r, "\nabla_b"] \arrow[d, "\gamma_{b,b}"'] & b \arrow[d, equal] \\
a \arrow[r, "\langle g,f\rangle"'] & b \oplus b \arrow[r, "\nabla_b"'] & b
\end{tikzcd}$$

commutes. The insight is that ∇_b is invariant after the braiding because its induced by the pair of maps $(1_b, 1_b)$ which is symmetric. We can do the same thing with the natural associator $\alpha_{a,b,c}$. To conclude that $(f+g)\circ h = f\circ h + g\circ h$ and the other way around, one can use naturality of $\nabla = \nabla_a : a \to a \oplus a$.

The interesting part is uniqueness. Suppose $f \cdot g$ is another **CMon**-enrichment with unit 1. By (2) and the fact that $\nabla_b \circ \langle f, 0\rangle = f$, it suffices to show that

$$\nabla_b \circ \langle f\cdot g, 0\rangle = \nabla_b \circ \langle f, g\rangle\,. \tag{$*$}$$

First we claim that in all hom-monoids $1 = 0$. Indeed by definition $0 = a \to 0 \to b$, thus $1 \cdot 0 = (1 \cdot 0_b) \circ (1 \cdot {}_a 0) = 1 \circ 1 = 1$, where the second equality is because there is only one map in $\mathrm{Hom}(a, 0)$ and $\mathrm{Hom}(0, b)$. But also $1 \cdot 0 = 0$, so $0 = 1$. Now we'll show $\langle f, 0\rangle \cdot \langle 0, g\rangle = \langle f, g\rangle$. By universal property:

$$p_1 \circ (\langle f, 0\rangle \cdot \langle 0, g\rangle) = (p_1 \circ \langle f, 0\rangle)\cdot(p_2 \circ \langle 0, g\rangle) = f \cdot 0 = f$$

since $0 = 1$. We do the same for p_2 and g, showing it is indeed $\langle f, g\rangle$. Finally we have:

$$\begin{aligned}\nabla_b \circ \langle f\cdot g, 0\rangle &= f\cdot g = (\nabla_b \circ \langle f, 0\rangle)\cdot(\nabla_b \circ \langle 0, g\rangle)\\ &= \nabla_b \circ (\langle f, 0\rangle \cdot \langle g, 0\rangle)\\ &= \nabla_b \circ \langle f, g\rangle\,,\end{aligned}$$

which is what we wanted. □

Let $よ\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}] =: \mathrm{PSh}(\mathcal{A})$ be the Yoneda embedding and $U\colon \mathbf{CMon} \to \mathbf{Set}$ be the forgetful functor. We have the induced functor

$$U_*\colon [\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}] \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}],$$

and notice that U_* is faithful because U is, and presheaf maps are checked for equality pointwise. Prop. 3.1 says that the Yoneda embedding factors through $[\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]$ so that we have a commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "よ^e"] \arrow[dr, "よ"'] & {[\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]} \arrow[d, "U_*"] \\ & \mathrm{PSh}(A) \end{tikzcd}$$

defining the functor $よ^e$, the *enriched* Yoneda embedding. We have an easy lemma:

Lemma 3.2 *Let A, B, C be categories and suppose*

$$\begin{tikzcd} A \arrow[r, "G"] \arrow[dr, "F"'] & B \arrow[d, "H"] \\ & C \end{tikzcd}$$

commutes where F is fully faithful and H is faithful. Then G is fully faithful.

Proof Suppose $f, g\colon a \to b$; if $Gf = Gg$ then applying H we get $Ff = Fg$ so $f = g$, i.e. it is faithful. On the other hand, if $f\colon Ga \to Gb$, then $Hf\colon Fa \to Fb$, so there is $g\colon a \to b$ such that $Fg = Hf$. But then $HGg = Hf$, so $Gg = f$ by faithfulness, so we have shown that G is full. □

Corollary 3.3 *For a semiadditive category $\mathcal{A}$, $よ^e\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]$ is fully faithful.*

Recall that a functor $F\colon \mathcal{A} \to \mathcal{B}$ between semiadditive categories is *additive* if it preserves finite biproducts (including the zero object). It is always sufficient to show that F preserves 0 and binary products. Note additionally that if C is a category and $\mathcal{A}$ is semiadditive, then $[C, \mathcal{A}]$ is semiadditive; this is immediate: (co)limits, and thus (co)products are computed pointwise, so it has biproducts and 0. In particular, $[\mathcal{A}^{\mathrm{op}}, \mathbf{CMon}]$ is semiadditive so the following proposition makes sense.

Proposition 3.4 *The enriched Yoneda embedding $よ^e$ is additive.*

Proof For 0, we can show $よ^e(0)$ is zero by showing it's initial, which we do by checking that $よ^e(0)_x = \mathcal{A}(x, 0)$ is the initial commutative monoid. But 0 is terminal so indeed $\mathcal{A}(x, 0) = *$. Next we need $よ^e(x \times y) = よ^e(x) \times よ^e(y)$. By evaluating, we need to show that

$$\mathcal{A}(a, x) \times \mathcal{A}(a, y) \longrightarrow \mathcal{A}(a, x \times y)$$

as an isomorphism. We already know it is bijective by definition of the product, so what remains is that it is a homomorphism. We check it on the inverse, i.e. that

$$\begin{aligned} \mathcal{A}(a, x \times y) &\to \mathcal{A}(a, x) \times \mathcal{A}(a, y) \\ f &\mapsto (p_1 \circ f, p_2 \circ f) \end{aligned}$$

respects addition. Equivalently we check $f \mapsto p_i \circ f$ respects addition, which is a consequence of the enrichment: $f + g \mapsto p_i \circ (f + g) = p_i \circ f + p_i \circ g$ by bilinearity of composition. □

A semiadditive category $\mathcal{A}$ is *additive* for any objects a and b, the monoid $\mathcal{A}(a, b)$ is an abelian group. Observe in this case that embedding $よ^e$ will also factor through presheaves of abelian groups. The same functor

$$よ^e\colon \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Ab}] = \widehat{\mathcal{A}}$$

thereto clearly will clearly also be fully faithful and additive. From now on, for a category C, let $\widehat{C} \coloneqq [C^{\mathrm{op}}, \mathbf{Ab}]$. We remarked above that $\widehat{C}$ is still semiadditive. It is a general fact that since $\mathbf{Ab}$ is abelian, $\widehat{C}$ is also abelian, and in particular additive.

Remark 3.5 The proof of Prop. 3.4 would have been purely categorical and more general if we were working with $\mathcal{A}$ additive and the functor $よ^e\colon \mathcal{A} \to \widehat{\mathcal{A}}$ to abelian group presheaves instead. This is because $\mathbf{Ab}$ is *balanced*, i.e. mono + epi = isomorphism (while $\mathbf{CMon}$ is not). Both $\mathbf{Ab}$ and $\mathbf{Set}$ have pullbacks and pushouts, and it follows that $\mathrm{PSh}(\mathcal{A})$ and $\widehat{\mathcal{A}}$ are balanced. U_* is faithful, so it reflects epimorphisms and monomorphisms, and therefore between balanced categories it is conservative. Conservative functors reflect all (co)limits which they preserve. Since $U\colon \mathbf{Ab} \to \mathbf{Set}$ partakes in a free-forgetful adjunction, it follows that U_* is a right adjoint, and in particular preserves small limits, and thus reflects small limits.[1]

All of that was just category theory, but now the result is trivial. We claim $よ^e(\varprojlim_i x_i) = \varprojlim_i よ^e(x_i)$. By the remark we can check after applying U, which commutes past the limit; by commutativity we have reduced to $よ(\varprojlim_i x_i) = \varprojlim_i よ(x_i)$ which is true since the Yoneda embedding always preserves limits. In particular $よ^e$ preserves 0 and the product.

Returning to considerations more relevant to Dold-Kan, we define the main objects of study. Fix an additive category $\mathcal{A}$, and write $\mathbf{s}\mathcal{A} = [\mathbf{\Delta}^{\mathrm{op}}, \mathcal{A}]$ for the category of simplicial $\mathcal{A}$-objects. $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ is the category of chain complexes of $\mathcal{A}$-objects defined the same way as above, which works since we have zero maps. We will construct the backwards functor $\mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A}$ first this time. We proceed like before.

Let $A = (A_*, \partial)$ be an object of $\mathrm{Ch}_{\geq 0}(\mathcal{A})$. Define the associated semisimplicial object $\mathrm{K}^+_{\mathcal{A}}(A) \colon \mathbf{\Delta}^{\mathrm{op}}_+ \to \mathbf{s}\mathcal{A}$ as follows: $K^+_{\mathcal{A}}(A)_n = A_n$, and given an injection $u \colon [m] \hookrightarrow [n]$ set $u^* \colon A_n \to A_m$ to 1_{A_n} if $m = n$ (and thus $u = \mathrm{id}$), ∂_n if $m = n - 1$ and $u = d^n$, and 0 otherwise. The verification that this is semisimplicial is as before.

Definition 3.2 We define the *Eilenberg-MacLane* functor

$$\mathrm{K}_{\mathcal{A}} \colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A}$$

as follows. For a complex $A = (A_*, \partial)$, take

$$\mathrm{K}_{\mathcal{A}}(A)_n = \bigoplus_{\alpha \colon [n] \twoheadrightarrow [k]} A_k$$

like before (which is a finite biproduct), taken over epis $[n] \twoheadrightarrow [k]$ of $\mathbf{\Delta}$. To define it on a morphism $u \colon [m] \to [n]$, consider one factor of the domain A_k corresponding to $\alpha \colon [n] \twoheadrightarrow [k]$. Form the epimonic splitting of the composite $\alpha \circ u$ giving

$$\begin{tikzcd} {[m]} \arrow[r, two heads, "\pi"] \arrow[d, "u"'] & {[r]} \arrow[d, hook, "i"] \\ {[n]} \arrow[r, "\alpha"] & {[k]} \end{tikzcd}$$

and then set $u^*_{\alpha,\pi} = \mathrm{K}^+_{\mathcal{A}}(A)(i)$, and let it be zero elsewhere. The family $\{u^*_{\alpha,\pi}\}$ specifies a map u^* between the coproducts. Checking that this defines a simplicial object, and that it's functorial, is the same as before.

Now we can state the main result of this section:

[1] Note: Might want to rework this remark. Even though a presheaf of commutative monoids which is epi and mono need not be an isomorphism, U_* should still be conservative since a bijective monoid homomorphism is an isomorphism.

Theorem 3.6 (Dold-Kan v2) *Let $\mathcal{A}$ be an additive category. Then the Eilenberg-MacLane functor*

$$\mathrm{K}_{\mathcal{A}} \colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A}$$

is fully faithful. If $\mathcal{A}$ is idempotent-complete, then $\mathrm{K}_{\mathcal{A}}$ is an equivalence of categories.

For the first part, we will need to reduce to the case $\mathcal{A} = \mathbf{Ab}$ proven above. For $f \colon \mathcal{A} \to \mathcal{B}$ an additive functor between additive categories, define the canonical functors

$$f_* \colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{B}) \qquad f_* \colon \mathbf{s}\mathcal{A} \to \mathbf{s}\mathcal{B}$$

which on the left is object-wise application and on the right is post-composition $- \circ f$. It defines a functor of chain complexes because it takes zero maps to zero maps (so $\partial^2 = 0$ is satisfied).

Lemma 3.7 *Let $f \colon \mathcal{A} \to \mathcal{B}$ an additive functor between additive categories. Then the square*

$$\begin{tikzcd}
\mathrm{Ch}_{\geq 0}(\mathcal{A}) \arrow[r, "\mathrm{K}_{\mathcal{A}}"] \arrow[d, "f_*"'] & \mathbf{s}\mathcal{A} \arrow[d, "f_*"] \\
\mathrm{Ch}_{\geq 0}(\mathcal{B}) \arrow[r, "\mathrm{K}_{\mathcal{B}}"] & \mathbf{s}\mathcal{B}
\end{tikzcd}$$

commutes up to natural isomorphism.

Proof For any chain complex $A = (A_*, \partial)$, we want $f_*(\mathrm{K}_{\mathcal{A}}(A) \cong \mathrm{K}_{\mathcal{B}}(f_*(A))$ naturally in A. On the LHS, evaluate at n to get

$$\begin{aligned} f_*(\mathrm{K}_{\mathcal{A}}(A))_n = f(\mathrm{K}_{\mathcal{A}}(A)_n) = f\Big(\bigoplus_{[n] \twoheadrightarrow [k]} A_k\Big) \\ \cong \bigoplus_{[n] \twoheadrightarrow [k]} f(A_k) = \mathrm{K}_{\mathcal{B}}(f_*(A))_n \end{aligned}$$

because f is additive. For naturality with respect to a chain map $\phi \colon A \to B$, unwinding the definitions gives the square

$$\begin{tikzcd}
f\Big(\bigoplus A_k\Big) \arrow[r, "\simeq"] \arrow[d, "f(\bigoplus \phi_k)"'] & \bigoplus f(A_k) \arrow[d, "\bigoplus f(\phi_k)"] \\
f\Big(\bigoplus B_k\Big) \arrow[r, "\simeq"] & \bigoplus f(B_k)
\end{tikzcd}$$

This commutes since f must preserve the maps the biproduct is equipped with. $\square$

Now we recall some basic category theory. Let C, D, E be categories. The cartesian closed structure in the 2-category of categories gives canonical isomorphisms of categories

$$[C, [D, E]] \cong [C \times D, E] \cong [D \times C, E] \cong [D, [C, E]] \tag{$*$}$$

Given a functor $f\colon C \to [D, E]$ the associated functor $\widetilde{f}\colon D \to [C, E]$ is just $\widetilde{f}(d)(c) = f(c)(d)$. The definition on natural transformations is obvious. We will denote the isomorphism $f \mapsto \widetilde{f}$ by μ (even for different categories C, D, E.

Now fix a category J. The first special case of $(*)$ is by taking $C = \mathbf{\Delta}^{\mathrm{op}}$, $D = J^{\mathrm{op}}$, $E = \mathbf{Ab}$, giving

$$\mu\colon [\mathbf{\Delta}^{\mathrm{op}}, [J^{\mathrm{op}}, \mathbf{Ab}]] = \mathbf{s}\widehat{J} \cong [J^{\mathrm{op}}, [\mathbf{\Delta}^{\mathrm{op}}, \mathbf{Ab}]] = [J^{\mathrm{op}}, \mathbf{sAb}] \tag{3}$$

Next, let $\mathbb{Z}_{\geq 0}$ be its namesake poset category, and consider a functor $\mathbb{Z}_{\geq 0}^{\mathrm{op}} \to \mathcal{A}$ for an additive category $\mathcal{A}$. This is exactly a sequence of arrows

$$\cdots \to A_3 \to A_2 \to A_1 \to A_0$$

in $\mathcal{A}$. It is a chain complex exactly when $A(n \to n-1) = 0$. Observe that a map of chain complexes is just any natural transformation of the functors; thus we can equivalently say:

Proposition 3.8 *For an additive category $\mathcal{A}$, chain complexes* $\mathrm{Ch}_{\geq 0}(\mathcal{A}) \hookrightarrow [\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathcal{A}]$ *are the full subcategory spanned by functors A such that $A(n \to k) = 0$ whenever $k \leq n-2$.*

The second special case of the isomorphism $(*)$ is

$$[\mathbb{Z}_{\geq 0}^{\mathrm{op}}, [J^{\mathrm{op}}, \mathbf{Ab}]] = [\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \widehat{J}] \cong [J^{\mathrm{op}}, [\mathbb{Z}_{\geq 0}^{\mathrm{op}}, \mathbf{Ab}]] \tag{4}$$

We know $\widehat{J}$ is additive (because abelian), and we claim this descends to chain complexes:

Lemma 3.9 *The isomorphism of categories (4) descends to*

$$\mu\colon \mathrm{Ch}_{\geq 0}(\widehat{J}) \cong [J^{\mathrm{op}}, \mathrm{Ch}_{\geq 0}(\mathbf{Ab})],$$

the restriction of μ.

Proof We have the correspondence

$$A\colon \mathbb{Z}_{\geq}^{\mathrm{op}} \to [J^{\mathrm{op}}, \mathbf{Ab}] \quad \longleftrightarrow \quad \widetilde{A}\colon J^{\mathrm{op}} \to [\mathbb{Z}_{\geq}^{\mathrm{op}}, \mathbf{Ab}],$$

and we have to check that A is a chain complex if and only if $\widetilde{A}(j)$ is a chain complex for all j. By Prop. 3.8, it suffices to show that $A(n \to k) = 0$ iff $\widetilde{A}(j)(n \to k) = 0$ for all j. But this is a tautology: by definition,

$$\widetilde{A}(j)(n \to k) = A_n(j) \xrightarrow{A(n \to k)_j} A_k(j),$$

and a natural transformation is zero exactly when its components are. $\square$

In other words, chain complexes of presheaves are presheaves of chain complexes. The two isomorphisms of (3) and Lemma 3.9 fit into a canonical commutative square relating to the Eilenberg-MacLane functor.

Proposition 3.10 *Let J be a category. The diagram*

$$\begin{tikzcd} \mathrm{Ch}_{\geq 0}(\widehat{J}) \arrow[r, "\mathrm{K}_{\widehat{J}}"] \arrow[d, "\mu"', "\simeq"] & \mathbf{s}\widehat{J} \arrow[d, "\simeq"', "\mu"] \\ {[J^{\mathrm{op}}, \mathrm{Ch}_{\geq 0}(\mathbf{Ab})]} \arrow[r, "\mathrm{K}_{\mathbf{Ab}}\circ -"'] & {[J^{\mathrm{op}}, \mathbf{sAb}]} \end{tikzcd}$$

commutes up to natural isomorphism.

Proof Let $F = (F_*, \partial)$ be a chain complex with terms in $\widehat{J}$. Either composite of the square is a presheaf $J^{\mathrm{op}} \to \mathbf{sAb}$, so we evaluate at any k, and then look at the n-simplices. On one hand

$$\mu(\mathrm{K}_{\widehat{J}}F)_j(n) = (\mathrm{K}_{\widehat{J}}F)_n(j) = \Big(\bigoplus_{[n]\twoheadrightarrow[k]} F_n\Big)(j) \cong \bigoplus_{[n]\twoheadrightarrow[k]} F_n(j)$$

where we have used the fact that (co)limits are computed pointwise in the presheaf category (in particular evaluation functors preserve limits). So far this is clearly natural. in F. On the other hand, $(\mathrm{K}_{\mathbf{Ab}}\circ)(\mu(F)) = \mathrm{K}_{\mathbf{Ab}} \circ \mu(F)$, and

$$(\mathrm{K}_{\mathbf{Ab}} \circ \mu(F))_j(n) = \mathrm{K}_{\mathbf{Ab}}(\mu(F)_j(n)) = \mathrm{K}_{\mathbf{Ab}}(F_n(j)) = \bigoplus_{[n]\twoheadrightarrow[k]} F_n(j)$$

proving the commutativity up to natural isomorphism. □

We need a quick categorical lemma, and then the full faithfulness of $\mathrm{K}_{\mathcal{A}}$ will follow easily. If $f\colon D \to E$ is a functor, then we denote for post-composition $f_* = f \circ -\colon [C, E] \to [C, D]$.

Lemma 3.11 *Let $f\colon D \to E$ be a functor and consider $f_*\colon [C, D] \to [C, E]$. Then the association $f \mapsto f_*$ preserves the following properties:*
(a) Being faithful.
(b) Being fully faithful.
(c) Being an equivalence.

Proof For (a), suppose $\alpha, \beta\colon F \to G$ such that $f_*\alpha = f_*\beta\colon f \circ F \to f \circ G$. Then

$$(f_*\alpha)_c = (f_*\beta)_c \quad \forall c \implies f(\alpha_c) = f(\beta_c) \quad \forall c$$

and $\alpha_c = \beta_c$ since f is faithful. For (b), suppose in addition that f is full. Let $\beta\colon f \circ F \to f \circ G$ be a map, then for each c we have $\beta_c\colon f(Fc) \to f(Gc)$, so since f is full there is $\alpha_c\colon Fc \to Gc$ such that $f(\alpha_c) = \beta_c$. Since β is natural, for any $u\colon c \to d$ the square

$$\begin{tikzcd} f(Fc) \arrow[r, "f(Fu)"] \arrow[d, "f(\alpha_c)"'] & f(Fd) \arrow[d, "f(\alpha_d)"] \\ f(Gc) \arrow[r, "f(Gu)"] & f(Gd) \end{tikzcd} \implies \begin{tikzcd} Fc \arrow[r, "Fu"] \arrow[d, "\alpha_c"] & Fd \arrow[d, "\alpha_d"] \\ Gc \arrow[r, "Gu"] & Gd \end{tikzcd}$$

commutes. Then the square on the right commutes by functoriality and since f is faithful. For (c), suppose that f is an equivalence of categories, then let $g\colon E \to D$ be the inverse. We claim g_* is an inverse to f. Indeed,

$$(g_* \circ f_*(F))c = gf(Fc) \cong Fc \implies (g_* \circ f_*)F \cong F,$$

naturally in F, where the implication uses naturality of $gf \cong \mathrm{id}$, and the other direction is the same. □

Remark 3.12 Notice how we needed faithfulness to prove that f_* was full, and an inverse to show f_* was an equivalence. In other words f_* need not be full if f is, and it need not be essentially surjective if f is.

Corollary 3.13 *Let J be a category. Then*

$$\mathrm{K}_{\widehat{J}}\colon \mathrm{Ch}_{\geq 0}(\hat{J}) \to \mathbf{s}\widehat{J}$$

is an equivalence of categories.

Proof Consider the commuting square in Prop. Now 3.10. Dold-Kan v1 (Thm. 2.2) shows that $\mathrm{K}_{\mathbf{Ab}}$ is an equivalence of categories, and Lemma 3.11 implies $\mathrm{K}_{\mathbf{Ab}} \circ -$ (the bottom map in the square) is an equivalence. The left and right sides are isomorphisms, so the top map $\mathrm{K}_{\widehat{J}}$ is an equivalence (it has an obvious inverse given that of the bottom). □

We bring everything so far in this section to get the first half of the generalized Dold-Kan correspondence, which works on any additive category.

Theorem 3.14 (Half of Dold-Kan v2) *Let $\mathcal{A}$ be an additive category. Then the Eilenberg-MacLane functor*

$$\mathrm{K}_{\mathcal{A}}\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A}$$

is fully faithful.

Proof We said as a corollary of Prop. 3.4 that $よ^e\colon \mathcal{A} \to \widehat{\mathcal{A}}$ is additive and fully faithful. By Lemma 3.7, we form the commutative square below:

$$\begin{tikzcd}
\mathrm{Ch}_{\geq 0}(\mathcal{A}) \arrow[r, "\mathrm{K}_{\mathcal{A}}"] \arrow[d, "よ^e_*"'] \arrow[dr, dashed, "\mathrm{K}_{\widehat{\mathcal{A}}} \circ よ^e_*"] & \mathbf{s}\mathcal{A} \arrow[d, "よ^e_*"] \\
\mathrm{Ch}_{\geq 0}(\widehat{\mathcal{A}}) \arrow[r, "\mathrm{K}_{\widehat{\mathcal{A}}}"'] & \mathbf{s}\widehat{\mathcal{A}}
\end{tikzcd}$$

We want to show the top is fully faithful, and we know the right is faithful by Lemma 3.11. By Lemma 3.2 it suffices to show that the diagonal map is fully faithful. The left side is a composite of a fully faithful (Lemma 3.11) functor - post-composition with $よ^e$ - and a full subcategory inclusion (Prop. 3.8), so it is fully faithful. But the bottom map is an equivalence of categories by Cor. 3.13, so it is fully faithful, and therefore the diagonal is fully faithful. □

4 The other half: Equivalence

In Cor. 3.13 we showed that the Eilenberg-MacLane functor induced an equivalence of categories

$$\mathrm{Ch}_{\geq 0}(\widehat{\mathcal{A}}) \simeq \mathbf{s}\widehat{\mathcal{A}}, \tag{$*$}$$

and the goal is to deduce an equivalence on $\mathcal{A}$. This is not immediate because it is not a priori clear how to recover $\mathcal{A}$ from $\widehat{\mathcal{A}}$; so in this section we will do this. First, we show that if a category J is *idempotent-complete*, then it can be recovered categorically as the full subcategory of atomic objects of $\mathrm{PSh}(J)$.

In that case we can at least say that $J \simeq J'$ iff $\mathrm{PSh}(J) \simeq \mathrm{PSh}(J')$. While this doesn't directly apply to the enriched case, we can use certain nice properties of the forgetful functor $U_*\colon \mathrm{PSh}(\mathcal{A}) \to \widehat{A}$ for a $\mathcal{A}$ additive. This will let us prove the Dold-Kan theorem.

4.1 Idempotent-completeness

For this section let J be an ordinary category and $\mathrm{PSh}(J) := [J^{\mathrm{op}}, \mathbf{Set}]$.

Definition 4.1 Let x, y be objects of J. We say y is a retract of x if there are maps $s\colon y \to x$ and $r\colon x \to y$ such that $rs = 1_y$.

Let $f\colon x \to y$, $f'\colon x' \to y'$ be morphisms in J. Then we say f' is a retract of f if it is so in the arrow category $\mathrm{Arr}(J)$.

Given these, we can ask if a full subcategory $J_0 \subset J$ is closed under retracts, or if a subset of morphisms $S \subset \mathrm{mor}(J)$ is stable under retracts.

For example, it is easy to see that the class of all isomorphisms in J is closed under retracts. For the definition we need to show that given a commutative diagram

$$\begin{tikzcd} y \arrow[r, "s"] \arrow[d, "f"] & x \arrow[r, "r"] \arrow[d, "g"] & y \arrow[d, "f"] \\ y' \arrow[r, "s'"] & x' \arrow[r, "r"] & y' \end{tikzcd}$$

where the rows compose to 1_y and $1_{y'}$, if g is an isomorphism then f is too. Indeed, let $h = rg^{-1}s'\colon y' \to y$. Then $hf = rg^{-1}s'f = rgg^{-1}s = 1$ and $fh = frg^{-1}s' = r'gg^{-1}s' = 1$.

Observe that if we have a retract diagram $y \xrightarrow{s} x \xrightarrow{r} y$, then $(sr)sr = sr$, so in particular we get an idempotent from every retraction/section pair. On the other hand, if we start with an idempotent $e\colon X \to X$ (so that $ee = e$), we say it *splits* if it comes from such a pair. Notice that these retraction pairs are 'absolute' in the sense that they are preserved by any functor.

Definition 4.2 An object x of J is *atomic* if the functor $\mathrm{Hom}_J(x, -)\colon J \to \mathbf{Set}$ preserves colimits. Let $J^{\mathrm{at}} \subset J^{\mathrm{at}}$ be the full subcategory spanned by the atomic objects of J.

For any J, the subcategory J^{at} will be our first example of a full subcategory closed under retracts. We need a quick categorical lemma first:

Lemma 4.1 *Let G, F be functors in $[J, \mathbf{Set}]$ such that F is a retract of G. Then the natural map*

$$\varinjlim_{i \in I} G(z_i) \to G(\varinjlim_{i \in I} z_i)$$

is a retract of the natural map $\varinjlim_i F(z_i) \to F(\varinjlim_i z_i)$, provided that the colimits exist.

Proof We have to show that

$$\begin{tikzcd}
\varinjlim_i F(z_i) \arrow[r] \arrow[d] & \varinjlim_i G(z_i) \arrow[r] \arrow[d] & \varinjlim_i F(z_i) \arrow[d] \\
F(\varinjlim_i z_i) \arrow[r] & G(\varinjlim_i z_i) \arrow[r] & F(\varinjlim_i z_i)
\end{tikzcd}$$

commutes and the rows compose to the identity. The latter part is immediate using the functoriality of $\varinjlim$ and the fact that F is a retract of G. Commutativity is from the more general fact that given $f\colon F \to G$, the diagram

$$\begin{tikzcd}
\varinjlim_i F(z_i) \arrow[r, "f"] \arrow[d] & \varinjlim_i G(z_i) \arrow[d] \\
F(\varinjlim_i z_i) \arrow[r, "f"] & G(\varinjlim_i z_i)
\end{tikzcd}$$

commutes. This is the case because there are two ways to get from $F(z_i)$ to $G(\varinjlim_i z_i)$ which respect the colimit maps (and are thus equal): we can either apply f before or after taking the colimit, and the relevant diagrams commute by naturality of f. □

Proposition 4.2 *$J^{\mathrm{at}} \subset J$ is closed under retracts.*

Proof Let y be a retract of x where x is atomic. Applying the contravariant Yoneda functor $J^{\mathrm{op}} \to [J, \mathbf{Set}]$, we deduce that $\mathrm{Hom}(y, -)$ is a retract of $\mathrm{Hom}(x -)$ in the functor category, so the natural map

$$\varinjlim_{i\in I} \mathrm{Hom}(y, z_i) \to \mathrm{Hom}(y, \varinjlim_{i\in I} z_i)$$

is a retract of

$$\varinjlim_{i\in I} \mathrm{Hom}(x, z_i) \to \mathrm{Hom}(x, \varinjlim_{i\in I} z_i)$$

by the lemma. But since x is atomic, then by definition the bottom map is an isomorphism, and retracts of isomorphisms are isomorphisms, so y is atomic. □

The canonical example of atomic objects are representable presheaves in $\mathrm{PSh}(J)$. Indeed:

$$\mathrm{Hom}(よ(x), \varinjlim_{i\in I} F_i) = \left(\varinjlim_{i\in I} F_i\right)(x) = \varinjlim_{i\in I} F_i(x) = \varinjlim_{i\in I} \mathrm{Hom}(よ(x), F_i)$$

by the Yoneda lemma and since (co)limits are evaluated pointwise. Now we move to studying idempotent-completeness. J is called *idempotent-complete* if all idempotents split. We can characterize this in a more categorical manner:

Proposition 4.3 *Let $e\colon x \to x$ be an idempotent in J. The following are equivalent:*

(i) *e splits.*
(ii) *The equalizer* $\mathrm{Eq}(e, 1_x)$ *exists.*
(iii) *The coequalizer* $\mathrm{Coeq}(e, 1_x)$ *exists.*

Proof Observe that (ii) and (iii) are equivalent since if e is idempotent in J then it is also one in J^{op}. For $(i) \Rightarrow (ii)$, suppose $e = x \xrightarrow{r} y \xrightarrow{s} x$ where $rs = 1$. The claim is that

$$\begin{tikzcd} z \arrow[d, dashed, "rg"'] \arrow[dr, "g"] & & \\ y \arrow[r, "s"] & x \arrow[r, shift left, "e"] \arrow[r, shift right, "1"'] & x \end{tikzcd}$$

is an equalizer diagram. Indeed, $es = srs = s$, and if $eg = g$ as above, then $rg \colon z \to y$ such that $g = s(rg) = eg = g$. It's unique since if $sh = g$, then $rsh = rg = h$.

On the other hand, if we already have the equalizer, then consider

$$\begin{tikzcd} x \arrow[d, dashed, "r"'] \arrow[dr, "e"] & & \\ \mathrm{Eq}(e,1) \arrow[r, "s"] & x \arrow[r, shift left, "e"] \arrow[r, shift right, "1"'] & x \end{tikzcd}$$

so that $sr = e$. Then $srs = es = s \circ 1_x$ implies that $rs = 1$, since equalizer maps (s) are always monic. $\square$

Corollary 4.4 *If J has equalizers or coequalizers then it is idempotent complete.*

Thus, in particular, most of our favorite categories, including presheaf categories, abelian categories, topos categories, etc. are all idempotent-complete.

Lemma 4.5 *If J is idempotent-complete, then the essential image of the Yoneda embedding $よ\colon J \to \mathrm{PSh}(J)$, denoted $よ(J)$, is closed under retracts.*

Proof Suppose we have a retract diagram

$$1 = Y \xrightarrow{s} よ(x) \xrightarrow{r} Y$$

in $\mathrm{PSh}(J)$. Now sr is an idempotent of $よ(J)$, and since J is idempotent-complete and $J \simeq よ(J)$, it splits so that $s'r' = sr$ and $r's' = 1$, i.e.

$$\begin{tikzcd} よ(x) \arrow[r, "r'"] \arrow[d, "r"] & よ(y) \arrow[d, "s'"] \\ Y \arrow[r, "s"] & よ(x) \end{tikzcd}$$

commutes for some y in J. But then

$$Y \cong \mathrm{Eq}(sr, 1) = \mathrm{Eq}(s'r', 1) \cong よ(y)$$

by the proposition, so we're done. $\square$

In the case that the category J is *not* idempotent-complete, we can complete it into a universal idempotent-complete category into which J is embedded.

Definition 4.3 The idempotent completion of J is an idempotent-complete category J^{idem} equipped with a fully faithful functor $y\colon J \to J^{\mathrm{idem}}$ such that for any idempotent-complete C, the induced map

$$- \circ y \colon [J^{\mathrm{idem}}, C] \to [J, C]$$

is an equivalence of categories.

We first have to show that the completion exists. To do so we will embed J into $\mathrm{PSh}(J)$ and take the 'retract-closure':

Lemma 4.6 *Let $J_0 \subset J$ be a full subcategory. Then:*

1. *There is a smallest full subcategory of J containing J_0 which is closed under retracts, written $\overline{J_0}$.*
2. *Each object of $\overline{J_0}$ is a retract of an object of J_0.*

Proof Let $\overline{J_0}$ be the full subcategory spanned by all retracts of objects of J_0. Then $J_0 \subset \overline{J_0}$ since all objects are retracts of themselves. Moreover clearly $\overline{J_0}$ is contained in any full subcategory containing J_0 which is closed under retracts. So showing $\overline{J_0}$ itself is closed will prove (1) and (2). Suppose y is a retract of $x \in \overline{J_0}$ via $rs = 1$; then x is a retract of some $w \in J_0$ via $r's' = 1$, so the outer square in

$$\begin{tikzcd} y \arrow[r, "s"] \arrow[d, equal] & x \arrow[r, "s'"] \arrow[d, equal] & w \arrow[d, equal] \\ y & x \arrow[l, "r"'] & w \arrow[l, "r'"'] \end{tikzcd}$$

gives y as a retract of w. $\square$

Proposition 4.7 *Let $\overline{\text{よ}(J)} \subset \mathrm{PSh}(J)$ be the retract-closure of the essential image of the Yoneda embedding. Then:*

1. *$\overline{\text{よ}(J)}$ is idempotent-complete.*
2. *The Yoneda embedding $\text{よ}\colon J \to \overline{\text{よ}(J)}$ exhibits $\overline{\text{よ}(J)}$ as the idempotent completion of J.*

Proof For (1), let $e\colon X \to X$ be an idempotent in $\overline{\text{よ}(J)}$. Since $\mathrm{PSh}(X)$ is idempotent-complete, this splits in $\mathrm{PSh}(X)$ as a retract of X onto some $Y \in \mathrm{PSh}(J)$, but then $Y \in \overline{\text{よ}(J)}$ since $\overline{\text{よ}(J)}$ is closed under retracts.

For (2), fix an idempotent-complete category C; we will construct an inverse

$$[J, C] \to [\overline{\text{よ}(J)}, C]$$

to $- \circ \text{よ}$. Let $F\colon J \to C$ be any functor, and define $\widetilde{F}\colon \overline{\text{よ}(J)} \to C$ as follows: On representable presheaves $\text{よ}(x)$ take $\widetilde{F}(\text{よ}(x)) = F(x)$, and any other object is a retract of representables:

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[dr, equal] & \text{よ}(x) \arrow[d, "r"] \\ & Y \end{tikzcd}$$

so define $\widetilde{F}(Y) = \mathrm{Eq}(\widetilde{F}(sr), 1)$, where $\widetilde{F}$ is defined on sr by the Yoneda lemma (it corresponds to an idempotent $y \to y$). It is easy to see that $F \mapsto \widetilde{F}$ is a (natural) inverse to $- \circ \text{よ}$ *if* it is well-defined; so we just have to show that.

In particular, suppose Y is a retract of $\text{よ}(x), \text{よ}(x')$:

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "s'"'] \arrow[dr, equal] & \text{よ}(x) \arrow[d, "r"] \\ \text{よ}(y) \arrow[r, "r'"'] & Y \end{tikzcd}$$

we must check $\mathrm{Eq}(\widetilde{F}(sr), 1) = \mathrm{Eq}(\widetilde{F}(s'r'), 1)$. It suffices to give an isomorphism $\text{よ}(x) \cong \text{よ}(y)$ making the diagram

$$\begin{tikzcd} \text{よ}(x) \arrow[r, "\simeq"] \arrow[d, "sr"] & \text{よ}(y) \arrow[d, "s'r'"] \\ \text{よ}(x) \arrow[r, "\simeq"] & \text{よ}(y) \end{tikzcd}$$

commute. But indeed $s'r$ is such an isomorphism, and its inverse is $r's$. □

Therefore the idempotent completion J^{idem} always exists and is unique up to a specified isomorphism. We now make a few useful observations. First, if J is idempotent-complete, then is is clear that $\mathrm{Id}_J \colon J \to J$ exhibits J as its own idempotent completion. But, more interestingly, passing to the presheaf category doesn't distinguish between J and its idempotent completion:

Proposition 4.8 *The inclusion $J \hookrightarrow J^{\mathrm{idem}}$ induces an equivalence of categories $\mathrm{PSh}(J^{\mathrm{idem}}) \simeq \mathrm{PSh}(J)$.*

Proof Let $y \colon J \to J^{\mathrm{idem}}$ be the idempotent-completion. Then we have the opposite functor $y^{\mathrm{op}} \colon J^{\mathrm{op}} \to (J^{\mathrm{idem}})^{\mathrm{op}}$, and observe that

$$- \circ y^{\mathrm{op}} \colon [(J^{\mathrm{op}})^{\mathrm{idem}}, C] \to [J^{op}, C]$$

is still an equivalence of categories since $- \circ y$ was. Therefore $(J^{\mathrm{op}})^{\mathrm{idem}} = (J^{\mathrm{idem}})^{\mathrm{op}}$ by the universal property. Since **Set** is idempotent-complete, a special case of the equivalence

$$[(J^{\mathrm{idem}})^{\mathrm{op}}, C] \simeq [J^{op}, C]$$

is $\mathrm{PSh}(J^{\mathrm{idem}}) \simeq \mathrm{PSh}(J)$. □

This statement says that we definitely can't recover a category which isn't idempotent-complete from its presheaf category. The next few results show that this is the only thing that can go wrong. We recall a quick fact about colimits in **Set** first.

Lemma 4.9 *Suppose $\varinjlim_i Z_i = Z$ is a colimit in* **Set** *with limiting cocone $\eta = \eta_i \colon Z_i \to Z$. Then for each $z \in Z$, there is an i and an $x \in Z_i$ such that $\eta_i(x) = z$.*

Proof Just as in any category, we can express the colimit as a coequalizer of coproducts,

$$\coprod_{i \to j} Z_i \rightrightarrows \coprod_i Z_i \longrightarrow \varinjlim_{i \in I} Z_i$$

where the last map is induced by the family $\{\eta_i \colon Z_i \to \varinjlim_i Z_i\}$. But coequalizers are always epi, and thus surjective in **Set**, and that this map is a surjection from the disjoint union is a rephrasing of what we wanted to show. □

Proposition 4.10 *A presheaf F in $\mathrm{PSh}(J)$ is atomic if and only if it is a retract of a representable presheaf.*

Proof We know that representables are atomic and that the atomic subcategory is closed under retracts (Prop. 4.2), which proves one direction. For the other, write an atomic presheaf F as the colimit of representables $F = \varinjlim_{i \in I} \text{よ}(x_i)$ by the density theorem. Then

$$\mathrm{Hom}(F, F) = \mathrm{Hom}(F, \varinjlim_{i \in I} \text{よ}(x_i)) = \varinjlim_{i \in I} \mathrm{Hom}(F, \text{よ}(x_i)),$$

and the colimit maps $\mathrm{Hom}(F, \text{よ}(x_i)) \to \mathrm{Hom}(F, F)$ are composition with the maps $f_i \colon \text{よ}(x_i) \to F$ from the density theorem. Applying Lem. 4.9 to the identity $1_F \colon F \to F$, there is some $i \in I$ and a map $g \colon F \to \text{よ}(x_i)$ such that $g \circ f_i = 1_F$. Thus F is a retract of $\text{よ}(x_i)$. □

Corollary 4.11 *The Yoneda embedding* $\text{よ} \colon J \to \mathrm{PSh}(J)^{\mathrm{at}}$ *exhibits the atomic presheaves* $\mathrm{PSh}(J)^{\mathrm{at}}$ *as the idempotent completion of* J*.*

Proof Apply Prop. 4.7, and recall by Lem. 4.6 that $\overline{\text{よ}(J)}$ is the subcategory of $\mathrm{PSh}(J)$ consisting of retracts of representable presheaves, which we now know is just $\mathrm{PSh}(J)^{\mathrm{at}}$. □

So, if J is idempotent-complete then $\text{よ} \colon J \simeq \mathrm{PSh}(J)^{\mathrm{at}}$. This is useful because of the following result:

Corollary 4.12 *Let* $f \colon J \to C$ *be a functor. Then the following are equivalent:*
(a) The induced functor $f^{\mathrm{idem}} \colon J^{\mathrm{idem}} \to C^{\mathrm{idem}}$ *is an equivalence.*
(b) The functor $f^* \colon \mathrm{PSh}(C) \to \mathrm{PSh}(C)$ *is an equivalence.*
Alternatively, $J^{\mathrm{idem}} \simeq C^{\mathrm{idem}}$ *if and only if* $\mathrm{PSh}(C) \simeq \mathrm{PSh}(J)$*.*

Proof Without maps, $\mathrm{PSh}(J) \simeq \mathrm{PSh}(J^{\mathrm{idem}}) \simeq \mathrm{PSh}(C^{\mathrm{idem}}) \simeq \mathrm{PSh}(C)$, and on the other hand if $\mathrm{PSh}(J) \simeq \mathrm{PSh}(C)$, then since the equivalence preserves colimits we have $\mathrm{PSh}(J)^{\mathrm{at}} \simeq \mathrm{PSh}(C)^{\mathrm{at}}$, so $J^{\mathrm{idem}} \simeq C^{\mathrm{idem}}$.

With maps, for the forward direction note

$$\begin{tikzcd} J \arrow[r, "f"] \arrow[d, hook] \arrow[dr, dashed] & C \arrow[d, hook] \\ J^{\mathrm{idem}} \arrow[r, "f^{\mathrm{idem}}"] & C^{\mathrm{idem}} \end{tikzcd}$$

commutes, and by taking precompositions and applying the universal property we see

$$\begin{tikzcd} \mathrm{PSh}(J) & \mathrm{PSh}(C) \arrow[l, "f^*"'] \\ \mathrm{PSh}(J^{\mathrm{idem}}) \arrow[u, hook] & \mathrm{PSh}(C^{\mathrm{idem}}) \arrow[l, "(f^{\mathrm{idem}})^*"'] \arrow[u, hook] \end{tikzcd}$$

commutes. But then the bottom map and both vertical ones are equivalences (Prop. 4.8), so f^* is one too. Conversely, we have a commutative diagram

$$\begin{tikzcd} \mathrm{PSh}(J)^{\mathrm{at}} \arrow[d, hook] & \mathrm{PSh}(C)^{\mathrm{at}} \arrow[l, "f^*"'] \arrow[d, hook] \\ \mathrm{PSh}(J) & \mathrm{PSh}(C) \arrow[l, "f^*"'] \end{tikzcd}$$

as the bottom equivalence restricts to one on the top. Then we can take (the restriction of) f^* to be f^{idem} so we're done. ☐

Now we have resolved the issue at the beginning of this section: While we can't generally recover a category from the presheaves over it, we can recover its idempotent-completion; hence if it is idempotent-complete, we can get the category itself. We almost have Dold-Kan now - what remains is to transport this result into back into the setting of $\widehat{\mathcal{A}} = [\mathcal{A}^{\text{op}}, \mathbf{Ab}]$, i.e. presheaves of abelian groups.

4.2 Finishing Dold-Kan

Based on the results prior, we have a commutative diagram of functors

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\text{よ}^e"] \arrow[d, "\text{よ}"] & & \widehat{\mathcal{A}} \arrow[d, "U_*"'] \\
\mathrm{PSh}(\mathcal{A})^{\text{at}} \arrow[rr, hook, "j"] & & \mathrm{PSh}(\mathcal{A})
\end{tikzcd} \tag{5}$$

Here よ^e is additive and fully faithful, j is fully faithful. よ is fully faithful, and an equivalence if $\mathcal{A}$ is idempotent-complete. We first summarize properties of U and the induced functor U_*:

Proposition 4.13 *The forgetful functors*

$$U\colon \mathbf{Ab} \to \mathbf{Set} \qquad U_*\colon \widehat{A} \to \mathrm{PSh}(A)$$

are faithful, conservative, and a right adjoint. In particular, they creates limits.

Proof This was Remark 3.5 - in particular, U_* inherits the the properties of U. Recall that a conservative functor reflects any preserved limits, so by adjointness U_* creates limits. ☐

In the idempotent-complete case, U_* will let us test whether, given an arbitrary presheaf of abelian groups $X \in \widehat{A}$, it is in fact in the essential image of よ^e.

Lemma 4.14 *Let $\mathcal{A}$ be idempotent-complete. For $X \in \widehat{A}$, there is an $x \in \mathcal{A}$ such that $\text{よ}^e(x) \cong X$ if and only if $U^*(X)$ is an atomic presheaf.*

Proof If $X \cong \text{よ}^e(x)$, then $U_*X \cong j\text{よ}(x)$, i.e. U_*X is isomorphic to an atomic object and hence is atomic. On the other hand, suppose U_*X is atomic, then since $\text{よ}\colon \mathcal{A} \to \mathrm{PSh}(\mathcal{A})^{\text{at}}$ is an equivalence there is x such that $f\colon j\text{よ}(x) \cong U_*X$ is an isomorphism. By commutativity of diagram (5), we have $f\colon U_*\text{よ}^e(x) \cong U_*X$. By conservativity of U_*, it will suffice to show f is of the form $U_*\widetilde{f}$, since then $\widetilde{f}\colon \text{よ}^e(x) \cong X$.

We show this in two steps

1. X is an additive presheaf.
2. f can be lifted to $\widetilde{f}$ using the Yoneda lemma.

For (1), it suffices to note that $X\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ preserves limits. Indeed, $よ(x)\colon \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ preserves limits, since

$$よ(x)(\varinjlim_{i\in I} a_i) \cong \mathrm{Hom}(\varinjlim_{i\in I} a_i, x) \cong \varprojlim_{i\in I} \mathrm{Hom}(a_i, x) \cong \varprojlim_{i\in I} よ(x)(a_i)$$

by the Yoneda lemma. By the isomorphism f, U_*X preserves limits, and using the definition

$$U(X(\varinjlim_{i\in I} a_i)) \cong \varprojlim_{i\in I} U(X(a_i)) \cong U(\varprojlim_{i\in I} X(a_i))$$

since U preserves limits. But it also reflects limits, hence X preserves limits as desired.

For (2), note that the isomorphism $f\colon よ(x) \to U_*X$ corresponds under the Yoneda lemma to some section $s \in U(X(x)) = X(x)$. So f must be of the following form

$$\begin{aligned} f = f_y\colon \mathrm{Hom}(y, x) &\to U(X(y)) \\ g &\mapsto Xg(s), \end{aligned}$$

and we may let $\widetilde{f}\colon よ^e(x) \to X$ be

$$\begin{aligned} \widetilde{f} = \widetilde{f}_y\colon \mathcal{A}(y, x) &\to X(y) \\ g &\mapsto Xg(s). \end{aligned}$$

If this is a valid morphism, then clearly $U_*(\widetilde{f}) = f$ so we will be done. Since X is an additive functor, it defines a homomorphism of abelian groups

$$\mathcal{A}(y, x) \to \mathbf{Ab}(X(y), X(x)).$$

Therefore, because of the group structure given to $\mathbf{Ab}(X(y), X(x))$, we have

$$\begin{aligned} \widetilde{f}_y(g+h) &= X(g+h)(s) = (X(g(s) + X(h)(s)) \\ &= X(g)s + X(h)s = \widetilde{f}_y(g) + \widetilde{f}_y(h) \end{aligned}$$

so $\widetilde{f}_y$ is a group homomorphism. To to check naturality, let $u\colon y \to z$ be a map of $\mathcal{A}$. Since f was natural we have

$$U(Xu) \circ U(\widetilde{f}_z) = U(\widetilde{f}_y) \circ U(よ^e(y)),$$

so $\widetilde{f}$ is natural by functoriality and faithfulness of U, hence is a morphism. $\square$

Remark 4.15 In the proof of Lemma 4.14, we implicitly proved that if X is an *additive* presheaf of abelian groups, then we have a natural bijection

$$\mathrm{Hom}_{\widehat{\mathcal{A}}}(よ^e(x), X) \cong \mathrm{Hom}_{\mathbf{PSh}A}(よ(x), X).$$

Using the Yoneda lemma, this gives a bijection

$$U\widehat{\mathcal{A}}(よ^e(x), X) \cong U(X_x),$$

and the map $\left(よ^e(x) \xrightarrow{f} X\right) \mapsto f_x(1_x)$ is also a homomorphism of abelian groups, so we get the natural isomorphism

$$\widehat{A}(よ^e(x), X) \cong X_x$$

in $\mathbf{Ab}$, i.e. the *enriched Yoneda lemma.*

Dold-Kan will now follow from a simple observation about the Eilenberg-MacLane functor: If $A_* \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ is a chain complex, then for each n, A_n is a retract of $\mathrm{K}_{\mathcal{A}}(A_*)_n$. Indeed, $\mathrm{K}_{\mathcal{A}}(A_*)_n$ is a finite biproduct with A_n a summand, hence the canonical injection and projection provides the retract diagram. Finally:

Theorem 4.16 (Dold-Kan v2) *Let $\mathcal{A}$ be an additive category. The Eilenberg-MacLane functor*

$$\mathrm{K}_{\mathcal{A}}\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{s}\mathcal{A}$$

is fully faithful. If $\mathcal{A}$ is idempotent-complete, then $\mathrm{K}_{\mathcal{A}}$ is an equivalence of categories.

Proof The first part was proven as Thm. 3.14. For the second, recall the diagram

$$\begin{tikzcd} \mathrm{Ch}_{\geq 0}(\mathcal{A}) \arrow[r, "\mathrm{K}_{\mathcal{A}}"] \arrow[d, "よ^e_*"'] & \mathbf{s}\mathcal{A} \arrow[d, "よ^e_*"] \\ \mathrm{Ch}_{\geq 0}(\widehat{\mathcal{A}}) \arrow[r, "\mathrm{K}_{\widehat{\mathcal{A}}}"] & \mathbf{s}\widehat{\mathcal{A}} \end{tikzcd}$$

which commutes up to natural isomorphism, where $\mathrm{K}_{\widehat{\mathcal{A}}}$ is an equivalence of categories and $\mathrm{K}_{\mathcal{A}}$ is fully faithful. To show it is essentially surjective, let $X \in \mathbf{s}\mathcal{A}$, and since the bottom is an equivalence choose a complex $(\widetilde{B}_*, \widetilde{\partial})$ so that $\mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{B}_*) \cong よ^e_*(X)$. It suffices to show that each $\widetilde{A}_k$ is of the form $よ^e(A_k)$ for $A_k \in \mathcal{A}$: indeed, by full faithfulness

$$\widetilde{\partial}_k = よ(\partial_k)\colon よ^e(A_k) \to よ(A_{k-1})$$

and $\partial^2 = 0$ since $よ^e$ reflects limits. So in that case we'll have $(A_*, \partial) \in \mathrm{Ch}_{\geq 0}(\mathcal{A})$ such that $よ^e_*(A_*) = \widetilde{A}_*$, and

$$よ^e_*(X) \cong \mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{A}_*) = \mathrm{K}_{\widehat{\mathcal{A}}} よ^e_*(\widetilde{A}_*) \cong よ^e_* \mathrm{K}_{\mathcal{A}}(A_*),$$

hence $X \cong \mathrm{K}_{\mathcal{A}}(A_*)$ by full faithfullness. Thus it remains to show that $\widetilde{A}_k$ is in the (essential) image of $よ^e$.

By Lem. 4.14, we only need to prove that $U_*(\widetilde{A}_k)$ is an atomic presheaf. But by assumption $\mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{A}_*)_n \cong よ^e(A_n)$, so

$$U_* \mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{A}_*)_n$$

is an atomic presheaf. Since $\widetilde{A}_n$ is a retract of $\mathrm{K}_{\widehat{\mathcal{A}}}(\widetilde{A}_*)_n$, $U_* \widetilde{A}_n$ is the retract of an atomic presheaf, so is still atomic by Prop. 4.2. □

5 Sources Consulted

Goerss-Jardine [1]; Cisinski [3].

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