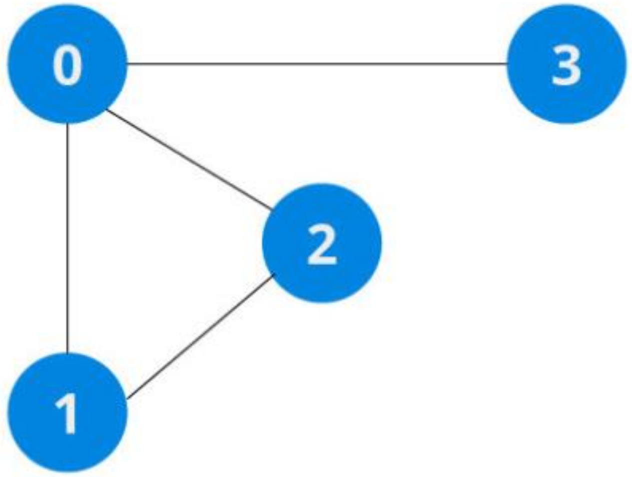
Graphs

CS223: Data Structures

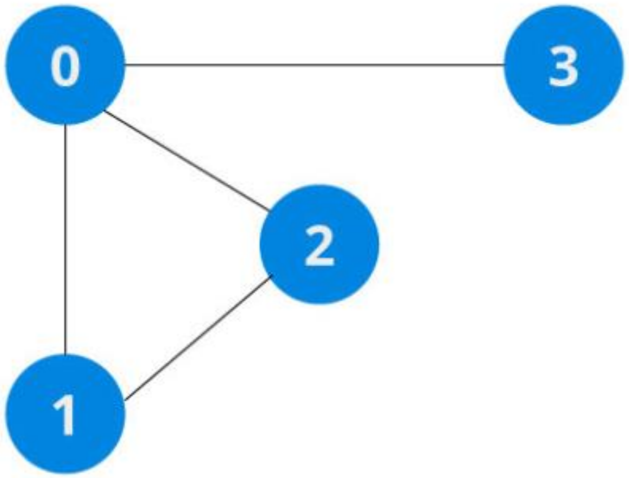
Definitions

- A graph is a non-linear data structure that enables representing relationships between different types of data.
- Real Applications:
  - Google maps, social media, web Page searching, City Planning, Traffic Control, Transportation & Navigation, Travelling Salesman Problems, GSM mobile phone networks
- There are two main parts of a graph G(V,E):
  - The vertices (nodes) where the data is stored (V).
  - The edges (connections) which connect the nodes (E).
    - represented as pairs of vertices (u,v), where $u,v \in V$.
- In the right graph:
  - V = {0, 1, 2, 3}
  - E = {(0,1), (0,2), (0,3), (1,2)}
  - G = {V, E}



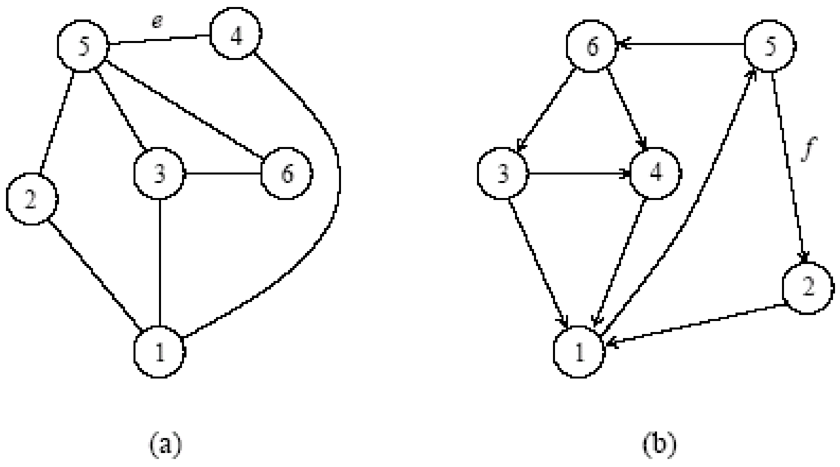
Definitions (Cont.)

- A **path** is a sequence of edges that allows you to go from a vertex, e.g., 0, to another vertex, e.g., 2.
  - Example: (0-1, 1-2) and (0-2) are paths from vertex 0 to vertex 2.
  - The length of a path is defined as the number of edges in the path.
- A vertex is said to be **adjacent** to another vertex if there is an edge connecting them. Vertices 2 and 3 are **not** adjacent because there is no edge between them.



<u>Definitions</u> (Cont.)

- Graphs can be **undirected** or **directed**
  - **Undirected graph:** The relationship exists in both directions (bi-directional) i.e. the edges do not point in any specific direction.
  - **Directed graph:** The relationships are based on the direction of the edges.
    - It can *be* a **one way relationship** *or* a **two-way relationship**, but it must be **explicitly** stated.
    - In a directed graph, the edge *e* is an ordered pair *(u,v)*.
    - An edge *(u,v)* is *incident from* vertex *u* and is *incident to* vertex *v*.



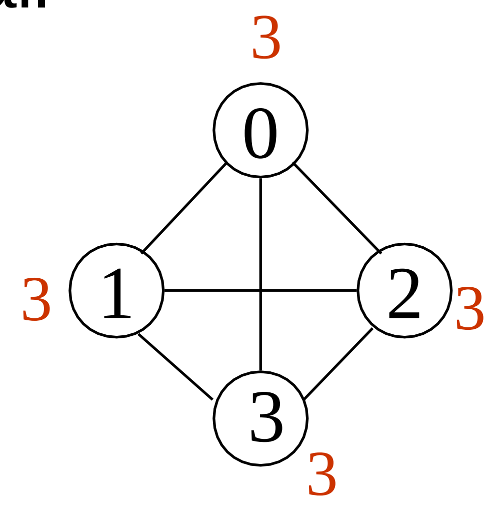
(a) An undirected graph and (b) a directed graph.

Definitions (Cont.)

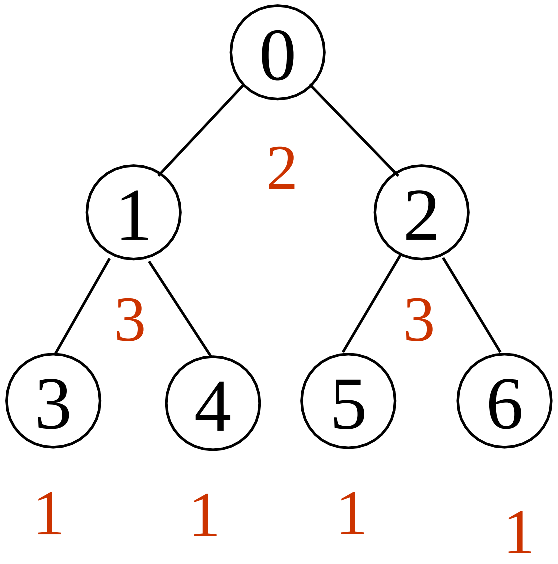
The degree of a vertex is the number of edges incident to that vertex

For directed graph,

- the in-degree of a vertex *v* is the number of edges that have *v* as the head
- the out-degree of a vertex *v* is the number of edges that have *v* as the tail

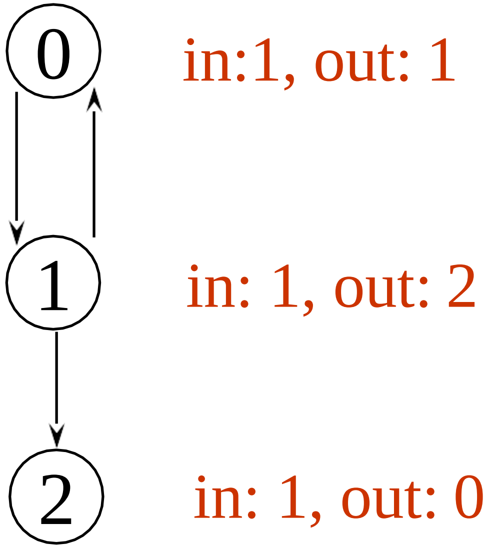


G1



G2

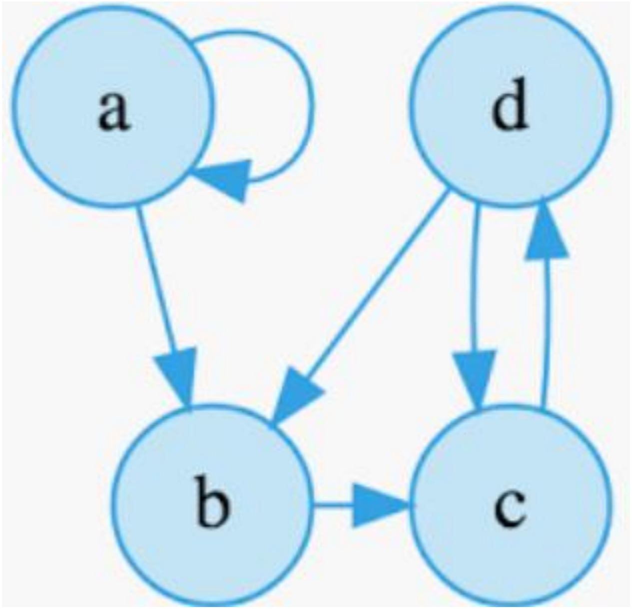
directed graph
in-degree
out-degree



G3

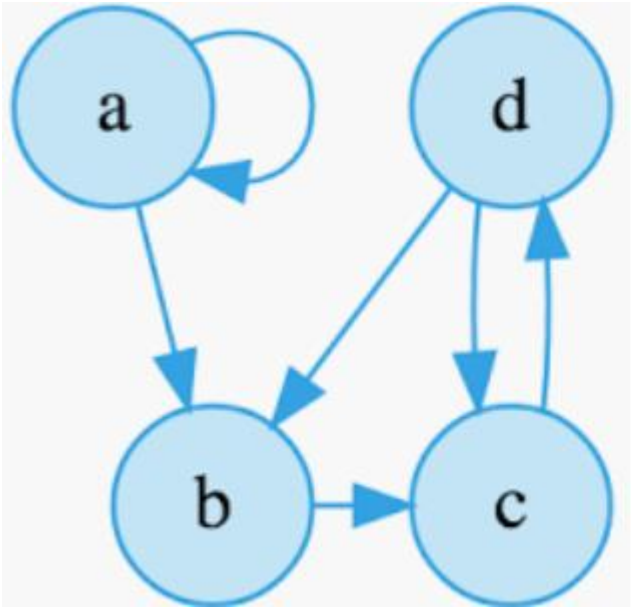
Definitions (Cont.)

- A **loop** is an edge from a vertex onto itself. It is denoted by (v, v) (e.g., (a,a)).
- A **simple path** is a path where no vertices are repeated along the path (dbc, abc).
- A **cycle** is a path with at least one edge such that the first and last vertices are the same (e.g., bcdb, cdbc, dbcd).
  - A cycle can have length one (i.e. a self loop).



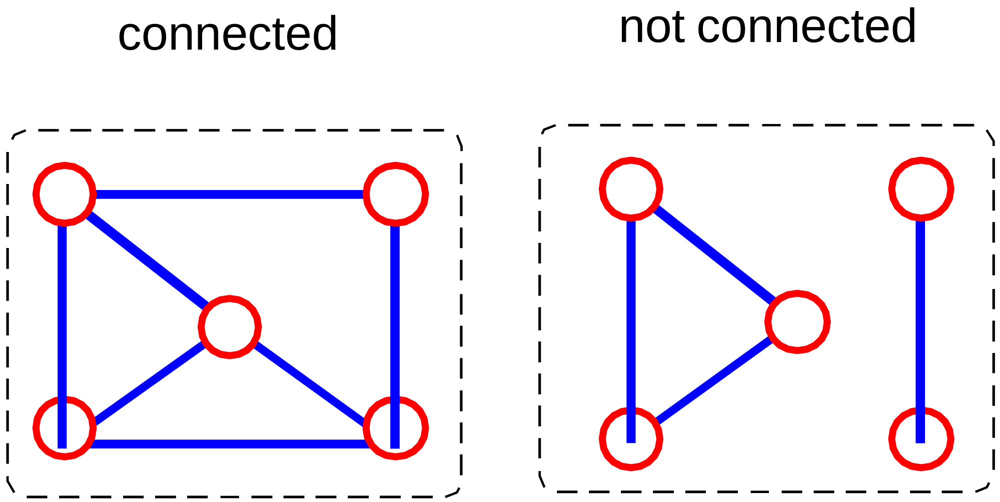
Definitions (Cont.)

- **Reachability** and **Connectivity**. A vertex v is reachable from a vertex u in G if there is a path starting at v and ending at u in G.
    - We use $R_G(v)$ to indicate the set of all vertices reachable from v in G.
    - e.g., $R_G(b) = \{c, d\}$



Definitions (Cont.)

- Connected graph: any two vertices are connected by some path
- An undirected graph is connected if all vertices are reachable from all other vertices.



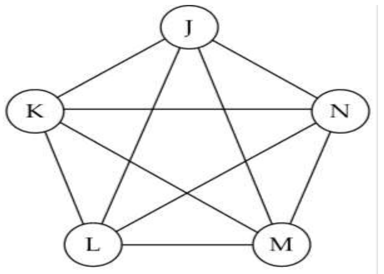
- A connected graph is strongly connected if it is a connected graph as well as a directed graph.
  - if all vertices are reachable from all other vertices.
- A connected graph is weakly connected if it is
  - a directed graph that is not strongly connected, but,
  - the underlying undirected graph is connected

Definitions (Cont.)

- Let **n** #vertices, and **m** = #edges
- **A complete graph**: one in which all pairs of vertices are adjacent
- *How many total edges in a complete graph?*
  - Undirected graph:
    - Each of the n vertices is incident to n-1 edges, however, we would have counted each edge twice! Therefore, intuitively, m = n(n -1)/2.
    - Therefore, if a graph is not complete, m < n(n -1)/2
  - Directed graph
    - m = n(n -1)
- Weighted graph: a graph in which each edge carries a value

$\mathbf{n} = 5$

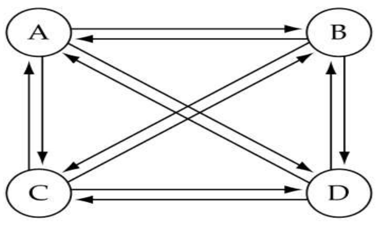
$\mathbf{m} = (5 * 4)/2 = 10$



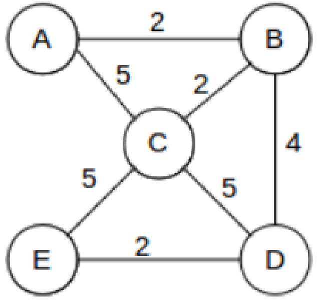
(b) Complete undirected graph.

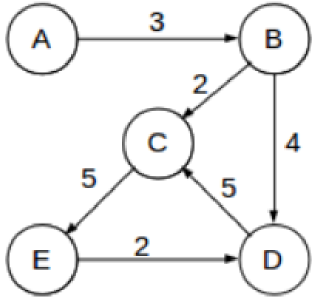
$\mathbf{n} = 4$

$\mathbf{m} = (4 * 3) = 12$



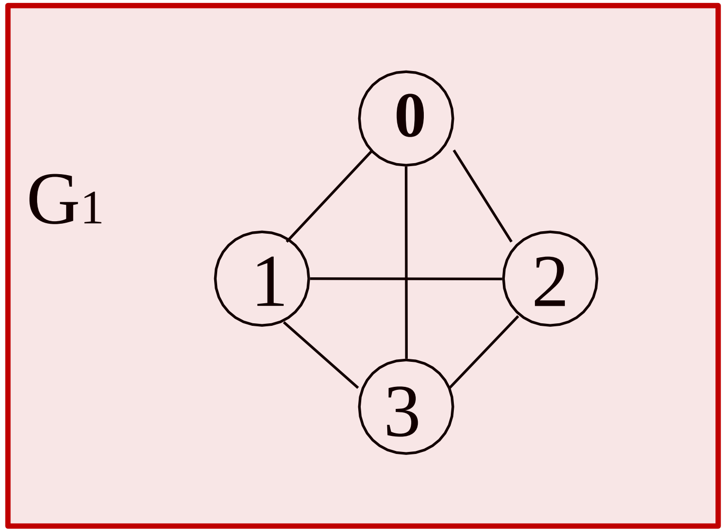
(a) Complete directed graph.

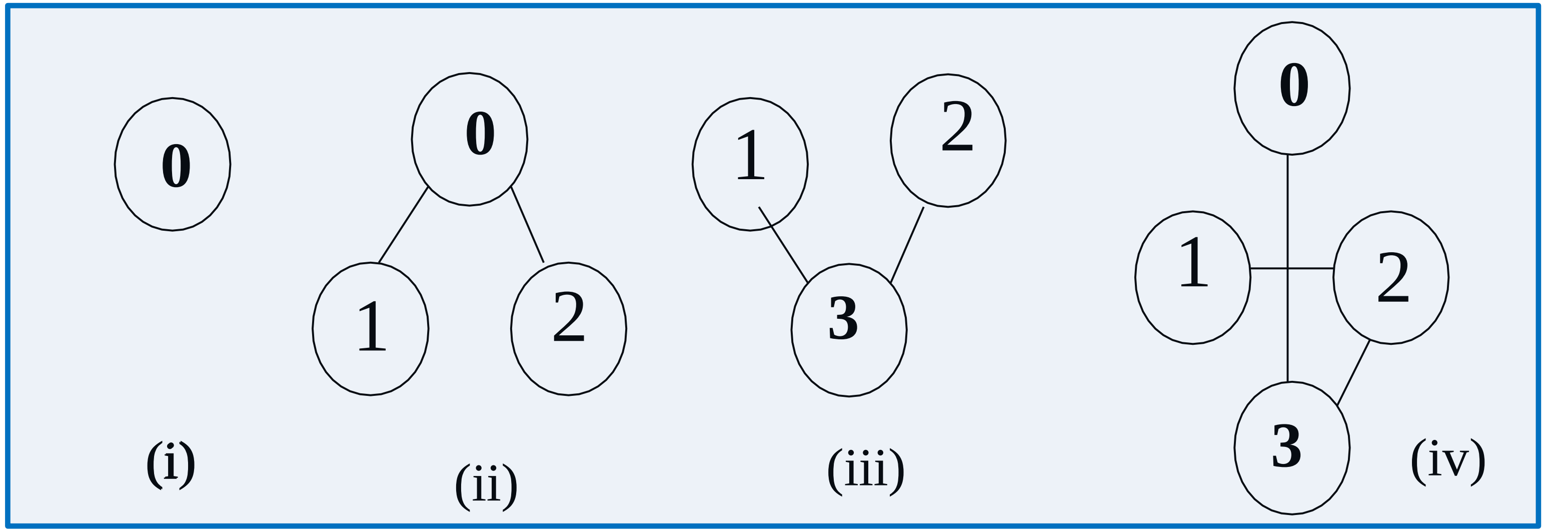




<u>Definitions</u> (Cont.)

- Subgraph: subset of vertices and edges forming a graph

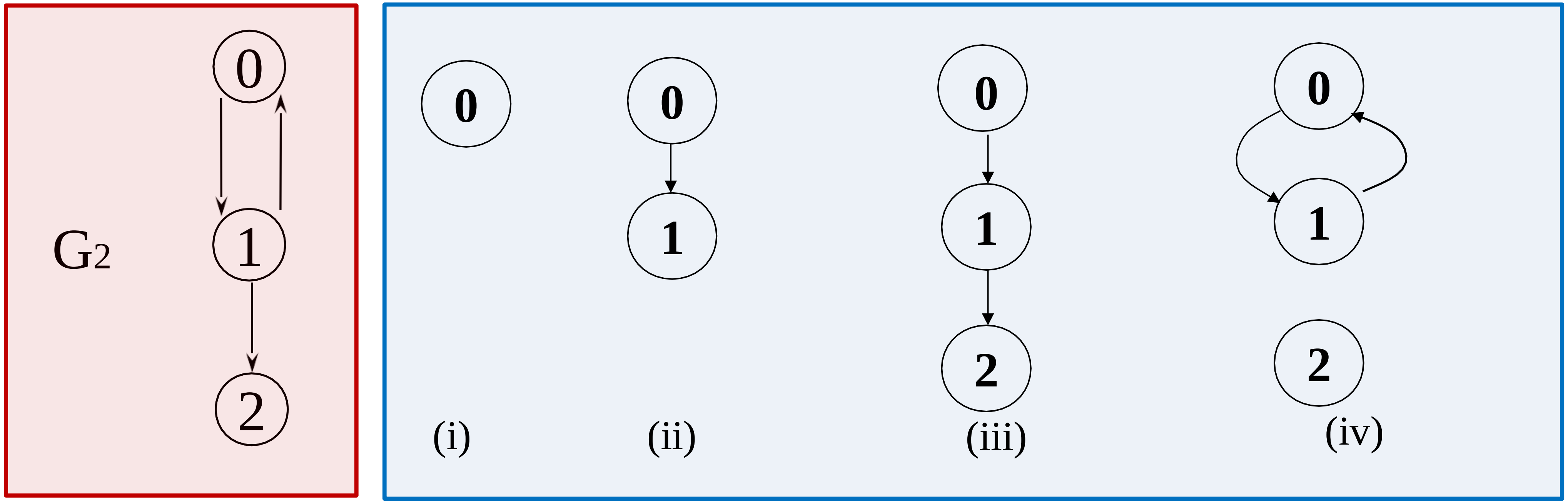




(i), (ii), (iii), and (iv) are subgraphs of G_1

Definitions (Cont.)

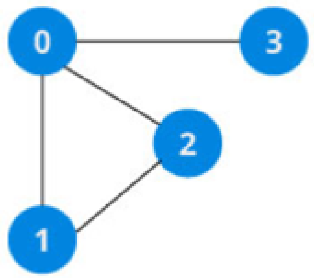
- Subgraph: subset of vertices and edges forming a graph



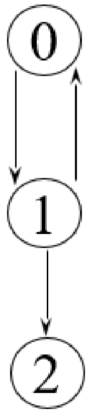
(i), (ii), (iii), and (iv) are subgraphs of G_2

Representation using a 2D Array

- A Graph can be represented by an adjacency matrix.
- An adjacency matrix is 2D array of V x V vertices. Each row and column represent a vertex.
- Adjacency matrices have a value $a_{i,j} = 1$ if nodes *i* and *j* share an edge; 0 otherwise.
  - If the value of any element a[i][j] is 1, it represents that there is an edge connecting vertex i and vertex j.
- The degree of a vertex is $\sum_{j=0}^{n-1} adj\_mat[i][j]$
- For a directed graph, the row sum is the out_degree, while the column sum is the in_degree.
  - In_d(v_i) = $\sum_{j=0}^{n-1} adj\_mat[j][i]$
  - Out_d(v_i) = $\sum_{j=0}^{n-1} adj\_mat[i][j]$
- The adjacency matrix for an undirected graph is symmetric;
- The adjacency matrix for a digraph need not be symmetric.



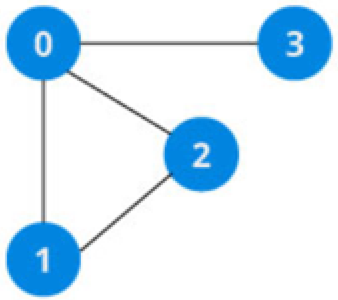
	0	1	2	3
0	0	1	1	1
1	1	0	1	0
2	1	1	0	0
3	1	0	0	0

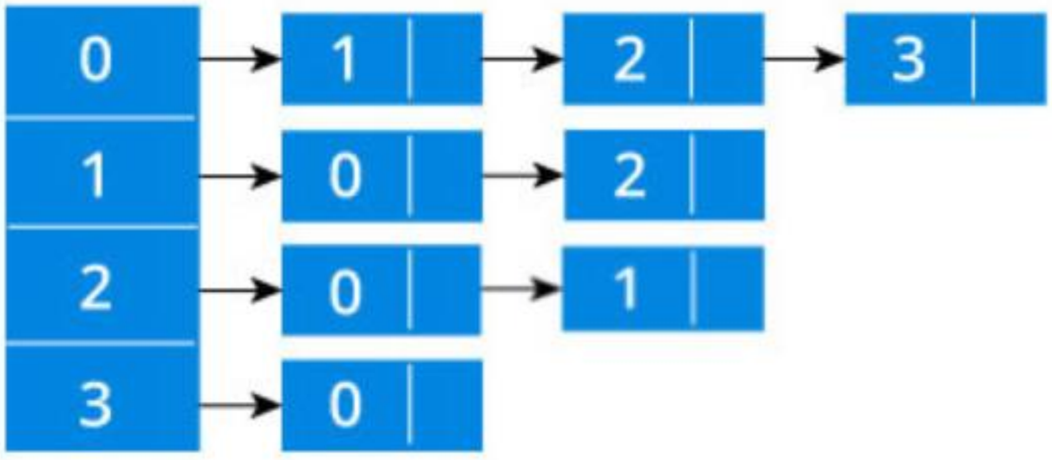


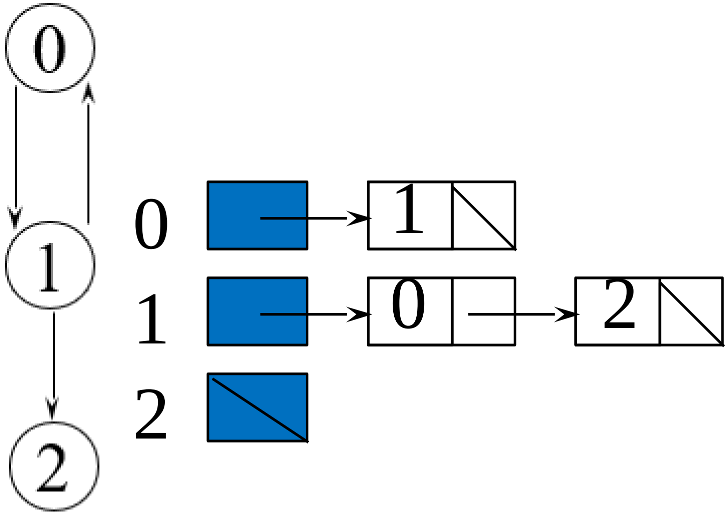
	0	1	2
0	0	1	0
1	1	0	1
2	0	0	0

Representation using a linked list

- An adjacency list represents a graph as an array of linked list.
- The index of the array represents a vertex and each element in its linked list represents the other vertices that form an edge with the vertex.
- Degree of a vertex in an undirected graph is # of nodes in adjacency list
- If |*E*| is close to the number of vertices then G is considered **Sparse** and an adjacency list is preferred.
- If |*E*| is close to the maximum number of edges in G then it is considered **Dense** and the adjacency matrix is preferred.







GRAPH TRAVERSAL/ SEARCHING

- **How to traverse a graph or find a path between two nodes of the graph?**
  - Breadth-First-Search (BFS)
  - Depth-First-Search (DFS)

Breadth First Traversal

- Similar to traversing a tree **level-by-level**
  - Nodes at each level
    - Visited from left to right
  - All nodes at any level *i*
    - Visited before visiting nodes at level *i* + one
- Start several paths at a time, and advance in each, one step at a time
  - Expand shallowest unexpanded node

BREADTH-FIRST-TRAVERSAL

- Can be implemented efficiently using a queue

```
Input: s as the source node

BFS (G, s)
let Q be queue.
Q.enqueue( s )

mark s as visited
while ( Q is not empty)
v = Q.dequeue( )

for all neighbors w of v in Graph G
if w is not visited
Q.enqueue( w )
mark w as visited
```

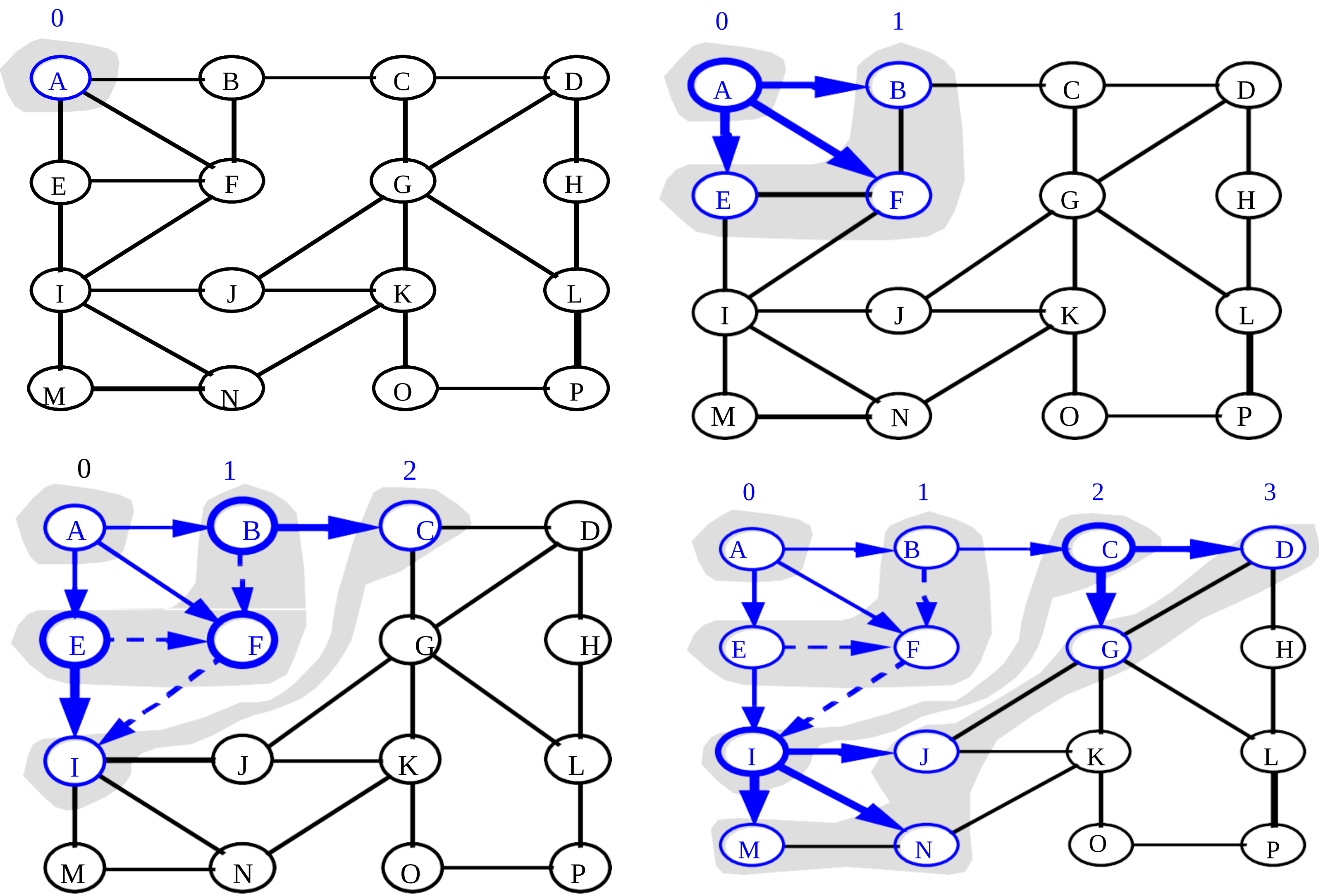
Breadth First Traversal

a)

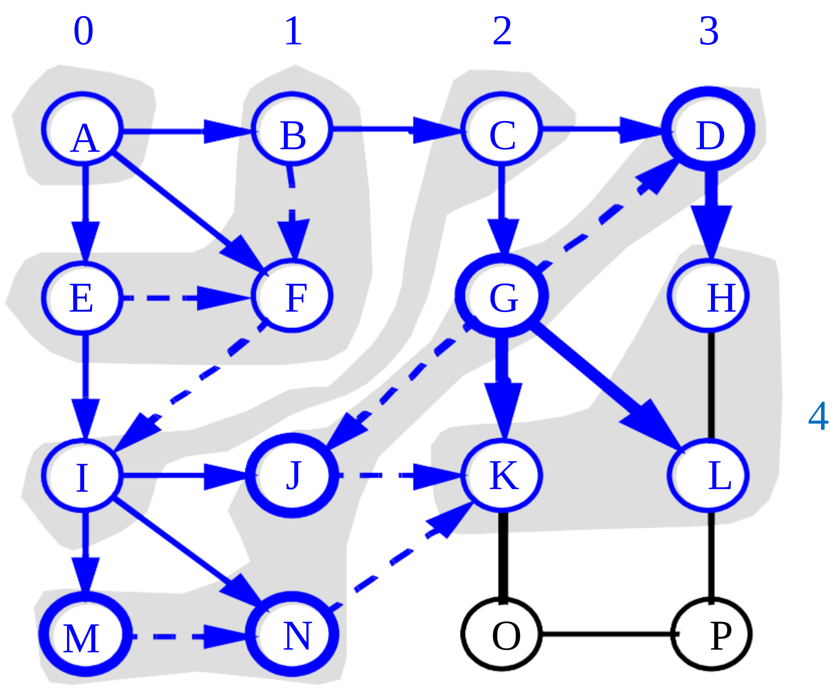
b)

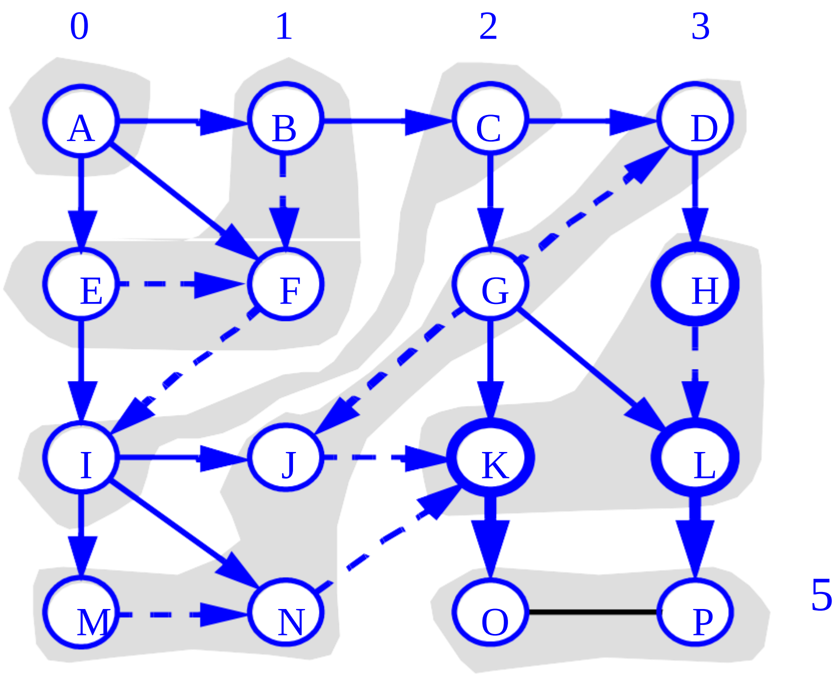
c)

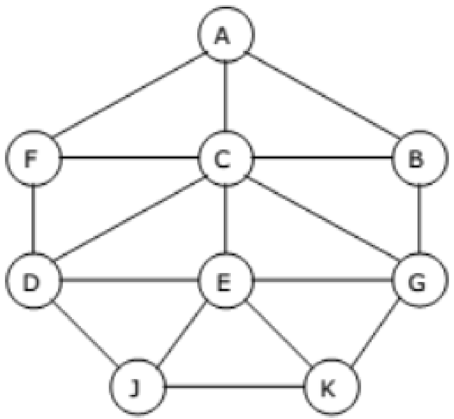
d)



Breadth First Traversal







A Graph G

Node	Adjacency List
A	F, C, B
B	A, C, G
C	A, B, D, E, F, G
D	C, F, E, J
E	C, D, G, J, K
F	A, C, D
G	B, C, E, K
J	D, E, K
K	E, G, J

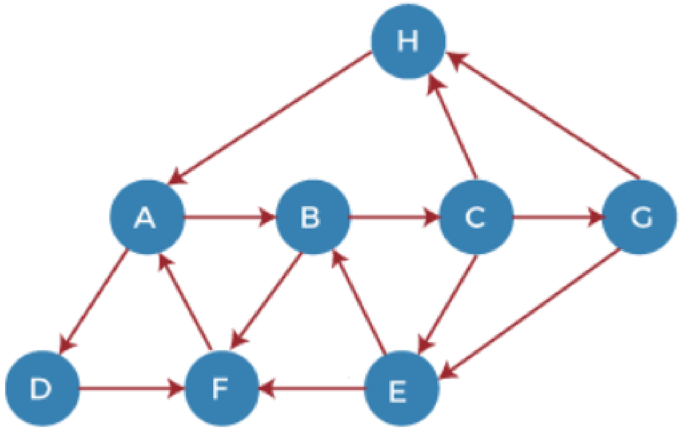
Adjacency list for graph G

Current Node	QUEUE	Processed Nodes	Status								
			A	B	C	D	E	F	G	J	K
			1	1	1	1	1	1	1	1	1
	A		2	1	1	1	1	1	1	1	1
A	F C B	A	3	2	2	1	1	2	1	1	1
F	C B D	A F	3	2	2	2	1	3	1	1	1
C	B D E G	A F C	3	2	3	2	2	3	2	1	1
B	D E G	A F C B	3	3	3	2	2	3	2	1	1
D	E G J	A F C B D	3	3	3	3	2	3	2	2	1
E	G J K	A F C B D E	3	3	3	3	3	3	2	2	2
G	J K	A F C B D E G	3	3	3	3	3	3	3	2	2
J	K	A F C B D E G J	3	3	3	3	3	3	3	3	2
K	EMPTY	A F C B D E G J K	3	3	3	3	3	3	3	3	3

For the above graph the breadth first traversal sequence is: ***A F C B D E G J K***.

Breadth First Traversal

- Example



Traversal starting from Node H.

HABDCFEG

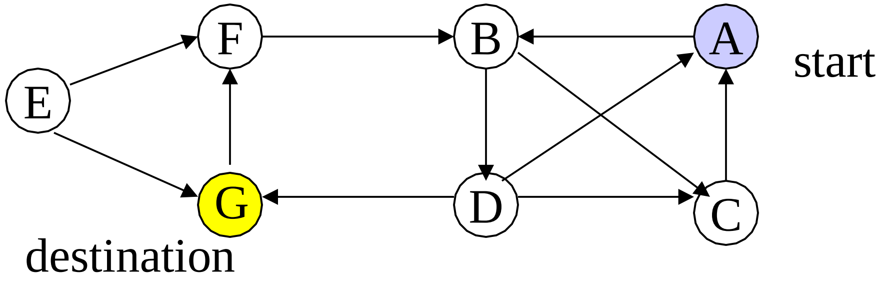
BREADTH-FIRST-SEARCH (BFS)

- BFS can be implemented efficiently using a queue

BFS-iterative

```
Set found to false
Enqueue(startVertex)
Do
     Dequeue(vertex)
     IF vertex == endVertex
             Set found to true
     ELSE
             Enqueue all adjacent vertices onto queue
WHILE !queue.IsEmpty() AND !found
IF(!found)
      Write "Path does not exist"
```

BREADTH-FIRST-SEARCH (BFS)



rear front

A

Add A to queue

rear front

B

Dequeue A
Add B

rear front

D C

Dequeue B
Add C, D

rear front

D

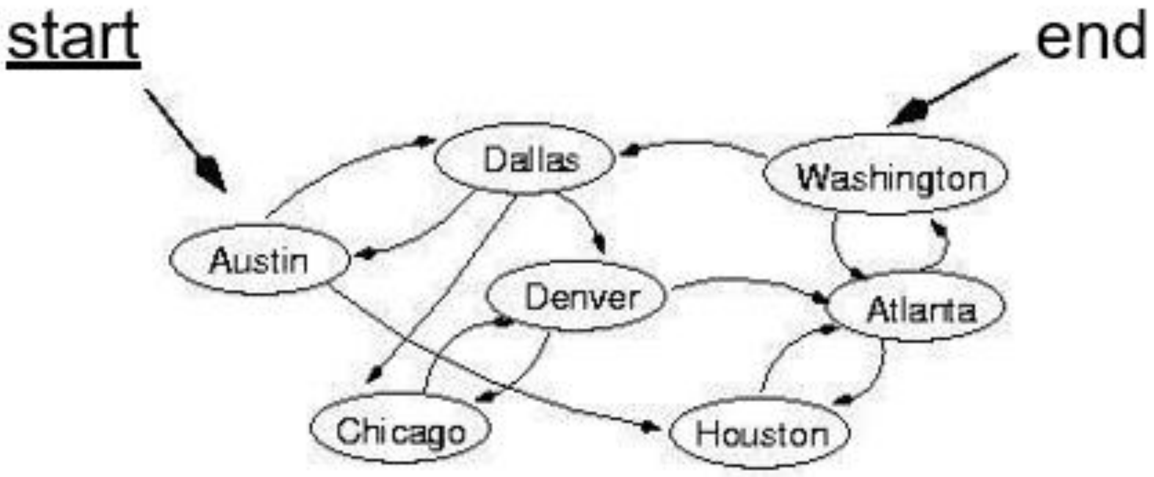
Dequeue C
Nothing to add

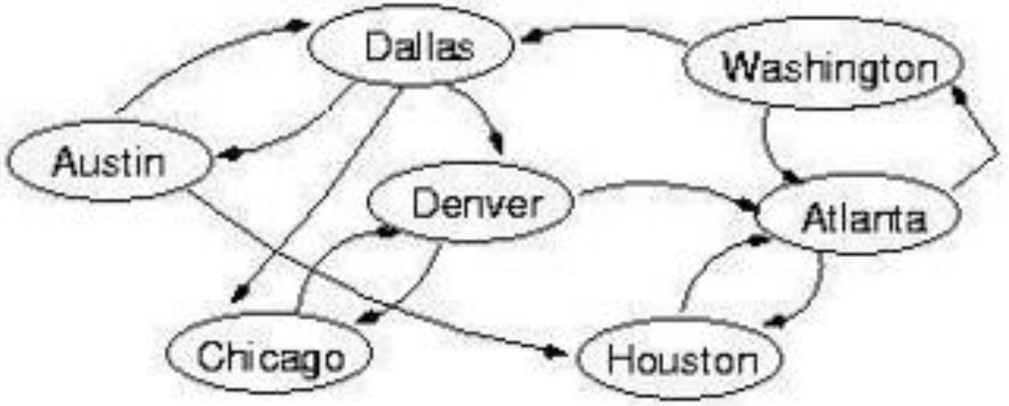
rear front

G

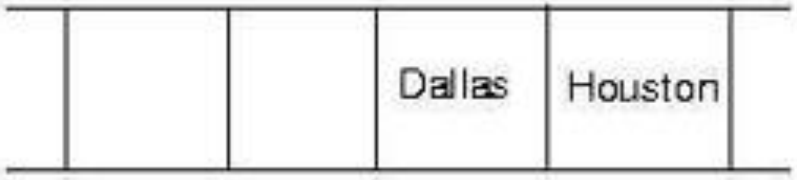
Dequeue D
Add G

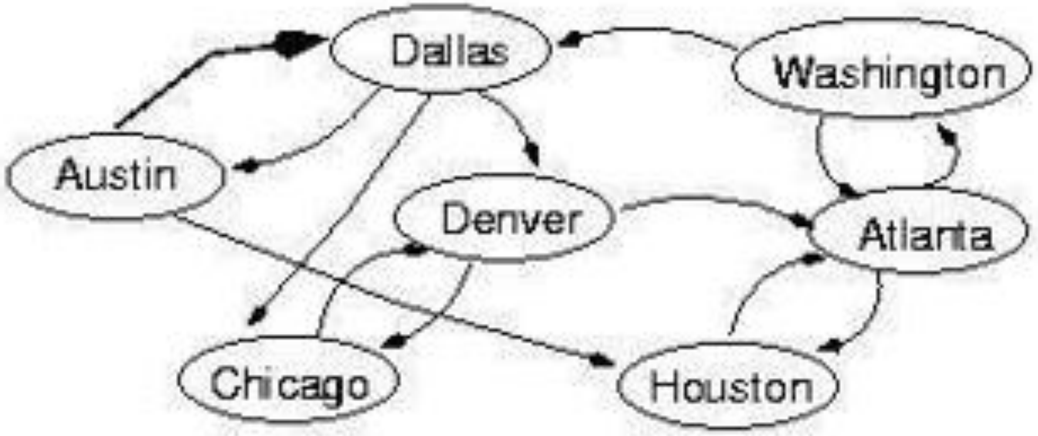
found destination - done!





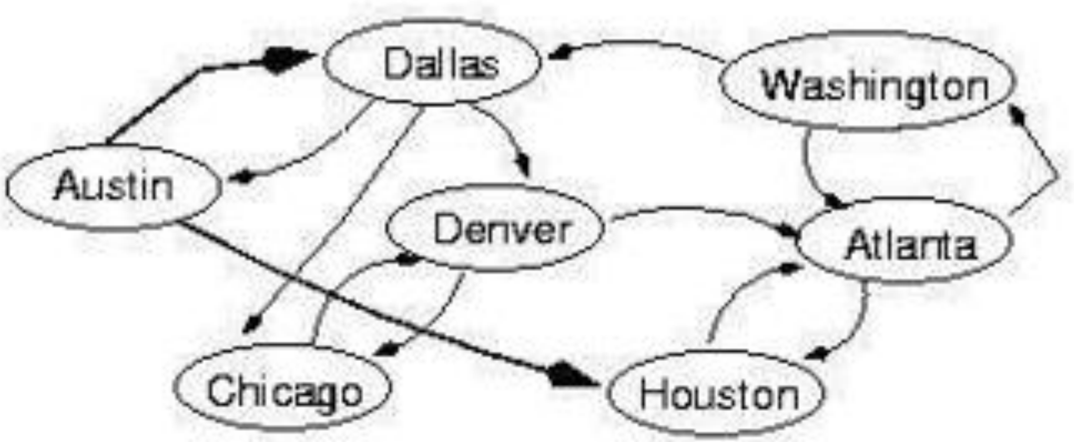
dequeue Austin





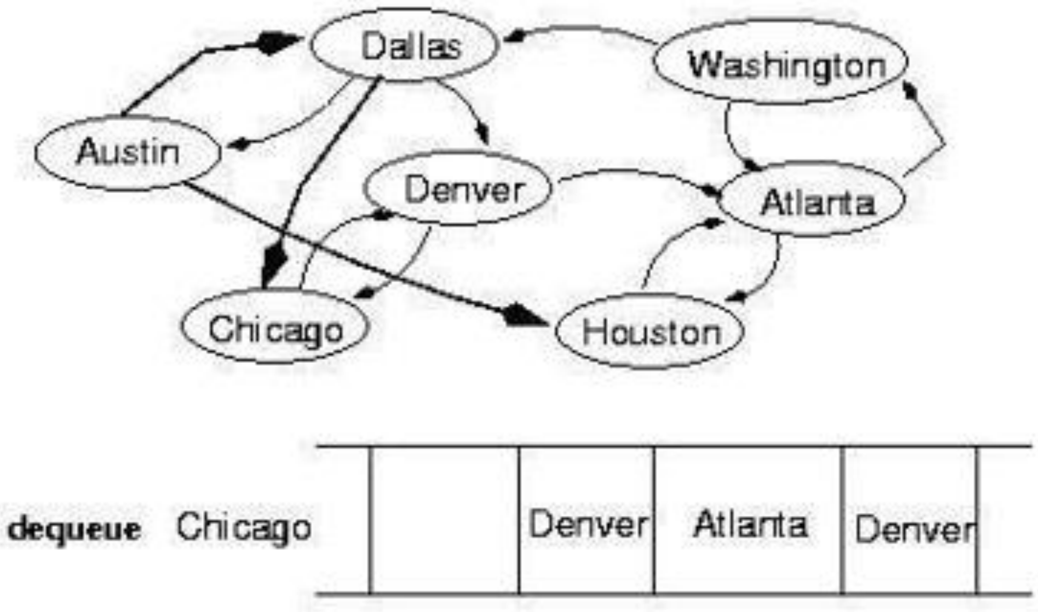
dequeue Dallas

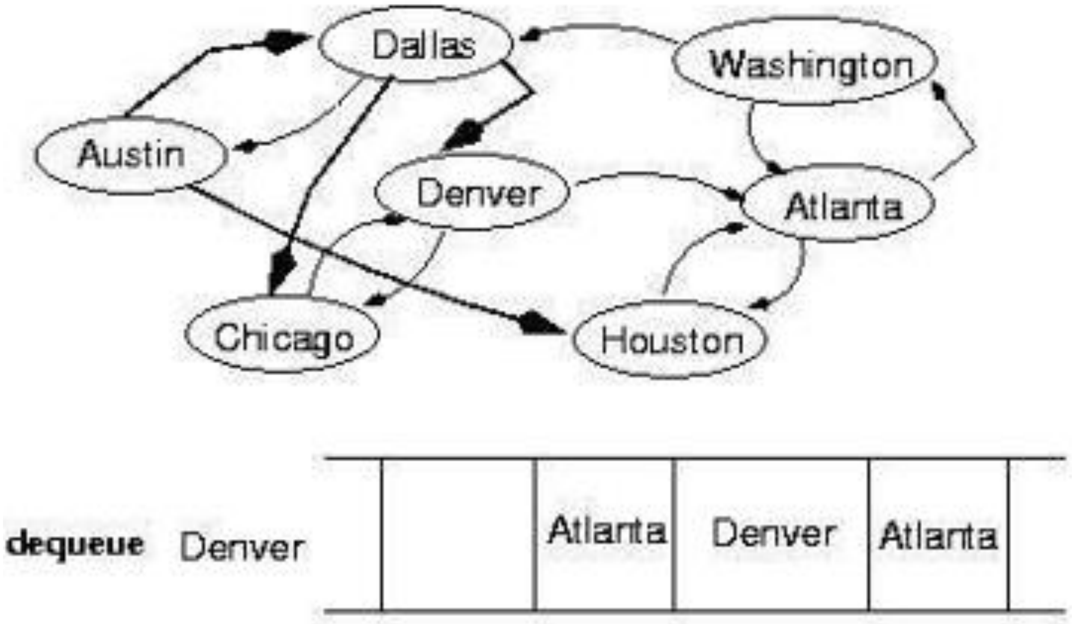
	Houston	Chicago	Denver



dequeue Houston

	Chicago	Denver	Atlanta


Dallas
Washington
Austin
Denver
Atlanta
Chicago
Houston
dequeue
Chicago
Denver
Atlanta
Denver


Dallas
Washington
Austin
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Chicago
Houston
dequeue
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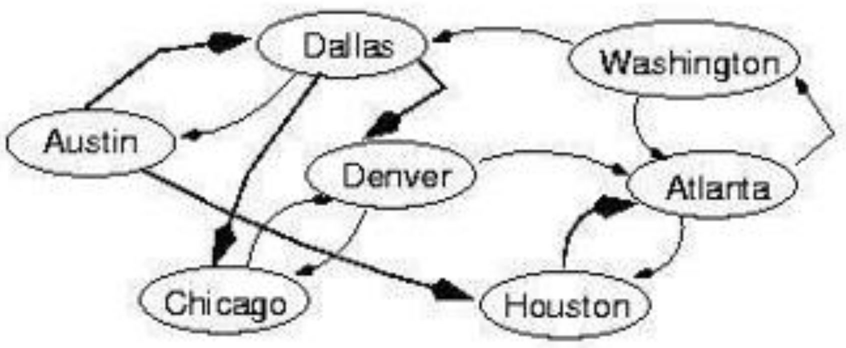
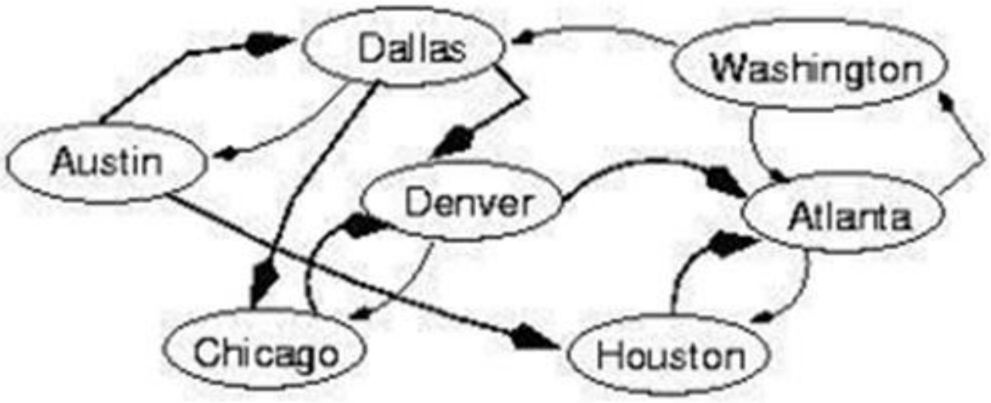
Denver
Atlanta



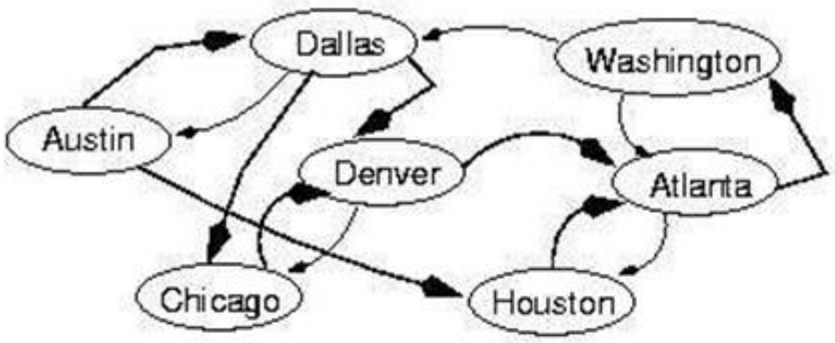
dequeue Atlanta

	Denver	Atlanta	Washington



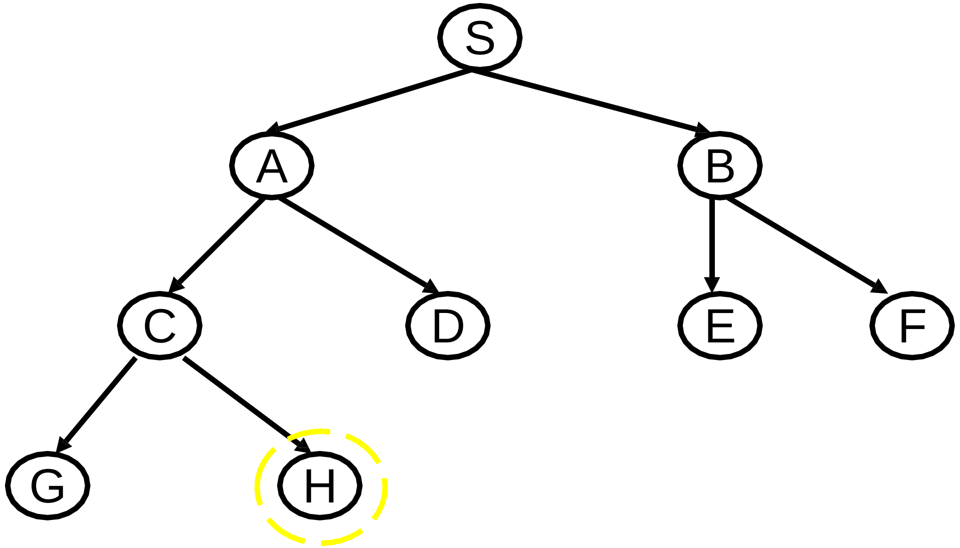
dequeue Denver, Atlanta

	Washington



dequeue Washington

BFS: EXAMPLE



Q

1

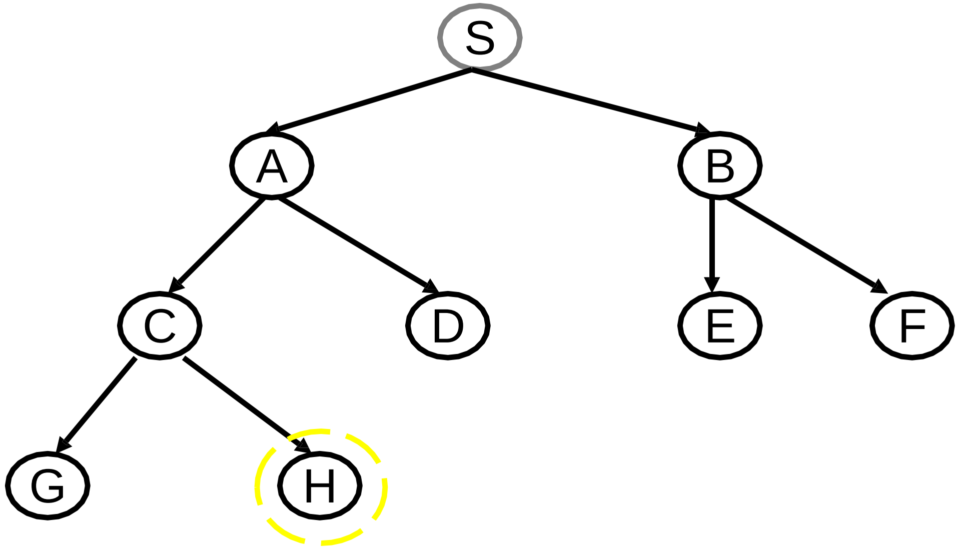
2

3

4

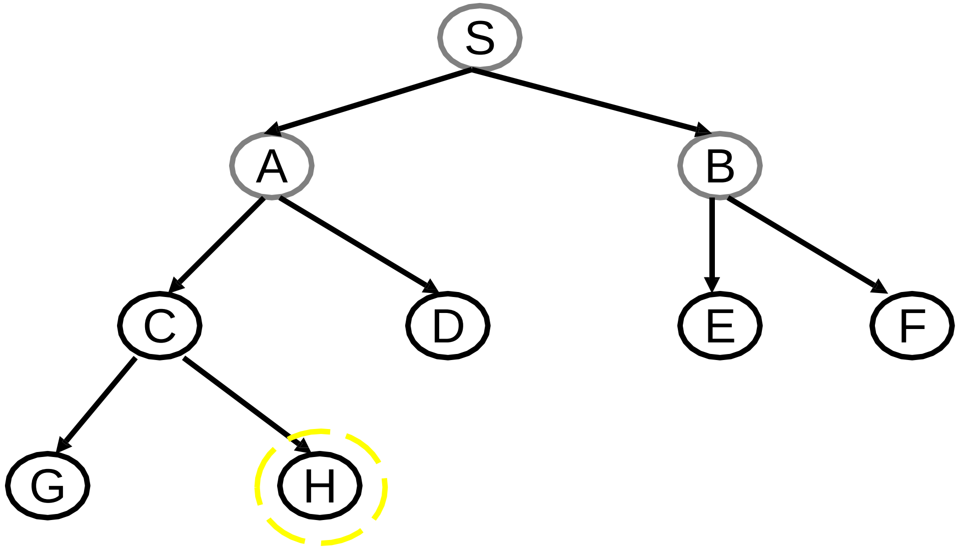
5

BFS: EXAMPLE



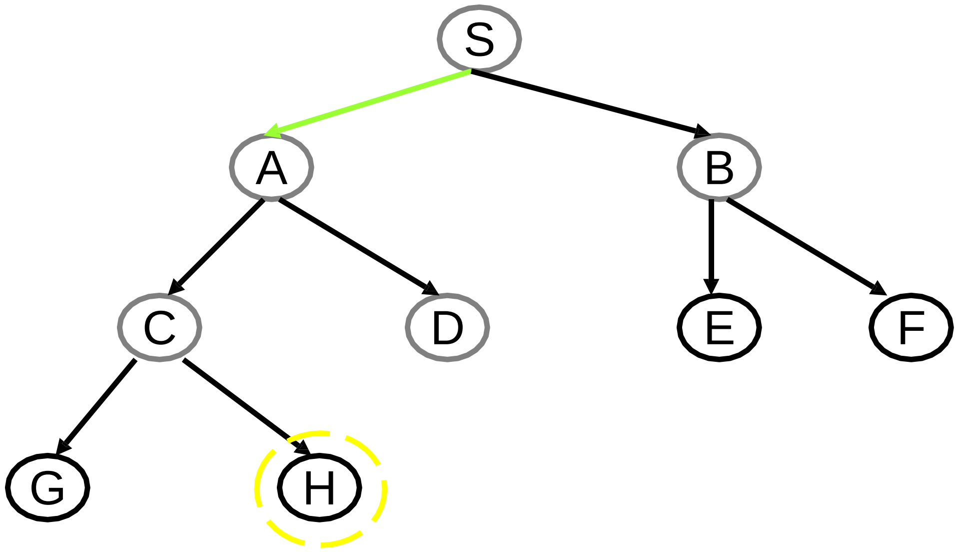
	Q
1	S
2	
3	
4	
5	

BFS: EXAMPLE



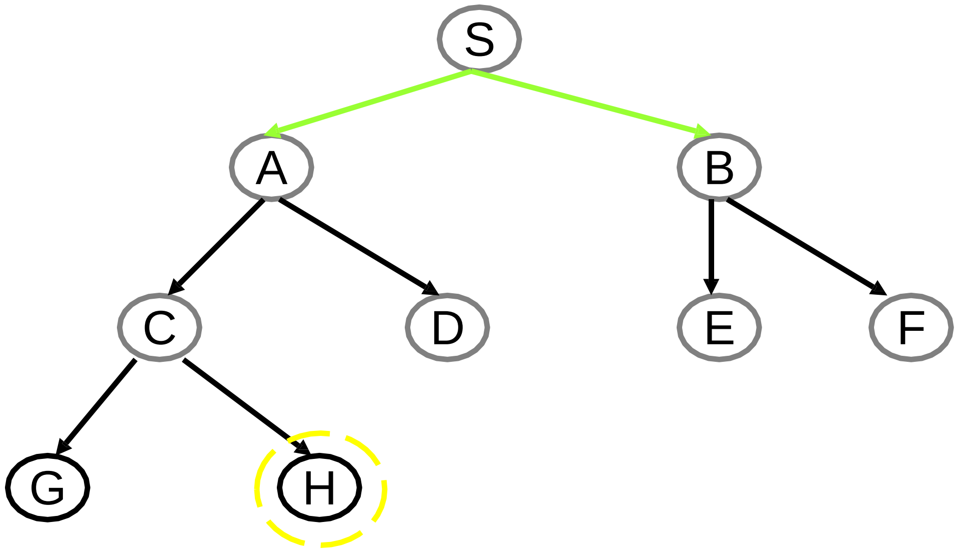
	Q
1	S
2	A,B
3	
4	
5	

BFS: EXAMPLE



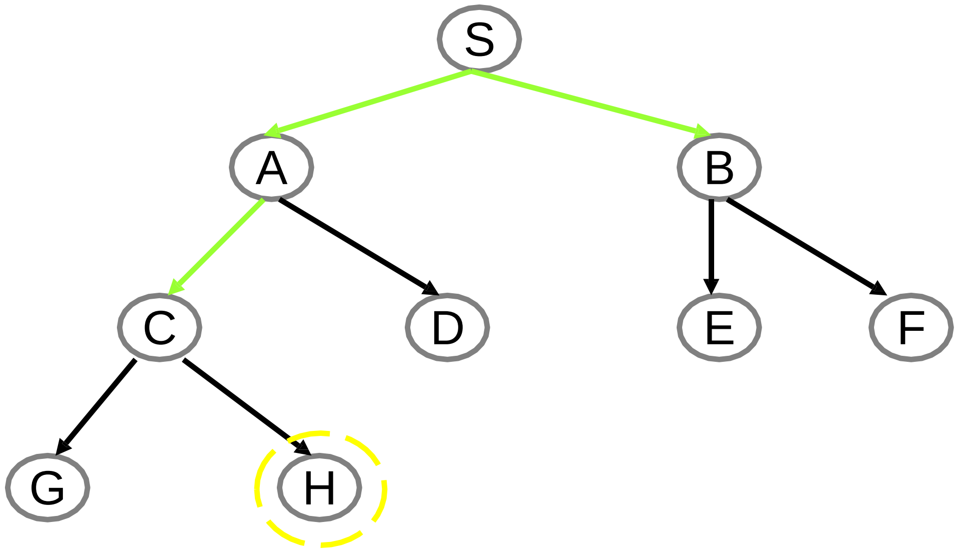
	Q
1	S
2	A,B
3	B,C,D
4	
5	

BFS: EXAMPLE



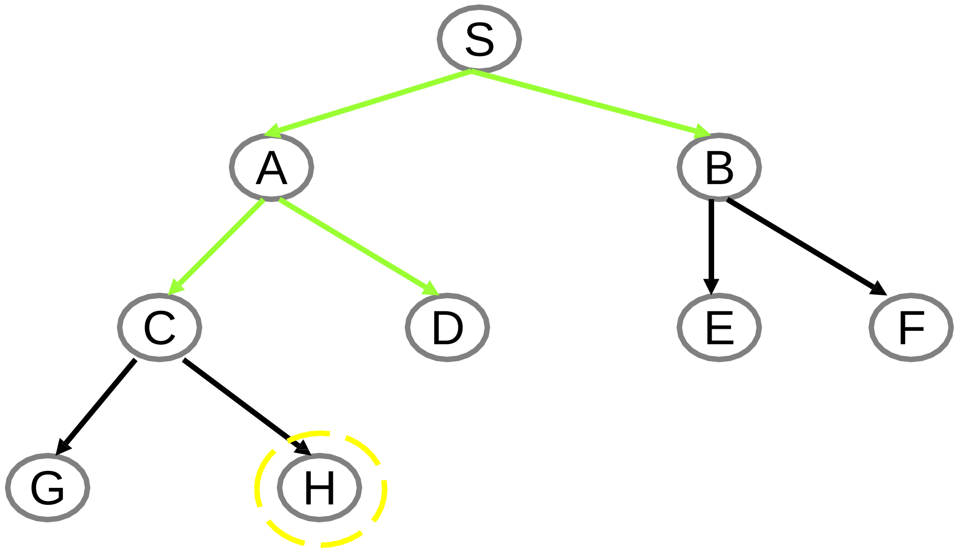
	Q
1	S
2	A,B
3	B,C,D
4	C,D,E,F
5	

BFS: EXAMPLE



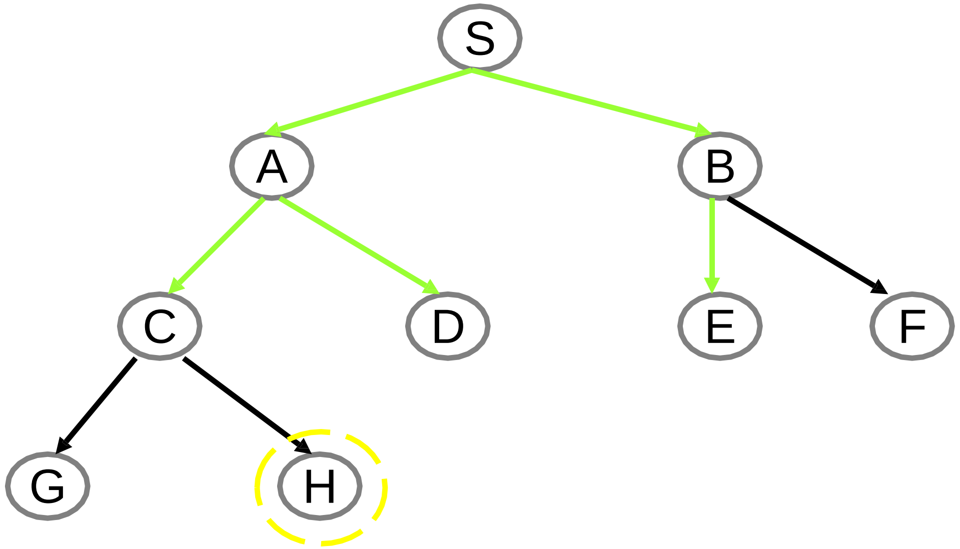
	Q
2	A,B
3	B,C,D
4	C,D,E,F
5	D,E,F,G,H
6	

BFS: EXAMPLE



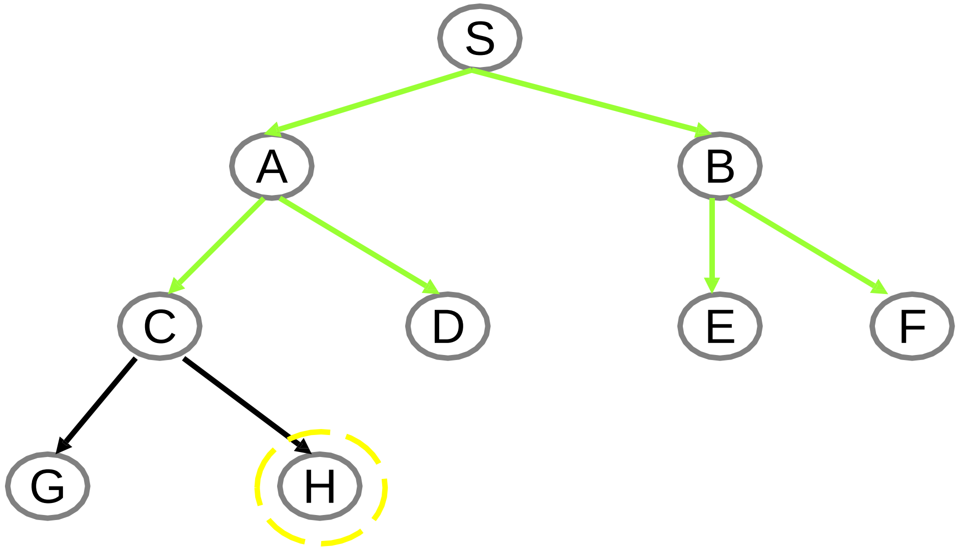
	Q
3	B,C,D
4	C,D,E,F
5	D,E,F,G,H
6	E,F,G,H
7	

BFS: EXAMPLE



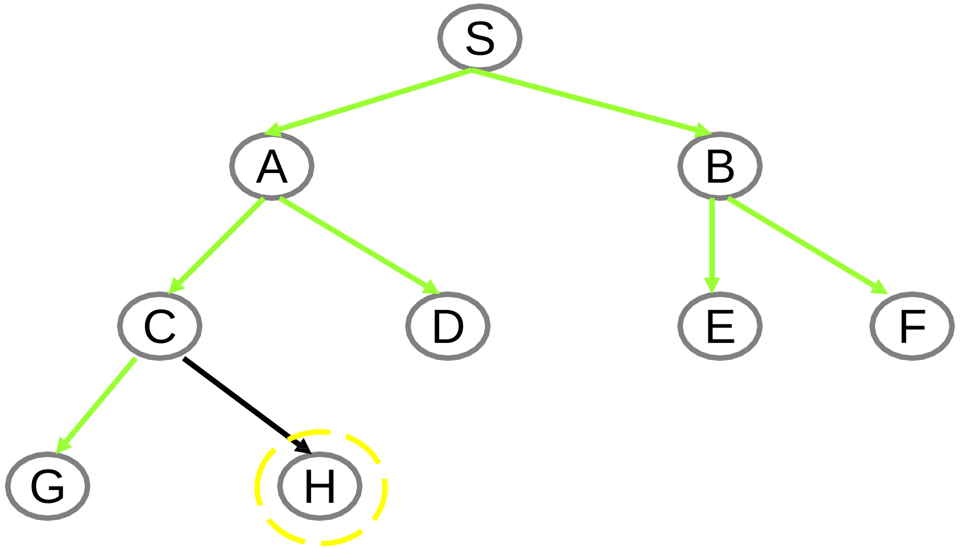
	Q
4	C,D,E,F
5	D,E,F,G,H
6	E,F,G,H
7	F,G,H
8	

BFS: EXAMPLE



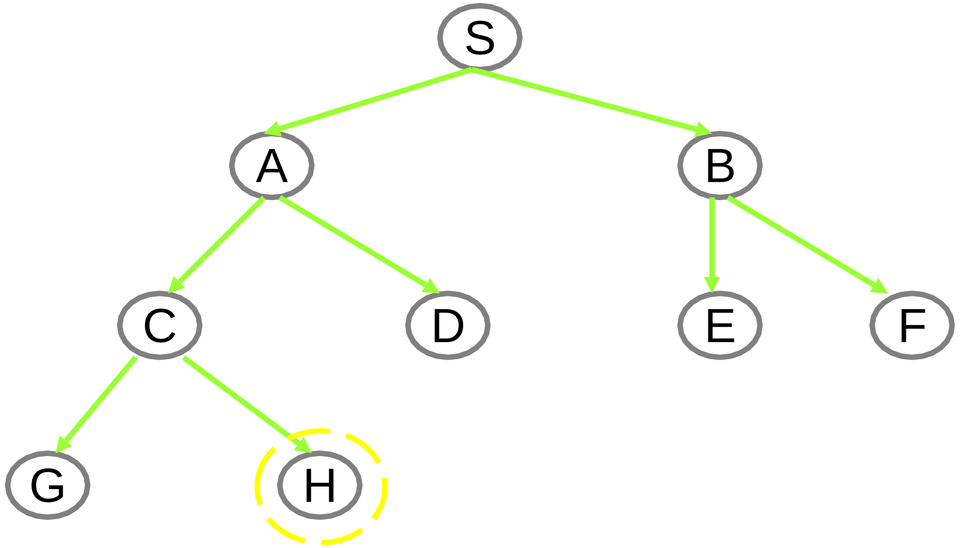
	Q
5	D,E,F,G,H
6	E,F,G,H
7	F,G,H
8	G,H
9	

BFS: EXAMPLE



	Q
6	E,F,G,H
7	F,G,H
8	G,H
9	H
10	

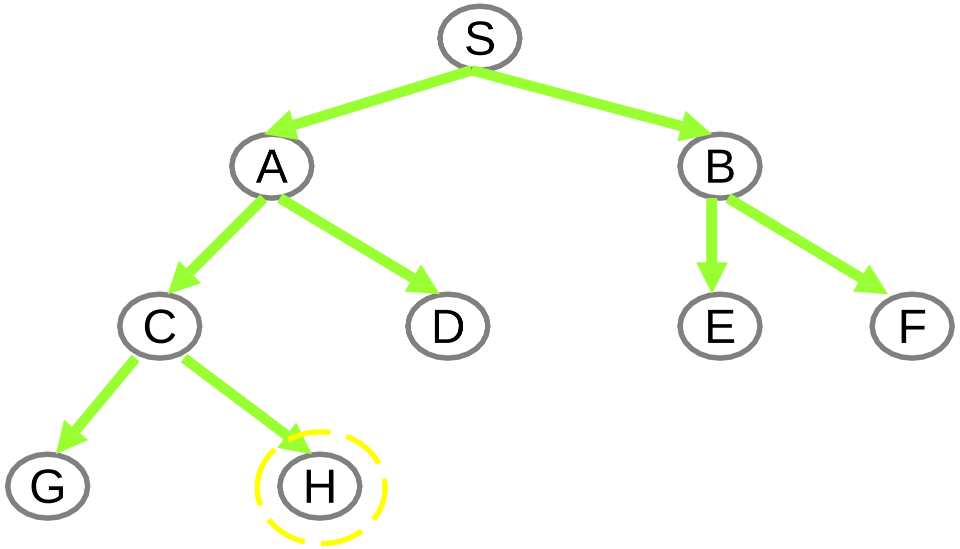
BFS: EXAMPLE



	Q
6	E,F,G,H
7	F,G,H
8	G,H
9	
10	

BFS: EXAMPLE

	Q
1	S
2	A,B
3	B,C,D
4	C,D,E,F
5	D,E,F,G,H
6	E,F,G,H
7	F,G,H
8	G,H
9	H
10	



Depth First Traversal

- Similar to binary tree preorder traversal
- Once a possible path is found, continue the search until the end of the path
- What is the idea behind DFS?
  - Travel as far as you can down a path
  - Back up *as little as possible* when you reach a "dead end" (i.e., next vertex has been "marked" or there is no next vertex)
  - Expand deepest unexpanded node
- General algorithm for depth first traversal at a given node *v (recursive)*

```
DFS-recursive(G, s):
    mark s as visited
    for all neighbours w of s in Graph G:
        if w is not visited:
            DFS-recursive(G, w)
```

Depth First Traversal

- General algorithm for depth first traversal at a given node *v (recursive)*

```
DFS-recursive(G, s):
    mark s as visited
    for all neighbours w of s in Graph G:
        if w is not visited:
            DFS-recursive(G, w)
```

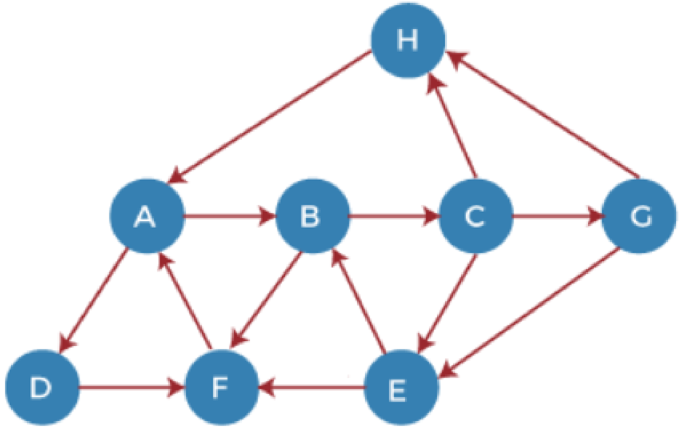
Depth First Traversal

- General algorithm for depth first traversal at a given node *v (non-recursive)*

```
DFS(G,v)   ( v is the vertex where the search starts )
       Stack S := {};   ( start with an empty stack )
       for each vertex u, set visited[u] := false;
       push S, v;
       while (S is not empty) do
          u := pop S;
          if (not visited[u]) then
             visited[u] := true;
             for each unvisited neighbour w of uu
                push S, w;
          end if
       end while
    END DFS()
```

Depth First Traversal

- Example

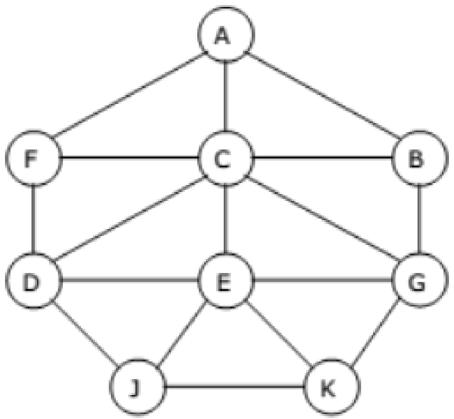


Adjacency Lists

A : B, D
B : C, F
C : E, G, H
G : E, H
E : B, F
F : A
D : F
H : A

Traversal starting from Node H.

HADFBCGE



A Graph G

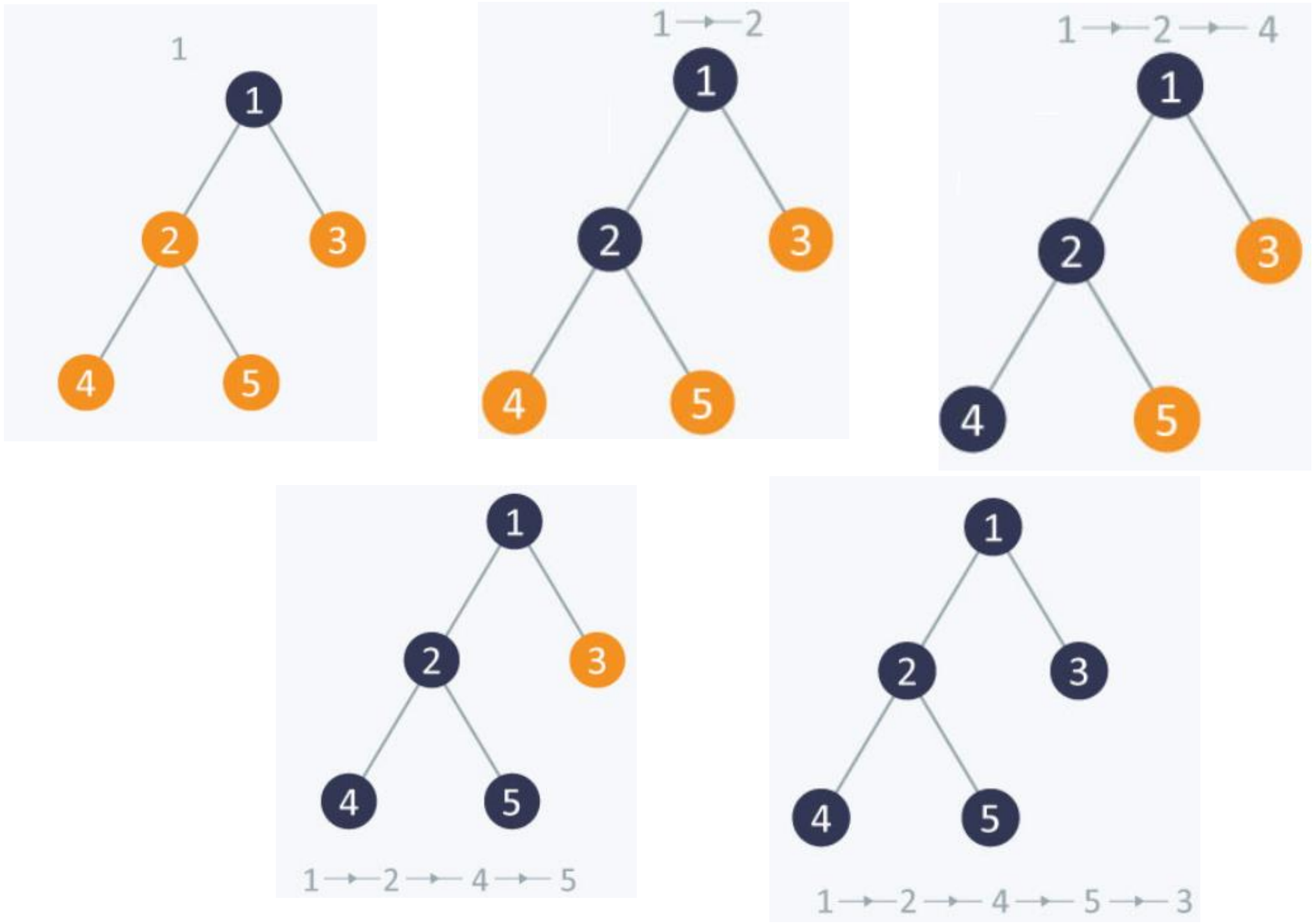
Node	Adjacency List
A	F, C, B
B	A, C, G
C	A, B, D, E, F, G
D	C, F, E, J
E	C, D, G, J, K
F	A, C, D
G	B, C, E, K
J	D, E, K
K	E, G, J

Adjacency list for graph G

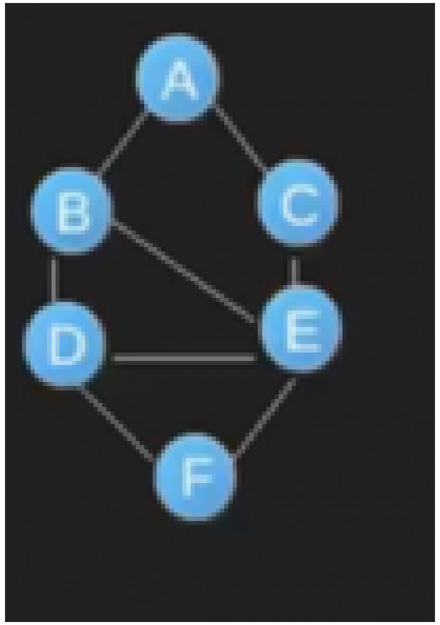
Current Node	Stack	Processed Nodes	Status								
			A	B	C	D	E	F	G	J	K
			1	1	1	1	1	1	1	1	1
	A		2	1	1	1	1	1	1	1	1
A	B C F	A	3	2	2	1	1	2	1	1	1
F	B C D	A F	3	2	2	2	1	3	1	1	1
D	B C E J	A F D	3	2	2	3	2	3	1	2	1
J	B C E K	A F D J	3	2	2	3	2	3	1	3	2
K	B C E G	A F D J K	3	2	2	3	2	3	2	3	3
G	B C E	A F D J K G	3	2	2	3	2	3	3	3	3
E	B C	A F D J K G E	3	2	2	3	3	3	3	3	3
C	B	A F D J K G E C	3	2	3	3	3	3	3	3	3
B	EMPTY	A F D J K G E C B	3	3	3	3	3	3	3	3	3

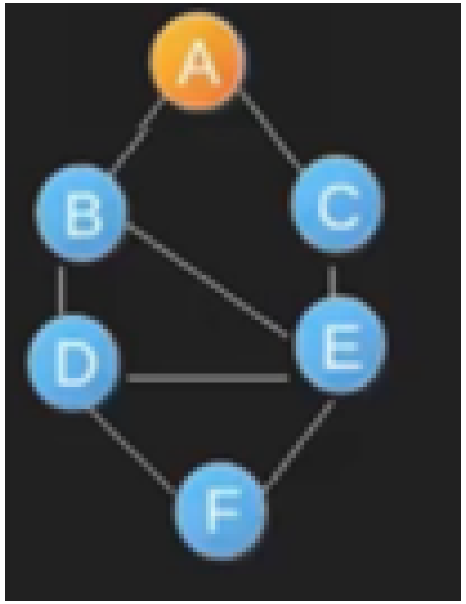
For the above graph the depth first traversal sequence is: ***A F D J K G E C B***.

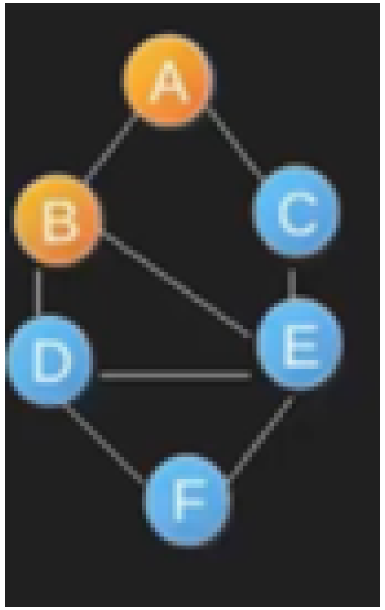
Depth First Traversal

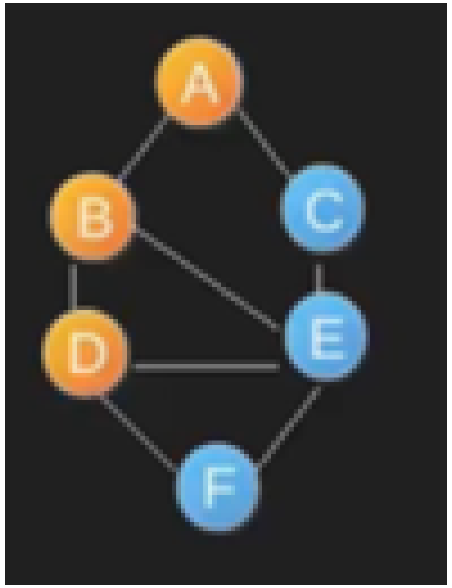


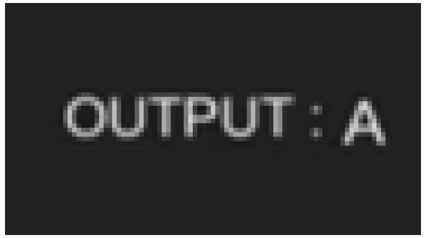
Depth First Traversal



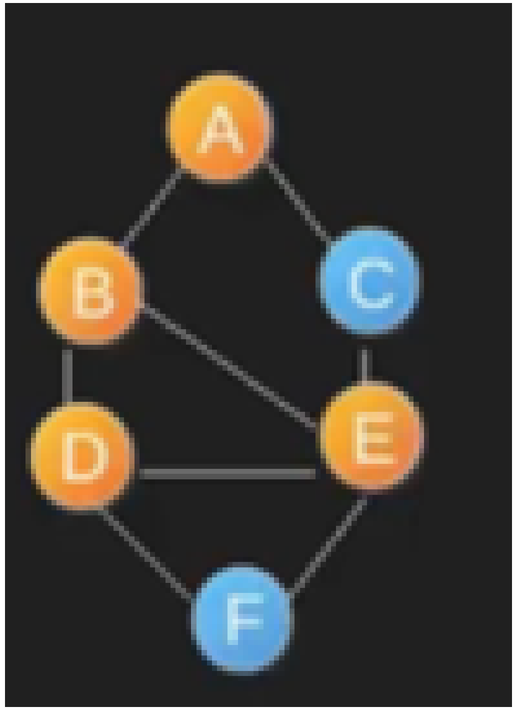




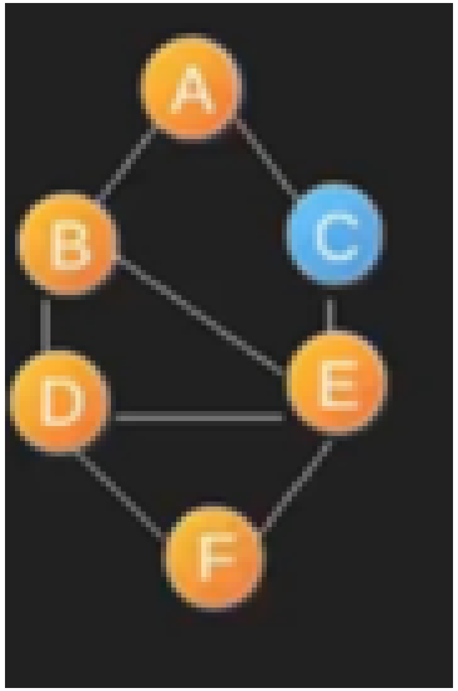




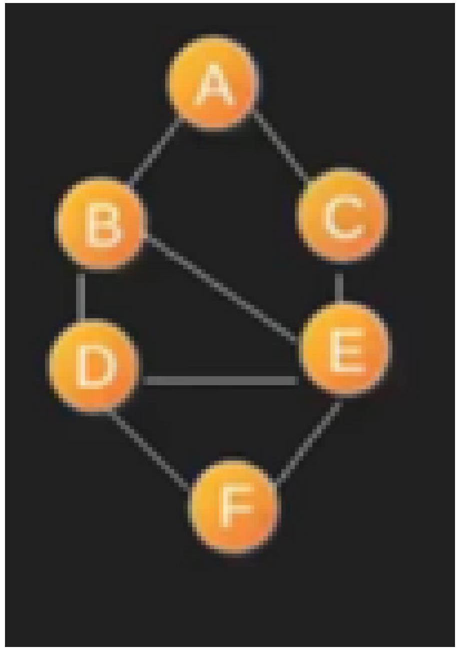
Depth First Traversal



OUTPUT : A B D E



OUTPUT : A B D E F



OUTPUT : A B D E F C

DEPTH-FIRST-SEARCH (DFS)

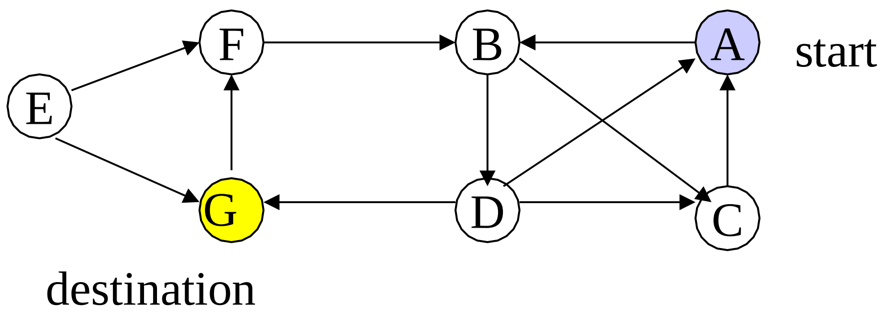
- DFS can be implemented efficiently using a *stack*

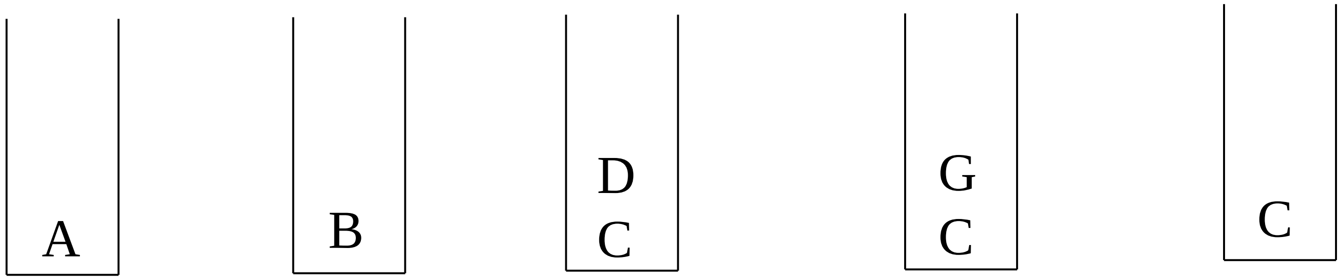
DFS-iterative

```
Set found to false
  Push(startVertex)
  DO
    Pop(vertex)
    IF vertex == endVertex
    Set found to true
    ELSE
      Push all adjacent vertices onto stack
  WHILE !stack.IsEmpty() AND !found

  IF(!found)
    Write "Path does not exist"
```

DEPTH-FIRST-SEARCH (DFS)





Push A

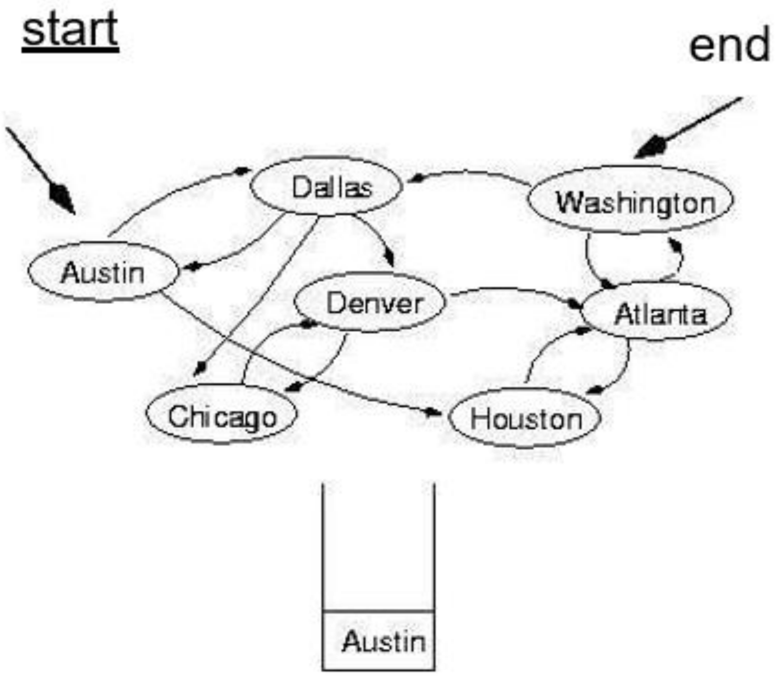
Pop A
Push B

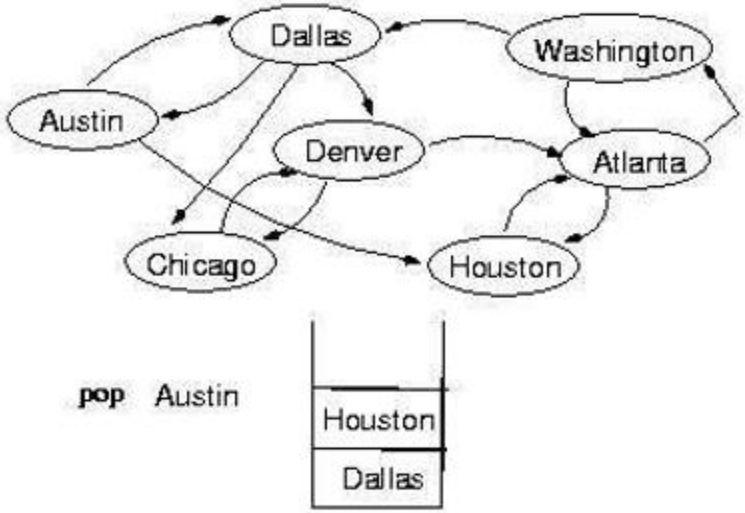
Pop B
Push C
Push D

Pop D
Push G

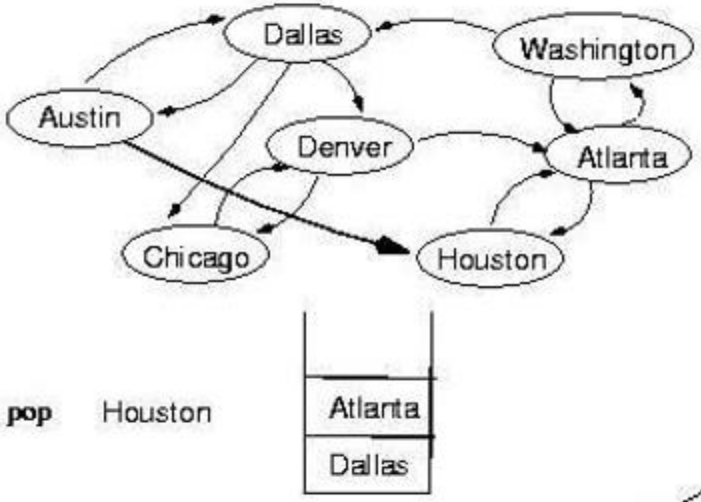
Pop G
found destination - done!

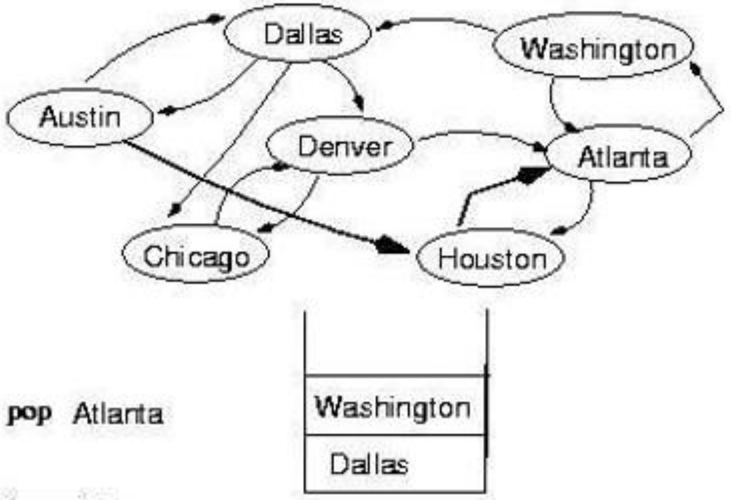
DEPTH-FIRST-SEARCH (DFS)

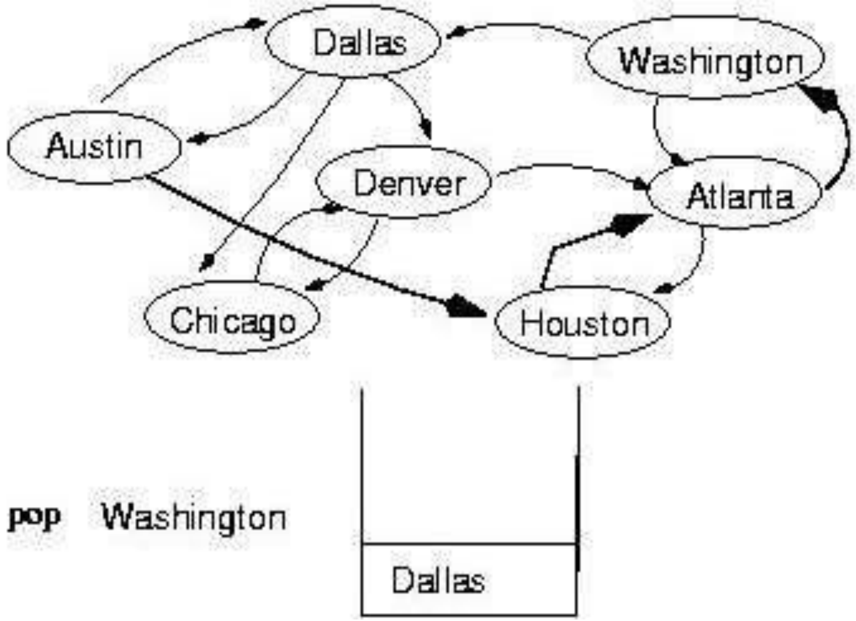




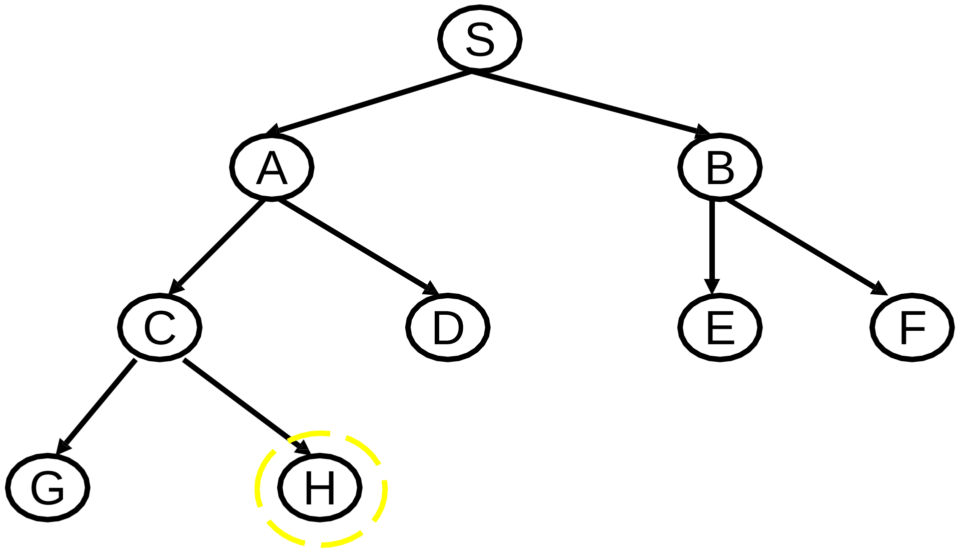
DEPTH-FIRST-SEARCH (DFS)







DFS: EXAMPLE



Q

1

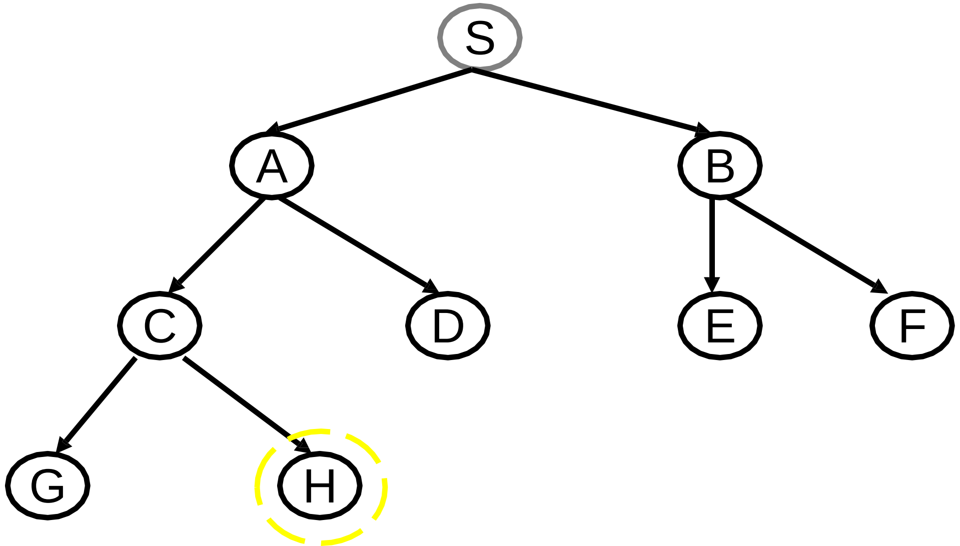
2

3

4

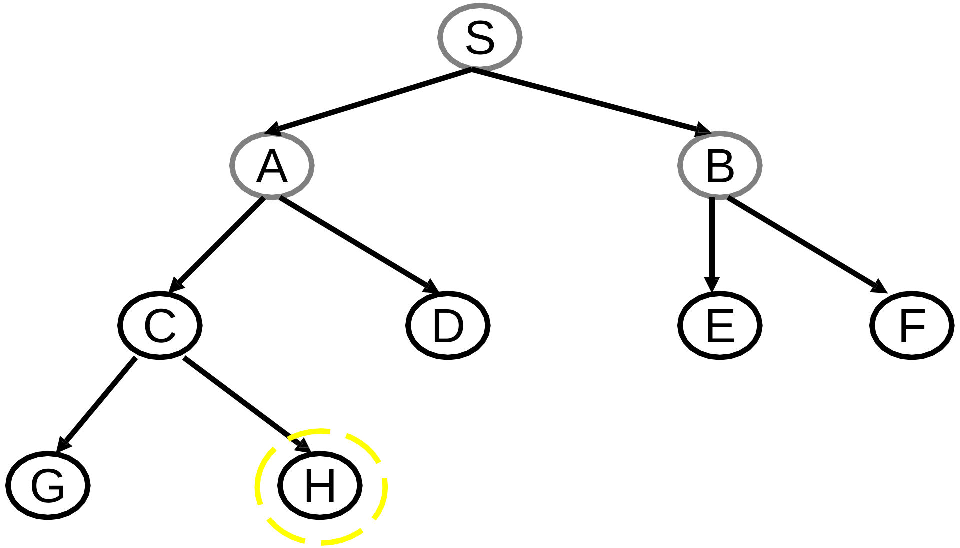
5

DFS: EXAMPLE



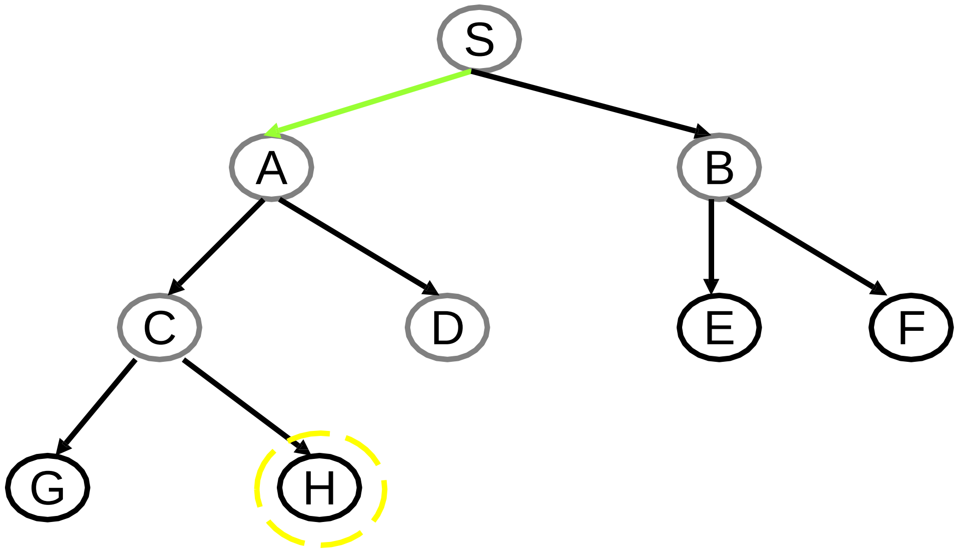
	Q
1	S
2	
3	
4	
5	

DFS: EXAMPLE



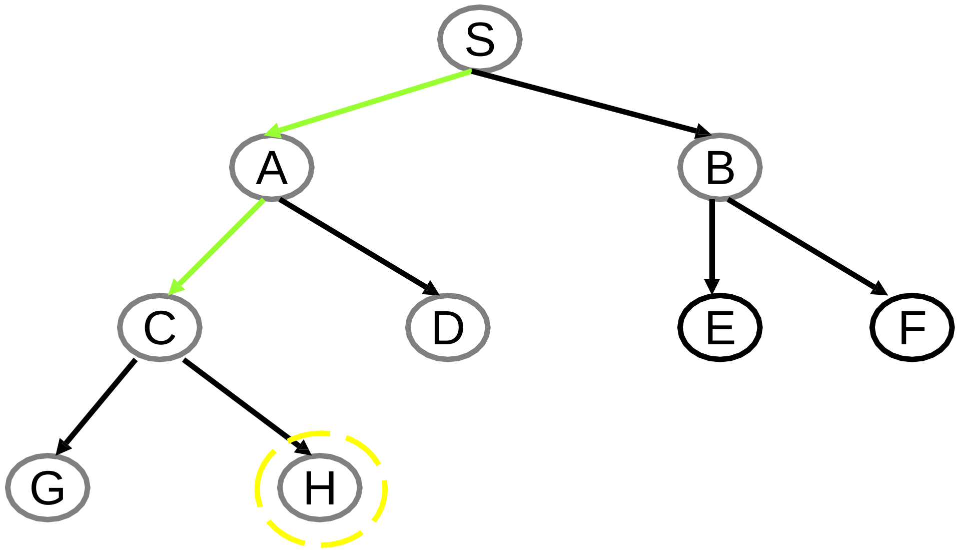
	Q
1	S
2	A,B
3	
4	
5	

DFS: EXAMPLE



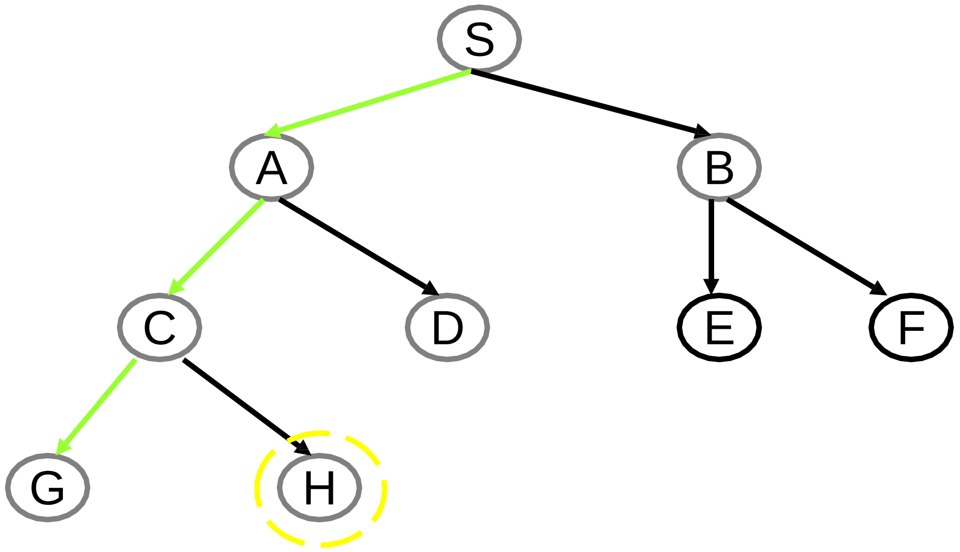
	Q
1	S
2	A,B
3	C,D,B
4	
5	

DFS: EXAMPLE



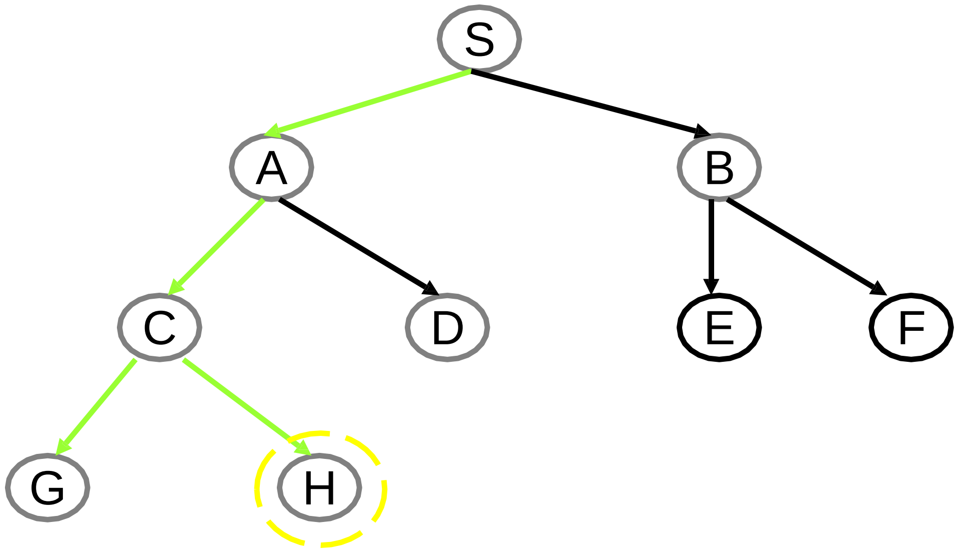
	Q
1	S
2	A,B
3	C,D,B
4	G,H,D,B
5	

DFS: EXAMPLE



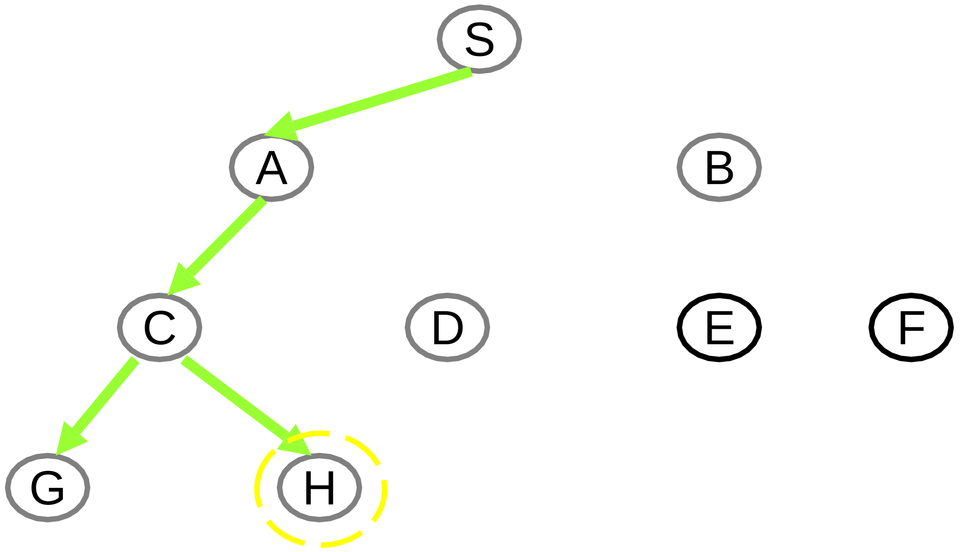
	Q
1	S
2	A,B
3	C,D,B
4	G,H,D,B
5	H,D,B

DFS: EXAMPLE



	Q
1	S
2	A,B
3	C,D,B
4	G,H,D,B
5	H,D,B
6	D,B

DFS: EXAMPLE



	Q
1	S
2	A,B
3	C,D,B
4	G,H,D,B
5	H,D,B
6	D,B