On the Motion of Ponytails

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1 Introduction

THE study of the dynamics of motion is arguably one of the most important aspects of applied mathematics. Our existence is purely governed by the physical laws of dynamics, both discovered and undiscovered. Throughout time, the goal of mankind has been to control the environment as much as possible. In order for this to be a possibility, we must first understand the laws of the physical plane and their interactions with each other.

2 Problem Statement

Curiosity is usually sparked when phenomena are observed and the desire to determine its causes is born. A common question involves movement that causes other, possibly unexpected, movement in another direction. One such example of this occurs when watching joggers, specifically those with long hair that is tied back into a ponytail. The vertical movement of the head as the jogger takes each step can lead to their ponytail swaying back and forth. The questions are posed: How does motion in one direction translate to motion in another? What are the requirements for this motion to occur?

2.1 Literature Review

The primary source for this work was Dr. Joseph Keller's *Ponytail Motion*, published in the Siam Journal of Mathematics in 2010[1]. In it he analyzes the problem using two different models: treating the ponytail as a rigid rod and treating the ponytail as a flexible, inelastic string. He finishes the paper by describing a third model based on a flexible rod with a small amount of bending stiffness. Keller believed this third model would be the most accurate of the three and laid the groundwork for further analysis along this path.

For our own analysis we used Keller's rigid rod model, albeit while showing more of the intermediate steps. In considering the stability of fixed points we followed Keller's lead, assuming relatively small and slow oscillations while the pendulum is near the bottom fixed point. However, Martin Beecher's book[2] *The Pendulum Paradigm: Variations on a Theme and the Measure of Heaven and Earth* has a wonderful section entitled "The Inverted Pendulum". In it, he describes three methods in which the higher fixed point can be stable: fast and small vertical oscillations (Kapitza frequencies), rapid horizontal oscillations (studied by Acheson for n-link pendulum suspension), and through torque applied on the pendulum (the so-called Furuta pendulum). Unfortunately, while these methods are very interesting in and of themselves, we determined that they were outside the scope of our analysis.

3 Analysis

3.1 Attaining the angle of displacement as a function of time

Our goal is to determine the necessary requirements for vertical motion to beget lateral oscillation.

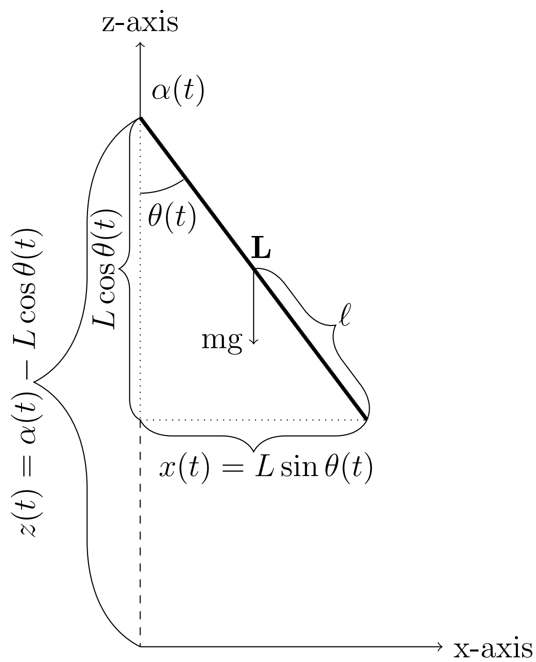

Figure 1: Rigid Rod Pendulum

Let us define the axes so that the z-axis is vertical, the y-axis is the direction the jogger is moving, and the x-axis is the direction in which the ponytail sways. Figure 1 is a 2 dimensional representation of this problem where we neglect the y-axis as it is the direction of forward motion, and does not have an effect on the pendulum in this example.

The function $\alpha(t)$ represents the position of the support point, that is the jogger's head bobbing as they run. The position of the end of the rod, the tip of the ponytail, can be can be represented by the functions $x(t) = L\sin\theta(t)$ for the horizontal position and $z(t) = \alpha(t) - L\cos\theta(t)$ for the vertical position. However, as we are considering this a rigid rod of uniform mass, our energy equations don't track the end of the rod, but the center of gravity at its midpoint. For $L = \frac{\ell}{2}$, our position functions become$x(t) = \ell\sin\theta(t)$ and $z(t) = \alpha(t) - \ell\cos\theta(t)$. Taking derivatives with respect to t yields the following.

$$\dot{x} = \ell\dot{\theta}\cos\theta \qquad \dot{z} = \dot{\alpha} + \ell\dot{\theta}\sin\theta$$

Using the principles of stationary action we have the following functions for kinetic energy, T, and potential energy due to gravity U_g.

$$
\begin{aligned}
T &= \frac{1}{2}mv^2 = \frac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big) = \frac{1}{2}m\big(\ell^2\dot{\theta}^2\cos^2\theta + \dot{\alpha}^2 + 2\dot{\alpha}\ell\dot{\theta}\sin\theta + \ell^2\dot{\theta}^2\sin^2\theta\big)\\
&= \frac{1}{2}m\big(\ell^2\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\alpha}\ell\dot{\theta}\sin\theta\big)\\
U_g &= mgz = mg\big(\alpha - \ell\cos\theta\big)
\end{aligned}
$$

From which we obtain the integral:

$$
\begin{aligned}
&\int_{t_1}^{t_2}\Big[\frac{m}{2}\big(\ell^2\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\alpha}\ell\dot{\theta}\sin\theta\big) - mg\big(\alpha - \ell\cos\theta\big)\Big]dt\\
&\qquad = \frac{m}{2}\int_{t_1}^{t_2}\Big(\ell^2\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\alpha}\ell\dot{\theta}\sin\theta - 2g\dot{\alpha} + 2\ell g\cos\theta\Big)dt
\end{aligned}
$$

We now set up the Euler-Lagrange Equation for:

$$
F\big[t, \theta, \dot{\theta}\big] = \ell^2\dot{\theta}^2 + \dot{\alpha}^2 + 2\dot{\alpha}\ell\dot{\theta}\sin\theta - 2g\dot{\alpha} + 2\ell g\cos\theta
$$

$$
\frac{\partial F}{\partial\theta} = 2\dot{\alpha}\ell\dot{\theta}\cos\theta - 2g\ell\sin\theta \qquad \frac{\partial F}{\partial\dot{\theta}} = 2\ell^2\dot{\theta} + 2\dot{\alpha}\ell\sin\theta \implies \frac{d}{dt}\Big(\frac{\partial F}{\partial\dot{\theta}}\Big) = 2\ell^2\ddot{\theta} + 2\ddot{\alpha}\ell\sin\theta + 2\dot{\alpha}\ell\dot{\theta}\cos\theta
$$

$$
\begin{aligned}
\frac{\partial F}{\partial\theta} - \frac{d}{dt}\Big(\frac{\partial F}{\partial\dot{\theta}}\Big) &= 2\dot{\alpha}\ell\dot{\theta}\cos\theta - 2g\ell\sin\theta - 2\ell^2\ddot{\theta} - 2\ddot{\alpha}\ell\sin\theta - 2\dot{\alpha}\ell\dot{\theta}\cos\theta\\
&= -2\ell^2\Big(\frac{g}{\ell}\sin\theta + \ddot{\theta} + \frac{\ddot{\alpha}}{\ell}\sin\theta\Big) = 0 \implies \ddot{\theta} + \frac{g + \ddot{\alpha}}{\ell}\sin\theta = 0\\
&\implies \ddot{\theta} = -\frac{g + \ddot{\alpha}}{\ell}\sin\theta \implies \ddot{\theta} = -\frac{2(g + \ddot{\alpha})}{L}\sin\theta
\end{aligned}
$$

By setting $\omega_\theta = \sqrt{\frac{2(g+\ddot{\alpha})}{L}}$ we enter the below ODE symbolically into `MATLAB`, under the assumption that $\omega_\theta \in \mathbb{R}$, and we receive the below solution where $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$:

$$
\ddot{\theta} = -\omega_\theta^2\sin\theta \implies \theta(t) = \theta_0\cos\omega_\theta t + \frac{\dot{\theta}_0\sin\omega_\theta t}{\omega_\theta}
$$

3.2 Stability of the system

To determine the stability of this situation we first linearize our ODE about θ_0.

$$\ddot{\theta} + \frac{2(g+\ddot{\alpha})}{L}\sin\theta \approx \ddot{\theta} + \frac{2(g+\ddot{\alpha})}{L}\Big[\sin\theta_0 + \cos\theta_0(\theta-\theta_0)\Big]$$

$$\text{For } \theta_0 = 0 \text{ we have: } \ddot{\theta} + \frac{2(g+\ddot{\alpha})}{L}\sin\theta \approx \ddot{\theta} + \frac{2(g+\ddot{\alpha})}{L}\theta \implies \dddot{\theta} + \frac{2(g+\ddot{\alpha})}{L}\dot{\theta} = 0$$

$$\implies \dot{\theta} = c_1 \sin\left(-\theta\sqrt{\frac{2(g+\ddot{\alpha})}{L}}\right) + c_2\cos\left(-\theta\sqrt{\frac{2(g+\ddot{\alpha})}{L}}\right) \text{ for some } c_1, c_2 \in \mathbb{R}.$$

$$\text{For } \theta_0 = \pi \text{ we have: } \ddot{\theta} + \frac{2(g+\ddot{\alpha})}{L}\sin\theta \approx \ddot{\theta} - \frac{2(g+\ddot{\alpha})}{L}\theta \implies \dddot{\theta} - \frac{2(g+\ddot{\alpha})}{L}\dot{\theta} = 0$$

$$\implies \dot{\theta} = c_1 \exp\left(\theta\sqrt{\frac{2(g+\ddot{\alpha})}{L}}\right) + c_2\exp\left(-\theta\sqrt{\frac{2(g+\ddot{\alpha})}{L}}\right) \text{ for some } c_1, c_2 \in \mathbb{R}.$$

Notice that for $\theta_0 = 0$, $\dot{\theta}$ has a sinusoidal equation while for $\theta_0 = \pi$ it has an exponential equation. This indicates that the pendulum is stable when it is close to hanging straight down and unstable when it is held straight up.

In order to further analyze the stability of the system we consider the differences between the acceleration due to gravity and that of a person's head while jogging. People typically jog at a constant speed, that is the striding motion that occurs with each step causing the vertical perturbations of the head, making it a roughly constant speed as well. This would mean that $\ddot{\alpha}$ is close to 0. We discuss the implications of this assumption in the next section.

3.3 Nondimensionalization and the relationship between the two angular frequencies

To understand an interesting property of this system, we will consider when $|\ddot{\alpha}|$ is small enough so that the ponytail pendulum is close enough to simple harmonic motion. That is $\omega_\theta = \sqrt{\frac{2g}{L}}$. We must also consider the angular frequency of α, which we will denote as ω_α. By dividing these squares of these frequencies, we attain the following dimensionless value.

$$\frac{\omega_\theta^2}{\omega_\alpha^2} = \frac{2g}{L\omega_\alpha^2}$$

Dr. Keller obtains the same parameter through analysis of the G.W. Hill's work and that it has a particularly interesting result. When $\frac{2g}{L\omega_\alpha^2} = \frac{k^2}{4}$ for some $k \in \mathbb{Z}$ it follows that

$\frac{k}{2}\omega_\alpha = \sqrt{\frac{2g}{L}} = \omega_\theta$. That is, the angular frequency of the α function is proportional to the natural angular frequency of the ponytail pendulum by some factor $\frac{k}{2}$. For example, when this condition is met, it is possible for the ponytail to oscillate half as often as the head. Thus we have periodic instability in the system.

4 Conclusions

4.1 Future Work

Joseph Keller's equation has tremendous application potential. When Keller derived this equation, he was only looking to calculate the forces that shape and move the jogger's ponytail. However, a hanging ponytail is not the only object that behaves this way. In fact, there are many other circumstances where an object may behave in the same way. For instance, recall a time driving in a large car on a bumpy and uneven street. The front portion of the car for the most part is moving up and down; however, the body of the car is moving side-to-side similarly to the jogger's hanging ponytail. Hence, a possible future application of this equation can be used to minimize lateral perturbations when dealing with up and down movement of large cargo trucks on bumpy and uneven roads. Moreover, Keller's equation could also be used to help create similar motion in video games and in animation.

References

[1] J. B. Keller. *Ponytail Motion*, SIAM J. Appl. Math., 70:7 (2010), pp. 2667-2672.

[2] Martin Beecher. *The Pendulum Paradigm: Variations on a Theme and the Measure of Heaven and Earth*, Brown Walker Press, 2014, pp. 209-213. Google Books. Web. 10 Dec. 2018