FLUID ROTATION

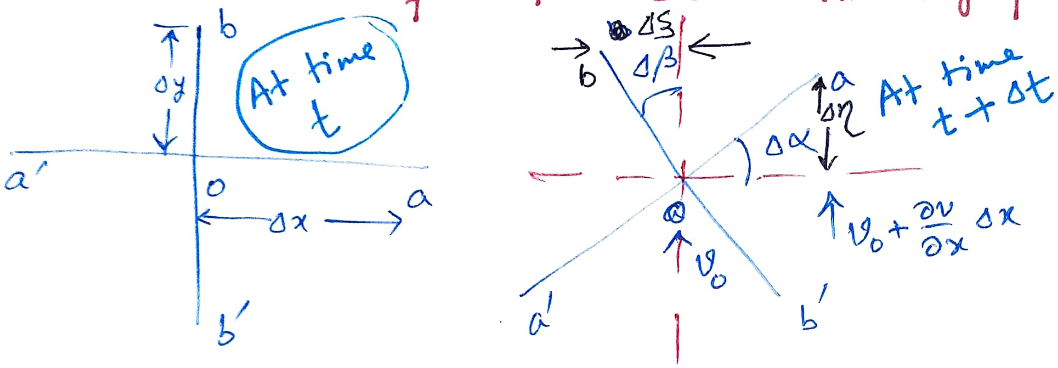
Angular velocity as a field variable $\vec{\omega} = \hat{i}\,\omega_x + \hat{j}\,\omega_y + \hat{k}\,\omega_z$

ω_x refers to the rotation about the x-axis

ω_y refers to the rotation about the y-axis

ω_z refers to the rotation about the z-axis

Consider motion of a fluid element in x-y plane



y-component of velocity at point 'o' is v_0

Then y-component of velocity at point a is $v_0 + \frac{\partial v}{\partial x}\delta x + \cdots$

By Taylor Series Expansion

Angular velocity of line oa is

$$\omega_{oa} = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\eta/\delta x}{\Delta t}$$

$\Delta\eta$ is the extra movement of 'a', over and above the movement of 'o'.

$$= \left(\frac{\partial v}{\partial x}\delta x\right)\Delta t$$

$$\Rightarrow \boxed{\omega_{oa} = \frac{\partial v}{\partial x}}$$

ROTATION OF LINE ob

x – component of velocity at point o is u_o

Then x – component of velocity at point b is $u_o + \frac{\partial u}{\partial y}\Delta y + \cdots$

By Taylor Series Expansion

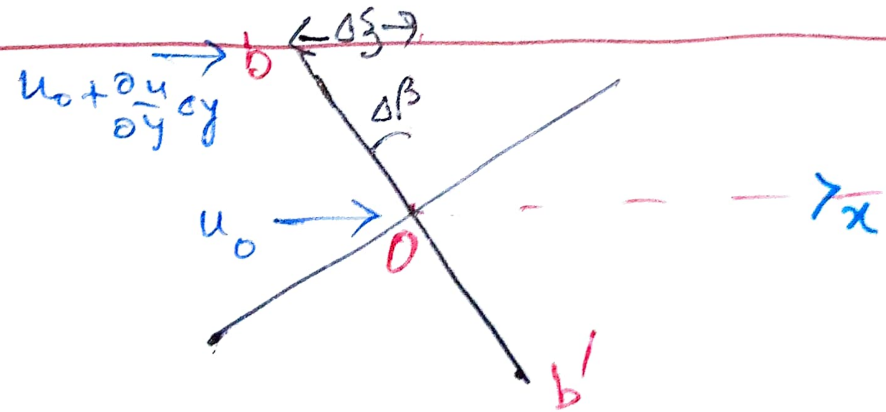
$$\Rightarrow \Delta \xi = -\frac{\partial u}{\partial y}\Delta y \, \Delta t$$

(Extra movement of 'b')

Negative, because the movement is in negative x direction.

$\Rightarrow$ The angular velocity of line ob is

$$\boxed{\omega_{ob} = \lim_{\Delta t \to 0} \frac{\Delta \beta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \xi / \Delta y}{\Delta t} = -\frac{\partial u}{\partial y}}$$



Rotation of fluid element about z-axis

= Average angular velocity of two mutually $\perp$ line elements oa and ob in x-y plane

$$\Rightarrow \boxed{\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}$$

Similarly ω_x = Rotation rate of pairs of $\perp$ line segments in y-z plane

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

and $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

$$\vec{\omega} = \frac{1}{2}\nabla \times \vec{V} = \frac{1}{2}\,\mathrm{Curl}\,\vec{V} = \frac{1}{2}\vec{\zeta} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Definition of <u>IRROTATIONAL FLOW</u>

Development of rotation requires shear stress on the surface. It cannot develop under the action of body force (gravity) or <u>normal</u> surface force (Pressure).

POTENTIAL FUNCTION & POTENTIAL LINES

for two dimensional, incompressible, irrotational flow.

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}$$

ϕ can exist, only if the flow is irrotational

i.e., $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

Because in that case, $\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0$

Note: Stream function exists when mass continuity is valid.

Potential function " " irrotationality is valid.

Potential lines are the lines along which potential function is constant.

$\Rightarrow$ Along potential lines, ϕ is constant, or $d\phi$ is zero.

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{\text{const. } \phi} = -\frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} = -\frac{u}{v}$$

Along streamlines, ψ is constant, or $d\psi$ is zero.

$$\Rightarrow \quad \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \quad \Rightarrow \quad \left.\frac{dy}{dx}\right|_{\text{Constant } \psi} = -\frac{\frac{\partial \psi}{\partial x}}{\frac{\partial \psi}{\partial y}} = \frac{v}{u}$$

Product of two slopes $= -1$ at a point.

$\Rightarrow$ Potential lines and streamlines are orthogonal.

Unsteady-state Bernoulli's Equation

For irrotational flow the momentum conservation is given by Euler Equation

$$\rho \frac{D\vec{V}}{Dt} = \rho g - \nabla P$$

$$\frac{D\vec{V}}{Dt} = \frac{\partial V}{\partial t} + (V \cdot \nabla) V = \frac{\partial V}{\partial t} + \nabla\left(\frac{1}{2}V^2\right) + \xi \times V$$

As per vector identity

Dividing both sides of Euler Equation by ρ

$$\frac{\partial V}{\partial t} + \nabla\left(\frac{1}{2}V^2\right) + \xi \times V + \frac{\nabla P}{\rho} - g = 0$$

The terms on left hand side represent $\left(\frac{\text{Force}}{\text{mass}}\right)$.

A dot product of (L.H.S.) with an arbitrary displacement vector dr gives work done or energy.

$$\left[\frac{\partial V}{\partial t} + \nabla\left(\frac{1}{2}V^2\right) + \xi \times V + \frac{1}{\rho}\nabla P - g\right] \cdot dr = 0$$

Without $(\xi \times V) \cdot dr$ term, L.H.S. becomes

$$\frac{\partial V}{\partial t} \cdot dr + d\left(\frac{1}{2}V^2\right) + \frac{dP}{\rho} + g\,dz = 0$$

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\Rightarrow \boxed{\int_1^2 \frac{\partial V}{\partial t}\,ds + \int_1^2 \frac{dP}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right) = 0}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

Between two points 1 and 2

$$\nabla(P) \cdot d\vec{r} = \left(\frac{\partial P}{\partial x}dx\right)\hat{i}\cdot\hat{i} + \left(\frac{\partial P}{\partial y}dy\right)\hat{j}\cdot\hat{j} + \left(\frac{\partial P}{\partial z}dz\right)\hat{k}\cdot\hat{k} = dP$$

Similarly $\nabla\left(\frac{1}{2}V^2\right) \cdot d\vec{r} = d\left(\frac{1}{2}V^2\right)$.

$$g = -g\hat{k}$$

$$g \cdot dr = -g\hat{k} \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) = -g\,dz$$

$(\xi \times V) \cdot dr$ can be zero, when

(1) V is zero ; No flow (Hydrostatics / Fluid statics)

(2) ξ is zero ; Irrotational flow

(3) $dr \parallel^{to} V$; Integration along a streamline

(4) $dr \perp^{to} \xi \times V$; Special and Rare case; No need to consider.

Bernoulli's Equation along a streamline

$$\int_1^2 \frac{\partial V}{\partial t} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}(V_2^2 - V_1^2) + g(z_2 - z_1) = 0$$

gives energy conservation between points 1 and 2 along a streamline, where ds is the arc length.

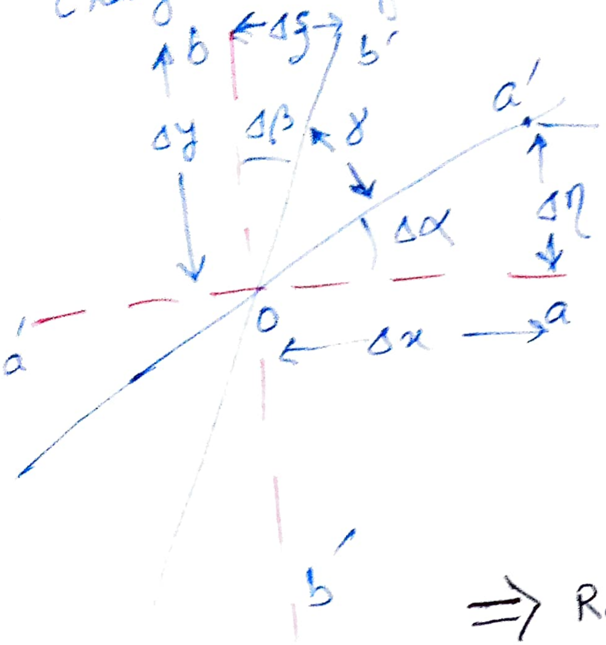
For incompressible and steady flow, $\frac{p}{\rho} + \frac{1}{2}V^2 + gz =$ Constant along a streamline.

constant may vary from streamline to streamline

For irrotational flow, integration can be performed between any two points on the flow field, and the constant will not vary.

Angular Deformation Rate

Change in angle between two mutually ⊥ line segments in the fluid.



Rate of angular deformation of the fluid element in x-y plane

= Rate of decrease of angle γ between lines oa and ob.

The change in angle γ over time interval δt is $\delta\gamma = \gamma - 90° = -(\delta\alpha + \delta\beta)$

$\Rightarrow$ Rate of angular deformation is $-\dfrac{d\gamma}{dt} = \dfrac{d\alpha}{dt} + \dfrac{d\beta}{dt}$

Now, $$\frac{d\alpha}{dt} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} = \lim_{\delta t \to 0} \frac{\delta\eta/\delta x}{\delta t} = \lim_{\delta t \to 0} \frac{\left\{\frac{\partial v}{\partial x}\delta x\, \delta t\right\}\frac{1}{\delta x}}{\delta t} = \frac{\partial v}{\partial x}$$

$$\frac{d\beta}{dt} = \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t} = \lim_{\delta t \to 0} \frac{\delta\xi/\delta y}{\delta t} = \lim_{\delta t \to 0} \frac{\left\{\left(\frac{\partial u}{\partial y}\right)\delta y\, \delta t\right\}\frac{1}{\delta y}}{\delta t} = \frac{\partial u}{\partial y}$$

Rate of angular deformation in the x-y plane is

$$-\dot{\gamma} = -\frac{d\gamma}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}.$$

Note the absence of $\left(\frac{1}{2}\right)$ term and the $(-)$ sign in front of $\frac{\partial u}{\partial y}$, in reference to rotational rate