Fluid Mechanics

Lectures 9 – 10

Continuity Equation

Equation of Motion

Navier-Stokes Equation

Equation of Continuity

$$[\partial\rho/\partial t + (\nabla \cdot \rho\mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):
$\dfrac{\partial\rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho v_x) + \dfrac{\partial}{\partial y}(\rho v_y) + \dfrac{\partial}{\partial z}(\rho v_z) = 0$
Cylindrical coordinates (r, θ, z):
$\dfrac{\partial\rho}{\partial t} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(\rho r v_r) + \dfrac{1}{r}\dfrac{\partial}{\partial\theta}(\rho v_\theta) + \dfrac{\partial}{\partial z}(\rho v_z) = 0$
Spherical coordinates (r, θ, ϕ):
$\dfrac{\partial\rho}{\partial t} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(\rho r^2 v_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\rho v_\theta \sin\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}(\rho v_\phi) = 0$

Equation of Motion for a Newtonian Fluid with Constant ρ and μ

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}]$$

Cartesian coordinates (x, y, z):

$$\rho\left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right] + \rho g_x$$

$$\rho\left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right] + \rho g_y$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Cylindrical coordinates (r, θ, z):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

Spherical coordinates (r, θ, ϕ):

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r}\right) = -\frac{\partial p}{\partial r}$$

$$+ \mu\left[\frac{1}{r^2}\frac{\partial^2}{\partial r^2}(r^2 v_r) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial v_r}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot\theta}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\theta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta \sin\theta)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] + \rho g_\theta$$

$$\rho\left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot\theta}{r}\right) = -\frac{1}{r \sin\theta}\frac{\partial p}{\partial \phi}$$

$$+ \mu\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi \sin\theta)\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin\theta}\frac{\partial v_r}{\partial \phi} + \frac{2\cot\theta}{r^2 \sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] + \rho g_\theta$$

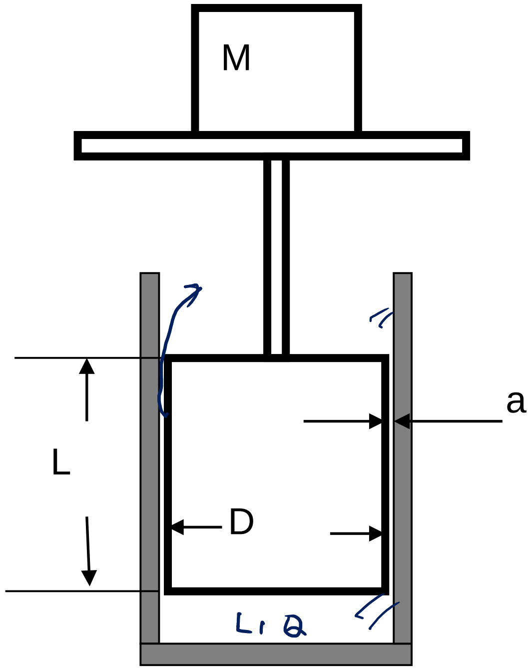
$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=-\frac{\partial P}{\partial x}+\mu\left(\frac{\partial^2 v_x}{\partial x^2}+\frac{\partial^2 v_x}{\partial y^2}+\frac{\partial^2 v_x}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial P}{\partial z}+\mu\left[\frac{\partial^2 v_z}{\partial x^2}+\frac{\partial^2 v_z}{\partial y^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+\rho g_z$$

$$\rho(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z})=\rho g_z-\frac{\partial p}{\partial z}+\mu\{\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_z}{\partial r})+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\}$$

$$\rho(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r\,v_\theta}{r})=-\frac{1}{r}\frac{\partial p}{\delta\theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\,v_\theta\right)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]+\rho g_\theta$$

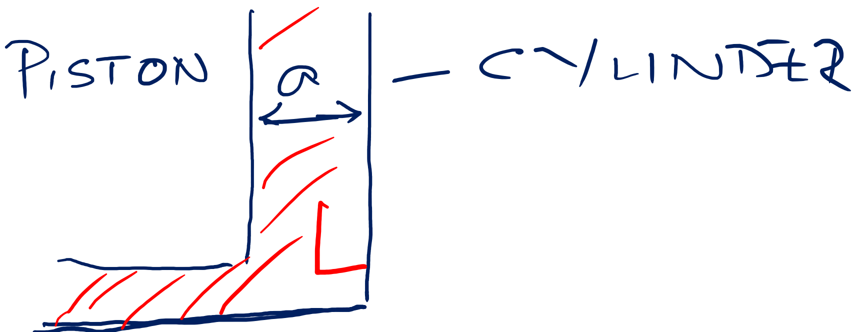
The basic component of a pressure gage tester consists of a piston-cylinder apparatus. The piston, 6 mm in diameter is loaded to develop a pressure of known magnitude. The radial clearance, a, is very small compared to the piston diameter D. The piston length, L, is 25 mm. Calculate the mass, M, required to produce 1.5 MPa (gage) in the cylinder. Determine the leakage flow rate as a function of radial clearance, a, for this load if the liquid is oil at 20°C (viscosity 0.42 N.s/m^2 and density 700 kg/m^3). Specify the maximum allowable radial clearance so that the vertical movement of the piston due to leakage will be less than 1 mm/min.

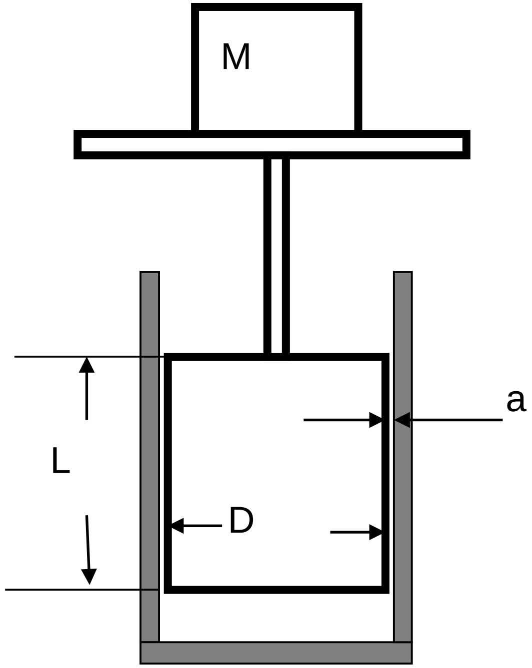


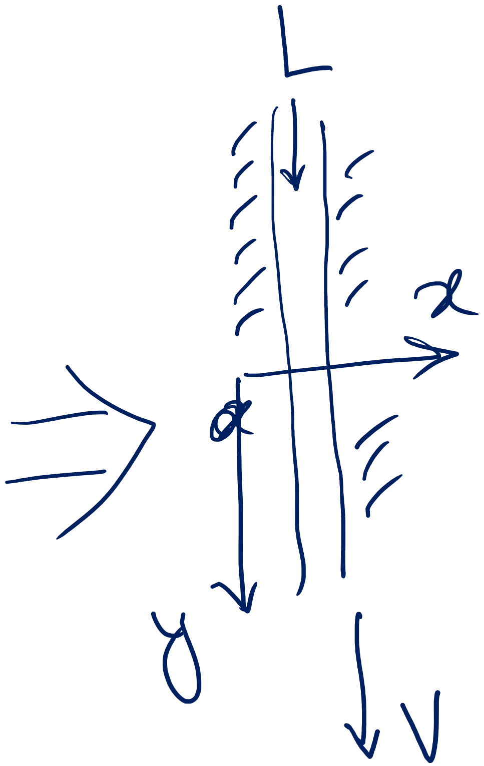
$$\frac{\pi D^2}{4}(p - p_{atm}) = Mg$$

1.5×10^6

$$M = 4.32\ kg$$







$\frac{\Delta p}{L}$:

$$\mu \frac{d^2 v_y}{dx^2} = \frac{dp}{dy} - \cancel{\rho g}$$

$v_y = 0 \quad x = 0$

$v_y = V, \quad x = a$

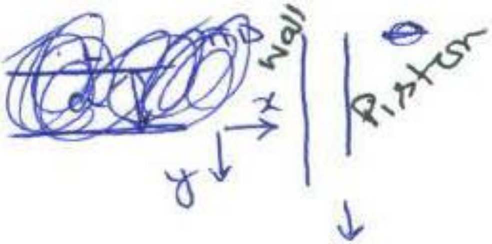
$$\rho\left(\cancel{\frac{\partial v_y}{\partial t}} + \cancel{v_x}\frac{\partial v_y}{\partial x} + v_y\cancel{\frac{\partial v_y}{\partial y}} + \cancel{v_z}\frac{\partial v_y}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v_y}{\partial x^2} + \cancel{\frac{\partial^2 v_y}{\partial y^2}} + \cancel{\frac{\partial^2 v_y}{\partial z^2}}\right] + \rho g_y$$

$$v_y = \frac{1}{2\mu}\frac{\Delta P}{L}\left[x^2 - \frac{x}{a}a^2\right] + V$$

$$\langle v_y \rangle = -\frac{1}{12\mu}\frac{\Delta P}{L}a^2 + \frac{Va}{2}$$

$$\boxed{a = 1.28 \times 10^{-5}\ m}$$

Since the gap between the cylinder and the piston is very small the flow can be treated as a flow between two parallel plates of length L and width πD, separated by 'a'.



$$\mu \frac{d^2 v_y}{dx^2} + \rho g - \frac{dp}{dx} = 0.$$

$\rho g \simeq 0$, quite small compared to $\frac{dp}{dx}$

$\rho \to O(10^3)$ $g \to O(10)$ $\rho g \to O(10^4)$

$$\frac{dp}{dx} = \frac{1.5\times10^6}{2.5\times10^{-3}} \simeq O(10^8)$$

$$\mu \frac{d^2 v_y}{dx^2} = \frac{dp}{dx} = \frac{\Delta p}{L}$$

$$v_y = \frac{1}{2\mu}\frac{\Delta p}{L} x^2 + c_1 x + c_2$$

at $x=0$, $v_y = 0 \Rightarrow c_2 = 0$.

at $x=a$, $v_y = V \Rightarrow c_1 = \frac{1}{a}\left[V - \frac{1}{2\mu}\frac{\Delta p}{L}a^2\right]$

$$v_y = \frac{1}{2\mu}\frac{\Delta p}{L}\left[x^2 - \frac{x}{a}a^2\right] + \cancel{\frac{V}{x}}\ V$$

$$\langle v_y \rangle = \frac{1}{a}\int_0^a v_y\,dx = -\frac{1}{12\mu}\frac{\Delta p}{L}a^2 + \frac{Va}{2}$$

... is quite small

$$\langle v_y \rangle = \frac{1}{a}\int_0^a v_y\,dx = -\frac{1}{12\mu}\frac{\Delta P}{L}a^2 + \frac{Va}{2}$$

Again $V = 1\,mm/min$ $\therefore$ 2nd term on rhs is quite small.

$$\therefore \langle v_y \rangle = -\frac{1}{12\mu}\frac{\Delta P}{L}a^2$$

$$Q = \langle v_y \rangle a\pi D = -\frac{1}{12\mu}\frac{\Delta P}{L}a^3\pi D,$$

The flow is in the -ve y directn as it should be.

For downward movement ($V\,m/s$) the vol. displaced is

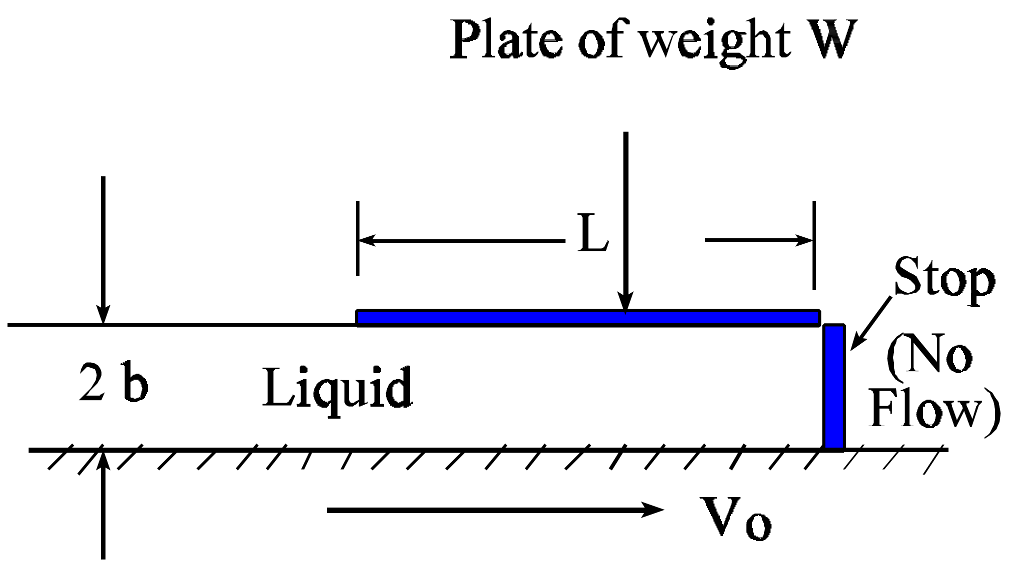
$$Q = \frac{\pi D^2}{4}V = \frac{\pi}{4}(0.006)^2 \times \frac{0.001}{60}\,\frac{m^3}{s} = 4.71\times10^{-10}\,m^3/s.$$

with $\mu = 0.42\,\frac{N\,s}{m^2}$

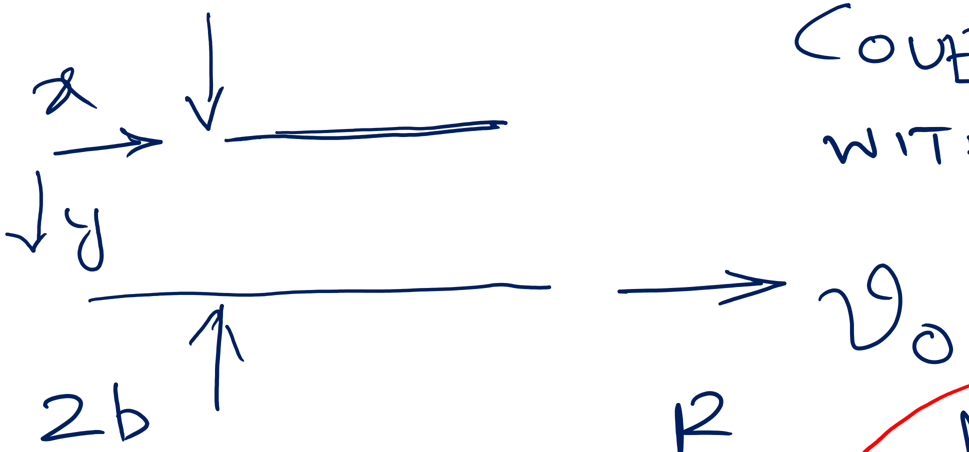
$$a = \left[\frac{12\mu Q L}{\pi D \Delta P}\right]^{1/3} = \left[\frac{12}{\pi}\times\frac{0.42\times4.71\times10^{-10}\times0.025}{0.006\times1.5\times10^6}\right]^{1/3}$$

$$\underline{a = 1.28\times10^{-5}\,m}$$

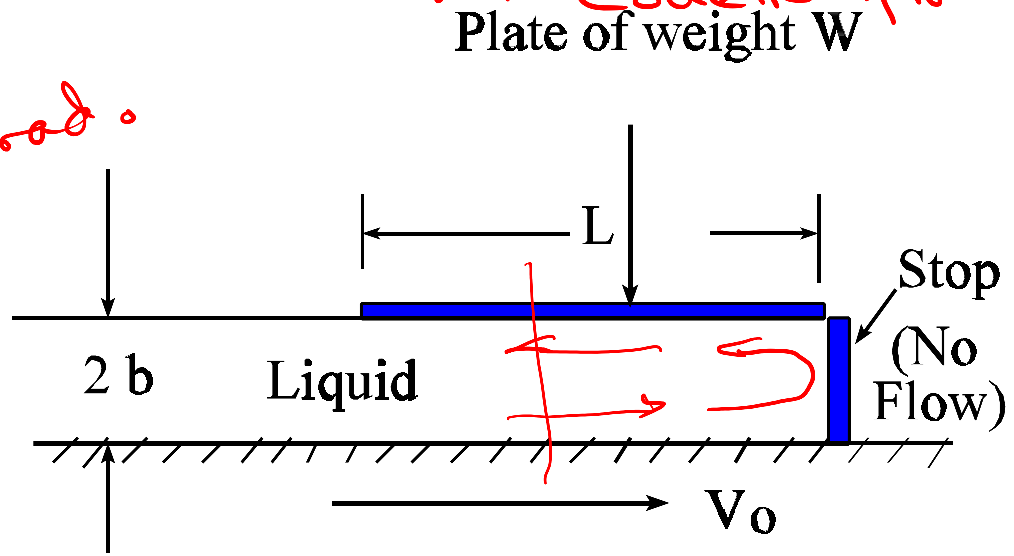
The lower plate of a lubricated thrust bearing moves to the right at velocity Vo . The stop at the right prevents any liquid flow beyond that point. Find the weight W that can be supported by the fluid (of viscosity μ and of density ρ) Assume the plate to be wide so that the end effects can be neglected. It can be assumed further that even if two unequal pressures act at the two ends (x=0 and x=L) of the plate it will not topple and the whole plate can be assumed to be acted on by an average of the two pressures at the two ends.



COUETTE FLOW WITH A ~~+~~Ve PR. GRAD.



The no-flow condition can be modelled if we assume an adverse pr. grad. with Couette flow.



$$0 = \mu \frac{d^2 u}{dy^2} - \frac{dp}{dx}$$

Adverse pr. grad.

$u = 0 \quad y = 0, \quad u = v_0, \; y = 2b$

$$u = \frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left[y^2 - 2by\right] + \frac{v_0}{2b} y$$

$$\langle u \rangle = \frac{1}{2b}\int_0^{2b} u \, dy \Rightarrow \langle u \rangle = \frac{1}{2b}\left[\frac{1}{2\mu}\left(\frac{dp}{dx}\right)\left(\frac{8}{3}b^3 - 4b^3\right) + v_0 b\right]$$

No FLOW $\langle u \rangle = 0$

$$\Rightarrow \frac{dp}{dx} = \frac{3\mu V_0}{2b^2}$$

$\frac{dp}{dx}$ is +ve.

Thus the pr. grad. opposes the flow by the motion of the bottom plate (Couette flow).

(The pr. is atmospheric at the other end)

$$\text{Av. } P_r = \frac{3\mu V_0}{4b^2} L$$

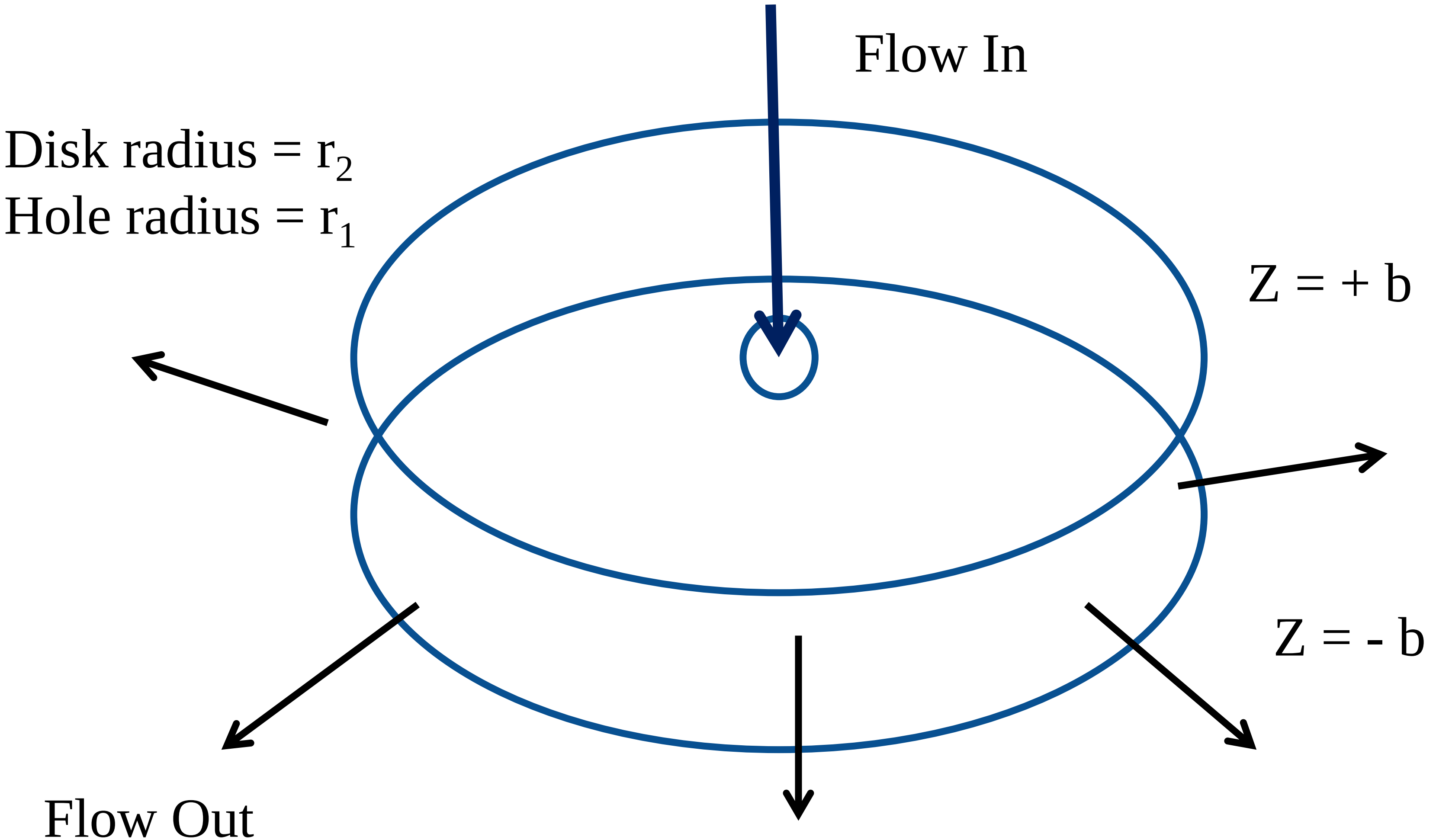
$$\text{LOAD} = P_{av} \times L \times 1$$

Order of magnitude analysis of NS Equation

Is it possible to identify the relative magnitudes of the different terms (even approximately)?

It may then be possible to neglect the term(s) that may not play a crucial role in the transport process thereby simplifying NS equations.

Flow between two parallel disks with liquid entry through a small hole at the centre of the top plate



The pressure is p_1 at $r = r_1$ and p_2 at $r = r_2$

$$\cancel{\frac{\partial \rho}{\partial t}}+\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r v_r\right)+\cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho v_\theta\right)}+\cancel{\frac{\partial}{\partial z}\left(\rho v_z\right)}=0$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r v_r\right)=0 \qquad \rightarrow \qquad r v_r = Constant; \quad v_r=\frac{\phi}{r}$$

As v_r is a function of r and z; ϕ must be a function of z.

$$\frac{\partial}{\partial r}\left(r v_r\right)=0$$

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}}+v_r\frac{\partial v_r}{\partial r}+\cancel{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}-\cancel{\frac{v_\theta^2}{r}}+\cancel{v_z\frac{\partial v_r}{\partial z}}\right)$$

$$=-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r v_r\right)\right)+\cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}-\cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}+\frac{\partial^2 v_r}{\partial z^2}\right]+\cancel{\rho g_r}$$

The governing equation

$$\rho\, v_r \frac{\partial v_r}{\partial r} = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2} \quad ; \quad v_r = \frac{\phi(z)}{r}$$

$$-\rho \frac{\phi^2}{r^3} = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 v_r}{\partial z^2}$$

The LHS refers convective transport of momentum, while the 2nd term on the RHS is the conductive, diffusive or viscous transport of momentum.

Major assumption on the nature of the flow – Convective effects may be small. The case when the entire LHS is set equal to zero is known as CREEPING FLOW.

$$0 = -\frac{dP}{dr} + \frac{\mu}{r}\frac{d^2\phi}{dz^2} \quad ; P \text{ is a } fn \text{ of } r \text{ only}, \phi = f(z) \text{ only}$$

Assume a constant applied pressure difference

$$\frac{\mu}{r}\frac{d^2\phi}{dz^2} = \frac{dp}{dr}, \quad \Delta P = p_1 - p_2$$

$$\downarrow \quad \mu \frac{d^2\phi}{dz^2}\int_{r_1}^{r_2}\frac{dr}{r} = \int_{p_1}^{p_2} dp$$

$$\mu \ln\frac{r_2}{r_1}\frac{d^2\phi}{dz^2} + \Delta p = 0$$

Boundary Conditions

$$\phi = -\frac{\Delta p z^2}{2\mu \ln r_2/r_1} + c_1 z + c_2$$

BC

1) $z = +b$

11) $z = -b$

$v = 0$

$$v_r = \frac{\phi}{r} =$$

$$v_r(r,z) = \frac{\Delta P b^2}{2\mu r \ln\frac{r_2}{r_1}}\left[1-\left(\frac{z}{b}\right)^2\right]$$

$$Q = 2\pi \int_{-b}^{+b} r\, v_r \, dz = 2\pi \int_{-b}^{+b} \phi(z)\, dz = \frac{4\pi\, \Delta P\, b^3}{3\mu \ln \frac{r_2}{r_1}}$$