Superposition of flow using complex potential

In a 2-D system, velocity components $\left(\vec{V} = u\hat{i} + v\hat{j}\right)$ can be related to stream function and potential function

$$\left.\begin{aligned} \frac{\partial \phi}{\partial x} &= u = \frac{\partial \psi}{\partial y} \\ \frac{\partial \phi}{\partial y} &= v = -\frac{\partial \psi}{\partial x} \end{aligned}\right\}$$

Complex Potential $F(z) = \phi(x,y) + i\psi(x,y)$

$$F'(z) = \frac{dF}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$= \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}$$

$$= u(x,y) - iv(x,y)$$

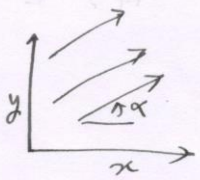
= Complex velocity, referred as $W(z)$.

Unit complex potentials

$F(z) = Uz \qquad W(z) = u - iv = U$

$F(z) = -iVz \qquad W(z) = u - iv = -iV$

$F(z) = \left(Ce^{-i\alpha}\right)z \qquad W(z) = u - iv = C\cos\alpha - iC\sin\alpha$



$$F(z) = C\ln z = C\ln\left(re^{i\theta}\right)$$
$$= C\ln r + iC\theta$$
$$= \phi + i\psi$$

$$F(z) = -iC\ln z = -iC\ln\left(re^{i\theta}\right)$$
$$= C\theta - iC\ln r$$
$$= \phi + i\psi$$

Velocity of source/sink

$$W(z) = \frac{dF}{dz} \Rightarrow u - iv = \frac{c}{z}$$

Further,
$$\left.\begin{aligned} u &= u_r \cos\theta - u_\theta \sin\theta \\ v &= u_r \sin\theta + u_\theta \cos\theta \end{aligned}\right\}$$

$$\begin{aligned}(u - iv) &= (u_r \cos\theta - u_\theta \sin\theta) - i(u_r \sin\theta + u_\theta \cos\theta) \\ &= u_r(\cos\theta - i\sin\theta) - i\,u_\theta(\cos\theta - i\sin\theta) \\ &= (u_r - i u_\theta)\,e^{-i\theta}\end{aligned} \Bigg\} \Rightarrow \left.\begin{aligned} u_r &= \frac{c}{r} \\ u_\theta &= 0\end{aligned}\right\}$$

$$= \frac{c}{z} = \frac{c}{re^{i\theta}} = \frac{c}{r}e^{-i\theta}$$

$$\text{Volume flow per unit depth} = \int_0^{2\pi} u_r\,(r\,d\theta) = 2\pi c$$

$$\Rightarrow F(z) = \frac{(-Q)}{2\pi}\ln z$$

when the source is located at position $z = z_0$

$$F(z) = \frac{(-Q)}{2\pi}\ln(z - z_0)$$

Superposition: source + uniform flow

$$F(z) = Uz + \frac{m}{2\pi} \ln z \qquad m \equiv (-Q)$$

$$\Rightarrow \phi = Ur\cos\theta + \frac{m}{2\pi} \ln r$$

$$\psi = Ur\sin\theta + \frac{m}{2\pi}\theta$$

$$w(z) = \frac{dF}{dz} = (u_r - iu_\theta)e^{-i\theta}$$

$$\frac{dF}{dz} = U + \frac{m}{2\pi z} = U + \frac{m}{2\pi r e^{i\theta}} = \left(Ue^{i\theta} + \frac{m}{2\pi r}\right)e^{-i\theta}$$

$$= \left[\left\{U\cos\theta + \frac{m}{2\pi r}\right\} - i\left\{-U\sin\theta\right\}\right]e^{-i\theta}$$

$$\Rightarrow u_r = U\cos\theta + \frac{m}{2\pi r} \qquad u_\theta = -U\sin\theta$$

Stagnation Point

$$u_r = 0, \quad u_\theta = 0 \quad \Rightarrow \quad \theta = \pi, \quad r = \frac{m}{2\pi U}$$

$$\psi\Big|_{\text{body streamline}} = U\left(\frac{m}{2\pi U}\right)\sin\pi + \frac{m}{2\pi}\pi = \frac{m}{2}$$

Any point on body streamline follows $\psi = \frac{m}{2}$

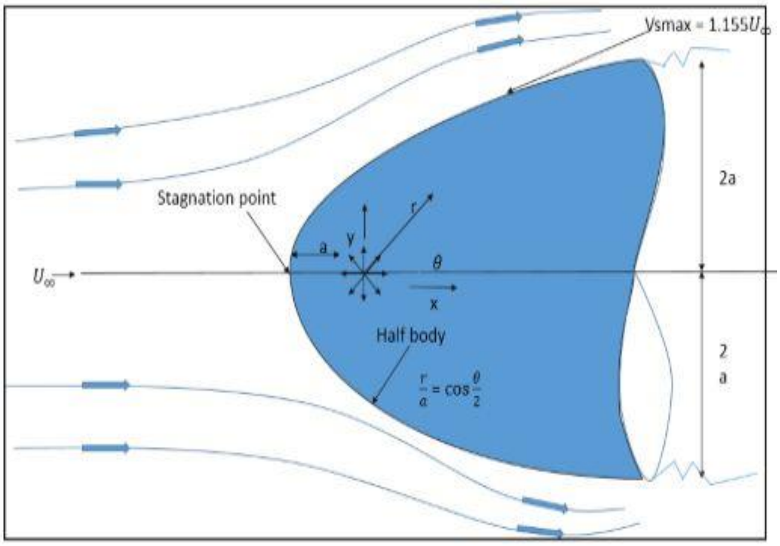
$$\Rightarrow r_s = \frac{m}{2\pi U} \frac{\pi - \theta_s}{\sin\theta_s}$$

$$\Rightarrow y_s = \frac{m}{2\pi U}(\pi - \theta_s)$$

$\Rightarrow$ Asymptotic half width of Rankine half body

$$y_s\Big|_{\theta=0} = \frac{m}{2U}$$

$\Rightarrow$ The point, directly above the origin $y_s\Big|_{\theta=\frac{\pi}{2}} = \frac{m}{4U}$


Vsmax = 1.155U_∞
2a
Stagnation point
r
y
a
θ
U_∞
x
Half body
$\frac{r}{a} = \cos\frac{\theta}{2}$
2
a