

Fluid Mechanics

Lectures 7 – 8

Continuity Equation

Equation of Motion

Navier-Stokes Equation

Increasingly difficult to visualize a shell, especially for situations where multi-dimensional effects are to be considered

Need for a generalized approach

Generalized Approach

- Concept of Partial, Total and Substantial Time Derivatives
- Equation of Continuity
- Equation of Motion
- Navier Stokes Equation
- Euler Equation

Conceptual derivation of the equation of continuity and equation of motion,
Navier-Stokes equation

See the attached file

Equation of Continuity

$$[\partial\rho/\partial t + (\nabla \cdot \rho\mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Cylindrical coordinates (r, θ , z):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

Spherical coordinates (r, θ , ϕ):

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}(\rho v_\phi) = 0$$

Equation of Motion for a Newtonian Fluid with Constant ρ and μ

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}]$$

Cartesian coordinates (x, y, z):

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho g_x$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho g_y$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z$$

Cylindrical coordinates (r, θ, z) :

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] + \rho g_r$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \rho g_\theta$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z$$

Spherical coordinates (r, θ, ϕ) :

$$\begin{aligned} \rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) &= -\frac{\partial p}{\partial r} \\ &+ \mu \left[\frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right] + \rho g_r \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r} \right) &= -\frac{1}{r} \frac{\partial p}{\partial \theta} \\ &+ \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right] + \rho g_\theta \\ \rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) &= -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} \\ &+ \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] + \rho g_\phi \end{aligned}$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial P}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z$$


$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = \rho g_z - \frac{\partial p}{\partial z} + \mu \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right\}$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = - \frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \rho g_\theta$$

Tangential Annular Flow of a Newtonian Fluid in Between two Cylinders. One rotating, the other remains stationary

Only one term remains in the NS equation

Solve with no slip
condition at the two walls
of the rotating cylinders at
 $r = R$ and $r = kR$



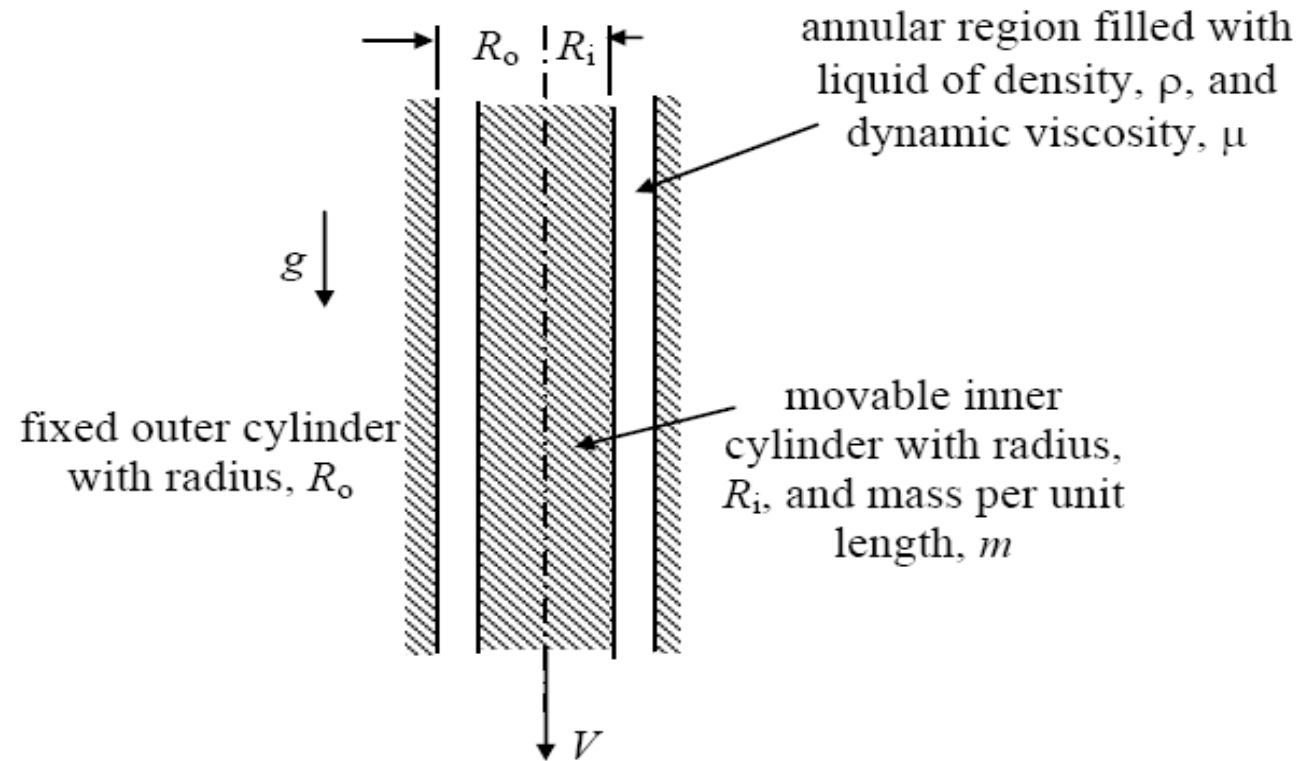
$$\cancel{\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right)} = \cancel{-\frac{1}{r} \frac{\partial p}{\partial \theta}} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \cancel{\rho g_\theta}$$

$$\rho \left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r} \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{v_z} \frac{\partial v_\theta}{\partial z} + \cancel{\frac{v_r v_\theta}{r}} \right) = -\cancel{\frac{1}{r} \frac{\partial p}{\partial \theta}} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \frac{2}{r^2} \cancel{\frac{\partial v_r}{\partial \theta}} \right] + \cancel{\rho g_\theta}$$

$$0 = \mu \left[\frac{d}{dr} \left(\frac{1}{r} \frac{d(r v_\theta)}{dr} \right) \right]$$

Practice Problem

Consider two concentric cylinders with a Newtonian liquid of constant density, ρ , and viscosity, μ , contained between them. The outer pipe, with radius, R_o , is fixed while the inner pipe, with radius, R_i , and mass per unit length, m , falls under the action of gravity at a constant speed. There is no pressure gradient within the flow and no swirl velocity component. Determine the vertical speed, V , of the inner cylinder as a function of the following parameters: g , R_o , R_i , m , ρ , and μ . The space between the two cylinders is **not** 'too small' as compared to the radii of the cylinders



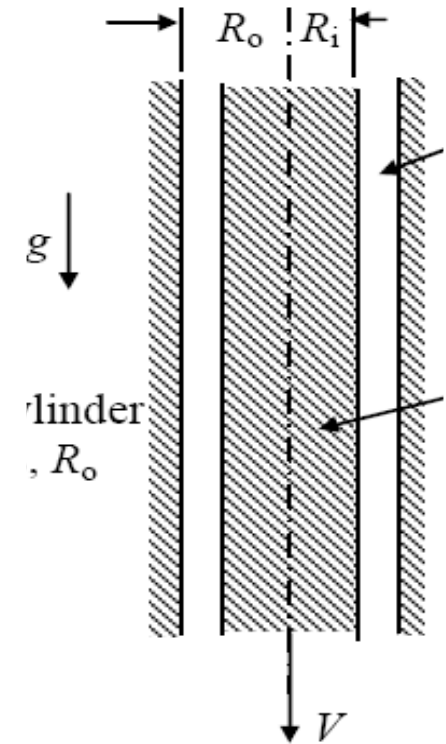
Solution

For the inner cylinder moving at constant velocity, the downward force is exactly balanced by the viscous force as

$$(\tau_w A_w) \big|_{Inner\ Cylinder} = m L g$$

The z component of the equation of motion in cylindrical coordinate is

$$\begin{aligned} & \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) \\ &= -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z \end{aligned}$$



Cancelling the terms with the observations i) steady state, ii) v_z is a function of r only, not of z or θ , iii) no applied pressure gradient

The following simplified form of the NS equation can be obtained.

Governing equation

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) = - \frac{\rho g}{\mu}$$

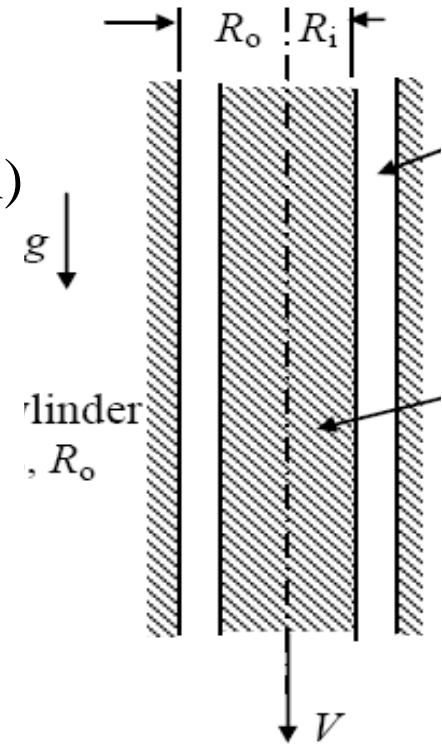
Upon integration (with C_1 and C_2 being the constants of integration)

$$v_z = - \frac{\rho g r^2}{4\mu} + C_1 \ln r + C_2$$

Boundary Conditions

$$v_z = V \text{ at } r = R_i \quad \Rightarrow \quad V = - \frac{\rho g}{4\mu} R_i^2 + C_1 \ln R_i + C_2$$

$$v_z = 0 \text{ at } r = R_o \quad \Rightarrow \quad 0 = - \frac{\rho g}{4\mu} R_o^2 + C_1 \ln R_o + C_2$$



Therefore
$$V = \frac{\rho g}{4\mu} (R_o^2 - R_i^2) + C_1 \ln \frac{R_i}{R_o}$$

$$C_1 = \frac{1}{\ln \frac{R_i}{R_o}} \left[V - \frac{\rho g}{4\mu} (R_o^2 - R_i^2) \right]$$

$$\frac{dv_z}{dr} = -\frac{\rho g r}{2\mu} + \frac{C_1}{r} \quad \tau_{rz} = \mu \frac{dv_z}{dr} = -\frac{\rho g r}{2} + \frac{C_1 \mu}{r}$$

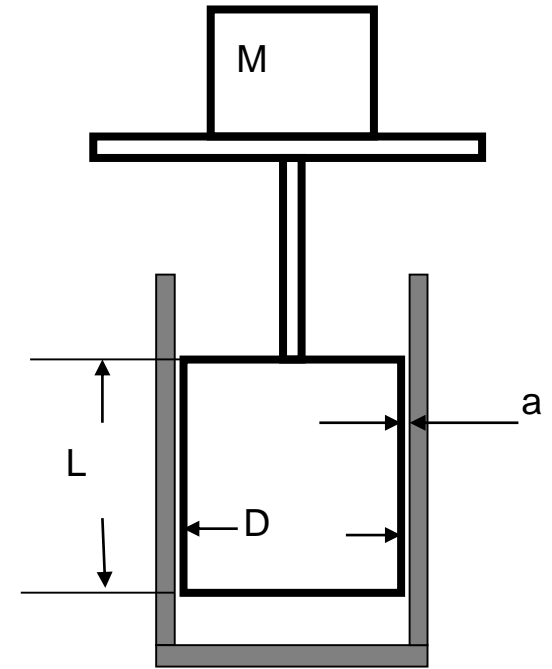
Since the force on the inner cylinder = force due to gravity

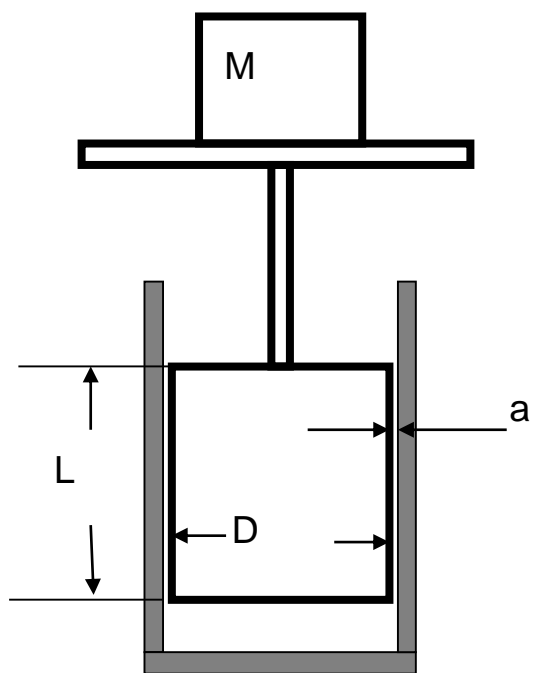
$$\tau \Big|_{r=R_i} 2\pi R_i L = m L g$$

Upon substitution for the expression of the shear stress

$$V = R_i \ln \frac{R_i}{R_o} \left(\frac{\rho g R_i}{2\mu} - \frac{m g}{2\pi R_i \mu} \right) - \frac{\rho g}{4\mu} (R_i^2 - R_o^2)$$

The basic component of a pressure gage tester consists of a piston-cylinder apparatus. The piston, 6 mm in diameter is loaded to develop a pressure of known magnitude. The radial clearance, a , is very small compared to the piston diameter D . The piston length, L , is 25 mm. Calculate the mass, M , required to produce 1.5 MPa (gage) in the cylinder. Determine the leakage flow rate as a function of radial clearance, a , for this load if the liquid is oil at 20°C (viscosity 0.42 N.s/m² and density 700 kg/m³). Specify the maximum allowable radial clearance so that the vertical movement of the piston due to leakage will be less than 1 mm/min.





$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho g_y$$

