

# Fluid Mechanics

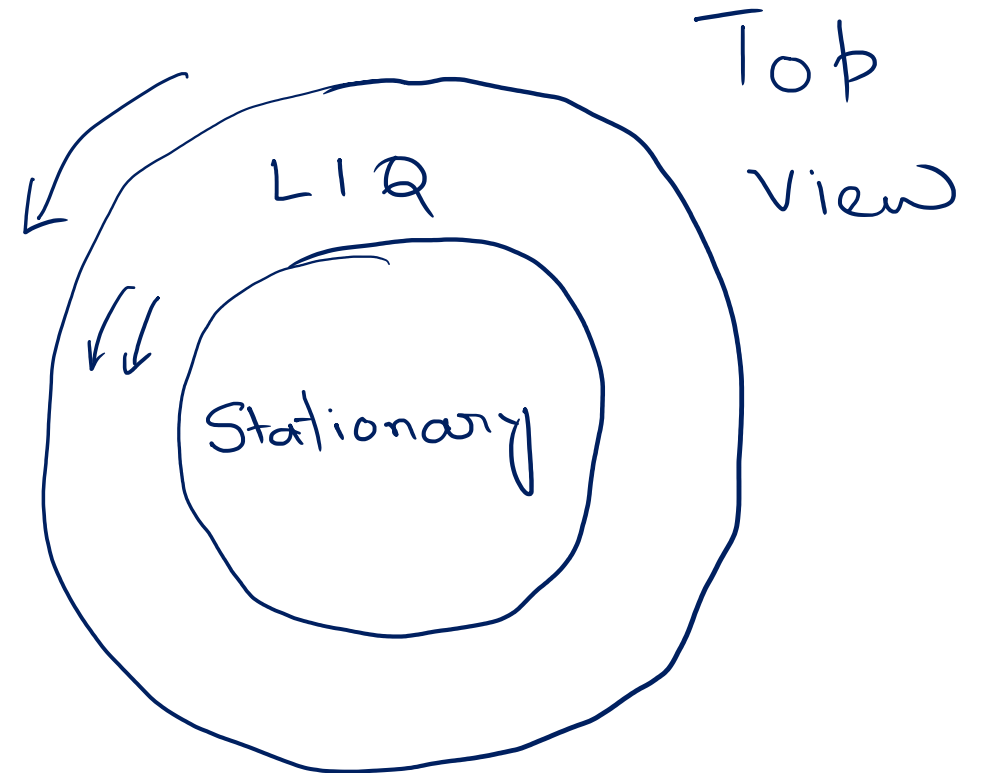
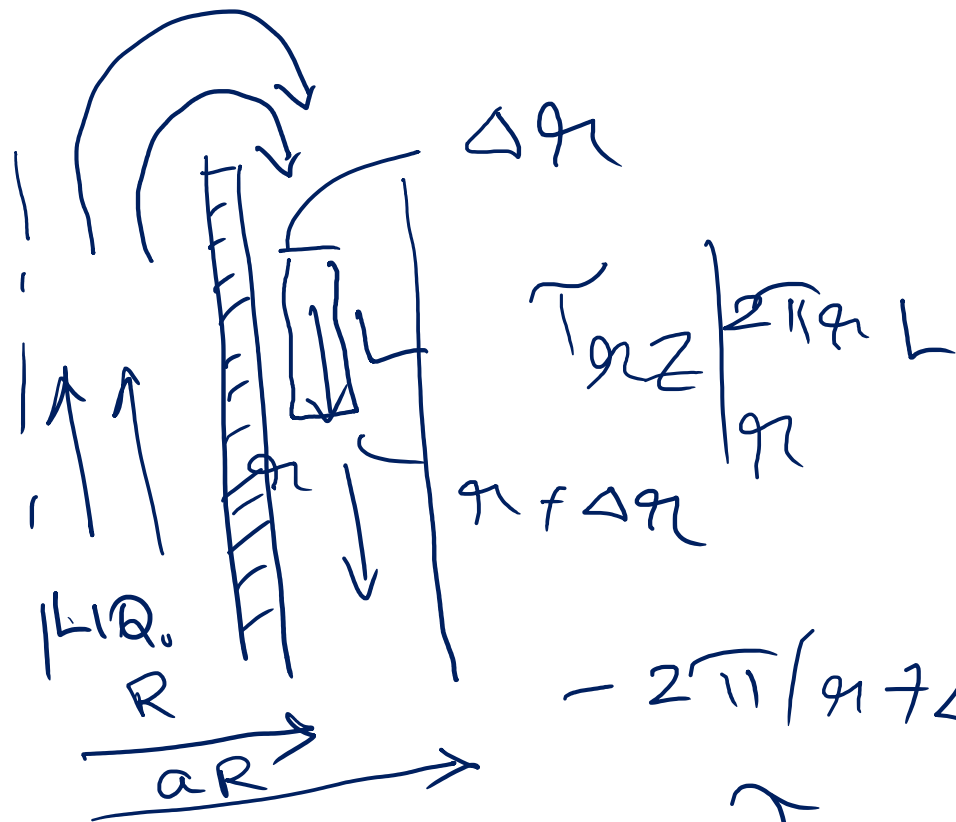
Lectures 5 – 6

Continuity Equation

Equation of Motion

Navier-Stokes Equation

## Limitations of Shell Momentum Balance

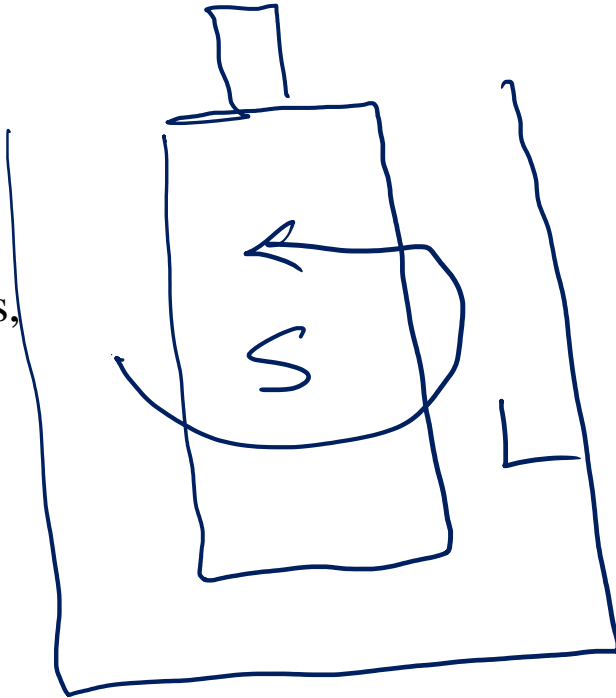


$$v_z \neq 0, v_r, v_\theta = 0$$

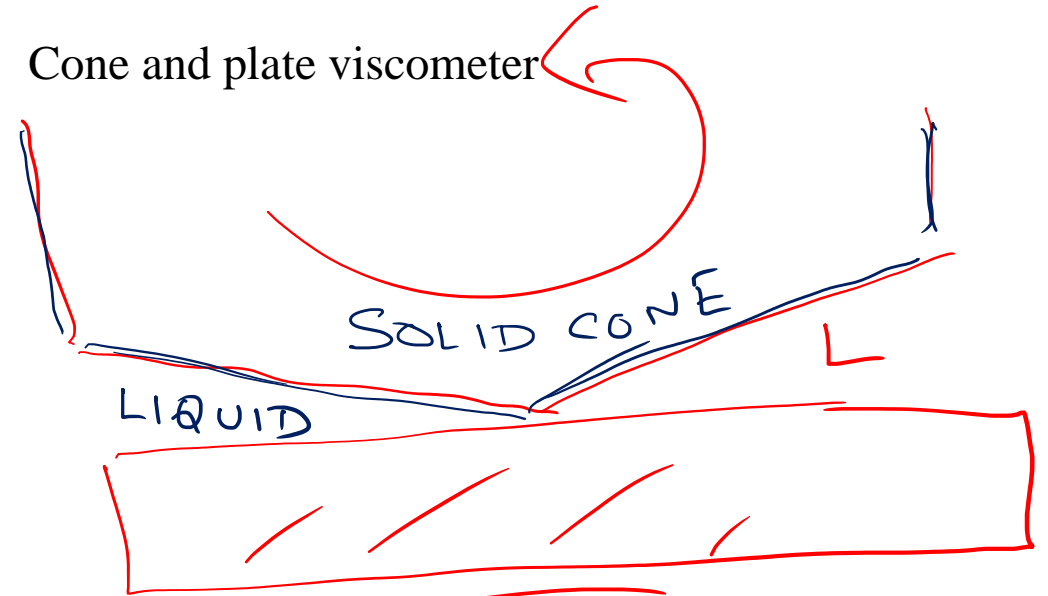
$$+ 2\pi r \Delta r L p_g = 0$$

# Situations of Increasingly Complex Geometry – Shell MM Balance??

One cylinder rotates,  
the other stationary



Cone and plate viscometer



A stationary plate  
suddenly set in  
motion

$$t \geq 0$$

$$v = V$$

$$z$$

$$L$$

$$x$$

$$v_a = (z, t)$$

$$t = 0$$

$$v = 0$$

Increasingly difficult to visualize a shell, especially for situations where multi-dimensional effects are to be considered

Need for a generalized approach

## Generalized Approach

- Concept of Partial, Total and Substantial Time Derivatives
- Equation of Continuity
- Equation of Motion
- Navier Stokes Equation
- Euler Equation

Conceptual derivation of the equation of continuity and equation of motion,  
Navier-Stokes equation

*See the attached file*

## Equation of Continuity

$$[\partial\rho/\partial t + (\nabla \cdot \rho\mathbf{v}) = 0]$$

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*Cartesian coordinates (x, y, z):*

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$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

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*Cylindrical coordinates (r,  $\theta$ , z):*

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$$\frac{\partial\rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

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*Spherical coordinates (r,  $\theta$ ,  $\phi$ ):*

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$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}(\rho v_\phi) = 0$$

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## Equation of Motion for a Newtonian Fluid with Constant $\rho$ and $\mu$

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}]$$

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*Cartesian coordinates (x, y, z):*

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$$\rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left[ \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho g_x$$

$$\rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[ \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho g_y$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[ \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z$$

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*Cylindrical coordinates  $(r, \theta, z)$ :*

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$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] + \rho g_r$$

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \rho g_\theta$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z$$

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Spherical coordinates  $(r, \theta, \phi)$ :

$$\begin{aligned} \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) &= -\frac{\partial p}{\partial r} \\ &+ \mu \left[ \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right] + \rho g_r \\ \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r} \right) &= -\frac{1}{r} \frac{\partial p}{\partial \theta} \\ &+ \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right] + \rho g_\theta \\ \rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) &= -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} \\ &+ \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] + \rho g_\phi \end{aligned}$$

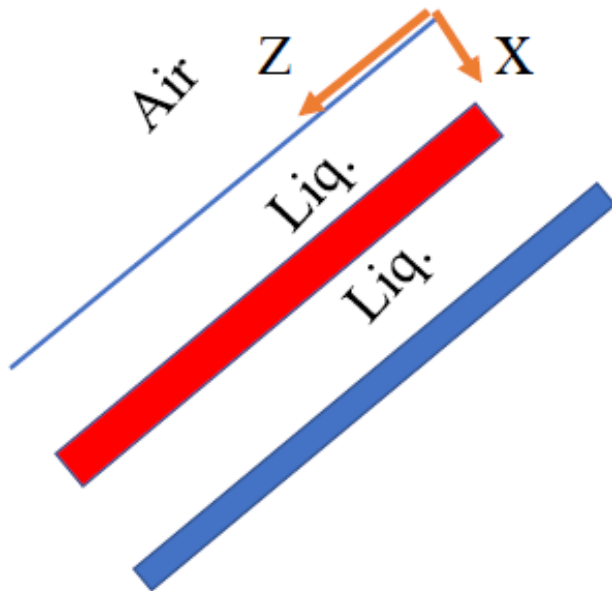
$$\rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial P}{\partial z} + \mu \left[ \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z$$

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right\}$$

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \rho g_\theta$$

## Solve flow along an inclined plate using NS equation

$$\rho \left( \cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_x} \cancel{\frac{\partial v_z}{\partial x}} + \cancel{v_y} \cancel{\frac{\partial v_z}{\partial y}} + \cancel{v_z} \cancel{\frac{\partial v_z}{\partial z}} \right) = - \cancel{\frac{\partial P}{\partial z}} + \mu \left[ \frac{\partial^2 v_z}{\partial x^2} + \cancel{\frac{\partial^2 v_z}{\partial y^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}} \right] + \rho g_z$$



SS

$$V_x = 0$$

$$V_y = 0$$

$V_z$  is not a function of  $z$

No applied pressure gradient

$V_z$  is not a function of  $y$

$V_z$  is not a function of  $z$

$$0 = \mu \left[ \frac{d^2 v_z}{dx^2} \right] + \rho g \cos \beta$$

## Solve the pipe flow problem using NS equation

$$\rho \left( \cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r \frac{\partial v_z}{\partial r}} + \cancel{\frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta}} + \cancel{v_z \frac{\partial v_z}{\partial z}} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \cancel{\frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}} \right\}$$

SS

$$V_r = 0$$

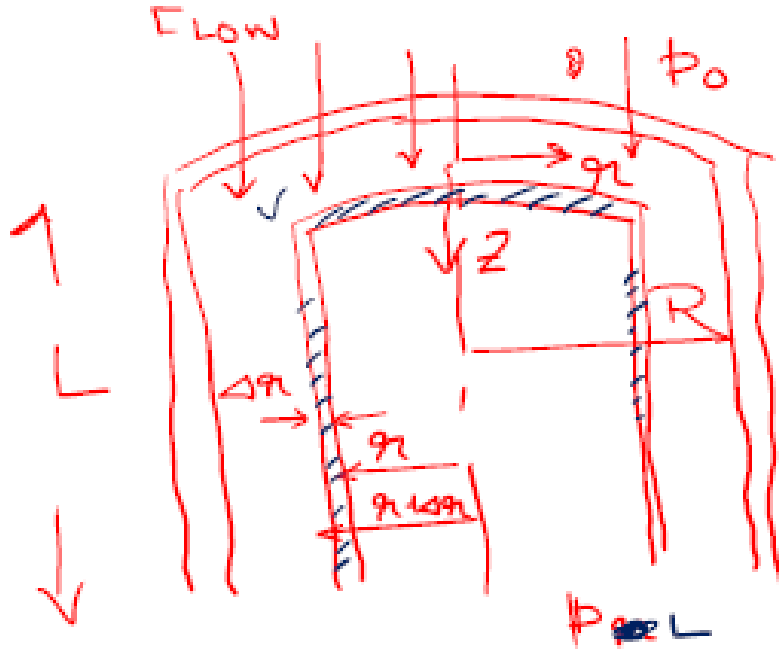
$$V_\theta = 0$$

$V_z$  is not a function of  $\theta$

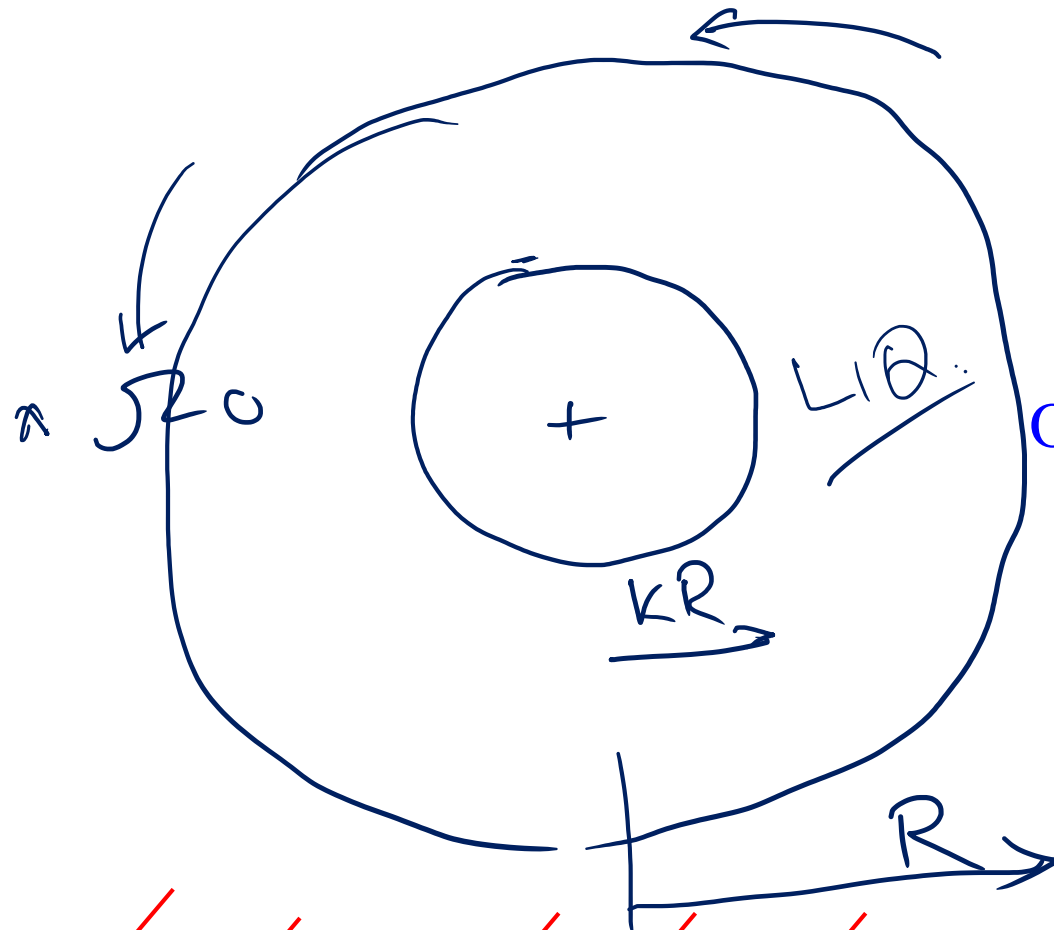
Presence of an applied pressure gradient

$V_z$  is not a function of  $\theta$

$V_z$  is not a function of  $z$



# Tangential Annular Flow of a Newtonian Fluid in Between two Cylinders. One rotating, the other remains stationary



Only one term remains in the NS equation

Solve with no slip  
condition at the two walls  
of the rotating cylinders at  
 $r = R$  and  $r = kR$

$$\cancel{\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right)} = \cancel{-\frac{1}{r} \frac{\partial p}{\partial \theta}} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \cancel{\rho g_\theta}$$

$$\rho \left( \cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r} \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z} \frac{\partial v_\theta}{\partial z} + \cancel{\frac{v_r v_\theta}{r}} \right) = -\cancel{\frac{1}{r} \frac{\partial p}{\partial \theta}} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \cancel{\frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancel{\frac{2}{r^2} \frac{\partial v_r}{\partial \theta}} \right] + \cancel{\rho g_\theta}$$

$$0 = \mu \left[ \frac{d}{dr} \left( \frac{1}{r} \frac{d(r v_\theta)}{dr} \right) \right]$$

BC

1. At  $r = kR$       $V_\theta = 0$
2. At  $R = R$       $V_\theta = \Omega_o R$

Solve for  $V_\theta$