INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR CLASS TEST / LABORATORY TEST	1 Signature of the Invigilator

EXAMINATION (Mid-Semester / End-Semester)				SEMESTER (Autumn / Spring)	
Roll Number		Section		Name	Koeli Ghoshal
Subject Number	MA20202	Subject Name		Transform Calculus	

Transform Calculus

Lectures-1 and 2 — 9.1.2023

Operator and linear operator

$$\phi y = y^2 \qquad \phi y = Dy$$

$$L(c_1 y_1 + c_2 y_2) = c_1 L y_1 + c_2 L y_2$$

Integral transform (Definition)

Let $K(s,t)$ be a f^n. of s and t, where s is a parameter (may be real or complex) independent of t. The f^n. $f(s)$ defined by the integral (assumed to be convergent)

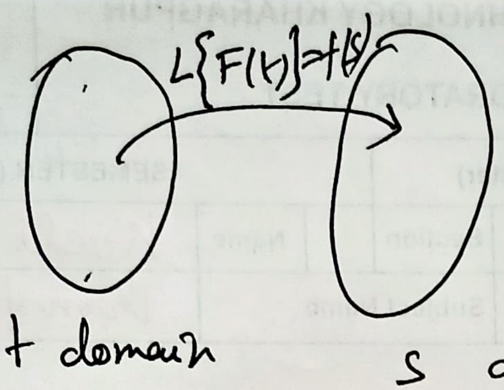
$$f(s) = \int_{-\infty}^{\infty} K(s,t)\, F(t)\, dt$$

is called the I. T. of the f^n. $F(t)$ and is denoted by $T\{F(t)\}$.

Def^n. of Laplace transform

If the kernel $K(s,t)$ is defined as $K(s,t) = \begin{cases} 0 & t<0 \\ e^{-st} & t \geq 0 \end{cases}$

then $f(s) = \int_0^{\infty} e^{-st} F(t)\, dt$. The f^n. $f(s)$ is called L. T. of $F(t)$ is is denoted by $L\{F(t)\}$.



Theorem

The Laplace transformation is a linear transformation.

i.e. $L\{a_1 F_1(t) + a_2 F_2(t)\} = a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$

Proof $\quad L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$

$$L\{a_1 F_1(t) + a_2 F_2(t)\} = \int_0^\infty e^{-st}\{a_1 F_1(t) + a_2 F_2(t)\} dt$$

$$= a_1 \int_0^\infty e^{-st} F_1(t) dt + a_2 \int_0^\infty e^{-st} F_2(t) dt$$

$$= a_1 L\{F_1(t)\} + a_2 L\{F_2(t)\}$$

Ex If $F(t) = 1$ for $t \geq 0$. Then find $L\{F(t)\}$.

Sol^n: $L\{F(t)\} = \int_0^\infty e^{-st} \cdot 1\, dt$

$$= \lim_{\tau \to \infty} \left[\frac{e^{-st}}{-s} \Big|_0^\tau \right]$$

$$= \lim_{\tau \to \infty} \left[\frac{e^{-s\tau}}{-s} + \frac{1}{s} \right] = \frac{1}{s}$$

if $s > 0$ (s is real). Thus $L(1) = \frac{1}{s}$, $s > 0$.

If $s \leq 0$, then the integral will diverge and there will be no resulting L.T.

Ex Attempt to find $L\left\{\frac{1}{t^2}\right\}$

Sol^n.

$$L\left\{\frac{1}{t^2}\right\} = \int_0^\infty \frac{e^{-st}}{t^2}\,dt$$

$$= \int_0^1 \frac{e^{-st}}{t^2}\,dt + \int_1^\infty \frac{e^{-st}}{t^2}\,dt$$

When $0 \le t \le 1$, $e^{-st} \geq e^{-s}$ if $s>0$

$$\int_0^\infty \frac{e^{-st}}{t^2}\,dt \geq \int_0^1 \frac{e^{-s}}{t^2}\,dt + \int_1^\infty \frac{e^{-st}}{t^2}\,dt$$

i.e. $$\int_0^\infty \frac{e^{-st}}{t^2}\,dt \geq e^{-s}\int_0^1 \frac{dt}{t^2} + \int_1^\infty \frac{e^{-st}}{t^2}\,dt$$

But $\int_0^1 \frac{dt}{t^2}$ diverges and hence $L\left\{\frac{1}{t^2}\right\}$ fails to converge. i.e. $\frac{1}{t^2}$ does not have a L.T.

Ex For the f^n. $f(t) = e^{t^2}$

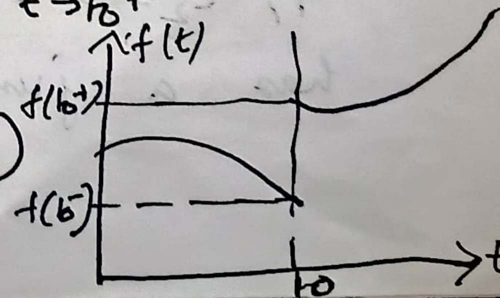
$$\lim_{\tau\to\infty}\int_0^\tau e^{-st}e^{t^2}\,dt = \lim_{\tau\to\infty}\int_0^\tau e^{t^2-st}\,dt = \infty$$

for any s since the integrand grows without bound as $\tau \to \infty$.

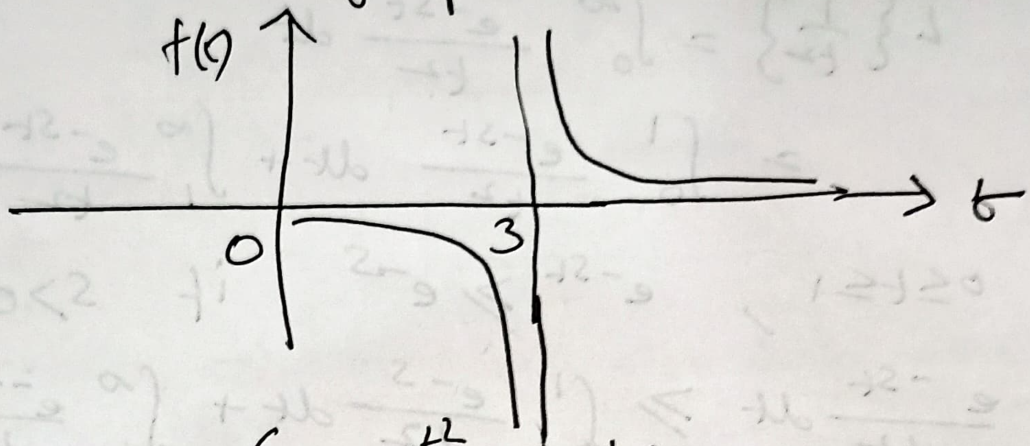
Def^n. of jump discontinuity

A f^n. f has a jump discontinuity at a pt. t_0 if both the limits $\lim_{t\to t_0^-} f(t) = f(t_0^-)$ and $\lim_{t\to t_0^+} f(t) = f(t_0^+)$ exist as finite numbers and

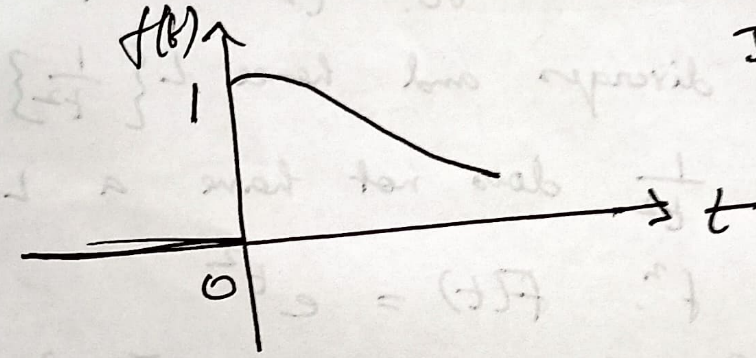
$$f(t_0^-) \neq f(t_0^+)$$



Ex $f(t) = \frac{1}{t-3}$ has a discontinuity at $t=3$ but it is not a jump discontinuity.



Ex $f(t) = \begin{cases} e^{-\frac{t^2}{2}} & t>0 \\ 0 & t<0 \end{cases}$

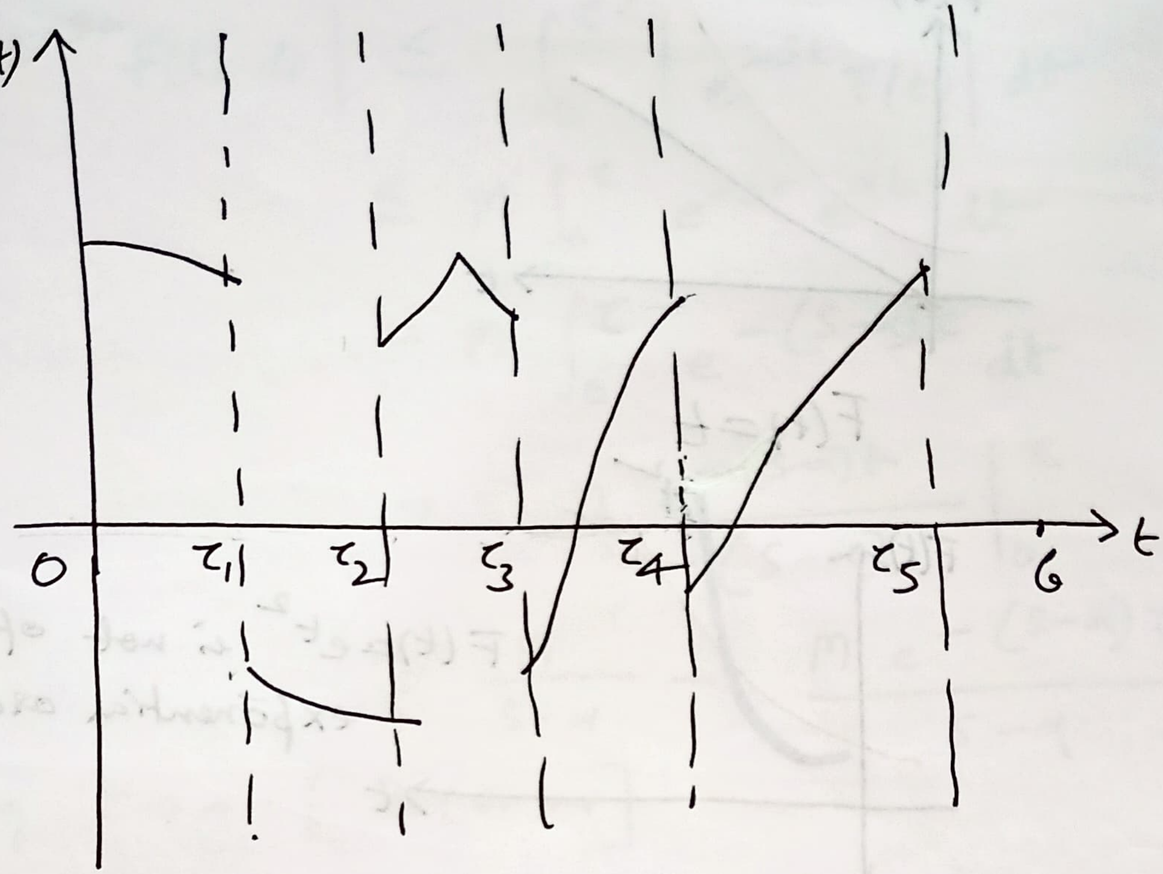


Jump discontinuity at $t=0$

Def^n. of PWC function

A f^n. f is PWC in $[0,\infty)$ if

(i) $\lim_{t\to 0+} f(t) = f(0+)$ exist and (ii) f is continuous on every finite interval $(0,b)$ except possibly at a finite no. of points $\tau_1, \tau_2, \ldots, \tau_n$ in $(a\ b)$ at which f has a jump discontinuity.

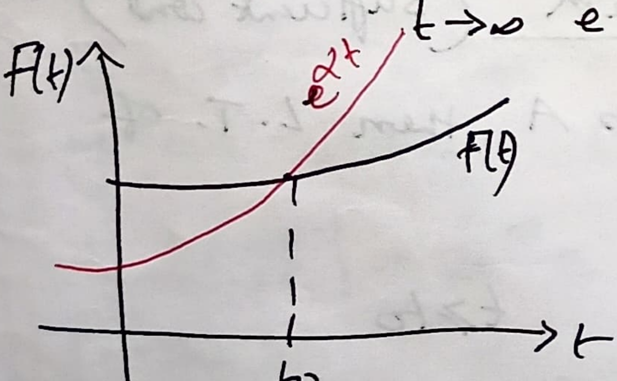


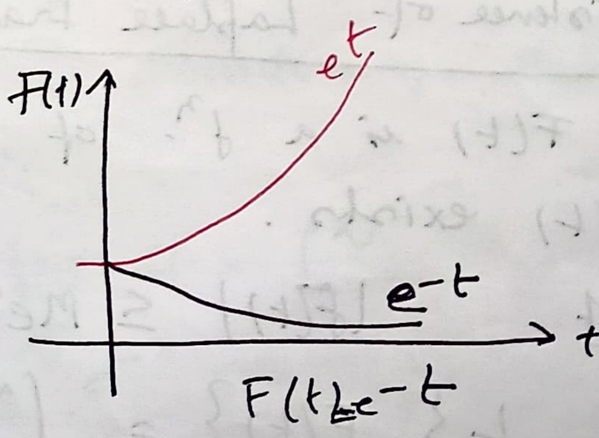
Exponential order

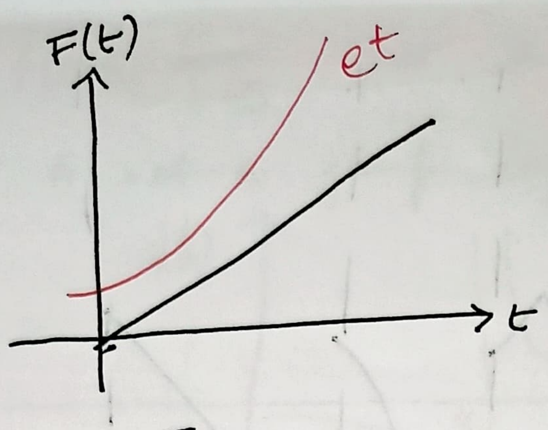
A f^n. F has exponential order α if $\exists$ constant $M > 0$ and α such that for some $t_0 \geq 0$

$$|F(t)| \leq M e^{\alpha t} \quad t \geq t_0$$

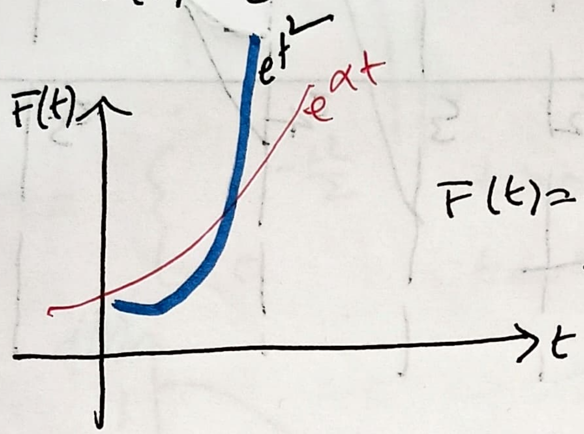
i.e. $\lim\limits_{t \to \infty} \dfrac{f(t)}{e^{\alpha t}}$ is finite







$F(t) = t$



$F(t) = e^{t^2}$ is not of exponential order

<u>Def^n</u>: Functions of class A

A f^n: $f(t)$ is said to be of class A if

(i) it is piecewise continuous over every finite interval in the range $t \geq 0$

(ii) $f(t)$ is of exponential order.

<u>Existence of Laplace transform</u> (Sufficient cond^n)

If $F(t)$ is a f^n. of class A, then L.T. of $F(t)$ exists.

<u>Proof</u> $|f(t)| \leq Me^{\alpha t}$ $t \geq t_0$

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

$$\left| \int_0^{\tau} e^{-st} F(t)\, dt \right| \le \int_0^{\tau} \left| e^{-st} F(t) \right| dt$$

$$\le M \int_0^{\tau} e^{-st} e^{\alpha t}\, dt$$

$$= M \int_0^{\tau} e^{-(s-\alpha)t}\, dt$$

$$= \left. \frac{M e^{-(s-\alpha)t}}{-(s-\alpha)} \right|_0^{\tau}$$

$$= \frac{M}{s-\alpha} - \frac{M e^{-(s-\alpha)\tau}}{s-\alpha}$$

Letting $\tau \to \infty$ $\left[Re(s) > \alpha \right]$

$$\int_0^{\infty} \left| e^{-st} f(t) \right| dt \le \frac{M}{s-\alpha}$$

Thus the Laplace integral converges absolutely ~~ex~~ and hence converges for $Re(s) > \alpha$.

Ex $\quad F(t) = \frac{1}{\sqrt{t}}$

It is not PWC on $[0, \infty)$ ~~sind~~ since $F(t) \to \infty$ as $t \to 0^+$ i.e. $t=0$ is not a jump discontinuity. But still we can compute $L\left[\frac{1}{\sqrt{t}}\right]$.

$$L\left[\frac{1}{\sqrt{t}}\right] = \int_0^{\infty} e^{-st} \frac{1}{\sqrt{t}}\, dt = \int_0^{\infty} e^{-st} t^{-1/2}\, dt$$

Put $st = x$, $dt = \frac{dx}{s}$

$$= \frac{1}{\sqrt{s}} \int_0^{\infty} e^{-x} x^{\frac{1}{2}-1}\, dx$$

$$= \frac{1}{\sqrt{s}} \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\frac{\pi}{s}}, \quad s > 0$$

Behaviour of $f(s)$ as $s \to \infty$

Theorem If F is PWC on $[0,\infty)$ and has exp. order α, then

$$f(s) = L\{F(t)\} \to 0 \text{ as } Re(s) \to \infty$$

Remark $\frac{s-1}{s+1}$, $\frac{e^s}{s}$ or s^2 cannot be L.T. of any f^n. F (class A).

Laplace transform of some elementary functions

Ex Find the L.T. of (i) 1 (ii) t (iii) t^n, n +ve integer (iv) e^{at} (v) $\sin at$ (vi) $\cos at$ (vii) $\sinh at$ (viii) $\cosh at$

Sol^n: (i) $L[1] = \int_0^\infty e^{-st}\, 1\, dt = \frac{1}{s}$ if $s>0$

(ii) $L[t] = \frac{1}{s^2}$

(iii) $L[t^n] = \int_0^\infty e^{-st} t^n\, dt$

$$= \cancel{n/} \left[-\frac{t^n}{s} e^{-st}\right]_0^\infty + \int_0^\infty \frac{nt^{n-1}}{s} e^{-st} dt$$

$$= \frac{n}{s} L[t^{n-1}]$$

If $n=2$, $L[t^2] = \frac{2}{s^3}$

$n=3$ $L[t^3] = \frac{3!}{s^4}$

If induction is applied

$$L[t^n] = \frac{n!}{s^{n+1}}$$

Aliter $\quad L[t^n] = \int_0^\infty e^{-st} t^n \, dt$

$$= \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}, \quad st = x$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-x} x^{n+1-1} \, dx$$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \quad \text{if } s>0, \; n+1>0$$

If n is a +ve integer, $\Gamma(n+1) = n!$

$$\therefore L[t^n] = \frac{n!}{s^{n+1}} \quad s>0$$

(iv) $L[e^{at}] = \frac{1}{s-a}, \quad s>a$

$$\int_0^\infty e^{-(s-a)t} dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}, \; s>a$$

(v) $L[\sin at] = \frac{a}{s^2+a^2}, \quad s>0$

(vi) $L[\cos at] = \frac{s}{s^2+a^2}, \quad s>0$

(vii) $L[\sinh at] = L\left[\frac{e^{at}-e^{-at}}{2}\right] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$

$s>|a|$

(viii) $L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}, \quad s>|a|$$

Lecture - 3

10.1.2023

Ex Find $L\{\sin t\cos t\}$

Sol^n: $L\{\sin t\cos t\} = L\{\frac{1}{2}\sin 2t\} = \frac{1}{2}L\{\sin 2t\}$

$= \frac{1}{2}\frac{2}{s^2+4} = \frac{1}{s^2+4}$, $s>0$

Ex Find $L\{F(t)\}$ where $F(t) = \begin{cases} 4 & 0<t<1 \\ 3 & t>1 \end{cases}$

Sol^n: $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$= \int_0^1 e^{-st}\,4\,dt + \int_1^\infty e^{-st}\,3\,dt$

$= \left[-\frac{4}{s}e^{-st}\right]_0^1 + \left[-\frac{3}{s}e^{-st}\right]_1^\infty$

$= \frac{1}{s}(4-e^{-s})$, $s>0$

Ex Find $L\{\sin^3 2t\}$

Sol^n: $\sin 3t = 3\sin t - 4\sin^3 t$

$\therefore \sin^3 t = \frac{3}{4}\sin t + \frac{1}{4}\sin 3t$

$\therefore \sin^3 2t = \frac{3}{4}\sin 2t - \frac{1}{4}\sin 6t$

$L\{\sin^3 2t\} = \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$

$= \frac{3}{4}\frac{2}{s^2+4} - \frac{1}{4}\frac{6}{s^2+36}$, $s>0$

$= \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$

$= \frac{48}{(s^2+4)(s^2+36)}$

E1 Find $L\{\sin at \sin bt\}$

sol^n $F(t) = \frac{1}{2}(2\sin at \sin bt)$

$$= \frac{1}{2}\left[\cos(at-bt) - \cos(at+bt)\right]$$

$$= \frac{1}{2}\cos(a-b)t - \frac{1}{2}\cos(a+b)t$$

$$L\{F(t)\} = \frac{1}{2}L\{\cos(a-b)t\} - \frac{1}{2}L\{\cos(a+b)t\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2}\right] - \frac{1}{2}\left[\frac{s}{s^2+(a+b)^2}\right] \quad s>0$$

$$= \frac{2abs}{\{s^2+(a-b)^2\}\{s^2+(a+b)^2\}}$$

E1 Find $L\{e^{at}\cos bt\}$ and $L\{e^{at}\sin bt\}$

sol^n: Let $F(t) = e^{(a+ib)t}$

$$L\{F(t)\} = \frac{1}{s-(a+ib)} = \frac{1}{s-a-ib}$$

$$= \frac{(s-a)+ib}{(s-a)^2+b^2}$$

$$e^{(a+ib)t} = e^{at}\left[\cos bt + i\sin bt\right]$$

$$= e^{at}\cos bt + ie^{at}\sin bt$$

$$L\{F(t)\} = L\{e^{at}\cos bt\} + iL\{e^{at}\sin bt\}$$

$$L\{e^{at}\cos bt\} + iL\{e^{at}\sin bt\} = \frac{(s-a)+ib}{(s-a)^2+b^2}$$

$$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

L.T. for an infinite series

Theorem

If $F(t) = \sum_{n=0}^{\infty} a_n t^n$ converges for $t \geq 0$ with

$|a_n| \leq \frac{K\alpha^n}{n!}$ $\forall n$ sufficiently large and $\alpha > 0, K > 0$

then $L[F(t)] = \sum_{n=0}^{\infty} a_n L[t^n] = \sum_{n=0}^{\infty} \frac{a_n n!}{s^{n+1}}$

$[Re(s) > \alpha]$

Ex Find $L\{\sin\sqrt{t}\}$

Solⁿ: $L\{\sin\sqrt{t}\}$

$$= L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \cdots\right\}$$

$$= L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \cdots\right\}$$

$$= L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \cdots$$

$$= \frac{\Gamma(\frac{3}{2})}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(\frac{5}{2})}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(\frac{7}{2})}{s^{7/2}} - \cdots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \cdots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}$$

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Roll Number	Section	Name: Koeli Ghoshal
Subject Number: MA20202	Subject Name: Transform Calculus

Lecture - 3 (continued) 10.1.2023

Ex Find the L.T. of $F(t) = \int_0^t \frac{\sin u}{u}\,du$

Sol^n: $\int_0^t \frac{\sin u}{u}\,du$

$$= \int_0^t \frac{1}{u}\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \cdots\right] du$$

$$= \int_0^t \left[1 - \frac{u^2}{3!} + \frac{u^4}{5!} - \frac{u^6}{7!} + \cdots\right] du$$

$$= t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \cdots$$

$$L\{F(t)\} = L\left\{t - \frac{t^3}{3.3!} + \frac{t^5}{5.5!} - \frac{t^7}{7.7!} + \cdots\right\}$$

$$= \frac{1}{s^2} - \frac{1}{3.3!}\frac{3!}{s^4} + \frac{1}{5.5!}\frac{5!}{s^6} - \cdots$$

$$= \frac{1}{s^2} - \frac{1}{3s^4} + \frac{1}{5s^6} - \frac{1}{7s^8} + \cdots$$

$$= \frac{1}{s}\left[\frac{1}{s} - \frac{(\frac{1}{s})^3}{3} + \frac{(\frac{1}{s})^5}{5} - \frac{(\frac{1}{s})^7}{7} + \cdots\right]$$

$$= \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Lectures 4 and 5 16.1.23

Elementary properties of Laplace Transform

Theorem 1

First translation (or shifting) theorem

If $L\{F(t)\} = f(s)$ $\quad s > \alpha$

then $L\{e^{at}F(t)\} = f(s-a)$, $\quad s > \alpha + a$

Proof $f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$$f(s-a) = \int_0^\infty e^{-(s-a)t} F(t)\,dt$$

$$= \int_0^\infty e^{-st} e^{at} F(t)\,dt$$

$$= L\{e^{at} F(t)\}$$

Ex Find $L\{t^3 e^{-3t}\}$

Solⁿ: $F(t) = t^3$ $\quad f(s) = \frac{3!}{s^4} = \frac{6}{s^4}$

$$L\{t^3 e^{-3t}\} = f(s-a) = \frac{6}{(s+3)^4}$$

Ex Find $L\{e^t \sin^2 t\}$

Solⁿ: $L\{\sin^2 t\} = L\{\frac{1}{2}(1-\cos 2t)\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+2^2}\right]$

$$= \frac{2}{s(s^2+4)} = f(s)$$

$$L\{e^t \sin^2 t\} = f(s-1) = \frac{2}{(s-1)\{(s-1)^2+4\}}$$

$$= \frac{2}{(s-1)(s^2-2s+5)}$$

Ex Find $L\{t\sin at\}$ and $L\{t\cos at\}$

Solⁿ: $L\{t\} = \frac{1}{s^2} = f(s)$

$$L\{te^{iat}\} = L\{t\cos at\} + iL\{t\sin at\}$$

Again. $L\{te^{iat}\} = f(s-a)$

$$= \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{[(s-ia)(s+ia)]^2}$$

$$= \frac{(s^2-a^2) + i(2as)}{(s^2+a^2)^2}$$

$$L\{t\cos at\} = \frac{s^2-a^2}{(s^2+a^2)^2} \qquad L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$$

Ex Find $L\{\sinh 3t \cos^2 t\}$

Solⁿ: $L\{\cos^2 t\} = \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 2t\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

$$= \frac{s^2+2}{s(s^2+4)}$$

$$L\{\sinh 3t \cos^2 t\} = L\left\{\frac{e^{3t}-e^{-3t}}{2}\cos^2 t\right\}$$

$$= \frac{1}{2}L\{e^{3t}\cos^2 t\} - \frac{1}{2}L\{e^{-3t}\cos^2 t\}$$

$$= \frac{1}{2}\left[\frac{(s-3)^2+2}{(s-3)[(s-3)^2+4]} - \frac{(s+3)^2+2}{(s+3)[(s+3)^2+4]}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-6s+11}{(s-3)(s^2-6s+13)} - \frac{s^2+6s+11}{(s+3)(s^2+6s+13)}\right]$$

Theorem 2

Second translation (or shifting theorem)

If $L\{F(t)\} = f(s)$ and G is a f^n. defined by

$$G(t) = \begin{cases} F(t-a) & t > a \\ 0 & t < a \end{cases}$$

then $L\{G(t)\} = e^{-as} f(s)$

Proof

$$L\{G(t)\} = \int_0^\infty e^{-st} G(t)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} F(t-a)\,dt$$

$$= \int_a^\infty e^{-st} F(t-a)\,dt \qquad t-a = x, \quad dt = dx$$

$$= \int_0^\infty e^{-s(a+x)} F(x)\,dx$$

$$= e^{-as} \int_0^\infty e^{-sx} F(x)\,dx$$

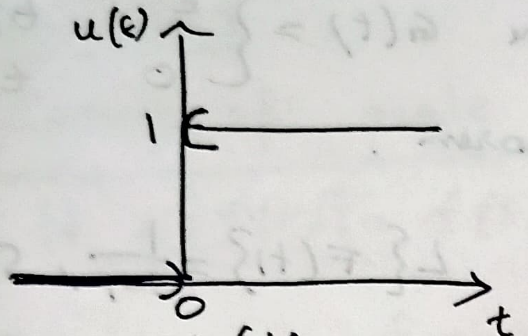
$$= e^{-as} \int_0^\infty e^{-st} F(t)\,dt$$

$$= e^{-as} f(s)$$

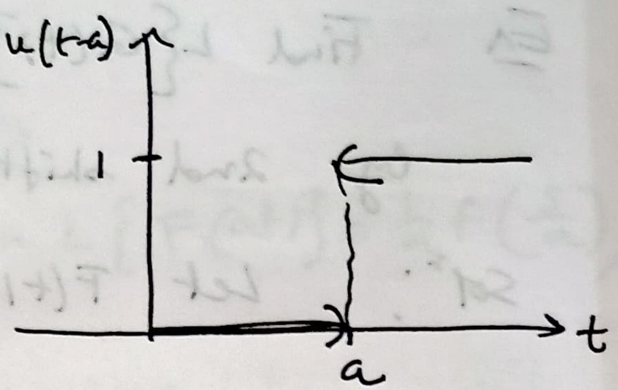
Unit step f^n. or Heaviside's unit function

$$u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$$

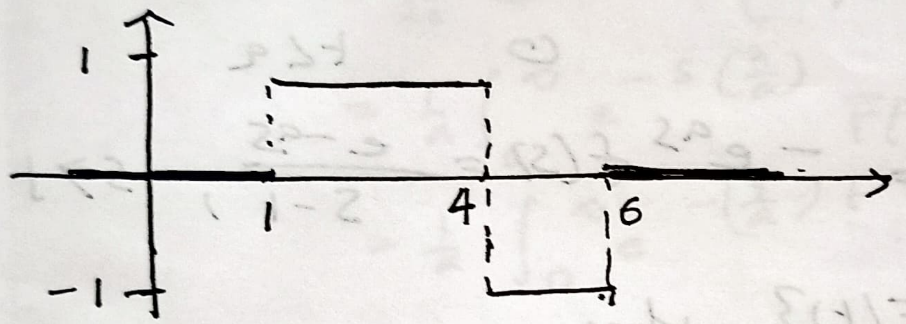
$$a \geq 0$$



$u(t)$



$u(t-a)$



Use many unit step f^n. to describe the figure.

$u(t-1) - 2u(t-4) + u(t-6)$

Alternative statement of 2nd shifting theorem

If $F(t)$ has the transform $f(s)$, then the shifted f^n.

$$F(t-a)\,u(t-a) = \begin{cases} 0 & \text{if } t<a \\ F(t-a) & \text{if } t>a \end{cases}$$

has the transform $e^{-as} f(s)$ i.e.

$$L\{F(t-a)\,u(t-a)\} = e^{-as} f(s)$$

Laplace transform of unit step f^n.

$$L\{u(t-a)\} = \int_0^\infty e^{-st}\,u(t-a)\,dt$$

$$= \int_0^a \quad + \int_a^\infty e^{-st}\,1\,dt$$

$$= \frac{e^{-as}}{s}$$

Ex Find $L\{G(t)\}$ where $G(t) = \begin{cases} e^{t-a} & t > a \\ 0 & t < a \end{cases}$

by 2nd shifting theorem.

Solⁿ: Let $F(t) = e^t$ $\quad L\{F(t)\} = \frac{1}{s-1}, \; s > 1$

$= f(s)$

$$G(t) = \begin{cases} F(t-a) = e^{t-a} & t > a \\ 0 & t < a \end{cases}$$

$$L\{G(t)\} = e^{-as} f(s) = \frac{e^{-as}}{s-1}, \quad s > 1$$

Ex Find $L\{F(t)\}$ where

$$F(t) = \begin{cases} \cos\left(t - \frac{2}{3}\pi\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$$

Solⁿ: Let $\varphi(t) = \cos t$ $\quad F(t) = \begin{cases} \varphi\left(t - \frac{2\pi}{3}\right) & t > \frac{2\pi}{3} \\ 0 & t < \frac{2\pi}{3} \end{cases}$

$$L\{\varphi(t)\} = \frac{s}{s^2+1} = f(s)$$

$$\therefore L\{F(t)\} = e^{-\frac{2\pi s}{3}} f(s)$$

$$= e^{-\frac{2\pi s}{3}} \frac{s}{s^2+1}$$

Theorem 3

Change of scale property

If $L\{F(t)\} = f(s)$, then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$, $a>$

Proof $L\{F(at)\}$

$$= \int_0^\infty e^{-st} F(at)\,dt \qquad at = x$$

$$= \frac{1}{a}\int_0^\infty e^{-s\left(\frac{x}{a}\right)} F(x)\,dx \qquad dt = \frac{1}{a}dx$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t} F(t)\,dt$$

$$= \frac{1}{a} f\left(\frac{s}{a}\right)$$

Ex Find $L\{\cos 5t\}$

Solⁿ: $L\{\cos t\} = \frac{s}{s^2+1} = f(s) \qquad s>0$

$$L\{\cos 5t\} = \frac{1}{5} f\left(\frac{s}{5}\right) = \frac{1}{5}\,\frac{\frac{s}{5}}{\left(\frac{s}{5}\right)^2+1} = \frac{s}{s^2+25}$$

Ex Find $L\{\sinh 3t\}$

Sol^n: $L\{\sinh t\} = \frac{1}{s^2-1} = f(s)$

$$L\{\sinh 3t\} = \frac{1}{3} f\left(\frac{s}{3}\right) = \frac{1}{3}\,\frac{1}{\left(\frac{s}{3}\right)^2-1} = \frac{3}{s^2-9}$$

L. T. of derivative of $F(t)$

Theorem 4

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

Proof $L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\, dt$

$$= \left[e^{-st} F(t)\right]_0^\infty + s\int_0^\infty e^{-st} F(t)\, dt$$

$$= \lim_{t\to\infty} e^{-st} F(t) - F(0) + sL\{F(t)\}$$

$|F(t)| \le Me^{at}$ $\forall$ $t \geq 0$ and for some a and M

$$|e^{-st} F(t)| \le e^{-st} Me^{at}$$

$$= Me^{-(s-a)t} \to 0 \text{ as } t\to\infty \text{ if } s>a$$

$\lim_{t\to\infty} e^{-st} F(t) = 0$ for $s > a$

$$\therefore L\{F'(t)\} = sL\{F(t)\} - F(0)$$

Generalized result

Th - 5

$$L\{F^n(t)\} = s^n L\{F(t)\} - s^{n-1} F(0) - s^{n-2} F'(0) \cdots - F^{n-1}(0)$$

$$L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) - F'(0)$$

Ex Find $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$

Solⁿ: Let $F(t) = \sin\sqrt{t}$

$$F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}} \qquad F(0) = 0$$

$$L\{F'(t)\} = sL\{F(t)\} - F(0)$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = sL\{\sin\sqrt{t}\} = s\frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}}$$

$$\therefore L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\,e^{-\frac{1}{4s}}$$

Theorem 6

Laplace transform of integrals

$$L\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{s}L\{F(t)\}$$

Proof Let $G(t) = \int_0^t F(x)\,dx$

$\therefore G(0) = 0$

$$G'(t) = \frac{d}{dt}\left[\int_0^t F(x)\,dx\right] = F(t)$$

$$L\{G'(t)\} = sL\{G(t)\} - G(0)$$

$$\therefore L\{F(t)\} = sL\{G(t)\} - 0 = sL\{G(t)\}$$

$$\therefore \frac{1}{s}f(s) = L\{G(t)\} = L\left\{\int_0^t F(x)\,dx\right\}$$

Multiplication by powers of t

Theorem - 7

If $L\{F(t)\} = f(s)$, then $L\{t\,F(t)\} = -f'(s)$

Proof $\quad f(s) = L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$

$$\frac{d}{ds}(f(s)) = \frac{d}{ds}\int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^\infty \frac{\partial}{\partial s}\{e^{-st} F(t)\}\, dt$$

$$= \int_0^\infty -t e^{-st} F(t)\, dt$$

$$= -\int_0^\infty e^{-st}\{t\,F(t)\}\, dt$$

$$= -L\{t\,F(t)\}$$

$$\therefore L\{t\,F(t)\} = -f'(s)$$

Generalized result

Theorem 8

If $L\{F(t)\} = f(s)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

Division by t

Theorem 9

~~Def~~ If $L\{F(t)\} = f(s)$

then $L\{\frac{1}{t}F(t)\} = \int_s^\infty f(x)\,dx$

provided $\lim_{t\to 0}\{\frac{1}{t}F(t)\}$ exists.

Proof Let $G(t) = \frac{1}{t}F(t)$

$\therefore F(t) = t\,G(t)$

$L\{F(t)\} = L\{tG(t)\} = -\frac{d}{ds}L\{G(t)\}$

$\therefore f(s) = -\frac{d}{ds}L\{G(t)\}$

Now integrating both sides w.r.t. s from s to ∞,

$$-\left[L\{G(t)\}\right]_s^\infty = \int_s^\infty f(s)\,ds$$

$$\Rightarrow -\lim_{s\to\infty} L\{G(t)\} + L\{G(t)\} = \int_s^\infty f(s)\,ds$$

$$\Rightarrow L\{G(t)\} = \int_s^\infty f(s)\,ds$$

$$\Rightarrow L\{\frac{1}{t}F(t)\} = \int_s^\infty f(x)\,dx$$

Ex Find $L\{t\cos at\}$

Sol^n: $L\{\cos at\} = \frac{s}{s^2+a^2}$

$L\{t\cos at\} = -\frac{d}{ds}L\{\cos at\} = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{s^2-a^2}{(s^2+a^2)^2}$

Ex Find $L\{t^2\sin at\}$

Sol^n. $L\{\sin at\} = \frac{a}{s^2+a^2}$

$L\{t^2\sin at\} = (-1)^2\frac{d^2}{ds^2}L\{\sin at\}$

$= \frac{d^2}{ds^2}\left\{\frac{a}{s^2+a^2}\right\} = \frac{d}{ds}\left\{-\frac{2as}{(s^2+a^2)^2}\right\} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$

Ex Use L.T. to prove that $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Sol^n: Let $F(t) = \sin t$ $f(s) = \frac{1}{s^2+1}$

$L\left\{\frac{\sin t}{t}\right\} = \int_0^\infty e^{-st}\frac{\sin t}{t}\,dt = \int_s^\infty f(x)\,dx = \int_s^\infty \frac{1}{x^2+1}\,dx$

$= \left[\tan^{-1}x\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s$

Taking limit $s \to 0$, $\int_0^\infty \frac{\sin t}{t}\,dt = \frac{\pi}{2}$

Ex Use L.T. to prove $\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt = \ln\frac{b}{a}$

Sol^n. $F(t) = e^{-at} - e^{-bt}$

$f(s) = L\{F(t)\} = \frac{1}{s+a} - \frac{1}{s+b}$

$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx$

$\Rightarrow \int_0^\infty e^{-st}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \int_s^\infty\left(\frac{1}{x+a} - \frac{1}{x+b}\right)dx$

$= \lim_{X\to\infty}\left[\ln(x+a) - \ln(x+b)\right]_s^X$

$= \lim_{X\to\infty}\left(\ln\frac{X+a}{X+b} - \ln\frac{s+a}{s+b}\right) = \lim_{X\to\infty}\ln\left(\frac{1+\frac{a}{X}}{1+\frac{b}{X}}\right) - \ln\frac{s+a}{s+b}$

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CLASS TEST / LABORATORY TEST

3

Signature of the Invigilator

EXAMINATION (Mid-Semester / End-Semester) SEMESTER (Autumn / Spring)

Roll Number		Section		Name	Koeli Ghoshal
Subject Number	MA20202	Subject Name			Transform Calculus

Lectures 4 and 5 (continued) 16.1.23

$$= \ln \frac{s+b}{s+a}$$

Taking limit as $s \to 0$

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \ln \frac{b}{a}$$

Ex Find $\int_0^\infty t e^{-3t} \sin t \, dt$ by L.T.

Solⁿ: $L\{t \sin t\} = -\frac{d}{ds} L\{\sin t\}$

$$\Rightarrow \int_0^\infty e^{-st} t \sin t \, dt = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{2s}{(s^2+1)^2}$$

Putting $s=3$

$$\int_0^\infty t e^{-3t} \sin t \, dt = \frac{3}{50}$$

Lecture - 6 17.1.23

Theorem 10

Initial value theorem

Let $F(t)$ be continuous $\forall$ $t \geq 0$ and be of exponential order as $t \to \infty$ and if $F'(t)$ is of class A, then $\lim_{t \to 0} F(t) = \lim_{s \to \infty} s f(s)$

Proof. We have

$$L\{F'(t)\} = s f(s) - F(0)$$

$$\underbrace{\lim_{s \to \infty} L\{F'(t)\}}_{0} = \lim_{s \to \infty} s f(s) - F(0)$$

$$\lim_{s \to \infty} s f(s) = F(0)$$

$$= \lim_{t \to 0} F(t)$$

Theorem - 11

Final value theorem

Let $F(t)$ be continuous $\forall$ $t \geq 0$ and be of exp. order as $t \to \infty$ and if $F'(t)$ is of class A, then $\lim_{t \to \infty} F(t) = \lim_{s \to 0} s f(s)$

Proof

$$L\{F'(t)\} = s f(s) - F(0)$$

or, $$\int_0^\infty e^{-st} F'(t)\,dt = s f(s) - F(0)$$

Making $s \to 0$,

$$\lim_{s \to 0} s f(s) - F(0) = \lim_{s \to 0} \int_0^\infty e^{-st} F'(t)\,dt$$

$$= \int_0^\infty \left(\lim_{s \to 0} e^{-st} \right) F'(t)\,dt$$

$$= \int_0^\infty F'(t)\,dt$$

$$= \left[F(t) \right]_{t=0}^{\infty}$$

$$= \lim_{t \to \infty} F(t) - F(0)$$

$$\lim_{s \to 0} s f(s) = \lim_{t \to \infty} F(t)$$

Periodic functions

Def^n.

If $F(t)$ is a f^n. that obeys the rule

$$F(t) = F(t+nT) \quad n=1,2,3, \ldots$$

for some real $T>0$ $\forall t$, then $F(t)$ is called a periodic f^n. with period T.

Theorem 12

If $F(t)$ be a periodic f^n. with period $T>0$

i.e. $F(t) = F(t+nT)$, then

$$L\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\,dt}{1-e^{-sT}}$$

Proof

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

$$= \int_0^T e^{-st} F(t)\,dt + \int_T^{2T} e^{-st} F(t)\,dt + \cdots$$

Put $t=u+T$

$t=u+2T$

in the 2nd, 3rd etc integral.

$$= \int_0^T e^{-st} F(t)\,dt + \int_0^T e^{-s(u+T)} F(u+T)\,du + \cdots$$

$$= \int_0^T e^{-st} F(t)\,dt + e^{-sT}\int_0^T e^{-su} F(u)\,du + \cdots$$

$$= \left[1+e^{-sT}+e^{-2sT}+\cdots\right]\int_0^T e^{-su} F(u)\,du$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st} F(t)\,dt$$

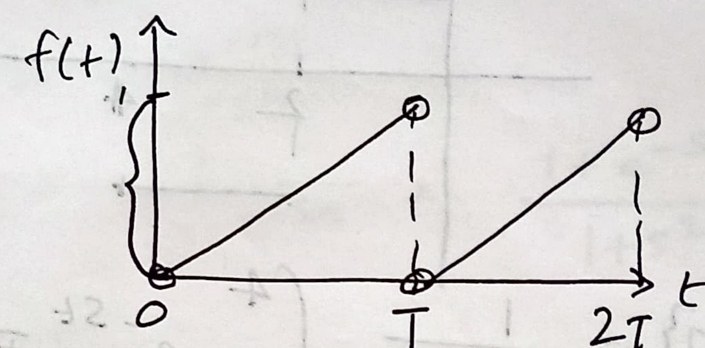
Problems on periodic f^ns.

Ex Find the L.T. of the saw-tooth f^n.

$$f(t) = \frac{t}{T} \text{ of period } T, \quad 0 < t < T$$

i.e. $f(t+T) = f(t)$

Sol^n:



$$L\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \frac{t}{T} \, dt$$

$$= \frac{1}{T(1-e^{-sT})} \int_0^T e^{-st} t \, dt$$

$$= \frac{1}{T(1-e^{-sT})} \left[\frac{te^{-st}}{-s} \Big|_0^T + \frac{1}{s} \int_0^T e^{-st} dt \right]$$

$$= \frac{1}{T(1-e^{-sT})} \left[\frac{Te^{-sT}}{-s} - \frac{1}{s^2} \left(e^{-sT} - 1 \right) \right]$$

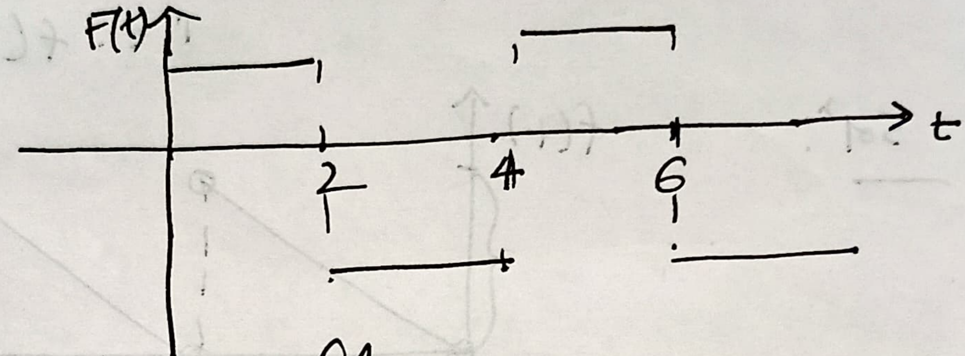
$$= \frac{1}{s^2 T} - \frac{e^{-sT}}{s(1-e^{-sT})}, \quad s > 0$$

Ex Find the L.T. of $F(t) = 1 \quad 0 \le t < 2$

$\qquad = -1 \quad 2 \le t < 4$

$$F(t+4) = F(t)$$

Sol^n:



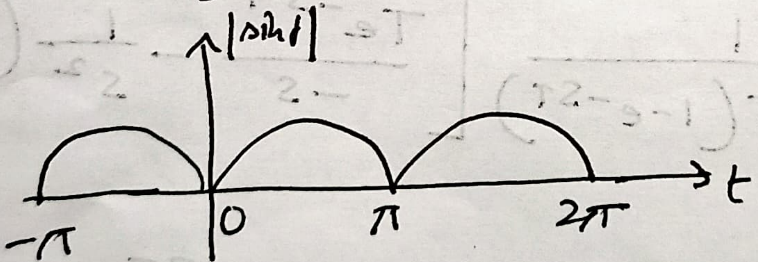
$$L\{F(t)\} = \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} F(t)\,dt$$

$$= \frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st} \cdot 1\,dt + \int_2^4 e^{-st}(-1)\,dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[-\frac{2e^{-2s}}{s} + \frac{e^{-4s}}{s} + \frac{1}{s} \right]$$

Ex Find $L[|\sin t|]$

Sol^n: $\qquad T = \pi$



$$L[|\sin t|] = \frac{1}{1-e^{-s\pi}} \underbrace{\int_0^\pi e^{-st} \sin t\,dt}_{I_1}$$

$$= \frac{1}{1-e^{-s\pi}} I_1$$

$$I_1 = \int_0^{\pi} e^{-st} \sin t \, dt$$

$$= \left[-e^{-st} \cos t\right]_0^{\pi} - s \int_0^{\pi} e^{-st} \cos t \, dt$$

$$= -\left[-e^{-s\pi} - 1\right] - s\left[e^{-st} \sin t\right]_0^{\pi} - s^2 \underbrace{\int_0^{\pi} e^{-st} \sin t \, dt}_{I_1}$$

$$(1+s^2) I_1 = 1 + e^{-\pi s}$$

$$I_1 = \frac{1+e^{-\pi s}}{1+s^2}$$

$$L\left[|\sin t|\right] = \frac{1}{1-e^{-s\pi}} \times \frac{1+e^{-s\pi}}{1+s^2}$$

Laplace transform of some special functions

1. Sine integral f^n. $\quad S_i(t) = \int_0^t \frac{\sin x}{x} \, dx$

Method I $\quad$ By infinite series (already done in lecture 3)

Method II $\quad$ Using $L\left\{\int_0^t F(x) \, dx\right\} = \frac{f(s)}{s}$ and

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(x) \, dx$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty} \frac{1}{x^2+1} \, dx = \left[\tan^{-1} x\right]_s^{\infty} = \frac{\pi}{2} - \tan^{-1} s$$

$$= \tan^{-1} \frac{1}{s}$$

$$L\left\{\int_0^t \frac{\sin x}{x} \, dx\right\} = \frac{1}{s} \tan^{-1} \frac{1}{s}$$

Method III   By Initial value theorem

Let $F(t) = \int_0^t \frac{\sin x}{x}\, dx$

$F(0) = 0 \qquad F'(t) = \frac{\sin t}{t}$

$\therefore t F'(t) = \sin t$

$L\{t F'(t)\} = L\{\sin t\}$

$\Rightarrow \quad -\frac{d}{ds}\{s f(s) - F(0)\} = \frac{1}{s^2+1}$

$\Rightarrow \quad -\frac{d}{ds}\{s f(s)\} = \frac{1}{s^2+1}$

$\Rightarrow \quad s f(s) = -\tan^{-1} s + c$

By initial value theorem,

$$\lim_{s \to \infty} s f(s) = \lim_{t \to 0} F(t) = F(0) = 0$$

$\therefore c - \tan^{-1}\infty = 0 \qquad \therefore c = \frac{\pi}{2}$

$s f(s) = \frac{\pi}{2} - \tan^{-1} s = \tan^{-1} \frac{1}{s}$

$f(s) = \frac{1}{s} \tan^{-1} \frac{1}{s}$

2. Cosine integral f^n:

$$Ci(t) = \int_t^\infty \frac{\cos x}{x}\,dx$$

Sol^n:

$$F(t) = \int_t^\infty \frac{\cos x}{x}\,dx$$

$$F'(t) = -\frac{\cos t}{t}$$

$$\therefore\ t\,F'(t) = -\cos t$$

$$\text{T.} \tfrac{d}{d}.\quad L\{t\,F'(t)\} = -L\{\cos t\}$$

$$\Rightarrow -\frac{d}{ds}\{s\,f(s) - F(0)\} = -\frac{s}{s^2+1}$$

$$\Rightarrow \frac{d}{ds}\{s f(s)\} = \frac{s}{s^2+1}$$

$$s f(s) = \frac{1}{2}\log(s^2+1) + c$$

By Final value theorem

$$\lim_{s\to 0} s f(s) = \lim_{t\to\infty} F(t) = 0 \quad \therefore c = 0$$

$$s f(s) = \frac{1}{2}\log(s^2+1)$$

$$\therefore f(s) = \frac{\log(s^2+1)}{2s}$$

3. Exponential integral function.

$$E(t) = \int_t^\infty \frac{e^{-x}}{x}\,dx$$

Let $F(t) = \int_t^\infty \frac{e^{-x}}{x}\,dx$

$$F'(t) = -\frac{e^{-t}}{t}$$

$$t\,F'(t) = -e^{-t}$$

$$L\{tF'(t)\} = -L\{e^{-t}\}$$

$$\Rightarrow -\frac{d}{ds}\{sf(s) - F(0)\} = -\frac{1}{s+1}$$

$$\Rightarrow \frac{d}{ds}\{sf(s)\} = \frac{1}{s+1}$$ $F(0) =$ constant value

$$\therefore sf(s) = \log(s+1) + c$$

$$\lim_{s\to 0} sf(s) = \lim_{s\to 0} \log(s+1) + c = 0 + c$$

By Final value theorem, $\lim_{s\to 0} sf(s) = \lim_{t\to\infty} F(t) = 0$

$\therefore c = 0$

$$\therefore sf(s) = \log(s+1)$$

$$f(s) = \frac{\log(s+1)}{s}$$

4. <u>Error f^n.</u>

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2}\,dx$$

$L\{\operatorname{erf}\sqrt{t}\} = ?$ $\quad L\{\operatorname{erf} t\} = ?$

$$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2}\,dx$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} \left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots\right] dx$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \cdots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}} \left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \cdots\right]$$

$$L\{\operatorname{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}} \left[\frac{\Gamma(\frac{3}{2})}{s^{3/2}} - \frac{\Gamma(\frac{5}{2})}{3.\,s^{5/2}} + \frac{\Gamma(\frac{7}{2})}{5.\,2!\,s^{7/2}} - \cdots\right]$$

$$= \frac{1}{s^{3/2}} \left[1 - \frac{1}{2}\frac{1}{s} + \frac{1.3}{2.4}\frac{1}{s^2} - \frac{1.3.5}{2.4.6}\frac{1}{s^3} \cdots\right]$$

$$= \frac{1}{s^{3/2}} \left(1 + \frac{1}{s}\right)^{-1/2}$$

$$= \frac{1}{s(s+1)^{1/2}}$$

$\operatorname{erf}(0) = 0 \qquad \operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1$

$$\operatorname{erf}_c(t) = \frac{2}{\sqrt{\pi}} \int_t^\infty e^{-x^2}\,dx = \frac{2}{\sqrt{\pi}} \left[\int_0^\infty e^{-x^2}\,dx - \int_0^t e^{-x^2}\,dx\right]$$

$$= 1 - \operatorname{erf}(t)$$

$$L\{\operatorname{erf}(t)\} = L\left\{ \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du \right\}$$

$$= \frac{2}{\sqrt{\pi}} L\left\{ \int_0^t e^{-u^2} du \right\}$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} L\left\{ e^{-u^2} \right\}$$

$$= \frac{2}{\sqrt{\pi}} \frac{1}{s} \int_0^\infty e^{-su} e^{-u^2} du$$

$$= \frac{2}{s\sqrt{\pi}} e^{\frac{s^2}{4}} \int_0^\infty e^{-\left(u+\frac{s}{2}\right)^2} du$$

Put $x = u + \frac{s}{2}$

$$L\{\operatorname{erf} t\} = \frac{2}{s\sqrt{\pi}} e^{\frac{s^2}{4}} \int_{s/2}^\infty e^{-x^2} dx$$

$$= \frac{1}{s} e^{\frac{s^2}{4}} \operatorname{erf}_c\left(\frac{s}{2}\right)$$

5. Unit step f^n.

Already done

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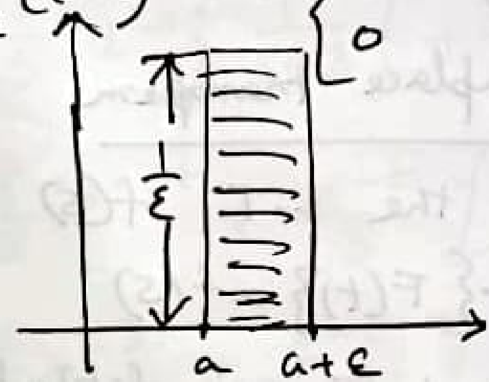
EXAMINATION (Mid-Semester / End-Semester)											SEMESTER (Autumn / Spring)
Roll Number									Section	Name	Koeli Ghoshal
Subject Number	M	A	2	0	2	0	2		Subject Name		Transform Calculus

Lectures 7 and 8 (continued) 23.1.2023

6. Unit impulse f^n. | Dirac delta f^n.

$$f_\varepsilon(t-a) = \begin{cases} \frac{1}{\varepsilon} & \text{if } a \le t \le a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$



$$I_\varepsilon = \int_0^\infty f_\varepsilon(t-a)\,dt = \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = 1$$

$$= \frac{1}{\varepsilon}[t]_a^{a+\varepsilon} = \frac{1}{\varepsilon}[\cancel{a}+\varepsilon-\cancel{a}] = \frac{\varepsilon}{\varepsilon} = 1$$

$$f_\varepsilon(t-a) = \frac{1}{\varepsilon}\left[u(t-a) - u\{t-(a+\varepsilon)\}\right]$$

$$L\{f_\varepsilon(t-a)\} = \frac{1}{\varepsilon s}\left[e^{-as} - e^{-(a+\varepsilon)s}\right]$$

$$= e^{-as}\,\frac{1-e^{-\varepsilon s}}{\varepsilon s} \quad \text{—(I)}$$

$$\lim_{\varepsilon \to 0} f_\varepsilon(t-a) = \delta(t-a) \rightarrow \text{Dirac delta} f^n$$

$$\delta(t-a) = \begin{cases} \infty & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$

$$L\{\delta(t-a)\} = e^{-as}$$

$$\int_0^\infty \delta(t-a)\,dt = 1$$

An ordinary function that is everywhere 0 except at a single pt. must have the integral 0. So Dirac delta fn. is not a fn. in the ordinary sense.

The inverse Laplace transform

If $F(t)$ has the L.T. $f(s)$ i.e

$$L\{F(t)\} = f(s)$$

then the inverse L.T. is defined by

$$L^{-1}\{f(s)\} = F(t)$$

Ex Find $L^{-1}\left\{\frac{s}{s^2+2} + \frac{6s}{s^2-16} + \frac{3}{s-3}\right\}$

$$= L^{-1}\left\{\frac{s}{s^2+(\sqrt{2})^2}\right\} + 6L^{-1}\left\{\frac{s}{s^2-4^2}\right\} + 3L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= \cos\sqrt{2}t + 6\cosh 4t + 3e^{3t}$$

Ex Find $L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\}$

$$= L^{-1}\left\{\frac{3s-2}{(s-2)^2+16}\right\}$$

$$= L^{-1}\left\{\frac{3(s-2)}{(s-2)^2+16} + \frac{4}{(s-2)^2+16}\right\}$$

Lecture-9

24.1.23

Convolution

Def^n

Let $F(t)$ and $G(t)$ be two f^n. of class A.
Then the convolution of the two f^ns. $F(t)$ and $G(t)$ denoted by $F * G$ is defined by the relation

$$F * G = \int_0^t F(x)\, G(t-x)\, dx$$

Property

$F * G$ is commutative

$$F * G = \int_0^t F(x)\, G(t-x)\, dx$$

Put $t - x = y$

$$= -\int_t^0 F(t-y)\, G(y)\, dy$$

$$= \int_0^t G(y)\, F(t-y)\, dy$$

Convolution theorem

Let $F(t)$ and $G(t)$ be two f^ns. of class A*
and let $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$.
Then $L^{-1}\{f(s)\, g(s)\} = \int_0^t F(x)\, G(t-x)\, dx = F * G$

i.e. $L\{F * G\} = f(s)\, g(s) = L\{F(t)\}\, L\{G(t)\}$

*(i.e. piecewise continuous function for $t \geq 0$ and of exponential order as $t \to \infty$)

$$= 3L^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2+4^2}\right\}$$

$$= 3e^{2t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$$

$$= 3e^{2t}\cos 4t + \cancel{4}e^{2t}\sin 4t$$

Ex Find $L^{-1}\left\{\dfrac{s-1}{(s+3)(s^2+2s+2)}\right\}$

$$= L^{-1}\left\{-\frac{4}{5(s+3)} + \frac{4s+1}{5(s^2+2s+2)}\right\}$$

$$= -\frac{4}{5}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{1}{5}\left\{\frac{4(s+1)-3}{(s+1)^2+1}\right\}$$

$$= -\frac{4}{5}e^{-3t} + \frac{e^{-t}}{5}\left[L^{-1}\left\{\frac{4s}{s^2+1} - \frac{3}{s^2+1}\right\}\right]$$

$$= -\frac{4}{5}e^{-3t} + \frac{e^{-t}}{5}\left[4\cos t - 3\sin t\right]$$

Ex Find $L^{-1}\left\{\dfrac{s}{(s+1)^{5/2}}\right\}$

$$= L^{-1}\left\{\frac{(s+1)-1}{(s+1)^{5/2}}\right\}$$

$$= e^{-t}L^{-1}\left\{\frac{s-1}{s^{5/2}}\right\}$$

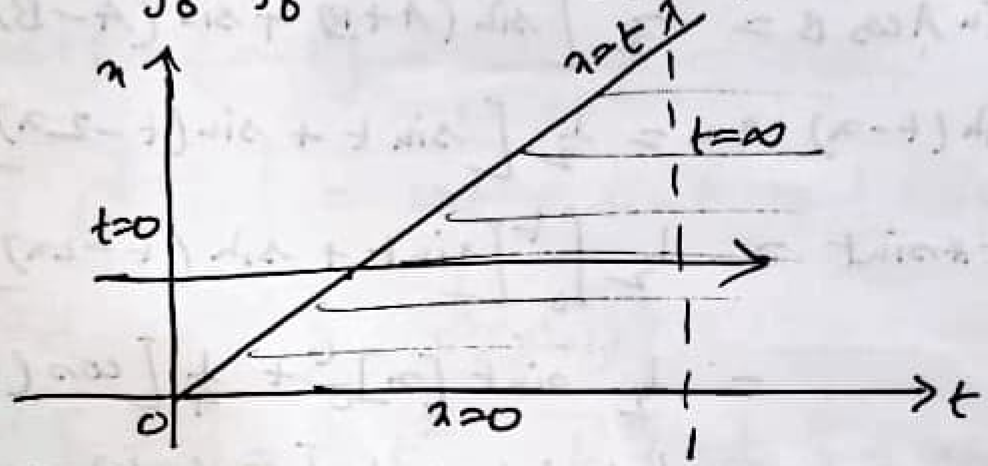
$$= e^{-t}L^{-1}\left\{\frac{1}{s^{3/2}} - \frac{1}{s^{5/2}}\right\}$$

$$= e^{-t}\frac{t^{3/2-1}}{\Gamma(\frac{3}{2})} - e^{-t}\frac{t^{5/2-1}}{\Gamma(\frac{5}{2})}$$

$$= 2e^{-t}\sqrt{\frac{t}{\pi}} - \frac{4}{3}e^{-t}t\sqrt{\frac{t}{\pi}} = \frac{2}{3}e^{-t}\sqrt{\frac{t}{\pi}}(3-2t)$$

Proof $L\{F(t)*G(t)\} = \int_0^\infty e^{-st}\left\{\int_0^t F(x)G(t-x)\,dx\right\}dt$

$$= \int_0^\infty \int_0^t e^{-st} F(x)G(t-x)\,dx\,dt$$



Changing the order of integration

$$L\{F(t)*G(t)\} = \int_0^\infty \int_{t=x}^\infty e^{-st} F(x)G(t-x)\,dt\,dx$$

$$= \int_0^\infty F(x)\left\{\int_x^\infty e^{-st} G(t-x)\,dt\right\}dx$$

Let $u = t-x$ $\therefore \int_x^\infty e^{-st} G(t-x)\,dt$

$$= \int_0^\infty e^{-s(u+x)} G(u)\,du$$

$$= e^{-sx}\int_0^\infty e^{-su} G(u)\,du$$

$$= e^{-sx} g(s)$$

$$L\{F*G\} = \int_0^\infty F(x)e^{-sx} g(s)\,dx$$

$$= g(s)f(s)$$

$$= f(s)g(s)$$

Ex Find the value of $\cos t * \sin t$

Solⁿ: $\cos t * \sin t = \int_0^t \cos x \sin(t-x)\,dx$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\sin(t-x)\cos x = \frac{1}{2}[\sin t + \sin(t-2x)]$

$\cos t * \sin t = \frac{1}{2}\int_0^t [\sin t + \sin(t-2x)]\,dx$

$= \frac{1}{2}\sin t\,[x]_0^t + \frac{1}{4}[\cos(t-2x)]_0^t$

$= \frac{1}{2}t\sin t + \frac{1}{4}[\cos(-t) - \cos t]$

$= \frac{1}{2}t\sin t$

Ex Find the value of $\sin t * t^2$

Solⁿ: $\sin t * t^2 = \int_0^t \sin x\,(t-x)^2\,dx$

$= [-(t-x)^2\cos x]_0^t - \int_0^t 2(t-x)\cos x\,dx$

$= t^2 - 2\left[\{(t-x)\sin x\}_0^t - \int_0^t \sin x\,dx\right]$

$= t^2 - 2\{0 + (-\cos x)_0^t\}$

$= t^2 + 2\cos t - 2$

Ex Find the value of $e^t * t$

Solⁿ: $e^t * t = \int_0^t e^x(t-x)\,dx$

$= te^x|_0^t - (xe^x - e^x)|_0^t$

$= e^t - t - 1$

Ex Find $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

Sol^n: $L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t \quad L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$$L\{\cos t\}\, L\{\sin t\} = \frac{s}{(s^2+1)^2}$$

$$L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \cos t * \sin t = \frac{1}{2} t \sin t$$

Ex Find $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

Sol^n: $L^{-1}\left\{\frac{1}{s^3}\right\} = \frac{1}{2}t^2 \quad L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \frac{1}{2} t^2 * \sin t$$

$$= \frac{1}{2}\left(t^2 + 2\cos t - 2\right)$$

Ex Find $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}$

Sol^n: $L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \quad L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = \int_0^t e^{-x} e^{2(t-x)}\,dx$$

$$= \int_0^t e^{2t} e^{-3x}\,dx$$

$$= e^{2t} \frac{e^{-3x}}{-3}$$

$$= \frac{1}{3}\left[e^{2t} - e^{-t}\right]$$

Ex Find $L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right]$

Solⁿ: 1st method $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$

$$L[e^{t}\operatorname{erf}\sqrt{t}] = \frac{1}{\sqrt{s}(s-1)}$$

$$L^{-1}\left[\frac{1}{\sqrt{s}(s-1)}\right] = e^{t}\operatorname{erf}(\sqrt{t})$$

2nd method (By convolution)

Let $f(s) = \frac{1}{\sqrt{s}}$ $\quad g(s) = \frac{1}{s-1}$

$G(t) = e^{t}$ $\quad L\{t^{-1/2}\} = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\} = \frac{1}{\sqrt{s}} \quad \therefore L^{-1}\left\{\frac{1}{\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} = F(t)$$

$$L^{-1}\{f(s)g(s)\} = \int_0^t F(x)\, G(t-x)\,dx$$

$$= \int_0^t \frac{1}{\sqrt{\pi x}} e^{(t-x)}\,dx$$

$$= \frac{e^{t}}{\sqrt{\pi}} \int_0^t \frac{e^{-x}}{\sqrt{x}}\,dx$$

Putting $x = u^2$

$$= \frac{e^{t}}{\sqrt{\pi}} \int_0^{\sqrt{t}} \frac{e^{-u^2}}{u} \cdot 2u\,du$$

$$= e^{t}\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-u^2}\,du$$

$$= e^{t}\operatorname{erf}\sqrt{t}$$

Lectures 10 and 11 30.1.2023

Ex Find $L^{-1}\left\{\frac{1}{s^2(s-1)}\right\}$

Solⁿ: $L\{t\} = \frac{1}{s^2}$ $L\{e^t\} = \frac{1}{s-1}$

$L^{-1}\left\{\frac{1}{s^2(s-1)}\right\} = t * e^t = e^t - t - 1$

Ex Apply Convolution theorem to prove that

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \quad m>0, n>0$$

Sol: Let $F(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx$

Let $F_1(t) = t^{m-1}$ $F_2(t) = t^{n-1}$

$F(t) = \int_0^t F_1(x)F_2(t-x)\,dx = F_1 * F_2$

$$\begin{aligned} L\{F(t)\} &= L\{F_1 * F_2\} \\ &= L\{F_1(t)\}.L\{F_2(t)\} \\ &= L\{t^{m-1}\}.L\{t^{n-1}\} \\ &= \frac{\Gamma(m)}{s^m}.\frac{\Gamma(n)}{s^n} = \frac{\Gamma(m)\Gamma(n)}{s^{m+n}} \end{aligned}$$

$$F(t) = \int_0^t x^{m-1}(t-x)^{n-1}\,dx = L^{-1}\left\{\frac{\Gamma(m)\Gamma(n)}{s^{m+n}}\right\}$$

Put $t=1$

$$= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\, t^{m+n-1}$$

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Applications

I Evaluation of improper integrals

Evaluate $\int_0^\infty e^{-x^2}\,dx$

Sol^n: Let $F(t) = \int_0^\infty e^{-tx^2}\,dx$

$$L\{F(t)\} = \int_0^\infty e^{-st}\left\{\int_0^\infty e^{-tx^2}\,dx\right\}dt$$

$$= \int_0^\infty \left\{\int_0^\infty e^{-st} e^{-tx^2}\,dt\right\}dx$$

$$= \int_0^\infty \left[L\{e^{-tx^2}\}\right]dx$$

$$= \int_0^\infty \frac{dx}{s+x^2} \qquad \left[L\{e^{at}\} = \frac{1}{s-a}\right]$$

$$= \left[\frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\right]_0^\infty = \frac{\pi}{2\sqrt{s}}$$

$$F(t) = \frac{\pi}{2} L^{-1}\left[\frac{1}{\sqrt{s}}\right]$$

$$= \frac{\pi}{2}\,\frac{1}{\sqrt{\pi t}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

$$\therefore \int_0^\infty e^{-tx^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}$$

Put $t=1$, $\therefore \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Ex Evaluate $\int_0^\infty \cos x^2\, dx$

Solⁿ: Let $F(t) = \int_0^\infty \cos tx^2\, dx$

$$L\{F(t)\} = \int_0^\infty e^{-st}\left\{\int_0^\infty \cos tx^2\, dx\right\} dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \cos tx^2\, dt\right] dx$$

$$= \int_0^\infty L\{\cos tx^2\}\, dx$$

$$= \int_0^\infty \frac{s}{s^2+x^4}\, dx$$

Put $x^2 = s\tan\theta$ $\quad x = \sqrt{s}\sqrt{\tan\theta}$

$$dx = \frac{s\sec^2\theta\, d\theta}{2\sqrt{(s\tan\theta)}}$$

$$I = \frac{1}{2\sqrt{s}}\int_0^{\pi/2} \frac{d\theta}{\sqrt{\tan\theta}}$$

$$= \frac{1}{2\sqrt{s}}\int_0^{\pi/2} \sin^{-\frac{1}{2}}\theta \cos^{\frac{1}{2}}\theta\, d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})}{2\Gamma(1)}$$

$$= \frac{1}{2\sqrt{s}} \frac{\Gamma(\frac{1}{4})\,\Gamma(1-\frac{1}{4})}{2}$$

$$= \frac{1}{4\sqrt{s}} \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2s}}$$

$$F(t) = \frac{\pi}{2\sqrt{2}} L^{-1}\left\{\frac{1}{\sqrt{s}}\right\}$$

$$= \frac{\pi}{2\sqrt{2}} \frac{t^{\frac{1}{2}-1}}{\Gamma(\frac{1}{2})} = \frac{1}{2}\sqrt{\frac{\pi}{2t}}$$

$t=1$ $\quad \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

$$\boxed{\begin{array}{l} \int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \dfrac{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})} \\ \because p>-1,\ q>-1 \\ \Gamma(n)\Gamma(1-n) = \dfrac{\pi}{\sin n\pi} \\ 0<n<1 \end{array}}$$

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EXAMINATION (Mid-Semester / End-Semester)	SEMESTER (Autumn / Spring)
Roll Number · Section · Name	Koeli Ghoshal
Subject Number MA20202 · Subject Name	Transform Calculus

Lectures 10 and 11 (continued) 30.1.2023

Ex Evaluate $\int_0^\infty \sin x^2\,dx$

Sol^n: Let $F(t) = \int_0^\infty \sin tx^2\,dx$

$$\therefore L\{F(t)\} = \int_0^\infty L\{\sin tx^2\}\,dx = \int_0^\infty \frac{x^2}{s^2+x^4}\,dx$$

Put $x = \sqrt{s\tan\theta}$ $\quad dx = \dfrac{s\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$

$$= \frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$$

$$= \frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sin^{\frac{1}{2}}\theta\cos^{-\frac{1}{2}}\theta\,d\theta$$

$$= \frac{1}{2\sqrt{s}}\,\frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{4})}{2\Gamma(1)}$$

$$= \frac{1}{2\sqrt{s}}\,\frac{\Gamma(1-\frac{1}{4})\,\Gamma(\frac{1}{4})}{2} = \frac{1}{4\sqrt{s}}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2s}}$$

$$\therefore F(t) = \frac{2}{2\sqrt{2}}\,\frac{1}{\sqrt{\pi t}}$$

$$\therefore \int_0^\infty \sin tx^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2t}}$$

Putting $t=1$, $\int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

II Solution of ODEs (IVP and BVP)

(a) Sol^n of linear ode with constant coefficients

(i) First order ODE

Ex Use L.T. to solve the IVP

$$\frac{dy}{dt} + 3y = 13 \sin 2t \qquad y(0) = 6$$

Sol^n: Taking L.T. of both the sides,

$$L\left\{\frac{dy}{dt}\right\} + 3L\{y\} = 13\, L\{\sin 2t\} \qquad L\{y(t)\} = Y(s)$$

$$\Rightarrow sY(s) - y(0) + 3L\{y\} = 13 \frac{2}{s^2+4}$$

$$\Rightarrow sY(s) - 6 + 3Y(s) = \frac{26}{s^2+4}$$

$$\Rightarrow (s+3)Y(s) = 6 + \frac{26}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{6}{s+3} + \frac{26}{(s^2+4)(s+3)}$$

$$= \frac{8}{s+3} - \frac{2s}{s^2+4} + \frac{6}{s^2+4}$$

Taking inverse L.T.

$$y(t) = 8e^{-3t} - 2\cos 2t + 3\sin 2t$$

(ii) Second order ODE

Ex. Solve $\frac{d^2y}{dt^2} + y = 0$, $y(0) = 1$, $y'(0) = 0$

Sol^n: $L\{y''\} + L\{y\} = 0$

$\Rightarrow s^2 L\{y\} - s\,y(0) - y'(0) + L\{y\} = 0$

$\Rightarrow s^2 L\{y\} - s + L\{y\} = 0$

$\Rightarrow L\{y\} = \frac{s}{s^2+1}$ $\quad \therefore y = L^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$

Ex. Solve $(D^2 - 2D + 2)y = 0$ $\quad y = Dy = 1$ at $t = 0$

Sol^n: $L\{y''\} - 2L\{y'\} + 2L\{y\} = 0$

$\Rightarrow s^2 L\{y\} - s\,y(0) - y'(0) - 2[s\,L\{y\} - y(0)] + 2L\{y\} = 0$

$\Rightarrow s^2 L\{y\} - s - 1 - 2s\,L\{y\} + 2 + 2L\{y\} = 0$

$\Rightarrow L\{y\} = \frac{s-1}{s^2-2s+2} = \frac{s-1}{(s-1)^2+1}$

$y = L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\}$

$= e^t L^{-1}\left\{\frac{s}{s^2+1}\right\}$

$\therefore y = e^t \cos t$

Ex Solve $(D^2+D)y = t^2 + 2t$ $y(0)=4,\ y'(0)=-2$

Solⁿ: $s^2 Y(s) - s\,y(0) - y'(0) + sY(s) - y(0) = \frac{2}{s^3} + \frac{2}{s^2}$

$\Rightarrow s^2 Y(s) - 4s + 2 + sY(s) - 4 = \frac{2(1+s)}{s^3}$

$\Rightarrow Y(s) = \left[\frac{2(1+s)}{s^3} + 4s + 2\right]\frac{1}{s(s+1)}$

$= \frac{2}{s^4} + 2\left(\frac{1}{s} + \frac{1}{s+1}\right)$

$y(t) = 2 + 2e^{-t} + \frac{t^3}{3}$

(b) Solⁿ of linear ODE with variable coefficients

Ex Solve $[tD^2 + (1-2t)D - 2]y = 0$
$y(0)=1 \quad y'(0)=2$

Solⁿ: $ty'' + y' - 2ty' - 2y = 0$

Taking L.T., $-\frac{d}{ds}[s^2 Y(s) - s\,y(0) - y'(0)]$
$+ [sY(s) - y(0)] + 2\frac{d}{ds}[sY(s) - y(0)]$
$- 2Y(s) = 0$

$\Rightarrow -\frac{d}{ds}[s^2 Y(s) - s - 2] + [sY(s) - 1]$
$+ 2\frac{d}{ds}[sY(s) - 1] - 2Y(s) = 0$

$\Rightarrow -s^2 \frac{dY(s)}{ds} - 2sY(s) + \cancel{1} + sY(s) - \cancel{1} + \cancel{2Y(s)}$
$+ 2s\frac{dY(s)}{ds} - \cancel{2Y(s)} = 0$

$\Rightarrow -(s^2 - 2s)\frac{dY(s)}{ds} - sY(s) = 0$

$\Rightarrow \frac{dY(s)}{ds} + \frac{1}{s-2}Y(s) = 0$

$$\ln Y(s) + \ln(s-2) = \ln c_1$$

$$\Rightarrow Y(s) = \frac{c_1}{s-2}$$

$$\therefore y(t) = c_1 L^{-1}\left\{\frac{1}{s-2}\right\} = c_1 e^{2t}$$

$y(0)=1$ $\quad c_1 = 1 \quad \therefore y(t) = e^{2t}$

Ex Solve $ty'' + y' + 4ty = 0 \quad y(0) = 3,\ y'(0) = 0$

Sol^n: $L\{ty''\} + L\{y'\} + 4L\{ty\} = 0$

$$\Rightarrow -\frac{d}{ds}\left[s^2Y(s) - s\,y(0) - y'(0)\right] + \left[sY(s) - y(0)\right] - 4\frac{dY(s)}{ds} = 0$$

$$\Rightarrow -\frac{d}{ds}\left[s^2Y(s) - 3s\right] + \left[sY(s) - 3\right] - 4\frac{dY(s)}{ds} = 0$$

$$\Rightarrow -(s^2+4)\frac{dY(s)}{ds} - sY(s) = 0$$

$$\Rightarrow \frac{dY(s)}{Y(s)} + \frac{s}{s^2+4}\,ds = 0$$

$$\therefore \ln Y(s) + \frac{1}{2}\ln(s^2+4) = \ln c_1$$

$$\therefore Y(s) = \frac{c_1}{\sqrt{s^2+4}}$$

$$y = c_1 L^{-1}\left\{\frac{1}{\sqrt{s^2+4}}\right\}$$

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Sol^n. of Bessel eqn for $n=0$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0$$

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$J_0(x)$ is called the Bessel f^n. of order zero.

Ex Prove that $L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$

and hence deduce that

(i) $L\{J_0(at)\} = \frac{1}{\sqrt{s^2+a^2}}$

(ii) $L\{t\, J_0(at)\} = \frac{s}{(s^2+a^2)^{3/2}}$

(iii) $L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{(s^2+2as+2a^2)}}$

and (iv) $\int_0^\infty J_0(t)\, dt = 1$

Sol^n: $J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2 \cdot 4^2} - \frac{t^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$

$$L\{J_0(t)\} = \frac{1}{s} - \frac{1}{2^2}\frac{2!}{s^3} + \frac{1}{2^2 \cdot 4^2}\frac{4!}{s^5} - \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\frac{1}{s^2} + \frac{1 \cdot 3}{2 \cdot 4}\left(\frac{1}{s^2}\right)^2 - \cdots\right]$$

$$= \frac{1}{s}\left[1 + \frac{1}{s^2}\right]^{-1/2}$$

$$= \frac{1}{\sqrt{1+s^2}}$$

The Bessel function

The diff. eqn.

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$$ where n is a non-ve const.

is known as Bessel's eqn. of order n.

When n is not integer or 0, the complete solⁿ. of Bessel's eqn. is given by

$$y = A\, J_n(x) + B\, J_{-n}(x)$$ $\because$ J_n, J_{-n} are independent

where $$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$$

$$= \frac{x^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{2(2n+2)} + \frac{x^4}{2.4.(2n+2)(2n+4)} \cdots\right]$$

is called the Bessel fⁿ. of 1st kind of order n.

When n is integer or 0, $J_n(x)$ and $J_{-n}(x)$ are not independent and $J_{-n}(x) = (-1)^n J_n(x)$

Complete solⁿ. $$y = A\, J_n(x) + B\, Y_n(x)$$

$\downarrow$

Bessel fⁿ. of 2nd kind of order n or Neumann fⁿ.

(i) $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

$\therefore L\{J_0(at)\} = \frac{1}{a} \frac{1}{\sqrt{1+\left(\frac{s}{a}\right)^2}} = \frac{1}{\sqrt{s^2+a^2}}$

(ii) $L\{t\, J_0(at)\} = -\frac{d}{ds} L\{J_0(at)\}$

$= -\frac{d}{ds}\left[\frac{1}{\sqrt{s^2+a^2}}\right]$

$= \frac{s}{(s^2+a^2)^{3/2}}$

(iii) $L\{e^{-at} F(t)\} = f(s+a)$

$\therefore L\{e^{-at} J_0(at)\} = \frac{1}{\sqrt{[(s+a)^2+a^2]}} = \frac{1}{\sqrt{(s^2+2as+2a^2)}}$

(iv) $L\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$

$\int_0^\infty J_0(t)\,dt = 1$ putting $s=0$

Ex. Prove that $L\{J_1(t)\} = 1 - \frac{s}{\sqrt{(s^2+1)}}$ where $J_1(t)$ is a Bessel f^n. of order one and hence deduce that

$L\{t\, J_1(t)\} = \frac{1}{(s^2+1)^{3/2}}$

Sol^n: $J_0'(t) = -J_1(t)$

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$$L\{J_1(t)\} = L\{-J_0'(t)\}$$
$$= -L\{J_0'(t)\}$$
$$= -\left[sL\{J_0(t)\} - J_0(0)\right]$$
$$= -\left[s\frac{1}{\sqrt{s^2+1}} - 1\right]$$
$$= 1 - \frac{s}{\sqrt{s^2+1}} \quad [J_0(0)=1]$$

$$L\{tJ_1(t)\} = -\frac{d}{ds}L\{J_1(t)\}$$
$$= -\frac{d}{ds}\left[1 - \frac{s}{\sqrt{s^2+1}}\right]$$
$$= \frac{1}{(s^2+1)^{3/2}}$$

Continuation of the solution of

$$ty'' + y' + 4ty = 0 \qquad y(0)=3,\ y'(0)=0$$

$$y(t) = c\,L^{-1}\left\{\frac{1}{\sqrt{s^2+4}}\right\}$$
$$= c\,J_0(2t)$$

$y(0)=3$ $\quad \therefore 3 = c\,J_0(0) = c$

$\therefore y = 3.J_0(2t)$

(c) Solⁿ. of simultaneous ODEs

(i) System of 1st order ODEs

Ex $\frac{dy}{dt} = -z$ with $y(0) = 1$

$\frac{dz}{dt} = y$ $\quad z(0) = 0$

Solⁿ: $L(y') = -L(z)$

$\Rightarrow sL(y) - 1 = -L(z)$

and $L(z') = L(y)$ i.e. $sL(z) = L(y)$

$$\left.\begin{aligned} sL(y) + L(z) &= 1 \\ L(y) - sL(z) &= 0 \end{aligned}\right\}$$

$sL(y) + L(z) = 1$

$sL(y) - s^2L(z) = 0$

$\underline{- \qquad +}$

$(s^2+1)\,L(z) = 1$

$L(z) = \frac{1}{s^2+1}$

$z = \sin t$

$y = \cos t$

Ex Solve $y' + z' + y + z = 1$ $\quad y(0) = -1$

$y' + z = e^t$ $\quad z(0) = 2$

Solⁿ: $sL(y) + 1 + sL(z) - 2 + L(y) + L(z) = \frac{1}{s}$

$sL(y) + 1 + L(z) = \frac{1}{s-1}$

Solving $L(y) = \frac{-s^2+s+1}{s(s-1)^2} = \frac{1}{s} - \frac{2}{s-1} + \frac{1}{(s-1)^2}$

$y = 1 - 2e^t + te^t$ $\qquad z = 2e^t - te^t$

(ii) system of 2nd order ODEs $\quad D \equiv \frac{d}{dt}$, $D^2 \equiv \frac{d^2}{dt^2}$

Ex Solve $(D^2-3)x - 4y = 0 \qquad x(0)=0 \quad x'(0)=2$

$x + (D^2+1)y = 0 \qquad y(0)=0 \quad y'(0)=0$

Solⁿ: $L\{x''\} - 3L\{x\} - 4L\{y\} = 0 \qquad \bar{x} = L\{x\}$

and $L\{x\} + L\{y''\} + L\{y\} = 0 \qquad \bar{y} = L\{y\}$

or, $s^2\bar{x} - s\,x(0) - x'(0) - 3\bar{x} - 4\bar{y} = 0$

and $\bar{x} + s^2\bar{y} - s\,y(0) - y'(0) + \bar{y} = 0$

or, $(s^2-3)\bar{x} - 4\bar{y} = 2$

$\bar{x} + (s^2+1)\bar{y} = 0$

Solving for $\bar{x}$ and $\bar{y}$

$$\bar{x} = \frac{2(s^2+1)}{(s^2-1)^2} = \frac{1}{(s-1)^2} + \frac{1}{(s+1)^2}$$

$$\bar{y} = \frac{-2}{(s+1)^2(s-1)^2} = \frac{1}{2}\left[-\frac{1}{s+1} + \frac{1}{s-1} - \frac{1}{(s+1)^2} - \frac{1}{(s-1)^2}\right]$$

$$x = e^t L^{-1}\left\{\frac{1}{s^2}\right\} + e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} = (e^t + e^{-t})\,t$$

$$y = \frac{1}{2}\left[-e^{-t} + e^t - te^{-t} - te^t\right]$$

$$\left.\begin{aligned} x &= (e^t + e^{-t})\,t \\ y &= \tfrac{1}{2}(1-t)e^t - \tfrac{1}{2}(1+t)e^{-t} \end{aligned}\right\}$$

Lectures 13 and 14 6.2.2023

Integral equation and integro-differential equation

An integral eqn. is an eqn. in which the unknown function, to be determined, appears under one or more integral signs. If the derivatives of the f^n. are involved, it is called an integro-differential eqn.

Volterra integral eqn.

$$F(x) = \varphi(x) + \lambda \int_a^x K(x,\xi) F(\xi)\, d\xi$$

Fredholm integral eqn

$$F(x) = \varphi(x) + \lambda \int_a^b K(x,\xi) F(\xi)\, d\xi$$

Find $1 * 1 * 1 \cdots$ n times, n is a +ve integer

Sol^n: $F(t) = 1 \quad G(t) = 1$

$$1 * 1 = F * G = \int_0^t F(x) G(t-x)\, dx = \int_0^t 1 \times 1\, dx = t$$

$$(1*1)*1 = t * 1 = \int_0^t x \cdot 1\, dx = \frac{t^2}{2}$$

$$(1*1*1)*1 = \frac{t^2}{2} * 1 \quad \int_0^t \frac{x^2}{2} \cdot 1\, dx = \frac{t^3}{3!}$$

$$1*1*1* \cdots \text{ n times} = \frac{t^{n-1}}{(n-1)!}$$

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Signature of the Invigilator

EXAMINATION (Mid-Semester / End-Semester)		SEMESTER (Autumn / Spring)	
Roll Number		Section		Name	Koeli Ghoshal
Subject Number	MA20202	Subject Name	Transform Calculus

Lectures 13 and 14 (continued) 6.2.2023

III Solⁿ. of integral equations (convolution type)

Ex Solve $F(t) = 1 + \int_0^t F(u)\sin(t-u)\,du$

Solⁿ: $F(t) = 1 + F(t) * \sin t$

$\therefore L\{F(t)\} = L\{1\} + L\{F(t) * \sin t\}$

$= \frac{1}{s} + L\{F(t)\}\frac{1}{s^2+1}$

$\Rightarrow \left[1 - \frac{1}{s^2+1}\right] L\{F(t)\} = \frac{1}{s}$

$\Rightarrow L\{F(t)\} = \frac{s^2+1}{s^3}$

$\therefore F(t) = L^{-1}\left\{\frac{s^2+1}{s^3}\right\} = L^{-1}\left\{\frac{1}{s}\right\} + L^{-1}\left\{\frac{1}{s^3}\right\}$

$= 1 + \frac{t^2}{2}$

Ex Solve $2F(t) = 2 - t + \int_0^t F(t-u)\, F(u)\,du$

Solⁿ: $2\,L\{F(t)\} = \frac{2}{s} - \frac{1}{s^2} + f(s)\cdot f(s)$

$\Rightarrow 2f(s) = \frac{2}{s} - \frac{1}{s^2} + f(s)\cdot f(s)$

$\Rightarrow \{f(s) - 1\}^2 = \frac{1}{s^2} - \frac{2}{s} + 1 = \frac{1-2s+s^2}{s^2}$

$\Rightarrow [f(s) - 1]^2 = \frac{(1-s)^2}{s^2}$

$\Rightarrow f(s) - 1 = \frac{1-s}{s} = \frac{1}{s} - 1$

$\Rightarrow f(s) = \frac{1}{s} \quad \therefore F(t) = 1$

IV Solⁿ of integro-differential eqn (Convolution type)

Ex $F'(t) = t + \int_0^t F(t-u)\cos u\,du$

$F(0) = 4$

Solⁿ: $F'(t) = t + F(t) * \cos t$

$L\{F'(t)\} = L\{t\} + L\{F(t)\}\, L\{\cos t\}$

$\Rightarrow s f(s) - 4 = \frac{1}{s^2} + f(s)\,\frac{s}{s^2+1}$

$\Rightarrow f(s)\,.\,\frac{s^3}{s^2+1} = \frac{1}{s^2} + 4$

$\Rightarrow f(s) = \frac{s^2+1}{s^5} + \frac{4(s^2+1)}{s^3} = \frac{5}{s^3} + \frac{1}{s^5} + \frac{4}{s}$

$F(t) = 4 + \frac{5t^2}{2} + \frac{t^4}{4!}$

Ex Solve $F'(t) = \sin t + \int_0^t F(t-u)\cos u \, du$

$F(0)=0$

Sol^n: $F'(t) = \sin t + F(t) * \cos t$

$L\{F'(t)\} = L\{\sin t\} + L\{F(t)\} L\{\cos t\}$

$\Rightarrow s f(s) - F(0) = \frac{1}{s^2+1} + f(s)\frac{s}{s^2+1}$

$\Rightarrow \left[s - \frac{s}{s^2+1} \right] f(s) = \frac{1}{s^2+1}$

$\Rightarrow f(s) = \frac{1}{s^3}$

$F(t) = \frac{t^2}{2}$

Q. Solve $y''(t) + y'(t) = \delta(t-4)$ $\quad y(0)=4$

$y'(0)=4$

Sol^n: $L\{\delta(t-4)\} = e^{-4s}$

$L\{y''(t)\} + L\{y'(t)\} = L\{\delta(t-4)\}$

$\Rightarrow s^2 L\{y(t)\} - s y(0) - y'(0) + sL\{y(t)\} - y(0) = e^{-4s}$

$\Rightarrow L\{y(t)\} = \frac{e^{-4s} + 4(s+1) + 4}{s(s+1)}$

$= \frac{e^{-4s}}{s(s+1)} + \frac{4}{s} + \frac{4}{s(s+1)}$

$\therefore y(t) = 4 + 4(1-e^{-t}) + H_4(t)\left[1 - e^{-(t-4)}\right]$

$H_4(t) = u(t-4) = \begin{cases} 0 & t<4 \\ 1 & t>4 \end{cases}$

Fourier Series

Euler's formula

$\infty \quad \alpha$

The F.S. for the f^n. $f(x)$ in the interval $\alpha < x < \alpha + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}^{\infty} a_n \cos nx + \sum_{n\geq 1}^{\infty} b_n \sin nx \quad \text{— (I)}$$

where $a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\, dx$ — (A)

$$a_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\, dx \quad \text{— (B)}$$

$$b_n = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\, dx \quad \text{— (C)}$$

∴ The values of a_0, a_n, b_n are known as Euler's formulae.

To establish these formulae, the following definite integrals will be required:

1. $\int_{\alpha}^{\alpha+2\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_{\alpha}^{\alpha+2\pi} = 0$

2. $\int_{\alpha}^{\alpha+2\pi} \sin nx\, dx = 0$

3. $\int_{\alpha}^{\alpha+2\pi} \cos mx \cos nx\, dx \qquad m \neq n$

$= 0$

4. $\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx = \pi$

5. $\int_{\alpha}^{\alpha+2\pi} \sin mx \cos nx\,dx = 0 \quad \forall\, m,n$

6. $\int_{\alpha}^{\alpha+2\pi} \sin nx \cos nx\,dx = 0$

7. $\int_{\alpha}^{\alpha+2\pi} \sin mx \sin nx\,dx = 0 \quad m \neq n$

8. $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\,dx = \pi \quad n \neq 0$

<u>Proof</u> Let $f(x)$ be represented in the interval $(\alpha, \alpha+2\pi)$ by the F.S.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (i)}$$

To find a_0, integrate both sides of (i) from $x=\alpha$ to $x=\alpha+2\pi$. Then

$$\int_{\alpha}^{\alpha+2\pi} f(x)\,dx = \frac{1}{2} a_0 \int_{\alpha}^{\alpha+2\pi} dx + \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} a_n \cos nx\right] dx$$

$$+ \int_{\alpha}^{\alpha+2\pi} \left[\sum_{n=1}^{\infty} b_n \sin nx\right] dx$$

$$= \frac{1}{2} a_0 (\alpha+2\pi-\alpha) + 0 + 0 \quad [\text{By (1) \& (2)}]$$

$$= a_0 \pi$$

$$\therefore a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

To find a_n, multiply each side of (i) by $\cos nx$ and integrate from $x=\alpha$ to $x=\alpha+2\pi$.

$$\int_\alpha^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{1}{2}a_0\int_\alpha^{\alpha+2\pi}\cos nx\,dx$$
$$+\int_\alpha^{\alpha+2\pi}\left[\sum_{n=1}^{\infty} a_n\cos nx\right]\cos nx\,dx$$
$$+\int_\alpha^{\alpha+2\pi}\left[\sum_{n=1}^{\infty} b_n\sin nx\right]\cos nx\,dx$$
$$=\pi a_n$$

$$\therefore a_n = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\cos nx\,dx$$

To find b_n, multiply each side of (i) by $\sin nx$ and integrate from $x=\alpha$ to $x=\alpha+2\pi$.

$$\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx = \frac{1}{2}a_0\int_\alpha^{\alpha+2\pi}\sin nx\,dx$$
$$+\int_\alpha^{\alpha+2\pi}\left[\sum_{n=1}^{\infty} a_n\cos nx\right]\sin nx\,dx$$
$$+\int_\alpha^{\alpha+2\pi}\left[\sum_{n=1}^{\infty} b_n\sin nx\right]\sin nx\,dx$$
$$=\pi b_n$$

$$b_n = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx$$

Cor - I $\quad a=0$ $\quad 0<x<2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Cor II $\quad \alpha = -\pi$ $\quad -\pi < x < \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Lecture - 15

7.2.2023

Convergence of Fourier series

Theorem (sufficient condition)

If a periodic function $f(x)$ with period 2π is PWC in the interval $-\pi \le x \le \pi$ and has a left hand derivative and right hand derivative at each point of that interval, the the F.S. (I) of $f(x)$ with coefficients (A), (B) and (C) is convergent. It's sum is $f(x)$, except at a point x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left and right hand limits of $f(x)$ at x_0.

Proof We prove the convergence for a continuous function $f(x)$ having continuous first and second derivatives.

Now $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \left.\frac{f(x)\sin nx}{n\pi}\right|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx\,dx$$

The 1st term on the right is zero.

Another integration by parts gives

$$a_n = \left.\frac{f'(x)\cos nx}{n^2\pi}\right|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx\,dx$$

The 1st term on the right is zero because of the periodicity and continuity of $f'(x)$. Since f'' is continuous in the interval of integration, we have

$|f''(x)| < M$ [A continuous f^n. on a closed bounded interval is bounded]

for a constant M. Furthermore, $|\cos nx| \le 1$

It follows that

$$|a_n| = \frac{1}{n^2\pi}\left|\int_{-\pi}^{\pi} f''(x)\cos nx\,dx\right| < \frac{1}{n^2\pi}\int_{-\pi}^{\pi} M\,dx = \frac{2M}{n^2}$$

$$\left[\left|\int_a^b f(x)g(x)\,dx\right| \le \int_a^b |f(x)||g(x)|\,dx\right]$$

Similarly $|b_n| < \frac{2M}{n^2}$ $\forall n$.

Hence the absolute value of each term of the F.S. of $f(x)$ is at most equal to the corresponding term of the series $|a_0| + 2M\left(1+1+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\cdots\right)$ which is convergent. $\left[\sum_{n=1}^{\infty}\frac{1}{n^p},\ p>1 \text{ is conv}\right]$

Hence the F.S. converges.

Lectures 16 & 17

27.2.2023

If in the interval $(\alpha, \alpha+2\pi)$ $f(x)$ is defined by

$$f(x) = \varphi(x) \qquad \alpha < x < c$$
$$= \psi(x) \qquad c < x < \alpha+2\pi$$

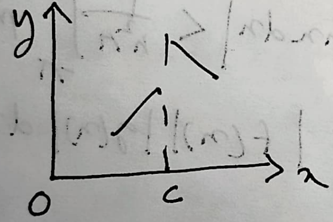
i.e. c is a point of discontinuity, then

$$a_0 = \frac{1}{\pi}\left[\int_\alpha^c \varphi(x)\,dx + \int_c^{\alpha+2\pi} \psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_\alpha^c \varphi(x)\cos nx\,dx + \int_c^{\alpha+2\pi} \psi(x)\cos nx\,dx\right]$$

and $$b_n = \frac{1}{\pi}\left[\int_\alpha^c \varphi(x)\sin nx\,dx + \int_c^{\alpha+2\pi} \psi(x)\sin nx\,dx\right]$$

At the point c, $f(x) = \frac{1}{2}\left[f(c-0) + f(c+0)\right]$



Ex Find the F.S. of the periodic f^n with period 2π defined as

$$f(x) = 0 \qquad -\pi < x \le 0$$
$$= x \qquad 0 \le x < \pi$$

Hence deduce that $1 + \frac{1}{3^2} + \cdots = \frac{\pi^2}{8}$

Sol^n: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}$

$a_n = \frac{1}{\pi}\int_0^\pi x\cos nx\,dx = 0$ for n even

$= \frac{1}{\pi n}\left[\frac{\cos nx}{n}\right]_0^\pi \qquad = -\frac{2}{\pi n^2}$ for n odd

$b_n = -\frac{\cos n\pi}{n} \qquad = -\frac{1}{n}$ n even

$\qquad\qquad\qquad\quad \frac{1}{n}$ n odd

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right\}$$
$$+ \left\{\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right\}$$

At $x=0$, $f(0)=0$

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right\}$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \cdots$$

Fourier series for even and odd functions

<u>Even f^ns</u>

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0$$

a_0 will be there.

So F.S of an even f^n. consists of terms of cosines only. $\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

<u>Odd f^ns.</u>

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$\therefore$ F.S. of an odd f^n. consists of sine terms only.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

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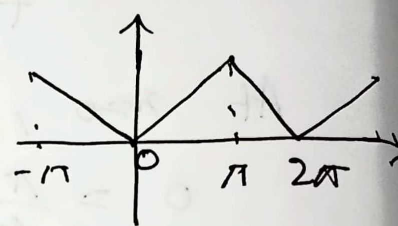
Ex Find the F.S. generated by the periodic f^n $|x|$ of period 2π $(-\pi, \pi)$ and compute the value of the series at 0, 2π.

Sol^n.

$$f(x) = \begin{cases} -x & -\pi \le x \le 0 \\ x & 0 \le x \le \pi \end{cases}$$



It is an even f^n and therefore F.S. will consists of cosine terms only.

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx = \frac{2}{\pi n^2}(\cos n\pi - 1)$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ -\frac{4}{\pi n^2} & \text{for } n \text{ odd} \end{cases}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Sum of the series at 2π is the same as at 0.

At 0, $$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right]$$

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Signature of the Invigilator

EXAMINATION (Mid-Semester / End-Semester)	SEMESTER (Autumn / Spring)
Roll Number	Section	Name: Koeli Ghoshal
Subject Number: MA20202	Subject Name: Transform Calculus

27.2.2023

Letures 16 & 17 (continued)

Functions having arbitrary periods

Let $f(x)$ be defined in $(-l, l)$ i.e. $f(x)$ is a periodic function having period $2l$ where l is any +ve number.

Let $z = \frac{\pi x}{l}$ Hence $x = \frac{lz}{\pi}$

When $x = -l$, $z = -\pi$

$x = l$, $z = \pi$

$f(x) = f\left(\frac{lz}{\pi}\right) = F(z)$ which is defined in $(-\pi, \pi)$

$\therefore$ F.S. of $F(z)$ is given by

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\,dz$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\cos nz\,dz$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(z)\sin nz\,dz$

$$\therefore f\left(\frac{lz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nz + b_n \sin nz)$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) dz$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right)\cos nz\,dz$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right)\sin nz\,dz$

We go back to the variable x by using $x = \frac{lz}{\pi}$

so that $dx = \frac{l}{\pi} dz$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \right]$$

where $a_0 = \frac{1}{l} \int_{-l}^{l} f(x)\, dx$

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

and $b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

The above formulae are valid for any interval of length $2l$ namely $(c, c+2l)$.

$$a_0 = \frac{1}{l} \int_{c}^{c+2l} f(x)\, dx$$

$$a_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \cos \frac{n\pi x}{l}\, dx$$

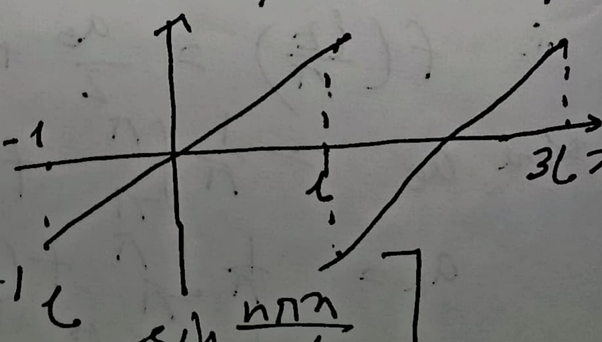
$$b_n = \frac{1}{l} \int_{c}^{c+2l} f(x) \sin \frac{n\pi x}{l}\, dx$$

Ex If $f(x) = x$ is defined in $-l < x < l$ with period $2l$, find the F.S. of $f(x)$.

Solⁿ: Since $f(x)$ is an odd f^n, $a_n = 0\ \forall n \geq 0$

$$b_n = \frac{2}{l} \int_0^l x \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2(-1)^{n+1} l}{n\pi}$$

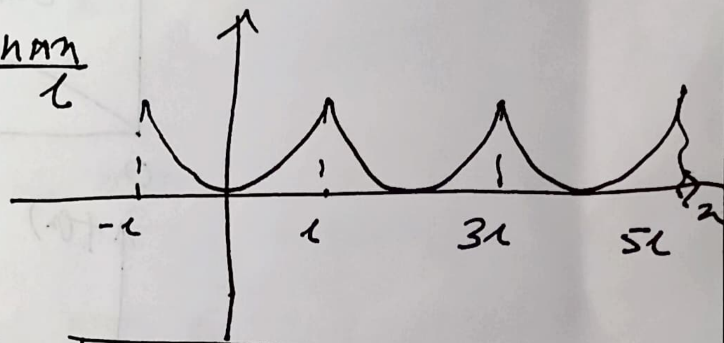


F.S. $f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{(-1)^{n+1} l}{n} \sin \frac{n\pi x}{l} \right]$

Ex Find a F.S. to represent a periodic f^n. x^2 in the interval $(-l, l)$.

Sol^n: Since $f(x)$ is even in $(-l, l)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$



$$a_0 = \frac{2}{l}\int_0^l x^2\,dx = \frac{2l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l x^2 \cos\frac{n\pi x}{l}\,dx$$

$$= 4l^2 \frac{(-1)^n}{n^2\pi^2}$$

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - - -$$

$$a_1 = -\frac{4l^2}{\pi^2}$$

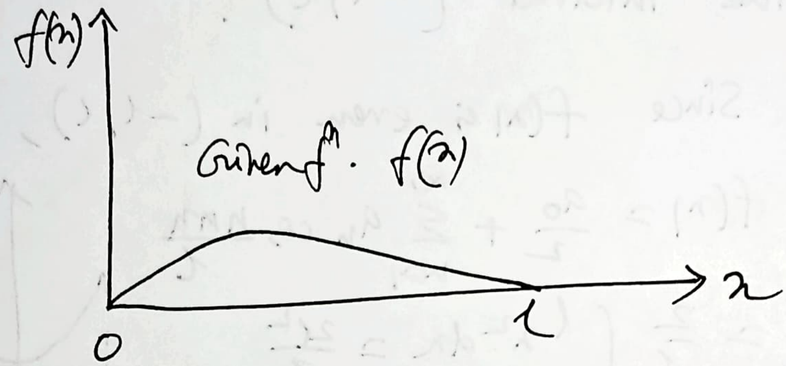
$$a_2 = \frac{4l^2}{2^2\pi^2}$$

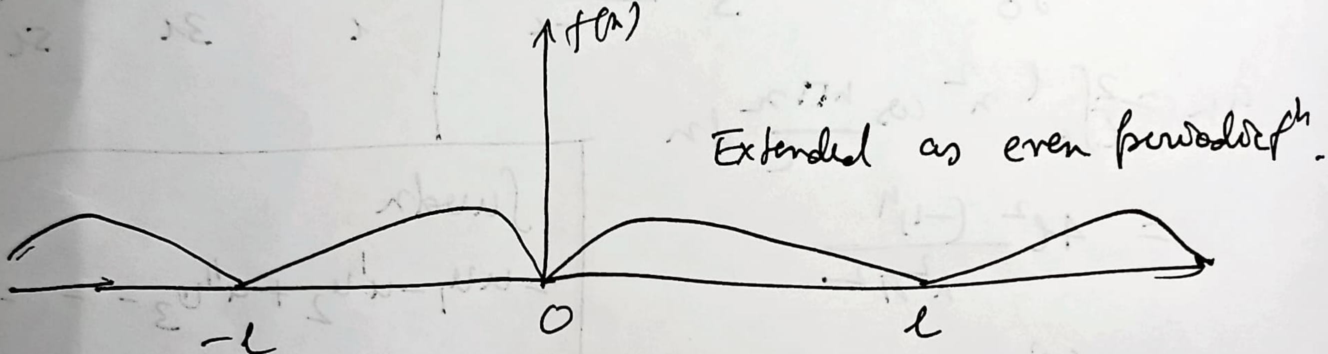
$$a_3 = -\frac{4l^2}{3^2\pi^2}$$

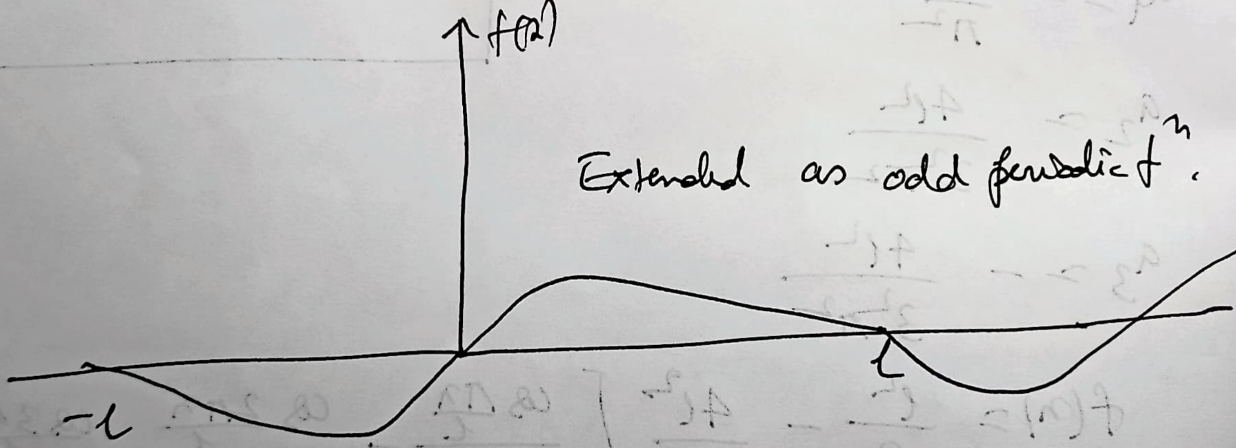
$$f(x) = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos\frac{\pi x}{l}}{1^2} - \frac{\cos\frac{2\pi x}{l}}{2^2} + \frac{\cos\frac{3\pi x}{l}}{3^2} + \cdots\right]$$

which is the reqd. F.S.

Half range Fourier series







Half range sine series

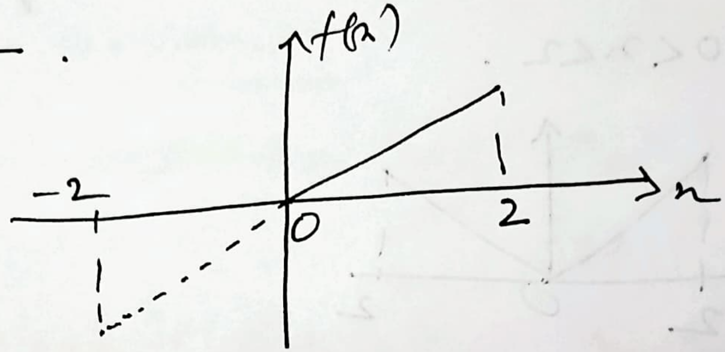
$$f(x) = \sum_{n\geq 1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\,dx$$

Half range cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\,dx$

Ex Express $f(x) = x$ as a half range sine series in $0 < x < 2$.



Sol^n:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

where $b_n = \frac{2}{2} \int_0^2 f(x) \sin \frac{n\pi x}{2} \, dx$

$$= \int_0^2 x \sin \frac{n\pi x}{2} \, dx$$

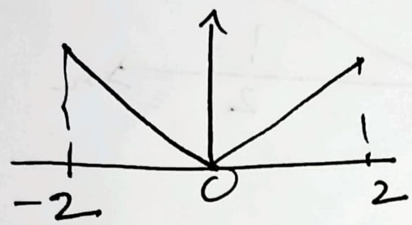
$$= - \frac{4(-1)^n}{n\pi}$$

$b_1 = \frac{4}{\pi}$, $b_2 = -\frac{4}{2\pi}$, $b_3 = \frac{4}{3\pi}$, $b_4 = -\frac{4}{4\pi}$ etc

$$f(x) = \frac{4}{\pi} \left[\sin \frac{\pi x}{2} - \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} - \cdots \right]$$

Ex Express $f(x) = x$ as a half range cosine series in $0 < x < 2$.

Sol^n:



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

where $a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = 2$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$a_1 = -\frac{8}{\pi^2}$, $a_2 = 0$, $a_3 = -\frac{8}{3^2\pi^2}$, $a_4 = 0$ - -

$$f(x) = 1 - \frac{8}{\pi^2}\left[\frac{\cos\frac{\pi x}{2}}{1^2} + \frac{\cos\frac{3\pi x}{2}}{3^2} + \cdots\right]$$

Lecture - 18 28.2.2023

Perseval's theorem

If $f(x)$ is continuous in $(-\pi, \pi)$ and is square integrable $\left[\text{i.e. } \int_{-\pi}^{\pi} [f(x)]^2 dx < \infty\right]$ and has Fourier coefficients a_n, b_n, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Proof F.S. of $f(x)$ in $(-\pi, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Multiplying both sides by $f(x)$ and integrating term by term from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0}{2}\int_{-\pi}^{\pi} f(x)dx + \sum_{n=1}^{\infty}\left[a_n \int_{-\pi}^{\pi} f(x)\cos nx\, dx + b_n \int_{-\pi}^{\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{a_0}{2}\pi a_0 + \sum_{n=1}^{\infty}\left[a_n(\pi a_n) + b_n(\pi b_n)\right]$$

$$= 2\pi\left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)\right]$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

Ex For the F.S. expansion of $f(x)=x^2$ in $-\pi<x<\pi$, prove that $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$.

Solⁿ: The F.S expansion of $f(x)=x^2$ in $(-\pi,\pi)$ is given by

$$x^2=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$a_0=\frac{2\pi^2}{3},\quad a_n=\frac{4(-1)^n}{n^2}\quad b_n=0$$

By Perseval's theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx=\frac{a_0^2}{4}+\frac{1}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi}x^4\,dx=\frac{\pi^4}{9}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4}$$

$$\Rightarrow \frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi}-\frac{\pi^4}{9}=8\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{8}\left[\frac{\pi^4}{5}-\frac{\pi^4}{9}\right]=\frac{\pi^4}{90}$$

Ex (a) Obtain the cosine series for $f(x)=x$ in $0<x<\pi$ and deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}$

(b) Obtain the sine series for $f(x)=x$ in $0<x<\pi$ and deduce that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

Complex form of Fourier series :

Let $f(x)$ be a periodic f^n. of period 2π. The F.S. is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad -(1)$$

We know $e^{inx} = \cos nx + i \sin nx$

and $e^{-inx} = \cos nx - i \sin nx$

$$\therefore \cos nx = \frac{1}{2} (e^{inx} + e^{-inx})$$

and $$\sin nx = \frac{1}{2i} (e^{inx} - e^{-inx})$$

Substituting these values in (1),

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{a_n}{2} (e^{inx} + e^{-inx}) + \frac{b_n}{2i} (e^{inx} - e^{-inx}) \right]$$

$$= c_0 + \sum_{n=1}^{\infty} (c_n e^{inx} + c_{-n} e^{-inx})$$

where $c_0 = \frac{a_0}{2}$, $c_n = \frac{1}{2}(a_n - ib_n)$, $c_{-n} = \frac{1}{2}(a_n + ib_n)$

Now $c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left[\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx - \frac{i}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \right]$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) [\cos nx - i \sin nx] \, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

Similarly $c_{-n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} \, dx$

$\therefore c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad \forall n = 0, \pm 1, \pm 2, \ldots$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$n = 0, \pm 1, \pm 2, \ldots$

$$\therefore\ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\text{where } c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\text{where } c_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-\frac{in\pi x}{L}}\, dx$$

Lectures 19 and 20 — 6. 3. 2023

Ex Find the complex form of the F.S. of $f(x) = e^{-x}$ of period 2 in $-1 < x < 1$.

Solⁿ: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$, $c_n = \frac{1}{2}\int_{-1}^{1} f(x) e^{-in\pi x}\, dx$

i.e. $$c_n = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{-(1+in\pi)}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)} - e^{(1+in\pi)}}{-(1+in\pi)}\right]$$

$$= \frac{e(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)}{2(1+in\pi)}$$

$$= \frac{(e - e^{-1})\cos n\pi}{2(1+in\pi)} = \frac{\sinh 1\, (-1)^n (1 - in\pi)}{1 + n^2\pi^2}$$

$$\therefore\ e^{-x} = \sinh 1 \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n (1 - in\pi)}{1 + n^2\pi^2}\right] e^{in\pi x}$$

Fourier integral representation

(From Fourier series to Fourier integral)

We consider a periodic f^n. $f_l(x)$ of period $2L$ that can be represented by a F.S.

$$f_L(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos\alpha_n x + b_n \sin\alpha_n x), \quad \alpha_n = \frac{n\pi}{l}$$

and find out what happens if we let $l \to \infty$. Here we expect an integral instead of a series involving $\cos\alpha x$ and $\sin\alpha x$ with α no longer restricted to integer multiples $\alpha = \alpha_n = \frac{n\pi}{l}$ of $\frac{\pi}{l}$ but taking all values.

If we insert a_0, a_n and b_n from Euler's formula, and denote the variable of integration by v, the F.S. of $f_l(x)$ becomes

$$f_l(x) = \frac{1}{2l}\int_{-l}^{l} f_l(v)\,dv + \frac{1}{l}\sum_{n=1}^{\infty}\Big[\cos\alpha_n x \int_{-l}^{l} f_l(v)\cos\alpha_n v\,dv$$
$$+ \sin\alpha_n x \int_{-l}^{l} f_l(v)\sin\alpha_n v\,dv$$

We now set $\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{(n+1)\pi}{l} - \frac{n\pi}{l} = \frac{\pi}{l}$

$\therefore \frac{1}{l} = \frac{\Delta\alpha}{\pi}$ and we may write the F.S. in the form

$$f_l(x) = \frac{1}{2l}\int_{-l}^{l} f_l(v)\,dv + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[(\cos\alpha_n x)\,\Delta\alpha\int_{-l}^{l} f_l(v)\cos\alpha_n v\,dv + (\sin\alpha_n x)\,\Delta\alpha\int_{-l}^{l} f_l(v)\sin\alpha_n v\,dv\Big] \quad \text{— (1)}$$

This representation is valid for any l, arbitrarily large, but finite.

We now let $l \to \infty$ and assume that the resulting function $f(x) = \lim_{l\to\infty} f_l(x)$ is absolutely integrable on the x-axis i.e. the following limit exist $\int_{-\infty}^{\infty} |f(x)|\,dx = \lim_{a\to-\infty}\int_a^0 |f(x)|\,dx + \lim_{b\to\infty}\int_0^b |f(x)|\,dx$

Thus $\frac{1}{l} \to 0$ and the value of the 1st term on the RHS of (1) approaches 0. Also $\Delta\alpha = \frac{\pi}{l} \to 0$ and the infinite series in (1) becomes an integral from 0 to ∞ which represents $f(x)$, namely

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\cos\alpha x\int_{-\infty}^{\infty} f(v)\cos\alpha v\,dv + \sin\alpha x\int_{-\infty}^{\infty} f(v)\sin\alpha v\,dv\Big]\,d\alpha \quad \text{— (2)}$$

INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

CLASS TEST / LABORATORY TEST

8

Signature of the Invigilator

EXAMINATION (Mid-Semester / End-Semester)

SEMESTER (Autumn / Spring)

Roll Number		Section		Name	Koeli Ghoshal
Subject Number	MA20202	Subject Name			Transform Calculus

Lectures 19 and 20 (continued) 6.3.2023

If we introduce the notations

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\alpha v\, dv$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\alpha v\, dv$$

we can write $f(x) = \int_0^{\infty}\left[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right]d\alpha$

This is called FIR of $f(x)$

Theorem

If $f(x)$ is PWC in every finite interval and has a right-hand derivative and a left-hand derivative at every point and if $f(x)$ is absolutely integrable for all real x, then $f(x)$ can be represented by a Fourier integral (2). At a point where $f(x)$ is discontinuous, the value of Fourier integral equals the average of the left and right limits of $f(x)$ at that point.

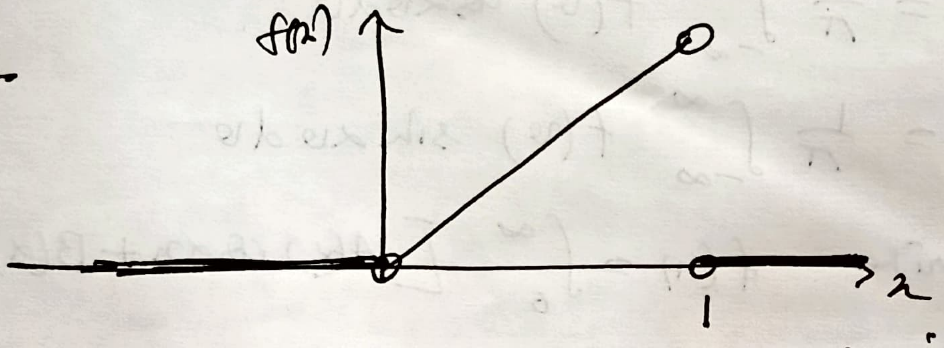
Ex Consider the function

$$f(x) = \begin{cases} 0 & \text{when } x<0 \\ x & \text{when } 0<x<1 \\ 0 & \text{when } x>1 \end{cases}$$

(a) Find the Fourier integral representing $f(x)$

(b) Determine the convergence of the integral at $x=1$.

Sol^n



The integral representation of f is

$$f(x) = \int_0^\infty \left[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right] d\alpha$$

$$A(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\alpha t\, dt$$

$$= \frac{1}{\pi}\left[\frac{\cos\alpha + \alpha\sin\alpha - 1}{\alpha^2}\right]$$

$$B(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\alpha t\, dt$$

$$= \frac{1}{\pi}\left[\frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\right]$$

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\cos\alpha + \alpha\sin\alpha - 1}{\alpha^2}\cos\alpha x + \frac{\sin\alpha - \alpha\cos\alpha}{\alpha^2}\sin\alpha x \right] d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\cos\alpha(1-x) + \alpha\sin\alpha(1-x) - \cos\alpha x}{\alpha^2}\, d\alpha$$

(b) At $x=1$, the f^n is not defined. The convergence is $\dfrac{f(1+) + f(1-)}{2} = \dfrac{0+1}{2} = \dfrac{1}{2}$

Fourier cosine and sine integrals

If $f(x)$ is an even f^n, then $B(\alpha) = 0$ and

$$A(\alpha) = \frac{2}{\pi}\int_0^\infty f(v)\cos\alpha v\, dv$$

$\therefore$ FI reduces to the Fourier cosine integral

$$f(x) = \int_0^\infty A(\alpha)\cos\alpha x\, d\alpha \quad (f \text{ even})$$

Similarly if $f(x)$ is odd, then $A(\alpha)=0$ and

$$B(\alpha) = \frac{2}{\pi}\int_0^\infty f(v)\sin\alpha v\, dv$$

$\therefore$ The F.I. reduces to the Fourier ~~cosine~~ sine integral

$$f(x) = \int_0^\infty B(\alpha)\sin\alpha x\, d\alpha$$

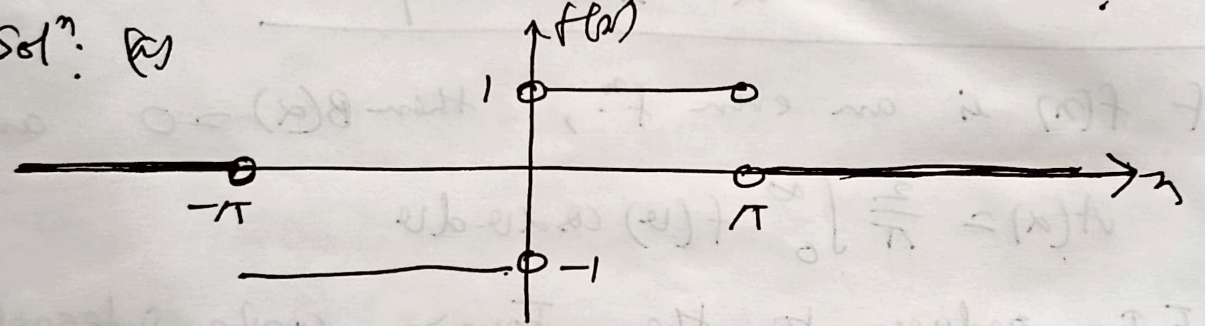
Ex Consider the f^n

$$f(x) = \begin{cases} 0 & \text{when } -\infty < x < -\pi \\ -1 & \text{when } -\pi < x < 0 \\ 1 & \text{when } 0 < x < \pi \\ 0 & \text{when } \pi < x < \infty \end{cases}$$

(a) Determine the Fourier integral for $f(x)$

(b) To what number does the integral found in (a) converge at $x = -\pi$?

Sol^n: (a)



$$B(\alpha) = \frac{2}{\pi} \int_0^\infty f(v) \sin \alpha v \, dv$$

$$= \frac{2}{\pi\alpha} (1 - \cos \alpha\pi)$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \left(\frac{1 - \cos \alpha\pi}{\alpha} \right) \sin \alpha x \, d\alpha$$

(b) The integral converges to $-\frac{1}{2}$ at $x = -\pi$.

Fourier cosine and sine transform

For an even f^n. $f(x)$, the Fourier integral is the FCI

$$f(x) = \int_0^\infty A(\alpha)\cos\alpha x\,d\alpha \quad \text{where } A(\alpha) = \frac{2}{\pi}\int_0^\infty f(v)\cos\alpha v\,dv$$

We now set $A(\alpha) = \sqrt{\frac{2}{\pi}}\,\hat{f}_c(\alpha)$

$$\hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\alpha x\,dx \quad \text{—(1)}$$

and $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(\alpha)\cos\alpha x\,d\alpha$ —(2)

(1) gives from $f(x)$ a new f^n. $\hat{f}_c(\alpha)$ called the Fourier cosine transform of $f(x)$ and (2) gives us back $f(x)$ from $\hat{f}_c(\alpha)$ and we call $f(x)$ the inverse Fourier cosine transform of $\hat{f}_c(\alpha)$.

Similarly for an odd f^n

$$f(x) = \int_0^\infty B(\alpha)\sin\alpha x\,d\alpha \quad \text{where } B(\alpha) = \frac{2}{\pi}\int_0^\infty f(v)\sin\alpha v\,dv$$

$B(\alpha) = \sqrt{\frac{2}{\pi}}\,\hat{f}_s(\alpha)$

$\therefore \hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\alpha x\,dx \rightarrow$ FST of $f(x)$

$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(\alpha)\sin\alpha x\,d\alpha \rightarrow$ IFST of $\hat{f}_s(\alpha)$

Lectures 21 and 22 13.3.23

The exponential Fourier integral

(The complex form of Fourier integral)

The Fourier integral for $f(x)$ can be expressed as

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\cos\alpha(x-v)\,dv\,d\alpha \quad \text{—(A)}$$

Now $\cos\alpha(x-v) = \dfrac{e^{i\alpha(x-v)} + e^{-i\alpha(x-v)}}{2}$ —(B)

Inserting (B) in (A),

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\,\frac{e^{i\alpha(x-v)} + e^{-i\alpha(x-v)}}{2}\,dv\,d\alpha$$

$$= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\,e^{i\alpha(x-v)}\,dv\,d\alpha + \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\,e^{-i\alpha(x-v)}\,dv\,d\alpha$$

If α is replaced by $-\alpha$ in the second integral, then

$$f(x) = \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\,e^{i\alpha(x-v)}\,dv\,d\alpha - \frac{1}{2\pi}\int_0^{-\infty} \int_{-\infty}^{\infty} f(v)\,e^{i\alpha(x-v)}\,dv\,d\alpha$$

$$= \frac{1}{2\pi}\int_0^\infty \int_{-\infty}^{\infty} f(v)\,e^{i\alpha(x-v)}\,dv\,d\alpha + \frac{1}{2\pi}\int_{-\infty}^{0} \int_{-\infty}^{\infty} f(v)\,e^{i\alpha(x-v)}\,dv\,d\alpha$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^{\infty} f(v)\,e^{i\alpha(x-v)}\,dv\,d\alpha \qquad -\infty < x < \infty \quad \text{(C)}$$

This form is known as the complex form of Fourier integral or the exponential Fourier integral representation of the function $f(x)$.

Fourier transform and its inverse

Writing (C) in the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v)\, e^{-i\alpha v}\, dv \right] e^{i\alpha x}\, d\alpha \quad \text{——— (D)}$$

The expression in brackets is a function of α and is denoted as $\hat{f}(\alpha)$ which is called the Fourier transform or exponential Fourier transform of $f(x)$.

Writing x for v

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx$$

With this, (D) becomes

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{i\alpha x}\, d\alpha$$

Alternate form

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{-i\alpha x}\, d\alpha$$

Ex Find the Fourier sine transform of the f^n.

$$f(x) = e^{-ax}$$

and hence evaluate $\int_0^\infty \frac{\alpha \sin \alpha x}{\alpha^2+a^2} \, d\alpha$

Sol^n: $\hat{f_s}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin \alpha x \, dx$

$$I = \int_0^\infty e^{-ax} \sin \alpha x \, dx$$

$$= \left[\sin \alpha x \frac{e^{-ax}}{-a} \right]_0^\infty - \alpha \int_0^\infty \cos \alpha x \frac{e^{-ax}}{-a} \, dx$$

$$= \frac{\alpha}{a} \left[\left[\cos \alpha x \frac{e^{-ax}}{-a} \right]_0^\infty + \alpha \int_0^\infty \sin \alpha x \frac{e^{-ax}}{-a} \, dx \right]$$

$$= \frac{\alpha}{a} \left[\frac{1}{a} - \frac{\alpha}{a} I \right]$$

$$I = \frac{\alpha}{\alpha^2+a^2}$$

$$\hat{f_s}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{\alpha}{\alpha^2+a^2}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f_s}(\alpha) \sin \alpha x \, d\alpha$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\alpha}{\alpha^2+a^2} \sin \alpha x \, d\alpha$$

$$\therefore \int_0^\infty \frac{\alpha}{\alpha^2+a^2} \sin \alpha x \, d\alpha = \frac{\pi}{2} f(x)$$

$$= \frac{\pi}{2} e^{-ax}, \quad x > 0$$

Ex Find the Fourier cosine transform of the f^n:

$$f(x) = e^{-ax}$$

and hence evaluate $\int_0^\infty \frac{\cos \alpha x}{\alpha^2+a^2}\, d\alpha$.

Sol^n. $\hat{f_c}(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos \alpha x \, dx$

$$I = \int_0^\infty e^{-ax} \cos \alpha x \, dx$$

$$= \left[\cos \alpha x \frac{e^{-ax}}{-a} \Big|_0^\infty + \alpha \int_0^\infty \sin \alpha x \frac{e^{-ax}}{-a} dx \right]$$

$$= \frac{1}{a} - \frac{\alpha}{a} \left[\sin \alpha x \frac{e^{-ax}}{-a} \Big|_0^\infty - \int_0^\infty \alpha \cos \alpha x \frac{e^{-ax}}{-a} dx \right]$$

$$= \frac{1}{a} - \frac{\alpha^2}{a^2} \int_0^\infty \cos \alpha x \, e^{-ax} dx$$

$$= \frac{1}{a} - \frac{\alpha^2}{a^2} I$$

$$I = \frac{a}{\alpha^2 + a^2}$$

$$\hat{f_c}(\alpha) = \sqrt{\frac{2}{\pi}} \frac{a}{\alpha^2 + a^2}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f_c}(\alpha) \cos \alpha x \, d\alpha$$

$$= \frac{2}{\pi} \int_0^\infty \frac{a}{\alpha^2 + a^2} \cos \alpha x \, d\alpha$$

$$\therefore \int_0^\infty \frac{\cos \alpha x}{\alpha^2 + a^2} d\alpha = \frac{\pi}{2a} f(x)$$

$$= \frac{\pi}{2a} e^{-ax}, \quad x > 0$$

Ex Find the F cosine transform of the f^n.

$$f(x) = \begin{cases} 1 & 0 < x < a \\ 0 & x > a \end{cases}$$

and hence evaluate $\displaystyle\int_0^\infty \frac{\sin a\alpha \cos \alpha x}{\alpha} \, d\alpha$

Sol^n: $\displaystyle\hat{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \frac{\sin a\alpha}{\alpha}$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{\sin a\alpha}{\alpha} \cos \alpha x \, d\alpha$$

$$\therefore \int_0^\infty \frac{\sin a\alpha}{\alpha} \cos \alpha x \, d\alpha = \frac{\pi}{2} f(x) = \left.\begin{matrix} \frac{\pi}{2} \\ 0 \end{matrix}\right\} \begin{matrix} 0 < x < a \\ x > a \end{matrix}$$

Ex Find the F sine transform of the f^n.

$$f(x) = 1 \quad 0 < x < a$$
$$\quad\quad 0 \quad x > a$$

and hence evaluate $\displaystyle\int_0^\infty \frac{(1 - \cos a\alpha) \sin \alpha x}{\alpha} \, d\alpha$

Sol^n: $\displaystyle\hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos a\alpha}{\alpha}$

$$\therefore f(x) = \frac{2}{\pi} \int_0^\infty \frac{1 - \cos a\alpha}{\alpha} \sin \alpha x \, d\alpha$$

$$\int_0^\infty \frac{1 - \cos a\alpha}{\alpha} \sin \alpha x \, d\alpha = \frac{\pi}{2} f(x)$$

$$= \left.\begin{matrix} \frac{\pi}{2} \\ 0 \end{matrix}\right\} \begin{matrix} 0 < x < a \\ x > a \end{matrix}$$

Ex. Find the F.T. of $f(x) = e^{-|x|}$

and hence find $\int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+\alpha^2} d\alpha$

Sol^n: $\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x} e^{i\alpha x} dx + \int_{0}^{\infty} e^{-x} e^{i\alpha x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(1+i\alpha)x} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(1-i\alpha)x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{(1+i\alpha)x}}{1+i\alpha} \right]_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-(1-i\alpha)x}}{-(1-i\alpha)} \right]_{0}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1+i\alpha} + \frac{1}{1-i\alpha} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1-i\alpha+i\alpha+1}{1+\alpha^2}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2} e^{-i\alpha x} d\alpha$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+\alpha^2} d\alpha$$

$\cancel{\pi e^{-|x|}}$

$$\therefore \int_{-\infty}^{\infty} \frac{e^{-i\alpha x}}{1+\alpha^2} d\alpha = \pi e^{-|x|}$$

Ex Find the F. T. of $f(x) = 1$ if $|x| < a$

$= 0$ if $|x| \geq a$

where a is a +ve real no. Hence deduce that

$$\int_0^\infty \frac{\sin a\alpha}{\alpha}\, d\alpha = \frac{\pi}{2} \text{ and } \int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

Soln:

$$f(x) = \begin{cases} 1 & -a < x < a \\ 0 & \text{otherwise} \end{cases}$$

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha x}}{i\alpha} \right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{i\alpha a} - e^{-i\alpha a}}{i\alpha} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2i \sin a\alpha}{i\alpha} \right]$$

$$\hat{f}(\alpha) = \sqrt{\frac{2}{\pi}}\, \frac{\sin a\alpha}{\alpha}$$

Lecture-26

21.3.23

Ex Deriving the F.S.T. of $f(x) = \frac{1}{x(a^2+x^2)}$, evaluate F.C.T. of $\frac{1}{x^2+a^2}$.

Solⁿ: $\hat{f}_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x(a^2+x^2)} \sin \alpha x \, dx$

Let $I = \int_0^\infty \frac{1}{x(a^2+x^2)} \sin \alpha x \, dx$ —— (1)

Differentiating w.r.t. α,

$$\frac{dI}{d\alpha} = \int_0^\infty \frac{x \cos \alpha x}{x(a^2+x^2)} dx = \int_0^\infty \frac{\cos \alpha x}{x^2+a^2} dx \text{ —— (2)}$$

$$\frac{d^2I}{d\alpha^2} = -\int_0^\infty \frac{x \sin \alpha x}{x^2+a^2} dx = -\int_0^\infty \frac{x^2 \sin \alpha x}{x(x^2+a^2)} dx$$

$$= -\int_0^\infty \frac{(x^2+a^2) - a^2}{x(x^2+a^2)} \sin \alpha x \, dx$$

$$= -\int_0^\infty \frac{\sin \alpha x}{x} dx + a^2 I = a^2 I - \frac{\pi}{2}$$

$\therefore \frac{d^2I}{d\alpha^2} - a^2 I = -\frac{\pi}{2}$

C.F. $c_1 e^{a\alpha} + c_2 e^{-a\alpha}$

$$P.I = \frac{-\frac{\pi}{2}}{D^2 - a^2} = \frac{\pi}{2a^2}\left(1 - \frac{D^2}{a^2}\right)^{-1}(1) = \frac{\pi}{2a^2}$$

$$I = c_1 e^{a\alpha} + c_2 e^{-a\alpha} + \frac{\pi}{2a^2} \text{ —— (3)}$$

$$\frac{dI}{d\alpha} = a(c_1 e^{a\alpha} - c_2 e^{-a\alpha}) \text{ —— (4)}$$

If $\alpha = 0$, (1) and (3) give $I = 0$

$$c_1 + c_2 + \frac{\pi}{2a^2} = 0$$

If $\alpha = 0$, (2) and (4) give $\frac{dI}{d\alpha} = \frac{\pi}{2a}$

$$\frac{\pi}{2a} = a(c_1 - c_2)$$

$\therefore c_1 = 0$, $c_2 = -\frac{\pi}{2a^2}$

$$I = \frac{\pi}{2a^2}\left(1 - e^{-a\alpha}\right)$$

$$\hat{f_s}(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{\pi}{2a^2}\left(1 - e^{-a\alpha}\right) = \sqrt{\frac{\pi}{2}}\,\frac{1}{a^2}\left(1 - e^{-a\alpha}\right)$$

$$\hat{f_c}(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{dI}{d\alpha} = \sqrt{\frac{2}{\pi}}\,\frac{\pi}{2a}e^{-a\alpha} = \sqrt{\frac{\pi}{2}}\,\frac{1}{a}e^{-a\alpha}$$

Ex Solve for $f(x)$, the integral eqn.

$$\int_0^\infty f(x)\sin(\alpha x)\,dx = \begin{cases} 1 & 0 \le \alpha < 1 \\ 2 & 1 \le \alpha < 2 \\ 0 & \alpha \ge 2 \end{cases}$$

Soln: $F_s[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\alpha x\,dx = \hat{f_s}(\alpha)$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f_s}(\alpha)\sin\alpha x\,d\alpha$$

$$\hat{f_s}(\alpha) = \begin{cases} \sqrt{\frac{2}{\pi}} & 0 \le \alpha < 1 \\ 2\sqrt{\frac{2}{\pi}} & 1 \le \alpha < 2 \\ 0 & \alpha \ge 2 \end{cases}$$

$$\frac{\pi}{2}f(x) = \int_0^1 \hat{f_s}(\alpha)\sin\alpha x\,d\alpha + \int_1^2 \hat{f_s}(\alpha)\sin\alpha x\,d\alpha$$

$$= \int_0^1 \sin\alpha x\,d\alpha + \int_1^2 2\sin\alpha x\,d\alpha$$

$$= \left[-\frac{\cos\alpha x}{x}\right]_0^1 + 2\left[-\frac{\cos\alpha x}{x}\right]_1^2 = \frac{1}{x}\left[(1-\cos x) + 2(\cos x - \cos 2x)\right]$$

$$f(x) = \frac{2}{\pi x}\left(1 + \cos x - 2\cos 2x\right)$$

Lectures 27 & 28

27.3.23

Solution of PDE by Laplace and Fourier transform

PDE of order one

$$f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad u \equiv u(x,y)$$

$$\frac{\partial u}{\partial x} \equiv p \qquad \frac{\partial u}{\partial y} \equiv q$$

Classification of 1st order PDE

1. Linear eqn.

$$P(x,y)\,p + Q(x,y)\,q = R(x,y)\,u + S(x,y)$$

Ex $\quad y^2 p - xy\,q = x^2 y^2 u + xy$

2. Semi-linear eqn.

$$P(x,y)\,p + Q(x,y)\,q = R(x,y,u)$$

Ex $\quad e^x x\,p + xy^2 q = xyu^3$

3. Quasilinear eqn.

$$P(x,y,u)\,p + Q(x,y,u)\,q = R(x,y,u)$$

Ex $\quad (x^2+u^3)\,p + x^2y^2u\,q = u^2x + y^2u$

4. Non-linear eqn.

$$pq = u$$

Linear / Semi-linear PDE of 2nd order in two independent variables

$$A\frac{\partial^2 u}{\partial x^2}+B\frac{\partial^2 u}{\partial x\partial y}+C\frac{\partial^2 u}{\partial y^2}+f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0$$

Classification

(a) elliptic if $B^2-4AC<0$

(b) hyperbolic if $B^2-4AC>0$

(c) parabolic if $B^2-4AC=0$

Wave, heat and Laplace equation

Wave eqn (Hyperbolic)

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \quad \text{(1 D wave eqn)}$$

Heat eqn (parabolic)

$$\frac{\partial u}{\partial t}=c^2\frac{\partial^2 u}{\partial x^2} \quad \text{(1D heat eqn)}$$

Laplace equation (elliptic)

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Application of Laplace transform in solving a PDE

Criteria for choosing a Laplace transform

(i) At least one of the independent variables should have the range from 0 to ∞. In case both have the range from 0 to ∞, we may apply L.T. w.r.t. either variable. In case, only one has the range from 0 to ∞, we can apply L.T. only w.r.t. that variable.
[The convention is to apply L.T. w.r.t. t]

(ii) Appropriate conditions must be specified at the lower limit of the variable which has the range from 0 to ∞.

§ If $y(x,t)$ is a function of x and t and

$$L\{y(x,t)\} = \bar{y}(x,s)$$

(a) $$L\left\{\frac{\partial y}{\partial t}\right\} = s\bar{y}(x,s) - y(x,0)$$

(b) $$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = s^2\bar{y}(x,s) - s\,y(x,0) - y_t(x,0)$$

(c) $$L\left\{\frac{\partial y}{\partial x}\right\} = \frac{d\bar{y}}{dx}$$

(d) $$L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \frac{d^2\bar{y}}{dx^2}$$

Proof (a) $L\left\{\frac{\partial y}{\partial t}\right\} = \int_0^\infty e^{-st} \frac{\partial y}{\partial t} dt$

$= \left[e^{-st} y(x,t)\right]_0^\infty + s\int_0^\infty e^{-st} y(x,t) dt$

$= s\int_0^\infty e^{-st} y(x,t) dt - y(x,0)$

$= s\bar{y}(x,s) - y(x,0)$

(b) Let $v = \frac{\partial y}{\partial t}$

$L\left\{\frac{\partial^2 y}{\partial t^2}\right\} = L\left\{\frac{\partial v}{\partial t}\right\} = sL\{v\} - v(x,0)$

$= s\left[sL\{y\} - y(x,0)\right] - y_t(x,0)$

$= s^2\bar{y}(x,s) - sy(x,0) - y_t(x,0)$

(c) $L\left\{\frac{\partial y}{\partial x}\right\} = \int_0^\infty e^{-st} \frac{\partial y}{\partial x} dt$

$= \frac{d}{dx}\int_0^\infty e^{-st} \cdot y\, dt$

$= \frac{d\bar{y}}{dx}$

(d) $L\left\{\frac{\partial^2 y}{\partial x^2}\right\} = L\left\{\frac{\partial U}{\partial x}\right\}$ $\quad U = \frac{\partial y}{\partial x}$

$= \frac{d}{dx} L\{U\}$

$= \frac{d}{dx} L\left\{\frac{\partial y}{\partial x}\right\}$

$= \frac{d}{dx}\left(\frac{d\bar{y}}{dx}\right) = \frac{d^2\bar{y}}{dx^2}$

Ex $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = x$ $\quad x>0 \quad t>0$

$u(0,t) = 0 \quad u(x,0) = 0$

Sol^n: Applying L.T. w.r.t. t,

$$\frac{d\bar{u}}{dx} + s\bar{u}(x,s) - u(x,0) = x \cdot \frac{1}{s}$$

$$\Rightarrow \frac{d\bar{u}}{dx} + s\bar{u} = x \cdot \frac{1}{s}$$

I.F. $\quad e^{\int s\,dx} = e^{sx}$

$$\bar{u}\,e^{sx} = \int e^{sx} \cdot x \cdot \frac{1}{s}\,dx + c$$

$$= \frac{1}{s}\left[x\frac{e^{sx}}{s} - \int \frac{e^{sx}}{s}\,dx\right] + c$$

$$= \frac{1}{s}\left[x\frac{e^{sx}}{s} - \frac{e^{sx}}{s^2}\right] + c$$

$$\bar{u} = \frac{1}{s}\,\frac{sx-1}{s^2} + ce^{-sx} = \frac{x}{s^2} - \frac{1}{s^3} + ce^{-sx}$$

$$\bar{u}(x,s) = \frac{x}{s^2} - \frac{1}{s^3} + ce^{-sx}$$

We have $u(0,t) = 0 \quad \bar{u}(0,s) = 0 \quad c = \frac{1}{s^3}$

$$\bar{u}(x,s) = \frac{x}{s^2} - \frac{1}{s^3} + \frac{1}{s^3}e^{-sx}$$

$$u(x,t) = xt - \cancel{\frac{1}{s^3}} - \frac{1}{2}t^2 + H(t-x)\,\frac{1}{2}(t-x)^2$$

$$u(x,t) = \begin{cases} xt - \frac{1}{2}t^2 & t<x \\ \frac{1}{2}x^2 & t>x \end{cases}$$

INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR

CLASS TEST / LABORATORY TEST

11

Signature of the Invigilator

EXAMINATION (Mid-Semester / End-Semester)									SEMESTER (Autumn / Spring)		
Roll Number									Section	Name	Koeli Ghoshal
Subject Number	M	A	2	0	2	0	2		Subject Name		Transform Calculus

Lectures 27 & 28 (continued) 27.3.23

E1

Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $x>0$ $t>0$

subject to $u(0,t) = u_0$ $t>0$

$u(x,0) = 0$ $x>0$

$\lim_{x\to\infty} u(x,t) = 0$ $t>0$

Given $L^{-1}\left[\frac{e^{-\sqrt{s}x}}{s}\right] = u_0 \, erf_c\left[\frac{x}{2\sqrt{t}}\right]$

Sol^n: Applying L.T. on both sides w.r.t. t

$$s\bar{u}(x,s) - u(x,0) = \frac{d^2\bar{u}}{dx^2}$$

i.e. $\frac{d^2\bar{u}}{dx^2} - s\bar{u} = 0$

$$\bar{u}(x,s) = Ae^{\sqrt{s}x} + Be^{-\sqrt{s}x}$$

As $x\to 0$, $u=0$ $\therefore$ As $x\to\infty$ $\bar{u}=0$ $\therefore A=0$

$$\therefore \bar{u}(x,s) = Be^{-\sqrt{s}x}$$

$$\bar{u}(0,s) = B$$

Given $u(0,t) = u_0$ $\bar{u}(0,s) = L[u_0] = \frac{u_0}{s}$

$$B = \frac{u_0}{s}$$

$$\bar{u}(x,s) = \frac{u_0}{s} e^{-\sqrt{s}x}$$

$$u(x,t) = u_0 L^{-1}\left[\frac{e^{-\sqrt{s}x}}{s}\right] = u_0 \, erf_c\left[\frac{x}{2\sqrt{t}}\right]$$

Ex Solve $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$ $x>0$ $t>0$

with $u(x,0)=0$ $u_t(x,0)=0$

$u(0,t)=F(t)$ $\lim_{x\to\infty} u(x,t)=0$

Solⁿ. Applying L.T. w.r.t. t

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = a^2 L\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow s^2\bar{u}(x,s) - s\,u(x,0) - u_t(x,0) = a^2 \frac{d^2\bar{u}}{dx^2}$$

$$\Rightarrow \frac{d^2\bar{u}}{dx^2} - \frac{s^2}{a^2}\bar{u} = 0$$

$$\bar{u}(x,s) = Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x}$$

$u=0$ as $x\to\infty$ $\bar{u}=0$ as $x\to\infty$

$\therefore A=0$

$$\bar{u}(x,s) = Be^{-\frac{s}{a}x}$$

$u(0,t)=F(t)$ $\bar{u}(0,s) = \int_0^\infty F(t)\,e^{-st}dt = \bar{f}(s)$

$$B = \bar{f}(s)$$

$$\bar{u}(x,s) = \bar{f}(s)\,e^{-\frac{s}{a}x}$$

$$u(x,t) = L^{-1}\left[e^{-\frac{x}{a}s}\,\bar{f}(s)\right]$$

$$= \begin{cases} F(t-\frac{x}{a}) & t>\frac{x}{a} \\ 0 & t<\frac{x}{a} \end{cases}$$

$$= F(t-\tfrac{x}{a})\,H(t-\tfrac{x}{a})$$

in terms of Heaviside unit step function

Application of Fourier transform in solving PDE

Criteria for choosing general F.T.

(i) One of the independent variables should have the range from $-\infty$ to $+\infty$ and we can apply F.T. w.r.t. that variable only.

(ii) Both u and $\frac{\partial u}{\partial x}$ must vanish as $x \to \pm\infty$

Criteria for choosing infinite sine or cosine transform

Sine transform

A) $\int_0^\infty \sin\alpha x \frac{\partial^2 u}{\partial x^2} dx$

$$= \left[\frac{\partial u}{\partial x} \sin\alpha x\right]_0^\infty - \alpha \int_0^\infty \cos\alpha x \frac{\partial u}{\partial x} dx$$

$$= -\alpha \int_0^\infty \cos\alpha x \frac{\partial u}{\partial x} dx \quad \text{if } \frac{\partial u}{\partial x} \to 0 \text{ as } x \to \infty$$

$$= -\alpha \left[\cos\alpha x . u \Big|_0^\infty + \alpha \int_0^\infty u \sin\alpha x\, dx\right]$$

$$= \alpha (u)_{x=0} - \alpha^2 \hat{u}_s(\alpha, t) \quad \text{if } u \to 0 \text{ as } x \to \infty$$

where $\hat{u}_s$ is the F.S.T. of u

(i) At least one of the independent variables should have the range from 0 to ∞ and we need to apply F.S.T. w.r.t. that variable only.

(ii) The value of the unknown f^n: $u(x,t)$ must be known at the lower limit of the variable which has the range from 0 to ∞.

(iii) The behaviour of $u(x,t)$ & $\frac{\partial u}{\partial x}$ at $x \to \infty$ should be known.

Cosine transform

B) $\int_0^\infty \cos\alpha x \frac{\partial^2 u}{\partial x^2} dx$

$= \left[\frac{\partial u}{\partial x} \cos\alpha x\right]_0^\infty + \alpha \int_0^\infty \sin\alpha x \frac{\partial u}{\partial x} dx$

$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} + \alpha\left[u \sin\alpha x\right]_0^\infty$
$- \alpha^2 \int_0^\infty u \cos\alpha x dx$

if $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

$= -\left(\frac{\partial u}{\partial x}\right)_{x=0} - \alpha^2 \hat{u}_c$ if $u \to 0$ as $x \to \infty$

where $\hat{u}_c$ is the F.C.T. of u.

(i) same as (i) of F.S.T.

(ii) The value of $\frac{\partial u}{\partial x}$ must be known at $x=0$

(iii) same as (iii) of F.S.T.

Lecture - 29

28.3.23

E1 Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $x>0$ $t>0$

s.t. (i) $u=0$ when $x=0$ $t>0$

(ii) $u = \begin{cases} 1 & 0<x<1 \\ 0 & x \geq 1 \end{cases}$ when $t=0$

(iii) u & $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

Solⁿ: Applying F.S.T. on both the sides w.r.t. x

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t} \sin\alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 u}{\partial x^2} \sin\alpha x \, dx$$

$$\Rightarrow \frac{d\hat{u}_s}{dt} = \sqrt{\frac{2}{\pi}} \left[\sin\alpha x \frac{\partial u}{\partial x}\Big|_0^\infty - \alpha \int_0^\infty \cos\alpha x \frac{\partial u}{\partial x} dx \right]$$

$$= -\alpha \sqrt{\frac{2}{\pi}} \int_0^\infty \cos\alpha x \frac{\partial u}{\partial x} dx$$

$$= -\sqrt{\frac{2}{\pi}} \alpha \left[\cos\alpha x \, u\Big|_0^\infty + \alpha \int_0^\infty \sin\alpha x \, u \, dx \right]$$

$$= -\alpha^2 \sqrt{\frac{2}{\pi}} \int_0^\infty \sin\alpha x \, u \, dx$$

$$= -\alpha^2 \hat{u}_s$$

$$\frac{d\hat{u}_s}{dt} + \alpha^2 \hat{u}_s = 0 \qquad \hat{u}_s = Ae^{-\alpha^2 t}$$

When $t=0$, $\hat{u}_s(\alpha,0) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,0) \sin\alpha x \, dx$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin\alpha x \, dx + \sqrt{\frac{2}{\pi}} \int_1^\infty 0 \cdot \sin\alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{\cos\alpha x}{\alpha} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \frac{1-\cos\alpha}{\alpha}$$

$$\hat{u}_s(\alpha,0) = A = \frac{1-\cos\alpha}{\alpha}\sqrt{\frac{2}{\pi}}$$

$$\hat{u}_s = \sqrt{\frac{2}{\pi}}\,\frac{1-\cos\alpha}{\alpha}\,e^{-\alpha^2 t}$$

$$\therefore u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_s \sin\alpha x\, d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty \frac{1-\cos\alpha}{\alpha}\,e^{-\alpha^2 t}\sin\alpha x\, d\alpha$$

Ex Solve $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ $\quad 0<x<\infty \quad t>0$

s.t.

(i) $u=0$ when $t=0$ $\quad x>0$

(ii) $\frac{\partial u}{\partial x} = -\mu$ (a constant) when $x=0$, $t>0$

(iii) u and $\frac{\partial u}{\partial x} \to 0$ as $x\to\infty$

Solⁿ. Taking F.C.T. on both sides w.r.t x

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos\alpha x\,dx = k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos\alpha x\,dx$$

$$\Rightarrow \frac{d\hat{u}_c}{dt} = k\sqrt{\frac{2}{\pi}}\left[\cos\alpha x\,\frac{\partial u}{\partial x}\Big|_0^\infty + \alpha\int_0^\infty \frac{\partial u}{\partial x}\sin\alpha x\,dx\right]$$

$$= -k\sqrt{\frac{2}{\pi}}\left(\frac{\partial u}{\partial x}\right)_{x=0} + k\alpha\sqrt{\frac{2}{\pi}}\left[u\sin\alpha x\right]_0^\infty$$

$$-k\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx$$

if $\frac{\partial u}{\partial x}\to 0$ as $x\to\infty$

$$\Rightarrow \frac{d\hat{u}_c}{dt} + k\alpha^2\hat{u}_c = \sqrt{\frac{2}{\pi}}\,k\mu \quad \text{if } u\to 0 \text{ as } x\to\infty$$

I. F. $\quad e^{\int k\alpha^2 dt} = e^{k\alpha^2 t}$

$$e^{k\alpha^2 t}\hat{u}_c = \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}e^{k\alpha^2 t} + A$$

When $t=0$, ~~u_c~~ $u=0$ (given)

$\therefore$ When $t=0$ $\quad \hat{u}_c = 0$

$$\therefore\ 0 = A + \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}$$

$$\therefore\ A = -\sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}$$

$$\hat{u}_c(\alpha,t) = \sqrt{\frac{2}{\pi}}\frac{\mu}{\alpha^2}\left(1-e^{-k\alpha^2 t}\right)$$

$$\therefore u = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_c \cos\alpha x \, d\alpha$$

$$= \frac{2\mu}{\pi}\int_0^\infty \frac{\cos\alpha x}{\alpha^2}\left(1-e^{-k\alpha^2 t}\right) d\alpha$$

Lectures 30 and 31 3.4.23

Ex Solve $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $x>0$ $t>0$

subject to $u_x(0,t)=0$, $u(x,0) = \begin{cases} x & 0\le x\le 1 \\ 0 & x>1 \end{cases}$

u and $\frac{\partial u}{\partial x} \to 0$ as $x\to\infty$.

Solⁿ: Applying F.C.T. w.r.t x

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos\alpha x\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos\alpha x\,dx$$

$$\Rightarrow \frac{d\hat{u}_c}{dt} = -\sqrt{\frac{2}{\pi}}\left(\frac{\partial u}{\partial x}\right)_{x=0} - \alpha^2\hat{u}_c$$

$$\Rightarrow \frac{d\hat{u}_c}{dt} + \alpha^2\hat{u}_c = 0$$

$$\hat{u}_c = Ae^{-\alpha^2 t}$$

At $t=0$, $\hat{u}_c = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\cos\alpha x\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 x\cos\alpha x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{x\sin\alpha x}{\alpha}\Big|_0^1 - \frac{1}{\alpha^2}\left(-\cos\alpha x\right)_0^1\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha - 1}{\alpha^2}\right] = A$$

$$\hat{u}_c = \sqrt{\frac{2}{\pi}}\left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha-1}{\alpha^2}\right]e^{-\alpha^2 t}$$

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{u}_c\cos\alpha x\,d\alpha$$

$$= \frac{2}{\pi}\int_0^\infty\left(\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha-1}{\alpha^2}\right)e^{-\alpha^2 t}\cos\alpha x\,d\alpha$$

Ex Solve $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\quad -\infty < x < \infty$

with (i) $u(x,0) = f(x)$ (ii) $u_t(x,0) = 0$

(iii) u & $\frac{\partial u}{\partial x} \to 0$ as $|x| \to \infty$

Solⁿ: Taking F.T. on both the sides w.r.t. x

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{i\alpha x} dx = c^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{i\alpha x} dx$$

$$\Rightarrow \frac{d^2 \hat{u}(\alpha,t)}{dt^2} = -c^2 \alpha^2 \hat{u}(\alpha,t)$$

$$e^{i\alpha x} \frac{\partial u}{\partial x} \Big|_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} e^{i\alpha x} \frac{\partial u}{\partial x} dx$$

$$= -i\alpha \left[e^{i\alpha x} u \Big|_{-\infty}^{\infty} - i\alpha \int_{-\infty}^{\infty} e^{i\alpha x} u \, dx \right]$$

$$= -\alpha^2 \int_{-\infty}^{\infty} e^{i\alpha x} u \, dx$$

$$\Rightarrow \frac{d^2 \hat{u}}{dt^2} + c^2 \alpha^2 \hat{u} = 0$$

$$\hat{u}(\alpha,t) = A \cos c\alpha t + B \sin c\alpha t$$

$$\frac{d\hat{u}}{dt} = -Ac\alpha \sin c\alpha t + Bc\alpha \cos c\alpha t$$

$\frac{\partial u}{\partial t} = 0$ at $t = 0$ (given)

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{i\alpha x} dx = \frac{d}{dt} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u e^{i\alpha x} dx = \frac{d\hat{u}}{dt} = 0$$

$\frac{d\hat{u}}{dt} = Bc\alpha$ if $t = 0$ $\quad \therefore Bc\alpha = 0 \quad \therefore B = 0$

$$\hat{u}(\alpha,t) = A \cos c\alpha t$$

$u(x,0) = f(x)$ (given) $\therefore \hat{u}(\alpha,0) = \hat{f}(\alpha) = A$

$$\hat{u}(\alpha,t) = \hat{f}(\alpha) \cos c\alpha t$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) \cos c\alpha t \, e^{-i\alpha x} d\alpha$$

Ex Solve the same (previous) problem with

$c = 1, \quad f(x) = e^{-|x|}$.

Soln: $\hat{u}(\alpha, 0) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{x} e^{i\alpha x} dx + \int_{0}^{\infty} e^{-x} e^{i\alpha x} dx \right]$

$= \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$

$A(\alpha) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2}$

$\hat{u}(\alpha, t) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2} \cos \alpha t$

$u(x, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+\alpha^2} \cos \alpha t \, e^{-i\alpha x} \, d\alpha$

Ex Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad -\infty < x < \infty$

$u(x, 0) = f(x)$

u is bounded as $y \to \infty$; u and $\frac{\partial u}{\partial x} \to 0$ as $|x| \to \infty$

Soln: $-\alpha^2 \hat{u}(\alpha, y) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_{yy} e^{i\alpha x} dx = 0$

$\Rightarrow \frac{d^2 \hat{u}(\alpha, y)}{dy^2} - \alpha^2 \hat{u}(\alpha, y) = 0$

$\hat{u}(\alpha, y) = A e^{\alpha y} + B e^{-\alpha y}$

$\hat{u}$ is also bdd as $y \to \infty$ $\quad \therefore A = 0$ for $\alpha > 0$

$B = 0$ for $\alpha < 0$

$\hat{u}(\alpha, y) = A e^{-|\alpha| y}$

$u(x, 0) = f(x) \quad \hat{u}(\alpha, 0) = \hat{f}(\alpha) \qquad A = \hat{f}(\alpha)$

$\hat{u}(\alpha, y) = \hat{f}(\alpha) e^{-|\alpha| y} \qquad u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{-|\alpha| y} e^{-i\alpha x}$

Solve by using L.T.

Ex $\dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} - 1 \qquad x>0 \; t>0$

s.t. $u(x,0) = 0 = u_t(x,0)$

$u(0,t) = 0 \qquad \lim\limits_{x\to\infty} u_x(x,t) = 0$

sol^n: $s^2 \bar{u}(x,s) - s u(x,0) - u_t(x,0) = a^2 \dfrac{d^2\bar{u}}{dx^2} - \dfrac{1}{s}$

$\Rightarrow s^2 \bar{u}(x,s) = a^2 \dfrac{d^2\bar{u}}{dx^2} - \dfrac{1}{s}$

$\Rightarrow \dfrac{d^2\bar{u}}{dx^2} - \dfrac{s^2}{a^2}\bar{u} = \dfrac{1}{a^2 s}$

$\bar{u} = Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x} + \dfrac{1}{a^2 s}\dfrac{1}{D^2 - \frac{s^2}{a^2}}(1)$

$= Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x} + \dfrac{1}{a^2 s}\dfrac{1}{\left(-\frac{s^2}{a^2}\right)}\left[1 - \dfrac{a^2}{s^2}D^2\right]^{-1}(1)$

$= Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x} - \dfrac{1}{s^3}$

$u(0,t) = 0 \quad \therefore \bar{u}(0,s) = 0 \qquad A + B - \dfrac{1}{s^3} = 0$

$u_x(x,t) = 0 \quad \bar{u}_x(x,s) = 0 \quad$ as $\quad x \to \infty$

$\bar{u}_x(x,s) = \dfrac{s}{a}\left[Ae^{\frac{s}{a}x} - Be^{-\frac{s}{a}x}\right]$

$\therefore A = 0 \qquad \therefore B = \dfrac{1}{s^3}$

$\bar{u}(x,s) = \dfrac{1}{s^3}e^{-\frac{s}{a}x} - \dfrac{1}{s^3}$

$u = -\dfrac{t^2}{2} + \dfrac{1}{2}\left(t - \dfrac{x}{a}\right)^2 H\left(t - \dfrac{x}{a}\right)$