VORTEX @ origin

Consider $F(z) = -ic \ln z$ where c is real

$$= -ic \ln(re^{i\theta})$$

$$= c\theta - ic \ln r$$

$$= \phi + i\psi$$

$\Longrightarrow$ Line of constant ψ is also the line along which r is constant

$\Longrightarrow$ Streamlines are circles and equipotential lines are radial lines. $\Rightarrow$ indicates vortex.

$$W(z) = \frac{dF}{dz} = u - iv = -i\frac{c}{z}$$

or $$(u_r - iu_\theta)e^{-i\theta} = -i\frac{c}{z}$$

$$= -i\frac{c}{re^{i\theta}}$$

$$= -i\frac{c}{r}e^{-i\theta}$$

$$\Rightarrow \left.\begin{array}{l} u_r = 0 \\ u_\theta = \dfrac{c}{r} \end{array}\right\}$$ $c > 0$ implies counter clockwise (positive) rotation.

Vortex @ origin ... contd.

Strength of a vortex is defined by circulation Γ.

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s} = \int_0^{2\pi} \left(u_r \hat{r} + u_\theta \hat{\theta}\right)\left(dr\, \hat{r} + (r\, d\theta)\, \hat{\theta}\right)$$

Among the dot products, $\hat{r} \cdot \hat{r}$ and $\hat{\theta} \cdot \hat{\theta}$ exist. $\hat{r} \cdot \hat{\theta}$ does not exist

$$\Rightarrow \Gamma = \int_0^{2\pi} \left(u_r\, dr + r\, u_\theta\, d\theta\right) = \int_0^{2\pi} (0 + c\, d\theta) = 2\pi c$$

$$\Rightarrow c = \frac{\Gamma}{2\pi} \qquad \Rightarrow F(z) = -i \frac{\Gamma}{2\pi} \ln z$$

$\Gamma > 0$ corresponds to counter-clockwise (positive) rotation

For vortex located at position $z = z_0$

$$F(z) = -i \frac{\Gamma}{2\pi} \ln (z - z_0)$$

Also note that in ~~cyl~~ cylindrical coordinates, vorticity is given by

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right] = \frac{1}{r}(0) \qquad \text{since } u_r = 0, \quad u_\theta = \frac{\Gamma}{2\pi r}$$

$\Rightarrow$ Vorticity is zero for all r except for $r = 0$ where all vorticity is concentrated at the centre (infinite vorticity)

This is definition of free vortex

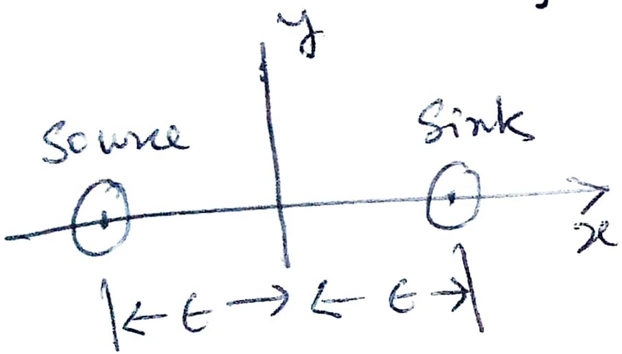
Alternatively, one can define a constant vorticity upto a radial distance R (referred as Forced vortex) and for $r > R$ it is free vortex. A rotational viscous core for $r \le R$ provides for the necessary vorticity. For example

$$u_\theta = \frac{\Gamma}{2\pi} \frac{r}{R^2} \quad \text{for } r \le R$$

$$u_\theta = \frac{\Gamma}{2\pi r} \quad \text{for } r > R$$

Doublet

Superposition of a source and a sink that are brought close together



Source of Strength m at $x = -\epsilon$

Sink " " $-m$ at $x = \epsilon$

$$F_{source}(z) = \frac{m}{2\pi} \ln (z + \epsilon)$$

$$F_{sink}(z) = -\frac{m}{2\pi} \ln (z - \epsilon)$$

Superposed $F(z) = \dfrac{m}{2\pi} \ln \left(\dfrac{z+\epsilon}{z-\epsilon} \right) = \dfrac{m}{2\pi} \ln \left(\dfrac{1 + \epsilon/z}{1 - \epsilon/z} \right)$

As source and sink approach each other, $\epsilon \to 0$

$$\Rightarrow \left(1 - \frac{\epsilon}{z}\right)^{-1} = 1 + \frac{\epsilon}{z} + \left(\frac{\epsilon}{z}\right)^2 + \cdots = 1 + \frac{\epsilon}{z}$$

By Binomial series expansion and by dropping higher order terms

$$\Rightarrow F(z) = \frac{m}{2\pi} \ln \left\{ \left(1 + \frac{\epsilon}{z}\right)\left(1 + \frac{\epsilon}{z}\right) \right\} = \frac{m}{2\pi} \ln \left(1 + \frac{2\epsilon}{z}\right)$$

Doublet .... Contd.

For small x, $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

$$\Rightarrow F(z) = \frac{m}{2\pi}\left\{2\frac{\epsilon}{z} + \cdots\right\} = \frac{m\epsilon}{\pi z} = \frac{\mu}{z} \quad \text{(say)}$$

$$= \frac{\mu}{r} \cdot e^{-i\theta} = \frac{\mu}{r}(\cos\theta - i\sin\theta)$$

$$= \frac{\mu}{r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + i\left[-\frac{\mu}{r}\frac{y}{\sqrt{x^2+y^2}}\right]$$

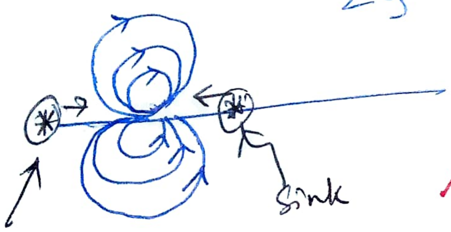
$$\Rightarrow \phi = \mu\left(\frac{x}{x^2+y^2}\right)$$

$$\psi = -\mu\left(\frac{y}{x^2+y^2}\right) \Rightarrow x^2 + \left(y + \frac{\mu}{2\psi}\right)^2 = \left(\frac{\mu}{2\psi}\right)^2$$

$\Rightarrow$ Lines of constant ψ are circles through origin with radius $\frac{\mu}{2\psi}$; Centre of circle is located at $y = \pm\frac{\mu}{2\psi}$

when $\psi > 0$, circles are in lower half plane

when $\psi < 0$, " " upper " "



μ is considered as the strength of the vortex

$$F(z) = \frac{\mu}{z}$$

For doublet located at $z = z_0$

$$F(z) = \frac{\mu}{z - z_0}$$

Doublet ..... Contd.

Complex velocity $W(z) = \frac{dF}{dz} = -\frac{\mu}{z^2} = -\frac{\mu}{r^2}e^{-i2\theta} = -\frac{\mu}{r^2}e^{-i\theta}e^{-i\theta}$

$$= -\frac{\mu}{r^2}(\cos\theta - i\sin\theta)e^{-i\theta}$$

$(u_r - i u_\theta)e^{-i\theta}$

$$\Rightarrow u_r = -\frac{\mu}{r^2}\cos\theta$$

$$u_\theta = -\frac{\mu}{r^2}\sin\theta$$

The velocity induced by doublet decreases as $\frac{1}{r^2}$, compared to $\frac{1}{r}$ for a source or vortex.

$$u_r = U\cos\theta + \frac{m}{2\pi r}$$
$$u_\theta = -U\sin\theta$$

$$u = u_r\cos\theta - u_\theta\sin\theta$$
$$v = u_r\sin\theta + u_\theta\cos\theta$$

$$\Rightarrow \quad u = \left[U\cos\theta + \frac{m}{2\pi r}\right]\cos\theta + U\sin^2\theta$$

$$v = \left[U\cos\theta + \frac{m}{2\pi r}\right]\sin\theta - U\sin\theta\cos\theta$$

$$= U\sin\theta\cos\theta + \frac{m}{2\pi r}\sin\theta - U\sin\theta\cos\theta$$

$$= \frac{m}{2\pi r}\sin\theta$$

$$U\cos^2\theta + U\sin^2\theta + \frac{m}{2\pi r}\cos\theta = U + \frac{m}{2\pi r}\cos\theta$$

$$w\bar{w} = u^2 + v^2 = \left(\frac{m}{2\pi r}\right)^2\sin^2\theta + U^2 + 2\frac{Um}{2\pi r}\cos\theta + \left(\frac{m}{2\pi r}\right)^2\cos^2\theta$$

$$V^2 = U^2 + \frac{mU}{\pi r}\cos\theta + \left(\frac{m}{2\pi r}\right)^2 = U^2 - \frac{mU}{\pi r}\cos\gamma + \left(\frac{m}{2\pi}\right)^2\frac{1}{r^2}$$

Putting $\gamma = \pi - \theta$

and $r = \frac{m}{2\pi U}\frac{\pi - \theta}{\sin\theta}$

$$= \frac{m}{2\pi U}\frac{\gamma}{\sin\gamma}$$

$$= U^2 - \frac{mU}{\pi \frac{m\gamma}{2\pi U\sin\gamma}}\cos\gamma + \left(\frac{m}{2\pi}\right)^2\left(\frac{2\pi U\sin\gamma}{m\gamma}\right)^2$$

$$= U^2 - \frac{2U^2\sin\gamma\cos\gamma}{\gamma} + \frac{U^2\sin^2\gamma}{\gamma^2}$$

Bernoulli's Eqn.

$$\frac{1}{2}\rho V_{surf}^2 + P = \frac{1}{2}\rho u^2 + P_\infty$$

Dimensionless Pressure Coefficient

$$C_p = \frac{P_s - P_\infty}{\frac{1}{2}\rho U^2} = 1 - \frac{V_s^2}{U^2}$$

$$= \frac{2}{\gamma}\sin\gamma\cos\gamma - \frac{1}{\gamma^2}\sin^2\gamma$$

— change in pressure head as a fraction of overall kinetic head.

At the stagnation point $C_p = 1$

$$P_s - P_\infty = \frac{1}{2}\rho U^2 \Rightarrow$$ Entire kinetic head is converted to pressure head.

At ∞, C_p approaches zero

C_p is negative $\Rightarrow$ $P_s < P_\infty$ $\Rightarrow$ velocity is increasing $\Rightarrow$ streamlines are converging

C_p is positive $\Rightarrow$ $P_s > P_\infty$ $\Rightarrow$ velocity is decreasing $\Rightarrow$ streamlines are diverging