Question: If $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ and

$C = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ then show that $AB = BA = 0$,

$AC = A$ and $CA = C$.

Answer:

$$AB = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+5 & 6+9-15 & 10+15-25 \\ 1+4-5 & -3-12+15 & -5-20+25 \\ -1-3+4 & 3+9-12 & 5+15-20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

$\therefore AB = 0$

$$BA = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3+5 & 3+12-15 & 5+15-20 \\ 2+3-5 & -3-12+15 & -5-15+20 \\ -2-3+5 & 3+12-15 & 5+15-20 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

$\therefore BA = 0$

$\therefore AB = BA = 0$

(showed)

Now,

$$AC = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3-5 & -4-9+10 & -8-12+15 \\ -2-4+5 & 2+12-10 & 4+6-15 \\ 2+3-4 & -2-9+8 & -4-12+12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$$

$= A$

$\therefore AC = A$ (showed)

Now,

$$CA = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+2-4 & -6+8+12 & -10-10+16 \\ -2-3+4 & 3+12-12 & 5+15-16 \\ 2+2-3 & -3-8+9 & -5-10+12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$= C$

$\therefore CA = C$ (showed)

Question: If $A = \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$.

Prove that $A^3 - 4A^2 - A + 4I = 0$

Answer:

$A^2 = AA$

$$= \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 16+6-6 & 24+18-24 & 24+12-18 \\ 4+3-2 & 6+9-8 & 6+6-6 \\ -4-4+3 & -6-12+12 & -6-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{bmatrix}$$

$A^3 = AA^2$

$$= \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 16 & 18 & 18 \\ 5 & 7 & 6 \\ -5 & -6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 64+30-30 & 72+42-36 & 72+36-30 \\ 16+15-10 & 18+21-12 & 18+18-10 \\ -16-20+15 & -18-28+18 & -18-24+15 \end{bmatrix}$$

$$= \begin{bmatrix} 64 & 78 & 78 \\ 21 & 27 & 26 \\ -21 & -28 & -27 \end{bmatrix}$$

Now,

$$A^3 - 4A^2 - A + 4I$$

$$= \begin{bmatrix} 64 & 78 & 78 \\ 21 & 27 & 26 \\ -21 & -28 & -27 \end{bmatrix} - \begin{bmatrix} 64 & 72 & 72 \\ 20 & 28 & 24 \\ -20 & -24 & -20 \end{bmatrix} - \begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 64-64-4+4 & 78-72-6+0 & 78-72-6+0 \\ 21-20-1+0 & 27-28-3+4 & 26-24-2+0 \\ -21+20+1+0 & -28+24+4+0 & -27+20+3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

$\therefore A^3 - 4A^2 - A + 4I = 0$ (proved)

Question: Verify that $(AB)^T = B^T A^T$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 4 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & -1 \\ -3 & 2 & 4 \\ 1 & 1 & 0 \end{bmatrix}$$

Answer:

$$AB = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 4 & 9 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ -3 & 2 & 4 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0-3+1 & 1+2+1 & -1+4+0 \\ 0-6+3 & 2+4+3 & -2+8+0 \\ 0-12+9 & 2+8+9 & -2+16+0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 4 & 3 \\ -3 & 9 & 6 \\ -3 & 19 & 14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} -2 & -3 & -3 \\ 4 & 9 & 19 \\ 3 & 6 & 14 \end{bmatrix}$$

Now,

$$B^T = \begin{bmatrix} 0 & -3 & 1 \\ 1 & 2 & 1 \\ -1 & 4 & 0 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

$$\therefore B^T A^T = \begin{bmatrix} 0 & -3 & 1 \\ 1 & 2 & 1 \\ -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0-3+1 & 0-6+3 & 0-12+9 \\ 1+2+1 & 2+4+3 & 2+8+9 \\ -1+4+0 & -2+8+0 & -2+16+0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -3 & -3 \\ 4 & 9 & 19 \\ 3 & 6 & 14 \end{bmatrix}$$

$$= (AB)^T$$

$$\therefore (AB)^T = B^T A^T$$

Question: Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

Answer:

Let D be the determinant of the matrix

$$\therefore D = \begin{vmatrix} 2 & -1 & -1 \\ 1 & -2 & 1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(-4+1) + (-1)(1-2) - 1(-1+2)$$

$$= -6 \neq 0$$

So, the determinate is non-zero. therefore A^{-1} exists.

Now, cofectors of A.

$$A_{11} = \begin{vmatrix} -2 & 1 \\ -1 & 2 \end{vmatrix} = -3$$

$$A_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

$$A_{13} = \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} = 1$$

$$A_{21} = -\begin{vmatrix} -1 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$A_{22} = \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5$$

$$A_{23} = -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = 1$$

$$A_{31} = \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix} = -3$$

$$A_{32} = -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -3$$

$$A_{33} = \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = -3$$

$$\therefore Adj.\ A = \begin{bmatrix} -3 & -1 & 1 \\ 3 & 5 & 1 \\ -3 & -3 & -3 \end{bmatrix}^T$$

$$= \begin{bmatrix} -3 & 3 & -3 \\ -1 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} Adj.A$$

$$= \frac{1}{-6}\begin{bmatrix} -3 & 3 & -3 \\ -1 & 5 & -3 \\ 1 & 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{6} & -\frac{5}{6} & \frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \end{bmatrix}$$

Question: If $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$

show that $(A+B)^2 \neq A^2 + 2AB + B^2$

Answer:

$$A+B = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & 7 \\ 6 & 6 & 6 \\ 5 & 8 & 4 \end{bmatrix}$$

$$\therefore (A+B)^2 = (A+B)(A+B)$$

$$= \begin{bmatrix} 3 & 5 & 7 \\ 6 & 6 & 6 \\ 5 & 8 & 4 \end{bmatrix} \begin{bmatrix} 3 & 5 & 7 \\ 6 & 6 & 6 \\ 5 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9+30+35 & 15+30+56 & 21+30+28 \\ 18+36+30 & 30+36+48 & 42+36+24 \\ 15+48+20 & 25+48+32 & 35+48+16 \end{bmatrix}$$

$$= \begin{bmatrix} 74 & 101 & 79 \\ 84 & 114 & 102 \\ 83 & 105 & 99 \end{bmatrix}$$

$$A^2 = AA$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4+12+8 & 6+9+16 & 8+6+12 \\ 8+12+4 & 12+9+8 & 16+6+6 \\ 4+16+6 & 6+12+12 & 8+8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 31 & 26 \\ 24 & 29 & 28 \\ 26 & 30 & 25 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \\ 2 & 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2+6+12 & 4+9+16 & 6+12+4 \\ 4+6+6 & 8+9+8 & 12+12+2 \\ 2+8+9 & 4+12+12 & 6+16+3 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 29 & 22 \\ 16 & 25 & 26 \\ 19 & 28 & 25 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 2+6+12 & 3+8+3 \\ 2+6+12 & 4+9+16 & 6+12+4 \\ 3+8+3 & 6+12+4 & 9+16+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 20 & 14 \\ 20 & 29 & 22 \\ 14 & 22 & 26 \end{bmatrix}$$

$$R.H.S = A^2 + 2AB + B^2$$

$$= \begin{bmatrix} 24 & 31 & 26 \\ 24 & 29 & 28 \\ 26 & 30 & 25 \end{bmatrix} + 2\begin{bmatrix} 20 & 29 & 22 \\ 16 & 25 & 26 \\ 19 & 28 & 25 \end{bmatrix} + \begin{bmatrix} 14 & 20 & 14 \\ 20 & 29 & 22 \\ 14 & 22 & 26 \end{bmatrix}$$

$$= \begin{bmatrix} 78 & 109 & 89 \\ 76 & 108 & 102 \\ 78 & 108 & 101 \end{bmatrix}$$

$$\therefore (A+B)^2 \neq A^2 + 2AB + B^2$$

(showed)

Question: Find the inverse of each of the following matrices by using only the elementary row operation (row equivalent canonical matrix)

(i). $\begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 3 & -1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 1 \\ -1 & 5 & 1 \end{bmatrix}$

Answer:

(i) $$[A \; I_3] = \left[\begin{array}{ccc:ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3/2 & -2 & 1/2 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ -3 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \quad R_1' = R_1/2$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 5 & -1/2 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \quad R_2' = R_2 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 5 & -1/2 & 1 & 0 \\ 3 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \quad \cancel{R_3' = R_3 - 3R_1}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 5 & -1/2 & 1 & 0 \\ 0 & -11/2 & 5 & -3/2 & 0 & 1 \end{array}\right] \quad R_3' = R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1 & 10 & -1 & 2 & 0 \\ 0 & -11/2 & 5 & -3/2 & 0 & 1 \end{array}\right] \quad R_2' = R_2 \times 2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1 & 10 & -1 & 2 & 0 \\ 0 & 0 & 60 & -7 & 11 & 1 \end{array}\right] \quad R_3' = R_3 - \tfrac{11}{2} R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1 & 10 & -1 & 2 & 0 \\ 0 & 0 & 1 & -\frac{7}{60} & 11/60 & 1/60 \end{array}\right] \quad R_3' = R_3/60$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3/2 & -2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/6 & 1/6 & -1/6 \\ 0 & 0 & 1 & -\frac{7}{60} & \frac{11}{60} & 1/60 \end{array}\right] \quad R_2' = R_2 - 10R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3/2 & 0 & \frac{4}{15} & \frac{11}{10} & \frac{1}{30} \\ 0 & 1 & 0 & 1/6 & 1/6 & -1/6 \\ 0 & 0 & 1 & -7/60 & 11/60 & 1/60 \end{array}\right] \quad R_1' = R_1 + 2R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/60 & 7/60 & 17/60 \\ 0 & 1 & 0 & 1/6 & 1/6 & -1/6 \\ 0 & 0 & 1 & -7/60 & 11/60 & 1/60 \end{array}\right] \quad R_1' = R_1 - \tfrac{3}{2}R_2$$

$$= [I_3, A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 1/60 & 7/60 & 17/60 \\ 1/6 & 1/6 & -1/6 \\ -7/60 & 11/60 & 1/60 \end{bmatrix}$$

$$= \frac{1}{60} \begin{bmatrix} 1 & 7 & 17 \\ 10 & 10 & -10 \\ -7 & 11 & 1 \end{bmatrix}$$

(ii) $[A\,I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 3 & -1 & 6 & 0 & 1 & 0 \\ -1 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 10 & -6 & -3 & 1 & 0 \\ -1 & 5 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2' = R_2 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -10 & -6 & -3 & 1 & 0 \\ 0 & 8 & 5 & 1 & 0 & 1 \end{array}\right] \quad R_3' = R_3 + R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3/5 & 3/10 & -1/10 & 0 \\ 0 & 8 & 5 & 1 & 0 & 1 \end{array}\right] \quad R_2' = R_2/-10$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3/5 & \frac{3}{10} & \frac{-1}{10} & 0 \\ 0 & 0 & 1/5 & -\frac{7}{5} & 4/5 & 1 \end{array}\right] \quad R_3' = R_3 - 8R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 3/5 & \frac{3}{10} & -\frac{1}{10} & 0 \\ 0 & 0 & 1 & -7 & 4 & 5 \end{array}\right] \quad R_3' = 5R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{9}{2} & -\frac{5}{2} & -3 \\ 0 & 0 & 1 & -7 & 4 & 5 \end{array}\right] \quad R_2' = R_2 - \frac{3}{5}R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 3 & 0 & 29 & -16 & 20 \\ 0 & 1 & 0 & \frac{9}{2} & -\frac{5}{2} & -3 \\ 0 & 0 & 1 & -7 & 4 & 5 \end{array}\right] \quad R_1' = R_1 - 4R_3$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 31/2 & -17/2 & -11 \\ 0 & 1 & 0 & 9/2 & -5/2 & -3 \\ 0 & 0 & 1 & -7 & 4 & 5 \end{array}\right] \quad R_1' = R_1 - 3R_2$$

$$= [I_3 \; A^{-1}]$$

$$\therefore A^{-1} = \begin{bmatrix} 31/2 & -17/2 & -11 \\ \frac{9}{2} & -5/2 & -3 \\ -7 & 4 & 5 \end{bmatrix}$$

Question: If $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ 1 & -1 & 1 \end{bmatrix}$ then find $A^3 - 2A^2 - 9A - 2I$ and hence find A^{-1}

Answer:

$A^2 = A.A$

$$= \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+2 & 3+0-2 & 2+9+2 \\ 2+0+3 & 6+0-3 & 4+0+3 \\ 1-2+1 & 3+0-1 & 2-3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 1 & 13 \\ 5 & 3 & 7 \\ 0 & 2 & 0 \end{bmatrix}$$

$A^3 = A^2 A$

$$= \begin{bmatrix} 9 & 1 & 13 \\ 5 & 3 & 7 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 3 \\ 1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9+2+13 & 27+0-13 & 18+13+13 \\ 5+6+7 & 15+0-7 & 10+9+7 \\ 0+4+0 & 0+0+0 & 0+6+0 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 14 & 34 \\ 18 & 8 & 26 \\ 4 & 0 & 6 \end{bmatrix}$$

$\therefore A^3 - 2A^2 - 4A - 2I$

$$= \begin{bmatrix} 24 & 14 & 34 \\ 18 & 8 & 26 \\ 4 & 0 & 6 \end{bmatrix} - \begin{bmatrix} 18 & 2 & 26 \\ 10 & 6 & 14 \\ 0 & 4 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 12 & 8 \\ 8 & 0 & 12 \\ 4 & -4 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 24-18-4-2 & 14-2-12-0 & 34-26-8-0 \\ 18-10-8-0 & 8-6-0-2 & 26-14-12-0 \\ 4-0-4-0 & 0-4+4-0 & 6-0-4-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$

$\therefore A^3 - 2A^2 - 4A - 2I = 0$

Now,

$$A^{-1}A^{3} - 2A^{2}A^{-1} - 4AA^{-1} - 2A^{-1} = 0$$

$$\Rightarrow A^{2} - 2A - 4I - 2A^{-1} = 0 \quad [\because AA^{-1} = I]$$

$$\therefore A^{-1} = \frac{1}{2}(A^{2} - 2A - 4I)$$

Now,

$$A^{2} - 2A - 4I$$

$$= \begin{bmatrix} 9 & 1 & 13 \\ 5 & 3 & 7 \\ 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 2 & 6 & 4 \\ 4 & 0 & 6 \\ 2 & -2 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9-2-4 & 1-6-0 & 13-4-0 \\ 5-4-0 & 3-0-4 & 7-6-0 \\ 0-2-0 & 2+2-0 & 0-2-4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -5 & 9 \\ 1 & -1 & 1 \\ -2 & 4 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -5 & 9 \\ 1 & -1 & 1 \\ -2 & 4 & -6 \end{bmatrix}$$

Question: Solve the following system of linear equation with the help of matrices:

$$3x + 2y - z = 20$$
$$2x + 3y - 3z = 7$$
$$x - y + 6z = 41$$

Answer:

Rearrange the given system of liner equation in matrix form:

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -3 \\ 1 & -1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 7 \\ 41 \end{bmatrix}$$

$\Rightarrow AX = L \;\; \ldots\ldots (1)$

where,

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -3 \\ 1 & -1 & 6 \end{bmatrix}; \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; L = \begin{bmatrix} 20 \\ 7 \\ 41 \end{bmatrix}$$

from $(1) \Rightarrow$

$$AX = L$$
$$AA^{-1}X = A^{-1}L$$
$$\therefore X = A^{-1}L$$

The argumented matrix of given linear equation

$$[AL] = \begin{bmatrix} 3 & 2 & -1 & : & 20 \\ 2 & 3 & -3 & : & 7 \\ 1 & -1 & 6 & : & 41 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & : & \frac{20}{3} \\ 2 & 3 & -3 & : & 7 \\ 1 & -1 & 6 & : & 41 \end{bmatrix} \qquad R_1' = R_1/3$$

$$\sim \begin{bmatrix} 1 & 2/3 & -1/3 & : & 20/3 \\ 0 & 5/3 & -1/7 & : & -19/3 \\ 1 & -1 & 6 & : & 41 \end{bmatrix} \qquad R_2' = R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 2/3 & -1/3 & : & 20/3 \\ 0 & 5/3 & -7/3 & : & -19/3 \\ 0 & -5/3 & 19/3 & : & 103/3 \end{bmatrix} \qquad R_3' = R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2/3 & -1/3 & : & 20/3 \\ 0 & 1 & -7/3 & : & -19/5 \\ 0 & -5/3 & 19/3 & : & 103/3 \end{bmatrix} \qquad R_1 = \frac{3}{5} R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 3/5 & 46/5 \\ 0 & 1 & -7/5 & -19/5 \\ 0 & -5/3 & 19/3 & 103/3 \end{array}\right] \quad R_1' = R_1 - \frac{2}{3}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 3/5 & 46/5 \\ 0 & 1 & -7/5 & -19/5 \\ 0 & 0 & 4 & 28 \end{array}\right] \quad R_3' = R_3 + \frac{5}{3}R_2$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 3/5 & 46/5 \\ 0 & 1 & -7/5 & -19/5 \\ 0 & 0 & 1 & 7 \end{array}\right] \quad R_3' = R_3/4$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 1 & -7/5 & -19/5 \\ 0 & 0 & 1 & 7 \end{array}\right] \quad R_1' = R_1 - 3/5\, R_3$$

$$\sim \left[\begin{array}{ccc:c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right] \quad R_2' = R_2 + \frac{7}{5} R_3$$

$\therefore \cancel{A^{-1}A} X = IX$

$\therefore X = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$

$\therefore x = 5, \; y = 6, \; z = 7$

Question: Solve the following system of linear equations with the help of matrices:

(i) $5x - 6y + 4z = 15$

$7x + 4y - 3z = 19$

$2x + y + 6z = 46$

(ii) $x + 2y - z = -1$

$3x + 8y + 2z = 28$

$4x + 9y - z = 14$

(i)

Rearrange the following systems of linear eqn. in matrix form:

$$\begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$\Rightarrow AX = L$

$\Rightarrow X = A^{-1} L$

where, $A = \begin{bmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $L = \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$

Let,

D be the determinant of the matrix then,

$$D = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix}$$

$$= 5(24+3) + 6(42+6) + 4(7-8)$$

$$= 419 \neq 0$$

So, the matrix is no-singular and A^{1} is exists

Now, the cofactors of D are

$$A_{11} = \begin{vmatrix} 4 & -3 \\ 1 & 6 \end{vmatrix} = 27$$

$$A_{12} = -\begin{vmatrix} 7 & -3 \\ 2 & 6 \end{vmatrix} = -48$$

$$A_{13} = \begin{vmatrix} 7 & 4 \\ 2 & 1 \end{vmatrix} = -1$$

$$A_{21} = -\begin{vmatrix} -6 & 4 \\ 1 & 6 \end{vmatrix} = 40$$

$$A_{22} = \begin{vmatrix} 5 & 4 \\ 2 & 6 \end{vmatrix} = 22$$

$$A_{23} = -\begin{vmatrix} 5 & -6 \\ 2 & 1 \end{vmatrix} = -17$$

$$A_{31} = \begin{vmatrix} -6 & 4 \\ 4 & -3 \end{vmatrix} = 2$$

$$A_{32} = -\begin{vmatrix} 5 & 4 \\ 7 & -3 \end{vmatrix} = 43$$

$$A_{33} = \begin{vmatrix} 5 & -6 \\ 7 & 4 \end{vmatrix} = 62$$

Now,

$$\text{Adj}.\, A = \begin{bmatrix} 27 & -48 & -1 \\ 40 & 22 & -17 \\ 2 & 43 & 62 \end{bmatrix}^T = \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} \text{Adj}.A$$

$$= \frac{1}{419} \begin{bmatrix} 27 & 40 & 2 \\ -48 & 22 & 43 \\ -1 & -17 & 62 \end{bmatrix}$$

$$X = A^{-1} L$$

$$= \begin{bmatrix} 27/419 & 40/419 & 2/419 \\ -48/419 & 22/419 & 43/419 \\ -1/419 & -17/419 & 62/419 \end{bmatrix} \begin{bmatrix} 15 \\ 19 \\ 46 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{405}{419} + \frac{760}{419} + \frac{92}{419} \\[2ex] -\frac{720}{419} + \frac{418}{419} + \frac{1978}{419} \\[2ex] -\frac{15}{419} - \frac{323}{419} + \frac{2852}{419} \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$\therefore x = 3, y = 4, z = 6$

(ii) Rearrange the following system of linear equation in matrix form

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 28 \\ 14 \end{bmatrix}$$

$$\Rightarrow AX = L \quad \dots (1)$$

where,

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{bmatrix} ; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; L = \begin{bmatrix} -1 \\ 28 \\ 14 \end{bmatrix}$$

From (1) $\Rightarrow$

$$AX = L$$

$$\Rightarrow X = A^{-1} L$$

Let, D be the determinant of the matrix, then

$$D = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 8 & 2 \\ 4 & 9 & -1 \end{vmatrix}$$

$$= 1(-8-18) - 2(-3-8) + 1(27-32)$$

$$= 1 \neq 0$$

So, A^{-1} exists.

Cofectors of D are,

$$A_{11} = \begin{vmatrix} 8 & 2 \\ 9 & -1 \end{vmatrix} = -26$$

$$A_{12} = -\begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix} = 11$$

$$A_{13} = \begin{vmatrix} 3 & 8 \\ 4 & 9 \end{vmatrix} = -5$$

$$A_{21} = -\begin{vmatrix} 2 & -1 \\ 9 & -1 \end{vmatrix} = -7$$

$$A_{22} = \begin{vmatrix} 1 & -1 \\ 4 & -1 \end{vmatrix} = 3$$

$$A_{23} = -\begin{vmatrix} 1 & 2 \\ 4 & 9 \end{vmatrix} = -1$$

$$A_{31} = \begin{vmatrix} 2 & -1 \\ 8 & 2 \end{vmatrix} = 12$$

$$A_{32} = -\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = -5$$

$$A_{33} = \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2$$

Therefore,

$$Adj.A = \begin{bmatrix} -26 & 11 & -5 \\ -7 & 3 & -1 \\ 12 & -5 & 2 \end{bmatrix}^T = \begin{bmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{D} Adj\, A$$

$$= \frac{1}{1} \begin{bmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1} L$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -26 & -7 & 12 \\ 11 & 3 & -5 \\ -5 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 28 \\ 14 \end{bmatrix}$$

$$= \begin{bmatrix} 26 - 196 + 168 \\ -11 + 84 - 70 \\ 5 - 28 + 28 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$$

$\therefore x = -2,\ y = 3,\ z = 5$

Question: Find the eigen values and eigen vectors for the matrices $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 5 & 4 \\ 4 & 4 & 3 \end{bmatrix}$

Answer:

The characteristic equation of the matrix is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & 5-\lambda & 4 \\ 4 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(5-\lambda)(3-\lambda)-16\} + 4\{-4(5-\lambda)\} = 0$$

$$\Rightarrow (1-\lambda)(15-3\lambda-5\lambda+\lambda^2-16) - 80 + 16\lambda = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2-8\lambda-1) - 80 + 16\lambda = 0$$

$$\Rightarrow \lambda^2 - 8\lambda - 1 - \lambda^3 + 8\lambda^2 + \lambda - 80 + 16\lambda = 0$$

$$\Rightarrow 9\lambda^2 - \lambda^3 + 9\lambda - 81 = 0$$

$$\Rightarrow \lambda^2(\lambda-9) - 9(\lambda-9) = 0$$

$$\Rightarrow (\lambda-9)(\lambda^2-9) = 0$$

$\therefore \lambda = 9, \pm 3$

which are the eigen value of the matrix.

for, $\lambda = 9$, we seek a non-vector v such that

$$\begin{bmatrix} -8 & 0 & 4 \\ 0 & -4 & 4 \\ 4 & 4 & -6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, the system eq becomes,

$$-8v_1 + 4v_2 = 0$$

$$-4v_2 + 4v_3 = 0 \quad \Rightarrow \quad v_2 - v_3 = 0$$

$$4v_1 + 4v_2 - 6v_3 = 0$$

Let,

$v_2 = a$

$\therefore v_3 = a$

$\therefore v_1 = \frac{v_3}{2} = \frac{a}{2}$

Therefore, $v = \begin{bmatrix} a/2 \\ a \\ a \end{bmatrix}$ is the eigen vectors corresponding to $\lambda = 9$

Let, $a = 2$, then $v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

for $\lambda = 3$, we seek a non-vector v such that

$$\begin{bmatrix} -2 & 0 & 4 \\ 0 & 2 & 4 \\ 4 & 4 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, the system becomes

$$-2v_1 + 4v_3 = 0 \Rightarrow v_1 - 2v_3 = 0$$

$$2v_2 + 4v_3 = 0 \Rightarrow v_2 + 2v_3 = 0$$

$$4v_1 + 4v_2 = 0 \Rightarrow v_1 = -v_2$$

let,

$$v_1 = a$$

$$\therefore v_2 = -a$$

$$v_3 = \frac{a}{2}$$

Therefore, $v = \begin{bmatrix} a \\ -a \\ a/2 \end{bmatrix}$ is the ~~corr~~ eigen vector for $\lambda = 3$

let, $a = 2$ then, $v = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

for $\lambda = 3$, wee seek a non-vector v such that

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 8 & 4 \\ 4 & 4 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, system becomes

$$4v_1 + 4v_3 = 0$$

$$\Rightarrow v_1 + v_3 = 0$$

$$8v_2 + 4v_3 = 0$$

$$\Rightarrow 2v_2 + v_3 = 0$$

$$4v_1 + 4v_2 + 6v_3 = 0$$

Let,

$$v_1 = 0$$

$$v_3 = a$$

$$v_2 = -\frac{a}{2}$$

Therefor, $v = \begin{bmatrix} a \\ -a/2 \\ -a \end{bmatrix}$ is the eigenvectors for $\lambda = -3$

Let, $a = 2$ then $v = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\therefore \lambda = 9, 3, -3$

and corresponding eigenvectors are

$\lambda = 9$ is $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$; $\lambda = 3$ is $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$

$\lambda = -3$ is $\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

Question: Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$

Answer:

The characteristics equation of the matrix is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(4-\lambda)(-5-\lambda)+18) + 3\{3(4-\lambda)-18\} + 3\{-18-6(-5-\lambda)\} = 0$$

$$\Rightarrow (1-\lambda)\{-20+5\lambda-4\lambda+\lambda^2+18\} + 3\{12-3\lambda-18\} + 3\{-18+30-6\lambda\} = 0$$

$$\Rightarrow (1-\lambda)(\lambda^2+\lambda-2) + 3(-9\lambda-54-54+90+18\lambda = 0$$

$$\Rightarrow \lambda^2-\lambda-2-\lambda^3-\lambda^2+2\lambda+9\lambda+18 = 0$$

$$\Rightarrow -\lambda^3+12\lambda+16 = 0$$

$\therefore \lambda = -2, 4$ which are the eigen values of the matrix.

for $\lambda = -2$, we seek a non-vector v such that

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now the system becomes,

$$3v_1 - 3v_2 + 3v_3 = 0$$

$$\Rightarrow v_1 - v_2 + v_3 = 0$$

$$3v_1 - 3v_2 + 3v_3 = 0$$

$$\Rightarrow v_1 - v_2 + v_3 = 0$$

$$6v_1 - 6v_2 + 6v_3 = 0$$

$$\Rightarrow v_1 - v_2 + v_3 = 0$$

The system is constant as it has more than one solution.

If $v_1 = 1, v_2 = 1$ then $v_3 = 0$

and $v_1 = 1, v_2 = 0$ then $v_3 = -1$

Therefore, $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are two eigen vectors corresponding to $\lambda = -2$

for $\lambda = 4$, we seek a non-vector v such that

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ -8 & -6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, the system becomes

$$v_1 + v_2 - v_3 = 0$$

$$v_1 - 3v_2 + v_3 = 0$$

$$v_1 - v_2 = 0$$

Let,

$v_1 = a$

$v_2 = a$

$v_3 = 2a$

Therefore $v = \begin{bmatrix} a \\ a \\ 2a \end{bmatrix}$ is the eigenvectors for $\lambda = 4$

Let $a = 1$ then, $v = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Eigen values are, $\lambda = -2, 4$

eigenvectors corresponding $\lambda = -2$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

and $\lambda = 4$ is $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Question: Find the characteristic equation of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$ and verify Cayley-Hamilton theorem for it.

Answer:

The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\{(1-\lambda)(-1-\lambda) - 4\} - 2\{2(1-\lambda) - 12\} + 3\{2 - 3(-1-\lambda)\} = 0$

$\Rightarrow (1-\lambda)\{-1-\lambda+\lambda+\lambda^2-4\} - 2\{2-2\lambda-12\} + 3\{2+2+3\lambda\} = 0$

$\Rightarrow (1-\lambda)(\lambda^2-5) - 4 + 4\lambda + 24 + 6 + 9 + 9\lambda = 0$

$\Rightarrow \lambda^2 - 5 - \lambda^3 + 5\lambda + 13\lambda + 35 = 0$

$\Rightarrow -\lambda^3 + \lambda^2 + 18\lambda + 30 = 0$

$\Rightarrow \lambda^3 - \lambda^2 - 18\lambda - 30 = 0$

which is the required polynomial equation

To verify Caylay-Hamilton theorem, we need to show that $A^3 - A^2 - 18A - 30I = 0$

Now,

$$A^2 = AA$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 2-2+3 & 3+8+3 \\ 2-2+12 & 4+1+4 & 6-4+4 \\ 3+2+3 & 6-1+1 & 9+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14+6+12 & 28-3+14 & 42+12+14 \\ 12+18+18 & 24-9+6 & 36+36+6 \\ 8+12+42 & 16-6+14 & 24+24+14 \end{bmatrix}$$

$$= \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix}$$

$\therefore A^3 - A^2 - 18A - 30I$

$$= \begin{bmatrix} 62 & 39 & 68 \\ 48 & 21 & 78 \\ 62 & 24 & 62 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 14 \\ 12 & 9 & 6 \\ 8 & 6 & 14 \end{bmatrix} - 18\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 30 \end{bmatrix}$$

$$= \begin{bmatrix} 62-14-18-30 & 39-3-36-0 & 68-14-54-0 \\ 48-12-36-0 & 21-9+18-30 & 78-6-72-0 \\ 62-8-54-0 & 24-6-18-0 & 62-14-18-30 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$

$\therefore A^3 - A^2 - 18A - 30I = 0$

(verified)

Question: Using Cayley-Hamilton theorem find the inverse of each of the following matrices:

(i) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 3 \\ -1 & -1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$

Answer:

(i) The characteristic equation of the matrix is

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 4-\lambda & 3 \\ -1 & -1 & 0-\lambda \end{vmatrix}=0$$

$\Rightarrow (2-\lambda)\{(4-\lambda)(-\lambda)+3\} -1(-\lambda+3) +1\{-1+(4-\lambda)\}=0$

$\Rightarrow (2-\lambda)\{-4\lambda+\lambda^2+3\} + \lambda -3+3-\lambda=0$

$\Rightarrow 2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda=0$

$\Rightarrow -\lambda^3+6\lambda^2-11\lambda+6=0$

$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0$

using cayley Hamilton theorem we got,

$$A^3 - 6A^2 + 11A - 6I = 0$$

$$\Rightarrow A^2 - 6A + 11I - 6A^{-1} = 0$$

$$\Rightarrow 6A^{-1} = A^2 - 6A + 11I$$

Now,

$$A^2 = AA$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 3 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 3 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1-1 & 2+4-1 & 2+3+0 \\ 2+4-3 & 1+16-3 & 1+12+0 \\ -2-1-0 & -1-4-0 & -1-3+0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 5 \\ 3 & 14 & 13 \\ -3 & -5 & -4 \end{bmatrix}$$

$$\therefore 6A^{-1} = \begin{bmatrix} 4 & 5 & 5 \\ 3 & 14 & 13 \\ -3 & -5 & -4 \end{bmatrix} - 6 \begin{bmatrix} 2 & 1 & 1 \\ 1 & 4 & 3 \\ -1 & -1 & 0 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 & -1 \\ -3 & 1 & -5 \\ 3 & 1 & 7 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{6} \begin{bmatrix} 3 & -1 & -1 \\ -3 & 1 & -5 \\ 3 & 1 & 7 \end{bmatrix}$$

(ii) The characteristics equation for the matrix

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & 3 \\ 1 & 3-\lambda & 3 \\ 1 & 2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\{(3-\lambda)(4-\lambda) - 6\} + 3(3-4+\lambda) + 3(2-3+\lambda) = 0$

$\Rightarrow (1-\lambda)(12-4\lambda-3\lambda+\lambda^2) + -3+3\lambda-3+3\lambda = 0$

$\Rightarrow (1-\lambda)(12-7\lambda+\lambda^2) + 6\lambda - 6 = 0$

$\Rightarrow 12 - 7\lambda + \lambda^2 - 12\lambda + 7\lambda^2 - \lambda^3 + 6\lambda - 6 = 0$

$\Rightarrow -\lambda^3 + 8\lambda^2 - 13\lambda + 6 = 0$

$\Rightarrow \lambda^3 - 8\lambda^2 + 13\lambda - 6 = 0$

using Caylay Hamilton theorem we get,

$$A^3 - 8A^2 + 13A - 6 = 0$$

$$\therefore A^2 - 8A + 13I - 6A^{-1} = 0$$

$$\therefore 6A^{-1} = A^2 - 8A + 13I$$

$A^2 = AA$

$$= \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3+3 & 3+9+6 & 3+9+12 \\ 1+3+3 & 3+9+6 & 3+9+12 \\ 1+2+4 & 3+6+8 & 3+6+16 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 18 & 24 \\ 7 & 18 & 24 \\ 7 & 17 & 25 \end{bmatrix}$$

$\therefore 6A^{-1} = A^2 - 8A + 13I$

$$= \begin{bmatrix} 7 & 18 & 24 \\ 7 & 18 & 24 \\ 7 & 17 & 25 \end{bmatrix} - \begin{bmatrix} 8 & 24 & 24 \\ 8 & 24 & 24 \\ 8 & 16 & 32 \end{bmatrix} + \begin{bmatrix} 13 & 0 & 0 \\ 0 & 13 & 0 \\ 0 & 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 7-8+13 & 18-24+0 & 24-24+0 \\ 7-8+0 & 18-24+13 & 24-24+0 \\ 7-8+0 & 17-16+0 & 25-32+13 \end{bmatrix}$$

$$A^{-1} = \frac{1}{6}\begin{bmatrix} 12 & -6 & 0 \\ -1 & 7 & 0 \\ -1 & 1 & 6 \end{bmatrix}$$

Question: Verify Calay-Hamilton theorem for the matrix A where $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 3 & 2 \end{bmatrix}$

Answer:

The characteristic equation for the matrix

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 & -2 \\ 1 & -\lambda & 3 \\ -2 & 3 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)\{(-\lambda)(2-\lambda) - 9\} + 1(-6 - 2 + \lambda) - 2(3 - 2\lambda) = 0$

$\Rightarrow (1-\lambda)(-2\lambda + \lambda^2 - 9) - 8 + \lambda - 6 + 4\lambda = 0$

$\Rightarrow -2\lambda + \lambda^2 - 9 + 2\lambda^2 - \lambda^3 + 9\lambda - 6 + 4\lambda = 0$

$\Rightarrow -\lambda^3 + 3\lambda^2 + 12\lambda - 23 = 0$

$\Rightarrow \lambda^3 - 3\lambda^2 - 12\lambda + 23 = 0$

using Caylay-Hamilton theorem we got,

$$A^3 - 3A^2 - 12A + 23I = 0$$

we need to showe that $A^3-3A^2-12A+23I=0$

Now,

$$A^2 = AA$$

$$= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+4 & 1+0-6 & -2+3-4 \\ 1+0-6 & 1+0+9 & -2+0+6 \\ -2+3-4 & -2+0+6 & 4+9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & -3 \\ -5 & 10 & 4 \\ -3 & 4 & 17 \end{bmatrix}$$

$$A^3 = A^2 A$$

$$= \begin{bmatrix} 6 & -5 & -3 \\ -5 & 10 & 4 \\ -3 & 4 & 17 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6-5+6 & 6+0-9 & -12-15-6 \\ -5+10-8 & -5+0+12 & 10+30+8 \\ -3+4-34 & -3+0+51 & -6+12+34 \end{bmatrix}$$

Now,

$\cancel{A^3 - 3A^2 - 12A + 23I}$

$$A^3 = \begin{bmatrix} 7 & -3 & -33 \\ -3 & 7 & 48 \\ -33 & 48 & 52 \end{bmatrix}$$

Now

$\therefore \quad A^3 - 3A^2 - 12A + 23I$

$$= \begin{bmatrix} 7 & -3 & -33 \\ -3 & 7 & 48 \\ -33 & 48 & 52 \end{bmatrix} - \begin{bmatrix} 18 & -15 & -9 \\ -15 & 30 & 12 \\ -9 & 12 & 51 \end{bmatrix} - \begin{bmatrix} 12 & 12 & -24 \\ 12 & 0 & 36 \\ -24 & 36 & 24 \end{bmatrix} + \begin{bmatrix} 23 & 0 & 0 \\ 0 & 23 & 0 \\ 0 & 0 & 23 \end{bmatrix}$$

$$= \begin{bmatrix} 7-18-12+23 & -3+15-12-0 & -33+9+24+0 \\ -3+15-12+0 & 7-30+0+23 & 48-12-36+0 \\ -33+9+24+0 & 48-12-36+0 & 52-51-24+23 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$= 0$ (verified)

Question: Find the characteristic values and characteristic vector of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Answer:

The characteristic equation for the matrix is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-2-\lambda)\{(1-\lambda)(-\lambda) - 12\} + 2(6+2\lambda) - 3(-4+1-\lambda)$$

$$\Rightarrow \cancel{(-2-2\lambda)} - 2\lambda^2 + 24 + \lambda^2 - \lambda^3 + 12\lambda + 12 + 4\lambda + 9 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\Rightarrow \lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\Rightarrow (\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$$

$$\Rightarrow (\lambda+3)(\lambda+3)(\lambda-5) = 0$$

$$\therefore \lambda = -3, -3, 5$$

for $\lambda = -3$, we seek a non vector v such that

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now the system becomes,

$$v_1 + 2v_2 - 3v_3 = 0$$

$$v_1 + 2v_2 - 3v_3 = 0$$

$$v_1 + 2v_2 - 3v_3 = 0$$

The system is constant as it has more than one solution

If $v_1 = -2$, $v_2 = 1$ then $v_3 = 0$

and $v_1 = 3$, $v_2 = 0$ then $v_3 = \cancel{3/3}\ 1$

therefore,

$$v = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \text{ and } v = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$ are corresponding

eigen ~~value~~ vectors for $\lambda = -3$

for $\lambda = \cancel{5}$, we seek a non-vector v such that

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\cancel{-7v_1 + 2v_2 - 3v_3 = 0}$

$\cancel{2v_1 - 4v_2 - 6v_3 = 0}$ . . . . (2)

$\cancel{-v_1 - 2v_2 - 5v_3 = 0}$ . . . . (3)

~~From (2) & (3) ⇒~~

$\cancel{v_3 = 0}$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & 0 & 4 \\ -1 & -2 & -5 \end{bmatrix} \quad R_2' = R_2 + 2R_3$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & 0 & 4 \\ -8 & 0 & -8 \end{bmatrix} \quad R_3' = R_3 + R_1$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & 0 & 4 \\ 1 & 0 & 1 \end{bmatrix} \quad R_3' = \frac{1}{8} R_3$$

Now the ~~system~~ system becomes,

$$-7v_1 + 2v_2 - 3v_3 = 0$$

$$4v_3 = 0$$

$$v_1 + v_3 = 0$$

Let,

$v_3 = a$

$v_1 = -a$

$v_2 = -2a$

$\therefore$ The eigen vectors $\begin{bmatrix} -a \\ -2a \\ a \end{bmatrix}$

Let, $a = -1$, then the vector will be $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\therefore$ Eingen ~~vectors~~ values are $\lambda = -3, -3, 5$

and the characteristics vectors are,

for $\lambda = -3$ is $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

and $\lambda = 5$ is $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Question: Find all characteristic values and characteristic vectors of the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Answer:

The characteristic equation for the matrix A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(3-\lambda)(2-\lambda)-2\} + 2(1-2+\lambda) + 1(2-3+\lambda) = 0$$

$$\Rightarrow (2-\lambda)(6-3\lambda-2\lambda+\lambda^2-2) + 2(-1+\lambda) + (-1+\lambda) = 0$$

$$\Rightarrow (2-\lambda)(6-5\lambda+\lambda^2-2) - 2 + 2\lambda - 1 + \lambda = 0$$

$$\Rightarrow 12 - 10\lambda + \lambda^2 - 4 - 6\lambda + 5\lambda - \lambda^3 - 2\lambda - 3 + 3\lambda = 0$$

$$\Rightarrow -\lambda^3 + 7\lambda^2 - 11\lambda + 5 = 0$$

$$\Rightarrow -(\lambda-1)(\lambda^2-6\lambda+5) = 0$$

$$\Rightarrow -(\lambda-1)(\lambda-1)(\lambda-5) = 0$$

$\therefore \lambda = 1, 1, 5$

for $\lambda = 1$, we seek a non-vector v such that

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2' = R_2 - R_1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3' = R_3 - R_1$$

Now, the system becomes,

$$v_1 + 2v_2 + v_3 = 0$$

Let,

$v_3 = 0$ then, $v_2 = a$, $v_1 = -2a$

$v_2 = 0$ then, $v_3 = a$, $v_1 = -a$

then, vectors will be $\begin{bmatrix} -2a \\ a \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -a \\ 0 \\ a \end{bmatrix}$

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Let, $a = -1$ then, vectors will be, $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Now,

$\lambda = 5$, we seek a non vector v such that

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The system becomes,

$$-3v_1 + 2v_2 + v_3 = 0$$

$$v_1 - 2v_2 + v_3 = 0$$

$$v_1 - 2v_3 - 3v_3 = 0$$

Let,

$v_3 = a$

$v_1 = a$

$v_2 = a$

$\therefore$ Vector will be $\begin{bmatrix} a \\ a \\ a \end{bmatrix}$

Let,

$a = 1$, we got $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$