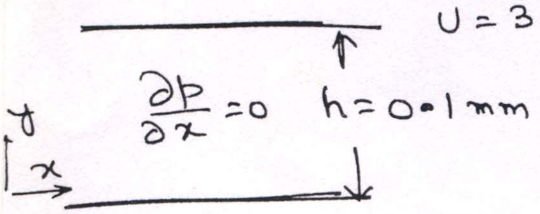
Extra Question

Q

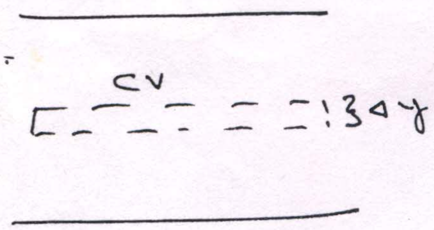
Flow between parallel plates.



$$\rho B_x = 800\ N/m^3$$

$$\mu = 0.02\ kg/m.s$$

$$\tau_{yx}\big|_{y} LW - \tau_{yx}\big|_{y+\Delta y} LW + \rho B_x LW\Delta y = 0.$$



$$\frac{d}{dy}(\tau_{yx}) = \rho B_x$$

$$\tau_{yx} = \rho B_x y + c_1.$$

$$-\mu \frac{d v_x}{dy} = \rho B_x y + c_1$$

$$v_x = -\frac{\rho B_x}{2\mu} y^2 - c_1 y + c_2.$$

$v_x = 0$ at $y = 0$ $\Rightarrow$ $c_2 = 0$.

$v_x = U$ at $y = h$ $\Rightarrow$ $U = -\frac{\rho B_x}{2\mu} h^2 - c_1 h$

$$c_1 = -\frac{1}{h}\left[U + \frac{\rho B_x}{2\mu} h^2\right]$$

$$\boxed{v_x = \frac{\rho B_x}{2\mu}(hy - y^2) + U\frac{y}{h}.}$$

$$Q = \int_A v_x\, dA = \int_0^h v_x\, b\, dy \quad \therefore \frac{Q}{b} = \int_0^h v_x\, dy = \frac{\rho B_x h^3}{12\mu} + \frac{Uh}{2}$$

$$\frac{Q}{b} = \frac{1}{12} \times 800\frac{N}{m^3} \times (10^{-4})^3 m^3 \times \frac{1}{0.02\ kg} m.s \cdot \frac{kg\ m}{N.s^2} + \frac{1}{2} \times 0.003\frac{m}{s} \times 10^{-4} m$$

$$\frac{Q}{b} = 1.53 \times 10^{-7}\ m^3/s/m$$

4(b)

2. Since the gap is so small, the situation can be approximated by flow ~~bet~~ between two parallel plates with one plate moving and the other stationary with no applied pressure difference.

$V = R\omega = 0.15\,m \times 3600\,\frac{rev}{min} \times 2\pi\,\frac{rad}{rev} \times \frac{1}{60}\,\frac{min}{sec}$

$= 56.5\,m/s.$ 1

$Re = \frac{\rho V a}{\mu} = \frac{V a}{\nu} = \frac{56.5\,\frac{m}{s} \times 0.5\times10^{-6}\,m \times \frac{s}{m^2}}{1.45\times10^{-5}} = 1.95$

For flow bet^n // plates with no pr grad applied.

$\tau = \mu\frac{du}{dy} = \mu\frac{V}{a}$ (linear profile for small gap)

1 $\tau = 1.78\times10^{-5}\,\frac{kg}{m\,sec} \times 56.5\,\frac{m}{sec} \times \frac{1}{0.5\times10^{-6}\,m}$

$= 2.01\,kN/m^2$

Force, $F = \tau A = \tau wL$ 3

Torque, $T = FR = \tau wLR$

Power dissipation rate

$P = ~~T\omega~~ = ~~FLRw~~$ 1

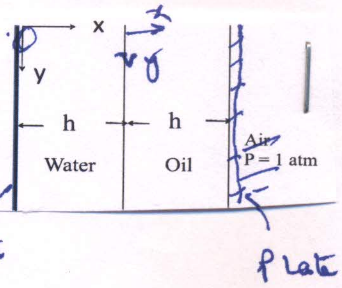
$P = T\omega$

$= \tau wLR\omega$

$= 2.01\times10^3 \times ~~0.1\times0.0~~\ 0.01\times0.01\times0.15 \times 3600\times2\pi\times\frac{1}{60}$

$\underline{P = 11.4\,W}$

Q1. Water and oil flow down a vertical plane as shown. The flow is steady, laminar and fully developed. Simplify the Navier-Stokes equations separately for water and oil films with the relevant boundary conditions. Obtain and sketch (qualitatively) the two velocity profiles clearly emphasizing the region near the oil-water interface. 2+2+1=5



Water

$$\mu_w \frac{d^2 v_{y_w}}{dx^2} + \rho_w g = 0$$

$$\mu_w \frac{d v_{y_w}}{dx} = -\rho_w g x + c_1$$

$$v_{y_w} = -\frac{\rho_w g}{\mu_w}\frac{x^2}{2} + \frac{c_1}{\mu_w} x + c_2$$

Oil

$$\mu_o \frac{d^2 v_{y_o}}{dx^2} + \rho_o g = 0.$$

$$\mu_o \frac{d v_{y_o}}{dx} = -\rho_o g x + c_3.$$

$$v_{y_o} = -\frac{\rho_o g}{\mu_o}\frac{x^2}{2} + \frac{c_3}{\mu_o} x + c_4$$

BC. I Continuity of stress at the L-L interface ($x=0$).

$$\mu_w \frac{d v_{y_w}}{dx} = \mu_o \frac{d v_{y_o}}{dx}.$$

$$c_1 = c_3.$$

BC II Continuity of velocity at the L-L interface ($x=0$).

$x=0$ $\quad v_{y_w} = v_{y_o}$

$$c_2 = c_4$$

BC III & IV No slip at $x = \pm h$.

$$0 = -\frac{\rho_w g}{\mu_w}\frac{h^2}{2} - \frac{c_1}{\mu_w} h + c_2 \quad \text{— A}$$

$$0 = -\frac{\rho_o g}{\mu_o}\frac{h^2}{2} + \frac{c_1}{\mu_o} h + c_2. \quad \text{— B}$$

A - B $\Rightarrow$

$$0 = g\left(\frac{\rho_o}{\mu_o} - \frac{\rho_w}{\mu_w}\right)\frac{h^2}{2} - c_1 h\left(\frac{1}{\mu_o} + \frac{1}{\mu_w}\right).$$

$$0 = g(\rho_o \mu_w - \rho_w \mu_o)\frac{h^2}{2} - c_1 h(\mu_o + \mu_w).$$

$$c_1 = \frac{g(\rho_o \mu_w - \rho_w \mu_o)}{2(\mu_o + \mu_w)} \cdot h.$$

Putting c_1 in (B)

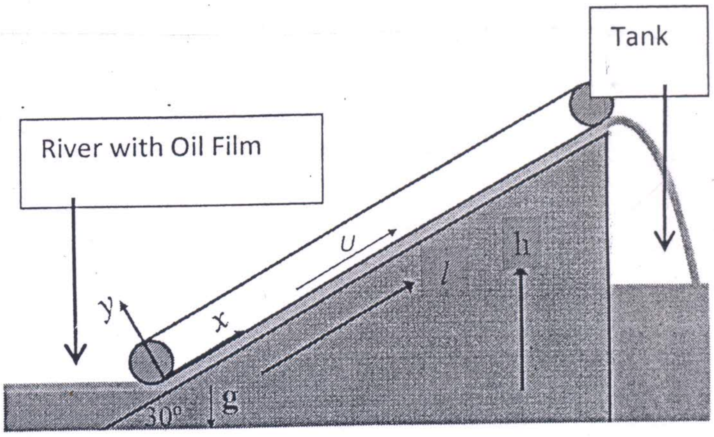
$$c_2 = \frac{\rho_o}{\mu_o} g \frac{h^2}{2} - \frac{c_1 h}{\mu_o}.$$

$$c_2 = \frac{\rho_o}{\mu_o} g \frac{h^2}{2} - \frac{g h^2(\rho_o \mu_w - \rho_w \mu_o)}{2(\mu_o + \mu_w)\mu_o} \quad \checkmark$$

Putting c_1 & c_2 in the velocity expressions the complete profile may be obtained.

While drawing the two profiles the continuity of stress and velocity at the L-L interfaces and no slip at the S-L interfaces must be shown clearly.

Q2 An oil skimmer uses a 5 m wide x 6 m long moving belt above a fixed platform ($\theta = 30°$) to skim oil off of rivers (T = 10°C). The belt travels at 3 m/s. The distance between the belt and the fixed platform is 2 mm. The belt discharges into an open tank on the ship. The fluid is actually a mixture of oil and water. To simplify the analysis, assume crude oil dominates. Find the discharge of oil into the tank on the ship, the force acting on the belt and the power required (kW) to move the belt.
For oil: $\rho = 860$ kg/m^3, viscosity, $\mu = 1\times10^{-2}$ N.s/m^2



~~4+3+2=9 Marks~~

NS eq^n reduces to

$$\frac{d^2u}{dy^2} = -\frac{\rho}{\mu} g_x$$

(No applied pr. grad, $u \neq f(x, z)$
SS

$$\frac{d^2u}{dy^2} = \frac{\rho g}{\mu} \sin\theta.$$

BC 1 At $y = 0$ $u = 0$

BC 2 At $y = h$, $u = U$

$$\therefore \frac{du}{dy} = \frac{\rho g \sin\theta}{\mu} y + A.$$

$$u = \left(\frac{\rho g}{\mu}\sin\theta\right)\frac{y^2}{2} + Ay + B.$$

Applying the BCs. $B = 0$ & $A = -\left(\frac{\rho g}{\mu}\sin\theta\right)\frac{h}{2} + \frac{U}{h}$

$$\therefore u = -\left(\frac{\rho g}{\mu}\sin\theta\right)\left(\frac{hy}{2} - \frac{y^2}{2}\right) + \frac{Uy}{h}.$$

Vol. flow rate per unit width of the fluid film in z direct

$$Q = \int_0^h u\,dy = -\int_0^h \left(\frac{\rho g}{\mu}\sin\theta\right)\left(\frac{hy}{2} - \frac{y^2}{2}\right)dy + \int_0^h \frac{U}{h} y\,dy$$

$$Q = -\frac{\rho g h^3}{12\mu}\sin\theta + \frac{Uh}{2}$$

$$Q = -\frac{860\times9.81\times0.002}{12\times10^{-2}}\sin30 + \frac{3\times0.002}{2} = 0.0027\ \text{m}^2/\text{s}$$

per unit width

$0.0027\ \text{m}^2/\text{s} \times 5\ \text{m} = 0.0135\ \text{m}^3/\text{s}.$

Evaluate $\tau = \mu \frac{du}{dy}$ at the moving belt

$$u = -\left(\frac{\rho g}{\mu}\sin\theta\right)\left(\frac{hy}{2} - \frac{y^2}{2}\right) + \frac{U}{h}y$$

$$\frac{du}{dy} = -\left(\frac{\rho g}{\mu}\sin\theta\right)\left(\frac{h}{2} - y\right) + \frac{U}{h}.$$

At the moving belt

$$\tau\Big|_{\text{at the belt}} = \mu\left(\frac{du}{dy}\right)_{y=h} = \left(\rho g \sin\theta\right)\frac{h}{2} + \frac{\mu U}{h}$$

$$\tau\Big|_{\text{at the belt}} = \mu \frac{du}{dy}\Big|_{y=h} = \left(\rho g \sin\theta\right)\frac{h}{2} + \frac{\mu U}{h}$$

$$\tau\Big|_{\text{at the belt}} = \frac{860 \times 9.8 \times 0.5 \times 0.002}{2} + \frac{10^{-2} \times 3}{0.002}$$

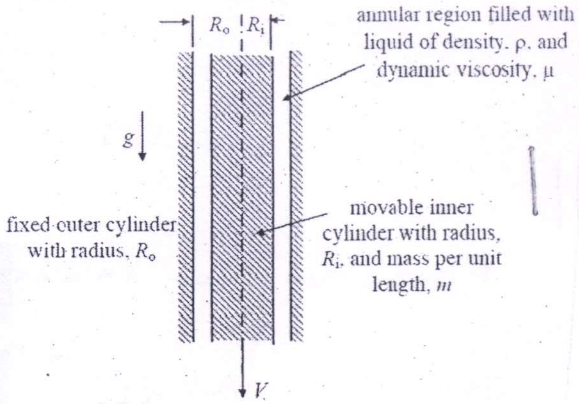
$$\tau\Big|_{\text{at the belt}} = 19.21\ N/m^2$$

$$\text{Power} = (\tau \times L \times W)U$$

$$= 19.21 \times 6 \times 5 \times 3$$

$$= 1.73\ KW.$$

Q2 Consider two concentric cylinders with a Newtonian liquid of constant density, ρ, and constant dynamic viscosity, μ, contained between them. The outer pipe, with radius, Ro, is fixed while the inner pipe, with radius, Ri, and mass per unit length, m, falls under the action of gravity at a constant speed. There is no pressure gradient within the flow and no swirl velocity component. Determine the vertical speed, V, of the inner cylinder as a function of the following (subset of) parameters: g, R_o, R_i, m, ρ, and μ. The space between the two cylinders is **not** 'too small' compared to the radii of the cylinders concerned. (Marks = 9)



For inner cylinder moving at constant velocity, the downward force is exactly balanced by the viscous force

$$\tau_w A_{w\,\substack{inner\\cylinder}} = mLg$$

z comp.

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \cancel{\frac{v_\theta}{r}}\frac{\partial v_z}{\partial z}\right\} = \rho g_z - \cancel{\frac{\partial p}{\partial z}} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right.$$

$$\left. + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right\}.$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = -\frac{\rho g}{\mu}.$$ 2

$$r\frac{dv_z}{dr} = -\frac{\rho g r^2}{2\mu} + c_1$$

$$\frac{dv_z}{dr} = \frac{-\rho g r}{2\mu} + \frac{c_1}{r}.$$

$$\boxed{v_z = -\frac{\rho g r^2}{4\mu} + c_1 \ln r + c_2.}$$

$v_z = V$ at $r = Ri$ | 2 $\quad V = -\frac{\rho g}{4\mu}Ri^2 + c_1 \ln Ri + c_2$ 7

$v_z = 0$ at $r = Ro$ $\quad 0 = -\frac{\rho g}{4\mu}\frac{Ro^2}{4} + c_1 \ln Ro + c_2$

$$\therefore \quad V = \frac{\rho g}{4\mu}\left(Ro^2 - Ri^2\right) + c_1 \ln\frac{Ri}{Ro}$$

$$c_1 = \frac{1}{\ln\frac{Ri}{Ro}}\left[V - \frac{\rho g}{4\mu}\left(Ro^2 - Ri^2\right)\right]$$

$$\frac{dv_z}{dr} = -\frac{\rho g r}{2\mu} + \frac{c_1}{r}$$

$$\tau_{rz} = \mu\frac{dv_z}{dr} = \frac{-\rho g r}{2} + \frac{c_1 \mu}{r}$$

$$c_1 = \frac{1}{\ln\frac{R_i}{R_o}}\left[V - \frac{\rho g}{4\mu}\left(R_o^2 - R_i^2\right)\right]$$

Force on the inner cylinder = gravity force

$$\tau\Big|_{r=R_i} 2\pi R_i \cancel{L} = m\cancel{L}\, g$$

$$2\pi R_i\left[\frac{-\rho g R_i}{2} + \frac{\mu}{R_i \ln R_i/R_o}\left\{V - \frac{\rho g}{4\mu}\left(R_o^2 - R_i^2\right)\right\}\right] = mg$$

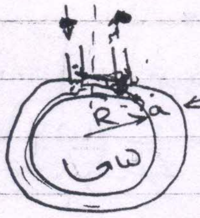
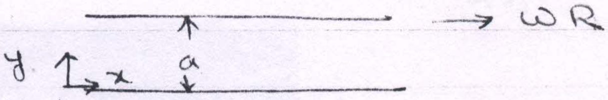
$$\cancel{\frac{\mu V}{R_i \ln R_i/R_o} = \frac{\rho g R_i}{2} - mg}$$

$$\frac{\mu}{R_i \ln R_i/R_o}\left\{V - \frac{\rho g}{4\mu}\left(R_o^2 - R_i^2\right)\right\} = \frac{\rho g R_i}{2} - \frac{mg}{2\pi R_i}$$

$$V - \frac{\rho g}{4\mu}\left(R_o^2 - R_i^2\right) = \left(R_i \ln\frac{R_i}{R_o}\right)\left(\frac{\rho g R_i}{2\mu} - \frac{mg}{2\pi R_i \mu}\right)$$

$$V = R_i \ln\frac{R_i}{R_o}\left(\frac{\rho g R_i}{2\mu} - \frac{mg}{2\pi R_i \mu}\right) - \frac{\rho g}{4\mu}\left(R_i^2 - R_o^2\right)$$

Correct.

2.  

$$0 = -\frac{dp}{dx} + \mu \frac{d^2 v_x}{dy^2}$$

$$\frac{dv_x}{dx} = \frac{1}{\mu}\left(\frac{dp}{dx}\right) y + c_1$$

$$v_x = \frac{1}{2\mu}\left(\frac{dp}{dx}\right) y^2 + c_1 y + c_2$$

$y = 0 \quad v_x = 0 \quad \Rightarrow \quad c_2 = 0$

$y = a, \quad v_x = \omega R \qquad c_1 = \frac{1}{a}\left[\omega R - \frac{1}{2\mu}\left(\frac{dp}{dx}\right) a^2\right]$

$$v_x = \frac{1}{2\mu}\left(\frac{dp}{dx}\right) y^2 - \frac{1}{2\mu}\left(\frac{dp}{dx}\right) a y + \frac{\omega R}{a} \cdot y$$

$$= \frac{1}{2\mu}\frac{dp}{dx} \cdot a^2 \left[\left(\frac{y}{a}\right)^2 - \frac{y}{a}\right] + \frac{\omega R}{a} y \quad 2$$

$$\langle v_x \rangle = \frac{1}{ab}\int_0^b \int_0^a v_x \, dy \, dz = \frac{1}{a}\int_0^a v_x \, dy$$

$$= \frac{1}{a}\left[\frac{1}{2\mu} \cdot \frac{dp}{dx} \cdot \left(\frac{y^3}{3} - \frac{a y^2}{2}\right) + \frac{\omega R}{a} \cdot \frac{y^2}{2}\right]_0^a$$

$$= \frac{1}{a}\left[-\frac{1}{12\mu}\frac{\Delta p}{L} \cdot a^3 + \frac{\omega R a}{2}\right]$$

$$= -\frac{1}{12\mu}\frac{\Delta p}{L} \cdot a^2 + \frac{\omega R}{2} \quad ✓ \; 2$$

$$Q = ab \langle v_x \rangle$$

$$\frac{Q}{b} = -\frac{1}{12\mu}\frac{\Delta p}{L} \cdot a^3 + \frac{\omega R a}{2}$$

$$\Delta p = \frac{12 \mu L}{a^3}\left[\frac{\omega R a}{2} - \frac{Q}{b}\right] \quad 2$$

$$\Delta p = \frac{6 \mu L \omega R}{a^2}\left[1 - \frac{2Q}{a b \omega R}\right] \quad ✓$$

$$\frac{dv_x}{dx} = \frac{1}{\mu}\frac{\Delta p}{L}\cdot y + c_1 \qquad c_1 = \frac{1}{a}\left[\omega R - \frac{1}{2\mu}\left(\frac{\Delta p}{L}\right)\cdot a^2\right]$$

$$= \frac{1}{\mu}\frac{\Delta p}{L}y - \frac{1}{2\mu}\left(\frac{\Delta p}{L}\right)\cdot a + \frac{\omega R}{a}$$

$$T\Big|_{\text{at } y=a} = +\mu\frac{dv_x}{dy}\Big|_{y=a} \qquad \frac{\Delta p}{L} = \frac{6\mu L\omega R}{a^2}\left[1 - \frac{2Q}{ab\omega R}\right]$$

at the surface

$$= \frac{\Delta p}{L}\cdot a - \frac{\Delta p}{L}\cdot\frac{a}{2} + \frac{\omega R\mu}{a}$$

$$= \frac{\Delta p}{L}\cdot\frac{a}{2} + \frac{\omega R\mu}{a}$$

$$= \frac{a}{2}\cdot\frac{6\mu L\omega R}{a^2 L}\left[1 - \frac{2Q}{ab\omega R}\right] + \frac{\omega R\mu}{a}$$

$$= \frac{3\mu\cancel{L}\omega R}{a\cancel{L}}\left[1 - \frac{2Q}{ab\omega R}\right] + \frac{\omega R\mu}{a}$$

$$\text{Torque} = T\cdot(LB)R \qquad \underset{(P)}{\text{Power}} = \text{Torque}\times\omega = T\,\omega RLb$$

$$P = \omega RLB\left[\frac{3\mu\omega R}{a}\left(1 - \frac{2Q}{ab\omega R}\right) + \frac{\omega R\mu}{a}\right]$$

$$= \mu\frac{LB(\omega R)^2}{a}\left[3 - \frac{6Q}{ab\omega R} + 1\right]$$

$$= \frac{\mu LB(\omega R)^2}{a}\left[4 - \frac{6Q}{ab\omega R}\right]$$

output power $= Q\Delta P$

efficiency $\eta = \frac{Q\Delta P}{P(\text{power})}$

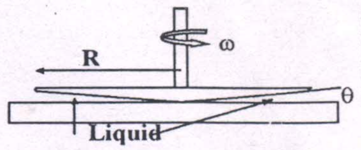
$$= \frac{2\pi R\mu B(\omega R)^2}{a}\left[4 - \frac{6Q}{ab\omega R}\right]$$

$$= \frac{2\pi R^2 b\mu\omega}{a}\left[4\omega R - \frac{6Q}{ab}\right]$$

$$=$$

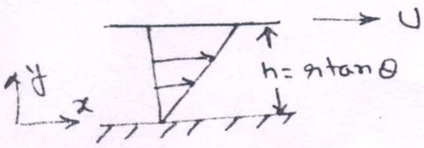
$$= \frac{6Q}{abR\omega}\frac{\left(1 - \frac{2Q}{abR\omega}\right)}{\left(4 - \frac{6Q}{abR\omega}\right)}$$

1 The cone and plate viscometer consists of a flat plate and a rotating cone with a very obtuse angle (typically θ less than 0.5 degrees). The apex of the cone just touches the plate surface and the liquid to be tested fills the narrow gap formed by the cone and the plate. Derive an expression for the shear rate in the liquid that fills the gap in terms of the geometry of the system. Evaluate the torque in terms of the shear rate and geometry of the system. Assume that there is no mixing in the r direction.



Since the angle θ is very small, the average gap width is also very small. Thus it is reasonable to assume a linear velocity profile (as is the case for flow betn || plates)

The shear is given by



$$\tau = \mu \frac{du}{dy}$$

$U = \omega r$ and $h = r \tan\theta$.

$$\therefore u = \frac{\omega r}{h} y = \frac{\omega r}{r\tan\theta} \cdot y = \frac{\omega}{\tan\theta} y.$$

$$\therefore \tau = \frac{\mu\omega}{\tan\theta}.$$

Since θ is very small $\tan\theta \simeq \theta$.

$$\therefore \tau = \frac{\mu\omega}{\theta}$$

— Thus shear rate is independent of r and the whole sample is subjected to the same shear.

The torque on the cone is given by

$$T = \tau_{yx} \int_0^R r \cdot 2\pi r \, dr$$

$$T = \frac{2}{3}\pi R^3 \tau_{yx}$$

$$\boxed{T = \frac{2}{3}\pi R^3 \frac{\mu\omega}{\theta}.}$$